Womersley Solution for Blood Flow in a Thin-walled Elastic Tube

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The Womersley theory provides a analytical solution for pulsatile flow in a thin-walled elastic tube which is assumed to be axisymmetric circular straight and the flow inside is fully developed.

The Womersley solutions contain the fluid pressure, the longitudinal and radial flow velocities, and the longitudinal and radial wall displacement. And the solutions can be divided to the steady and oscillatory parts. The steady part is exactly the solution for the rigid tube, namely the Poiseuille flow.

$$p_s = p_0 + k_s x,$$
$$u_s = \frac{k_s}{4\mu}(r^2 - R^2).$$

To solve the oscillatory part of the solution, we first write down the governing equations for the fluid domain. With the assumptions of axisymmetric straight tube and incompressible flow, the Navier-Stokes equations and the continuity equation can be write as

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r}\right) + \frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right),$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial r}\right) + \frac{\partial p}{\partial r} = \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right),$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0.$$

Considering a further simplification of the governing equations, we introduce the long-wave approximation which assumes that the length of the propagating wave is much higher than the tube radius, and wave speed is much higher than the average flow velocity within the tube.

$$\frac{R}{L}, \frac{\overline{u}}{c_0} \ll 1.$$

With dimension analysis, the comparisons of terms in the governing equations are shown as follows,

$$u\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial r} \ll \frac{\partial u}{\partial t},$$
$$u\frac{\partial v}{\partial x}, v\frac{\partial v}{\partial r} \ll \frac{\partial v}{\partial t},$$
$$\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial r^2},$$
$$\frac{\partial^2 v}{\partial x^2} \ll \frac{\partial^2 v}{\partial r^2}.$$

Thus, the simplified governing equations are,

$$\rho\frac{\partial u}{\partial t}+\frac{\partial p}{\partial x}=\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}\right),$$

$$\rho\frac{\partial v}{\partial t}+\frac{\partial p}{\partial r}=\mu\left(\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}\right),$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial r}+\frac{v}{r}=0.$$

The governing equations for the solid domain are the wall motion equations with the assumptions of linear elasticity and thin-walled tube theories.

Cauchy's equation of motion is

$$\rho_s\frac{\partial^2\vec{\mathbf{x}}}{\partial t^2}=\vec{\mathbf{F}}+\nabla\cdot\bar{\bar{\boldsymbol{\sigma}}}.$$

The wall displacement vector $\vec{\mathbf{x}}$ has longitudinal component ξ and radial component η.

The fictitious body force $\vec{\mathbf{F}}$ has radial and longitudinal components coming from the pressure and shear stresses acting on the lateral fluid boundary. That is,

$$F_r=\frac{p}{h},$$

$$F_x=\frac{\tau}{h}=\frac{\mu}{h}\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right),$$

The stress tensor $\bar{\bar{\boldsymbol{\sigma}}}$ is defined as the linear combination of two stress states: internal pressurization with no axial strain and axial force with no internal pressure, which has two non-zero components,

$$\sigma_{\theta\theta}=\frac{E}{1-\nu^2}\left(\frac{\eta}{R}+\nu\frac{\partial^2\xi}{\partial t^2}\right),$$

$$\sigma_{xx}=\frac{E}{1-\nu^2}\left(\nu\frac{\eta}{R}+\frac{\partial\xi}{\partial x}\right).$$

Above all, the wall motion equations are written as,

$$\frac{\partial^2\xi}{\partial t^2}=-\frac{\mu}{\rho_s h}\left(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x}\right)+\frac{E_\nu}{\rho_s}\left(\frac{\partial^2\xi}{\partial x^2}+\frac{\nu}{R}\frac{\partial\eta}{\partial x}\right),$$

$$\frac{\partial^2\eta}{\partial t^2}=\frac{p}{\rho_s h}-\frac{E_\nu}{\rho_s}\left(\frac{\eta}{R^2}+\frac{\nu}{R}\frac{\partial\xi}{\partial x}\right),$$

where,

$$E_\nu=\frac{E}{1-\nu^2}.$$

Using the method of separation of variables and the exponential form of Fourier series, we design the formulation of the oscillatory solutions as,

$$u(x,r,t) = \sum_{n=1}^{\infty} U_n(r) e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$v(x,r,t) = \sum_{n=1}^{\infty} V_n(r) e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$p(x,r,t) = \sum_{n=1}^{\infty} P_n(r) e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$\xi(x,R,t) = C_n e^{i\omega n\left(t-\frac{x}{c_n}\right)}, \quad (C_n\ is\ constant)$$

$$\eta(x,R,t) = D_n e^{i\omega n\left(t-\frac{x}{c_n}\right)}, \quad (D_n\ is\ constant)$$

to make the derivation simply, the Fourier mode is set to be 1, but the final results will be expressed in general form.

