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A NEW FINITE ELEMENT FORMULATION FOR COMPUTATIONAL FLUID DYNAMICS: V. CIRCUMVENTING THE BABUŠKA–BREZZI CONDITION: A STABLE PETROV–GALERKIN FORMULATION OF THE STOKES PROBLEM ACCOMMODATING EQUAL-ORDER INTERPOLATIONS*

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A new Petrov–Galerkin formulation of the Stokes problem is proposed. The new formulation possesses better stability properties than the classical Galerkin/variational method. An error analysis is performed for the case in which both the velocity and pressure are approximated by C^0 interpolations. Combinations of C^0 interpolations which are unstable according to the Babuška–Brezzi condition (e.g., equal-order interpolations) are shown to be stable and convergent within the present framework. Calculations exhibiting the good behavior of the methodology are presented.

Tertullian's rule of faith:
Certum est, quia impossibile est

1. Introduction

The Stokes equations govern the incompressible creeping flow of viscous fluids, and are form-identical to the equations of classical isotropic incompressible elasticity theory which govern the linear response of rubber-like materials, solid rocket propellants, etc. In addition, the Stokes operator is a constituent of more complicated models of physical phenomena such as the incompressible Navier–Stokes equations. Consequently, the numerical solution of many problems of engineering interest is contingent upon the ability to solve the Stokes equations. It is thus no wonder that the Stokes equations have been a focal point of finite element research for over twenty years.

The dependent variables in Stokes flow are the velocity and pressure. In a finite element setting, the classical Galerkin/variational formulation, which was first proposed by Herrmann [5] and may be viewed as a particular case of the Hellinger–Reissner principle [4, 10], naturally gives rise to what is termed a 'mixed method'. It was recognized fairly early on that the success of a formulation of this type was strongly dependent upon the particular pair of velocity and pressure interpolations employed. In many cases, seemingly natural combinations produced violently oscillating pressures. The mathematical framework for understanding the behavior of mixed methods for the Stokes problem was provided by Babuška [1] and Brezzi

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[2]. The key requirement boils down to satisfaction of a *stability condition* which involves both velocity and pressure spaces. This condition is one of the most celebrated results in the mathematical theory of finite elements and has become known as the *Babuška–Brezzi condition*. Although numerous convergent combinations of velocity and pressure (i.e., 'elements') have been developed, it is fair to say that most, if not all, involve interpolation patterns which are inconvenient from an implementational standpoint. Three-dimensional elements have been particularly hard to come by and only rather elaborate constructions have been found to pass the Babuška–Brezzi test. Implementationally attractive combinations, such as, for example, equal-order interpolations, fail to satisfy the Babuška–Brezzi condition.

In recent years, the senior author and his colleagues have been engaged in the development of Petrov–Galerkin formulations of various flow problems. *A contemporary point of view is that Petrov–Galerkin formulations are devices for enhancing stability without upsetting consistency*. In this work we attempt to exploit this idea in the context of the Stokes problem. We develop a new formulation for the Stokes problem, which consists of the Galerkin formulation plus some additional terms emanating from a 'perturbation' to the weighting function. The new formulation has improved stability compared with the Galerkin formulation. Furthermore, we are able to prove convergence for rather general C^0 combinations of velocity and pressure. In particular, the results apply to equal-order interpolations. The new formulation thus opens the way to the use of computationally desirable velocity-pressure pairs which have heretofore been dismissed as unstable. We believe that our new formulation may have a significant impact upon the way incompressible flows, and related phenomena, are computed henceforth.

An outline of the remainder of the paper follows: In Section 2 we review the boundary value problem of Stokes flow. Our new finite element formulation is introduced in Section 3 and an error analysis is presented in Section 4. The matrix formulation is discussed in Section 5, some preliminary numerical results are presented in Section 6, and conclusions are drawn in Section 7.

2. Stokes flow

Let Ω be an open bounded region in $\mathbb{R}^{n_{sd}}$, where n_{sd} is the number of space dimensions. We assume $n_{sd} \geq 2$. The boundary of Ω is denoted by Γ and is assumed to be piecewise smooth. We further assume that Γ is decomposed into two subregions as follows:

$$\Gamma = \overline{\Gamma_g \cup \Gamma_h}\,, \tag{1}$$

$$\emptyset = \Gamma_g \cap \Gamma_h\,. \tag{2}$$

The unit outward normal vector to Γ is denoted by $\boldsymbol{n}$.

The equations of Stokes flow are

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \quad \text{(momentum balance)}\,, \tag{3}$$

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{(incompressibility condition)}, \tag{4}$$

where

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{(Cauchy stress tensor)}\,. \tag{5}$$

The notation is as follows: $\boldsymbol{f}$ is the body force vector; $\boldsymbol{u}$ is the velocity vector; p is the pressure; $\boldsymbol{I}$ is the identity tensor; μ is the dynamic viscosity; and $\boldsymbol{\varepsilon}(\boldsymbol{u})$ is the symmetrical part of the velocity gradient.

The boundary value problem consists of finding $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ and $p = p(\boldsymbol{x})$ satisfying (3)–(5) $\forall \boldsymbol{x} \in \Omega$ and the prescribed boundary conditions, which are assumed to take the form

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma_g\,, \tag{6}$$

$$(\boldsymbol{\sigma}\cdot\boldsymbol{n})(\boldsymbol{x}) = \boldsymbol{h}(\boldsymbol{x}) \quad \forall \boldsymbol{x} \in \Gamma_h\,, \tag{7}$$

where $\boldsymbol{g}$ and $\boldsymbol{h}$ are given functions.

Assuming Γ_h is not empty and the given data are sufficiently regular, a unique solution of the boundary value problem may be shown to exist. If Γ_h is empty, in which case we have the *Dirichlet problem*, existence requires that $\boldsymbol{g}$ satisfy

$$\int_\Gamma \boldsymbol{g}\cdot\boldsymbol{n}\,\mathrm{d}\Gamma = 0\,. \tag{8}$$

In this case the solution is nonunique, the pressure being determined only up to an arbitrary additive constant.

REMARK 2.1. The equations of Stokes flow are form-identical to the equations of classical isotropic infinitesimal elasticity theory. Only the physical interpretation changes. In elasticity, $\boldsymbol{u}$ is the displacement vector and μ is the shear modulus.

