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A NEW FINITE ELEMENT FORMULATION FOR COMPUTATIONAL FLUID DYNAMICS: VII. THE STOKES PROBLEM WITH VARIOUS WELL-POSED BOUNDARY CONDITIONS: SYMMETRIC FORMULATIONS THAT CONVERGE FOR ALL VELOCITY/PRESSURE SPACES*

Thomas J.R. HUGHES and Leopoldo P. FRANCA**
Institute for Computer Methods in Applied Mechanics and Engineering, Division of Applied Mechanics, Stanford University, Stanford, CA 94305, U.S.A.

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Symmetric finite element formulations are proposed for the primitive-variables form of the Stokes equations and shown to be convergent for any combination of pressure and velocity interpolations. Various boundary conditions, such as pressure, are accommodated.

1. Introduction

In [11], (hereafter referred to as Part V) we developed a finite element method for the Stokes problem in primitive variables which was proved convergent for any combination of continuous interpolations. This is an advantage compared with the classical Galerkin formulation which requires satisfaction of the Babuška–Brezzi condition (see [2, 3]). The Babuška–Brezzi condition precludes the use of many seemingly natural combinations of velocity and pressure interpolations.

In this paper we continue our investigation of the Stokes problem and generalize and improve upon the formulation presented in Part V. The following shortcomings of Part V were noted: For all but the simplest elements, the matrix problem was nonsymmetric; and the formulation apparently did not improve upon the stability of the classical Galerkin formulation when the pressure was assumed discontinuous. In the approach proposed herein symmetry is attained for all elements, and convergence is proved for all velocity and pressure interpolations. Specifically, a pressure interpolation of arbitrary order, continuous or discontinuous, may be combined with a velocity interpolation of any other order, and the resulting method is stable and convergent.

We apply our approach to two different boundary-value problems for the Stokes equations. The first problem considers boundary conditions recently proposed and studied by Pironneau

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[16]. The variational formulation of this problem accommodates pressure as a weakly (or strongly) enforced boundary condition, and seems particularly well suited to fluid mechanics. Velocity and vorticity boundary conditions are also permitted. The second problem is the traditional one involving velocity and traction boundary conditions. It is identical to the standard displacement-pressure formulation of classical isotropic incompressible elasticity theory (due to Herrmann [9] and derived from the Hellinger–Reissner principle; see [8, 17]), although it has been used extensively for fluid mechanics as well.

An outline of the paper follows. In Section 2 we present the Stokes boundary value problem for Pironneau's boundary conditions. In Section 3 we present the finite element formulation and in Section 4 we derive error estimates. In Section 5 we present the traditional Stokes boundary value problem, its finite element formulation and error estimates. In Section 6 we draw conclusions.

The symbol □ is used to denote the end of a proof.

2. Stokes equations with velocity and pressure boundary conditions

Let Ω be an open, bounded region of $\mathbb{R}^d$, where $d = 2$ or 3, with piecewise smooth boundary Γ. Let $\{\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4\}$ be a partition of Γ. The unit outward normal vector to Γ is denoted by $\boldsymbol{n}$.

Let $\boldsymbol{w}$ be a vector field on Ω. The *vector Laplacian* is defined by (see [12, pp. 88, 116, 118]):

$$\Delta \boldsymbol{w} = \operatorname{grad} \operatorname{div} \boldsymbol{w} - \operatorname{curl} \operatorname{curl} \boldsymbol{w} . \tag{1}$$

The equations of Stokes flow are,

$$\mu \, \Delta \boldsymbol{u} - \operatorname{grad} p + \boldsymbol{f} = \boldsymbol{0} , \tag{2}$$

$$\operatorname{div} \boldsymbol{u} = 0 , \tag{3}$$

where μ is the viscosity (assumed constant), $\boldsymbol{u}$ is the velocity, p is the pressure, and $\boldsymbol{f}$ is the body force.

Vector fields $\boldsymbol{w}$ defined on Γ may be written as,

$$\boldsymbol{w} = \boldsymbol{w}_n + \boldsymbol{w}_s , \tag{4}$$

where

$$\boldsymbol{w}_n = w_n \boldsymbol{n} = (\boldsymbol{n} \cdot \boldsymbol{w})\boldsymbol{n} \quad \text{(normal component)} , \tag{5}$$

$$\boldsymbol{w}_s = \boldsymbol{n} \times (\boldsymbol{w} \times \boldsymbol{n}) \quad \text{(tangential component)} . \tag{6}$$

We consider the following boundary conditions:

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_1 , \tag{7}$$

$$\left.\begin{array}{l} \boldsymbol{n}\cdot\boldsymbol{u}=g_n \\ \boldsymbol{n}\times((\operatorname{curl}\boldsymbol{u})\times\boldsymbol{n})=\boldsymbol{\omega}_s \end{array}\right\} \quad \text{on } \Gamma_2\,, \tag{8}$$

$$\left.\begin{array}{l} p=\psi \\ \boldsymbol{n}\times(\boldsymbol{u}\times\boldsymbol{n})=\boldsymbol{g}_s \end{array}\right\} \quad \text{on } \Gamma_3\,, \tag{9}$$

$$\left.\begin{array}{l} p=\psi \\ \boldsymbol{n}\times((\operatorname{curl}\boldsymbol{u})\times\boldsymbol{n})=\boldsymbol{\omega}_s \end{array}\right\} \quad \text{on } \Gamma_4\,. \tag{10}$$

