A Simulation Test for Fluid-Structure-Interaction with Womersley Theory
Lu Qingshuang Report in January 2023

The Womersley theory provides a analytical solution for pulsatile flow in a thin-walled elastic tube which is assumed to be axisymmetric circular straight and the flow inside is fully developed.

The Womersley solutions contain the fluid pressure, the longitudinal and radial flow velocities, and the longitudinal and radial wall displacement. And the expression of the solutions are discussed in the last report.

The input pressure is normally known or specified. The given pressure can be derived by the Fourier decomposition, and the series of Fourier coefficients is denoted as B_n. The Womersley solutions are as follows:

$$p(x,r,t) = p_0 + k_s x + \sum_{n=1}^{\infty} B_n e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$u(x,r,t) = \frac{k_s}{4\mu}(r^2 - R^2) + \sum_{n=1}^{\infty} \frac{B_n}{\rho c_n}\left(1 - G_n \frac{J_0(\zeta_n)}{J_0(\Lambda_n)}\right) e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$v(x,r,t) = \sum_{n=1}^{\infty} \frac{i\omega n R B_n}{2\rho c_n^2}\left(\frac{r}{R} - G_n \frac{2J_0(\zeta_n)}{\Lambda_n J_0(\Lambda_n)}\right) e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$\xi(x,R,t) = \sum_{n=1}^{\infty} \frac{iB_n}{\omega n \rho c_n}(G_n - 1) e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

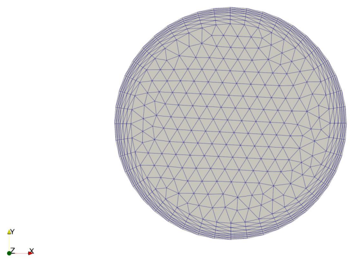
$$\eta(x,R,t) = \sum_{n=1}^{\infty} \frac{RB_n}{2\rho c_n^2}(1 - G_n g_n) e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

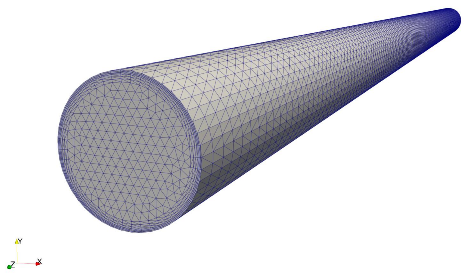
where,

$$G_n = \frac{2 + z_n(2\nu - 1)}{z_n(2\nu - g_n)}.$$

We plan to use the fluid-structure-interaction modal to get a numerical result and then compare it with the analytical solution, in order to see if there is any difference between two results.

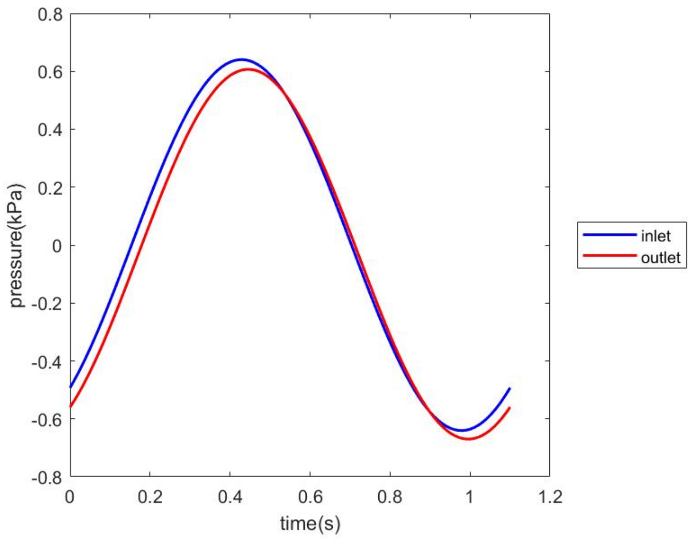
To simplify the simulation, we consider the single mode case. We set the pipe radius $R = 0.3$, and pipe length $L = 15.0$. The geometry and mesh are shown as following figure.