3. Finite element formulation

Let Ω^e denote the interior of the eth element. We assume Ω is discretized into n_{el} element domains such that

$$\overline{\Omega} = \bigcup_{e=1}^{n_{\mathrm{el}}} \overline{\Omega^e}\,, \tag{9}$$

$$\emptyset = \bigcap_{e=1}^{n_{\mathrm{el}}} \Omega^e\,. \tag{10}$$

Let Γ^e be the boundary of Ω^e. We define the discretization's *interior boundary* by

$$\Gamma_{\mathrm{int}} = \bigcup_{e=1}^{n_{\mathrm{el}}} \Gamma^e - \Gamma\,. \tag{11}$$

Let $\mathcal{V}^h \subset (H^1(\Omega))^{n_{\mathrm{sd}}}$ denote the space of velocity weighting functions. We assume that $\mathcal{V}^h$ consists of typical C^0 finite element functions, and that if $\boldsymbol{w}^h \in \mathcal{V}^h$, then

$$w^h(x) = 0 \quad \forall x \in \Gamma_g . \tag{12}$$

Let $\mathcal{S}^h \subset (H^1(\Omega))^{n_{sd}}$ denote the set of trial velocity solutions. We assume that $\mathcal{S}^h$ also consists of typical C^0 finite element functions, and that if $u^h \in \mathcal{S}^h$, then

$$u^h = v^h + g^h , \tag{13}$$

where $v^h \in \mathcal{V}^h$ and $g^h \in \mathcal{S}^h$ is a fixed extension of g, that is,

$$g^h(x) = g(x) \quad \forall x \in \Gamma_g . \tag{14}$$

Let $\mathcal{P}^h \subset L_2(\Omega)$ denote the sets of pressure weighting functions and trial solutions. In general, members of $\mathcal{P}^h$ may be discontinuous. However, the error estimates that we subsequently obtain will employ the assumption that $\mathcal{P}^h$ also consists of typical C^0 finite element interpolations. For the Dirichlet problem, $\mathcal{P}^h \subset L_2(\Omega)/\mathbb{R}$, the quotient space consisting of equivalence classes of $L_2(\Omega)$-functions which differ by an arbitrary additive constant.

Let $(\cdot,\cdot)$ denote the $L_2(\Omega)$ inner product. Likewise, let $(\cdot,\cdot)_D$ represent the $L_2(D)$ inner product, where D may be Γ_h, Γ_{int}, Ω^e, etc.

Consider the following Petrov–Galerkin formulation of the Stokes problem: Given f, g, and h, find $v^h \in \mathcal{V}^h$ and $p^h \in \mathcal{P}^h$, such that for all $w^h \in \mathcal{V}^h$ and $q^h \in \mathcal{P}^h$,

$$B_\alpha(w^h, q^h; v^h, p^h) = L_\alpha(w^h, q^h) , \tag{15}$$

where

$$B_\alpha(w^h, q^h; v^h, p^h) = (\varepsilon(w^h), 2\mu\varepsilon(v^h)) - (\operatorname{div} w^h, p^h) + (q^h, \operatorname{div} v^h) + \sum_{e=1}^{n_{el}} \left(\frac{\alpha^e (h^e)^2}{2\mu} \nabla q^h, \nabla p^h - 2\mu \operatorname{div} \varepsilon(v^h) \right)_{\Omega^e} , \tag{16}$$

$$\begin{aligned} L_\alpha(w^h, q^h) = {} & \sum_{e=1}^{n_{el}} \left(w^h + \frac{\alpha^e (h^e)^2}{2\mu} \nabla q^h, f \right)_{\Omega^e} + (w^h, h)_{\Gamma_h} \\ & - (\varepsilon(w^h), 2\mu\varepsilon(g^h)) - (q^h, \operatorname{div} g^h) \\ & + \sum_{e=1}^{n_{el}} \left(\frac{\alpha^e (h^e)^2}{2\mu} \nabla q^h, 2\mu \operatorname{div} \varepsilon(g^h) \right)_{\Omega^e} , \end{aligned} \tag{17}$$

$$h^e = \operatorname{dia}(\Omega^e)/\sqrt{n_{sd}} , \tag{18}$$

$$\alpha = \{\alpha^1, \alpha^2, \ldots, \alpha^{n_{el}}\} . \tag{19}$$

The α^e are nondimensional, nonnegative 'stability constants' which depend upon element type. Selection of the α^e will be discussed in subsequent sections. If $\alpha = \{0, 0, \ldots, 0\}$, then (15)–(17) reduce to the standard Galerkin variational formulation (see, e.g., [9]). Note that $B_\alpha(\cdot,\cdot;\cdot,\cdot)$ is a bilinear form defined on the produce space $\mathcal{V}^h \times \mathcal{P}^h$,

$$B_{\alpha}:(\mathcal{V}^h \times \mathcal{P}^h)\times(\mathcal{V}^h \times \mathcal{P}^h)\rightarrow \mathbb{R}\,. \tag{20}$$

Likewise, $L_{\alpha}(\cdot,\cdot)$ is linear form defined on $\mathcal{V}^h \times \mathcal{P}^h$,

$$L_{\alpha}:\mathcal{V}^h \times \mathcal{P}^h \rightarrow \mathbb{R}\,. \tag{21}$$

Integrating (15) by parts yields the *Euler–Lagrange form of the variational equation*:

$$\begin{aligned}0 = \sum_{e=1}^{n_{\mathrm{el}}} (\tilde{\boldsymbol{w}}^h, \underbrace{-\nabla p^h + 2\mu \operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{u}^h) + \boldsymbol{f}}_{\substack{\text{momentum-}\\ \text{balance residual}}})_{\Omega^e} - (q^h, \underbrace{\operatorname{div} \boldsymbol{u}^h}_{\substack{\text{incompressibility-}\\ \text{condition residual}}})\\ - (\boldsymbol{w}^h, \underbrace{\boldsymbol{\sigma}^h \cdot \boldsymbol{n} - \boldsymbol{h}}_{\substack{\text{traction boundary-}\\ \text{condition residual}}})_{\Gamma_h} - (\boldsymbol{w}^h, \underbrace{[\boldsymbol{\sigma}^h \cdot \boldsymbol{n}]}_{\substack{\text{traction continuity-}\\ \text{condition residual}}})_{\Gamma_{\mathrm{int}}}\,,\end{aligned} \tag{22}$$

where

$$\tilde{\boldsymbol{w}}^h = \boldsymbol{w}^h + \frac{\alpha^e (h^e)^2}{2\mu} \nabla q^h\,, \tag{23}$$

$$\boldsymbol{\sigma}^h = -p^h \boldsymbol{I} + 2\mu \boldsymbol{\varepsilon}(\boldsymbol{u}^h)\,, \tag{24}$$

and $[\boldsymbol{\sigma}^h \cdot \boldsymbol{n}]$ denotes the jump in $\boldsymbol{\sigma}^h \cdot \boldsymbol{n}$ across Γ_{int} (see, e.g., [6]).