By virtue of the identity,

$$\begin{aligned} \boldsymbol{n}\times\boldsymbol{w}_s &= \boldsymbol{n}\times(\boldsymbol{n}\times(\boldsymbol{w}\times\boldsymbol{n})) \\ &= \boldsymbol{n}\times(\boldsymbol{w}-w_n\boldsymbol{n}) \\ &= \boldsymbol{n}\times\boldsymbol{w}\,, \end{aligned} \tag{11}$$

the tangential velocity and vorticity boundary conditions may be expressed in the following alternative forms, respectively:

$$\boldsymbol{n}\times\boldsymbol{u}=\boldsymbol{n}\times\boldsymbol{g}_s\,, \tag{12}$$

$$\boldsymbol{n}\times\operatorname{curl}\boldsymbol{u}=\boldsymbol{n}\times\boldsymbol{\omega}_s\,. \tag{13}$$

Physical and mathematical justification for the boundary conditions on $\Gamma_2\cup\Gamma_3$ is presented in [16].

The object of the Stokes problem is to find $\boldsymbol{u}$ and p which satisfy (2), (3), and (7)–(10). In subsequent sections it is convenient to employ the notations,

$$\Gamma_\omega=\Gamma_2\cup\Gamma_4\,, \tag{14}$$

$$\Gamma_\psi=\Gamma_3\cup\Gamma_4\,. \tag{15}$$

Remark

The well-posedness of the boundary value problem has only been rigorously established in specific instances. For example, Pironneau [16] presents the following result: If Ω is bounded and simply-connected, with piecewise C^2 boundary, if $\Gamma-\Gamma_\omega$ is not empty, if Γ_3 is simply-connected and C^2, and if Γ_4 is empty, then for sufficiently regular given data, there exists a unique solution $\{\boldsymbol{u}, p\}\in H^1(\Omega)^3\times L^2(\Omega)$. (If Γ_3 is empty, the pressure is unique up to a constant.)

3. Finite element formulation

Consider an element partition of Ω. Let Ω^e be the interior of the eth element, let Γ^e be its boundary, and

$$\tilde{\Omega} = \bigcup_e \Omega^e \qquad \text{(element interiors)}\,, \tag{16}$$

$$\tilde{\Gamma} = \bigcup_e \Gamma^e - \Gamma \quad \text{(element interfaces)}\,. \tag{17}$$

Let $\mathcal{V}^h$ be the set of velocity weighting functions, let $\mathcal{S}^h$ be the set of velocity trial solutions, and let $\mathcal{P}^h$ be the set of pressure weighting functions and trial solutions. $\mathcal{V}^h$ and $\mathcal{S}^h$ consist of typical C^0 finite element interpolations. In addition,

$$\boldsymbol{w}^h = \boldsymbol{0} \quad \text{on } \Gamma_1\,, \qquad \boldsymbol{n}\cdot\boldsymbol{w}^h = 0 \quad \text{on } \Gamma_2\,, \qquad \boldsymbol{n}\times\boldsymbol{w}^h = \boldsymbol{0} \qquad \text{on } \Gamma_3 \quad \forall \boldsymbol{w}^h \in \mathcal{V}^h\,, \tag{18}$$

$$\boldsymbol{u}^h = \boldsymbol{g} \quad \text{on } \Gamma_1\,, \qquad \boldsymbol{n}\cdot\boldsymbol{u}^h = g_n \quad \text{on } \Gamma_2\,, \qquad \boldsymbol{n}\times\boldsymbol{u}^h = \boldsymbol{n}\times\boldsymbol{g}_s \quad \text{on } \Gamma_3 \quad \forall \boldsymbol{u}^h \in \mathcal{S}^h\,. \tag{19}$$

Functions in $\mathcal{P}^h$ may be continuous or discontinuous across element interfaces.

The finite element problem is to find $\boldsymbol{U}^h = \{\boldsymbol{u}^h, p^h\} \in \mathcal{S}^h \times \mathcal{P}^h$, such that for all $\boldsymbol{W}^h = \{\boldsymbol{w}^h, q^h\} \in \mathcal{V}^h \times \mathcal{P}^h$,

