the Viscoelastic Arterial Wall Behavior under the Forces of Pulsatile Flow

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Many studies on the arterial hemodynamics in the past, such as the Womersley solution, would considered solving the Navier-Stokes equations in fluid domain by coupling with the motion of equation of the vessel wall with the assumptions of the thin wall theory and linear elastic material. However, the assumption is not sure to be reasonable in practice.

The blood flow in the arteries is pulsating. And under the oscillatory forces of pulsatile flow, the arterial wall suffers considerable shear stresses and strains in the longitudinal direction, which are ignored on the thin wall assumption.

Recent studies tend to consider the vessel wall being viscoelastic wall with a finite thickness, and the relative proportions of viscous and elastic content within the wall will changes because of the disease or aging. The viscoelastic material model more accord with physiological reality. There are numerical experiments indicate that the viscoelastic parameters of the wall material have an impact on the hemodynamics quantity, especially the wall deformation and the fluid pressure.

We consider a thick-walled, straight and cylindrical tube. The wall materials is assumed to be incompressible and viscoelastic. With the axisymmetric assumption, the angular component of velocity and all derivatives in the angular direction are zero. That is,

$$w = 0, \quad \frac{\partial}{\partial \theta} = 0.$$

The Cauchy equation of the wall motion is

$$\rho_s \frac{\partial^2 \vec{\mathbf{x}}}{\partial t^2} = \vec{\mathbf{F}}_{body} + \nabla \cdot \bar{\bar{\boldsymbol{\sigma}}}.$$

And the longitudinal and radial components of the equation can be written as,

$$\rho_s \frac{\partial^2 \xi}{\partial t^2} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{rx}}{\partial r} + \frac{\sigma_{rx}}{r},$$

$$\rho_s \frac{\partial^2 \eta}{\partial t^2} = \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr}}{r} + \frac{\partial \sigma_{xr}}{\partial x} - \frac{\sigma_{\theta\theta}}{r}.$$

And the incompressibility gives

$$\frac{\partial \xi}{\partial x} + \frac{\partial \eta}{\partial r} + \frac{\eta}{r} = 0.$$

The stress−strain relations are given by

$$\sigma_{xx} = -\frac{\sigma_{xx} + \sigma_{rr} + \sigma_{\theta\theta}}{3} + 2E^* \frac{\partial \xi}{\partial x},$$

$$\sigma_{rr} = -\frac{\sigma_{xx} + \sigma_{rr} + \sigma_{\theta\theta}}{3} + 2E^* \frac{\partial \eta}{\partial r},$$

$$\sigma_{\theta\theta} = -\frac{\sigma_{xx} + \sigma_{rr} + \sigma_{\theta\theta}}{3} + 2E^* \frac{\eta}{r},$$

$$\sigma_{rx} = \sigma_{xr} = E^* \left(\frac{\partial \xi}{\partial r} + \frac{\partial \eta}{\partial x} \right),$$

where E^* is the complex modulus of elasticity related to the fundamental frequency of the oscillatory flow ω, the expression is

$$E^*(\omega) = E_0 + \eta_1 \omega^\alpha e^{i\pi\frac{\alpha}{2}} + \eta_2 \omega^\beta e^{i\pi\frac{\beta}{2}},$$

where E_0 is the static elastic modulus, and η_1 and η_2 are the viscosity parameters, α and β are the derivative orders.