REMARK 3.1. (1) The purpose of the $\boldsymbol{\alpha}$-term is to improve upon the *stability* of the Galerkin formulation without compromising *consistency*. That consistency has been maintained is suggested by the fact that the $\boldsymbol{\alpha}$-modified method is a *residual method*, that is, if $\boldsymbol{u}^h$ and p^h are replaced by their exact counterparts, then (15) is satisfied as can be immediately seen from (22).

(2) By virtue of the fact that the weighting function which multiplies the momentum-balance residual (i.e., (23)) is not simply $\boldsymbol{w}^h$, the present formulation may be considered a *Petrov–Galerkin method*. The ∇q^h-term in (23) is sometimes referred to as the 'perturbation' of the Galerkin weighting function.

(3) Johnson and Saranen [8] have included a $\delta \nabla q^h$-perturbation to the Galerkin weighting function in their 'streamline diffusion' formulation of the incompressible Navier–Stokes equations (see also [7]). They only consider the convection-dominated case and conclude that δ should be $O(h^e)$ to achieve the best rate of convergence. It will be confirmed in the next section that our choice (i.e., $\delta = O((h^e)^2)$, see (23)), is the correct one for the diffusion-dominated limit. Combining these results suggests an improved methodology for the incompressible Navier–Stokes equations valid over the entire range of convective-diffusive phenomena. (In Johnson's papers, discrete velocity fields are employed which are assumed to be *exactly* divergence-free. This is a serious impediment to the practical utility of the approach. However, it appears that a slightly modified formulation can be developed in which the assumption of divergence-free velocity is no longer essential.)

4. Error analysis

We need to introduce the following notations:

$$\|\cdot\| = L_2(\Omega)\text{-norm}\,, \qquad \|\cdot\|_{\Omega^e} = L_2(\Omega^e)\text{-norm},$$

$$\|\cdot\|_s = H^s(\Omega)\text{-norm}\,, \qquad h = \max_{1\leq e\leq n_{\mathrm{el}}} (h^e)\,.$$

Theorem 4.1. Consider the Dirichlet problem for which $\Gamma_h = \emptyset$ and $\Gamma_g = \Gamma$. We may write the exact velocity in the following way:

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{g}^h\,, \tag{25}$$

where

$$\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{0} \quad \forall \boldsymbol{x} \in \Gamma\,. \tag{26}$$

Assume:

(i) *μ is a positive constant.*

(ii) *$\mathcal{V}^h$ and $\mathcal{P}^h$ consist of C^0 finite element functions for which the following interpolation estimates hold: Let $\tilde{\boldsymbol{v}}^h \in \mathcal{V}^h$ and $\tilde{p}^h \in \mathcal{P}^h$ denote the interpolants of $\boldsymbol{v}$ and p, respectively. Then $\boldsymbol{\eta}_v = \tilde{\boldsymbol{v}}^h - \boldsymbol{v}$ and $\eta_p = \tilde{p}^h - p$ satisfy*

$$\|\boldsymbol{\eta}_v\| + h\|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\| + h^2\sum_{e=1}^{n_{\mathrm{el}}}\|\mathrm{div}\, \boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\|_{\Omega^e} \leq Ch^{k+1}\|\boldsymbol{v}\|_{k+1}\,, \tag{27}$$

$$\|\eta_p\| + h\|\nabla\eta_p\| \leq Ch^{l+1}\|p\|_{l+1}\,. \tag{28}$$

(iii) *The following inverse estimate holds*:

$$\|\mathrm{div}\, \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\|_{\Omega^e} \leq \bar{C}^e(h^e)^{-1}\|\boldsymbol{\varepsilon}(\boldsymbol{w}^h)\|_{\Omega^e} \quad \forall \boldsymbol{w}^h \in \mathcal{V}^h\,, \tag{29}$$

where $\bar{C}^e$ is a nondimensional constant which depends only upon element type.

(iv) *$0 < \alpha \leq \alpha^e \leq (\bar{C}^e)^{-2}$, $e = 1, 2, \ldots, n_{el}$, where α is a constant.*

(v) *$0 < r \leq h^e/h$, $e = 1, 2, \ldots, n_{\mathrm{el}}$, where r is a constant.*

Let

$$|||\boldsymbol{w}, q|||^2 = \mu\|\boldsymbol{\varepsilon}(\boldsymbol{w})\|^2 + \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e(h^e)^2}{4\mu}\|\nabla q\|^2_{\Omega^e}\,. \tag{30}$$

Clearly, $|||\cdot,\cdot|||$ defines a mesh-dependent norm on the product space $(H^1_0(\Omega))^{n_{\mathrm{sd}}} \times H^1(\Omega)/\mathbb{R}$. Then

$$|||\boldsymbol{e}_v, e_p|||^2 \leq 2\mu C_1 h^{2k}\|\boldsymbol{v}\|^2_{k+1} + (C_2/2\mu)h^{2(l+1)}\|p\|^2_{l+1}\,, \tag{31}$$

in which $\boldsymbol{e}_v = \boldsymbol{v}^h - \boldsymbol{v}$ and $e_p = p^h - p$ are the errors in the finite element solution and C_1 and C_2 are nondimensional constants independent of the exact solution and h.

REMARK 4.2. Restricting our attention to the Dirichlet problem and assuming μ is a constant simplify the proof, but are not essential for obtaining the error estimate.

Before proving Theorem 4.1. we shall establish two preliminary results.

LEMMA 4.3. (*orthogonality of the error*).

$$B_\alpha(w^h, q^h; e_v, e_p) = 0 \quad \forall w^h \in \mathcal{V}^h\,, \quad \forall q^h \in \mathcal{P}^h\,. \tag{32}$$

PROOF. This is immediate from (15)–(17).

