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MECH 501 Foundations of Solid Mechanics

Course Description:

Introduction of the basic concepts, equations and methods used in solid mechanics; stress, strain and displacement analysis of a deformed continuum (solid); constitutive equations; field equations and solutions for typical solid mechanics problems.

Prerequisites or background: MECH 101, MECH 202.

Textbook:

1. Y. C. Fung, **First Course in Continuum Mechanics**, 3d edition, Prentice Hall International, Inc., 1994.

Reference books:

1. S. Timoshenko and J.N. Goodier, **Theory of Elasticity**, McGraw-Hill, New York, N.Y., 1970.
2. Zhilun Xu, **Applied Elasticity**, John Wiley & Sons, 1992.
3. Y. C. Fung, **Foundations of Solid Mechanics,** Prentice Hall International, Inc., 1977.

Instructor: Prof. Qing-Ping SUN (e-mail: meqpsun; Tel: 8655; Office: Rm. 2545)

Grade Policy: (Homework required, Mid-term 50%; and final-exam 50%)

Contents

1. Introduction (Concepts, definition and basic laws in solid mechanics)
2. Vectors and tensors
3. Stress and stress analysis
4. Analysis of deformation
5. Constitutive equations
6. Isotropy and mechanical properties of several solids
7. Derivation of field equations
8. Solution for some problems in solid mechanics

Chapter 1. Introduction

Most engineering mechanics problems can be reduced to certain differential equations and boundary conditions. By formulating and solving such equations, we obtain precise quantitative solutions of these problems.

1.1 **The objective of this course** is to learn how to formulate problems in ***mechanics*** and how to reduce vague questions and ideas into precise mathematical statements, as well as to cultivate a habit of questioning, analyzing, and inventing in engineering and science. Our objective is formulation: the formal reduction of general ideas to a mathematical form.

1.2 **What is mechanics?** Mechanics is the study of the motion (or equilibrium) of matter and the forces that cause such motion (or equilibrium).

1.3 **What is continuum?**

- The classical definition or mathematical definition: The real number system is a *continuum*, thus we can identify *time* and *space* together as a 4-dimensional *continuum.*
- A *material continuum* is a material for which *the density of mass, momentum, and energy* exist in the *mathematical sense.* For example, we consider a continuous distribution of matter in space -------- the concept of *density* at a point P

$$\rho(P) = \lim_{\substack{n\to\infty \\ V_n \to 0}} \frac{M_n}{V_n}$$

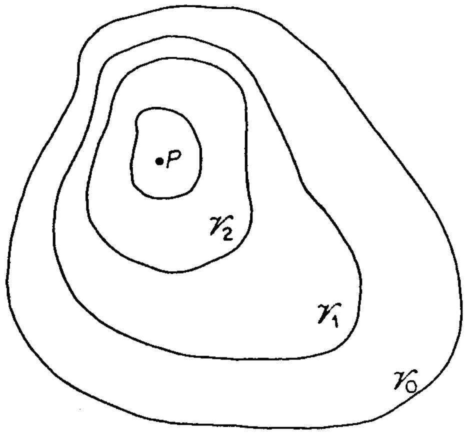


Figure 1.1 A sequence of spatial domains converging on P.

- Our definition of *continuum* for *a real-world system.*

- The *mechanics of such a material continuum* is <u>*continuum mechanics.*</u>

1.4 The concept of stress in continuum

--- *Stress vector or traction (Fig. 1.2)*

$$\overset{\nu}{\mathbf{T}} = \frac{d\mathbf{F}}{d\mathbf{S}}$$

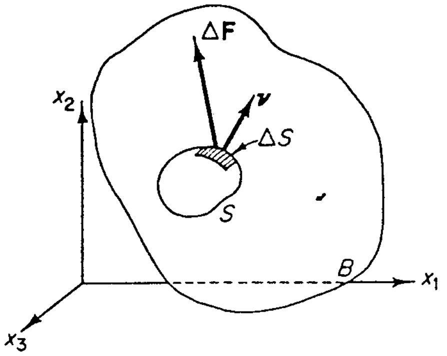


Figure 1.2 Stress principle.

Stress principle of Euler and Cauchy

1.5 Continuum mechanics

- Idealization of real continuum
- *The determination of the internal condition of a body in response to external forces is the objective of the continuum mechanics.*

1.6 Axioms of continuum mechanics:

- *Newton's laws of motion, first and second law of thermodynamics*
- *A material continuum remains continuum under the action of forces*
- *Stress and strain can be defined everywhere in the body*
- *Stress at a point is related* ***only*** *to strain and its rate with respect to time at the same point*

1.7 Some fundamental laws

- ***Newton's laws of motion***

(1) Consider a particle of mass m, let **x**, **v**, and **a** represents the position, velocity and acceleration, respectively, all defined in inertial frame of reference, then

$$\mathbf{v} = \frac{d\mathbf{x}}{dt}, \qquad \mathbf{a} = \frac{d\mathbf{v}}{dt}$$

If the total force **F** acting on the particle, **F**=0, then *Newton's first law* states that

$$\mathbf{v} = const.$$

If **F**≠0, then *Newton's second law* states that

$$\frac{d}{dt} m\mathbf{v} = \mathbf{F}, \qquad or \quad \mathbf{F} = m\mathbf{a}.$$

Or

$$\mathbf{F} + (-m\mathbf{a}) = 0.$$

(D'Alembert's principal)

(2) Consider a system of particles that interact with each other. Let $\mathbf{F_{IJ}}$ denote the force of interaction exerted by particle number J on particle I (If I=J, then we set $\mathbf{F_{II}}$=0). Then *Newton's third law* states that

$$\mathbf{F_{IJ}} = -\mathbf{F_{JI}}, \qquad or \quad \mathbf{F_{IJ}} + \mathbf{F_{JI}} = 0.$$

The total force $\mathbf{F_I}$ acting on particle I consists of an external force $\mathbf{F_I}^{(e)}$, such as gravity, and internal force that is the results of mutual interaction between the particles, thus

$$\mathbf{F}_I = \mathbf{F}_I^{(e)} + \sum_{J=1}^{K} \mathbf{F}_{IJ}.$$

K is the total number of particles. The equation of motion of the *Ith* particle is, therefore,

$$\frac{d}{dt} m_I \mathbf{v}_I = \mathbf{F}_I^{(e)} + \sum_{J=1}^{K} \mathbf{F}_{IJ}, \qquad (I = 1,2,\ldots., K).$$

Each particle is described by such an equation and a total of K equations describes the motion of the system.

- ***Equilibrium*** A *special motion is equilibrium*, i.e., one in which there is no acceleration for any particle of the system. At equilibrium , the above equation becomes

$$\mathbf{F}_I^{(e)} + \sum_{J=1}^{K} \mathbf{F}_{IJ} = 0, \qquad (I = 1,2,\ldots., K).$$

Summing over I from 1 to K, we have

$$\sum_{I=1}^{K} \mathbf{F}_I^{(e)} + \sum_{I=1}^{K} \sum_{J=1}^{K} \mathbf{F}_{IJ} = 0.$$

$$\Downarrow$$

$$\sum_{I=1}^{K} \mathbf{F}_I^{(e)} = 0$$

That is, *for a body in equilibrium, the summation of all external forces acting on the body is zero.*

- *Read* the 8 examples and *practise* the derivation of the formulas by using the concept of equilibrium — Free-Body Diagram Method.

Chapter 2. Vectors and Tensors

Tensor is the language of Mechanics.

2.1 Vectors

A vector in a three-dimensional Euclidean space is defined as a direct line segment with a given magnitude and a given direction. Vectors will be denoted by boldface letters such as **u, v, F, T**.

Their magnitude are denoted by

$$|\mathbf{u}|, |\mathbf{T}|$$

- *Equal vectors and zero vectors*
- *Vector summation and subtraction ---- parallelogram law*
- *A vector multiplied by a number yields another vector*
- *Every vector in a 3-D Euclidean space with coordinate axes x_1,x_2,x_3 can be represented by*

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the unit vectors in the directions of the positive x_1,x_2,x_3 axes, respectively. Then we have

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

- *Scalar (or dot) product of* $\mathbf{u}$ *and* $\mathbf{v}$

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta = u_1 v_1 + u_2 v_2 + u_3 v_3$$

- *Vector (or cross) product of* $\mathbf{u}$ *and* $\mathbf{v}$ *is a vector* $\mathbf{w}$ *(right-handed system)*

$$\mathbf{w} = \mathbf{u} \times \mathbf{v},$$
$$|\mathbf{w}| = |\mathbf{u}||v|\sin\theta$$

2.2 Vector equations

- A plane with unit normal vector **n** can be represented by the equation for the vector **r** as (p is a constant)

$$\mathbf{r} \cdot \mathbf{n} = p$$

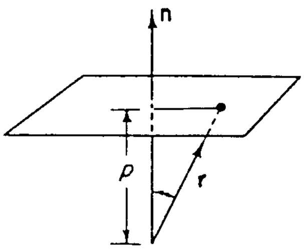


Figure 2.1 Equation of a plane, $\mathbf{r} \cdot \mathbf{n} = p$.

- *Equilibrium condition* we have a particle on which the forces $\mathbf{F}^{(1)}$, $\mathbf{F}^{(2)}$, ···, $\mathbf{F}^{(n)}$ act, then the condition of equilibrium for this particle can be expressed as

$$\mathbf{F}^{(1)} + \mathbf{F}^{(2)} + ... + \mathbf{F}^{(n)} = \mathbf{0}$$

- *The above vector equations can be conveniently expressed in analytic form by their components*

$$ax + by + cz = p,$$

$$\sum_{i=1}^{n} F_x^{(i)} = 0, \quad \sum_{i=1}^{n} F_y^{(i)} = 0, \quad \sum_{i=1}^{n} F_z^{(i)} = 0,$$

- Try to give other examples of vector equations.

2.3 The summation convention and the Kronecker delta

- *Index and summation convention* ----- The repetition of an index in a term will denote a summation with respect to that index over its range. *Dummy* and *free* index.

- **For example:**

(1) The plane equation can be expressed as

$$\sum_{i=1}^{3} a_i x_i = p \quad \Leftrightarrow \quad a_i x_i = p$$

(2) Let the direction cosines of a unit vector **v** defined as

$$\alpha_1 = \cos(\mathbf{v}, x), \quad \alpha_2 = \cos(\mathbf{v}, y), \quad \alpha_{31} = \cos(\mathbf{v}, z),$$

$$\Rightarrow \quad (\alpha_1)^2 + (\alpha_2)^2 + (\alpha_3)^2 = 1, \Leftrightarrow \quad \alpha_i \alpha_i = 1$$

(3) The length *ds* of a line element with components *dx, dy, dz* in a Cartesian coordinates *x, y, z* is

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 . \Leftrightarrow \quad (ds)^2 = \delta_{ij} dx_i dx_j$$

The symbol

$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

is called the ***Kronecker delta***.