$$B(\boldsymbol{W}^h, \boldsymbol{U}^h) = L(\boldsymbol{W}^h)\,, \tag{20}$$

where

$$\begin{aligned} B(\boldsymbol{W}^h, \boldsymbol{U}^h) = {} & (\operatorname{curl} \boldsymbol{w}^h, \mu \operatorname{curl} \boldsymbol{u}^h)_\Omega + (\operatorname{div} \boldsymbol{w}^h, \mu \operatorname{div} \boldsymbol{u}^h)_\Omega - (\operatorname{div} \boldsymbol{w}^h, p^h)_\Omega \\ & - (q^h, \operatorname{div} \boldsymbol{u}^h)_\Omega - \left(\frac{\alpha h^2}{2\mu}(\mu\, \Delta \boldsymbol{w}^h - \operatorname{grad} q^h),\, \mu\, \Delta \boldsymbol{u}^h - \operatorname{grad} p^h\right)_{\tilde{\Omega}} \\ & - \left(\frac{\beta h}{2\mu} [\![q^h]\!], [\![p^h]\!]\right)_{\tilde{\Gamma}} - \left(\frac{\beta h}{2\mu} q^h, p^h\right)_{\Gamma_\psi}\,, \end{aligned} \tag{21}$$

$$\begin{aligned} L(\boldsymbol{W}^h) = {} & \left(\boldsymbol{w}^h + \frac{\alpha h^2}{2\mu}(\mu\, \Delta \boldsymbol{w}^h - \operatorname{grad} q^h),\, \boldsymbol{f}\right)_{\tilde{\Omega}} - (\boldsymbol{w}^h, \mu \boldsymbol{n} \times \boldsymbol{\omega}_s)_{\Gamma_\omega} \\ & - \left(\boldsymbol{n}\cdot\boldsymbol{w}^h + \frac{\beta h}{2\mu} q^h, \psi\right)_{\Gamma_\psi}\,, \end{aligned} \tag{22}$$

in which $(\cdot\,,\cdot)_D$ denotes the $L^2(D)$ inner product, h is the mesh parameter, $[\![\cdot]\!]$ is the jump operator, and α and β are nondimensional stability constants. We assume h is defined locally.

The formal consistency of the method is apparent from the Euler–Lagrange form of (20):

$$\begin{aligned} 0 = {} & -\left(\boldsymbol{w}^h + \frac{\alpha h^2}{2\mu}(\mu\, \Delta \boldsymbol{w}^h - \operatorname{grad} q^h),\, \mu\, \Delta \boldsymbol{u}^h - \operatorname{grad} p^h + \boldsymbol{f}\right)_{\tilde{\Omega}} - (q^h, \operatorname{div} \boldsymbol{u}^h)_\Omega \\ & + (\boldsymbol{n}\cdot\boldsymbol{w}^h, \mu \operatorname{div} \boldsymbol{u}^h)_{\Gamma_\psi} - (\boldsymbol{w}^h, \mu(\boldsymbol{n}\times\operatorname{curl} \boldsymbol{u}^h - \boldsymbol{n}\times\boldsymbol{w}_s))_{\Gamma_\omega} + (\boldsymbol{n}\cdot\boldsymbol{w}^h, \mu[\![\operatorname{div} \boldsymbol{u}^h]\!])_{\tilde{\Gamma}} \\ & - (\boldsymbol{w}^h, \mu \boldsymbol{n} \times [\![\operatorname{curl} \boldsymbol{u}^h]\!])_{\tilde{\Gamma}} \\ & - \left(\boldsymbol{n}\cdot\boldsymbol{w}^h + \frac{\beta h}{2\mu} q^h, p^h - \psi\right)_{\Gamma_\psi} - \left(\boldsymbol{n}\cdot\boldsymbol{w}^h + \frac{\beta h}{2\mu} [\![q^h]\!], [\![p^h]\!]\right)_{\tilde{\Gamma}}\,. \end{aligned} \tag{23}$$

In obtaining (23) we have employed the following integration-by-parts formulas:

$$
\begin{aligned}
&(\operatorname{curl} \boldsymbol{w}^h, \operatorname{curl} \boldsymbol{u}^h)_\Omega + (\operatorname{div} \boldsymbol{w}^h, \operatorname{div} \boldsymbol{u}^h)_\Omega = \\
&\quad -(\boldsymbol{w}^h, \Delta \boldsymbol{u}^h)_{\tilde{\Omega}} + (\boldsymbol{n} \cdot \boldsymbol{w}^h, \operatorname{div} \boldsymbol{u}^h)_{\Gamma_\psi} - (\boldsymbol{w}^h, \boldsymbol{n} \times \operatorname{curl} \boldsymbol{u}^h)_{\Gamma_\omega} + (\boldsymbol{n} \cdot \boldsymbol{w}^h, [\![\operatorname{div} \boldsymbol{u}^h]\!])_{\tilde{\Gamma}} \\
&\quad -(\boldsymbol{w}^h, \boldsymbol{n} \times [\![\operatorname{curl} \boldsymbol{u}^h]\!])_{\tilde{\Gamma}} ,
\end{aligned}
\tag{24}
$$

$$
-(\operatorname{div} \boldsymbol{w}^h, p^h)_\Omega = (\boldsymbol{w}^h, \operatorname{grad} p^h)_{\tilde{\Omega}} - (\boldsymbol{n} \cdot \boldsymbol{w}^h, [\![p^h]\!])_{\tilde{\Gamma}} - (\boldsymbol{n} \cdot \boldsymbol{w}^h, p^h)_{\Gamma_\psi} . \tag{25}
$$

Remarks

(1) Note that the pressure and tangential vorticity boundary conditions are satisfied weakly.