LEMMA 4.4. (*stability*).

$$B_\alpha(w^h, q^h; w^h, q^h) \geq |||w^h, q^h|||^2 \quad \forall w^h \in \mathcal{V}^h, \quad \forall q^h \in \mathcal{P}^h\,. \tag{33}$$

PROOF. Direct calculation yields

$$\begin{aligned} B_\alpha(w^h, q^h; w^h, q^h) = 2\mu\|\boldsymbol{\varepsilon}(w^h)\|^2 &+ \sum_{e=1}^{n_{\rm el}} \frac{\alpha^e(h^e)^2}{2\mu} \|\nabla q^h\|^2_{\Omega^e} \\ &- \sum_{e=1}^{n_{\rm el}} \alpha^e(h^e)^2(\nabla q^h, \operatorname{div} \boldsymbol{\varepsilon}(w^h))_{\Omega^e}\,. \end{aligned} \tag{34}$$

The last term may be estimated as follows:

$$\begin{aligned} |\alpha^e(h^e)^2(\nabla q^h, \operatorname{div} \boldsymbol{\varepsilon}(w^h))_{\Omega^e}| &\leq \tfrac{1}{2}\alpha^e(h^e)^2\Big(\frac{1}{2\mu}\|\nabla q^h\|^2_{\Omega^e} + 2\mu\|\operatorname{div} \boldsymbol{\varepsilon}(w^h)\|^2_{\Omega^e}\Big) \\ &\leq \tfrac{1}{2}\alpha^e(h^e)^2\Big(\frac{1}{2\mu}\|\nabla q^h\|^2_{\Omega^e} + 2\mu(\bar{C}^e(h^e)^{-1})^2\|\boldsymbol{\varepsilon}(w^h)\|^2_{\Omega^e}\Big) \\ &\leq \tfrac{1}{2}\Big(\frac{\alpha^e(h^e)^2}{2\mu}\|\nabla q^h\|^2_{\Omega^e} + 2\mu\|\boldsymbol{\varepsilon}(w^h)\|^2_{\Omega^e}\Big)\,. \end{aligned} \tag{35}$$

We have used assumptions (iii) and (iv) in obtaining (35). Combining (34) and (35),

$$B_\alpha(w^h, q^h; w^h, q^h) \geq \mu\|\boldsymbol{\varepsilon}(w^h)\|^2 + \sum_{e=1}^{n_{\rm el}} \frac{\alpha^e(h^e)^2}{4\mu}\|\nabla q^h\|^2_{\Omega^e}\,. \tag{36}$$

REMARK 4.5. Lemma 4.4 justifies the terminology 'stability constants' for the α^e. It may also be seen that the selection of the power $(h^e)^2$ in the perturbation term is dictated by the stability condition.

PROOF OF THEOREM 4.1. Let $e_v^h = v^h - \tilde{v}^h$ and $e_p^h = p^h - \tilde{p}^h$. It is helpful to record two results which are used in the sequel:

$$(e_p^h, \operatorname{div} \boldsymbol{\eta}_v) = -(\nabla e_p^h, \boldsymbol{\eta}_v)\,, \tag{37}$$

$$\|\operatorname{div} \boldsymbol{e}_v^h\| \leqslant \sqrt{n_{\mathrm{sd}}}\,\|\boldsymbol{\varepsilon}(\boldsymbol{e}_v^h)\|\,. \tag{38}$$

The first follows from the fact that $\boldsymbol{\eta}_v(\boldsymbol{x}) = \boldsymbol{0}$ on Γ and the second is trivial.

Let β and γ denote nondimensional constants. Then

$$\begin{aligned}
|||\boldsymbol{e}_v^h, e_p^h|||^2 &\leqslant B_{\boldsymbol{\alpha}}(\boldsymbol{e}_v^h, e_p^h; \boldsymbol{e}_v^h, e_p^h) && \text{(by Lemma 4.4)}\\
&= B_{\boldsymbol{\alpha}}(\boldsymbol{e}_v^h, e_p^h; \boldsymbol{e}_v - \boldsymbol{\eta}_v, e_p - \eta_p) && \text{(by definition of } \boldsymbol{e}_v^h \text{ and } e_p^h\text{)}\\
&= -B_{\boldsymbol{\alpha}}(\boldsymbol{e}_v^h, e_p^h; \boldsymbol{\eta}_v, \eta_p) && \text{(by Lemma 4.3)}\\
&= -\Big\{2\mu(\boldsymbol{\varepsilon}(\boldsymbol{e}_v^h), \boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)) - (\operatorname{div} \boldsymbol{e}_v^h, \eta_p) + (e_p^h, \operatorname{div} \boldsymbol{\eta}_v)\\
&\qquad + \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e (h^e)^2}{2\mu} (\nabla e_p^h, \nabla \eta_p)_{\Omega^e}\\
&\qquad - \sum_{e=1}^{n_{\mathrm{el}}} \alpha^e (h^e)^2 (\nabla e_p^h, \operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{\eta}_v))_{\Omega^e}\Big\} && \text{(by definition of } B_{\boldsymbol{\alpha}}\text{)}\\
&= -\Big\{2\mu(\beta^{-1/2}\boldsymbol{\varepsilon}(\boldsymbol{e}_v^h), \beta^{1/2}\boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)) - \Big(\sqrt{2\mu}\,\beta^{-1/2} \operatorname{div} \boldsymbol{e}_v^h, \frac{\beta^{1/2}}{\sqrt{2\mu}}\,\eta_p\Big)\\
&\qquad - \sum_{e=1}^{n_{\mathrm{el}}} \Big(\sqrt{\frac{\alpha^e (h^e)^2}{2\mu}}\,\gamma^{-1/2} \nabla e_p^h, \sqrt{\Big(\frac{\alpha^e (h^e)^2}{2\mu}\Big)^{-1}}\,\gamma^{1/2} \boldsymbol{\eta}_v\Big)_{\Omega^e}\\
&\qquad + \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e (h^e)^2}{2\mu} (\gamma^{-1/2} \nabla e_p^h, \gamma^{1/2} \nabla \eta_p)_{\Omega^e}\\
&\qquad - \sum_{e=1}^{n_{\mathrm{el}}} \alpha^e (h^e)^2 \Big(\frac{\gamma^{-1/2}}{\sqrt{2\mu}}\, \nabla e_p^h, \sqrt{2\mu}\,\gamma^{1/2} \operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\Big)_{\Omega^e}\Big\} && \text{(by (37))}\\
&\leqslant \tfrac{1}{2}\Big\{2\mu\beta^{-1}\|\boldsymbol{\varepsilon}(\boldsymbol{e}_v^h)\|^2 + 2\mu\beta\|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\|^2 + 2\mu\beta^{-1}\|\operatorname{div} \boldsymbol{e}_v^h\|^2 + \frac{\beta}{2\mu}\|\eta_p\|^2\\
&\qquad + \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e (h^e)^2}{2\mu} \gamma^{-1} \|\nabla e_p^h\|_{\Omega^e}^2 + \sum_{e=1}^{n_{\mathrm{el}}} \Big(\frac{\alpha^e (h^e)^2}{2\mu}\Big)^{-1} \gamma \|\boldsymbol{\eta}_v\|_{\Omega^e}^2\\
&\qquad + \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e (h^e)^2}{2\mu} \gamma^{-1} \|\nabla e_p^h\|_{\Omega^e}^2 + \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e (h^e)^2}{2\mu} \gamma \|\nabla \eta_p\|_{\Omega^e}^2\\
&\qquad + \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e (h^e)^2}{2\mu} \gamma^{-1} \|\nabla e_p^h\|_{\Omega^e}^2 + \sum_{e=1}^{n_{\mathrm{el}}} 2\mu\alpha^e (h^e)^2 \gamma \|\operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\|_{\Omega^e}^2\Big\}\\
&\leqslant \tfrac{1}{2}\Big\{(n_{\mathrm{sd}} + 1) 2\mu\beta^{-1}\|\boldsymbol{\varepsilon}(\boldsymbol{e}_v^h)\|^2 + 3\gamma^{-1} \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e (h^e)^2}{2\mu} \|\nabla e_p^h\|_{\Omega^e}^2\Big\}
\end{aligned}$$