(4) The differential of a function $f(x_1, x_2, .., x_n)$ can be written as

$$df = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + ... + \frac{\partial f}{\partial x_n} dx_n = \frac{\partial f}{\partial x_i} dx_i$$

2.4 Matrices and Determinants

- An $m{\times}n$ matrix $\mathbf{A}$ is defined as

$$\mathbf{A} = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

- A transpose of $\mathbf{A}$ is another matrix, denoted by $\mathbf{A}^{\mathrm{T}}$

$$\mathbf{A}^T = (a_{ij})^T = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{m1} \\ a_{21} & a_{22} & \dots & a_{m2} \\ \dots & \dots & \dots & \dots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{pmatrix}$$

- The product of two 3 × 3 matrices $\mathbf{A}$ and $\mathbf{B}$ is a 3 × 3 square matrix defined as

$$(\mathbf{A}\cdot\mathbf{B})_{ij} = (a_{ik}b_{kj})$$

- By this rule, the scalar product of two vectors $\mathbf{u}$ and $\mathbf{v}$ can be written as

$$\mathbf{u}\cdot\mathbf{v} = (u_i)(v_i)^T = u_i v_i$$

- The ***permutation symbol*** and the *determinant* of a square matrix

  (1) The *permutation symbol* is defined as

$$\varepsilon_{rst} = \begin{cases} 1, & for \quad \varepsilon_{123}, \quad \varepsilon_{231}, \quad \varepsilon_{312}, \\ -1, & for \quad \varepsilon_{213}, \quad \varepsilon_{321,} \quad \varepsilon_{132}, \\ 0, & for \quad others \end{cases}$$

(2) By using the *permutation symbol,* the determinant of a 3 × 3 matrix **A** can be expressed as

$$\det \mathbf{A} = \det(a_{ij}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$$= \varepsilon_{rst}a_{r1}a_{s2}a_{t3}$$

(3) The vector product of **u** and **v** can be expressed as

$$\mathbf{u} \times \mathbf{v} = \varepsilon_{rst}u_s v_t \mathbf{e_r}$$

(4) The relation between *permutation symbol* and *Kronecker delta*

$$\varepsilon_{ijk}\varepsilon_{ist} = \delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}$$

2.5 Transform of coordinates

- **2-D space** *coordinate translation* Consider two sets of rectangular Cartesian frames of reference $O'x'y'$ and Oxy on a plane. If $O'x'y'$ *is obtained by a shift of origin without rotation and the coordinates of the new origin O' are (h, k) relative to Oxy, then the coordinates of any point P : (x, y) and (x', y') with respect to the old and new frames have the following relations:*

$$\begin{cases} x = x' + h \\ y = y' + k \end{cases} \quad or \quad \begin{cases} x' = x - h \\ y' = y - k \end{cases}$$

- **2-D space** *coordinate rotation* If the origin remains fixed, and the new axes are obtained by rotating Ox and Oy through an angle θ in the counterclockwise direction (Fig. 2.2). *Then the coordinates of any point P : (x, y) and (x', y') with respect to the old and new frames have the following relations:*

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

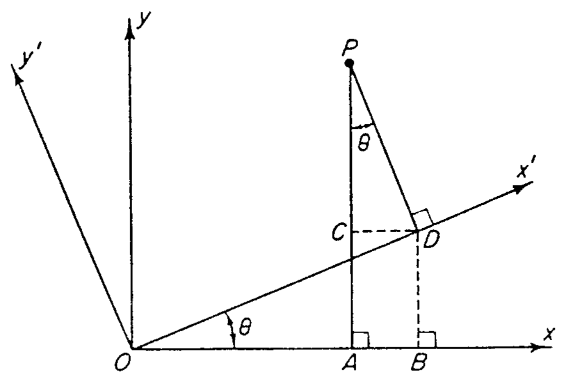


Figure 2.2 Rotation of coordinates.

Using the index notation (i.e., let x_1 , x_2 replace x, y, etc), then the above equation can be represented by

$$x'_i = \beta_{ij} x_j, \qquad (i = 1,2)$$

$$x_i = \beta_{ji} x'_j, \qquad (i = 1,2)$$

where

$$(\beta_{ij}) = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

$$(\beta_{ji}) = (\beta_{ij})^T = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

On the other hand, (β_{ji}) must be identified as the inverse of the matrix (β_{ij}), so we have

$$(\beta_{ji}) = (\beta_{ij})^{-1} = (\beta_{ij})^T$$

Thus matrix (β_{ij}) is an orthogonal matrix. It has the property

$$(\beta_{ij})(\beta_{ij})^T = (\beta_{ij})(\beta_{ij})^{-1} = (\delta_{ij})$$

i.e., the component of the matrix $(\beta_{ij})(\beta_{ij})^T$ is

$$\beta_{ik}\beta_{jk} = \delta_{ij}$$

$$(\beta \cdot \alpha)_{ij} = \beta_{ik}\alpha_{kj} = \beta_{ik}\beta_{jk}(\leftrightarrow \beta^T_{kj}) = \delta_{ij}$$

- **3-D space** The above coordinate transfer relations can be easily extended to the 3-D case: Let **x** be the position vector of any point P in the two Cartesian coordinate systems (see Fig. 2.3) with the same origin O. Let $\mathbf{e}_i$ denote the unit vector (or base vector) in the direction of the positive *i*th axis. Since the coordinates are orthogonal, we have

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \qquad \mathbf{e}_i' \cdot \mathbf{e}_j' = \delta_{ij}.$$

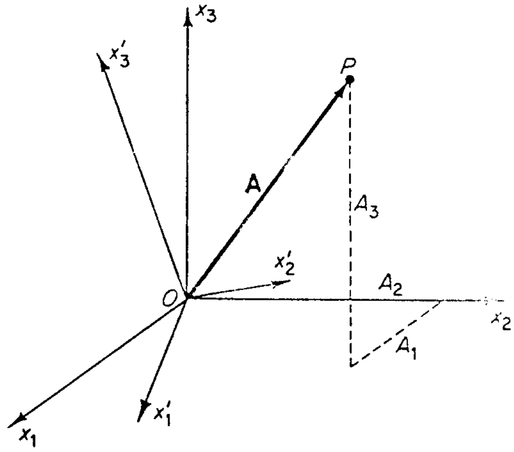


Figure 2.3 Radius vector and coordinates.

The vector **x** can be expressed as

$$\mathbf{x} = x_j \mathbf{e}_j = x_j' \mathbf{e}_j'$$

This leads to

$$x_i = x_j\delta_{ij} = x_j(\mathbf{e}_j \cdot \mathbf{e}_i) = x_j'(\mathbf{e}_j' \cdot \mathbf{e}_i) = (\mathbf{e}_j' \cdot \mathbf{e}_i)x_j'$$

$$\Downarrow$$

$$x_i = (\mathbf{e}_j' \cdot \mathbf{e}_i)x_j'$$

$$\Downarrow$$

$$define \quad (\mathbf{e}_j' \cdot \mathbf{e}_i) = \beta_{ji};$$

$$\Downarrow$$

$$x_i = \beta_{ji}x_j', \quad (i, j = 1,2,3)$$

$$Similarly\,we \quad have \quad x_i' = \beta_{ij}x_j, \quad (i, j = 1,2,3)$$

<u>*Proof:*</u>

$$|\beta_{ij}| \equiv \begin{vmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{vmatrix} = 1.$$

- **Coordinate transformation in general** Given any two coordinate frames

$$(x_1, x_2, x_3) \qquad and \quad (\bar{x}_1, \bar{x}_2, \bar{x}_3)$$

The condition for the reversible and one to one correspondence — significance of the *Jacobian determinant J.* The *admissible* and *proper* transformations.

2.6. Analytical definition of Scalars, Vectors, and Cartesian Tensors

Given two rectangular Cartesian coordinate frames

$$(x_1, x_2, x_3) \quad and \quad (\bar{x}_1, \bar{x}_2, \bar{x}_3)$$

- **A system of quantities is called a <u>*scalar*</u>** — if it has *only a single component* in the corresponding variables space and the components in the two frames are numerically *equal* at the same point, i.e.,

$$\Phi(x_1, x_2, x_3) = \overline{\Phi}(\bar{x}_1, \bar{x}_2, \bar{x}_3).$$

- **A system of quantities is called a <u>*vector field*</u> or a <u>*tensor field of rank 1*</u>** — if it has *three components* in variable space and the components in different coordinate spaces are related by the characteristic law

$$\bar{\xi}_i(\bar{x}_1, \bar{x}_2, \bar{x}_3) = \xi_k(x_1, x_2, x_3)\beta_{ik},$$
$$\xi_i(x_1, x_2, x_3) = \bar{\xi}_k(\bar{x}_1, \bar{x}_2, \bar{x}_3)\beta_{ki}.$$

- **A system of quantities is called a <u>*tensor field of rank 2*</u>** — if it has *9 components t_{ij} (i,j=1,2,3)* in variable space and the components in different coordinate spaces are related by the characteristic law

$$\bar{t}_{ij}(\bar{x}_1, \bar{x}_2, \bar{x}_3) = t_{mn}(x_1, x_2, x_3)\beta_{im}\beta_{jn},$$
$$t_{ij}(x_1, x_2, x_3) = \bar{t}_{mn}(\bar{x}_1, \bar{x}_2, \bar{x}_3)\beta_{mi}\beta_{nj}.$$

Similarly, high rank tensors can be defined. These tensors are called *Cartesian tensors* and can be denoted by either Boldface or indices notation.

- **The significance of tensor equations** — It *does not change its form* with the change of the coordinate system obtained by the *admissible* transformation. Tensor analysis is as important as *dimensional analysis* in formulation of physical relations.

- **The Quotient rule** A method that enables us to determine whether a function is a tensor without using the law of transformation directly. For example, consider a set of n^3 functions $A(i, j, k)$ with indices i, j, k ranging over 1, 2, ⋯, n. Let $\xi_i(x)$ be a vector. Suppose that the product $A(i, j, k)\ \xi_i$ (summation convention used over i) is known to yield a tensor of the type $A_{ij}(x)$ *(rank two), i.e.,*

$$A(i,j,k)\xi_i = A_{jk}$$

  Then we can prove that $A(i, j, k)$ is a tensor of the type $A_{ijk}(x)$ (i.e., rank 3 tensor).
  (Proof: ……………)

- **Partial derivatives** *When only Cartesian coordinates* are considered, the partial derivatives of any tensor field behave like the components of a *Cartesian tensor*. For example,

$$\Phi_{,i} \qquad \xi_{i,j} \qquad \sigma_{ij,k}$$

are tensors of rank 1, 2, and 3, respectively, provided *that* Φ, ξ, *and* σ_{ij} *are tensors with rank 0, 1 and 2 respectively.*

Chapter 3. Stress

We discuss the properties of the stress tensors in this chapter.

3.1 Stress vector and notation of stress tensor Consider a continuum in a rectangular parallelepiped in the rectangular Cartesian coordinate frame with axes x_1, x_2, x_3 as shown (Fig. 3.1). Let the surface ΔS_1 be the surface of the parallelepiped with outer normal vector pointing in the positive direction of the x_1 axis. The ***stress vector*** on this surface is denoted by $\overset{1}{\mathbf{T}}$ with three components $\overset{1}{T}_1, \overset{1}{T}_2, \overset{1}{T}_3$ in the directions of the coordinates axes x_1, x_2, x_3, respectively. We introduce a new set of symbols for these stress components:

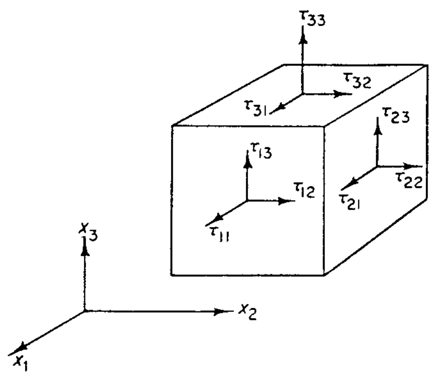


Figure 3.1 Notations of stress components.

$$\overset{1}{T}_1 = \sigma_{11}, \qquad \overset{1}{T}_2 = \sigma_{12}, \qquad \overset{1}{T}_3 = \sigma_{13}.$$

Similarly we can list the stress components on the other two surfaces ΔS_2 and ΔS_3, so finally we obtain

	Components of stress vectors along axis 1	2	3
Surface normal to x_1	σ_{11}	σ_{12}	σ_{13}
Surface normal to x_2	σ_{21}	σ_{22}	σ_{23}
Surface normal to x_3	σ_{31}	σ_{32}	σ_{33}

The components σ_{ii} $(i=1,2,3)$ are called normal stresses, and the remaining $\sigma_{ij}\,(i\neq j)$ are called shear stresses.

