Parameter estimation using the quasi-linear viscoelastic model proposed by Fung

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Parameter Estimation Using the Quasi-Linear Viscoelastic Model Proposed by Fung

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Using the quasi-linear viscoelastic model proposed by Fung for the description of the viscoelastic properties of soft biological tissues, the parameters governing their time-dependent behavior are commonly estimated from relaxation experiments. Exact quantification is possible from the response to a step change in the strain. Since it is physically impossible to realize a true step change in the strain, in practice the response to a steplike strain change is used. In the present study the discrepancies between the exact and the estimated parameter values are investigated using a hypothetical quasi-linear viscoelastic material. The parameter τ_1, governing the fast viscous phenomena, is found to be subject to the largest errors. Methods for obtaining better estimates of τ_1 are outlined in a number of special cases.

Introduction

The quasi-linear viscoelastic model proposed by Fung [1] provides a relatively easily handled and useful tool for the description of the behavior of many soft biological tissues [2-11]. The relation between stress σ and strain ϵ in simple elongation is given by

$$\sigma(t) = \int_{\tau=0}^{t} G(t-\tau)(d\sigma^e/d\epsilon)(d\epsilon/d\tau)d\tau \qquad (1)$$

with

$$\sigma(t)=0 \text{ and } \epsilon(t) = 0 \text{ for } t < 0$$

$$G(t=0)=1$$

$G(t)$ and $\sigma^e(\epsilon)$ represent the "reduced relaxation function" and the "elastic response," respectively. A polynomial or exponential function of strain ϵ may be chosen for the elastic response, whereas the reduced relaxation function can be formulated in terms of a relaxation spectrum. Fung [1] proposed a relaxation spectrum of the form

$$S(\tau) = \begin{cases} C/\tau & \text{for } 0<\tau_1 \leq \tau \leq \tau_2 \quad \text{with } \tau_1 << \tau_2 \\ 0 & \text{elsewhere} \end{cases} \qquad (2)$$

which, for the reduced relaxation function, yields

$$G(t) = [1 + C\{E_1(t/\tau_2) - E_1(t/\tau_1)\}]/[1 + C\ln(\tau_2/\tau_1)] \qquad (3)$$

with

$E_1(x) =$

$\int_{y=x}^{\infty} (\exp(-y)/y)dy$, the exponential integral function [13].

The dimensionless positive constant C determines the degree to which viscous effects are present whereas the time constants τ_1 and τ_2 govern the fast and slow viscous phenomena, respectively, [12]. Theoretically the values of C, τ_1 and τ_2 can be determined from the stress response to a step change in the strain ϵ from zero to ϵ_o at $t = 0$ (Fig. 1(*a*)). $\sigma^e(\epsilon_o) = \sigma(0^+)$ and $G(t) = \sigma(t)/\sigma(0^+)$ apply in this case.

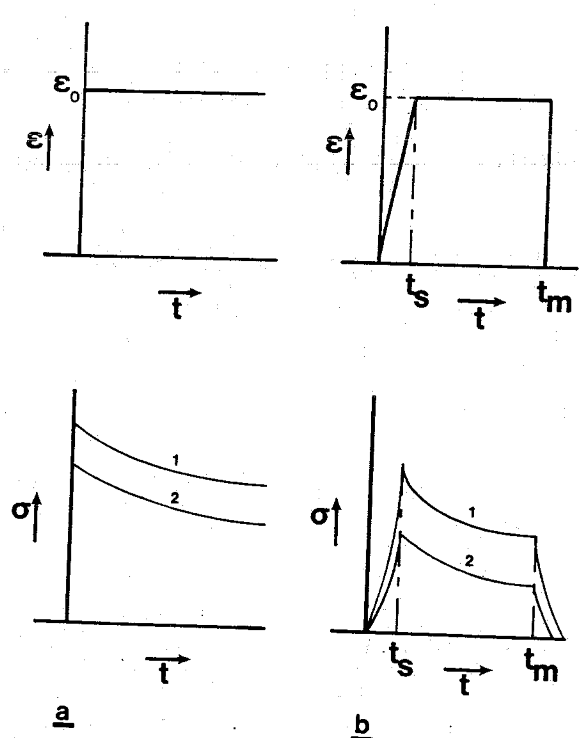

Fig. 1 Schematic diagrams showing the course of stress and strain with time for (*a*) step change of the strain; (*b*) steplike change of the strain: 1 stress response for a strongly nonlinear elastic response (large value of the constant *B* in equation (11)), 2 idem for a weakly nonlinear elastic response (i.e., small value of *B*)

