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A NEW FINITE ELEMENT FORMULATION FOR COMPUTATIONAL FLUID DYNAMICS: VIII. THE GALERKIN/LEAST-SQUARES METHOD FOR ADVECTIVE-DIFFUSIVE EQUATIONS*

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Galerkin/least-squares finite element methods are presented for advective-diffusive equations. Galerkin/least-squares represents a conceptual simplification of SUPG, and is in fact applicable to a wide variety of other problem types. A convergence analysis and error estimates are presented.

1. Introduction

SUPG finite element methods (also known as streamline-diffusion methods) have been shown to be effective for hyperbolic and advective-diffusive systems (see [6, 9, 11–18, 22]). In this paper we present Galerkin/least-squares finite element methods for advective-diffusive equations. These methods coincide with SUPG for hyperbolic cases, but are conceptually simpler when diffusion is present. We have developed successful finite element methods using the Galerkin/least-squares approach for a variety of elliptic and second-order hyperbolic problems (see [2, 3, 7, 8, 10, 19–21]). Thus Galerkin/least-squares may be viewed as a general methodology for obtaining convergent finite element methods accomodating a much wider class of interpolations than the classical Galerkin method. The way Galerkin/least-squares works is as follows: Least-squares forms of residuals are added to the Galerkin method. These terms enhance the stability of the Galerkin method without degrading accuracy. The result is that practically convenient interpolations, which are unstable within the Galerkin framework, become convergent.

In this paper we apply the Galerkin/least-squares method to steady and unsteady advective-diffusive systems. We begin in Section 2 with a statement of a class of boundary value problems for the scalar steady advection-diffusion equation. A fairly general spectrum of possible boundary conditions, leading to well-posed variational problems, is considered. The boundary value problem is specialized to the hyperbolic case for completeness. Galerkin,

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SUPG, and Galerkin/least-squares finite element methods are presented and contrasted. A detailed global error analysis of Galerkin/least-squares is presented. In Section 3, the developments for the steady case are generalized to the unsteady case by way of a space-time formulation employing the discontinuous Galerkin technique with respect to time. In Section 4, symmetric advective-diffusive systems are considered. The set-up so closely follows the scalar case that we are able to present completely analogous results. In order to fully comprehend these developments, the reader is urged to first carefully study the scalar case, as the system case is presented in virtually equation-for-equation form with little amplification. Conclusions are drawn in Section 5.

2. The scalar steady advection-diffusion equation

2.1. Preliminaries

Let Ω be an open, bounded region in $\mathbb{R}^d$, where d is the number of space dimensions. The boundary of Ω is denoted by Γ and is assumed smooth. The unit outward normal vector to Γ is denoted by $\boldsymbol{n} = (n_1, n_2, \ldots, n_d)$. Let $\boldsymbol{a}$ denote the given flow velocity, assumed solenoidal, i.e., $\nabla \cdot \boldsymbol{a} = 0$. The following notations prove useful:

$$a_n = \boldsymbol{n} \cdot \boldsymbol{a}\,, \tag{1}$$

$$a_n^+ = (a_n + |a_n|)/2\,, \tag{2}$$

$$a_n^- = (a_n - |a_n|)/2\,. \tag{3}$$

Let $\{\Gamma^-, \Gamma^+\}$ and $\{\Gamma_g, \Gamma_h\}$ be partitions of Γ, where

$$\Gamma^- = \{\boldsymbol{x} \in \Gamma \mid a_n(\boldsymbol{x}) < 0\} \quad \text{(inflow boundary)}\,, \tag{4}$$

$$\Gamma^+ = \Gamma - \Gamma^- \quad \text{(outflow boundary)}\,. \tag{5}$$

The following subsets are also needed (see Fig. 1):

$$\Gamma_g^\pm = \Gamma_g \cap \Gamma^\pm\,, \tag{6}$$

$$\Gamma_h^\pm = \Gamma_h \cap \Gamma^\pm\,. \tag{7}$$

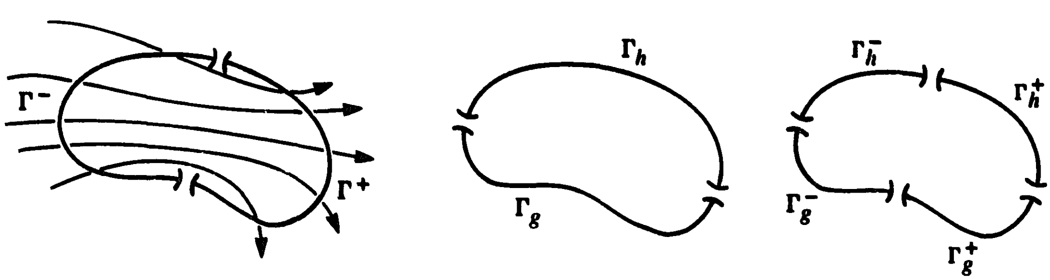

Fig. 1. Illustration of boundary partitions.

Let $\kappa = \text{const.} > 0$ denote the diffusivity. Various fluxes are employed in the sequel:

$$\boldsymbol{\sigma}^a(u) = -\boldsymbol{a}u \quad \text{(advective flux)}\,, \tag{8}$$

$$\boldsymbol{\sigma}^d(u) = \kappa\nabla u \quad \text{(diffusive flux)}\,, \tag{9}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^a + \boldsymbol{\sigma}^d \quad \text{(total flux)}\,, \tag{10}$$

$$\sigma_n^a = \boldsymbol{n}\cdot\boldsymbol{\sigma}^a\,, \tag{11}$$

$$\sigma_n^d = \boldsymbol{n}\cdot\boldsymbol{\sigma}^d\,, \tag{12}$$

$$\sigma_n = \boldsymbol{n}\cdot\boldsymbol{\sigma}\,. \tag{13}$$

Let D denote a domain (e.g. Ω, Γ, etc.). The $L_2(D)$ inner product and norm are denoted by $(\cdot,\cdot)_D$ and $\|\cdot\|_D$, respectively.