- *Direction of positive stress components (Fig. 3.2)*

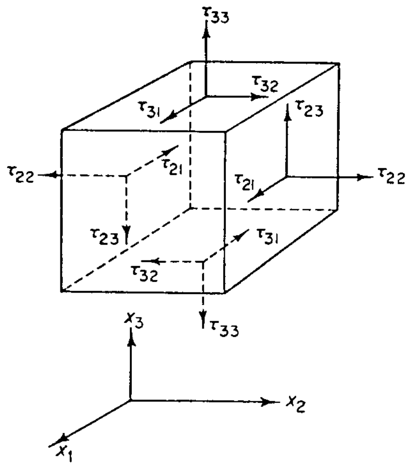


Figure 3.2 Directions of positive stress components.

3.2 The laws of motion Continuum mechanics is founded on *Newton's laws of motion.* Consider a material body which occupies a space *B(t)* at time *t* in the Cartesian coordinate frame (Fig. 3.3). Let **r** be the position vector of a point, ρ the density of the material, *dv* the infinitesimal element enclosing the point at **r** and **V** the velocity at **r**.

$$\int_B \mathbf{X}\,dv.$$

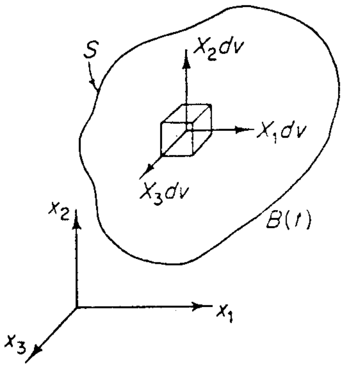


Figure 3.3 Body forces.

Then the *linear momentum of the element* is *(ρdv)***V**. The *linear momentum of the body B(t)* at time *t* is the integral

$$\mathbf{P}(t) = \int_{B(t)} \mathbf{V} \rho dv$$

The *moment of momentum* of the body is the integral

$$\mathbf{H}(t) = \int_{B(t)} \mathbf{r} \times \mathbf{V} \rho dv$$

So *Newton's law of motion* (as stated by Euler for a continuum) assert that *the rate of change of the linear momentum is equal to the total external applied force **F** acting on the body*, i.e.,

$$\dot{\mathbf{P}} = \mathbf{F}$$

and *that the rate of change of moment of momentum is equal to the total applied torque* **L** *about the origin,* i.e.,

$$\dot{\mathbf{H}} = \mathbf{L}$$

The above two derivatives refer to the time rate of change of **P** and **H** of <u>*a fixed set of material particles*</u>. We shall denote them by *DP/Dt* and *DH/Dt*, respectively. There are *two types* of external forces acting on material bodies in the mechanics of continuum media: (1*) Body force*

$$\int_B \mathbf{X} dv, \qquad (X_i = \rho g_i)$$

and (2) *surface forces*

$$\oint_S \overset{v}{\mathbf{T}} dS$$

So the total external force is

$$\mathbf{F} = \oint_S \overset{v}{\mathbf{T}} dS + \int_B \mathbf{X} dv,$$

Similarly we can get the expression of torque about the origin

$$\mathbf{L} = \oint_{S(t)} \mathbf{r} \times \overset{v}{\mathbf{T}} dS + \int_{B(t)} \mathbf{r} \times \mathbf{X} dv$$

We can finally have the equations of motion

$$\oint_S \overset{v}{\mathbf{T}} dS + \int_B \mathbf{X} dv = \frac{D}{Dt} \int_{B(t)} \mathbf{V} \rho dv$$

$$\oint_S \mathbf{r} \times \overset{v}{\mathbf{T}} dS + \int_B \mathbf{r} \times \mathbf{X} dv = \frac{D}{Dt} \int_{B(t)} \mathbf{r} \times \mathbf{V} \rho dv$$

Note that *B(t)* must consists of the *same material particles* at all times.

3.3 Cauchy’ s formula

From the equation of motion, we first derive the relation of stress vectors $\mathbf{T}^{(+)}$ and $\mathbf{T}^{(-)}$ (see the Fig. 3.4)

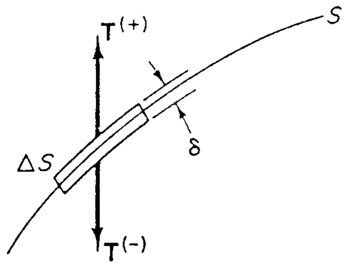


Figure 3.4 Equilibrium of a "pillbox" across a surface *S*.

$$\mathbf{T}^{(-)} = -\mathbf{T}^{(+)}$$

Another way of stating this result is to say that *stress vector is a function of the normal vector to a surface.* When the sense of direction of the normal vector reverse, that of the stress vector reverse also.

Second we have the *Cauchy’ s formula*: the relation between the *stress vector* $\overset{\nu}{\mathbf{T}}$ (components $\overset{\nu}{T}_1, \overset{\nu}{T}_2, \overset{\nu}{T}_3$) acting on any surface with unit outer normal $\mathbf{v}$ (component ν_1, ν_2, ν_3) and the *stress tensor* (σ_{ij})

$$\overset{\nu}{T}_i = \nu_j \sigma_{ij} = \sigma_{ij} \nu_j$$

Proof. Cauchy’ s formula can be derived in several ways.

Cauchy’ s formula assures us that the nine components of stresses σ_{ij} are <u>*necessary and sufficient*</u> to define the *traction across any surface element* in a body. Hence, the stress state in a body is characterized <u>*completely*</u> by the set of quantities σ_{ij}. Since $\overset{\nu}{T}_i$ is a vector and the above relation is valid for *an arbitrary* vector ν_i, it follows that σ_{ij} is a tensor. Henceforth, σ_{ij} will be called <u>*a stress tensor.*</u>

3.4 Equation of equilibrium

We can transform the above <u>*equation of motion*</u> into differential equations (see chapter 10), but here we give it by elementary method to assure physical clarity.

- First we consider the static equilibrium state of an infinitesimal parallelepiped with surfaces parallel to the coordinate planes. The stresses acting on various surfaces are shown in Fig. 3.6.

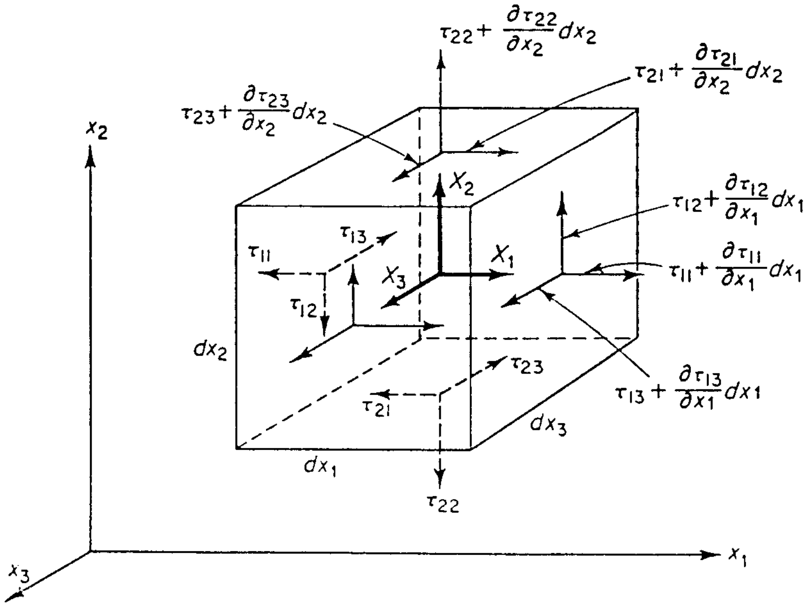


Figure 3.6 Equilibrating stress components on an infinitesimal parallelepiped.

The *static force equilibrium* of the parallelepiped along x_1 direction (Fig. 3.7) gives

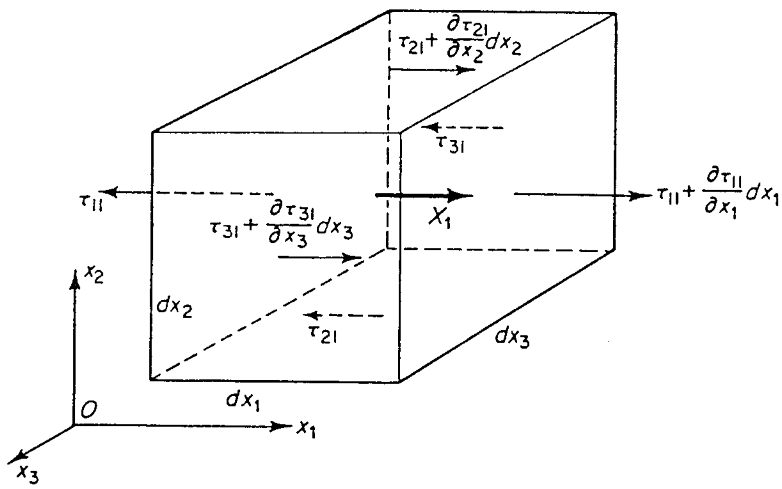


Figure 3.7 Components of tractions in x_1-direction.

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{21}}{\partial x_2} + \frac{\partial \sigma_{31}}{\partial x_3} + X_1 = 0, \qquad or \quad \frac{\partial \sigma_{1j}}{\partial x_j} = 0.$$

Similar equations in x_2- and x_3- directions can be obtained and finally the three equations can be expressed concisely as

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = 0. \qquad (i = 1,2,3)$$

The *static moment equilibrium* of the parallelepiped (Fig. 3.8) leads to the important conclusion that *the stress tensor is symmetric*, i.e.,

$$\sigma_{ij} = \sigma_{ji}$$

(Prove this).

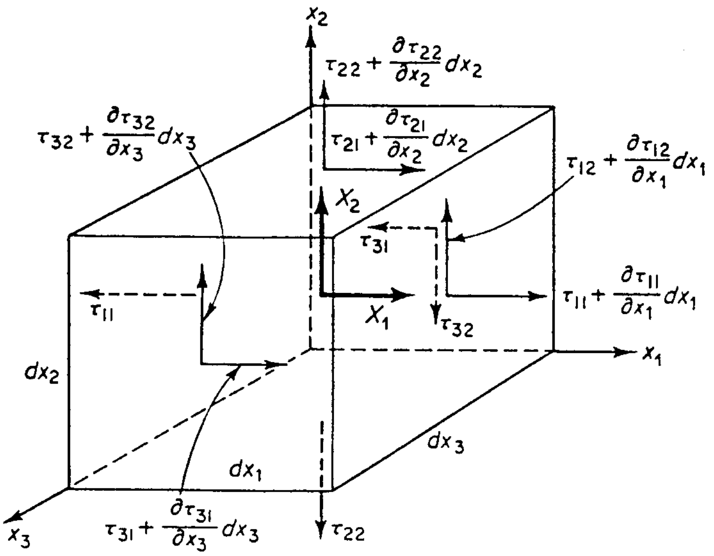


Figure 3.8 Components of tractions that contribute moment about the Ox_3-axis.