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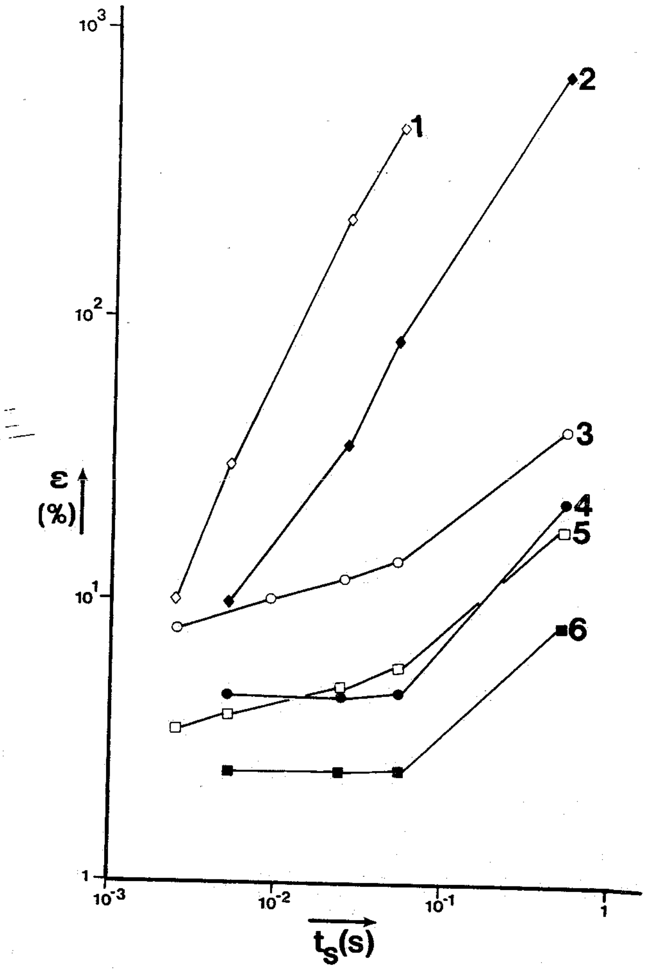

Fig. 2 The relative errors as a function of t_s with B as parameter. The different curves represent: 1) $\epsilon_{\tau 1}$ for B = 0.1, 2) $\epsilon_{\tau 1}$ for B = 100, 3) $\epsilon_{\tau 2}$ for B = 0.1, 4) $\epsilon_{\tau 2}$ for B = 100, 5) ϵ_C for B = 0.1, 6) ϵ_C for B = 100.

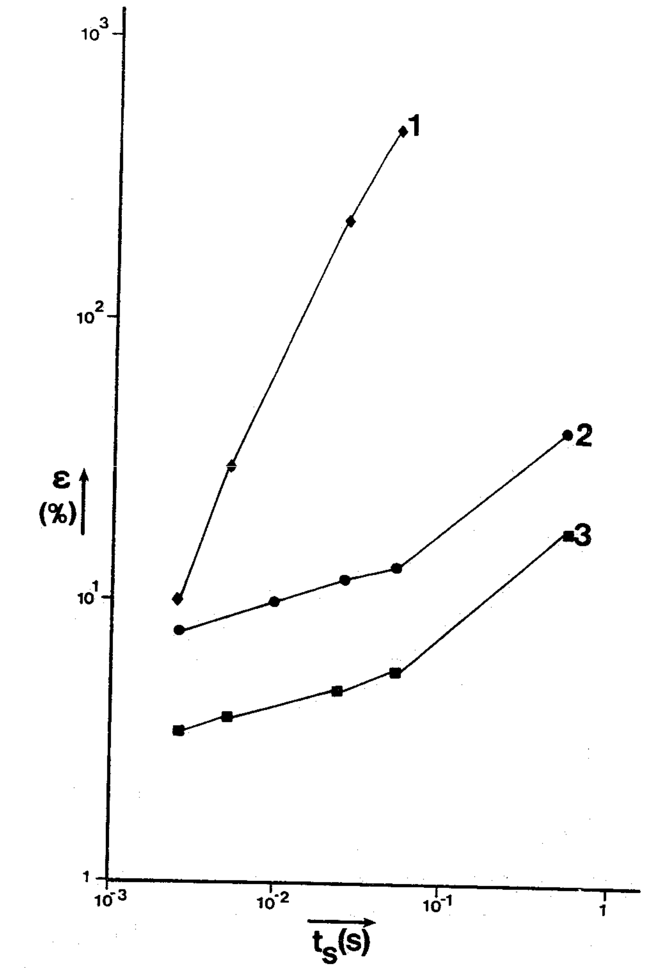

Fig. 3 The relative errors as a function of t_s with C as parameter. The different curves represent: 1) $\epsilon_{\tau 1}$ for C = 0.005, 0.05 and 0.5, 2) $\epsilon_{\tau 2}$ for C = 0.005, 0.05 and 0.5, 3) ϵ_C for C = 0.005, 0.05 and 0.5.

However, it is physically impossible to realize a true step change in the strain. It is therefore assumed in practice that the stress response to a fast steplike change in the strain (Fig. 1(*b*)) can be used as a fair approximation of the response to a true step change. The steplike change in strain in an experiment is realized by straining a sample from $\epsilon = 0$ to $\epsilon = \epsilon_o$ at a high strain rate within a time interval $[0, t_s]$, followed by maintaining $\epsilon = \epsilon_o$ during the time interval $[t_s, t_m]$ (Fig. 1(*b*)). The parameter quantification follows the lines depicted in a previous paper [11], using the approximations

$$G^*(t) = \sigma(t)/\sigma(t_s) \text{ for } t \geq t_s, \tag{4}$$
$$G^*(\infty) = G^*(t_m).$$

The parameter values thus obtained will be approximations of the true values. In this paper some investigations are presented as to the influence of the t_s and t_m values upon the discrepancies between the true constants C, τ_1 and τ_2 and the approximations C^*, τ_1^*, τ_2^*. Furthermore, methods for obtaining a better approximation of τ_1 will be outlined for some special cases.

