Fredholm Operators over Hilbert C^*-Modules

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Chapter 1

Introduction

Nam dui ligula, fringilla a, euismod sodales, sollicitudin vel, wisi. Morbi auctor lorem non justo. Nam lacus libero, pretium at, lobortis vitae, ultricies et, tellus. Donec aliquet, tortor sed accumsan bibendum, erat ligula aliquet magna, vitae ornare odio metus a mi. Morbi ac orci et nisl hendrerit mollis. Suspendisse ut massa. Cras nec ante. Pellentesque a nulla. Cum sociis natoque penatibus et magnis dis parturient montes, nascetur ridiculus mus. Aliquam tincidunt urna. Nulla ullamcorper vestibulum turpis. Pellentesque cursus luctus mauris.

Chapter 2

K-theory of Banach Algebras

2.1 General portrait of homological theories

A homological theory for a category $\mathcal{C}$ consist in a sequence of covariant functors $H_n : \mathcal{C} \longrightarrow \mathbf{GrpAb}$ for each $n \in \mathbb{N}$ which satisfies some set of axioms, which depends on what theory one is interested. For example, if $\mathcal{C}$ contains a nice homotopical concept, its rather common to ask for homotopical invariance. If exact sequences naturally pops in the domain encoding a lot of information, some other axioms are required to obtain long exact sequences. The usual notation is:

$$\begin{tikzcd}
H_n\colon \mathcal{C} \arrow[r] & \mathbf{GrpAb} \\
A \arrow[r, mapsto] \arrow[d, "\phi"'] & H_n(A) \arrow[d, "\phi_n"'] \\
B \arrow[r, mapsto] & H_n(B)
\end{tikzcd}$$

We also need a way to translate short exact sequences of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

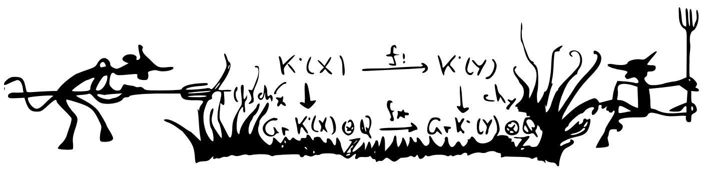
from the original category to higher counterparts obtained by H_n, hence, every homology theory seeks to define a connecting morphism $\delta_n : H_n(C) \longrightarrow H_{n+1}(A)$ into a long exact sequence:

$$\begin{tikzcd}
H_0(A) \arrow[r] & H_0(B) \arrow[r] & H_0(C) \arrow[dll, rounded corners, "\delta_0"] \\
H_1(A) \arrow[r] & H_1(C) \arrow[r] & H_1(B) \arrow[dll, rounded corners, "\delta_1"] \\
H_2(A) \arrow[r] & H_2(B) \arrow[r] & \cdots
\end{tikzcd}$$

In the other hand, as everything containing the prefix "co", cohomology theories are consisted of contra-variant functors $(H^n)_n$ with the same pay-off. The position of the index on the notation usually indicates what sort of theory one is dealing with.

Here we are concerned with a homology theory for complex Banach Algebras $\boldsymbol{\mathscr{B}}$-**Alg** or, more popularly, for C^*-algebras $\boldsymbol{C^*}$- **Alg**, a.k.a., K-theory for Operator Algebras. It is the mirror image of Topological K-theory, in light of *Gelfand Duality* connecting the category of Locally Compact Hausdorff spaces and complex abelian C^*-algebras, but not restricted to commutative spaces, which is often referred to as the "Non-Commutative Topology".

In his work to reformulate Riemman-Roch theorem [4], *A. Grothendieck* introduces the group $K(A)$ associated to a subcategory of an abelian category, which nowadays, it is the so called "Grothendieck's group". That's where is from the letter K, which he had chosen for "Klassen". His reformulation famously contains his legendary drawing:



Riemann-Roch Theorem: the new black: The diagram

[...] *is commutative!*

I would need to misuse about $2h$ *of my listener's time in order to impart only an approximal understanding of this statement for* $f : X \longrightarrow Y$.

In cold print (as in Springer's lecture notes) this would take around 400-500 *pages.*

A thrilling example of how our urge for knowledge and discoveries decays into a lifeless and ideological delirium while life itself goes thousandfold to the devil - and is threatened with final destruction.

It's high time to change our course!

(16.12.1971) Alexander Grothendieck

With his work settled in what is about to be algebraic K-theory, topological K-theory would be a product of *M. Atiyah* and *F. Hirzebruch* replicating Grothendieck's construction for topological vector bundles over compact Hausdorff spaces.