- Second, the above two equations can be extended to the dynamic case as (ρ is the density, **a** is the acceleration with components a_1, a_2, a_3)

$$\frac{\partial \sigma_{ij}}{\partial x_j} + X_i = \rho a_i. \qquad (i = 1,2,3)$$

$$\sigma_{ij} = \sigma_{ji}$$

3.5 Change of stress components in transformation of coordinates

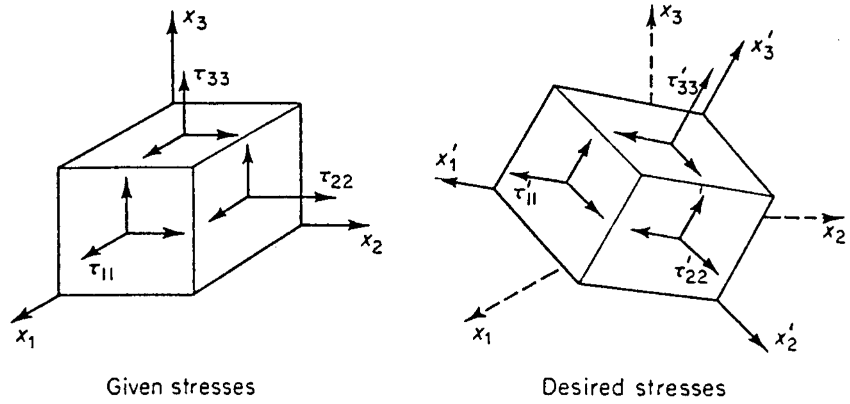


Figure 3.9 Transformation of stress components under rotation of coordinate systems.

Consider the rotation of Cartesian coordinate frame from x_1, x_2, x_3 to x'_1, x'_2, x'_3 (see Fig. 3.9). How to obtain σ'_{ij} from σ_{ij}? There are two methods:

- By the transformation law at once, since σ_{ij} is a tensor.

$$\sigma'_{km} = \sigma_{ji}\beta_{kj}\beta_{mi}.$$

where $\beta_{kj} = \mathbf{e}'_{\mathrm{k}} \cdot \mathbf{e}_{\mathrm{j}}$.

- By the Cauchy's formula, we can also obtain

$$\sigma'_{km} = \sigma_{ji}\beta_{kj}\beta_{mi}.$$

(<u>*Prove this*</u>) So we can see that the stress components transform like a Cartesian tensor of rank 2. Thus the physical concept of a stress agrees with the mathematical definition of a rank 2 tensor in Euclidean space.

3.6 Stress components in orthogonal curvilinear coordinates

As an example, consider the two *orthogonal coordinate frames* shown in Fig. 3.10. They are related by

$$\begin{cases} x = r\cos\theta, \qquad y = r\sin\theta, \quad z = z, \\ \\ \theta = \tan^{-1}\dfrac{y}{x}, \quad r^2 = x^2 + y^2, \qquad z = z. \end{cases}$$

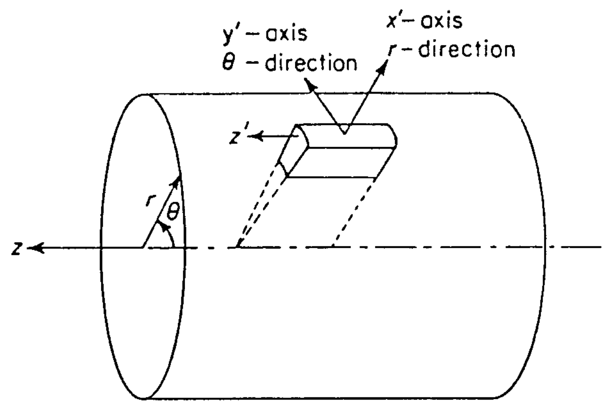


Figure 3.10 Stress components in cylindrical polar coordinates.

For the same material point, to relate the stress components in the *x-y-z* coordinate $(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}, \sigma_{xy}, \ldots$ *etc.*) with the stress components in the *r-θ-z* coordinate $(\sigma_{rr}, \sigma_{\theta\theta}, \ldots$ *etc.*). (*Prove Eq. (3.6-5)*)

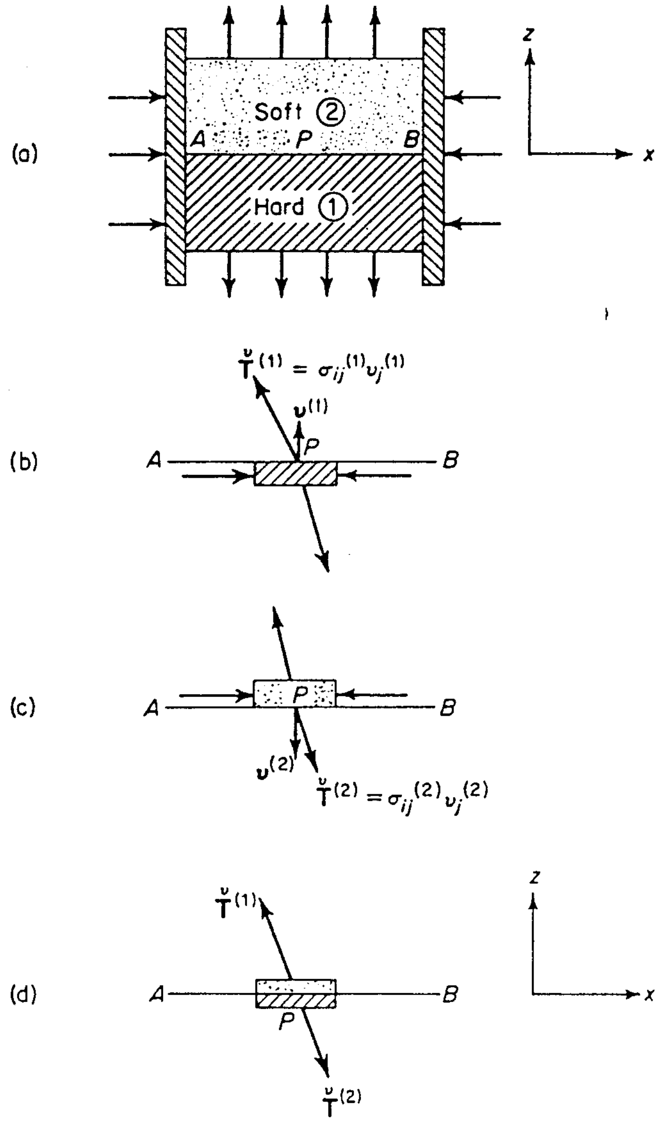
This approach can be used for the transformation of stress components between Cartesian and (or among) spherical or other orthogonal curvilinear coordinates.

For coordinate transfer of non-orthogonal coordinate frames, please refer to the << Tensor Analysis >>.

3.7 Boundary conditions, Stress boundary conditions

The problems in solid mechanics usually appear this way: we know something about *forces* or *velocities* or *displacement* on the ***surface*** of a solids, and inquire into what happens inside the body, like what is the stress, displacement distribution (or fields). If a solution satisfy all the field equations and boundary conditions, then complete information is obtained for the entire interior of the body.

On the *surface* of a body or at an *interface* between two bodies, *the traction (force per unit area) acting on the surface must be the same on both sides of the surface,* which is the basic concept of stress (see Fig. 3.11).



(e)

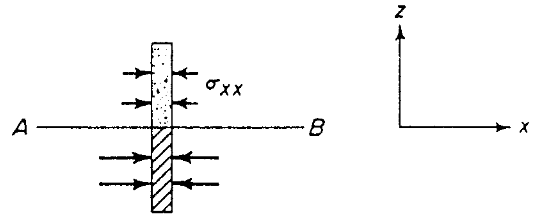


Figure 3.11 Derivation of the stress boundary condition at an interface between two materials. (a) An interface AB between two continuous media 1 and 2. (b) Free-body diagram of a small element of material No. 1 at a point P on the interface. The stress vector $\overset{\nu}{\mathbf{T}}{}^{(1)}$ acts on the surface AB of this element. (c) Free-body diagram of a small element of material No. 2 at P. (d) Free-body diagram of a small flat element including both materials. (e) Free-body diagram of a small vertical element, showing that σ_{xx} can be discontinuous at the interface.

Chapter 4. Principal stresses and principal axes

Principal stresses, stress invariants, stress deviations, and the maximum shear are important concepts. They tell us the state of stress in the simplest numerical way. They are directly related to the strength of materials. One has to evaluate them frequently; therefore, we devote a chapter to them.

4.1 Introduction

- Symmetric stress tensor with 6 independent components

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}, (\sigma_{ij} = \sigma_{ji}),$$

- We will show that: because the stress tensor is **symmetric**, a set of coordinates can be found with respect to which the matrix of stress components can be reduced to a diagonal matrix of the form.

$$\sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

- Principal axes, principal stresses, principal planes

- We shall show that the symmetry of the stress tensor is the basic reason for the existence of principal axes, such as the strain tensor. This is also true for n-dimensions.

- Symmetry is thus a great asset.

4.2 Plane State of Stress

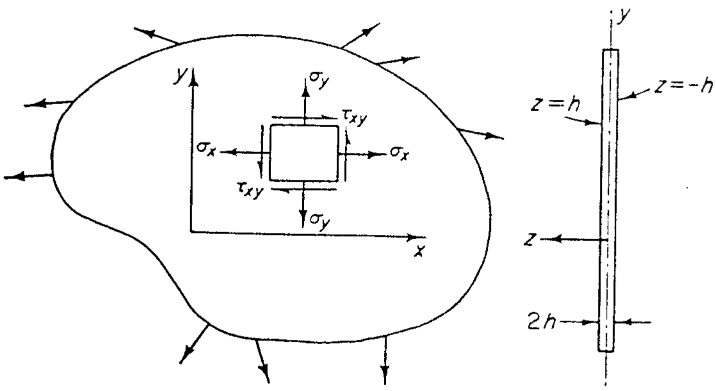


Figure 4.1 An approximate plane state of stress.

- The stress tensor only have three non-zero components and after rotation of coordinates frame to $x'y'$, we have

$$\begin{pmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} \sigma_{x'} & \tau_{x'y'} & 0 \\ \tau_{x'y'} & \sigma_{y'} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The transform matrix depends on angle θ (as shown):

$$\begin{pmatrix} \beta_{11} & \beta_{12} & \beta_{13} \\ \beta_{21} & \beta_{22} & \beta_{23} \\ \beta_{31} & \beta_{32} & \beta_{33} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

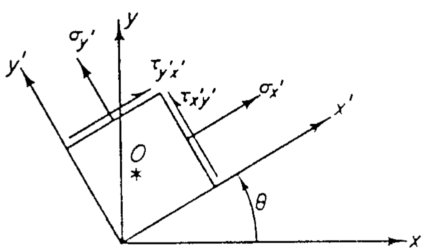


Figure 4.2 Change of coordinates in plane state of stress.