2.2. *Problem statement*

The problem consists of finding $u = u(\boldsymbol{x})\ \forall \boldsymbol{x} \in \bar{\Omega}$, such that

$$\mathcal{L}u \equiv -\nabla\cdot\boldsymbol{\sigma}(u) = f \quad \text{in } \Omega\,, \tag{14}$$

$$u = g \quad \text{on } \Gamma_g\,, \tag{15}$$

$$-a_n^- u + \sigma_n^d(u) = h \quad \text{on } \Gamma_h\,, \tag{16}$$

where $f : \Omega \rightarrow \mathbb{R}$, $g : \Gamma_g \rightarrow \mathbb{R}$, and $h : \Gamma_h \rightarrow \mathbb{R}$ are prescribed data. Equation (14) is a parabolic equation. The boundary condition (16) can be better understood by letting

$$h = \begin{cases} h^- & \text{on } \Gamma_h^-\,, \\ h^+ & \text{on } \Gamma_h^+\,. \end{cases} \tag{17}$$

Thus we may write (16) in the equivalent form

$$\sigma_n(u) = h^- \quad \text{on } \Gamma_h^- \quad \text{(total flux b.c.)}\,, \tag{18}$$

$$\sigma_n^d(u) = h^+ \quad \text{on } \Gamma_h^+ \quad \text{(diffusive flux b.c.)}\,. \tag{19}$$

2.3. *Variational formulation*

The variational form of the boundary value problem is stated in terms of the following function spaces:

$$\mathcal{S} = \{u \in H^1(\Omega) \mid u = g \text{ on } \Gamma_g\}\,, \tag{20}$$

$$\mathcal{V} = \{w \in H^1(\Omega) \mid w = 0 \text{ on } \Gamma_g\}\,. \tag{21}$$

The objective is to find $u \in \mathcal{S}$, such that:

$$B(w, u) = L(w) \quad \forall w \in \mathcal{V}\,, \tag{22}$$

where

$$B(w, u) \equiv (\nabla w, \boldsymbol{\sigma}(u))_\Omega + (w, a_n^+ u)_{\Gamma_h}\,, \tag{23}$$

$$L(w) \equiv (w, f)_\Omega + (w, h)_{\Gamma_h}\,. \tag{24}$$

The *formal consistency* of (22) with the strong form of the boundary value problem, i.e. (14)–(16), may be verified as follows:

$$\begin{aligned} 0 &= B(w, u) - L(w) \\ &= -(w, \nabla\cdot \boldsymbol{\sigma}(u))_\Omega + (w, \sigma_n(u))_{\Gamma_h} + (w, a_n^+ u)_{\Gamma_h} - (w, f)_\Omega - (w, h)_{\Gamma_h} \\ &= -(w, \nabla\cdot \boldsymbol{\sigma}(u) + f)_\Omega + (w, -a_n^- u + \sigma_n^d(u) - h)_{\Gamma_h}\,. \end{aligned} \tag{25}$$

Stability, or *coercivity*, is established as follows:

$$\begin{aligned} B(w, w) &= (\nabla w, -\boldsymbol{a} w + \kappa \nabla w)_\Omega + (w, a_n^+ w)_{\Gamma_h} \\ &= -\tfrac{1}{2}(w, a_n w)_{\Gamma_h} + \kappa \|\nabla w\|_\Omega^2 + (w, a_n^+ w)_{\Gamma_h} \\ &= \kappa \|\nabla w\|_\Omega^2 + \tfrac{1}{2}\||a_n|^{1/2} w\|_{\Gamma_h}^2 \quad \forall w \in \mathcal{V}\,. \end{aligned} \tag{26}$$

For future reference we define

$$|||w|||^2 = B(w, w)\,. \tag{27}$$

Finally, we wish to investigate the *global conservation of flux*. Consider the case in which $\Gamma_g = \emptyset$. Set $w \equiv 1$ in (22):

$$\begin{aligned} 0 &= B(1, u) - L(1) \\ &= \int_{\Gamma^+} a_n^+ u \, \mathrm{d}\Gamma - \int_\Omega f \, \mathrm{d}\Omega - \int_\Gamma h \, \mathrm{d}\Gamma\,, \end{aligned} \tag{28}$$

which may be written equivalently as

$$0 = \int_{\Gamma^-} h^- \, \mathrm{d}\Gamma + \int_\Omega f \, \mathrm{d}\Omega + \int_{\Gamma^+} (-a_n u + h^+) \, \mathrm{d}\Gamma\,. \tag{29}$$

This confirms the conservation property for the case assumed. If $\Gamma_g \neq \emptyset$, "consistent" fluxes on Γ_g may be defined via a mixed variational formulation which automatically attains global conservation. See [5, p. 107; 4] for background.

2.4. Hyperbolic case

In the absence of diffusion we cannot specify a boundary condition on the outflow boundary. The equations of the boundary value problem are

$$\mathscr{L}u \equiv -\nabla \cdot \boldsymbol{\sigma}^a(u) = f \quad \text{on } \Omega\,, \tag{30}$$

$$u = g \quad \text{on } \Gamma_g^-\,, \tag{31}$$

$$\sigma_n^a(u) = h^- \quad \text{on } \Gamma_h^-\,. \tag{32}$$

The variational operators are defined as

$$B(w, u) \equiv (\nabla w, \boldsymbol{\sigma}^a(u))_\Omega + (w, a_n^+ u)_\Gamma\,, \tag{33}$$

$$L(w) \equiv (w, f)_\Omega + (w, h^-)_{\Gamma_h^-}\,. \tag{34}$$

Consistency, stability and conservation are established as follows:

Consistency:

$$\begin{aligned} 0 &= B(w, u) - L(w) \\ &= -(w, \nabla \cdot \boldsymbol{\sigma}^a(u))_\Omega + (w, -a_n u)_\Gamma + (w, a_n^+ u)_\Gamma - (w, f)_\Omega - (w, h^-)_{\Gamma_h^-} \\ &= -(w, \nabla \cdot \boldsymbol{\sigma}^a(u) + f)_\Omega + (w, -a_n^- u - h^-)_{\Gamma_h^-}\,. \end{aligned} \tag{35}$$

