

```

In[*]:= M2 = {
  {c*ℓ - c*Subscript[r, x], -c*b, c*Subscript[r, x]},
  {α*b*(1 + c*ℓ - c*Subscript[r, x]),
  -Subscript[r, y]*α - c*α*b^2,
  c*Subscript[r, x]*α*b},
  {β*(1 + c*ℓ - c*Subscript[r, x]),
  -c*b*β,
  -β + β*c*Subscript[r, x]}
};

```

```

M2 // MatrixForm

```

```

F = CharacteristicPolynomial[M2, λ]

```

```

Solve[F == 0, λ][[2]] (* The eigenvalue of M2 *)

```

Out[*]//MatrixForm=

$$\begin{pmatrix} c\ell - cr_x & -bc & cr_x \\ b\alpha(1 + c\ell - cr_x) & -b^2c\alpha - \alpha r_y & bc\alpha r_x \\ \beta(1 + c\ell - cr_x) & -bc\beta & -\beta + c\beta r_x \end{pmatrix}$$

Out[*]=

$$-b^2c\alpha\beta - b^2c\alpha\lambda + c\ell\beta\lambda - b^2c\alpha\beta\lambda + c\ell\lambda^2 - b^2c\alpha\lambda^2 - \beta\lambda^2 - \lambda^3 - c\lambda^2r_x + c\beta\lambda^2r_x + c\ell\alpha\beta r_y + c\ell\alpha\lambda r_y - \alpha\beta\lambda r_y - \alpha\lambda^2r_y - c\alpha\lambda r_x r_y + c\alpha\beta\lambda r_x r_y$$

Out[*]=

$$\left\{ \lambda \rightarrow \frac{1}{3} \left(c\ell - b^2c\alpha - \beta - cr_x + c\beta r_x - \alpha r_y \right) - \left((1 + i\sqrt{3}) \left(- (c\ell - b^2c\alpha - \beta - cr_x + c\beta r_x - \alpha r_y)^2 - 3 \left(-b^2c\alpha + c\ell\beta - b^2c\alpha\beta + c\ell\alpha r_y - \alpha\beta r_y - c\alpha r_x r_y + c\alpha\beta r_x r_y \right) \right) \right) / \left(3 \times 2^{2/3} \left(-2c^3\ell^3 + 9b^2c^2\ell\alpha + 6b^2c^3\ell^2\alpha - 9b^4c^2\alpha^2 - 6b^4c^3\ell\alpha^2 + 2b^6c^3\alpha^3 - 3c^2\ell^2\beta + 18b^2c\alpha\beta + 6b^2c^2\ell\alpha\beta - 3b^4c^2\alpha^2\beta + 3c\ell\beta^2 - 3b^2c\alpha\beta^2 + 2\beta^3 + 6c^3\ell^2r_x - 9b^2c^2\alpha r_x - 12b^2c^3\ell\alpha r_x + 6b^4c^3\alpha^2r_x - 3c^2\ell\beta r_x - 6c^3\ell^2\beta r_x + 12b^2c^2\alpha\beta r_x + 12b^2c^3\ell\alpha\beta r_x - 6b^4c^3\alpha^2\beta r_x + 6c\beta^2r_x + 3c^2\ell\beta^2r_x - 3b^2c^2\alpha\beta^2r_x - 6c\beta^3r_x - 6c^3\ell r_x^2 + 6b^2c^3\alpha r_x^2 + 6c^2\beta r_x^2 + 12c^3\ell\beta r_x^2 - 12b^2c^3\alpha\beta r_x^2 - 12c^2\beta^2r_x^2 - 6c^3\ell\beta^2r_x^2 + 6b^2c^3\alpha\beta^2r_x^2 + 6c^2\beta^3r_x^2 + 2c^3r_x^3 - 6c^3\beta r_x^3 + 6c^3\beta^2r_x^3 - 2c^3\beta^3r_x^3 - 3c^2\ell^2\alpha r_y - 9b^2c\alpha^2r_y - 3b^2c^2\ell\alpha^2r_y + 6b^4c^2\alpha^3r_y - 12c\ell\alpha\beta r_y - 6b^2c\alpha^2\beta r_y - 3\alpha\beta^2r_y + 6c^2\ell\alpha r_x r_y + 3b^2c^2\alpha^2r_x r_y - 6c\alpha\beta r_x r_y - 6c^2\ell\alpha\beta r_x r_y - 3b^2c^2\alpha^2\beta r_x r_y + 6c\alpha\beta^2r_x r_y - 3c^2\alpha r_x^2r_y + 6c^2\alpha\beta r_x^2r_y - 3c^2\alpha\beta^2r_x^2r_y + 3c\ell\alpha^2r_y^2 + 6b^2c\alpha^3r_y^2 - 3\alpha^2\beta r_y^2 - 3c\alpha^2r_x r_y^2 + 3c\alpha^2\beta r_x r_y^2 + 2\alpha^3r_y^3 + \sqrt{\left(-2c^3\ell^3 + 9b^2c^2\ell\alpha + 6b^2c^3\ell^2\alpha - 9b^4c^2\alpha^2 - 6b^4c^3\ell\alpha^2 + 2b^6c^3\alpha^3 - 3c^2\ell^2\beta + 18b^2c\alpha\beta + 6b^2c^2\ell\alpha\beta - 3b^4c^2\alpha^2\beta + 3c\ell\beta^2 - 3b^2c\alpha\beta^2 + 2\beta^3 + 6c^3\ell^2r_x - 9b^2c^2\alpha r_x - 12b^2c^3\ell\alpha r_x + 6b^4c^3\alpha^2r_x - 3c^2\ell\beta r_x - 6c^3\ell^2\beta r_x + 12b^2c^2\alpha\beta r_x + 12b^2c^3\ell\alpha\beta r_x - 6b^4c^3\alpha^2\beta r_x + 6c\beta^2r_x + 3c^2\ell\beta^2r_x - 3b^2c^2\alpha\beta^2r_x - 6c\beta^3r_x - 6c^3\ell r_x^2 + 6b^2c^3\alpha r_x^2 + 6c^2\beta r_x^2 + 12c^3\ell\beta r_x^2 - 12b^2c^3\alpha\beta r_x^2 - 12c^2\beta^2r_x^2 - 6c^3\ell\beta^2r_x^2 + 6b^2c^3\alpha\beta^2r_x^2 + 6c^2\beta^3r_x^2 + 2c^3r_x^3 - 6c^3\beta r_x^3 + 6c^3\beta^2r_x^3 - 2c^3\beta^3r_x^3 \right)} \right\}$$