The new stress components:

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$
$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$
$$\tau_{x'y'} = (-\sigma_x + \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

since

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta), \cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

we may write the preceding equation as

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta,$$
$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$
$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta.$$

From these equations, it follows that

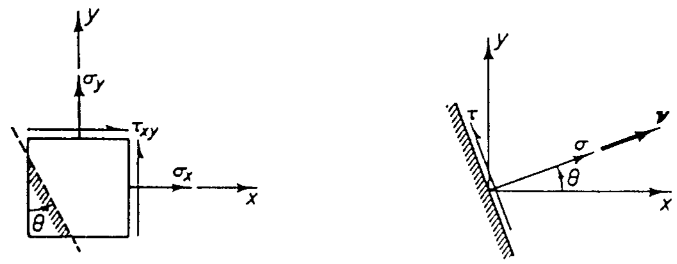
$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y,$$
$$\frac{\partial \sigma_{x'}}{\partial \theta} = 2\tau_{x'y'}, \qquad \frac{\partial \sigma_{y'}}{\partial \theta} = -2\tau_{x'y},$$
$$\tau_{x'y'} = 0 \qquad when \qquad \tan 2\theta = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}.$$

- The directions of the $x'-y'$ axes corresponding to particular values of θ with $\tau_{x'y'} = 0$ are called *Principal directions*, and this $x'-y'$ axes are called *Principal axes*. The corresponding normal stresses are called *Principal stresses*.

$$\sigma_{\max} \Big/ \sigma_{\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2{}_{xy}}$$

$$\tau_{\max} = \frac{\sigma_{\max} - \sigma_{\min}}{2} = \sqrt{\left(\frac{\sigma_z - \sigma_y}{2}\right)^2 + \tau^2_{xy}}\,.$$

- **Geometric approach for plane stress state — Mohr's circle**



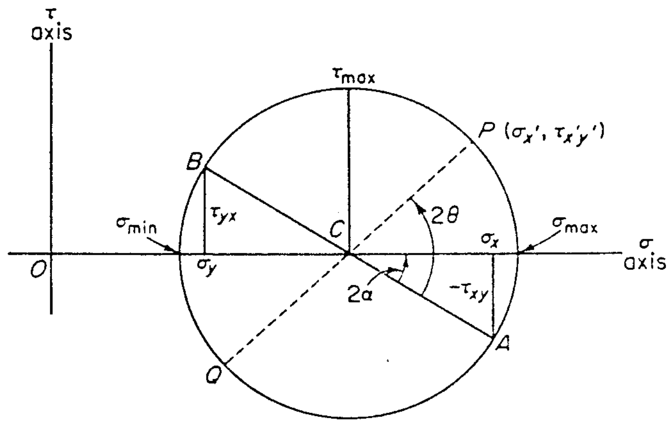


Figure 4.3 Mohr's circle for plane stress.

A geometric representation of the above analysis on principal stress was developed by Otto Mohr (1882). From above figures, it is seen

$$\overline{OC} = \frac{\sigma_x + \sigma_y}{2}, \text{ and a radius } \bar{A}C = \bar{C}P = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau^2_{xy}}\,.$$

From the figures , we see that the abscissa of P is

$$\sigma_{x'} = \overline{OC} + \overline{CP}\cos(2\theta - 2\alpha)$$
$$= \overline{OC} + \overline{CP}(\cos 2\theta \cos 2\alpha + \sin 2\theta \sin 2\alpha)$$

But we see also from the diagram that

$$\cos 2\alpha = \frac{\sigma_x - \sigma_y}{2\overline{CP}}, \sin 2\alpha = \frac{\tau_{xy}}{\overline{CP}},$$

we get,

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta,$$

Which is the same as the analytical form before. Similarly, we have the ordinate of P:

$$\begin{aligned}\tau_{x'y'} &= \overline{CP}\sin(2\theta - 2\alpha) = \overline{CP}(\sin 2\theta\cos 2\alpha - \cos 2\theta \sin 2\alpha) \\ &= \frac{\sigma_{xx} - \sigma_y}{2}\sin 2\theta - \tau_{xy}\cos 2\theta,\end{aligned}$$

Hence and the validity of Mohr's circle in 2-D is proved.

4.3 Analysis for the three-dimensional stress states

- **3-D Mohr's circle** Otto Mohr has shown the interesting result that if the normal stress $\sigma_{(n)}$ and the shearing stress τ acting on any section are plotted on a plane, with $\sigma_{(n)}$ and τ as coordinates as shown, they will necessarily fall in a closed domain represented by the shaded area bounded by the three circles with centers on the σ-axis .

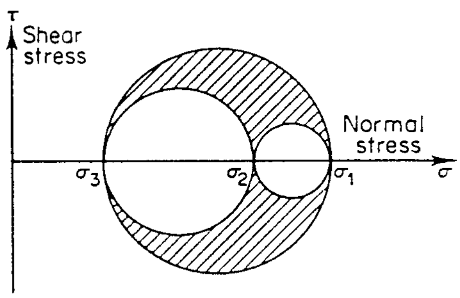


Figure 4.4 Mohr's circles.

- **Analytical formula for *Principal stresses* in 3-D**

At a given point in a body, the angle between the *stress vector* and the normal ν varies with the orientation of the surface. We shall show that we can always find a surface so oriented that the *stress vector is exactly normal to it*. In fact, we shall show that there are at least *three* mutually orthogonal surfaces.

Let ν be a unit vector in the direction of a principal axis, and let σ be the corresponding principal stress, then we have

$$(\tau_{ji} - \sigma\delta_{ji})\nu_j = 0, \quad (i = 1, 2, 3).$$

Since ν is a unit vector, we must find a set of nontrivial solutions for which $\nu_1^2 + \nu_1^2 + \nu_3^2 = 1$. Thus, the above equations pose an eigenvalue problem. Since τ_{ij} as a matrix is real valued and symmetric, we need only to recall a result in the theory of matrices to assert that there exist three real-valued principal stresses and a set of orthonormal principal axes. Whether the principal stresses are all positive, all negative, or mixed depends on whether the quadratic form $\tau_{ij}\chi_i\chi_j$ is positive definite, negative definite, or uncertain, respectively.

The system of equation has a set of nonvanishing solutions ν_1, ν_2, ν_3 if and only if the determinant of the coefficients vanishes, i.e.,

$$|\tau_{ij} - \sigma\delta_{ij}| = 0.$$

$$|\tau_{ij} - \sigma\delta_{ij}| = \begin{vmatrix} \tau_{11} - \sigma & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} - \sigma & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} - \sigma \end{vmatrix}$$

$$= -\sigma^3 + I_1\sigma^2 - I_2\sigma + I_3 = 0$$

where

$$I_1 = \tau_{11} + \tau_{22} + \tau_{33}$$

$$I_2 = \begin{vmatrix} \tau_2 & \tau_{23} \\ \tau_{32} & \tau_{33} \end{vmatrix} + \begin{vmatrix} \tau_{33} & \tau_{31} \\ \tau_{13} & \tau_{11} \end{vmatrix} + \begin{vmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{vmatrix}$$

$$I_3 = \begin{vmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{vmatrix}$$

I_1, I_2, I_3 are called the <u>*invariants of the stress tensor*</u> *with respect to rotation of coordinates*.

<u>*Prove:*</u>

(1)
$$\begin{aligned} I_1 &= \sigma_1 + \sigma_2 + \sigma_3, \\ I_2 &= \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1, \\ I_3 &= \sigma_1\sigma_2\sigma_3. \end{aligned}$$

When $\sigma_1 \neq \sigma_2 \neq \sigma_3$, we can prove that

$$\overset{1}{v}_i \overset{2}{v}_i = 0, \qquad \overset{2}{v}_i \overset{3}{v}_i = 0, \qquad \overset{3}{v}_i \overset{1}{v}_i = 0$$

i.e. the principle vectors are mutually orthogonal to each other. If $\sigma_1 = \sigma_2 \neq \sigma_3, \overset{3}{v}_i$ will be fixed, but we can determine an infinite number of pairs of vectors $\overset{1}{v}_i$ *and* $\overset{2}{v}_i$ orthogonal to $\overset{3}{v}_i$. If $\sigma = \sigma_2 = \sigma_3$ then any set of orthogonal axes may be taken as the principal axes.

4.4 shearing stresses

On an element of surface with unit outer normal ν (with components $_i$),

there acts a traction $\overset{\nu}{\mathbf{T}}$ ($\overset{\nu}{T}_i = \tau_{ji}\nu_j$). The component of $\overset{\nu}{\mathbf{T}}$ in the direction of ν is the normal stress $\sigma_{(n)}$ acting on the surface element.

$$\sigma_{(n)} = \overset{\nu}{T}_i \, \nu_i = \tau_{ij}\nu_i\nu_j$$

On the other hand, $\overset{\nu}{\mathbf{T}}$ can be decomposed into two orthogonal components $\sigma_{(n)}$ and τ, where denotes the shearing stress tangent to the surface (as shown). So the magnitude of the shearing stress τ on a surface having normal ν is given by the equation

$$\tau = \left|\overset{\nu}{T}_i\right|^2 - \sigma_{(n)}^2$$

4.5 Stress-deviation tensor

The tensor

$$\tau'_{ij} = \tau_{ij} - \sigma_0 \delta_{ij}$$

is called the *stress-deviation tensor*, and

$$\sigma_0 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3}(\tau_{11} + \tau_{22} + \tau_{33}) = \frac{1}{3} I_1,$$

I_1 is the first invariant of stress tensor σ. The separation of τ_{ij} into a hydrostatic part and the deviatoric part is very important in describing the plastic behavior of metals.

- The principal stress of the stress-deviation tensor is determined from the similar determinantal equation

$$\left| \tau'_{ij} - \sigma' \delta_{ij} \right| = 0$$

which can be expressed as

$$\sigma'^3 - J_2 \sigma' - J_3 = 0.$$

- The invariants of the stress-deviation tensor:

The first invariant always vanishes

$$I'_1 = \tau'_{11} + \tau'_{22} + \tau'_{33} = 0.$$

The second and third invariants can be related to I_1, I_2 as

$$J_2 = 3\sigma_0^2 - I_2,$$
$$J_3 = I_3 + J_2 \sigma_0 - \sigma_0^3,$$

Also J_2 can be expressed as

$$J_2 = \frac{1}{2}\tau'_{ij}\tau'_{ij}$$

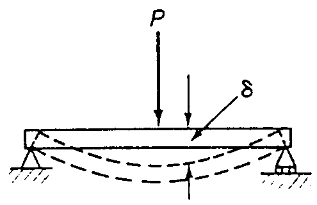
we observe that the principal axes of the stress tensor and the stress-deviation tensor coincide. Then if $\sigma'_1, \sigma'_2, \sigma'_3$, are the principal stress deviations, we have

$$\sigma'_1 = \sigma_1 - \sigma_0, \sigma'_2 = \sigma_2 - \sigma_0, \sigma'_3 = \sigma_3 - \sigma_0$$

$$J_2 = -\left(\sigma'_1\sigma'_2 + \sigma'_2\sigma'_3 + \sigma'_3\sigma'_1\right)$$
$$J_3 = \sigma'_1\sigma'_2\sigma'_3$$

- Example of Application: In testing of material in a pressurized chamber, the test results indicate that *the P-δ curve of steel beam is virtually unaffected by the hydrostatic pressure.* Yielding of most materials is related not to τ_{ij}, but to τ'_{ij}.

(a)



(b)

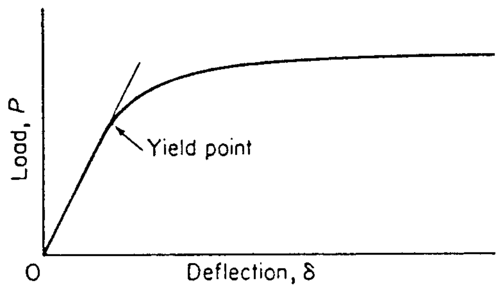


Figure 4.6 Testing of a steel bean in a pressurized chamber. (a) The beam. (b) The load-deflection curve.