Substituting into the fluid governing equations leads to an ordinary differential equations system, namely,

$$\frac{d^2U}{dr^2} + \frac{1}{r}\frac{dU}{dr} - \frac{i\omega\rho}{\mu}U = -\frac{i\omega}{\mu c}P,$$

$$\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \left(\frac{1}{r^2} + \frac{i\omega\rho}{\mu}\right)V = -\frac{1}{\mu}\frac{dP}{dr},$$

$$\frac{dV}{dr} + \frac{1}{r}V - \frac{i\omega}{c}U = 0.$$

Here we introduce a non-dimensional number called Womersley number,

$$\Omega_n = R\sqrt{\frac{\rho w n}{\mu}}, \quad for\ n = 1,2,\dots.$$

and the variable ζ for constructing the Bessel equations,

$$\zeta = \Lambda\frac{r}{R},$$

where

$$\Lambda = \Omega\frac{i-1}{\sqrt{2}}.$$

Then the equation is expressed as

$$\frac{d^2U}{d\zeta^2} + \frac{1}{\zeta}\frac{dU}{d\zeta} + U = -\frac{1}{\rho c}P,$$

$$\frac{d^2V}{d\zeta^2} + \frac{1}{\zeta}\frac{dV}{d\zeta} + \left(1 - \frac{1}{\zeta^2}\right)V = -\frac{i\Lambda}{\omega\rho R}\frac{dP}{d\zeta},$$

$$\frac{dV}{d\zeta} + \frac{1}{\zeta}V - \frac{i\omega R}{c\Lambda}U = 0.$$

The required boundary conditions are zero velocities at the tube wall and finite velocity at the tube center, that is,

$$for\, r = R, \zeta = \Lambda, \quad U = V = 0,$$
$$for\, r = 0, \zeta = 0, \quad |U|, |V| < \infty.$$

Then the solutions of the Bessel equation system can be written as,

$$U = AJ_0(\zeta) + \frac{B}{\rho c} J_0(\frac{\gamma}{\Lambda}\zeta),$$
$$V = A\frac{\gamma}{\Lambda} J_1(\zeta) + \frac{B}{\rho c} J_1(\frac{\gamma}{\Lambda}\zeta),$$
$$P = BJ_0(\frac{\gamma}{\Lambda}\zeta),$$

where A, B are constants and

$$\gamma = \frac{i\omega R}{c}.$$

The solutions can be simplified by

$$\gamma = \frac{i\omega R}{c} \sim (\frac{R}{L}) \ll 1,$$
$$J_0(\frac{\gamma}{\Lambda}\zeta) = J_0(\gamma\frac{r}{R}) \approx 1,$$
$$J_1(\frac{\gamma}{\Lambda}\zeta) \approx \frac{1}{2}\frac{\gamma}{\Lambda}\zeta,$$

that is,

$$U = AJ_0(\zeta) + \frac{B}{\rho c},$$
$$V = A\frac{\gamma}{\Lambda} J_1(\zeta) + \frac{1}{2}\frac{B}{\rho c}\frac{\gamma}{\Lambda}\zeta = A\frac{\gamma}{\Lambda} J_1(\zeta) + B\frac{i\omega R}{2\rho c^2},$$
$$P = B. \quad (B\ is\ constant)$$

Now the oscillatory solutions can be written as follows,

$$u(x, r, t) = \left(AJ_0(\zeta) + \frac{B}{\rho c}\right) e^{i\omega\left(t - \frac{x}{c}\right)},$$

$$v(x, r, t) = \left(A\frac{\gamma}{\Lambda} J_1(\zeta) + B\frac{i\omega R}{2\rho c^2}\right) e^{i\omega\left(t - \frac{x}{c}\right)},$$

$$p(x, r, t) = Be^{i\omega\left(t - \frac{x}{c}\right)},$$

$$\xi(x, R, t) = Ce^{i\omega\left(t - \frac{x}{c}\right)},$$

$$\eta(x, R, t) = De^{i\omega\left(t - \frac{x}{c}\right)},$$

where $A, B, C\ and\ D$ are arbitrary constants.