Stability:

$$\begin{aligned} B(w, w) &= (\nabla w, -\boldsymbol{a} w)_\Omega + (w, a_n^+ w)_\Gamma \\ &= -\tfrac{1}{2}(w, a_n w)_\Gamma + (w, a_n^+ w)_\Gamma \\ &= \tfrac{1}{2}\| |a_n|^{1/2} w \|_\Gamma^2\,. \end{aligned} \tag{36}$$

Conservation ($\Gamma_g^- = \emptyset$):

$$\begin{aligned} 0 &= B(1, u) - L(1) \\ &= \int_\Gamma a_n^+ u \,\mathrm{d}\Gamma - \int_\Omega f \,\mathrm{d}\Omega - \int_{\Gamma_h^-} h \,\mathrm{d}\Gamma\,. \end{aligned} \tag{37}$$

Equivalently,

$$0 = \int_{\Gamma^-} h^- \,\mathrm{d}\Gamma + \int_\Omega f \,\mathrm{d}\Omega + \int_{\Gamma^+} (-a_n u) \,\mathrm{d}\Gamma\,. \tag{38}$$

2.5. Finite element formulations

Consider a partition of Ω into finite elements. Let Ω^e be the interior of the eth element, let Γ^e be its boundary, and

$$\tilde{\Omega} = \bigcup_e \Omega^e \qquad \text{(element interiors)}\,, \tag{39}$$

$$\tilde{\Gamma} = \bigcup_e \Gamma^e - \Gamma \qquad \text{(element interfaces)}\,. \tag{40}$$

Let $\mathcal{S}^h \subset \mathcal{S}$, $\mathcal{V}^h \subset \mathcal{V}$ be finite element spaces consisting of *continuous* piecewise polynomials of order k. As a point of departure we consider the classical *Galerkin method*.

Find $u^h \in \mathcal{S}^h$, such that:

$$B(w^h, u^h) = L(w^h) \quad \forall w^h \in \mathcal{V}^h\,. \tag{41}$$

Remark

The element Peclet number is defined by $\alpha = h|\boldsymbol{a}|/(2\kappa)$. We are interested in the entire range of α, i.e. $0 < \alpha < \infty$. The advection-dominated case (i.e. α large) is viewed as "hard". The Galerkin method possesses poor stability properties for this case. Spurious oscillations are generated by unresolved internal and boundary layers.

Methods with improved stability properties are:

SUPG:

$$B_{\text{SUPG}}(w^h, u^h) = L_{\text{SUPG}}(w^h)\,, \tag{42}$$

$$B_{\text{SUPG}}(w^h, u^h) \equiv B(w^h, u^h) + (\tau \boldsymbol{a} \cdot \nabla w^h, \mathcal{L}u^h)_{\tilde{\Omega}}\,, \tag{43}$$

$$L_{\text{SUPG}}(w^h) \equiv L(w^h) + (\tau \boldsymbol{a} \cdot \nabla w^h, f)_{\tilde{\Omega}}\,. \tag{44}$$

Galerkin/least-squares:

$$B_{\text{GLS}}(w^h, u^h) = L_{\text{GLS}}(w^h)\,, \tag{45}$$

$$B_{\text{GLS}}(w^h, u^h) \equiv B(w^h, u^h) + (\tau \mathcal{L} w^h, \mathcal{L}u^h)_{\tilde{\Omega}}\,, \tag{46}$$

$$L_{\text{GLS}}(w^h) \equiv L(w^h) + (\tau \mathcal{L} w^h, f)_{\tilde{\Omega}}\,. \tag{47}$$

Remarks

(1) τ is a positive parameter having dimensions of time. It will be described in detail later.

(2) In the hyperbolic case, or for piecewise linear elements in the general case, SUPG and Galerkin/least-squares become identical.

(3) SUPG and Galerkin/least-squares are *residual methods*, i.e. (42) and (45) are satisfied if u^h is replaced by u, the exact solution of the boundary value problem.

2.6. Error analysis

The SUPG method has been analyzed in [16, 22]. In this section we perform a global error analysis of Galerkin/least-squares.

Let $e = u^h - u$ denote the error in the finite element solution. By Remark (3) above,

$$B_{\text{GLS}}(w^h, e) = 0 \quad \forall w^h \in \mathcal{V}^h\,. \tag{48}$$

This is referred to as the *consistency condition* for Galerkin/least-squares.

Let

$$|||w^h|||^2_{\mathrm{GLS}} = |||w^h|||^2 + \|\tau^{1/2}\mathscr{L}w^h\|^2_{\tilde{\Omega}} \,. \tag{49}$$

By (46) and (49),

$$B_{\mathrm{GLS}}(w^h, w^h) = |||w^h|||^2_{\mathrm{GLS}} \quad \forall w^h \in \mathscr{V}^h \,. \tag{50}$$

This is the *stability condition* for Galerkin/least-squares.

Remarks

(1) Stability is less straightforward for SUPG. One needs to invoke an "inverse estimate" and specific properties of τ. These assumptions are seen to be unnecessary for establishing the stability of Galerkin/least-squares. However, they resurface in the convergence analysis.

(2) A term of the form $\|w^h\|^2_{\Omega}$ can be added to (49) by employing a change of variables (see [16, 22] for further discussion).

Let $\tilde{u}^h \in \mathscr{V}^h$ denote an interpolant of u. The interpolation error is denoted by $\eta = \tilde{u}^h - u$. Thus, $e = e^h + \eta$, where $e^h \in \mathscr{V}^h$.