Methods

For a hypothetical quasi-linear viscoelastic material, an expression for the true stress response during the constant strain phase ($t_s \leq t \leq t_m$), preceeded by a strain change realized at a constant high strain rate $v = \epsilon_o/t_s$ in the time interval $[0, t_s]$, as depicted in Fig. 1(*b*), was determined from equation (1) after substitution of equation (3) and an elastic response of the form

$$\sigma^e(\epsilon) = A[\exp(B\epsilon) - 1] \tag{5}$$

yielding

$$\begin{aligned}\sigma(t) = {} & A[\exp(B\epsilon_o) - 1]G(t - t_s) \\ & - [CA/\{1 + C\ln(\tau_2/\tau_1)\}][\{E_1((t - t_s)^* \\ & ^*(1/\tau_2 + Bv)) - E_1((t - t_s)(1/\tau_1 + Bv)) \\ & + E_1(t(1/\tau_1 + Bv)) - E_1(t(1/\tau_2 + Bv))\}\exp(B\epsilon_o) \\ & + E_1(t/\tau_2) - E_1(t/\tau_1) + E_1((t - t_s)/\tau_1) \\ & - E_1((t - t_s)/\tau_2)] \quad \text{for } t \geq t_s.\end{aligned} \tag{6}$$

The $\sigma(t)$ values were calculated for given values of A, B, C, τ_1, τ_2, ϵ_o, t_s and t_m and subsequently fed into a computer program, previously developed to fit the model to experimental data [11], with approximations C^*, τ_1^* and τ_2^* as the outcome. These approximations are thus the result of treating the $\sigma(t)$ values in the interval $[t_s, t_m]$ as the response to a true step change in the strain as is commonly done with the results of real experiments. The constants C^*, τ_1^* and τ_2^* were calculated from the function $G^*(t)$ (equation (4)) by means of the method presented in [11]. These were compared with the true values in terms of relative errors defined as

$$\epsilon_p = (p^* - p)/p^* 100 \text{ percent with } p = \tau_1 \text{ or } \tau_2 \tag{7}$$
$$\epsilon_C = (C - C^*)/C^* 100 \text{ percent}$$

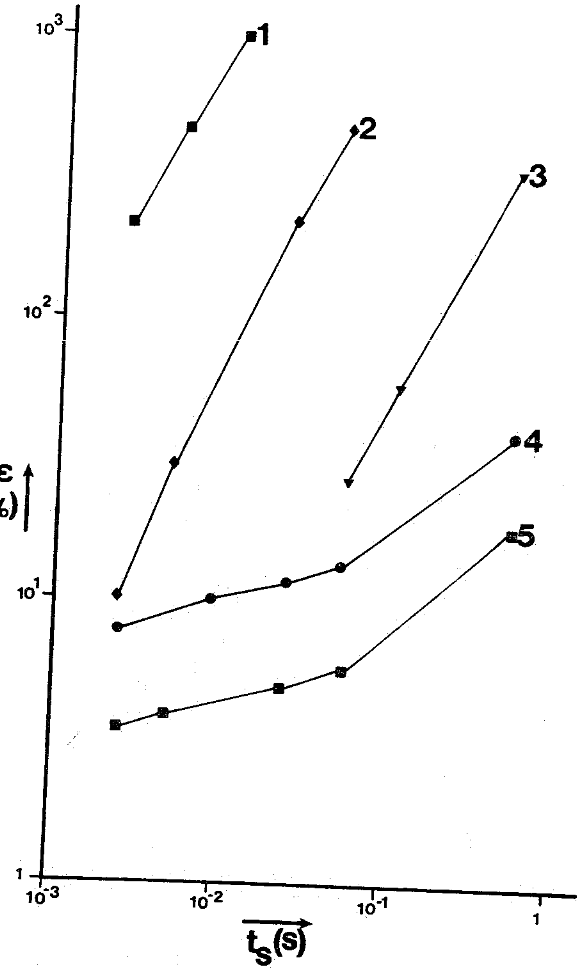

Fig. 4 The relative errors as a function of t_s with τ_1 as parameter. The different curves represent: 1) $\epsilon_{\tau 1}$ for τ_1 = 0.0005 s, 2) $\epsilon_{\tau 1}$ for τ_1 = 0.005 s, 3) $\epsilon_{\tau 1}$ for τ_1 = 0.05 s, 4) $\epsilon_{\tau 2}$ for τ_1 = 0.0005, 0.05 and 0.5 s, 5) ϵ_C for τ_1 = 0.0005, 0.05 and 0.5 s.