$$\begin{aligned}
& 6c^2\beta^3r_x^2 + 2c^3r_x^3 - 6c^3\beta r_x^3 + 6c^3\beta^2r_x^3 - 2c^3\beta^3r_x^3 - 3c^2\ell^2\alpha r_y - 9b^2c \\
& \alpha^2r_y - 3b^2c^2\ell\alpha^2r_y + 6b^4c^2\alpha^3r_y - 12c\ell\alpha\beta r_y - 6b^2c\alpha^2\beta r_y - 3\alpha\beta^2 \\
& r_y + 6c^2\ell\alpha r_x r_y + 3b^2c^2\alpha^2r_x r_y - 6c\alpha\beta r_x r_y - 6c^2\ell\alpha\beta r_x r_y - 3b^2 \\
& c^2\alpha^2\beta r_x r_y + 6c\alpha\beta^2r_x r_y - 3c^2\alpha r_x^2r_y + 6c^2\alpha\beta r_x^2r_y - 3c^2\alpha\beta^2r_x^2r_y + \\
& 3c\ell\alpha^2r_y^2 + 6b^2c\alpha^3r_y^2 - 3\alpha^2\beta r_y^2 - 3c\alpha^2r_x r_y^2 + 3c\alpha^2\beta r_x r_y^2 + 2\alpha^3r_y^3)^2 + \\
& 4\left(-\left(c\ell - b^2c\alpha - \beta - c r_x + c\beta r_x - \alpha r_y\right)^2 - 3\left(-b^2c\alpha + c\ell\beta - \right.\right. \\
& \left.\left. b^2c\alpha\beta + c\ell\alpha r_y - \alpha\beta r_y - c\alpha r_x r_y + c\alpha\beta r_x r_y\right)\right)^3\Big)^{1/3} + \\
& \frac{1}{6 \times 2^{1/3}} \left(1 - i\sqrt{3}\right) \left(-2c^3\ell^3 + 9b^2c^2\ell\alpha + 6b^2c^3\ell^2\alpha - 9b^4c^2\alpha^2 - 6b^4c^3\ell\alpha^2 + \right. \\
& 2b^6c^3\alpha^3 - 3c^2\ell^2\beta + 18b^2c\alpha\beta + 6b^2c^2\ell\alpha\beta - 3b^4c^2\alpha^2\beta + \\
& 3c\ell\beta^2 - 3b^2c\alpha\beta^2 + 2\beta^3 + 6c^3\ell^2r_x - 9b^2c^2\alpha r_x - 12b^2c^3\ell\alpha r_x + \\
& 6b^4c^3\alpha^2r_x - 3c^2\ell\beta r_x - 6c^3\ell^2\beta r_x + 12b^2c^2\alpha\beta r_x + \\
& 12b^2c^3\ell\alpha\beta r_x - 6b^4c^3\alpha^2\beta r_x + 6c\beta^2r_x + 3c^2\ell\beta^2r_x - \\
& 3b^2c^2\alpha\beta^2r_x - 6c\beta^3r_x - 6c^3\ell r_x^2 + 6b^2c^3\alpha r_x^2 + 6c^2\beta r_x^2 + \\
& 12c^3\ell\beta r_x^2 - 12b^2c^3\alpha\beta r_x^2 - 12c^2\beta^2r_x^2 - 6c^3\ell\beta^2r_x^2 + \\
& 6b^2c^3\alpha\beta^2r_x^2 + 6c^2\beta^3r_x^2 + 2c^3r_x^3 - 6c^3\beta r_x^3 + 6c^3\beta^2r_x^3 - \\
& 2c^3\beta^3r_x^3 - 3c^2\ell^2\alpha r_y - 9b^2c\alpha^2r_y - 3b^2c^2\ell\alpha^2r_y + 6b^4c^2\alpha^3r_y - \\
& 12c\ell\alpha\beta r_y - 6b^2c\alpha^2\beta r_y - 3\alpha\beta^2r_y + 6c^2\ell\alpha r_x r_y + \\