We assume τ possesses the following properties:

$$\tau = \mathrm{O}\left(\frac{h}{|\boldsymbol{a}|}\right), \quad \alpha \text{ large}, \tag{51}$$

$$\tau = \mathrm{O}\left(\frac{h^2}{\kappa}\right), \quad \alpha \text{ small}. \tag{52}$$

A specific choice of τ satisfying these properties is given by

$$\tau = \tfrac{1}{2}\,\frac{h}{|\boldsymbol{a}|}\,\zeta(\alpha)\,, \tag{53}$$

where $\zeta(\alpha)$ is illustrated in Fig. 2. (See [13, Appendix I] for some other possibilities.)

For sufficiently smooth u, standard interpolation theory (see e.g. [1]) and the above asymptotic properties of τ enable us to establish the following *interpolation estimate*:

$$2\|\tau^{-1/2}\eta\|^2_{\Omega} + \kappa\|\nabla\eta\|^2_{\Omega} + \||a_n|^{1/2}\eta\|^2_{\Gamma_h} + \|\tau^{1/2}\mathscr{L}\eta\|^2_{\tilde{\Omega}} \le c_u h^{2l}\,, \tag{54}$$

$$2l = \begin{cases} 2k+1\,, & \alpha \text{ large}\,, \\ 2k\,, & \alpha \text{ small}\,, \end{cases} \tag{55}$$

where c_u is a function of u. The notation c_u is used subsequently, it being understood that in each instance its value may change by a multiplicative constant.

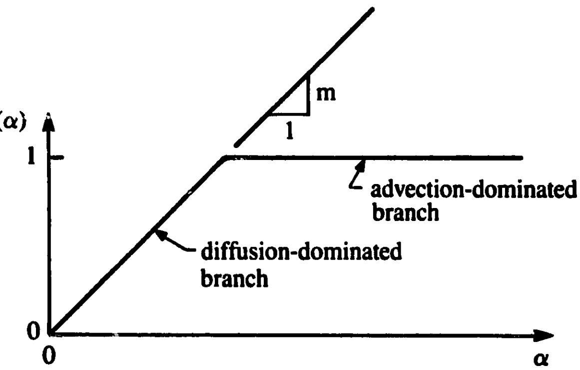

Fig. 2. Definition of $\zeta(\alpha)$.

We also need to introduce an *inverse estimate*. The appropriate form in the present circumstances is

$$\|\Delta w^h\|_{\tilde{\Omega}} \leq ch^{-1}\|\nabla w^h\|_{\Omega} \quad \forall w^h \in \mathcal{V}^h\,, \tag{56}$$

where c is a nondimensional constant. (See [1, pp. 140–146] for results of this kind.)

THEOREM 2.1. Assume the consistency condition (48), *stability condition* (50), *and interpolation estimate* (54) *hold. Assume the slope m in the definition of* $\zeta(\alpha)$ *satisfies* $m \leq 4/c^2$, *where c is the constant in the inverse estimate* (56). *Then the error estimate for the Galerkin/least-squares method is*

$$|||e|||^2_{\mathrm{GLS}} \leq c_u h^{2l}\,. \tag{57}$$

PROOF. We first estimate e^h:

$$\begin{aligned}
|||e^h|||^2_{\mathrm{GLS}} &= B_{\mathrm{GLS}}(e^h, e^h) && \text{(stability)}\\
&= B_{\mathrm{GLS}}(e^h, e-\eta)\\
&= -B_{\mathrm{GLS}}(e^h, \eta) && \text{(consistency)}\\
&\leq |B_{\mathrm{GLS}}(e^h, \eta)|\\
&= |-(\boldsymbol{a}\cdot\nabla e^h, \eta)_{\Omega} + \kappa(\nabla e^h, \nabla\eta)_{\Omega}\\
&\qquad + (e^h, a_n^+\eta)_{\Gamma_h} + (\tau\mathscr{L}e^h, \mathscr{L}\eta)_{\tilde{\Omega}}| && \text{(definition of } B_{\mathrm{GLS}}(\cdot,\cdot))\\
&= |-(\mathscr{L}e^h, \eta)_{\tilde{\Omega}} - \kappa(\Delta e^h, \eta)_{\tilde{\Omega}} + \kappa(\nabla e^h, \nabla\eta)_{\Omega} + (e^h, a_n^+\eta)_{\Gamma_h}\\
&\qquad + (\tau\mathscr{L}e^h, \mathscr{L}\eta)_{\tilde{\Omega}}|\\
&\leq \tfrac14\|\tau^{1/2}\mathscr{L}e^h\|^2_{\tilde{\Omega}} + \|\tau^{-1/2}\eta\|^2_{\Omega} + \tfrac14\kappa^2\|\tau^{1/2}\,\Delta e^h\|^2_{\tilde{\Omega}}\\
&\qquad + \|\tau^{-1/2}\eta\|^2_{\Omega} + \tfrac14\kappa\|\nabla e^h\|^2_{\Omega} + \kappa\|\nabla\eta\|^2_{\Omega}\\
&\qquad + \tfrac14\||a_n|^{1/2}e^h\|^2_{\Gamma_h} + \||a_n|^{1/2}\eta\|^2_{\Gamma_h}\\
&\qquad + \tfrac14\|\tau^{1/2}\mathscr{L}e^h\|^2_{\tilde{\Omega}} + \|\tau^{1/2}\mathscr{L}\eta\|^2_{\tilde{\Omega}}\,.
\end{aligned} \tag{58}$$

To proceed further we need to invoke the bound on m:

$$\begin{aligned}\kappa\tau &= \frac{\kappa h}{2|\boldsymbol{a}|}\,\zeta(\alpha)\\ &= \tfrac{1}{4}\,h^2\frac{\zeta(\alpha)}{\alpha}\\ &\leq \frac{h^2}{c^2}\,.\end{aligned} \tag{59}$$

Combining (59) with the inverse estimate yields

$$\kappa^2\|\tau^{1/2}\,\Delta e^h\|^2_{\tilde{\Omega}} \leq \kappa\|\nabla e^h\|^2_{\Omega}\,. \tag{60}$$