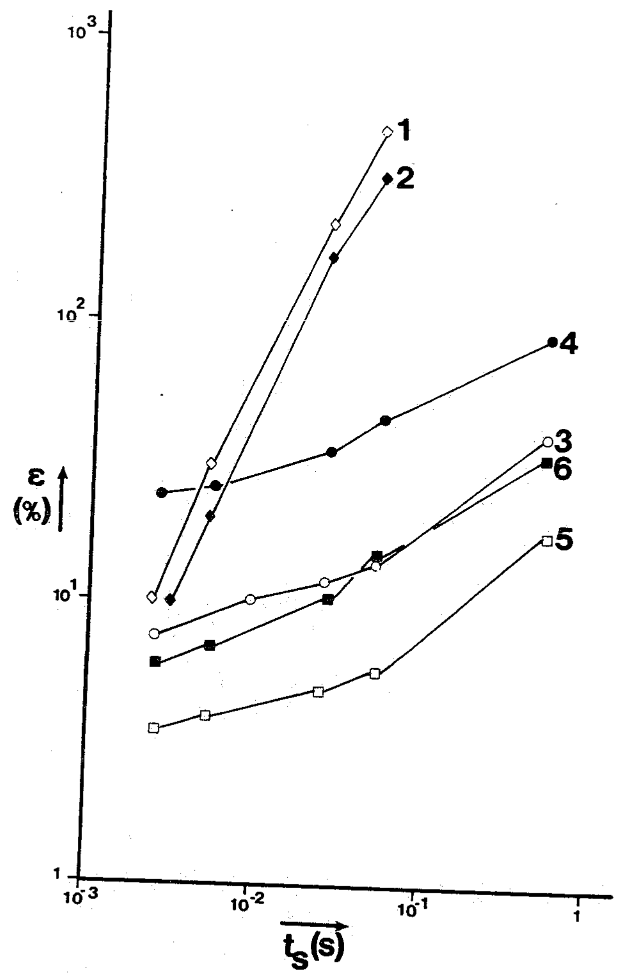

Fig. 5 The relative errors as a function of t_s with τ_2 as parameter. The different curves represent: 1) $\epsilon_{\tau 1}$ for τ_2 = 50 s, 2) $\epsilon_{\tau 1}$ for τ_2 = 5 s, 3) $\epsilon_{\tau 2}$ for τ_2 = 50 s, 4) $\epsilon_{\tau 2}$ for τ_2 = 5 s, 5) ϵ_C for τ_2 = 50 s, 6) ϵ_C for τ_2 = 5 s.

These errors were evaluated as functions of t_s and t_m. B, C, τ_1 and τ_2 were varied around the standard values $B = 0.1$, $C = 0.05$, $\tau_1 = 0.005$ s and $\tau_2 = 50$ s, while A and ϵ_o remained constant (1 N/m² and 0.1, respectively). Furthermore, $t_m = 100$ s was used unless otherwise stated.

Results

In Figs. 2–5 the relative errors in τ_1, τ_2 and C are given as functions of t_s with B, C, τ_1 and τ_2 as parameters. Figures 2 and 5 show the course of the errors as a function of t_s for two different values of B and τ_2, respectively, while in Figs. 3 and 4 similar data are presented for the different values of C and τ_1. It is clear that, with increasing values of t_s the relative errors increase. In the presentation and discussion of the results, attention will be mainly focused on what happens at t_s – values of about 0.1 s, as values of this order are mostly obtained in experiments [6, 7, 10]. From Figs. 2–5 $\epsilon_{\tau 1}$ can be seen to attain rather dramatic values, whereas $\epsilon_{\tau 2}$ and ϵ_C are about a factor of 10 smaller than $\epsilon_{\tau 1}$. Figure 2 shows that the weaker the nonlinearity of the elastic response (decreasing B), the larger are the errors which are found. Variation of C has no effect on the relative errors (Fig. 3). The influence of variations of τ_1 on $\epsilon_{\tau 2}$ and ϵ_C is negligible, but $\ln(\epsilon_{\tau 1})$ appears to be almost inversely proportional to τ_1 (Fig. 4). Variation of τ_2 results in only slight variations of $\epsilon_{\tau 1}$, $\epsilon_{\tau 2}$ and ϵ_C (Fig. 5). Apart from the risetime t_s, the duration $t_m - t_s$ of the constant strain phase is also likely to have some influence on the accuracy of the approximations. In Fig. 6 the relative errors are presented as functions of t_m ($t_s = 0.1$ s; $10 \leq t_m \leq 1000$ s). $\epsilon_{\tau 1}$ appears to be independent of t_m whereas $\epsilon_{\tau 2}$ and ϵ_C are only independent of t_m for $t_m > 100$ s and $t_m > 200$ s, respectively.

Summarizing, it can be stated that, for the range of parameter values considered here, ϵ_C does not exceed 20 percent for $t_s = 0.1$ s and the maximum value of $\epsilon_{\tau 2}$ amounts to 90 percent (Fig. 5). The errors involved with τ_1 are often more considerable, reaching values of 1000 percent or even more.

Discussion of the Results

The independence of $\epsilon_{\tau 1}$ and t_m is consistent with the importance of τ_1 for the description of fast viscous phenomena [12]. $\epsilon_{\tau 2}$ and ϵ_C, however, do not become constant until t_m exceeds a certain value. Thus an increase in the duration of an experiment will no longer reduce the values of $\epsilon_{\tau 2}$ and ϵ_C when t_m has passed the value at which each stabilizes. Bearing in mind the definition (7) of ϵ_C and the values of ϵ_C in Figs. 2–5, it is seen that C^* is an underestimate of C. This can be explained from the results of the sensitivity analysis mentioned earlier [12]. There it was demonstrated that an increase of C corresponds to an increase of stress relaxation. As stress relaxation already occurs during the straining phase in an experiment ($0 \leq t \leq t_s$), the total stress relaxation, and consequently C, is underestimated on the basis of the relaxation relative to $\sigma(t_s)$.