(2) If $\alpha = \beta = 0$, we reduce to a Galerkin formulation. In general we assume $\alpha \geq 0$ and $\beta \geq 0$.

4. Error analysis

Let $\boldsymbol{U} = \{\boldsymbol{u}, p\}$ denote the solution of the Stokes problem (2), (3), and (7)–(10). It follows from (23) that,

$$
B(\boldsymbol{W}^h, \boldsymbol{U}) = L(\boldsymbol{W}^h) , \tag{26}
$$

and thus, by (20),

$$
B(\boldsymbol{W}^h, \boldsymbol{E}) = 0 , \tag{27}
$$

where $\boldsymbol{E} = \{\boldsymbol{e}_u, e_p\} = \boldsymbol{U}^h - \boldsymbol{U}$ is the error in the finite element solution. We refer to (27) as the *consistency condition*.

For quasi-uniform mesh refinements, it considerably simplifies the error analysis if we assume h in (21) and (22) is constant for each mesh. This represents no loss in generality.

We will need to assume the following *inverse estimate*:

$$
h^2 \|\Delta \boldsymbol{w}^h\|^2_{\tilde{\Omega}} \leq \bar{c}(\|\operatorname{curl} \boldsymbol{w}^h\|^2_\Omega + \|\operatorname{div} \boldsymbol{w}^h\|^2_\Omega) \quad \forall \boldsymbol{w}^h \in \mathcal{V}^h , \tag{28}
$$

where $\bar{c}$ is a constant.

Let

$$
\begin{aligned}
|||\boldsymbol{W}^h|||^2 = {}& \frac{\mu}{2} (\|\operatorname{curl} \boldsymbol{w}^h\|^2_\Omega + \|\operatorname{div} \boldsymbol{w}^h\|^2_\Omega) + \frac{\alpha h^2}{2\mu} \|\operatorname{grad} q^h\|^2_{\tilde{\Omega}} \\
&+ \frac{\beta h}{2\mu} (\|[\![q^h]\!]\|^2_{\tilde{\Gamma}} + \|q^h\|^2_{\Gamma_\psi}) .
\end{aligned}
\tag{29}
$$

LEMMA 4.1. Let $\bar{W}^h = \{w^h, -q^h\} \in \mathcal{V}^h \times \mathcal{P}^h$. *Assume the inverse estimate holds and* $0 < \alpha \leq \bar{c}^{-1}$, *where* $\bar{c}$ *is the constant in* (28). *Then,*

$$B(\bar{W}^h, W^h) \geq |||W^h|||^2 \quad \forall W^h \in \mathcal{V}^h \times \mathcal{P}^h . \tag{30}$$

We refer to (23) *as the* stability condition (*also known as the* coercivity condition).

PROOF.

$$\begin{aligned} B(\bar{W}^h, W^h) &= \mu \|\text{curl } w^h\|_\Omega^2 + \mu \|\text{div } w^h\|_\Omega^2 - \frac{\mu}{2} \alpha h^2 \|\Delta w^h\|_{\tilde{\Omega}}^2 \\ &\quad + \frac{\alpha h^2}{2\mu} \|\text{grad } q^h\|_{\tilde{\Omega}}^2 + \frac{\beta h}{2\mu} (\|[\![q^h]\!]\|_{\tilde{\Gamma}}^2 + \|q^h\|_{\Gamma_\psi}^2) \\ &\geq \mu\Big(1 - \frac{\alpha \bar{c}}{2}\Big)(\|\text{curl } w^h\|_\Omega^2 + \|\text{div } w^h\|_\Omega^2) \\ &\quad + \frac{\alpha h^2}{2\mu} \|\text{grad } q^h\|_{\tilde{\Omega}}^2 + \frac{\beta h}{2\mu} (\|[\![q^h]\!]\|_{\tilde{\Gamma}}^2 + \|q^h\|_{\Gamma_\psi}^2) \\ &\geq |||W^h|||^2 . \qquad \square \end{aligned} \tag{31}$$

Let $\tilde{U}^h \in \mathcal{S}^h \times \mathcal{P}^h$ denote an interpolant of U. The interpolation error is denoted by

$$H = \tilde{U}^h - U = \{\eta_u, \eta_p\} . \tag{32}$$

Thus,

$$E = E^h + H , \tag{33}$$

where

$$E^h = \{e_u^h, e_p^h\} \in \mathcal{V}^h \times \mathcal{P}^h . \tag{34}$$

We need to assume the following *interpolation estimates*:

$$\|\text{curl } \eta_u\|_\Omega^2 + \|\text{div } \eta_u\|_\Omega^2 + h^{-2}\|\eta_u\|_\Omega^2 + h^{-1}\|n \cdot \eta_u\|_{\tilde{\Gamma} \cup \Gamma_\psi}^2 + h^2 \|\Delta \eta_u\|_{\tilde{\Omega}}^2 \leq c_u h^{2k} , \tag{35}$$

$$\|\eta_p\|_\Omega^2 + h^2\|\text{grad } \eta_p\|_{\tilde{\Omega}}^2 + h(\|[\![\eta_p]\!]\|_{\tilde{\Gamma}}^2 + \|\eta_p\|_{\Gamma_\psi}^2) \leq c_p h^{2(l+1)} , \tag{36}$$

where c_u and c_p are functions of u and p, respectively. These notations are used subsequently, it being understood that the values may change in each instance.

