Womersley Solution for Blood Flow in an Rigid Tube

Lu Qingshuang Report in October 2022

To study the fluid dynamics of the pulsatile flow in the tube, the governing equations generally requires the Navier-Stokes equations and the continuity equation in the polar coordinate system. Assuming that the tube is a rigid axisymmetric circular straight tube with no external forces to cause flow rotation, the angular component of velocity and all derivatives in the angular direction are zero. That is,

$$w = 0, \quad \frac{\partial}{\partial \theta} = 0.$$

If the flow is incompressible, then the equations can be written as,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r}\right) + \frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right),$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r}\right) + \frac{\partial p}{\partial r} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right),$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0.$$

Considering the flow field is fully developed, there is no velocity change along the longitudinal axis, thus,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0.$$

The continuity equation is simplified as

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0,$$

which can be solved to give $vr = constant$, since r is a variation, we have $v \equiv 0$.

The momentum equations is rewritten as

$$\rho\frac{\partial u}{\partial t} + \frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right),$$

$$\frac{\partial p}{\partial r} = 0.$$

where

$$u = u(r, t), \quad p = p(x, t).$$

1. Steady-State: Poiseuille Flow

At steady-state, the equation becomes

$$\frac{dp_s}{dx} = \mu\left(\frac{d^2 u_s}{dr^2} + \frac{1}{r}\frac{du_s}{dr}\right), \quad u_s = u(r), \quad p_s = p(x).$$

Denote that

$$k_s = \frac{dp_s}{dx} = \frac{p(L) - p(0)}{L},$$

then

$$p_s = p(0) + k_s x,$$

$$u_s = \frac{k_s}{4\mu} r^2 + A \ln r + B.$$

Given the boundary conditions

$$u(R) = 0, \quad |u(0)| < \infty,$$

we get

$$A = 0, \quad B =- \frac{k_s}{4\mu} R^2.$$

Conclusionally,

$$u_s = u(r) = \frac{k_s}{4\mu}(r^2 - R^2).$$

This solution is called Poiseuille flow, which is the classical solution for the steady flow in a tube.

2. Oscillatory Flow

With using the feature that the momentum equation is linear in velocity and pressure field, the equation can be separated to the steady and oscillatory parts of the flow, and these two parts are independent of each other.

The steady part of the flow is solved in the last section, and the oscillatory parts will be treated as an equation dependent on both time and spatial position.

The momentum equation of the oscillatory part is shown as follows,

$$\rho \frac{\partial u_\varphi}{\partial t} + \frac{\partial p_\varphi}{\partial x} = \mu \left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r} \frac{\partial u_\varphi}{\partial r} \right),$$

where u_φ and p_φ is the oscillatory part of the longitudinal velocity and pressure.

Also the pressure gradient has the oscillatory part k_φ and the relation is

$$k_\varphi(t) = \frac{\partial p_\varphi}{\partial x},$$

Then the equation become

$$\mu \left(\frac{\partial^2 u_\varphi}{\partial r^2} + \frac{1}{r} \frac{\partial u_\varphi}{\partial r} \right) - \rho \frac{\partial u_\varphi}{\partial t} = k_\varphi(t).$$

Using the method of separation of variables and the exponential form of Fourier series, we design the formulation of the solution as

$$k_\varphi(t) = \sum_{n=1}^{\infty} k_n e^{i\omega n t},$$

$$u_\varphi(r,t) = \sum_{n=1}^{\infty} U_n(r) e^{i\omega n t}.$$

Substituting into the equation leads to a set of ordinary differential equations about

$U_n(r)$ only,

$$\frac{d^2U_n(r)}{dr^2}+\frac{1}{r}\frac{dU_n(r)}{dr}-\frac{i\rho wn}{\mu}U_n(r)=\frac{k_n}{\mu},\quad for\, n=1,2,\ldots.$$

Here we introduce a non-dimensional number called Womersley number,

$$\Omega_n=R\sqrt{\frac{\rho wn}{\mu}},\quad for\, n=1,2,\ldots.$$

We rewrite the ordinary differential equation as a form of Bessel equation such that it has a known general solution.

The Bessel equation is about a new variable ζ, witch is a complex variable related to the radial coordinate r. For different n,

$$\zeta_n(r)=\Lambda_n\frac{r}{R},$$

where

$$\Lambda_n(r)=\Omega_n\frac{i-1}{\sqrt{2}}.$$

Then the equation is expressed as

$$\frac{d^2U_n(\zeta_n)}{d{\zeta_n}^2}+\frac{1}{\zeta_n}\frac{dU_n(\zeta_n)}{dr}+U_n(\zeta_n)=\frac{ik_nR^2}{\mu\Omega_n^2},\quad for\, n=1,2,\ldots.$$

And the solution of the equation is

$$U_n(\zeta_n)=\frac{ik_nR^2}{\mu\Omega_n^2}+A_nJ_0(\zeta_n)+B_nY_0(\zeta_n).$$

Given the boundary conditions

$$for\, r=R,\zeta_n=\Lambda_n,\quad U_n=0,$$
$$for\, r=0,\zeta_n=0,\quad |U_n|<\infty,$$

we can get

$$B_n=0,\quad A_n=-\frac{ik_nR^2}{\mu\Omega_n^2J_0(\Lambda_n)}.$$

Thus, the solution is

$$U_n(\zeta_n)=\frac{ik_nR^2}{\mu\Omega_n^2}\left(1-\frac{J_0(\zeta_n)}{J_0(\Lambda_n)}\right).$$

Conclusionally, the complete expression of the flow velocity is

$$u(r,t)=u_s(r)+u_\varphi(r,t)=\frac{k_s}{4\mu}(r^2-R^2)+\sum_{n=1}^{\infty}\frac{ik_nR^2}{\mu\Omega_n^2}\left(1-\frac{J_0(\zeta_n)}{J_0(\Lambda_n)}\right)e^{i\omega nt}.$$

This is a classical analytical solution for oscillatory flow in a rigid tube, usually called the Womersley Solution, witch is widely used in the study of the blood flow problems.

Reference:

[1]Zamir, M. (2000). The Physics of Pulsatile Flow. Biological Physics Series.

[2]Lan, I.S., Liu, J., Yang, W., & Marsden, A.L. (2022). A reduced unified continuum formulation for vascular fluid–structure interaction. Computer Methods in Applied Mechanics and Engineering.

[3]Figueroa, C.A., Vignon-Clementel, I.E., Jansen, K.E., Hughes, T.J., & Taylor, C.A. (2006). A coupled momentum method for modeling blood flow in three-dimensional deformable arteries. Computer Methods in Applied Mechanics and Engineering, 195, 5685-5706.