& 3b^2c^2\alpha^2r_x r_y - 6c\alpha\beta r_x r_y - 6c^2\ell\alpha\beta r_x r_y - 3b^2c^2\alpha^2\beta r_x r_y + \\
& 6c\alpha\beta^2r_x r_y - 3c^2\alpha r_x^2r_y + 6c^2\alpha\beta r_x^2r_y - 3c^2\alpha\beta^2r_x^2r_y + 3c\ell\alpha^2r_y^2 + \\
& 6b^2c\alpha^3r_y^2 - 3\alpha^2\beta r_y^2 - 3c\alpha^2r_x r_y^2 + 3c\alpha^2\beta r_x r_y^2 + 2\alpha^3r_y^3)^2 + \\
& \sqrt{\left((-2c^3\ell^3 + 9b^2c^2\ell\alpha + 6b^2c^3\ell^2\alpha - 9b^4c^2\alpha^2 - 6b^4c^3\ell\alpha^2 + 2b^6c^3\alpha^3 - 3c^2\ell^2\beta + \right. \\
& 18b^2c\alpha\beta + 6b^2c^2\ell\alpha\beta - 3b^4c^2\alpha^2\beta + 3c\ell\beta^2 - 3b^2c\alpha\beta^2 + 2\beta^3 + \\
& 6c^3\ell^2r_x - 9b^2c^2\alpha r_x - 12b^2c^3\ell\alpha r_x + 6b^4c^3\alpha^2r_x - 3c^2\ell\beta r_x - \\
& 6c^3\ell^2\beta r_x + 12b^2c^2\alpha\beta r_x + 12b^2c^3\ell\alpha\beta r_x - 6b^4c^3\alpha^2\beta r_x + 6c\beta^2r_x + \\
& 3c^2\ell\beta^2r_x - 3b^2c^2\alpha\beta^2r_x - 6c\beta^3r_x - 6c^3\ell r_x^2 + 6b^2c^3\alpha r_x^2 + 6c^2\beta r_x^2 + \\
& 12c^3\ell\beta r_x^2 - 12b^2c^3\alpha\beta r_x^2 - 12c^2\beta^2r_x^2 - 6c^3\ell\beta^2r_x^2 + 6b^2c^3\alpha\beta^2r_x^2 + \\
& 6c^2\beta^3r_x^2 + 2c^3r_x^3 - 6c^3\beta r_x^3 + 6c^3\beta^2r_x^3 - 2c^3\beta^3r_x^3 - 3c^2\ell^2\alpha r_y - \\
& 9b^2c\alpha^2r_y - 3b^2c^2\ell\alpha^2r_y + 6b^4c^2\alpha^3r_y - 12c\ell\alpha\beta r_y - 6b^2c\alpha^2\beta r_y - 3\alpha \\
& \beta^2r_y + 6c^2\ell\alpha r_x r_y + 3b^2c^2\alpha^2r_x r_y - 6c\alpha\beta r_x r_y - 6c^2\ell\alpha\beta r_x r_y - 3b^2 \\
& c^2\alpha^2\beta r_x r_y + 6c\alpha\beta^2r_x r_y - 3c^2\alpha r_x^2r_y + 6c^2\alpha\beta r_x^2r_y - 3c^2\alpha\beta^2r_x^2r_y + \\
& 3c\ell\alpha^2r_y^2 + 6b^2c\alpha^3r_y^2 - 3\alpha^2\beta r_y^2 - 3c\alpha^2r_x r_y^2 + 3c\alpha^2\beta r_x r_y^2 + 2\alpha^3r_y^3)^2 + \\
& 4\left(-\left(c\ell - b^2c\alpha - \beta - c r_x + c\beta r_x - \alpha r_y\right)^2 - 3\left(-b^2c\alpha + c\ell\beta - \right.\right. \\
& \left.\left. b^2c\alpha\beta + c\ell\alpha r_y - \alpha\beta r_y - c\alpha r_x r_y + c\alpha\beta r_x r_y\right)\right)^3\Big)^{1/3}\Big\}
\end{aligned}$$