$$+\tfrac{1}{2}\left\{\gamma \sum_{e=1}^{n_{\mathrm{el}}} \left(\frac{\alpha^e (h^e)^2}{2\mu}\right)^{-1} \|\boldsymbol{\eta}_v\|^2_{\Omega^e} + 2\mu\beta\|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\|^2 \right.$$

$$+ \gamma \sum_{e=1}^{n_{\mathrm{el}}} 2\mu\alpha^e (h^e)^2 \|\operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\|^2_{\Omega^e} + \frac{\beta}{2\mu}\|\eta_p\|^2$$

$$\left. + \gamma \sum_{e=1}^{n_{\mathrm{el}}} \frac{\alpha^e (h^e)^2}{2\mu} \|\nabla \eta_p\|^2_{\Omega^e}\right\}. \tag{39}$$

Note that if we select $\beta = 2(n_{\mathrm{sd}}+1)$ and $\gamma = 6$, then the first term on the right-hand side becomes half the left-hand side. Consequently,

$$|||\boldsymbol{e}^h_v, e^h_p||| \le 2\mu\left\{6 \sum_{e=1}^{n_{\mathrm{el}}} \frac{1}{\alpha^e (h^e)^2} \|\boldsymbol{\eta}_v\|^2_{\Omega^e} + 2(n_{\mathrm{sd}}+1)\|\boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\|^2 \right.$$

$$\left. + 6 \sum_{e=1}^{n_{\mathrm{el}}} \alpha^e (h^e)^2 \|\operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{\eta}_v)\|^2_{\Omega^e}\right\}$$

$$+ \frac{1}{2\mu}\left\{2(n_{\mathrm{sd}}+1)\|\eta_p\|^2 + 6 \sum_{e=1}^{n_{\mathrm{el}}} \alpha^e (h^e)^2 \|\nabla \eta_p\|^2_{\Omega^e}\right\}$$

$$\le \mu C_1 h^{2k} \|\boldsymbol{v}\|^2_{k+1} + \frac{1}{4\mu} C_2 h^{2(l+1)} \|p\|^2_{l+1}, \tag{40}$$

where C_1 and C_2 are constants. We have used assumptions (ii), (iv), and (v) in obtaining (40).

Noting also that

$$|||\boldsymbol{\eta}_v, \eta_p||| \le \mu C_1 h^{2k} \|\boldsymbol{v}\|^2_{k+1} + \frac{1}{4\mu} C_2 h^{2(l+1)} \|p\|^2_{l+1}, \tag{41}$$

the theorem follows from (40) and from the triangle inequality in the form

$$|||\boldsymbol{e}_v, e_p||| \le |||\boldsymbol{e}^h_v, e^h_p||| + |||\boldsymbol{\eta}_v, \eta_p|||. \tag{42}$$

4.1. Remarks

(1) Our error estimate holds for all C^0 continuous velocity and pressure interpolations. Within the present framework, a velocity field of one order may be mixed with a pressure field of any other order, thus circumventing restrictions which pertain to the usual Galerkin formulation.

(2) If $k = l + 1$ (i.e., velocity one order higher than pressure), then the velocity gradient and pressure gradient converge at optimal rate. Thus for elements which already converge within the Galerkin framework such as the biquadratic velocity/bilinear pressure quadrilateral (i.e., 'Q2/Q1') and the quadratic velocity/linear pressure triangle (i.e., 'P2/P1'), the optimal rate of convergence is maintained by the present formulation.

(3) Perhaps the most potentially useful elements are those which employ the same interpolation on both velocity and pressure (i.e., $k = l$). From an implementational standpoint, elements of this type are no doubt the most convenient. In this case, the gradient of

velocity converges at optimal rate (i.e., k), whereas the gradient of pressure converges at a suboptimal rate (i.e., $k-1$). Due to the coupling of velocity and pressure errors, it is probably unreasonable to expect any better. For example, the elements Q2/Q2 and P2/P2 are convergent within the present formulation. These elements are notoriously unstable within the Galerkin formulation.

(4) The simplest elements within the present framework are Q1/Q1 and P1/P1. Unfortunately, when the velocity field is complete through only first-order polynomials, our error estimate only tells us that the gradient of pressure is bounded by a constant. (The velocity gradient, of course, still converges optimally.) We conjecture that, under reasonable hypotheses, one can show that the pressure converges at rate 1 in the $L_2(\Omega)$-norm. There is already one element for which this is known to be the case, namely, the P1/P1-triangle. For this element our formulation reduces to the 'modified discrete-incompressibility approach' of Brezzi and Pitkäranta [3] modulo additions to $L_\alpha(\cdot,\cdot)$.

(5) The present formulation is applicable to any number of space dimensions. This is particularly noteworthy because three-dimensional elements which satisfy the Babuška–Brezzi condition tend to be extremely complicated. The present formulation opens the way to successful use of the simplest interpolations. It would seem that the equal-order trilinear brick and linear tetrahedron would be particularly attractive in three dimensions.

(6) The added stability of our method should translate into more rapid convergence than the Galerkin formulation when using iterative solvers.

(7) There are two main places in the error analysis where the assumption of C^0 continuity of pressure is used. The first is in the definition of the *norm*, (30), on the product space $(H_0^1(\Omega))^{n_{sd}} \times (H^1(\Omega)/\mathbb{R})$. If discontinuous pressures were allowed we would need to extend (30) to the larger product space $(H_0^1(\Omega))^{n_{sd}} \times H^1(\bigcup_{e=1}^{n_{el}} \Omega^e)/\mathbb{R}$. In this case, (30) would constitute a *seminorm* rather than a norm. The second place is in (37). Without continuity of pressure, we would have to account for jump terms in e_p^h across Γ_{int}. Nevertheless, in a formal sense, the present formulation may be applied to discontinuous pressure fields and, clearly, in most cases some additional stability is gained. However, it is also clear that additional conditions will be needed to establish convergence. In particular cases it is apparent that the present formulation is identical to the Galerkin formulation, namely, for constant pressure elements, and therefore no advantage with respect to the Galerkin formulation can be claimed. Whether or not it is necessary to satisfy the Babuška–Brezzi condition in all cases of discontinuous pressure is an open question.