Employing this result in (58) leads to

$$\tfrac{1}{2}|||e^h|||^2_{\mathrm{GLS}} \leq 2\|\tau^{-1/2}\eta\|^2_{\Omega} + \kappa\|\nabla\eta\|^2_{\Omega} + \||a_n|^{1/2}\eta\|^2_{\Gamma_h} + \|\tau^{1/2}\mathscr{L}\eta\|^2_{\tilde{\Omega}}\,. \tag{61}$$

Therefore, by the interpolation estimate,

$$|||e^h|||^2_{\mathrm{GLS}} \leq c_u h^{2l}\,. \tag{62}$$

Likewise,

$$|||\eta|||^2_{\mathrm{GLS}} \leq c_u h^{2l}\,, \tag{63}$$

and so, by the triangle inequality,

$$|||e|||^2_{\mathrm{GLS}} \leq c_u h^{2l}\,. \tag{64}$$

This completes the proof of the theorem. □

3. The scalar unsteady advection-diffusion equation: Space-time formulation

The initial/boundary value problem consists of finding $u(\boldsymbol{x}, t)$ $\forall \boldsymbol{x} \in \bar{\Omega}$ $\forall t \in [0, T]$, such that

$$\mathscr{L}_t u \equiv \dot{u} + \mathscr{L}u = f \quad \text{in } \Omega \times]0, T[\,, \tag{65}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \qquad \forall \boldsymbol{x} \in \Omega\,, \tag{66}$$

$$u = g \qquad \text{on } \Gamma_g \times]0, T[\,, \tag{67}$$

$$-a_n^- u + \sigma_n^d(u) = h \quad \text{on } \Gamma_h \times]0, T[\,, \tag{68}$$

where $\dot{u} = \partial u/\partial t$, and $u_0 : \Omega \rightarrow \mathbb{R}$, $f : \Omega \times]0, T[\rightarrow \mathbb{R}$, $g : \Gamma_g \times]0, T[\rightarrow \mathbb{R}$, and $h : \Gamma_h \times]0, T[\rightarrow \mathbb{R}$ are prescribed data.

The procedures we advocate are based upon the *discontinuous Galerkin method in time*. (See [15], and references therein, for a description of the discontinuous Galerkin method.) Space-time (i.e. $\Omega \times]0, T[$) is divided into *time slabs* $\Omega \times]t_n, t_{n+1}[$, where $0 = t_0 < t_1 < \cdots < t_N = T$. Each time slab is discretized by space-time finite elements. The finite element spaces consist of piecewise polynomials of order k in x and t, continuous in x, but *discontinuous* across time slabs. Again, as a point of departure, we will first present the *Galerkin method*:

$$B(w^h, u^h)_n = L(w^h)_n\,, \quad n = 0, 1, \ldots, N-1\,, \tag{69}$$

$$B(w^h, u^h)_n \equiv \int_{t_n}^{t_{n+1}} (-(\dot{w}^h, u^h)_\Omega + B(w^h, u^h))\,\mathrm{d}t + (w^h(t_{n+1}^-), u^h(t_{n+1}^-))_\Omega\,, \tag{70}$$

$$L(w^h)_n \equiv \int_{t_n}^{t_{n+1}} L(w^h)\,\mathrm{d}t + (w^h(t_n^+), u^h(t_n^-))_\Omega\,, \tag{71}$$

where

$$u^h(t_n^\pm) = u^h(x, t_n^\pm)\,, \tag{72}$$

$$u^h(x, t_0^-) \equiv u_0(x)\,. \tag{73}$$

Remark

Continuity of the solution across time slabs is seen to be *weakly* enforced.

Generalization of SUPG and Galerkin/least-squares proceeds analogously to the steady case.

SUPG:

$$B_{\mathrm{SUPG}}(w^h, u^h)_n = L_{\mathrm{SUPG}}(w^h)_n\,, \quad n = 0, 1, \ldots, N-1\,, \tag{74}$$

$$B_{\mathrm{SUPG}}(w^h, u^h)_n \equiv B(w^h, u^h)_n + \int_{t_n}^{t_{n+1}} (\tau(\dot{w}^h + \boldsymbol{a} \cdot \nabla w^h), \mathscr{L}_t u^h)_{\tilde{\Omega}}\,\mathrm{d}t\,, \tag{75}$$

$$L_{\mathrm{SUPG}}(w^h)_n \equiv L(w^h)_n + \int_{t_n}^{t_{n+1}} (\tau(\dot{w}^h + \boldsymbol{a} \cdot \nabla w^h), f)_{\tilde{\Omega}}\,\mathrm{d}t\,. \tag{76}$$

Galerkin/least-squares:

$$B_{\mathrm{GLS}}(w^h, u^h)_n = L_{\mathrm{GLS}}(w^h)_n\,, \quad n = 0, 1, \ldots, N-1\,, \tag{77}$$

$$B_{\mathrm{GLS}}(w^h, u^h)_n \equiv B(w^h, u^h)_n + \int_{t_n}^{t_{n+1}} (\tau\mathscr{L}_t w^h, \mathscr{L}_t u^h)_{\tilde{\Omega}}\,\mathrm{d}t\,, \tag{78}$$

$$L_{\mathrm{GLS}}(w^h)_n \equiv L(w^h)_n + \int_{t_n}^{t_{n+1}} (\tau\mathscr{L}_t w^h, f)_{\tilde{\Omega}}\,\mathrm{d}t\,. \tag{79}$$

Remarks

(1) In the unsteady case, h represents a space-time mesh parameter.

(2) The issue of the time integration method is obviated by the choice of space-time interpolation. Unconditional stability is achieved in all cases. On each time slab a system of linear algebraic equations needs to be solved.