We'll construct functors $K_n : \mathscr{B}\text{-}\mathbf{Alg} \longrightarrow \mathbf{GrpAb}$ and the connecting maps will be call *index map*, denoted by ∂. A remarkable aspect of operator K-theory is the *Bott periodicity*: $K_n \simeq K_{n+2}$, which then describes for any short exact sequence $0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$, where $I \lhd A$, a six-term exact sequence:

$$\begin{tikzcd} K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[d, "\partial"] \\ K_1(A/I) \arrow[u, "\partial"] & K_1(A) \arrow[l] & K_1(I) \arrow[l] \end{tikzcd}$$

The details will be spared in what is outside of our scope, which will include the definition of the groups K_0 and K_1 for complex Banach algebras, and the index map mentioned. The connecting map will be used in the classfication of finite rank modules, and later on, the index of our Fredholm operators. Hence it is important to define it in a helpfull way.

2.2 The K_0-group

Our object is to deal with Hilbert C^*-modules, witch are right A-modules with a generalized A-valued inner product for a given C^*-algebra A (plus some other details), generalizing the concept of Hilbert space. Therefore, it is reasonable to understand some K-theory for C^*-algebras, as they are our underlying space. Unfortunately, as we'll see later on, there is no *Riesz representation lemma* and, there exists bounded linear operators that aren't adjointable between Hilbert modules. Hence, dealing with self-adjoint operators is a restriction for sure. Thankfully, the K-theory for Banach algebras is good enougth in order to fill our needs.

In topological K-theory, in order to define the $0^{\underline{\text{th}}}$ K group, one would consider a complex vector bundle E over $X \in \mathbf{CHaus}$ and take the right $C(X)$-module $\Gamma(X, E)$ of continuous sections $s : X \longrightarrow E$ with pointwise scalar multiplication[1]. Compactness of X implies that $\Gamma(X, E)$ is a projective $C(X)$-module, and *Serre-Swan* theorem [14, Thr. 6.18] states that $E \longmapsto \Gamma(X, E)$ induces an equivalence between the category of complex vector bundles and finitely generated projective $C(X)$-modules. Hence, $K^0(X)$ is the *Grothendieck group* of the set of equivalence classes of isomorphisms between vector bundles over X.

For a given Banach Algebra A, the following definitions and constructions mimics the above paragraph, by replacing vector bundles by finitely generated projective A-modules.

Definition 2.2.1. In any given Banach algebra A, for two idempotent elements x and y, define the following notions of equivalence:

[1] That is to say, for $f \in C(X)$ and $s \in \Gamma(X, E)$, let $x \longmapsto s(x)f(x)$.

(i) **Murray-von-Neuman equivalent**: There are elements $p, q \in A$ such that $x = pq$ and $y = qp$.

(ii) **Similarly equivalent**: There exists an invertible[2] element $u \in \mathrm{GL}(A)$ such that $x = u^{-1}yu$.

(iii) **Homotopic**: There are a continuous path $\gamma \in C([0,1], A)$ of idempotents between x and y, i.e.,

$$\gamma(0) = x, \gamma(1) = y \quad \text{and} \quad \forall\, t \in [0,1], \gamma(t)^2 = \gamma(t).$$

If A is assured to be a C^*-algebra, those definitions are concerned with self-adjoint idempotent elements, a.k.a., projections. Two projections x, y are equivalent if there exists a *partial isometry* u such that $x = u^*u$ and $y = uu^*$.

For the canonical embedding $x \longmapsto \operatorname{diag}(x, 0)$ over matrices, consider the inductive limit $\mathbb{M}_\infty(A) := \varinjlim_{n\in\mathbb{N}} \mathbb{M}_n(A)$, which can be seen as the set of infinite matrices over A but only finitely many of the entries are non-zero.

***Remark* 2.2.2.** Note that $\mathbb{M}_\infty(A)$ contains no unity, but that doesn't stop us to declaring two elements x, y to be similar when they are similar in some square matrix space $\mathbb{M}_n(A)$. Therefore, all equivalence relations listed in the definition 2.2.1 coincide in $\mathbb{M}_\infty(A)$.

Simply shouting "Let A be an C^*-algebra" in the crowd is a powerfull classification tool, whenever is a mathematicians crowd[3].

(i) If you hear in response "unital or not?", you know that there is some C^*-algebraic fellow around you.

(ii) If the crowd contains mathematicians and no-one ask wetter A contains a unity or not, no C^*-algebraist is contained in the crowd. They are instantly assuming the unity is there.

This is because dealing without unital rings outside C^*-theories are usually simple. Just unitize and go on. However, the presence of unity in C^*-algebras is crucial to determine theyr underlying hidden topology, as explicitly is made in *Gelfand's duality* theorem.

