

# Lecture 1

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STATS 67 - UCI

***APPROVED***

# Statistics

- Statistics is the mathematical science of learning from data, and of measuring, controlling, and communicating uncertainty.
- It is concerned with developing methods for collecting and analyzing empirical data.
- In many fields of the physical and social sciences, empirical data will naturally have variability and randomness.
- Probability theory provides a substantial part of the underlying framework used to describe variability and randomness, and therefore provides a foundation for the tools developed in statistics.

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# Probability

- In a *frequency* framework, probability of an event ( $P$ ) is defined to be the proportion of times the event is observed under repeated observation.
- Assume we conduct an experiment of flipping a coin  $n$  times. Let the number of heads,  $X$ , be recorded.
- The probability of getting a head from flipping the coin is  $P = \lim_{n \rightarrow \infty} \frac{X}{n}$ .
- It can be viewed as the long run average of the number of success'.
- If we flip the coin a very large number of times, the proportion of success' ( $\frac{X}{n}$ ) will converge to the true probability of a single success. This is a loose statement of the *law of large numbers*.
- When the event cannot be repeated, it is a little difficult to intuitively view probability from a frequency standpoint.
  - An example is if it will rain on a specific day.
- As such, there is another interpretation of probability referred to as the *Bayesian* interpretation.
- In this framework, the probability of an event is the degree of one's belief (between 0 and 1) the event will occur.
  - Example: There is a 24% chance it will rain tomorrow.
  - Example: There is a 99% chance that a certain subject will recover from surgery.

# Probability

For now, let us only concern ourselves with discrete and categorical outcomes.

- The set of all possible outcomes in a random experiment is the **Sample Space**,  $S$ .
- Determine the sample space for the following situations.
  - Example: Flip a coin once.
  - Example: Roll a die once.
  - Example: Flip a coin twice.
  - Example: Flip a coin 3 times.

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

- Example: Roll a pair of dice.

$$S = \{(1, 1), (1, 2), (1, 3), \dots, (6, 5), (6, 6)\}.$$

or

$$S = \{(x, y) : x = 1, 2, \dots, 6, y = 1, 2, \dots, 6\}.$$

# Probability

An **event**,  $A$ , is a subset of the sample space. Also known as a *sample point*.

**Examples** Determine the event space for the following situations.

- Assume you roll a die one time. Let the event  $A$  be the event that the number the die lands on is even.
- Assume you roll a die one time. Let the event  $A$  be the event that the number the die lands on is greater than 4.
- Assume you flip a coin three times. Let the event  $B$  be the event that more than 1 tail appears.
- Assume you flip a coin three times. Let the event  $B$  be the event that the first coin flip is a head.

# Set Theory Basics

Let  $A$  and  $B$  be sets.

- $\emptyset$  is the **Empty Set**. A set that has no elements in it. I.e.  $\emptyset = \{\}$ .
- $A$  is a **Subset** of  $B$  if  $s \in A$  implies  $s \in B$ . That is to say whatever is in  $A$  is also in  $B$ .
  - Notationally this is presented as  $A \subset B$ .
  - Example:  $A = \{1, 2, 3\}$  and  $B = \{1, 2, 3, 4\}$ .
- The **Union** of  $A$  and  $B$  is denoted as  $A \cup B$ . When translating we say  $A$  or  $B$ .
  - If  $s \in A$  or  $s \in B$ , then  $s \in A \cup B$ .
  - Example: Let  $S = \{1, 2, 3, 4, 5, 6\}$ ,  $A = \{1, 2, 3\}$ , and  $B = \{3, 4, 5\}$
  - Find  $A \cup B$
- The **Intersection** of  $A$  and  $B$  is denoted as  $A \cap B$ . It is the overlap of the two sets. When translating we say  $A$  and  $B$ .
  - If  $s \in A \cap B$ , then  $s \in A$  and also  $s \in B$ .
  - Example: Let  $S = \{1, 2, 3, 4, 5, 6\}$ ,  $A = \{1, 2, 3\}$ , and  $B = \{3, 4, 5\}$
  - Find  $A \cap B$

## Set Theory Basics

Let  $A$  and  $B$  be sets.

- The **Complement** of  $A$  is denoted as  $A^c$ . It is the collection of elements that are not in  $A$ . When translating we say not in  $A$ .

- If  $s \in A$ , then  $s \notin A^c$ .
- Example: Let  $S = \{1, 2, 3, 4, 5, 6\}$ ,  $A = \{1, 2, 3\}$ , and  $B = \{3, 4, 5\}$
- Find  $A^c$

- Note  $(A^c)^c = A$  and  $A \cup A^c = S$ .

- $(A \cup B)^c = A^c \cap B^c$        $(A^c \cup B)^c = A \cap B^c$        $(A^c \cup B^c)^c = A \cap B$

- Example: Let  $S = \{1, 2, 3, 4, 5, 6\}$ ,  $A = \{1, 2, 3\}$ , and  $B = \{3, 4, 5\}$
- Find  $B^c$

- Find  $A^c \cap B^c$

- Find  $(A \cup B)^c$

- $(A \cap B)^c = A^c \cup B^c$        $(A^c \cap B)^c = A \cup B^c$        $(A^c \cap B^c)^c = A \cup B$

## Set Theory Basics

$$(A \cup B)^c = A^c \cap B^c$$

- Example: Assume you roll a die one time. Let  $A$  be the event you roll a 1 or 2 on a die, and  $B$  is the event you roll a 3 or a 4.
  - The event you don't roll a 1 or a 2 NOR a 3 or a 4 is  $(A \cup B)^c$ .
- Example: Let  $A$  be the event someone has blue eyes and  $B$  be the event they are a computer science major.
  - The event that someone is not blue eyed nor a computer science major is  $(A \cup B)^c$

$$(A \cap B)^c = A^c \cup B^c$$

- Example: Let  $A$  be the event someone has blue eyes and  $B$  be the event they are a computer science major.
  - The event that someone is not blue eyed and a computer science major is  $(A \cap B)^c = A^c \cup B^c$ .