$$\begin{aligned}
A2 = & -2 c^3 \ell^3 + 9 b^2 c^2 \ell \alpha + 6 b^2 c^3 \ell^2 \alpha - 9 b^4 c^2 \alpha^2 - 6 b^4 c^3 \ell \alpha^2 + 2 b^6 c^3 \alpha^3 - 3 c^2 \ell^2 \beta + \\
& 18 b^2 c \alpha \beta + 6 b^2 c^2 \ell \alpha \beta - 3 b^4 c^2 \alpha^2 \beta + 3 c \ell \beta^2 - 3 b^2 c \alpha \beta^2 + 2 \beta^3 + 6 c^3 \ell^2 r_x - 9 b^2 c^2 \alpha r_x - \\
& 12 b^2 c^3 \ell \alpha r_x + 6 b^4 c^3 \alpha^2 r_x - 3 c^2 \ell \beta r_x - 6 c^3 \ell^2 \beta r_x + 12 b^2 c^2 \alpha \beta r_x + 12 b^2 c^3 \ell \alpha \beta r_x - \\
& 6 b^4 c^3 \alpha^2 \beta r_x + 6 c \beta^2 r_x + 3 c^2 \ell \beta^2 r_x - 3 b^2 c^2 \alpha \beta^2 r_x - 6 c \beta^3 r_x - 6 c^3 \ell r_x^2 + 6 b^2 c^3 \alpha r_x^2 + \\
& 6 c^2 \beta r_x^2 + 12 c^3 \ell \beta r_x^2 - 12 b^2 c^3 \alpha \beta r_x^2 - 12 c^2 \beta^2 r_x^2 - 6 c^3 \ell \beta^2 r_x^2 + 6 b^2 c^3 \alpha \beta^2 r_x^2 + 6 c^2 \beta^3 r_x^2 + \\
& 2 c^3 r_x^3 - 6 c^3 \beta r_x^3 + 6 c^3 \beta^2 r_x^3 - 2 c^3 \beta^3 r_x^3 - 3 c^2 \ell^2 \alpha r_y - 9 b^2 c \alpha^2 r_y - 3 b^2 c^2 \ell \alpha^2 r_y + \\
& 6 b^4 c^2 \alpha^3 r_y - 12 c \ell \alpha \beta r_y - 6 b^2 c \alpha^2 \beta r_y - 3 \alpha \beta^2 r_y + 6 c^2 \ell \alpha r_x r_y + 3 b^2 c^2 \alpha^2 r_x r_y - \\
& 6 c \alpha \beta r_x r_y - 6 c^2 \ell \alpha \beta r_x r_y - 3 b^2 c^2 \alpha^2 \beta r_x r_y + 6 c \alpha \beta^2 r_x r_y - 3 c^2 \alpha r_x^2 r_y + 6 c^2 \alpha \beta r_x^2 r_y - \\
& 3 c^2 \alpha \beta^2 r_x^2 r_y + 3 c \ell \alpha^2 r_y^2 + 6 b^2 c \alpha^3 r_y^2 - 3 \alpha^2 \beta r_y^2 - 3 c \alpha^2 r_x r_y^2 + 3 c \alpha^2 \beta r_x r_y^2 + 2 \alpha^3 r_y^3;
\end{aligned}$$