(8) Note that the upper bound on the α^e is provided by the constant in the inverse estimate (see assumptions (iii) and (iv)).

5. Matrix formulation

The matrix problem for the present formulation may be written as

$$\begin{bmatrix} \boldsymbol{K} & \boldsymbol{G} \\ \boldsymbol{L}-\boldsymbol{G}^{t} & \boldsymbol{M} \end{bmatrix} \begin{Bmatrix} \boldsymbol{d} \\ \boldsymbol{p} \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{F} \\ \boldsymbol{H} \end{Bmatrix}. \tag{43}$$

In (43) $\boldsymbol{d}$ is a vector of unknown nodal velocity degrees of freedom and $\boldsymbol{p}$ is the vector of nodal pressure degrees of freedom; $\boldsymbol{K}$ is the viscosity matrix; $\boldsymbol{G}$ is the gradient matrix; $\boldsymbol{G}^{t}$ is the divergence matrix; $\boldsymbol{M}$ is the 'stabilization matrix' (i.e., negative discrete Laplacian); and $\boldsymbol{L}$ is the 'consistency matrix'.

In the Galerkin formulation $\boldsymbol{L}$ and $\boldsymbol{M}$ are absent. Assuming appropriately specified boundary conditions[1], the coefficient matrix is positive-definite for velocity and pressure interpolations of class C^0. This follows from the stability estimate of the previous section, which also ensures that $\boldsymbol{L}$ is 'small' and may be relegated to the right-hand side for purposes of iteration. The matrices $\boldsymbol{K}$ and $\boldsymbol{M}$ are symmetric and positive-definite. If $\boldsymbol{L}$ is omitted, the coefficient matrix may be symmetrized by multiplying the p-equation by -1. If the velocity interpolation is linear, such as for the P1/P1 element, $\boldsymbol{L}=\boldsymbol{0}$. We conjecture that for the Q1/Q1 element $\boldsymbol{L}$ can be neglected without affecting convergence rate.

The definitions of the arrays are given as follows:

$$\boldsymbol{K}=\mathop{\mathbf{A}}_{e=1}^{n_{\mathrm{el}}}(\boldsymbol{k}^e)\,,\qquad \boldsymbol{G}=\mathop{\mathbf{A}}_{e=1}^{n_{\mathrm{el}}}(\boldsymbol{g}^e)\,,\qquad \boldsymbol{L}=\mathop{\mathbf{A}}_{e=1}^{n_{\mathrm{el}}}(\boldsymbol{l}^e)\,,$$
$$\boldsymbol{M}=\mathop{\mathbf{A}}_{e=1}^{n_{\mathrm{el}}}(\boldsymbol{m}^e)\,,\qquad \boldsymbol{F}=\mathop{\mathbf{A}}_{e=1}^{n_{\mathrm{el}}}(\boldsymbol{f}^e)\,,\qquad \boldsymbol{H}=\mathop{\mathbf{A}}_{e=1}^{n_{\mathrm{el}}}(\boldsymbol{h}^e)\,, \tag{44}$$

where $\mathbf{A}_{e=1}^{n_{\mathrm{el}}}$ denotes the finite element assembly operator which adds elemental contributions to the appropriate locations in the global arrays, and

$$\boldsymbol{k}^e=[k^e_{pq}]\,,\qquad \boldsymbol{g}^e=[g^e_{p\tilde{b}}]\,,\qquad \boldsymbol{l}^e=[l^e_{\tilde{a}q}]\,,$$
$$\boldsymbol{m}^e=[m^e_{\tilde{a}\tilde{b}}]\,,\qquad \boldsymbol{f}^e=\{f^e_p\}\,,\qquad \boldsymbol{h}^e=\{h^e_{\tilde{a}}\}\,,$$

$$k^e_{pq}=\boldsymbol{e}_i^{\mathrm{t}}\int_{\Omega^e}\boldsymbol{B}_a^{e^{\mathrm{t}}}\boldsymbol{D}\boldsymbol{B}_b^e\,\mathrm{d}\Omega\,\boldsymbol{e}_j\,,\qquad g^e_{p\tilde{b}}=-\int_{\Omega^e}N^e_{a,i}\tilde{N}^e_{\tilde{b}}\,\mathrm{d}\Omega\,,$$

$$l^e_{\tilde{a}q}=-\tfrac{1}{2}\alpha^e(h^e)^2\int_{\Omega^e}\sum_{k=1}^{n_{\mathrm{sd}}}(\tilde{N}^e_{\tilde{a},j}N^e_{b,kk}+\tilde{N}^e_{\tilde{a},k}N^e_{b,jk})\,\mathrm{d}\Omega\,,$$

$$m^e_{\tilde{a}\tilde{b}}=\frac{\alpha^e(h^e)^2}{2\mu}\int_{\Omega^e}\sum_{k=1}^{n_{\mathrm{sd}}}(\tilde{N}^e_{\tilde{a},k}\tilde{N}^e_{\tilde{b},k})\,\mathrm{d}\Omega\,, \tag{45^2}$$

$$f^e_p=\int_{\Omega^e}N^e_a f_i\,\mathrm{d}\Omega+\int_{\Gamma^e_h}N^e_a h_i\,\mathrm{d}\Gamma-\sum_{q=1}^{n_{ee}}k^e_{pq}g^e_q\,,\qquad \Gamma^e_h=\Gamma_h\cap\Gamma^e\,,$$

$$h^e_{\tilde{a}}=\frac{\alpha^e(h^e)^2}{2\mu}\int_{\Omega^e}\sum_{k=1}^{n_{\mathrm{sd}}}\tilde{N}^e_{\tilde{a},k}f_k\,\mathrm{d}\Omega+\sum_{p=1}^{n_{ee}}g^e_{p\tilde{a}}g^e_p-\sum_{q=1}^{n_{ee}}l^e_{\tilde{a}q}g^e_q\,,$$

$$g^e_q=\begin{cases}g_j(\boldsymbol{x}^e_b)\,, & \text{if } \boldsymbol{x}^e_b\in\Gamma_g\,,\\ 0\,, & \text{otherwise}\,.\end{cases}$$

[1]One nodal pressure must also be specified in the case of the Dirichlet problem.
[2]The viscosity, μ, is assumed constant in obtaining these expressions.

$$\begin{gathered}1 \leq p, q \leq n_{ee} = n_{en} \cdot n_{sd}, \qquad p = n_{sd}(a-1)+i, \qquad q = n_{sd}(b-1)+j, \\ 1 \leq i, j \leq n_{sd}, \qquad 1 \leq a, b \leq n_{en}, \qquad 1 \leq \tilde{a}, \tilde{b} \leq \tilde{n}_{en},\end{gathered} \tag{46}$$

in which

e_i = the ith Cartesian unit basis vector of $\mathbb{R}^{n_{sd}}$;
n_{en} = the number of velocity-element nodes;
$\tilde{n}_{en}$ = the number of pressure-element nodes;
n_{ee} = the number of velocity-element degrees of freedom;
p, q = the local velocity-element equation numbers;
a, b = the local velocity-element node numbers;
$\tilde{a}, \tilde{b}$ = the local pressure-element equation/node numbers;
N_a^e = shape function associated with velocity-element node a; and
$\tilde{N}_{\tilde{a}}^e$ = shape function associated with pressure-element node $\tilde{a}$.