## Set Theory Basics

Let  $A$  and  $B$  be sets.

- We say that sets (or events) are **Mutually Exclusive** if the two sets (or events) cannot occur at the same time.
- Notationally this is  $A \cap B = \emptyset$ .
  - Example: Assume you roll one die once. Let  $A$  be the event that the number showing on the die is odd and  $B$  be the event that the number is a 2. Find  $A \cap B$
- We say events  $A$  and  $A^c$  form a *partition* of the sample space if they are mutually exclusive and if  $A \cup A^c = S$ .
- If  $A$  and  $A^c$  partition their sample space, then the event  $B$  can be written as follows:  $B = (B \cap A) \cup (B \cap A^c)$ . To be loose with notation, can think of it as  $B = (B \cap A) + (B \cap A^c)$ .

## Set Theory Basics

Let  $A_1, A_2, A_3, \dots$  be sets. (You can also think of sets  $A, B, C, \dots$ ).

- We say that sets (or events) are mutually exclusive if the intersection between any of these two sets is the null set.
- Notationally  $A_i \cap A_j = \emptyset$  for all  $i \neq j$ .
- **Example:** Let  $A_1$  be the event it rains tomorrow,  $A_2$  be the event you have an exam tomorrow, and  $A_3$  be the event you are late to class.
- Write in set notation the event that it does not rain tomorrow and you have an exam and you are late to class.
- What does  $A_1 \cup (A_2 \cap A_3)$  mean in words?

## Probability Theory Basics

A **probability distribution** is the rule that assigns a number ( $P(\cdot)$ ) to each possible outcome in the sample space ( $s \in S$ ), with the following conditions.

- $0 \leq P(s) \leq 1$  for all  $s \in S$
- $\sum_{s \in S} P(s) = 1$
- As an example, assume you roll a die one time. Each event in  $S = \{1, 2, 3, 4, 5, 6\}$  has probability of  $1/6$ .
  - Note: The sum of all the probabilities is equal to 1  $\left(\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}\right)$ .
  - Thus, we call this a valid probability distribution.

Let  $A$  and  $B$  be events in the sample space  $S$ .

- $P(S) = 1$ .
- If  $A \subset B$  then  $P(A) \leq P(B)$ .
- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .
  - In the equation above, solve for  $P(A \cap B)$ .

- $P(A) + P(A^c) = 1$ 
  - As a result:  $P(A) = 1 - P(A^c)$ .

## Probability Theory Basics

Let  $A$  and  $B$  be events in the sample space  $S$ .

- $P(\emptyset) = 0$ .
- Note that this means that  $A$  and  $B$  are mutually exclusive if and only if  $P(A \cap B) = 0$ .
- If the sets  $A$  and  $A^c$  partition their respective sample space, then  $P(B) = P(B \cap A) + P(B \cap A^c)$ .

EXAMPLE: Let  $A$  and  $B$  be events in the sample space  $S$ .

Draw a venn-diagram for each probability:

$$P(B)$$

$$P(A)$$

$$P(A \cap B)$$

$$P(A \cup B)$$

EXAMPLE: Let  $A$  and  $B$  be events in the sample space  $S$ . Draw a venn-diagram for each probability:

$$P(A^c)$$

$$P(A \cap B^c)$$

$$P(A^c \cap B)$$

$$P(A \cup B)^c$$

$$P(A \cap B)^c$$

$$A_1, A_2, A_3 \text{ Partition } S$$

## Probability Theory Basics

Example: Assume we sample UCI Information and Computer Science students. Let  $P(A) = 0.7$  where  $A$  is the event someone is an Undergrad. Let  $P(B) = 0.8$  where  $B$  is the event someone is a computer science major. And let  $P(A \cap B) = 0.6$ .

- What is the probability someone is a grad student?
- What is the probability someone is an undergrad or a computer science major?
- What is the probability someone is a grad student and a computer science major?
- What is the probability someone is a computer science grad student?

## Probability Theory Basics

Let sets  $A_1, A_2, A_3, \dots, A_M$  (or can think of sets  $A, B, C, \dots, M$ ) be mutually exclusive events.

- $P(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_M) = P(A_1) + P(A_2) + P(A_3) + \dots + P(A_M)$ .
- If  $A_1, A_2, A_3, \dots, A_M$  form a partition of the sample space, then  $P(A_1 \cup A_2 \cup A_3 \cup \dots \cup A_M) = P(S) = 1$ .

**Example:** Assume we roll a die once and  $A_1 = \{1, 2\}$ ,  $A_2 = \{3, 4\}$ , and  $A_3 = \{5, 6\}$ .

- Use the addition formula to find the probability of rolling a 1, 2, 3, or 4? Make sure to use proper statistical notation.