1 Random Walk

We investigate the gamblers ruin problem on a random walk via the first step analysis, we define the probability of getting to given state conditioned by the current position, this is a function depending on the current position

$$f(k) = p \cdot f(k+1) + q \cdot f(k-1).$$

Then using the increment operator $\Delta f(k) = f(k+1) - f(k)$ we can rewrite the equation as

$$\Delta f(k) = (q/p)\Delta f(k-1).$$

Then by iterating it and using the boundaries $f(A) = 1$ and $f(-B) = -1$ then we get to

$$P(S_t \text{ hits A before -B}|S_0 = 0) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}.$$

This same strategy can be applied to compute the expected time to hit the boundaries. Considering that $g(k)$ is the expected time to hit the boundaries given we are at k. Then

$$g(k) = pg(k+1) + qg(k-1) + 1.$$

We also use the probability generating function and the first step analysis to find the expected time of the game on gamblers ruin.

$$\phi(z) = E[z^X] = \sum_{x=0}^{\infty} p(x)z^x$$

2 First Martingale Steps

A sequence of random variables M_n is a martingale with respect to X_n an other sequence of random variables, if exists $f_n : \mathbb{R}^n \to \mathbb{R}$ such that $M_n = f_n(X_1, \dots, X_n)$, and

$$\mathbb{E}[M_n|X_1, \dots, X_{n-1}] = M_{n-1}.$$

We define stopping times and with them we see the Doob inequalities and the martingales convergence theorems.

One key aspect of the martingales is that one can transform them using an predictable process, a non-anticipating process, to generate new martingales, this can be helpful to prove equality, inequality or bounds for non trivial martingales that we encounter like, combination of martingales or by adding stopping times to our martingales, so we can use the known theorems on these new martingales.

See Probability Theory by Achim Klenke for a deeper and more rigorous treatment of the subject.

3 Brownian Motion

A stochastic process B_t is a Brownian motion if

1. $B_0 = 0$,
2. It has independent increments,
3. For any $0 \leq s \leq t \leq T$ the increment $B_t - B_s$ has a Gaussian distribution with mean 0 and variance $t - s$,
4. It's continuous almost everywhere.

The definition already gives us a lot of information about the process, as the increments being Gaussian and independent imply that the distributions of the increments are Gaussian random variables with a diagonalizable covariance matrix.

An other important feature is that it's a Gaussian process, as the Gaussian characteristic function is determined by only it's first two moments it's quite useful for many problems.

The Covariance of two instances B_s and B_t of a Brownian motion is $Cov(B_s, B_t) = \min(s,t)$.

Finally we construct the Brownian motion by approximating it with wavelet series on $\mathbb{L}^2$, first by approximating a standard Gaussian process and the by integrating the series, as the Brownian motion is the integral of such process.

This construction is useful to simulate computationally the path of the Brownian motion.

4 Martingales: The Next Steps

Define the **conditional expectation** for random variables and derive its basic properties like linearity, tower property, ...

Define **uniform integrability** and use it to appreciate that the conditional expectation is a contraction. And let us construct a link between convergence in probability an convergence in mean (L^1).

We now can define a **continuous time martingale** by a process $\{X_t\}_{t\in\mathbb{R}}$ adapted to a *filtration* $\{\mathcal{F}_t\}_{t\in\mathbb{R}}$

$$E(|X_t|) < \infty, \ \forall t \in [0, \infty), \tag{1}$$

$$E(X_t|\mathcal{F}_s) = X_s, \ \forall 0 \leq s < t < \infty. \tag{2}$$

We prove the **Doop's theorems** and the **martingale convergence theorems** for continuous martingales.

We prove the ruin theorems for the Brownian Motion using its representation as a martingale. The results are the same as for the random walk, recall that we prove those using random walk properties and discrete martingales by separate.

5 Richness of Paths

Brownian motion is **continuous everywhere** with probability one but is **not differentiable anywhere** with probability one.

Donsker Thoerem gives a formal transition from the random walk to the Brownian motion, this gives us the idea that properties can be translated from one to the other.

Two **embedding theorems** consolidate the relations between the random walk and the Brownian motion by showing that we can find random walks inside the Brownian motion, so the relation is bidirectional.