Continuum formulation for finite strain viscoelasticity and its implementation with finite element method

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Within the context of viscoelasticity, the free energy function is considered to consist of a volumetric and isochoric elastic response function and a configuration free energy, which is responsible for the viscoelastic response. Based on this structure, the stress response is usually decoupled into equilibrium and non-equilibrium parts. The non-equilibrium stresses derive the viscoelastic response and are governed by rate equation motivated by the linear theory.

In the following, we first discuss the constitutive response of the viscoelastic continuum. Secondly, we derive the evolution integration of the internal variables. Finally, we summarize the time integration algorithm.

1 Mathematical viscoelastic model

1.1 Constitution relation for viscoelasticity

To characterize viscoelastic isothermal processes, we postulate the following isotropic constitutive model [1]

$$G = G(\boldsymbol{C}, \boldsymbol{\Gamma}^\alpha) := G^\infty_{\text{vol}}(P) + G^\infty_{\text{ich}}(\tilde{\boldsymbol{C}}) + \Upsilon(\tilde{\boldsymbol{C}}, \boldsymbol{\Gamma}^\alpha) \quad \text{in } \Omega_X \times [0, T] \tag{1.1}$$

where $\{\boldsymbol{\Gamma}^\alpha\}(\alpha = 1, ..., m)$ are a set of internal variables, which are not observable quantities in phenomenological experiments. Each $\boldsymbol{\Gamma}^\alpha$ may be regarded as a strain tensor analogous to the strain tensor $\tilde{\boldsymbol{C}}$. The relaxational behavior is modeled by $m > 1$ relaxation process with corresponding relaxation time $\tau_a \in (0, \infty)(\alpha = 1, ..., m)$. The free energy satisfies the normalization condition that $G = G(\tilde{\boldsymbol{C}} = \boldsymbol{I}, \boldsymbol{\Gamma}^\alpha = \boldsymbol{I}) \equiv 0$. Further, $G^\infty_{\text{vol}}(P)$ is a convex, which is obtained by the Legendre transformation of the volumetric part of Helmholtz free energy. The superscript ∞ denotes the hyperelastic behavior of very slow process.

One form of the second law of thermodynamics under isothermal process is the Clausius-Plank inequality,

$$\mathcal{D}_{\text{int}} := \boldsymbol{S} : \dot{\boldsymbol{C}}/2 - \dot{G} \geq 0 \tag{1.2}$$

where $\mathcal{D}_{\text{int}}$ is the internal dissipation (or local entropy production). By the structure (1.1), we have the fundamental constitutive hyperelastic equation for the convected stresses $\boldsymbol{S}$ and a remainder inequality for the dissipation:

$$\boldsymbol{S} = 2\partial_{\boldsymbol{C}} G\left(\boldsymbol{C}, \boldsymbol{\Gamma}^\alpha\right), \quad \mathcal{D}_{\text{int}} = \sum_{\alpha=1}^{m} \left[-\partial_{\boldsymbol{\Gamma}^\alpha} G\left(\boldsymbol{C}, \boldsymbol{\Gamma}^\alpha\right)\right] : \dot{\boldsymbol{\Gamma}}^\alpha \geq 0, \tag{1.3}$$

in which the stresses consist of a purely volumetric and an isochoric contribution

$$\boldsymbol{S} = \boldsymbol{S}_{vol}(J) + \boldsymbol{S}_{ich}\left(\tilde{\boldsymbol{C}}, \boldsymbol{\Gamma}^\alpha\right)$$

The split is based on the following definitions:

$$\boldsymbol{S}^\infty_{\text{vol}} := -JP\boldsymbol{C}^{-1}, \quad \boldsymbol{S}_{\text{iso}} := \boldsymbol{S}^\infty_{\text{iso}} + \sum_{\alpha=1}^{m} \boldsymbol{Q}^\alpha,$$

with

$$\boldsymbol{S}_{\text{ich}}^{\infty} = J^{-2/3}\mathbb{P} : \tilde{\boldsymbol{S}}_{ich}^{\infty}, \quad \tilde{\boldsymbol{S}}_{ich}^{\infty} := 2\partial_{\tilde{\boldsymbol{C}}} G^{\infty}(\tilde{\boldsymbol{C}}),$$
$$\boldsymbol{Q}^{\alpha} = J^{-2/3}\mathbb{P} : \tilde{\boldsymbol{Q}}^{\alpha}, \quad \tilde{\boldsymbol{Q}}^{\alpha} := 2\partial_{\tilde{\boldsymbol{C}}} \Upsilon(\tilde{\boldsymbol{C}}, \boldsymbol{\Gamma}^{\alpha}),$$

where $\mathbb{P} := \mathbb{I} - \frac{1}{3}\boldsymbol{C}^{-1} \otimes \boldsymbol{C}$ is a projector tensor of fourth order. The internal variables $\{\boldsymbol{Q}^{\alpha}\}(\alpha = 1, ..., m)$ are interpreted as the non-equilibrium stress in the sense of thermodynamics. Motivated by the standard linear solid, $\boldsymbol{Q}^{\alpha}$ is defined to be variables conjugate to $\boldsymbol{\Gamma}^{\alpha}$ with the relation $\boldsymbol{Q}_{\alpha} := -\partial_{\boldsymbol{\Gamma}^{\alpha}} \Upsilon(\tilde{\boldsymbol{C}}, \boldsymbol{\Gamma}^{\alpha})$. This relation restricts the internal configuration free energy Γ as discussed in the [2]. With this relation, the internal dissipation is denoted by $\mathcal{D}_{\text{int}} = \sum_{\alpha=1}^{m} \boldsymbol{Q}^{\alpha} : \dot{\boldsymbol{\Gamma}}^{\alpha} \geq 0$. A suitable evolution equation for the internal strains is given as

$$\dot{\boldsymbol{\Gamma}}^{\alpha} = \mathbb{V}(\tilde{\boldsymbol{C}}, \boldsymbol{\Gamma}^{\alpha}) : \boldsymbol{Q}^{\alpha},$$

where $\mathbb{V}$ is a positive definite fourth-order tensor which contains the inverse viscosity.