In two- and three-dimensions, the arrays $\boldsymbol{B}_a^e$ and $\boldsymbol{D}$ are given as follows:

$$\boldsymbol{B}_a^e = \begin{pmatrix} N_{a,1}^e & 0 \\ 0 & N_{a,2}^e \\ N_{a,2}^e & N_{a,1}^e \end{pmatrix}, \qquad \boldsymbol{D} = \mu \operatorname{diag}(2,2,1) \quad \text{for } n_{sd} = 2, \tag{47}$$

$$\boldsymbol{B}_a^e = \begin{pmatrix} N_{a,1}^e & 0 & 0 \\ 0 & N_{a,2}^e & 0 \\ 0 & 0 & N_{a,3}^e \\ 0 & N_{a,3}^e & N_{a,2}^e \\ N_{a,3}^e & 0 & N_{a,1}^e \\ N_{a,2}^e & N_{a,1}^e & 0 \end{pmatrix}, \qquad \boldsymbol{D} = \mu \operatorname{diag}(2,2,2,1,1,1) \quad \text{for } n_{sd} = 3. \tag{48}$$

REMARK 5.1. Note that the $\boldsymbol{L}$ matrix entails *second derivatives* of the shape functions.

6. Numerical results

We have programmed the Q1/Q1 and Q2/Q2 elements. For the Q1/Q1 element we have set $\boldsymbol{L} = \boldsymbol{0}$. We performed a convergence study of Couette flow with Q1/Q1 elements, setting $\alpha = 1$, and found asymptotically

$$\|\boldsymbol{e}_v\| = \mathrm{O}(h^2), \qquad \|e_p\| = \mathrm{O}(h^{1.5}), \qquad \|\nabla e_p\| = \mathrm{O}(h^{0.5}).$$

Couette flow involves C^∞ velocity and pressure fields, consequently, it is useful for a convergence rate study. The velocity gradient converges in L_2 at optimal rate. The result for the pressure gradient is one half order higher than anticipated from the error estimate.

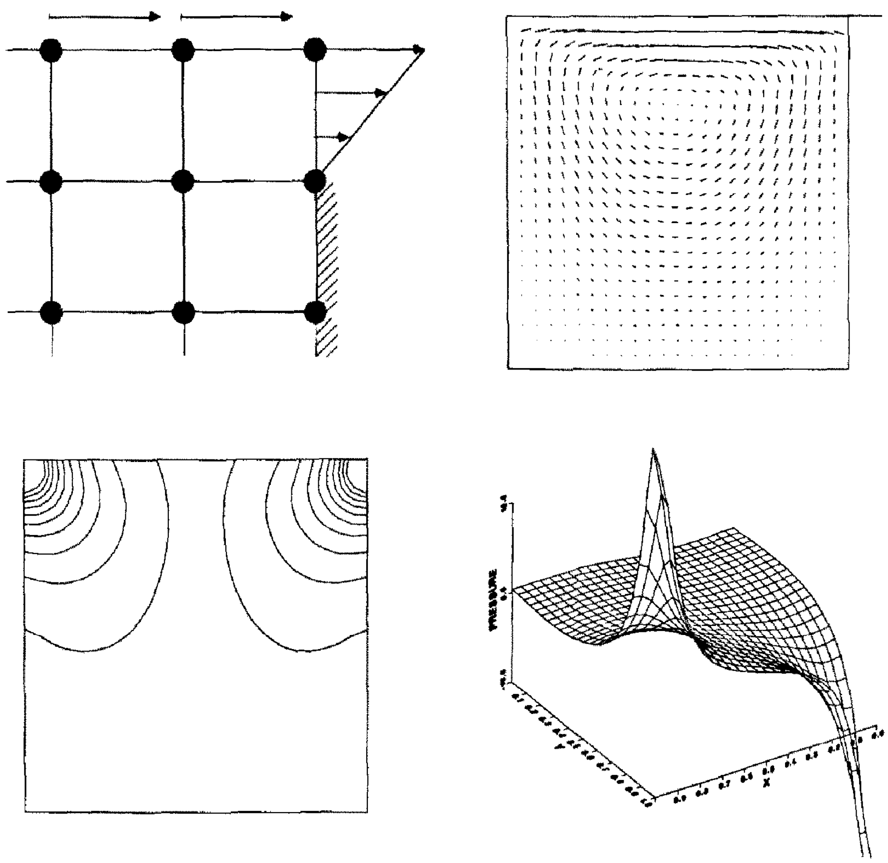

Fig. 1. Driven cavity flow with 'leaky lid' boundary condition; 24×24 mesh of Q1/Q1 elements; $\alpha = 0.5$; (a) corner boundary condition; (b) velocity vectors; (c) pressure contours; (d) pressure elevation.

It is well known that even unstable elements (i.e., those that fail the Babuška–Brezzi condition) often perform very well when the exact solution is sufficiently smooth. As a severe test of the present formulation, we have investigated the driven cavity with various discrete treatments of the corner velocity singularity (e.g., 'leaky lid', 'ramp', etc.). We have experimented with the following values of α: 0.001, 0.01, 0.1, 0.5, 1.0. For the Q1/Q1 element we found that for values of $\alpha \leq 0.01$, the pressure solution was highly oscillatory, whereas for the values of $\alpha \geq 0.1$, no oscillations were apparent. These results pertained to all treatments of the corner velocity boundary condition. The velocities appeared satisfactory for all values of α considered. Qualitatively similar results were obtained for Q2/Q2; the pressure was oscillatory for $\alpha = 0.001$, whereas for all values of $\alpha \geq 0.01$ no oscillations were discernible.

Some sample results of the cavity computations are presented in Figs. 1–3.