The next definition is in charge to define the functor K_0 for both cases, but some intermediate steps are required from one to another.

Definition 2.2.3. Let A be a Banach algebra. The set of equivalence classes over $\mathbb{M}_\infty(A)$ considering any relation $\sim$ contained in 2.2.1 is an abelian semigroup with $[x] + [y] := [\operatorname{diag}(x, y)]$. Before defining K_0, in order to include

[2] Assuming that A is unital.

[3] Otherwise, you are just playing creepy at dinner table again.

the non necessarily unital algebras, it is needed to be considered an auxiliar functor K_{00} much closer to the topological counterpart K^0. This is necessary in order to obtain the Bott periodicity result for Banach algebras, and other good functorial properties.

(*i*) $\boldsymbol{K_{00}}$: It is the Grothendieck group construction associated with the semi-group $V(A) := \mathbb{M}_\infty(A)/\!\sim$ where addition is given by $[x]+[y] := [\operatorname{diag}(x,y)]$, generalising the construction of $\mathbb{Z}$ from $\mathbb{N}$ considering formal differences. In lighter sheets, for pairs (a,b) and (c,d) of elements in $V(A)$, let $(a,b) \equiv (c,d)$ whenever there exists[4] $z \in V(A)$ such that $a+d+z = c+b+z$. This is an equivalence relation over the pais, and $[\,\cdot\,]_{00}$ will denote the related equivalence class.

We are mimicking the formal differences construction, so it's natural to define the addition operation coordinate-wise and let $x - y := [(x,y)]_{00}$. Therefore, it is well defined the folowing covariant functor:

$$
\begin{array}{rccc}
K_{00}\colon & \mathscr{B}\textbf{-Alg} & \longrightarrow & \textbf{GrpAb} \\
& A & \longmapsto & V(A)\times V(A)/\!\equiv \qquad x-y \\
& \phi\big\downarrow & & \phi_{00}\big\downarrow \qquad\qquad \big\downarrow \\
& B & \longmapsto & V(B)\times V(B)/\!\equiv \qquad \phi(x)-\phi(y)
\end{array}
$$

Since every element in $V(A)$ is the class of some idempotent matrix p, we can state that every element in $K_{00}(A)$ is on the form $[p]_{00} - [q]_{00}$. Two formal differences $[p]_{00} - [q]_{00}$ and $[x]_{00} - [y]_{00}$ coincide in $K_{00}(A)$ precisely when the operators $\operatorname{diag}(p,y)$ and $\operatorname{diag}(x,q)$ are *stably* homotopic.

(*ii*) $\boldsymbol{K_0}$: In our next step, it's crucial to know exactly who $K_{00}(\mathbb{C})$ is. Hence, remeber that two idempotents in $\mathbb{M}_n(\mathbb{C})$ are similar if, and only if, their images has the same dimension. Therefore $V(\mathbb{C}) \simeq \mathbb{N}$, and by historical nightmares with Analysis I exercises constructing the integer numbers, it is easy to infer that $K_{00}(\mathbb{C}) = \mathbb{Z}$.

For non necessarily unital A, consider $\widetilde{A} := A \oplus \mathbb{C}$ the *unifization* of A and the complex projection $\varepsilon : \widetilde{A} \twoheadrightarrow \mathbb{C}$, which induces the short exact sequence:

$$
0 \longrightarrow A \hookrightarrow \widetilde{A} \xrightarrow{\ \varepsilon\ } \mathbb{C} \longrightarrow 0
$$

The urge to obtain Bott periodicity theorem for Banach algebras, which is a relation between K_0 and K_1 in the presence of short exact

[4]Since it is only a semi-group, the cancelation property do not hold necessarily over $V(A)$. One might check that this is the case if, and only if, the inclusion of $V(A)$ at the Grothendieck's associated group is injective.

sequences, will obligate the exactness of the following:

$$0 \longrightarrow K_0(A) \hookrightarrow K_0(\widetilde{A}) \xrightarrow{\varepsilon_0} K_0(\mathbb{C}) \longrightarrow 0$$

Since it is a morphism between unital Banach algebras, the induced map $\varepsilon_{00} : K_{00}(\widetilde{A}) \longrightarrow \mathbb{Z}$ is a well defined morphism, hence, it is possible to define the following:

$$\begin{array}{rccc} K_0\colon & \mathscr{B}\textbf{-Alg} & \longrightarrow & \textbf{GrpAb} \end{array}$$

$$\begin{array}{ccccc} A & \longmapsto & \ker(K_{00}(\widetilde{A}) \to \mathbb{Z}) & & a+z \\ {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\phi_0}\big\downarrow & & \big\downarrow \\ B & \longmapsto & \ker(K_{00}(\widetilde{B}) \to \mathbb{Z}) & & \phi(a)+z \end{array}$$

Notice that $K_0(A)$ is precisely the set of elements $[p]_0 - [q]_0 \in K_{00}(\widetilde{A})$ such that $\varepsilon(p) \sim \varepsilon(q)$. If A is allready unital, it is possible to show that $K_0(A) \simeq K_{00}(A)$.