The following integration-by-parts formula is also used subsequently:

$$(e_p^h, \text{div } \eta_u)_\Omega = -(\text{grad } e_p^h, \eta_u)_{\tilde{\Omega}} + ([\![e_p^h]\!], n \cdot \eta_u)_{\tilde{\Gamma}} + (e_p^h, n \cdot \eta_u)_{\Gamma_\psi} . \tag{37}$$

THEOREM 4.2. Assume the inverse estimate and the interpolation estimates hold; assume the consistency and stability conditions also hold. Then,

$$|||\boldsymbol{E}|||^2 \leq c_u h^{2k} + c_p h^{2(l+1)} . \tag{38}$$

PROOF. Let $\bar{\boldsymbol{E}}^h = \{\boldsymbol{e}_u^h, -e_p^h\} \in \mathcal{V}^h \times \mathcal{P}^h$, and let a, b, and c be constants whose values will be assigned later. We estimate $|||\boldsymbol{E}^h|||$ as follows:

$$\begin{aligned}
|||\boldsymbol{E}^h|||^2 &\leq B(\bar{\boldsymbol{E}}^h, \boldsymbol{E}^h) \qquad \text{(stability)}\\
&= B(\bar{\boldsymbol{E}}^h, \boldsymbol{E} - \boldsymbol{H})\\
&= -B(\bar{\boldsymbol{E}}^h, \boldsymbol{H}) \qquad \text{(consistency)}\\
&\leq |B(\bar{\boldsymbol{E}}^h, \boldsymbol{H})|\\
&= |\mu(\text{curl}\, \boldsymbol{e}_u^h, \text{curl}\, \boldsymbol{\eta}_u)_\Omega + \mu(\text{div}\, \boldsymbol{e}_u^h, \text{div}\, \boldsymbol{\eta}_u)_\Omega - (\text{div}\, \boldsymbol{e}_u^h, \eta_p)_\Omega\\
&\quad - (\text{grad}\, e_p^h, \boldsymbol{\eta}_u)_{\tilde{\Omega}} + ([\![e_p^h]\!], \boldsymbol{n}\cdot\boldsymbol{\eta}_u)_{\tilde{\Gamma}} + (e_p^h, \boldsymbol{n}\cdot\boldsymbol{\eta}_u)_{\Gamma_\psi}\\
&\quad + \frac{\alpha h^2}{2\mu}\{(\text{grad}\, e_p^h, \text{grad}\, \eta_p)_{\tilde{\Omega}} - \mu^2(\Delta\boldsymbol{e}_u^h, \Delta\boldsymbol{\eta}_u)_{\tilde{\Omega}} - \mu(\text{grad}\, e_p^h, \Delta\boldsymbol{\eta}_u)_{\tilde{\Omega}}\\
&\quad + \mu(\Delta\boldsymbol{e}_u^h, \text{grad}\, \eta_p)_{\tilde{\Omega}}\} + \frac{\beta h}{2\mu}\{([\![e_p^h]\!], [\![\eta_p]\!])_{\tilde{\Gamma}} + (e_p^h, \eta_p)_{\Gamma_\psi}\}|
\end{aligned}$$

(by definition of B and the integration-by-parts formula (37))