4.6 Lame's stress ellipsoid

Let the principal axes of stress be chosen as the coordinate axes x_1, x_2, x_3, and let the principal stresses be written as $\sigma_1, \sigma_2, \sigma_3$. Then we see that the components of $\overset{\nu}{T}_i$ satisfy the equation

$$\frac{\left(\overset{\nu}{T}_1\right)^2}{(\sigma_1)^2}+\frac{\left(\overset{\nu}{T}_2\right)^2}{(\sigma_2)^2}+\frac{\left(\overset{\nu}{T}_3\right)^2}{(\sigma_3)^2}=1$$

which is the equation of an ellipsoid with reference to a system of rectangular coordinates with axes labeled $\overset{\nu}{T}_1, \overset{\nu}{T}_2, \overset{\nu}{T}_3$. This ellipsoid is the locus of the end points of vectors $\overset{\nu}{T}$ issuing from a common center.

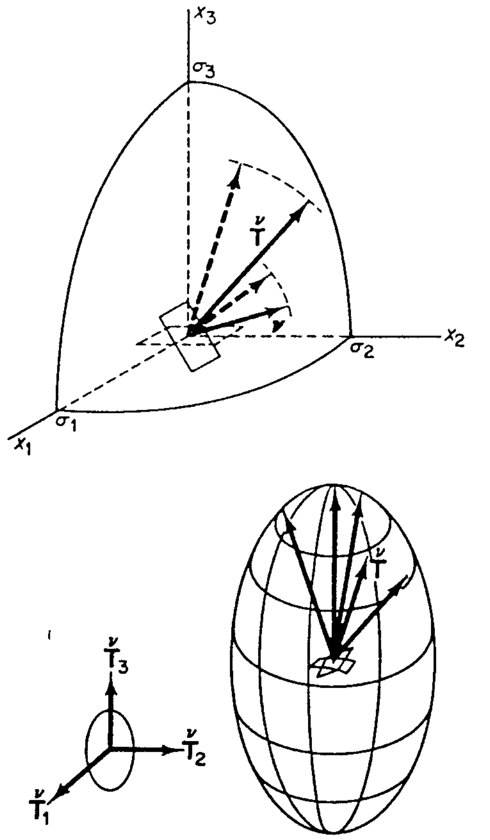


Figure 4.7 Stress ellipsoid as the locus of the end of the vector $\mathbf{T}$ as $\boldsymbol{\nu}$ varies.

Chapter 5 Analysis of Deformation

5.1 Deformation

- **Strain measures** Consider a string of an initial length L_0 and it is stretched to a length L. The ratio $\frac{L}{L_0}$ is called the stretch ratio and is denoted by the symbol λ. The ratios

$$\in = \frac{L - L_0}{L_0}, \qquad \in' = \frac{L - L_0}{L}$$

are strain measures. We can introduce other measures like

$$e = \frac{L^2 - L_0^2}{2L^2}, \qquad \varepsilon = \frac{L^2 - L_0^2}{2L_0^2}$$

But for *infinitesimal elongation* of the string, all of these strain measures are approximately *equal*. In finite elongations, however, they are different.

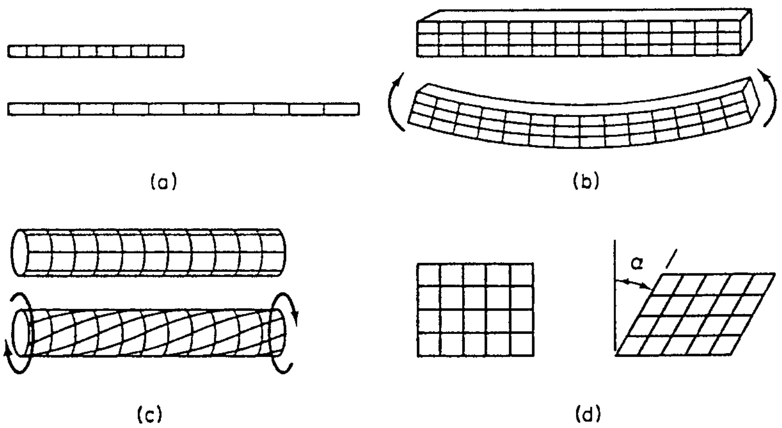


Figure 5.1 Patterns of deformation. (a) Stretching. (b) Bending. (c) Twisting. (d) Simple shear.

The selection of proper measures of strain is dictated basically by the stress-strain relationship (i.e., the constitutive equation of the material). The case of *infinitesimal strain* is simple because the *different measures* of strain just presented *all coincide*.

$$\sigma = Ee$$

where E is a constant called *Young's modulus*. A material obeying it is said to be a *Hookean material.*

Shear modulus or modulus of rigidity: $\tau = G \tan \alpha$

- **Mathematical description of deformation**. Consider the deformation of the object as shown.

$$x_1 - a_1, \quad x_2 - a_2, \quad x_3 - a_3.$$

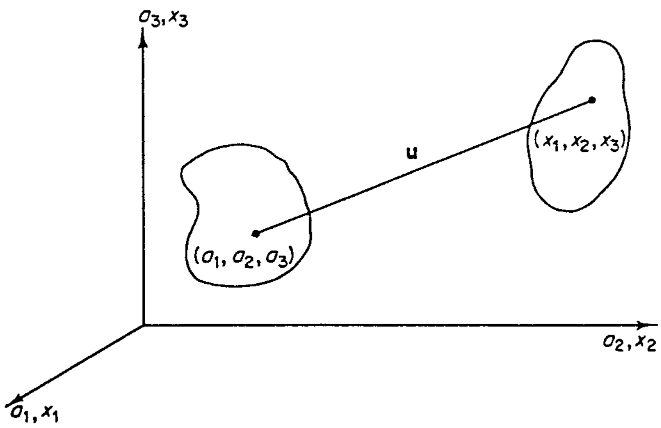


Figure 5.2 Displacement vector.

The vector $\overrightarrow{PQ}$ is called the displacement vector. A deformation can be described by a displacement field. Then the deformation of the body is known if x_1, x_2, x_3 are known functions of a_1,a_2, a_3:

$$X_I = x_I(a_1, a_2, a_3)$$

This is a *mathematical description of deformation*: transformation (*mapping*) from a_1, a_2, a_3 , to x_1, x_2, x_3. The functions for x_I are single valued, continuous and have the *unique inverse* **

$$a_i = a_i(x_1, x_2, x_3)$$

for *every point* in the body.

- **The displacement vector u** is then defined by its components

$$u_i = x_i - a_i$$

If a displacement vector is associated with every particle in the *original position,* we may write

$$u_i(a_1, a_2, a_3) = x_i(a_1, a_2, a_3) - a_i$$

If that displacement is associated with the particle in the *deformed position,* we write

$$u_i(x_1, x_2, x_3) = x_i - a_i(x_1, x_2, x_3)$$

** The condition for the function to be single valued, continuous and to have the *unique inverse is* : the Jacobian $|\partial x_i / \partial a_j|$ must not vanish in the space occupied by the body.

5.2 The Strain

The idea that the stress in a body is related to the strain was first announced by Robert Hooke (1635-1703). A rigid-body motion induces no stress. We must consider the stretching and distortion of the body.

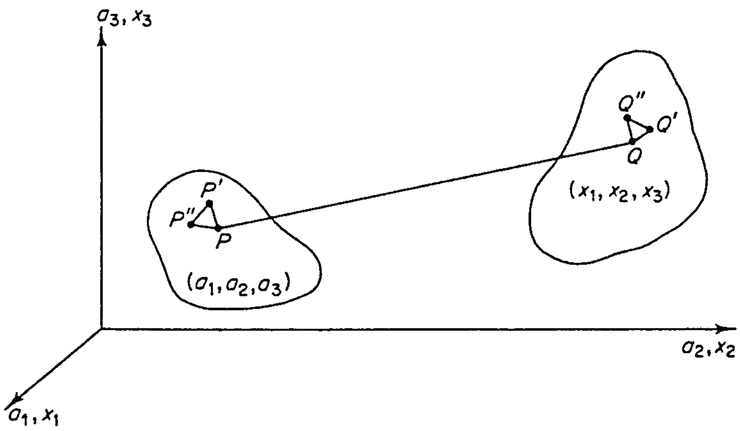


Figure 5.3 Deformation of a body.

The description of the change in distance between any two points of the body is the key to the analysis of deformation. The square of the length ds_0 of PP' in the original configurations given by

$$ds_0^2 = da_1^2 + da_2^2 + da_3^2.$$

When P and P' are deformed to the points $Q(x_1, x_2, x_3)$ and $Q'(x_1+dx_1, x_2+dx_2, x_3+dx_3)$, respectively, the square of the length ds of the new element QQ' is

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

then,

$$dx_i = \frac{\partial x_i}{\partial a_j} da_j, da_i = \frac{\partial a_i}{\partial x_j} dx_j$$

Hence, on introducing the Kronecker delta, we may write

$$ds_0^2 = \delta_{ij} da_i da_j = \delta_{ij} \frac{\partial a_i}{\partial x_l} \frac{\partial a_j}{\partial x_m} dx_l dx_m$$

$$ds^2 = \delta_{ij} dx_i dx_j = \delta_{ij} \frac{\partial x_i}{\partial a_l} \frac{\partial x_j}{\partial a_m} da_l da_m$$

The *difference between the squares of the length elements* may be written, after several changes in the symbols for dummy indices, either as

$$ds^2 - ds_0^2 = \left(\delta_{\alpha\beta} \frac{\partial x_\alpha}{\partial a_i} \frac{\partial x_\beta}{\partial a_j} - \delta_{ij} \right) da_i da_j$$

or as $$ds^2 - ds_0^2 = \left(\delta_{ij} - \delta_{\alpha\beta} \frac{\partial a_\alpha}{\partial x_i} \frac{\partial a_\beta}{\partial x_j} \right) dx_i dx_j$$

We define the *strain tensors*

$$E_{ij} = \frac{1}{2}\left(\delta_{\alpha\beta} \frac{\partial x_\alpha}{\partial a_i} \frac{\partial x_\beta}{\partial a_j} - \delta_{ij} \right)$$

$$e_{ij} = \frac{1}{2}\left(\delta_{ij} - \delta_{\alpha\beta} \frac{\partial a_\alpha}{\partial x_l} \frac{\partial a_\beta}{\partial x_j} \right)$$

so that

$$ds^2 - ds_0^2 = 2E_{ij}\, da_i\, da_{j,}$$

$$ds^2 - ds_0^2 = 2e_{ij}\, dx_i\, dx_j$$

The strain tensor E_{ij} was introduced by Green and St.-Venant and is called *Green's strain tensor*. The strain tensor e_{ij} was introduced by Cauchy for infinitesimal strains and by Almansi and Hamel for finite strains and is known as *Almansi's strain tensor*. In analogy with terminology in hydrodynamics, E_{ij} is often referred to as *Lagrangian* and e_{ij} as *Eulerian*.