Let

$$|w^h|^2 \equiv \tfrac{1}{2} \sum_{n=1}^{N-1} \|[\![w^h(t_n)]\!]\|_\Omega^2 + \tfrac{1}{2}(\|w^h(T^-)\|_\Omega^2 + \|w^h(0^+)\|_\Omega^2) + \int_0^T |||w^h|||^2 \,\mathrm{d}t\,, \tag{80}$$

where $[\![w^h(t_n)]\!] = w^h(t_n^+) - w^h(t_n^-)$. It is a simple exercise to show that

$$\sum_{n=0}^{N-1} B(w^h, w^h)_n - \sum_{n=1}^{N-1} \left(w^h(t_n^+), w^h(t_n^-)\right)_\Omega = |w^h|^2\,, \tag{81}$$

where we have used the constraint $w^h(t_0^-) = w^h(0^-) = 0$.

Likewise,

$$\sum_{n=0}^{N-1} B_{\mathrm{GLS}}(w^h, w^h)_n - \sum_{n=1}^{N-1} \left(w^h(t_n^+), w^h(t_n^-)\right) = |w^h|_{\mathrm{GLS}}^2$$

$$= |w^h|^2 + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\tau^{1/2} \mathscr{L}_t w^h\|_{\tilde{\Omega}}^2 \,\mathrm{d}t\,. \tag{82}$$

The following error estimate, analogous to the steady case, can be established for the space-time Galerkin/least-squares method:

$$|e|_{\mathrm{GLS}}^2 \leq c_u h^{2l}\,. \tag{83}$$

Remark

The hypotheses to prove (83) are virtually identical to those for the steady case. On the other hand, the space-time SUPG method requires a special inverse estimate involving the interpolation error (see [9] for details).

4. Symmetric advective-diffusive systems

The previous developments for the scalar advection-diffusion equation may be generalized to *symmetric* advective-diffusive systems. The equations are (see also [9, 11])

$$\mathscr{L}_t V \equiv A_0 V_{,t} + \mathscr{L} V = \mathscr{F}\,, \tag{84}$$

$$\mathscr{L} V \equiv \tilde{A} \cdot \nabla V - \nabla \cdot \tilde{K} \nabla V\,, \tag{85}$$

$$V = (V_1, V_2, \ldots, V_m)^{\mathrm{t}}\,, \tag{86}$$

$$\tilde{A}^{t} = [\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_d] , \tag{87}$$

$$\tilde{K} = \begin{bmatrix} \tilde{K}_{11} & \cdots & \tilde{K}_{1d} \\ \vdots & \ddots & \vdots \\ \tilde{K}_{d1} & \cdots & \tilde{K}_{dd} \end{bmatrix} , \tag{88}$$

$$\tilde{A} \cdot \nabla V = \tilde{A}^{t} \nabla V = \tilde{A}_i V_{,i} = \tilde{A}_1 \frac{\partial V}{\partial x_1} + \cdots + \tilde{A}_d \frac{\partial V}{\partial x_d} , \tag{89}$$

in which A_0 is an $m \times m$ symmetric, positive-definite matrix; $\tilde{A}_i$ is an $m \times m$ symmetric matrix, $1 \leq i \leq d$; and $\tilde{K}$ is an $(m \cdot d) \times (m \cdot d)$ symmetric, positive-definite matrix. (The case in which $\tilde{K}$ is positive-*semidefinite* is more interesting physically, but complicates the specification of boundary conditions.)

Corresponding to the developments for the scalar case, we have the following:

$$\tilde{A}_n = n_i \tilde{A}_i , \tag{90}$$

$$\tilde{A}_n^+ = \tfrac{1}{2}(\tilde{A}_n + |\tilde{A}_n|) , \tag{91}$$

$$\tilde{A}_n^- = \tfrac{1}{2}(\tilde{A}_n - |\tilde{A}_n|) , \tag{92}$$

$$U(V) = A_0 V \qquad \text{(temporal flux)} , \tag{93}$$

$$F_i^a(V) = -\tilde{A}_i V \quad \text{(advective flux)} , \tag{94}$$

$$F_i^d(V) = \tilde{K}_{ij} V_{,j} \quad \text{(diffusive flux)} , \tag{95}$$

$$F_i = F_i^a + F_i^d \qquad \text{(total flux)} , \tag{96}$$

$$F_n^a = n_i F_i^a , \tag{97}$$

$$F_n^d = n_i F_i^d , \tag{98}$$

$$F_n = n_i F_i . \tag{99}$$

For simplicity, we assume that for $x \in \Gamma$, $\tilde{A}_n(x)$ is either positive- or negative-definite. This will allow a concise statement of boundary conditions analogous to the scalar case. For situations in which A_n is indefinite, boundary condition specification is more complex, necessitating component by component specification. Let

$$\Gamma^- = \{x \in \Gamma \mid \tilde{A}_n(x) < 0\} , \tag{100}$$

$$\Gamma^+ = \Gamma - \Gamma^- , \tag{101}$$

$$\Gamma_{\mathscr{G}}^{\pm} = \Gamma_{\mathscr{G}} \cap \Gamma^{\pm} , \tag{102}$$

$$\Gamma_{\mathscr{H}}^{\pm} = \Gamma_{\mathscr{H}} \cap \Gamma^{\pm} . \tag{103}$$

4.1. Boundary value problem

$$\mathscr{L}V = -\nabla \cdot F = -F_{i,i} = \mathscr{F} \quad \text{on } \Omega , \tag{104}$$

$$V = \mathscr{G} \quad \text{on } \Gamma_{\mathscr{G}} , \tag{105}$$

$$-\tilde{A}_n^- V + F_n^d(V) = \mathscr{H} \quad \text{on } \Gamma_{\mathscr{H}} . \tag{106}$$