Since the pulsatile flow can be represented by Fourier decomposition. Using the method of separation of variables and the exponential form of Fourier series, we can design the formulation of the solution as

$$\xi(x,r,t) = \sum_{n=1}^{\infty} \bar{\xi}_n(r) e^{i\omega n\left(t-\frac{x}{c}\right)},$$

$$\eta(x,r,t) = \sum_{n=1}^{\infty} \bar{\eta}_n(r) e^{i\omega n\left(t-\frac{x}{c}\right)}.$$

Denote the wave number as $\gamma = \omega/c$. For convenience we take n=1, with introducing the general solution of the Modified Bessel equations, the solution of the displacement amplitudes $\bar{\xi}$ and $\bar{\eta}$ are given by

$$\bar{\xi}(r) =- i\gamma A_1 I_0(\gamma r) - i\gamma A_2 K_0(\gamma r) + \lambda_s A_3 I_1(i\lambda_s r) + \lambda A_4 K_1(i\lambda_s r),$$
$$\bar{\eta}(r) = \gamma A_1 I_1(\gamma r) - \gamma A_2 K_1(\gamma r) + \gamma A_3 I_1(i\lambda_s r) + \gamma A_4 K_1(i\lambda_s r).$$

where

$$\lambda_s^2 = \frac{\rho_s \omega^2}{E^*} - \gamma^2,$$

and the constants A_{1-4} are determined by the boundary conditions.

The boundary conditions given on the inner and outer wall surfaces are both described by displacement. For the inner boundary, the wall displacement is set to be the same as the fluid at that position, that is

$$\xi(x,R,t) = \xi_0 e^{i\omega\left(t-\frac{x}{c}\right)},$$

$$\eta(x,R,t) = \eta_0 e^{i\omega\left(t-\frac{x}{c}\right)}.$$

For the outer boundary, we assume that the wall is tethered to surrounding tissue, and the boundary condition is depended on the degree of tethering δ_x and δ_r,

$$\xi(x,R+h,t) = (1-\delta_x)\xi_0 e^{i\omega\left(t-\frac{x}{c}\right)},$$

$$\eta(x,R+h,t) = (1-\delta_r)\eta_0 e^{i\omega\left(t-\frac{x}{c}\right)},$$

where $\delta_x = \delta_r = 0$ for the free wall and $\delta_x = \delta_r = 1$ for the fully tethered case.

Then we consider the fluid-wall coupling. Here we use the classical solution for pulsatile flow in a viscoelastic tube given by Womersley, namely,

$$p(x,r,t) =- i\rho_f \omega B J_0(\mathrm{i}\gamma r) e^{i\omega\left(t-\frac{x}{c}\right)},$$

$$u(x,r,t) = \left(-i\gamma B J_0(\mathrm{i}\gamma r) - i\lambda_f C J_0(\mathrm{i}\lambda_f r)\right) e^{i\omega\left(t-\frac{x}{c}\right)},$$

$$v(x,r,t) = \left(-i\gamma B J_1(\mathrm{i}\gamma r) - i\gamma C J_1(\mathrm{i}\lambda_f r)\right) e^{i\omega\left(t-\frac{x}{c}\right)},$$

where

$$\lambda_f^2 = \frac{i\rho_f\omega^2}{\mu} - \gamma^2,$$

and B and C are constants to be determined.

At the fluid-wall interface, the boundary conditions are the same velocity and the force balance, that is

$$u = \frac{\partial\xi}{\partial t},\ \ v = \frac{\partial\eta}{\partial t},$$

$$\sigma_{xr} = \mu\left(\frac{\partial u}{\partial r} + \frac{\partial v}{\partial x}\right) = E^*\left(\frac{\partial\xi}{\partial r} + \frac{\partial\eta}{\partial x}\right),$$

$$\sigma_{rr} =- p + 2\mu\frac{\partial v}{\partial r} =- \frac{\sigma_{xx} + \sigma_{rr} + \sigma_{\theta\theta}}{3} + 2E^*\frac{\partial\eta}{\partial r}.$$

Above all we have six equations with six unknown constants A_{1-4}, B and C, which provide a system of homogeneous equations and can be write as a six-by-six matrix. The determinant of the matrix is defined as the wave speed c.

Once the wave speed is given, these six constants are then determined, and the displacements of the wall can be solved. From the constitutive relations given, the corresponding stress components are determined as well.

Reference:

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