Coupling the fluid and solid governing equations requires the boundary conditions on the fluid solid interface. Motion of the tube wall is coupled to the motion of the fluid through the action of fluid pressure and shear stress on the tube wall, and the two motions should be matched at the interface between the fluid and the inner surface of the tube wall. Namely,

$$\frac{\partial^2\xi}{\partial t^2}=-\frac{\mu}{\rho_s h}(\frac{\partial u}{\partial r}+\frac{\partial v}{\partial x})+\frac{E_v}{\rho_s}(\frac{\partial^2\xi}{\partial x^2}+\frac{\nu}{R}\frac{\partial\eta}{\partial x}),$$

$$\frac{\partial^2\eta}{\partial t^2}=\frac{p}{\rho_s h}-\frac{E_v}{\rho_s}(\frac{\eta}{R^2}+\frac{\nu}{R}\frac{\partial\xi}{\partial x}),$$

$$\frac{\partial\xi}{\partial t}=u,$$

$$\frac{\partial\eta}{\partial t}=v.$$

Then we can derive an equation system for the parameters $A, B, C\ and\ D$:

$$\left(\frac{\mu\Lambda J_1(\Lambda)}{\rho_s hR}\right)A+\left(1-\frac{E_v}{\rho_s c^2}\right)\omega^2 C-\left(\frac{i\omega\nu E_v}{\rho_s cR}\right)D=0$$

$$\left(\frac{1}{h}\right)B+\left(\frac{i\omega\nu E_v}{cR}\right)C-\left(\frac{E_v}{R^2}\right)D=0$$

$$J_0(\Lambda)A+\left(\frac{1}{\rho c}\right)B-i\omega C=0$$

$$\left(\frac{i\omega R J_1(\Lambda)}{c\Lambda}\right)A+\left(\frac{i\omega R}{2\rho c^2}\right)B-i\omega D=0$$

The linear equation system is homogeneous, a nontrivial solution is obtained only if the determinant of the coefficients is zero. By simplifying the determinant equation we get the equation as follows,

$$g+\frac{2\rho_s h}{\rho R}+\left[\frac{\rho_s h}{\rho R}(g-1)+\left(2\nu-\frac{1}{2}\right)g-2\right]z+\left[(g-1)(\nu^2-1)\right]z^2=0,$$

where,

$$g=\frac{2J_1(\Lambda)}{\Lambda J_0(\Lambda)},$$

$$z=\frac{E_v h}{\rho c^2 R}.$$

The wave speed can be determined by solving the above equations, commonly known as the frequency equation.

Since the input oscillatory pressure is normally known or specified, whose amplitude is exactly the coefficient B,

$$p(x,r,t)=p_0+k_s x+\sum_{n=1}^{\infty}B_n e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

then the other coefficients $A, C\ and\ D$ and be calculated.

The final results of the Womersley solution for thin-walled elastic tubes are,

$$u(x,r,t)=\frac{k_s}{4\mu}(r^2-R^2)+\sum_{n=1}^{\infty}\frac{B_n}{\rho c_n}\left(1-G_n\frac{J_0(\zeta_n)}{J_0(\Lambda_n)}\right)e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$v(x,r,t)=\sum_{n=1}^{\infty}\frac{i\omega nRB_n}{2\rho c_n^2}\left(\frac{r}{R}-G_n\frac{2J_0(\zeta_n)}{\Lambda_n J_0(\Lambda_n)}\right)e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$\xi(x,R,t)=\sum_{n=1}^{\infty}\frac{iB_n}{\omega n\rho c_n}(G_n-1)e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

$$\eta(x,R,t)=\sum_{n=1}^{\infty}\frac{RB_n}{2\rho c_n^2}(1-G_n g_n)e^{i\omega n\left(t-\frac{x}{c_n}\right)},$$

where,

$$G_n=\frac{2+z_n(2\nu-1)}{z_n(2\nu-g_n)}.$$

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