From the results the largest errors appear to be involved in

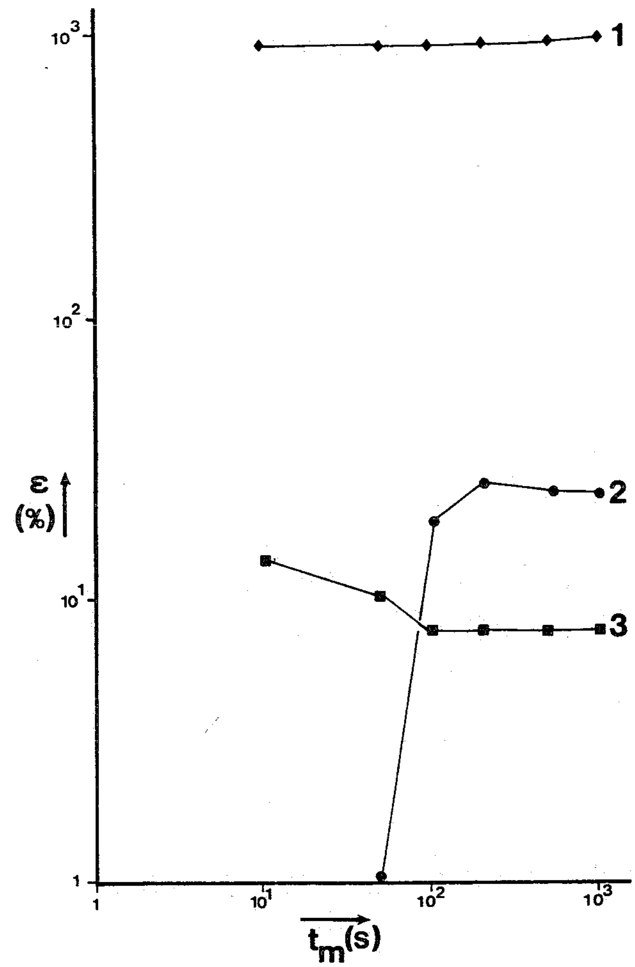

Fig. 6 Illustration of the influence of t_m on the relative errors. Curves 1, 2 and 3 represent $\epsilon_{\tau 1}$, $\epsilon_{\tau 2}$ and ϵ_C, respectively.

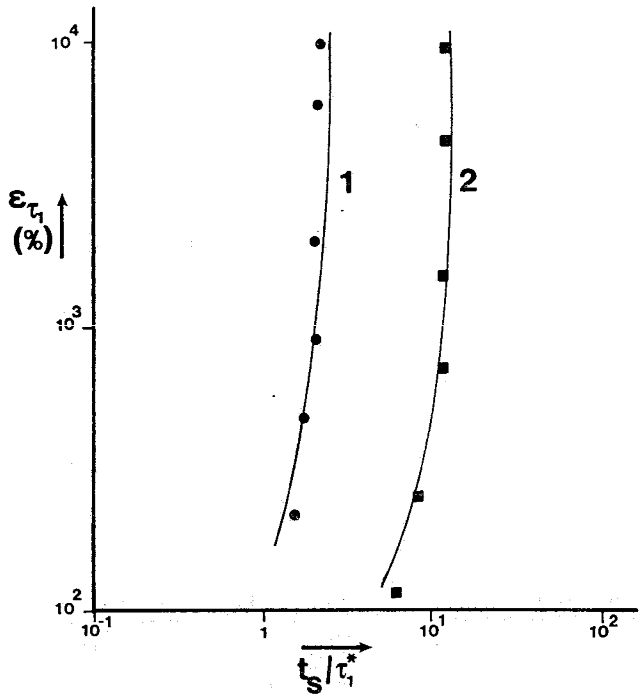

Fig. 7(*a*) The relative error $\epsilon_{\tau 1}$ as a function of t_s/τ_1^* for a weakly and highly nonlinear material. Curves 1 and 2 represent $\epsilon_{\tau 1}$ for $B=0.1$ and $B=100$, respectively.

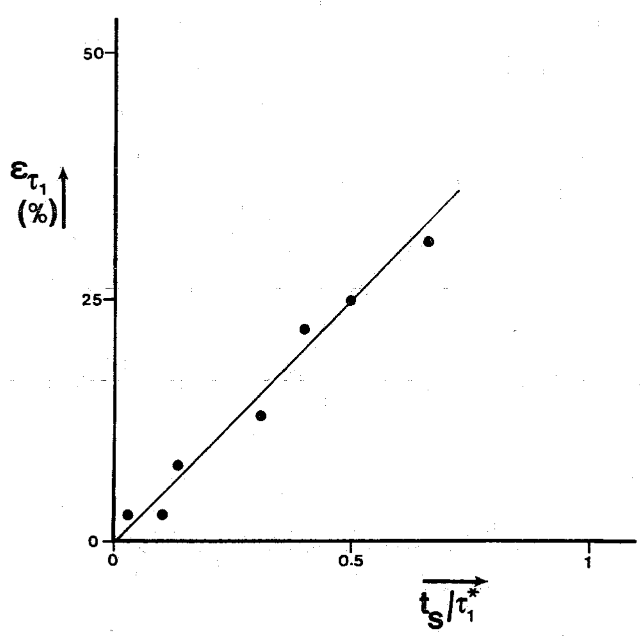

Fig. 7(*b*) The relative error $\epsilon_{\tau 1}$ for small values of t_s/τ_1^*

the determination of τ_1. They are mainly governed by the ratio t_s/τ_1^* and the nonlinearity of the elastic response, that is the value of B. The value of A (equation (5)) does not influence the errors of τ_1, τ_2 and C as it follows from equations (4) and (6) that $G^*(t)$ is not a function of A. When $B\epsilon_o \ll 1$, it follows from equation (5) that $\sigma^e(\epsilon_o) \approx AB\epsilon_o$ and then the two elastic parameters have no influence on the function $G^*(t)$, as may be seen from equations (4) and (6). In this case, there is in fact only one elastic parameter, as $\sigma^e(\epsilon) \approx AB\epsilon = K\epsilon$. When $B\epsilon_o \gg 1$ the function $G^*(t)$ is affected by the value of B and ϵ_o, which implies that τ_1^*, τ_2^* and C^* are functions of B and ϵ_o. On the other hand, it can be seen from Fig. 2 that for high values of $B\epsilon_o$, the errors are significantly smaller than for low values of $B\epsilon_o$. The errors involved in the determination of τ_1 will be investigated further in the next section.