Remark 1. *The condition for thermodynamic equilibrium implies that for $t \to \infty$ the internal stresses reach equilibrium which means that $\boldsymbol{Q}^{\alpha}|_{t\to\infty} \equiv \boldsymbol{0}$. And the internal dissipation is zero at equilibrium, which is characteristic for purely elastic materials.*

1.2 Rate equation

Motivated by the rate equation for one-dimensional generalized Maxwell model with linear geometry, the evolution equations for the $\boldsymbol{Q}^{\alpha}$ is

$$\begin{cases} \dot{\boldsymbol{Q}}^{\alpha} + \frac{\boldsymbol{Q}^{\alpha}}{\tau_{\alpha}} = \frac{d}{dt}\left(J^{-2/3}\mathbb{P} : 2\partial_{\tilde{\boldsymbol{C}}} G^{\alpha}(\tilde{\boldsymbol{C}})\right), \\ \boldsymbol{Q}^{\alpha}|_{t=0} = \boldsymbol{Q}_0^{\alpha}, \end{cases} \tag{1.4}$$

where $G^{\alpha}(\tilde{\boldsymbol{C}})$ denotes the free energy of the body, corresponding to the α-relaxation process with relaxation time $\tau_{\alpha} > 0$. The initial state of $\boldsymbol{Q}_0^{\alpha}$ is given by $\boldsymbol{Q}_0^{\alpha} = J_0^{-2/3}\mathbb{P}_0 : 2\partial_{\tilde{\boldsymbol{C}}_0} G^{\alpha}(\tilde{\boldsymbol{C}}_0)$. We have the closed-form solution of the linear dissipative equations in the following convolution form

$$\boldsymbol{Q}^{\alpha} = \exp(-t/\tau_{\alpha})\boldsymbol{Q}_0^{\alpha} + \int_{0^+}^{t} \beta_{\infty}^{\alpha} \exp\left(-(t-s)/\tau_{\alpha}\right) \frac{d}{ds}\left(J^{-2/3}\mathbb{P} : 2\partial_{\tilde{\boldsymbol{C}}} G^{\infty}(\tilde{\boldsymbol{C}})\right). \tag{1.5}$$

In the above expression we have the assumption of identical polymer chains as

$$G^{\alpha}(\tilde{\boldsymbol{C}}) = \beta_{\infty}^{\alpha} G^{\infty}(\tilde{\boldsymbol{C}}) \quad (\alpha = 1, ..., m).$$

Herein β_{∞}^{α} are given free energy factors associated with $\tau_{\alpha} > 0$.

1.3 Time integration algorithm

Consider a partition $\cup_{n=0}^{M}[t_n, t_{n+1}]$ of the time interval $[0^+, T]$ of interest, where $0^+ = t_0 < ... < t_{M+1} = T$. Now we focus on a typical time interval $[t_n, t_{n+1}]$, with time step size $\Delta t := t_{n+1} - t_n \in \mathbb{R}_+$. Assume that the solution Y_n is known at a certain time t_n, we need to find the solutions Y_{n+1} by Newton-Raphson method.

The evaluation of the viscoelastic stress Q_{n+1}^{α} plays a significant role in viscoelastic cases. To this end, we split the convolution integral in (1.5) into two parts $\int_0^{t_{n+1}}(\cdot)ds = \int_0^{t_n}(\cdot)ds + \int_{t_n}^{t_{n+1}}(\cdot)ds$. The internal variables at t_{n+1} are recovered by using the second-order accurate midpoint rule on the $\int_{t_n}^{t_{n+1}} ds$ term. After some algebraic manipulations, we have the recursive formula for the internal stresses

$$Q_{n+1}^{\alpha} = \exp(\xi^{2\alpha})\boldsymbol{Q}_n^{\alpha} + \int_{t_n}^{t_{n+1}} \beta_{\infty}^{\alpha} \exp\left(-(t_{n+1}-s)/\tau_{\alpha}\right) \dot{\boldsymbol{S}}_{iso}^{\infty} ds \tag{1.6}$$

$$= \beta_{\infty}^{\alpha} \exp(\xi^{\alpha}) \boldsymbol{S}_{iso\ n+1}^{\infty} + \mathcal{H}_n^{\alpha} \tag{1.7}$$

where

$$\mathcal{H}_n^{\alpha} := J_n^{-2/3}\mathbb{P}_n : \tilde{\mathcal{H}}_n^{\alpha}, \quad \tilde{\mathcal{H}}_n^{\alpha} := \exp(\xi^{\alpha})\left(\exp(\xi^{\alpha})\tilde{\boldsymbol{Q}}_n^{\alpha} - \beta_{\infty}^{\alpha}\tilde{\boldsymbol{S}}_{\text{iso}\ n}^{\infty}\right), \xi^{\alpha} := -\Delta t/2\tau_{\alpha}.$$

References

[1] G.A. Holzapfel. On large strain viscoelasticity: continuum formulation and finite element applications to elastomeric structures. *International Journal for Numerical Methods in Engineering*, 39(22):3903–3926, 1996.

[2] G.A. Holzapfel and J.C. Simo. A new viscoelastic constitutive model for continuous media at finite thermomechanical changes. *International Journal of Solids and Structures*, 33(20-22):3019–3034, 1996.