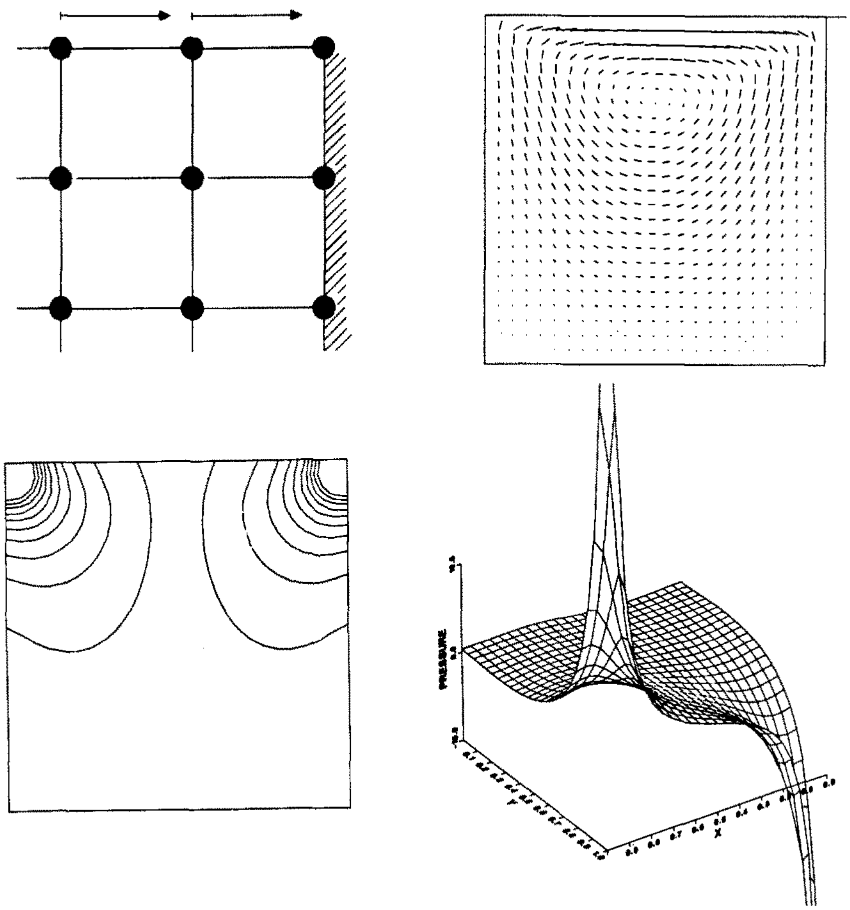

Fig. 2. Driven cavity flow with 'ramp over one element' boundary condition; 24×24 mesh of Q1/Q1 elements; $\alpha = 0.5$; (a) corner boundary condition; (b) velocity vectors; (c) pressure contours; (d) pressure elevation.

7. Concluding remarks

In this paper we have proposed a new finite element formulation of the Stokes problem. The formulation has been shown to be convergent for rather general C^0 combinations of velocity and pressure and thus allows us to circumvent restrictions of the Babuška–Brezzi condition. In particular, equal-order interpolations, which are convenient from a computational standpoint, are accommodated. In this regard, some successful numerical experiments were performed with the Q1/Q1 and Q2/Q2 elements.

Several issues warrant further investigation: The theoretical L_2 convergence rates of the velocity and pressure need to be established.[3] The best value of the stability constant α needs to be determined for elements of interest. It is of practical importance to know the consequences of ignoring the consistency matrix, $\boldsymbol{L}$, for various elements. It would also be worthwhile to know what, if anything, is gained by the present formulation when discontinuous pressure interpolations are employed. Finally, considerable practical experience needs to be gained with the new formulation. In any event, it appears that a new door has been opened to the subject of Stokes flow and related phenomena.

[3]Note added in proof: Franco Brezzi has proved the anticipated one-order improvement in L_2 convergence rates (private communication).

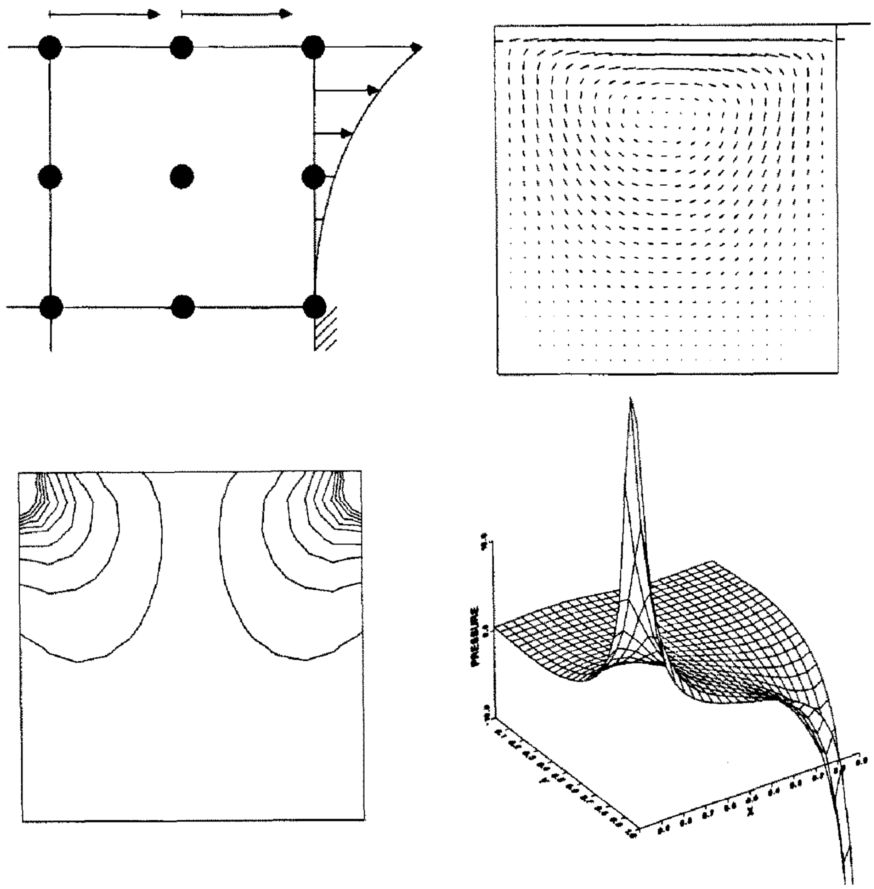

Fig. 3. Driven cavity flow with 'parabolic leaky lid' boundary condition; 12×12 mesh of Q2/Q2 elements; $\alpha = 0.01$; (a) corner boundary condition; (b) velocity vectors; (c) pressure contours; (d) pressure elevation.

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