Further Inquiries of $\epsilon_{\tau 1}$

Since τ_1 mainly governs the fast relaxation phenomena [12], the main cause of the errors involved in its determination will be obvious. During the straining phase ($0 \leq t \leq t_s$) a considerable amount of relaxation can come about. Consequently, the use of $\sigma(\epsilon_o, t_s)$ as an approximation of $\sigma^e(\epsilon_o)$ on determining $G^*(t)$ will result in an underestimation of the elastic response, so that

$$\sigma(t_s) = \beta\sigma^e(\epsilon_o) \text{ with } \beta \leq 1. \tag{8}$$

On the assumption that t_m is sufficiently large, hence that $\sigma(t_m) = G(\infty)^* \sigma^e(\epsilon_o)$ (equation (6)), and $\tau_2^* \approx \tau_2$, $C^* \approx C$, it can be derived that τ_1, τ_1^* and β are related (see Appendix A) as

$$\ln(\tau_2/\tau_1^*) \approx (\beta-1)/C + \beta\ln(\tau_2/\tau_1). \tag{9}$$

$\epsilon_{\tau 1}$ will be further investigated below in four cases (see Table 1), in all of which it will be assumed that $t_s/\tau_2 \ll 1$. In the subdivision into cases, a distinction is made between weakly and highly nonlinear materials ($B\epsilon_o \ll 1$ and $B\epsilon_o \gg 1$, respectively), which have been subjected to relatively slow and to fast strain changes ($t_s/\tau_1 \gg 1$ and $t_s/\tau_1 \ll 1$, respectively). An approximation of β is derived for each case (Appendix B) which, together with equation (9), leads to an expression for $\epsilon_{\tau 1}$ (Appendix C). These expressions are given in Table 1.

In case 1 (equations (10) and (11) in Table 1) it is seen that $\epsilon_{\tau 1}$ approaches infinity for $t_s/\tau_1^* \approx 1.5$. In a qualitative sense

Table 1 Expressions for β and $\epsilon_{\tau 1}$ in four cases

Case	t_s/τ_1	$B\epsilon_o$	$\beta, \epsilon_{\tau 1}$	
1	$>> 1$	$<< 1$	$\beta \approx [1 + C(1 - \gamma + \ln(\tau_2/t_s) - \tau_1/t_s)]/[1 + C\ln(\tau_2/\tau_1)]$	(10)
			$\epsilon_{\tau 1} \approx [\tau_1^*/\{t_s(1 - \gamma - \ln(t_s/\tau_1^*))\} - 1]$	(11)
2	$<< 1$	$<< 1$	$\beta \approx [1 + C\{\ln(\tau_2/\tau_1) - 0.5t_s/\tau_1\}]/[1 + C\ln(\tau_2/\tau_1)]$	(12)
			$\epsilon_{\tau 1} \approx 0.5t_s/\tau_1^*$	(13)
3 4	$>> 1$ $<< 1$	$>> 1$ $>> 1$	$\beta \approx [1 + C\ln((1 + Bv\tau_2)/(1 + Bv\tau_1))]/[1 + C\ln(\tau_2/\tau_1)]$	(14)
			$\epsilon_{\tau 1} \approx (t_s/\tau_1^*)/(B\epsilon_o - t_s/\tau_1^*)$	(15)

this result is in accordance with the asymptotic behavior of $\epsilon_{\tau 1}$ shown in Fig. 7(*a*). The aforementioned discrepancy between the value 1.5 and the value 2.0 (for $B = 0.1$) in Fig. 7(*a*) is probably due to the approximations used for the derivation of β and $\epsilon_{\tau 1}$. Equation (11) may provide a starting point for the determination of a better approximation of τ_1. Using the error definition (7), equation (11) may be rewritten as

$$\ln(t_s/\tau_1^*) \approx 1 - \gamma - \tau_1/t_s. \tag{16}$$

where $\gamma \approx 0.5772\ldots$ is Euler's constant.

Equation (16) states that, after performing experiments with different t_s values, τ_1 may be computed from the slope of the $\ln(t_s/\tau_1^*) - (1/t_s)$ curve.

In case 2 from equations (13) (see Table 1) and (7) it can be found that

$$\tau_1^* \approx \tau_1 + 0.5t_s. \tag{17}$$

As $t_s/\tau_1 << 1$, it can be seen that $\tau_1^* \approx \tau_1$, so that $\epsilon_{\tau 1}$, as is to be expected, will be relatively small (Fig. 7(*b*)) because in this case hardly any relaxation will come about during the constant-strain-rate phase ($0 \le t \le t_s$). Study of cases 3 and 4 shows that in equation (15) (see Table 1), $\epsilon_{\tau 1}$ will certainly be small for low values of t_s/τ_1^*, whereas $\epsilon_{\tau 1}$ will be infinitely large for t_s/τ_1^* values approaching $B\epsilon_o$ which is in good agreement with the results shown in Fig. 7(*a*). Equation (15) can be rewritten as