The tensors E_{ij} and e_{ij} are obviously symmetric; i.e.,

$$E_{ij}=E_{ji} \ , \ e_{ij}=e_{ji}$$

The necessary and sufficient condition that a deformation of a body be a rigid-body motion is that *all components of the strain tensor E_{ij} or e_{ij} be **zero** throughout the body.*

5.3 Strain components in terms of displacements

Introduce the *displacement vector* **u**

$$u_\alpha = x_\alpha - a_\alpha \qquad (\alpha=1,2,3)$$

then

$$\frac{\partial x_\alpha}{\partial a_i} = \frac{\partial u_\alpha}{\partial a_i} + \delta_{\alpha i}, \qquad \frac{\partial a_\alpha}{\partial x_i} = \delta_{\alpha i} - \frac{\partial u_\alpha}{\partial x_i}$$

and the strain tensors reduce to the simple form:

$$E_{ij} = \frac{1}{2}\left[\delta_{\alpha\beta}\left(\frac{\partial u_\alpha}{\partial a_i} + \delta_{\alpha i}\right)\left(\frac{\partial u_\beta}{\partial a_j} + \delta_{\beta j}\right) - \delta_{ij}\right]$$

$$= \frac{1}{2}\left[\frac{\partial u_j}{\partial a_i} + \frac{\partial u_i}{\partial a_j} + \frac{\partial u_\alpha}{\partial a_i}\frac{\partial u_\alpha}{\partial a_j}\right]$$

and
$$e_{ij} = \frac{1}{2}\left[\delta_{ij} - \delta_{\alpha\beta}\left(-\frac{\partial u_\alpha}{\partial x_i} + \delta_{\alpha i}\right)\left(-\frac{\partial u_\beta}{\partial x_j} + \delta_{\beta j}\right)\right]$$

$$= \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} - \frac{\partial u_\alpha}{\partial x_i}\frac{\partial u_\alpha}{\partial x_j}\right]$$

We have the typical terms:

$$E_{aa} = \frac{\partial u}{\partial a} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial a}\right)^2 + \left(\frac{\partial v}{\partial a}\right)^2 + \left(\frac{\partial w}{\partial a}\right)^2\right]$$

$$e_{xx} = \frac{\partial u}{\partial x} - \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

$$E_{ab} = \frac{1}{2}\left[\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a} + \left(\frac{\partial u}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial v}{\partial b} + \frac{\partial w}{\partial a}\frac{\partial w}{\partial b}\right)\right]$$

$$e_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \left(\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right)\right]$$

Note that when the <u>*Lagrangian strain tensor*</u> is evaluated, *u, v, w* are considered functions of a, b, c, the position of points in the body in unstrained configuration; whereas they are considered functions of x, y, z, the position of points in the strained configuration, when the <u>*Eulerian strain tensor*</u> is evaluated.

e_{ij} reduces to <u>Cauchy's infinitesimal strain tensor,</u>

$$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$$

In unabridged notation,

$$e_{xx} = \frac{\partial u}{\partial x}, \qquad e_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) = e_{yx}$$

$$e_{yy} = \frac{\partial v}{\partial y}, \qquad e_{xz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = e_{zx}$$

$$e_{zz} = \frac{\partial w}{\partial z}, \qquad e_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = e_{zy}$$

In the case of infinitesimal displacement, the distinction between the *Lagrangian* and *Eulerian* strain tensor disappears, since then it is immaterial whether the *derivatives of the displacements* are calculated at the position of a point before or after deformation.

(Warning: Notation for Shear Strain ------- In most books and papers, the strain components are defined as

$$e_x = \frac{\partial u}{\partial x}, \qquad \gamma_{xy} = 2e_{xy} = \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$e_y = \frac{\partial v}{\partial y}, \qquad \gamma_{yz} = 2e_{yz} = \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$e_z = \frac{\partial w}{\partial z}, \qquad \gamma_{zx} = 2e_{zx} = \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

we shall not use this notation, because the components e_x, γ_{xy}, etc., together <u>*do not form a tensor*</u>*).*

5.4 Geometric interpretation of strain components

- <u>**Infinitesimal**</u> **strain** — Consider a line element dx ($dy=dz=0$), we have

$$ds^2 - ds_0^2 = 2e_{xx}(dx)^2$$

Hence,

$$ds - ds_0 = \frac{2e_{xx}(dx)^2}{ds + ds_0}, \qquad \frac{ds - ds_0}{ds} = e_{xx}$$

$\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}$ represents the change in the angle xOy.

$$e_{xy} = \frac{1}{2}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right] = \frac{1}{2}\tan(\text{change of angle xOy})$$

In engineering usage, the strain components e_{ij} $(i \neq j)$ doubled ,*i.e.*, $2e_{ij}$, are called the shearing strains or detrusions. The case 3 of Fig. 5.4 is called the case of *simple shear*.

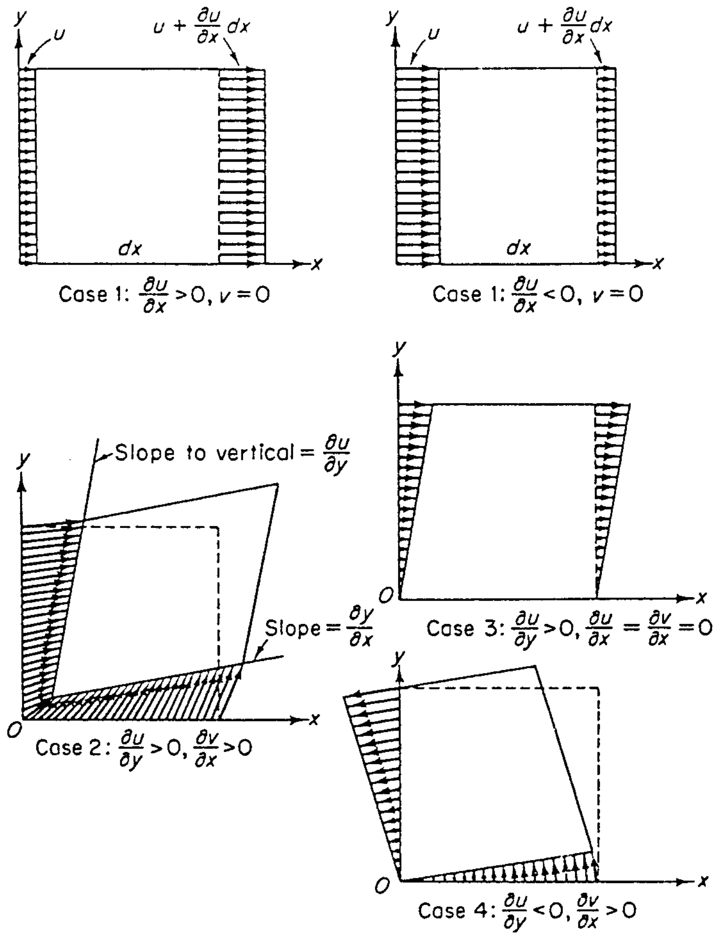


Figure 5.4 Deformation gradients and interpretation of infinitesimal strain components.

- **Infinitesimal Rotation** —Consider an infinitesimal displacement **u**=**u**(**x**), form a Cartesian tensor:

$$\omega_{ij} = \frac{1}{2}\left[\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}\right]$$

which is antisymmetric, *i.e.,*

$$\omega_{ij} = -\omega_{ji}$$

Hence, the tensor ω_{ij} has only three independent components. Define *dual vector* and *Dual tensor* as

$$\omega_k = \frac{1}{2} \in_{kij} \omega_{ij}$$

$$\omega_{ij} = \in_{ijk} \omega_k$$

We shall call ω_k and ω_{ij} , respectively, the *rotation vector* and *rotation tensor* of the displacement field u_i. The vanishing of the symmetric strain tensor E_{ij} *or* e_{ij} is a *necessary and sufficient condition* for a neighborhood of a particle to be moved like *a rigid body*. *A rigid-body motion* consists of a translation and a rotation. The translation is u_I and what is *rotation*?

$$du_i = \frac{\partial u_i}{\partial x_i} dx_j \Rightarrow$$

$$du_i = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)dx_j + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)dx_j$$

so

$$du_i = -\omega_{ij} dx_j = \omega_{ji} dx_j = -\in_{ijk} \omega_k dx_j = (\omega \times \mathbf{dx})_i$$

- **<u>Finite strain</u> components** — When the strains are not small, it is also easy to give simple geometric interpretations for strain components. Consider first the line element $da_1=ds_0$, the traditional extension of element is:

$$E_1 = \frac{ds - ds_0}{ds_0}$$

or

$$ds = (1 + E_1)ds_0$$

On the other hand,

$$ds^2 - ds_0^2 = 2E_{ij}da_i da_j = 2E_{11}(da_1)^2$$

so

$$(1 + E_1)^2 - 1 = 2E_{11}$$

which gives the meaning of E_{11} in terms of E_1. Conversely,

$$E_1 = \sqrt{1 + 2E_{11}} - 1$$

This reduces to

$$E_1 = E_{11}$$

when E_{11} is small compared to 1.

To get the Physical significance of the component E_{12}, consider two perpendicular line elements

$$\mathbf{ds_0}: \qquad da_1 = ds_0, da_2 = 0, da_3 = 0$$

$$\mathbf{d\bar{s}_0}: \qquad da_1 = 0, da_2 = d\bar{s}_0, da_3 = 0$$

$$(\mathbf{ds}) \bullet (\mathbf{d\bar{s}}) = ds\,d\bar{s}\cos\theta = dx_k d\bar{x}_k$$

$$= \frac{\partial x_k}{\partial a_i}da_i \frac{\partial x_k}{\partial a_j}d\bar{a}_j = \frac{\partial x_k}{\partial a_1}\frac{\partial x_k}{\partial a_2}ds_0 d\bar{s}_0$$

So, $$E_{12} = \frac{1}{2}\frac{\partial x_k}{\partial a_1}\frac{\partial x_k}{\partial a_2}.$$

Hence, $$ds\,d\bar{s}\cos\theta = 2E_{12}ds_0 d\bar{s}_0$$

But, as $$ds = \sqrt{1+2E_{11}}ds_0, d\bar{s} = \sqrt{1+2E_{22}}d\bar{s}_0$$

hence,
$$\cos\theta = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}}$$
$$\sin\alpha_{12} = \frac{2E_{12}}{\sqrt{1+2E_{11}}\sqrt{1+2E_{22}}} \quad where\, \alpha_{12} = \frac{\pi}{2} - \theta$$

These equations exhibits the relationship of E_{12} to the angles θ and α_{12}. The interpretation is not as simple as infinitesimal strain.

<u>Similar analysis can be done for the *Eulerian strain*</u>: extension e_1 per unit deformed length as

$$e_1 = \frac{ds - ds_0}{ds}$$

$$e_1 = 1 - \sqrt{1-2e_{11}}$$

Furthermore, we have

$$\sin\beta_{12} = \frac{2e_{12}}{\sqrt{1-2e_{11}}\sqrt{1-2e_{22}}}$$

In case of **<u>infinitesimal strains</u>** , we all get :

$$e_1 = e_{11}, \qquad E_1 = E_{11}, \qquad \alpha_{12} = 2E_{12}, \qquad \beta_{12} = 2e_{12}$$

5.5 Principal Strains: Mohr's Circle

Because the strain tensor concerned is symmetric, we similarly have:

(a) $$\left| e_{ij} - e\delta_{ij} \right| = 0$$

(b) $$(e_{ij} - e_1\delta_{ij})v_j^{(1)} = 0, (i = 1,2,3)$$

(c) In principal plane, the strain components have the canonical form

$$\begin{pmatrix} e_1 & 0 & 0 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$$

(d) <u>*Strain deviation tensor*</u> $e'_{ij} = e_{ij} - \frac{1}{3}e_{\alpha\alpha}\delta_{ij}$. Tensors e_{ij} and e'_{ij} have the following <u>*independent strain invariants:*</u>

$$I_1 = e_{ij}\delta_{ij}, \qquad J_1 = e'_{ij}\delta_{ij} = 0$$

$$I_2 = \frac{1}{2}e_{ik}e_{ik}, \qquad J_2 = \frac{1}{2}e'_{ik}e'_{ik}$$

$$I_3 = \frac{1}{3}e_{ik}e_{km}e_{mi}, \qquad J_3 = \frac{1}{3}e'_{ik}e'_{km}e'_{mi}$$

(f) <u>*Lame's ellipsoid*</u> is also applicable to strain

5.6 Infinitesimal Strain components in Polar coordinates

- **Analytical approach:** In polar coordinates r, θ, z, then strain components may be designated $e_{rr}, e_{\theta\theta}, e_{zz}, e_{r\theta}, e_{rz}, e_{z\theta}$, and by the tensor transformation law, they are related to $e_{xx}, e_{yy}, e_{zz}, e_{xy}, e_{yz}, e_{zx}$, as in the cases of stresses. However, the *strain-displacement relations may appear quite different* from the corresponding formulas in rectangular coordinates.

