A framework for the construction of the fiber directions in vessels

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1. Centerline computation

Centerline can be defined as the line drawn between two sections of a tubular structure which maximize the distance from the boundary. The problem of centerline computation inside an object $\Omega \subset \Re^3$ can therefore be formulated as looking for a path $\mathbf{C} = \mathbf{C}(s)$ traced between two points $\mathbf{p}_1$ and $\mathbf{p}_2$ for which the following functional

$$E_{\text{centerline}}(\mathbf{C}) = \int_{0=\mathbf{C}^{-1}(\mathbf{p}_0)}^{L=\mathbf{C}^{-1}(\mathbf{p}_1)} F(\mathbf{C}(s))ds$$

is minimal, where $F(\mathbf{x})$ is a scalar field which is lower for more internal points, for example a decreasing function of the distance transform associated with Ω , defined as

$$\mathrm{DT}(\mathbf{x}) = \min_{\mathbf{y}\in\partial\Omega}\{|\mathbf{x},\mathbf{y}|\}$$

where $|,|$ denotes the Euclidean distance, and $\partial\Omega$ the boundary of Ω . It is possible to demonstrate [1] that by choosing $F(\mathbf{x}) = \mathrm{DT}^{-1}(\mathbf{x})$, centerlines defined as in Equation 1 lie on the medial axis of Ω, $\mathrm{MA}(\Omega)$, defined as the locus of centers of the maximal inscribed balls in Ω , where an inscribed ball is maximal if it is not strictly contained in any other inscribed ball. Dealing with piecewise linear approximations of $\partial\Omega$, a method to obtain an approximation of the medial axis of Ω is to compute the embedded Voronoi diagram of a point set P densely sampling $\partial\Omega$ [2]. The Voronoi diagram of P is defined as

$$\mathrm{Vor}(P) = \bigcup_{\mathbf{p}\in P} \partial V(\mathbf{p})$$

where $V(\mathbf{p})$ is the Voronoi region associated with point $\mathbf{p}$, defined as

$$V(\mathbf{p}) = \left\{\mathbf{x} \in \Re^3 : |\mathbf{p},\mathbf{x}| \leq |\mathbf{q},\mathbf{x}| \forall \mathbf{q} \in P\right\}$$

In 3D, the Voronoi diagram is a non-manifold surface made up of convex polygons, whose vertices are the centers of the maximal emtpy balls with respect to point set P , whose radius is indicated by $R(\mathbf{x})$. Computation of the embedded Voronoi diagram was performed by first computing the Delaunay tessellation of $P, \mathrm{Del}(P)$, removing the tetrahedra whose circumcenter falls outside the object (using outward surface normals) and then constructing only those Voronoi polygons whose vertex loops are complete.

We then solved the problem in Equation 1 on the embedded Voronoi diagram, taking $F(\mathbf{x}) = R^{-1}(\mathbf{x})$, with an approach similar to that presented in [4] for the computation of centerlines in 3D images. As shown in [3], the strong formulation of Equation 1 is the Eikonal equation

$$|\nabla T(\mathbf{x})| = F(\mathbf{x})$$

with boundary condition $T(\mathbf{p}_0) = 0$. Equation 5 is a nonlinear partial hyperbolic equation that models first arrival times of a wavefront propagating over the domain with speed $F^{-1}(\mathbf{x})$. A very effecient method for the solution of the Eikonal equation is the Fast Marching Method ([5]), based on upwind finitedifference approximation, originally developed for orthogonal grids and successively extended to triangulated manifolds ([6]). In order to solve the problem on the Voronoi diagram, we extended the Fast Marching Method to polygonal non-manifolds [1], in which more than two polygons can share a point or an edge. Once the Eikonal equation is solved over the whole Voronoi diagram with boundary condition $T(\mathbf{p}_0) = 0$, centerlines are obtained by backtracing a path from $\mathbf{p}_1$ along the direction of maximum descent of $T(\mathbf{x})$. The resulting centerline is a piecewise linear line defined on $\mathrm{Vor}_E(P)$, whose vertices lie on Voronoi polygon boundaries. Moreover, values of Voronoi sphere radius $R(\mathbf{x})$ are defined on centerlines, so that centerline points are associated with maximal inscribed spheres.

References

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