$$\tau_1^* \approx \tau_1 + t_s/(B\epsilon_o). \tag{18}$$

A better approximation of τ_1 follows immediately from the extrapolation of the $\tau_1^* - t_s$ curve to $t_s = 0$. Moreover, for known ϵ_o the constant B can be estimated from the slope S of this curve, taking

$$B \approx 1/[\epsilon_o \tan(S)]. \tag{19}$$

Concluding Remarks

Only when there is a true step change in the strain it is possible to separate the elastic and time dependent effects in accordance with $\sigma_{\text{step}}(t) = \sigma^e(\epsilon_o)\ G(t)$. The accuracy of the determination from a relaxation experiment of the constants τ_1, τ_2 and C, which describe the time-dependent behavior depends greatly upon the time required to accomplish a sudden change in strain. This is not too unexpected because, during straining within a finite interval $[0, t_s]$, a certain amount of relaxation can come about. The stress response during the relaxation phase ($t > t_s$) will then also be governed by the elastic material properties, as can be seen from the occurrence of the elastic constant B in equation (6) (the elastic constant A cancels out when use is made of expression (4) for the reduced relaxation function G^* in experiments). The present study reveals a high nonlinearity, i.e., a high value of B, to correspond to relatively small errors involved in the quantification of the parameters. For almost linear materials the deviations are significantly larger. In practice this is not expected to give rise to problems because the model is meant for the description of clearly nonlinear material properties. Under all circumstances the determination of τ_1 will be subject to the largest errors. As was illustrated in the preceding section, improved approximations of τ_1 may be obtained from relaxation experiments for one strain level and different t_s values. Apart from τ_1 the value of B is also found to be obtainable from these data in the case of highly nonlinear materials. Once B is known, the constant A may be determined using relation (6) for $t \to \infty$. This will not be discussed here as the main aim of this paper is to investigate the errors involved in the determination of the constants τ_1, τ_2 and C. Estimation of the elastic constants and experimental testing of the theoretical considerations outlined in the foregoing will be subjects for forthcoming research.

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APPENDIX A

Derivation of an Expression Relating τ_1, τ_1^* and β

From the response to a steplike change in strain ϵ, we obtain the approximation

$$G^*(\infty) \approx \sigma(t_m)/\sigma(t_s) \tag{20}$$

On the assumption that t_m is sufficiently large, equation (6) yields

$$\sigma(t_m) \approx G(\infty)\sigma^e(\epsilon_o);\ t_m \to \infty \tag{21}$$

After substitution of equations (8) and (21) into equation (20), we obtain

$$G^*(t_m) \approx G(\infty)/\beta. \tag{22}$$

With $G^*(\infty) = 1/[1 + C^*\ln(\tau_2^*/\tau_1^*)]$ and $G(\infty) = 1/[1 + C \ln(\tau_2/\tau_1)]$ it follows from equation (22) after some manipulation that

$$\ln(\tau_2^*/\tau_1^*) \approx (\beta-1)/C^* + \beta(C/C^*)\ln(\tau_2/\tau_1). \quad (23)$$

If it is assumed that $\tau_2^* \approx \tau_2$ and $C^* \approx C$ the eventual result is

$$\ln(\tau_2/\tau_1^*) \approx (\beta-1)/C + \beta\ln(\tau_2/\tau_1). \quad (9)$$

APPENDIX B

Derivation of Expressions for β in Four Cases

From equation (6), it follows that

$$\begin{aligned}\sigma(\epsilon_o,t_s) = & A[\exp(B\epsilon_o)-1] \\ & -[CA/(1+C\ln(\tau_2/\tau_1))][\{E_1(t_s(1/\tau_1+Bv)) + \\ & -E_1(t_s(1/\tau_2+Bv)) \\ & +\ln((\tau_2/\tau_1)(1+Bv\tau_1)/(1+Bv\tau_2))\}\exp(B\epsilon_o) + \\ & -E_1(t_s/\tau_1)+E_1(t_s/\tau_2)-\ln(\tau_2/\tau_1)]. \quad (24)\end{aligned}$$

Case 1: $t_s/\tau_1 >> 1; t_s/\tau_2 >> 1; B\epsilon_o << 1$
Here it applies that

$$\exp(B\epsilon_o)-1 \approx B\epsilon_o \quad (25)$$

$$\exp(B\epsilon_o)E_1(t_s(1/\tau_1+Bv)) \approx E(t_s/\tau_1) + B\epsilon_o[E_1(t_s/\tau_1)-(\tau_1/t_s)\exp(-t_s/\tau_1)] \quad (26)$$

$$\exp(B\epsilon_o)E_1(t_s(1/\tau_2+Bv)) \approx E_1(t_s/\tau_2) + B\epsilon_o[E_1(t_s/\tau_2)-(\tau_2/t_s)\exp(-t_s/\tau_2)] \quad (27)$$

$$\exp(B\epsilon_o)\ln((\tau_2/\tau_1)(1+Bv\tau_1)/(1+Bv\tau_2)) \approx \ln(\tau_2/\tau_1)+B\epsilon_o(\tau_1-\tau_2). \quad (28)$$

These equations can be obtained from a series expansion of the different terms around $B\epsilon_o \approx 0$. Substitution of equations (25)–(28) into equation (24) gives

$$\begin{aligned}\sigma(\epsilon_o,t_s) \approx & AB\epsilon_o - [CAB\epsilon_o/(1+C\ln(\tau_2/\tau_1))\{E_1(t_s/\tau_1) \\ & -E_1(t_s/\tau_2)+ \\ & +\ln(\tau_2/\tau_1)+(\tau_1/t_s)(1-\exp(-t_s/\tau_1)) \\ & -(\tau_2/t_s)(1-\exp(t_s/\tau_2))\}] \quad (29)\end{aligned}$$

As $t_s/\tau_2 << 1$ it is true that

$$E_1(t_s/\tau_2) \approx -\gamma-\ln(t_s/\tau_2)([13]) \quad (30)$$

$$(\tau_2/t_s)(1-\exp(-t_s/\tau_2)) \approx 1 \quad (31)$$

where $\gamma \approx 0.5772\ldots$ is Euler's constant.