Equation (106) is equivalent to

$$F_n(V) = \mathscr{H}^- \quad \text{on } \Gamma_{\mathscr{H}}^- \quad \text{(total flux b.c.)} , \tag{107}$$

$$F_n^d(V) = \mathscr{H}^+ \quad \text{on } \Gamma_{\mathscr{H}}^+ \quad \text{(diffusive flux b.c.)} ; \tag{108}$$

variational formulation:

$$\mathscr{S} = \{V \in H^1(\Omega)^m \mid V = \mathscr{G} \text{ on } \Gamma_{\mathscr{G}}\} , \tag{109}$$

$$\mathscr{V} = \{W \in H^1(\Omega)^m \mid W = 0 \text{ on } \Gamma_{\mathscr{H}}\} , \tag{110}$$

$$B(W, V) \equiv (\nabla W, F(V))_\Omega + (W, \tilde{A}_n^+ V)_{\Gamma_{\mathscr{H}}} , \tag{111}$$

$$L(W) \equiv (W, \mathscr{F})_\Omega + (W, \mathscr{H})_{\Gamma_{\mathscr{H}}} , \tag{112}$$

$$\begin{aligned} 0 &= B(W, V) - L(W) \\ &= -(W, \nabla \cdot F(V) + \mathscr{F})_\Omega + (W, -\tilde{A}_n^- V + F_n^d(V) - \mathscr{H})_{\Gamma_{\mathscr{H}}} \\ &\quad \text{(formal consistency)} , \end{aligned} \tag{113}$$

$$\begin{aligned} &B(W, W) = \||\tilde{K}|^{1/2} \nabla W\|_\Omega^2 + \tfrac{1}{2}\||\tilde{A}_n|^{1/2} W\|_{\Gamma_{\mathscr{H}}}^2 \quad \forall W \in \mathscr{V} \\ &\quad \text{(stability)} , \end{aligned} \tag{114}$$

$$|||W|||^2 \equiv B(W, W) , \tag{115}$$

$$\begin{aligned} 0 &= B(1, V) - L(1) \\ &= -\left(\int_{\Gamma^-} \mathscr{H}^- \, d\Gamma + \int_\Omega \mathscr{F} \, d\Omega + \int_{\Gamma^+} (-\tilde{A}_n V + \mathscr{H}^+) \, d\Gamma\right) \\ &\quad \text{(conservation for } \Gamma_{\mathscr{G}} = \emptyset) ; \end{aligned} \tag{116}$$

hyperbolic case:

$$-\nabla\cdot \boldsymbol{F}^a(\boldsymbol{V}) = \mathscr{F} \quad \text{on } \Omega\,, \tag{117}$$

$$\boldsymbol{V} = \mathscr{G}^- \quad \text{on } \Gamma_{\mathscr{G}^-}\,, \tag{118}$$

$$\boldsymbol{F}^a_n(\boldsymbol{V}) = \mathscr{H}^- \quad \text{on } \Gamma^-_{\mathscr{H}}\,, \tag{119}$$

$$B(\boldsymbol{W},\boldsymbol{V}) \equiv (\nabla\boldsymbol{W}, \boldsymbol{F}^a(\boldsymbol{V}))_\Omega + (\boldsymbol{W}, \tilde{\boldsymbol{A}}^+_n\boldsymbol{V})_{\Gamma^+}\,, \tag{120}$$

$$L(\boldsymbol{W}) \equiv (\boldsymbol{W}, \mathscr{F})_\Omega + (\boldsymbol{W}, \mathscr{H}^-)_{\Gamma^-_{\mathscr{H}}}\,, \tag{121}$$

$$\begin{aligned} 0 &= B(\boldsymbol{W},\boldsymbol{V}) - L(\boldsymbol{W}) \\ &= -(\boldsymbol{W}, \nabla\cdot\boldsymbol{F}^a(\boldsymbol{V}) + \mathscr{F})_\Omega + (\boldsymbol{W}, -\tilde{\boldsymbol{A}}^-_n\boldsymbol{V} - \mathscr{H}^-)_{\Gamma^-_{\mathscr{H}}} \\ &\quad \text{(formal consistency)}\,, \end{aligned} \tag{122}$$

$$\begin{aligned} &B(\boldsymbol{W},\boldsymbol{W}) = \tfrac{1}{2}\| |\tilde{\boldsymbol{A}}_n|^{1/2}\boldsymbol{W}\|^2_{\Gamma_{\mathscr{H}}} \quad \forall \boldsymbol{W}\in\mathscr{V} \\ &\quad \text{(stability)}\,, \end{aligned} \tag{123}$$

$$\begin{aligned} &0 = \int_{\Gamma^-} \mathscr{H}^-\,\mathrm{d}\Gamma + \int_\Omega \mathscr{F}\,\mathrm{d}\Omega + \int_{\Gamma^+} -\tilde{\boldsymbol{A}}_n\boldsymbol{V}\,\mathrm{d}\Gamma \\ &\quad \text{(conservation for } \Gamma^-_{\mathscr{G}} = \emptyset)\,; \end{aligned} \tag{124}$$

finite element formulations:

$$B_{\mathrm{SUPG}}(\boldsymbol{W}^h, \boldsymbol{V}^h) = L_{\mathrm{SUPG}}(\boldsymbol{W}^h)\,, \tag{125}$$

$$B_{\mathrm{SUPG}}(\boldsymbol{W}^h, \boldsymbol{V}^h) \equiv B(\boldsymbol{W}^h, \boldsymbol{V}^h) + (\boldsymbol{\tau}\tilde{\boldsymbol{A}}\cdot\nabla\boldsymbol{W}^h, \mathscr{L}\boldsymbol{V}^h)_{\tilde{\Omega}}\,, \tag{126}$$

$$L_{\mathrm{SUPG}}(\boldsymbol{W}^h) \equiv L(\boldsymbol{W}^h) + (\boldsymbol{\tau}\tilde{\boldsymbol{A}}\cdot\nabla\boldsymbol{W}^h, \mathscr{F})_{\tilde{\Omega}}\,, \tag{127}$$

$$B_{\mathrm{GLS}}(\boldsymbol{W}^h, \boldsymbol{V}^h) = L_{\mathrm{GLS}}(\boldsymbol{W}^h)\,, \tag{128}$$

$$B_{\mathrm{GLS}}(\boldsymbol{W}^h, \boldsymbol{V}^h) \equiv B(\boldsymbol{W}^h, \boldsymbol{V}^h) + (\boldsymbol{\tau}\mathscr{L}\boldsymbol{W}^h, \mathscr{L}\boldsymbol{V}^h)_{\tilde{\Omega}}\,, \tag{129}$$

$$L_{\mathrm{GLS}}(\boldsymbol{W}^h) \equiv L(\boldsymbol{W}^h) + (\boldsymbol{\tau}\mathscr{L}\boldsymbol{W}^h, \mathscr{F})_{\tilde{\Omega}}\,; \tag{130}$$