$$\begin{aligned}
&\leq \frac{1}{2}\Big\{\frac{\mu}{2} b^{-1}\|\text{curl}\, \boldsymbol{e}_u^h\|_\Omega^2 + 2\mu b\|\text{curl}\, \boldsymbol{\eta}_u\|_\Omega^2 + \frac{\mu}{4} b^{-1}\|\text{div}\, \boldsymbol{e}_u^h\|_\Omega^2\\
&\quad + 4\mu b\|\text{div}\, \boldsymbol{\eta}_u\|_\Omega^2 + \frac{\mu}{4} b^{-1}\|\text{div}\, \boldsymbol{e}_u^h\|_\Omega^2 + \frac{4}{\mu} b\|\eta_p\|_\Omega^2\\
&\quad + \frac{\alpha h^2}{2\mu} a^{-1}\|\text{grad}\, e_p^h\|_{\tilde{\Omega}}^2 + \left(\frac{\alpha h^2}{2\mu}\right)^{-1} a\|\boldsymbol{\eta}_u\|_\Omega^2 + \frac{\beta h}{2\mu} c^{-1}\|e_p^h\|_{\Gamma_\psi}^2\\
&\quad + \left(\frac{\beta h}{2\mu}\right)^{-1} c\|\boldsymbol{n}\cdot\boldsymbol{\eta}_u\|_{\Gamma_\psi}^2 + \frac{\beta h}{2\mu} c^{-1}\|[\![e_p^h]\!]\|_{\tilde{\Gamma}}^2 + \left(\frac{\beta h}{2\mu}\right)^{-1} c\|\boldsymbol{n}\cdot\boldsymbol{\eta}_u\|_{\tilde{\Gamma}}^2\\
&\quad + \frac{\alpha h^2}{2\mu} a^{-1}\|\text{grad}\, e_p^h\|_{\tilde{\Omega}}^2 + \frac{\alpha h^2}{2\mu} a\|\text{grad}\, \eta_p\|_{\tilde{\Omega}}^2 + \frac{\mu}{2} h^2 b^{-1}\|\Delta\boldsymbol{e}_u^h\|_{\tilde{\Omega}}^2\\
&\quad + \frac{\mu}{2}\alpha^2 h^2 b\|\Delta\boldsymbol{\eta}_u\|_{\tilde{\Omega}}^2 + \frac{\alpha h^2}{2\mu} a^{-1}\|\text{grad}\, e_p^h\|_{\tilde{\Omega}}^2 + \frac{\mu}{2}\alpha h^2 a\|\Delta\boldsymbol{\eta}_u\|_{\tilde{\Omega}}^2\\
&\quad + \frac{\mu}{2} h^2 b^{-1}\|\Delta\boldsymbol{e}_u^h\|_{\tilde{\Omega}}^2 + \frac{\alpha^2 h^2}{2\mu} b\|\text{grad}\, \eta_p\|_{\tilde{\Omega}}^2 + \frac{\beta h}{2\mu} c^{-1}\|[\![e_p^h]\!]\|_{\tilde{\Gamma}}^2\\
&\quad + \frac{\beta h}{2\mu} c\|[\![\eta_p]\!]\|_{\tilde{\Gamma}}^2 + \frac{\beta h}{2\mu} c^{-1}\|e_p^h\|_{\Gamma_\psi}^2 + \frac{\beta h}{2\mu} c\|\eta_p\|_{\Gamma_\psi}^2\Big\}
\end{aligned}$$

$$\leq \frac{1}{2}\left\{\frac{\mu}{2} b^{-1}(1+2\bar{c})(\|\text{curl } \boldsymbol{e}_u^h\|_\Omega^2 + \|\text{div } \boldsymbol{e}_u^h\|_\Omega^2) + \frac{\alpha h^2}{2\mu} 3a^{-1}\|\text{grad } e_p^h\|_{\tilde{\Omega}}^2\right.$$

$$\left. + \frac{\beta h}{2\mu} 2c^{-1}(\|[\![e_p^h]\!]\|_{\tilde{\Gamma}}^2 + \|e_p^h\|_{\Gamma_\psi}^2)\right\} + \mu\left\{b\|\text{curl } \boldsymbol{\eta}_u\|_\Omega^2 + 2b\|\text{div } \boldsymbol{\eta}_u\|_\Omega^2\right.$$

$$\left. + \frac{a}{\alpha h^2}\|\boldsymbol{\eta}_u\|_\Omega^2 + \frac{c}{\beta h}\|\boldsymbol{n}\cdot\boldsymbol{\eta}_u\|_{\tilde{\Gamma}\cup\Gamma_\psi}^2 + \frac{\alpha h^2}{4}(a+\alpha b)\|\Delta\boldsymbol{\eta}_u\|_{\tilde{\Omega}}^2\right\}$$

$$+ \frac{1}{4\mu}\{8b\|\eta_p\|_\Omega^2 + \alpha h^2(a+\alpha b)\|\text{grad } \eta_p\|_{\tilde{\Omega}}^2 + \beta hc(\|[\![\eta_p]\!]\|_{\tilde{\Gamma}}^2 + \|\eta_p\|_{\Gamma_\psi}^2)\}\,. \tag{39}$$

By selecting $a = 3$, $b = 1 + 2\bar{c}$, and $c = 2$, the first term on the right-hand side of (39) becomes $\frac{1}{2}|||\boldsymbol{E}^h|||^2$. The second and third terms may be estimated by (35) and (36), respectively. Therefore,

$$|||\boldsymbol{E}^h|||^2 \leq c_u h^{2k} + c_p h^{2(l+1)}\,. \tag{40}$$

Likewise, by (29), (35), and (36),

$$|||\boldsymbol{H}|||^2 \leq c_u h^{2k} + c_p h^{2(l+1)}\,. \tag{41}$$

This completes the proof of the theorem. □

Remarks

(1) The theorem is valid for $0 < \alpha < \bar{c}^{-1}$, and $0 < \beta$. Note that there is no upper bound imposed upon β. To balance errors, we may as well take $\beta = O(1)$. The upper bound on α emanates from the necessity of controlling $\Delta\boldsymbol{e}_u^h$ on element interiors. If the velocity is piecewise linear, this term vanishes identically, and the upper bound on α is no longer necessary. In this case we may also take $\alpha = O(1)$.