With the equations (30) and (31) equation (29) may be rewritten as

$$\sigma(\epsilon_o,t_s) \approx [AB\epsilon_o/(1+C\ln(\tau_2/\tau_1))][1+C\{\ln(\tau_2/t_s) - E_1(t_s/\tau_1)-(\tau_1/t_s)(1-\exp(-t_s/\tau_1))+1-\gamma\}]. \quad (32)$$

From $t_s/\tau_1 >> 1$ it follows [13] that

$$E_1(t_s/\tau_1) \approx (\tau_1/t_s)(\exp(-t_s/\tau_1)) \quad (33)$$

With the equations (32), (33), (25), (5) and (8), β can be calculated:

$$\beta \approx [1+C\{1-\gamma+\ln(\tau_2/t_s)-\tau_1/t_s]/[1+C\ln(\tau_2/\tau_1)]. \quad (10)$$

Case 2: $t_s/\tau_1 << 1; t_s/\tau_2 << 1; B\epsilon_o << 1$
Equation (32) has been derived without assumptions as to the value of the ratio t_s/τ_1, so that this equation can be used in this case, too. As $t_s/\tau_1 << 1$ it can be stated that

$$E_1(t_s/\tau_1) \approx -\gamma-\ln(t_s/\tau_1)+t_s/\tau_1 \quad (34)$$

$$\exp(-t_s/\tau_1) \approx 1-t_s/\tau_1+0.5(t_s/\tau_1)^2. \quad (35)$$

Equations (32), (34), (35), (25), (5) and (8) will result in

$$\beta \approx [1+C\{\ln(\tau_2/\tau_1)-0.5t_s/\tau_1\}]/[1+C\ln(\tau_2/\tau_1)]. \quad (12)$$

Case 3: $t_s/\tau_1 >> 1; t_s/\tau_2 << 1; B\epsilon_o >> 1$
As $B\epsilon_o >> 1$, the following approximations can be used:

$$E_1(t_s(1/\tau_1+Bv)) \approx 0 \quad (36)$$

$$E_1(t_s(1/\tau_2+Bv)) \approx 0 \quad (37)$$

$$\exp(B\epsilon_o)-1 \approx \exp(B\epsilon_o). \quad (38)$$

Substitution of equations (36), (37) and (38) in equation (24) gives

$$\sigma(\epsilon_o,t_s) \approx A\exp(B\epsilon_o)[1-\{C/(1+C\ln(\tau_2/\tau_1))\}\{\ln(\tau_2/\tau_1)(1+Bv\tau_1)/(1+bv\tau_2))\}] \quad (39)$$

so that we eventually obtain

$$\beta \approx [1+C\ln((1+Bv\tau_2)/(1+Bv\tau_1))]/[1+C\ln(\tau_2/\tau_1)]. \quad (14)$$

Case 4: $t_s/\tau_1 << 1; t_s/\tau_2 << 1; B\epsilon_o >> 1$
In this case also equation (14) is found to apply.

APPENDIX C

Derivation of Expressions for $\epsilon_{\tau 1}$ in Four Cases

Case 1. From substitution of equation (10) into equation (9), after some elaboration, it follows that

$$\ln(t_s/\tau_1^*) \approx 1-\gamma-\tau_1/t_s \quad (40)$$

or

$$\tau_1 \approx t_s(1-\gamma-\ln(t_s/\tau_1^*)). \quad (41)$$

Substitution of equation (41) into the error definition (7) will result in equation (11).

Case 2. Substitution of equation (12) into equation (9) will lead to

$$\ln(t_s/\tau_1^*) \approx -0.5t_s/\tau_1. \quad (42)$$

As $t_s/\tau_1 << 1$, equation (42) can be rewritten as

$$\tau_1/\tau_1^* \approx \exp(-0.5t_s/\tau_1) \approx 1-0.5t_s/\tau_1. \quad (43)$$

From this equation it can be seen that $\tau_1 \approx \tau_1^*$ as $t_s/\tau_1 << 1$, so that $\epsilon_{\tau 1}$ will be small in this case. Even then τ_1 can be calculated from equation (43), resulting in

$$\tau_1 \approx \tau_1^* - 0.5t_s. \quad (44)$$

Equation (13) can easily be obtained with the error definition (7).

Cases 3 and 4. From equations (14) and (9) it follows that

$$\ln(\tau_2/\tau_1^*) \approx \ln((t_s+B\epsilon_o\tau_2)/(t_s+B\epsilon_o\tau_1)) \quad (45)$$

which can be rewritten, with $B\epsilon_o\tau_2 >> t_s$, as

$$\tau_1^* \approx \tau_1 + t_s/(B\epsilon_o). \quad (46)$$

Combination of equations (46) and (7) will lead to equation (15).