$$B2 = c \ell - b^2 c \alpha - \beta - c r_x + c \beta r_x - \alpha r_y;$$

$$C2 = -b^2 c \alpha + c \ell \beta - b^2 c \alpha \beta + c \ell \alpha r_y - \alpha \beta r_y - c \alpha r_x r_y + c \alpha \beta r_x r_y;$$

$$E2 = 4 (B2^2 + 3 C2)^3 - A2^2 / . \{b \rightarrow \text{Sqrt}[3 \ell r_y]\};$$

[平方根]

(* Corresponding to $27c^4(\ell-r_x)^2E2$ in the paper,
i.e., it has a factor $27c^4(\ell-r_x)^2$.)

$$D2 = (A2 + \text{Sqrt}[E2])^2;$$

[平方根] [虚数单位]

$$\text{vars} = \{\beta, \text{Subscript}[r, y]\};$$

[下角标]

$$\text{rules11} = \text{CoefficientRules}[\text{Expand}[E2], \text{vars}];$$

[系数规则] [展开]

$$\text{rules12} = \text{Select}[\text{rules11}, \text{Total}[\text{First}@\#] \leq 2 \&];$$

[选择] [总计] [第一个]

$$P1 = \text{FromCoefficientRules}[\text{rules12}, \text{vars}];$$

[根据系数规则构建多项式]

(* The term of E2 whose order w.r.t r_y is not more than 2. *)

$$\text{Factor}[P1]$$

[因式分解]

$$\text{rules21} = \text{CoefficientRules}[\text{Expand}[A2 / . \{b \rightarrow \text{Sqrt}[3 \ell r_y]\}], \text{vars}];$$

[系数规则] [展开] [平方根]

$$\text{rules22} = \text{Select}[\text{rules21}, \text{Total}[\text{First}@\#] \leq 1 \&];$$

[选择] [总计] [第一个]

$$P2 = \text{FromCoefficientRules}[\text{rules22}, \text{vars}]$$

[根据系数规则构建多项式]

(* The term of A2 whose order w.r.t r_y is not more than 1. *)

Out[]=

$$27 c^4 (\ell - r_x)^2 (\ell^2 \beta^2 + 4 \ell^2 \alpha \beta r_y - 10 \ell \alpha \beta r_x r_y + 4 \ell^2 \alpha^2 r_y^2 + 4 \ell \alpha^2 r_x r_y^2 + \alpha^2 r_x^2 r_y^2)$$

Out[]=

$$\begin{aligned}
& -2 c^3 \ell^3 + 6 c^3 \ell^2 r_x - 6 c^3 \ell r_x^2 + 2 c^3 r_x^3 + \\
& \beta (-3 c^2 \ell^2 - 3 c^2 \ell r_x - 6 c^3 \ell^2 r_x + 6 c^2 r_x^2 + 12 c^3 \ell r_x^2 - 6 c^3 r_x^3) + \\
& (24 c^2 \ell^2 \alpha + 18 c^3 \ell^3 \alpha - 21 c^2 \ell \alpha r_x - 36 c^3 \ell^2 \alpha r_x - 3 c^2 \alpha r_x^2 + 18 c^3 \ell \alpha r_x^2) r_y
\end{aligned}$$

```
p1 = Factor[P2 /. {ry → 0, β → 0}];
```

因式分解

```
p2 = Factor[P2 - p1] (* The term of A2 whose order w.r.t ry is exactly 1. *)
```

因式分解

Out[]:=

$$3 c^2 (\ell - r_x) (-\ell \beta - 2 \beta r_x - 2 c \ell \beta r_x + 2 c \beta r_x^2 + 8 \ell \alpha r_y + 6 c \ell^2 \alpha r_y + \alpha r_x r_y - 6 c \ell \alpha r_x r_y)$$

```
In[ ]:= vecs = Eigenvectors[M];
```

特征向量

```
roots = Union@Cases[vecs[[2]], _Root, All];
```

并集

模式匹配

根

全部

```
v = vecs[[2]] /. Thread[roots → {λ}];
```

逐项作用

```
(* The form of eigenvector corresponding to the eigenvalue λ we chose. *)
```

```
V = {1, v[[2]] / v[[1]], v[[3]] / v[[1]]} // Cancel
```

约简

Out[]:=

$$\left\{ 1, \frac{c \ell \beta + c \ell \lambda - \beta \lambda - \lambda^2 - c \lambda r_x + c \beta \lambda r_x}{b c (\beta + \lambda)}, \frac{\beta (1 + \lambda)}{\beta + \lambda} \right\}$$