(2) By a generalized Poincaré–Friedrichs inequality, proved by Arnold [1], which in the present circumstances can be written as,

$$h^2\|e_p\|_\Omega^2 \leq c_\Omega(h^2\|\text{grad } e_p\|_{\tilde{\Omega}}^2 + h\|[\![e_p]\!]\|_{\tilde{\Gamma}}^2 + h\|e_p\|_{\Gamma_\psi}^2)\,, \tag{42}$$

we may obtain an $L^2(\Omega)$ convergence result for pressure, namely,

$$\|e_p\|_\Omega^2 \leq c_u h^{2(k-1)} + c_p h^{2l}\,. \tag{43}$$

This is suboptimal and not useful when either $k = 1$ or $l = 0$. In many circumstances (43) is pessimistic because, for appropriate hypotheses, duality arguments can be used to attain the improved estimate:

$$\|e_p\|_\Omega^2 \leq c_u h^{2k} + c_p h^{2(l+1)} . \tag{44}$$

(3) Within the proposed formulation we may distinguish between four different types of methods:

Type I: $\alpha = 0$, $\beta = 0$ (Galerkin). Classical convergence theory requires satisfaction of the Babuška–Brezzi condition. Various texts summarize known examples (e.g., see [4, 7, 10, 18]). Among the most popular are the continuous pressure elements P2P1 and Q2Q1, and the discontinuous pressure element Q2P1.

Type II: $\alpha > 0$, $\beta = 0$. All continuous-pressure elements are convergent. Equal-order elements (i.e., P*k*P*k* and Q*k*Q*k*, $k \geq 1$) are particularly attractive due to implementational considerations.

Type III: $\alpha = 0$, $\beta > 0$. The β terms are sufficient to render convergent certain low order, discontinuous pressure elements such as P1P0 and Q1P0.

Type IV: $\alpha > 0$, $\beta > 0$. The α and β terms are both required for higher-order discontinuous-pressure elements. Equal-order elements again appear to be the most attractive. This is of course the general case and is applicable to *any* element.

(4) The β terms necessitate a nonstandard finite element assembly algorithm. For this reason, continuous pressure elements, which only require the presence of the α terms, may be more convenient in practice.

(5) The traction vector along 'no-slip' walls may be calculated from

$$\boldsymbol{t} = -p\boldsymbol{n} + \boldsymbol{t}_\mu , \tag{45}$$

$$\boldsymbol{t}_\mu = -\mu \boldsymbol{n} \times \operatorname{curl} \boldsymbol{u} . \tag{46}$$

Accurate calculation of this quantity is facilitated by the following post-processing procedure (see [10, p. 107] for an elementary description of the basic ideas): Let $\boldsymbol{w}_\Gamma^h$ denote weighting functions expanded in terms of velocity basis functions which do *not* vanish on Γ_1. Let $\boldsymbol{t}_\mu^h$ denote a vector field expanded in terms of the same basis functions. Define $\boldsymbol{t}_\mu^h$ by:

$$\begin{aligned}(\boldsymbol{w}_\Gamma^h, \boldsymbol{t}_\mu^h)_{\Gamma_1} &= (\operatorname{curl} \boldsymbol{w}_\Gamma^h, \mu \operatorname{curl} \boldsymbol{u}^h)_\Omega + (\operatorname{div} \boldsymbol{w}_\Gamma^h, \mu \operatorname{div} \boldsymbol{u}^h - p^h)_\Omega \\ &\quad - (\tfrac{1}{2}\alpha h^2 \Delta \boldsymbol{w}_\Gamma^h, \mu \Delta \boldsymbol{u}^h - \operatorname{grad} p^h + \boldsymbol{f})_{\tilde{\Omega}} - (\boldsymbol{w}_\Gamma^h, \boldsymbol{f})_{\tilde{\Omega}} .\end{aligned} \tag{47}$$

In forming the right-hand-side, only those elements with a boundary segment contained in Γ_1 need to be included. The approximate traction is then given by:

$$\boldsymbol{t}^h = -p^h \boldsymbol{n} + \boldsymbol{t}_\mu^h . \tag{48}$$

It is interesting that, despite traction not being a boundary condition for the problem considered, a natural post-processing procedure for wall traction nevertheless ensues.

5. Stokes equations with velocity and traction boundary conditions

Let $\{\Gamma_g, \Gamma_h\}$ be a partition of Γ. This time we work with the following form of the Stokes equations:

$$\operatorname{div} \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} \qquad \text{on } \Omega\,, \tag{49}$$

$$\operatorname{div} \boldsymbol{u} = 0 \qquad \text{on } \Omega\,, \tag{50}$$

$$\boldsymbol{\sigma} = -p\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{on } \Omega\,, \tag{51}$$

$$\boldsymbol{u} = \boldsymbol{g} \qquad \text{on } \Gamma_g\,, \tag{52}$$

$$\boldsymbol{n} \cdot \boldsymbol{\sigma} = \boldsymbol{h} \qquad \text{on } \Gamma_h\,, \tag{53}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{I}$ is the identity tensor, $\varepsilon(\boldsymbol{u})$ is the symmetric part of the velocity gradient, and $\boldsymbol{h}$ is the boundary traction. This form of the Stokes problem is more useful in the case of incompressible elasticity.