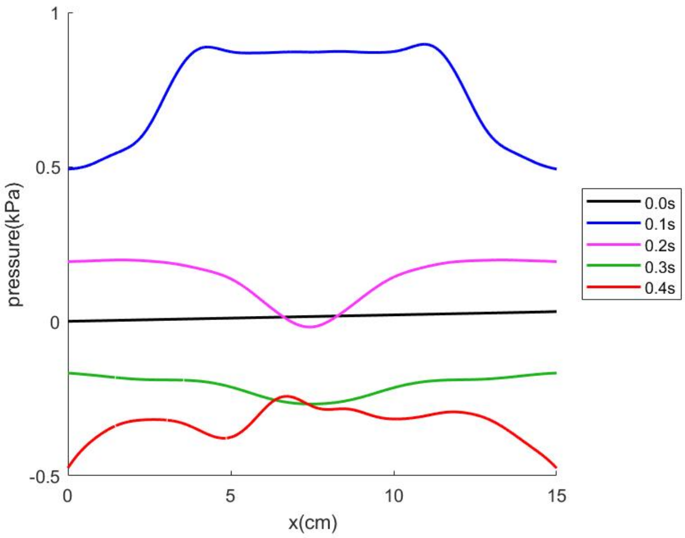


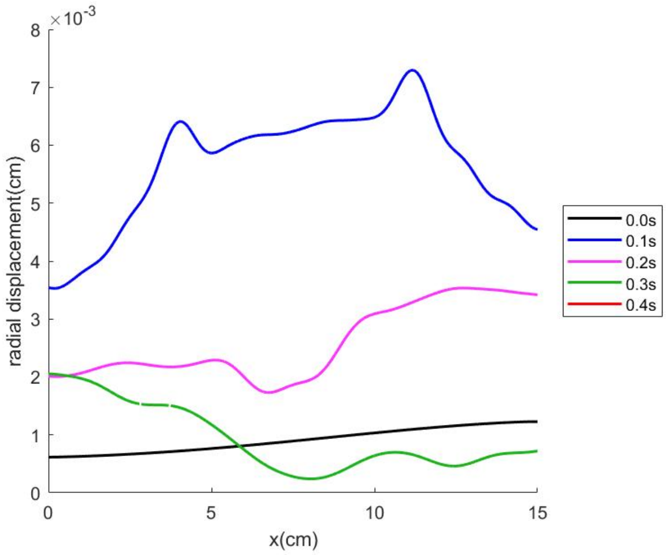
We set fluid density $\rho_f = 1.0$ and viscosity $\mu = 0.04$, period $T_p = 1.1$, reference pressure $p_0 = 0.0$, and complex coefficients $k_s =- 21.0496$, and $B =- 33.0102 + 42.9332i$. And we considered uniform wall properties, including a wall density $\rho_s = 1.0$, Poisson's ratio $\nu = 0.3$, thickness $h = 0.0015$, which is set as 5% of the radius to match the thin-wall assumption, and Young's modulus $E = 1.0 \times 10^7$.

The boundary conditions of the inlet and outlet of the tube is given by the analytical solution of pressure. It can be obtain that the pressure wave has a small attenuation and dispersion.



The simulation is failed at $t = 0.3$, and the output results are shown as follows. The reason may be that the given inlet and outlet boundary conditions are both in terms of pressure, which is equivalent to the inputs of two different pressure waves from two sides of the tube, leading to the propagation of these two pressure waves from the boundary to the interior of the flow field. The simulation of the interior of the flow field becomes difficult and unreasonable, so the predicted results cannot be obtained.





In the following work, we plan to change the inlet boundary condition to flow rate and the outlet boundary condition to pressure or RCR, and then conduct a test again to see whether simulation results can be obtained.

Reference:

[1]Zamir, M. (2000). The Physics of Pulsatile Flow. Biological Physics Series.

[2]Lan, I.S., Liu, J., Yang, W., & Marsden, A.L. (2022). A reduced unified continuum formulation for vascular fluid-structure interaction. Computer Methods in Applied Mechanics and Engineering.