($\boldsymbol{\tau}$ is a symmetric, positive-definite matrix generalizing the scalar τ. See [11] for elaboration.);

$$\begin{aligned} |||\boldsymbol{W}^h|||^2_{\mathrm{GLS}} &\equiv B_{\mathrm{GLS}}(\boldsymbol{W}^h, \boldsymbol{W}^h) \\ &= |||\boldsymbol{W}^h|||^2 + \|\boldsymbol{\tau}^{1/2}\mathscr{L}\boldsymbol{W}^h\|^2_{\tilde{\Omega}}\,, \end{aligned} \tag{131}$$

$$|||\boldsymbol{E}|||^2_{\mathrm{GLS}} \leq C_V h^{2l} \tag{132}$$

4.2. Initial/boundary value problem

$$\mathscr{L}_t V \equiv \dot{U}(V) + \mathscr{L}V = \mathscr{F} \quad \text{in } \Omega \times]0, T[\,, \tag{133}$$

$$U(V(x,0)) = U(V_0(x)) \quad \forall x \in \Omega\,, \tag{134}$$

$$V = \mathscr{G} \quad \text{on } \Gamma_{\mathscr{G}} \times]0, T[\,, \tag{135}$$

$$-\tilde{A}_n^- V + F_n^d = \mathscr{H} \quad \text{on } \Gamma_{\mathscr{H}} \times]0, T[\,; \tag{136}$$

finite element formulations:

$$B(W^h, V^h)_n \equiv \int_{t_n}^{t_{n+1}} ((-\dot{W}^h, U(V^h))_\Omega + B(W^h, V^h))\, dt + (W^h(t_{n+1}^-), U(V^h(t_{n+1}^-)))_\Omega\,, \tag{137}$$

$$L(W^h)_n \equiv \int_{t_n}^{t_{n+1}} L(W^h)\, dt + (W^h(t_n^+), U(V^h(t_n^-)))_\Omega\,, \tag{138}$$

$$\begin{aligned}|W^h|^2 &\equiv \sum_{n=0}^{N-1} B(W^h, W^h)_n - \sum_{n=1}^{N-1} (W^h(t_n^+), U(W^h(t_n^-)))_\Omega \\ &= \tfrac{1}{2} \sum_{n=1}^{N-1} \|A_0^{1/2} [\![W^h(t_n)]\!] \|_\Omega^2 \\ &\quad + \tfrac{1}{2}(\|A_0^{1/2}\, W^h(T^-)\|_\Omega^2 + \|A_0^{1/2}\, W^h(0^+)\|_\Omega^2) + \int_0^T |||W^h|||^2\, dt\,,\end{aligned} \tag{139}$$

$$B_{\text{SUPG}}(W^h, V^h)_n = L_{\text{SUPG}}(W^h)_n\,, \quad n = 0, 1, \ldots, N-1\,, \tag{140}$$

$$B_{\text{SUPG}}(W^h, V^h)_n \equiv B(W^h, V^h)_n + \int_{t_n}^{t_{n+1}} (\tau(A_0 \dot{W}^h + \tilde{A} \cdot \nabla W^h), \mathscr{L}_t V^h)_{\tilde{\Omega}}\, dt\,, \tag{141}$$

$$L_{\text{SUPG}}(W^h)_n \equiv L(W^h)_n + \int_{t_n}^{t_{n+1}} (\tau(A_0 \dot{W}^h + \tilde{A} \cdot \nabla W^h), \mathscr{F})_{\tilde{\Omega}}\, dt\,, \tag{142}$$

$$B_{\text{GLS}}(W^h, V^h)_n = L_{\text{GLS}}(W^h)_n\,, \quad n = 0, 1, \ldots, N-1\,, \tag{143}$$

$$B_{\text{GLS}}(W^h, V^h)_n \equiv B(W^h, V^h)_n + \int_{t_n}^{t_{n+1}} (\tau \mathscr{L}_t W^h, \mathscr{L}_t V^h)_{\tilde{\Omega}}\, dt\,, \tag{144}$$

$$L_{\mathrm{GLS}}(W^h)_n \equiv L(W^h)_n + \int_{t_n}^{t_{n+1}} (\tau \mathcal{L}_t W^h, \mathcal{F})_{\tilde{\Omega}} \, \mathrm{d}t \,, \tag{145}$$

$$\begin{aligned} |W^h|^2_{\mathrm{GLS}} &\equiv \sum_{n=0}^{N-1} B_{\mathrm{GLS}}(W^h, W^h)_n - \sum_{n=1}^{N-1} (W^h(t_n^+), U(W^h(t_n^-)))_\Omega \\ &= |W^h|^2 + \sum_{n=0}^{N-1} \int_{t_n}^{t_{n+1}} \|\tau^{1/2} \mathcal{L}_t W^h\|^2_{\tilde{\Omega}} \, \mathrm{d}t \,, \end{aligned} \tag{146}$$

$$|E|^2_{\mathrm{GLS}} \leq C_V h^{2l} \,. \tag{147}$$

6. Conclusions

In this paper we have presented the Galerkin/least-squares finite element method for advective-diffusive equations. The Galerkin/least-squares method is closely related to SUPG, but represents a conceptually simpler and more general methodology, applicable to a wide variety of problem classes. A detailed global convergence analysis of the steady scalar advection-diffusion equation was presented, and analogous results were quoted for the unsteady case, as well as steady and unsteady advective-diffusive systems.

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