Cylindrical-polar and rectangular coordinates:

$$\begin{cases} x = r\cos\theta, & \theta = \tan^{-1}\dfrac{y}{x} \\ y = r\sin\theta, & r^2 = x^2 + y^2 \\ z = z \end{cases}$$

$$\frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta, \qquad \frac{\partial r}{\partial y} = \frac{y}{r} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = -\frac{y}{r^2} = -\frac{\sin\theta}{r}, \qquad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

Transformed to:

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

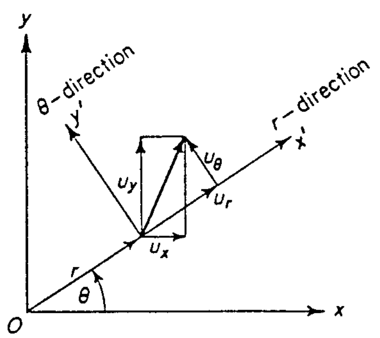


Figure 5.5 Displacement vector in polar coordinates.

It is seen that these displacements are related by the equations

$$
\begin{aligned}
u_x &= u_r \cos\theta - u_\theta \sin\theta \\
u_y &= u_r \sin\theta + u_\theta \cos\theta \\
u_z &= u_z
\end{aligned}
$$

The strain components in polar coordinates are designated as

$$
\begin{pmatrix}
e_{rr} & e_{r\theta} & e_{rz} \\
e_{\theta r} & e_{\theta\theta} & e_{\theta z} \\
e_{zr} & e_{z\theta} & e_{zz}
\end{pmatrix}
$$

	x	y	z
r or x'	cosθ	sinθ	0
θ or y'	-sinθ	cosθ	0
z or z'	0	0	1

The tensor transformation law holds, and we have

$$
\begin{aligned}
e_{rr} &= e_{xx} \cos^2\theta + e_{yy} \sin^2\theta + e_{xy} \sin 2\theta \\
e_{\theta\theta} &= e_{xx} \sin^2\theta + e_{yy} \cos^2\theta - e_{xy} \sin 2\theta \\
e_{r\theta} &= (e_{yy} - e_{xx}) \cos\theta \sin\theta + e_{xy} (\cos^2\theta - \sin^2\theta) \\
e_{zr} &= e_{zx} \cos\theta + e_{zy} \sin\theta \\
e_{z\theta} &= -e_{zx} \sin\theta + e_{zy} \cos\theta \\
e_{zz} &= e_{zz}
\end{aligned}
$$

using

$$
\begin{aligned}
e_{xx} &= \frac{\partial u_x}{\partial x}, e_{yy} = \frac{\partial u_y}{\partial y}, e_{zz} = \frac{\partial u_z}{\partial z} \\
e_{xy} &= \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\
e_{zx} &= \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)
\end{aligned}
$$

we have

$$e_{xx} = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)(u_r\cos\theta - u_\theta\sin\theta)$$

$$= \cos^2\theta\frac{\partial u_r}{\partial r} + \sin^2\theta\left(\frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}\right) - \cos\theta\sin\theta\left(\frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{r\partial\theta} - \frac{u_\theta}{r}\right)$$

$$e_{yy} = \sin^2\theta\frac{\partial u_r}{\partial r} + \cos^2\theta\left(\frac{u_r}{r} + \frac{\partial u_\theta}{r\partial\theta}\right) + \cos\theta\sin\theta\left(\frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{r\partial\theta} - \frac{u_\theta}{r}\right)$$

$$e_{xy} = \frac{\sin^2\theta}{2}\left(\frac{\partial u_r}{\partial r} - \frac{\partial u_\theta}{r\partial\theta} - \frac{u_r}{r}\right) + \frac{\cos^2\theta}{2}\left(\frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{r\partial\theta} - \frac{u_\theta}{r}\right)$$

Finally we have

$$e_{rr} = \frac{\partial u_r}{\partial r}$$

$$e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$e_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$e_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial\theta} + \frac{\partial u_\theta}{\partial z}\right)$$

$$e_{zz} = \frac{\partial u_z}{\partial z}$$

Thus , we see that method of transformation of coordinates is tedious but straightforward.

- **Direct Derivation (Geometric approach)** — From the geometric definition of infinitesimal strain, we have the radial strain (see the figure below)

$$e_{rr} = \frac{u_r + (\partial u_r / \partial r) - u_r}{dr} = \frac{\partial u_r}{\partial r}$$

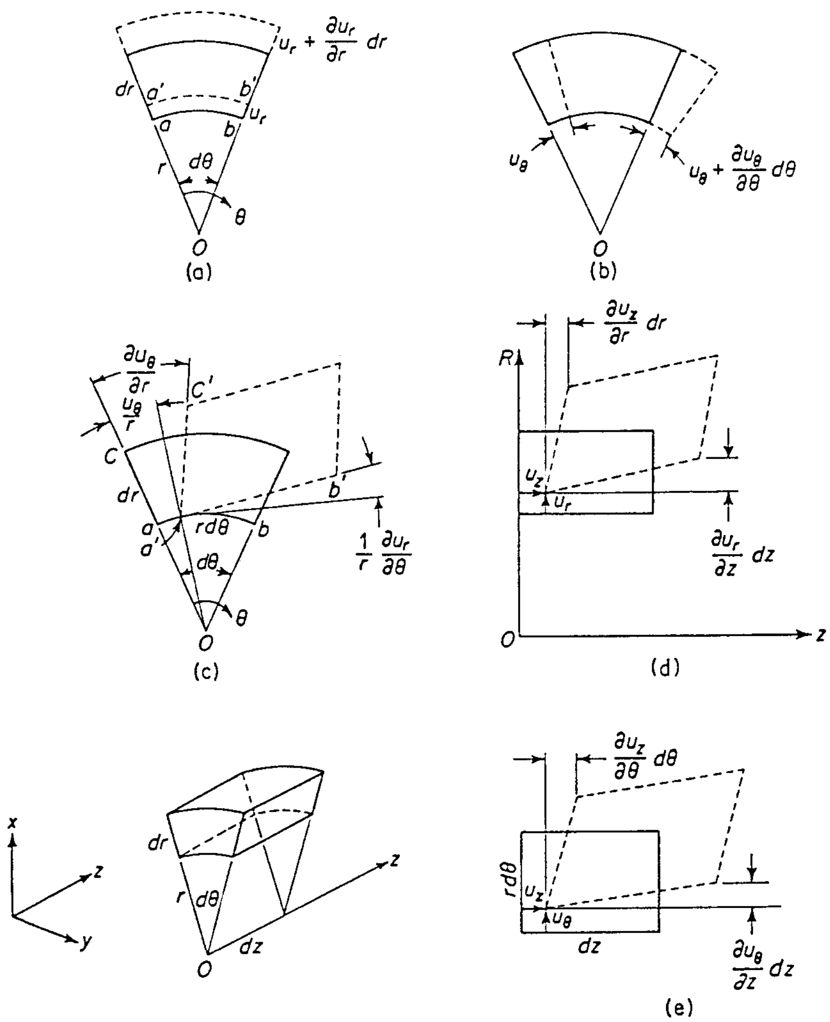


Figure 5.6 Displacement in cylindrical polar coordinates. (From E. E. Sechler, *Elasticity in Engineering*, Courtesy Mrs. Magaret Sechler.) A free-body diagram of an infinitesimal element of material and two systems of coordinates are shown at the lower left corner. (a) Radial strain due to variation of the radial displacement field in the radial direction. (b) Circumferential strain due to variation of circumferential displacement in the circumferential direction. (c) $\partial u_\theta/\partial r$ and $(1/r)\partial u_r/\partial\theta$ cause shear strain $e_{r\theta}$. (d) $\partial u_z/\partial r$ and $\partial u_r/\partial z$ cause shear strain e_{rz}. (e) $(1/r)\partial u_z/\partial\theta$ and $\partial u_\theta/\partial z$ cause shear strain $e_{z\theta}$.

The tangential strain are due to both radial and tangential displacements, the later is,

$$e_{\theta\theta}^{(2)} = \frac{u_\theta + (\partial u_\theta / \partial\theta) - u_\theta}{r d\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

The total tangential strain is

$$e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta}$$

The normal strain in the axial direction is

$$e_{r\theta} = \frac{\partial u_z}{\partial z}$$

Similarly we have

$$e_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right)$$

$$e_{z\theta} = \frac{1}{2}\left[\frac{(\partial u_z / \partial\theta) d\theta}{r d\theta} + \frac{(\partial u_\theta / \partial z) dz}{dz}\right] = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_z}{d\theta} + \frac{\partial u_\theta}{\partial z}\right]$$

$$e_{zr} = \frac{1}{2}\left[\frac{(\partial u_r / \partial z) dz}{dz} + \frac{(\partial u_z / \partial r)\partial r}{dr}\right] = \frac{1}{2}\left[\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right]$$

Indeed, the above <u>*direct geometric method*</u> of derivation provides a much clearer mental picture that <u>*the algebraic method*</u> of the preceding section.

5.7 Other strain measures

The above strain definition is the most natural one when we base our analysis of deformation on the change of *the square of the distances between any two particles*. By Pythagoras's theorem (that the square of the hypothenuse of a right triangle is equal to the sum of the squares of the legs), we have

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

However deformation *does not have to be* described this way. For example, we introduce *deformation gradients* (tensor):

$$u_{i,j} = \frac{\partial u_i}{\partial x_j} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z} \end{pmatrix} +$$

$$\begin{pmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ -\frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) & -\frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) & 0 \end{pmatrix}$$

The infinitesimal strain tensor is the symmetric part of the deformation gradient and the antisymmetric part represents the rotation tensor.

Other well-known strain measures are *Cauchy's strain tensors*

$$C_{ij} = \frac{\partial a_k}{\partial x_i}\frac{\partial a_k}{\partial x_j}, \qquad \bar{C}_{ij} = \frac{\partial x_k}{\partial a_i}\frac{\partial a_j}{\partial a_j}$$

and *Finger's strain tensors*

$$B_{ij} = \frac{\partial x_i}{\partial a_k}\frac{\partial x_j}{\partial a_k}, \qquad \bar{B}_{ij} = \frac{\partial a_i}{\partial x_k}\frac{\partial a_j}{\partial x_k}$$

For these tensors, the absence of strain is indicated , not by the vanishing of C_{ij}, *or* B_{ij} , but by $C_{ij} = \delta_{ij}, B_{ij} = \delta_{ij}$. We shall not discuss these strain measures any further, except to note that they may be convenient for some special purposes in advanced theories of continua.

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