A finite element formulation of (49)–(53) analogous to the one presented in Section 3 is given as follows: We require all $\boldsymbol{u}^h \in \mathcal{S}^h$ to satisfy $\boldsymbol{u}^h = \boldsymbol{g}$ on Γ_g, and all $\boldsymbol{w}^h \in \mathcal{V}^h$ to satisfy $\boldsymbol{w}^h = \boldsymbol{0}$ on Γ_g. Find $\boldsymbol{U}^h \in \mathcal{S}^h \times \mathcal{P}^h$ such that for all $\boldsymbol{W}^h \in \mathcal{V}^h \times \mathcal{P}^h$,

$$B(\boldsymbol{W}^h, \boldsymbol{U}^h) = L(\boldsymbol{W}^h)\,, \tag{54}$$

where

$$\begin{aligned} B(\boldsymbol{W}^h, \boldsymbol{U}^k) = {} & (\boldsymbol{\varepsilon}(\boldsymbol{w}^h), 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}^h))_\Omega - (\operatorname{div} \boldsymbol{w}^h, p^h)_\Omega - (q^h, \operatorname{div} \boldsymbol{u}^h)_\Omega \\ & - \left(\frac{\alpha h^2}{2\mu} \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{W}^h), \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{U}^h)\right)_{\tilde{\Omega}} - \left(\frac{\beta h}{2\mu} [\![q^h]\!], [\![p^h]\!]\right)_{\tilde{\Gamma}}\,, \end{aligned} \tag{55}$$

$$L(\boldsymbol{W}^h) = \left(\boldsymbol{w}^h + \frac{\alpha h^2}{2\mu} \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{W}^h), \boldsymbol{f}\right)_{\tilde{\Omega}} + (\boldsymbol{w}^h, \boldsymbol{h})_{\Gamma_h}\,. \tag{56}$$

The Euler–Lagrange form of (54) is

$$\begin{aligned} 0 = {} & -\left(\boldsymbol{w}^h + \frac{\alpha h^2}{2\mu} \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{W}^h), \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{U}^h) + \boldsymbol{f}\right)_{\tilde{\Omega}} + (\boldsymbol{w}^h, \boldsymbol{n} \cdot \boldsymbol{\sigma}(\boldsymbol{U}^h) - \boldsymbol{h})_{\Gamma_h} \\ & + (\boldsymbol{w}^h, \boldsymbol{n} \cdot [\![\boldsymbol{\sigma}(\boldsymbol{U}^h)]\!])_{\tilde{\Gamma}} - (q^h, \operatorname{div} \boldsymbol{u}^h)_\Omega - \left(\frac{\beta h}{2\mu} [\![q^h]\!], [\![p^h]\!]\right)_{\tilde{\Gamma}}\,. \end{aligned} \tag{57}$$

For the Dirichlet problem we are able to prove the following error estimate:

$$\mu \|\boldsymbol{\varepsilon}(\boldsymbol{e}_u)\|^2_\Omega + \frac{\alpha h^2}{2\mu} \|\operatorname{grad} e_p\|^2_{\tilde{\Omega}} + \frac{\beta h}{2\mu} \|[\![e_p]\!]\|^2_{\tilde{\Gamma}} \leq c_u h^{2k} + c_p h^{2(l+1)}\,. \tag{58}$$

To obtain (58) we need to assume $\alpha < 1/(2\tilde{c})$, where $\tilde{c}$ is the constant in the inverse estimate,

$$h^2\|\operatorname{div} \boldsymbol{\varepsilon}(\boldsymbol{w}^h)\|_{\tilde{\Omega}}^2 \leq \tilde{c}\|\boldsymbol{\varepsilon}(\boldsymbol{w}^h)\|_{\Omega}^2 \quad \forall \boldsymbol{w}^h \in \mathcal{V}^h . \tag{59}$$

The analysis is closely related to that presented in Section 4 and so it is omitted. (A proof for the case $\beta = 0$ may be found in [5].) We are unable to prove convergence for the case in which Γ_h is not empty, but presume that this case is also well behaved.

6. Conclusions

The symmetric methods presented herein were proved to be convergent for arbitrary combinations of velocity and pressure interpolations. Two modifications of the classical Galerkin formulation are required to achieve this result. Both involve the addition of 'least-squares' forms of residuals: one is the momentum equation residual on element interiors, the other is the pressure continuity residual on element boundaries. The bilinear form of the modified formulation is rendered coercive, in contrast to the classical Galerkin formulation, and this enables us to *circumvent* the Babuška–Brezzi test. (We have also employed similar techniques in the development of mixed methods which *satisfy* the Babuška–Brezzi condition for a wider range of functions than the classical Galerkin formulation; for applications to elasticity and various structural model problems, see [5, 6, 13–15]). The methodology appears general and offers a new line of attack on the problem of developing convergent mixed finite element methods.

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