***Remark* 2.2.4.** The argument to show that $V(\mathbb{C}) \simeq \mathbb{N}$ is equivalent for compact operators in a infinite-dimensional Hilbert space H, i.e., $V(\mathscr{K}(H)) \simeq \mathbb{N}$, hence $K_0\mathscr{K}(H) = \mathbb{Z}$. On the other hand, any two infinite rank projections in $\mathscr{B}(H)$ are equivalent, hence $V\mathscr{B}(H) \simeq \mathbb{N}\cup\{\infty\}$, which is a semi-group without the cancellation property. Since everyone is equivalent to ∞, it is obtained that $K_{00}\mathscr{B}(H) \simeq 0$. The semi-group $V(A)$ posses the cancellation property if, and only if, the inclusion $V(A) \hookrightarrow K_{00}(A)$ is injective.

Proposition 2.2.5 (Standard portrait of K_0)**.** Every element of $K_0(A)$ can be written as $[x+p_n]_0 - [p_n]_0$.

Proof. Let $p, q \in \mathbb{M}_\infty(\widetilde{A})$ be some idempotent square matrices with order no longer than n, such that $\varepsilon_0([p]_0 - [q]_0) = 0$, i.e., $[p]_0 - [q]_0 \in K_0(A)$. Matrices $p \in \mathbb{M}_n(\widetilde{A})$ can be written as $(p_A, p_\mathbb{C}) \in \mathbb{M}_n(A) \oplus \mathbb{M}_n(\mathbb{C})$, i.e., an algebraic part p_A and a scalar part $p_\mathbb{C}$. Stating that $\varepsilon_0([p]_0 - [q]_0) = 0$ means that the scalar parts of p and q coincide.

The identity $I_n \in \mathbb{M}_\infty(\widetilde{A})$ can be seen as the projection operator of the first n-th coordinates, by filling it with 0's, but to avoid confusions, let it be p_n. With $y \leqslant x$ be given by $xy = yx = y$, one may see that $p < p_n$ and $y < p_n$. Notice that $\mathrm{diag}(0,p) \in \mathbb{M}_{2n}(\widetilde{A})$ is similar to p and orthogonal to I_n, i.e.,

$$\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}\begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} = 0.$$

Hence, $x \coloneqq \mathrm{diag}(-q,p)$ is such that $x + p_n$ is an idempotent operator and:

$$\begin{aligned} [x+p_n]_0 - [p_n]_0 &= [\mathrm{diag}(0,p)]_0 + [p_n - q]_0 - ([p_n - q]_0 + [q]_0) \\ &= [p]_0 - [q]_0\,. \end{aligned}$$

□

2.3 The K_1-group

While K_0 is build upon equivallence classes of idempotent, K_1 uses invertible elements, but simpler. Therefore, let $\mathrm{GL}_\infty(A) \coloneqq \varinjlim_{n\in\mathbb{N}} \mathrm{GL}_n(A)$ considering the embedding $x \longmapsto \operatorname{diag}(x, 1)$. Calculus is back, and we shall consider exponentials inside a unital algebra A:

$$\exp(a) \coloneqq \sum_{n=0}^{\infty} \frac{a^n}{n!} \quad \text{and} \quad \log(1+a) \coloneqq \sum_{n=1}^{\infty} -\frac{a^n}{n} \qquad (a \in A)$$

where the log is defined whenever $\|a\| < 1$. This is the case since elements of the form $z - a$ for complex z are invertible if $\|a\| \leqslant |z|$. If a and b doesn't commute, $\exp(a)\exp(b) \neq \exp(a+b)$, which means that the set of exponentials isn't closed by multiplications.

Lemma 2.3.1. For a unital Banach algebra A, the component of the unity is the group generated by $\{\exp(a) \mid a \in A\} \subset \mathrm{GL}(A)$, denoted by $\exp(A)$.

Proof. Let $\mathrm{GL}^{(0)}(A)$ be the refered set of connected components of 1. Notice that $t \longmapsto \exp(tb)$ for $t \in [0,1]$ is a continuous path of invertible elements between 1 and $\exp(b)$ for any b, hence $\exp(A) \subset \mathrm{GL}^{(0)}(A)$. It remais only to show the converse inclusion.

For some a with $\|1 - a\| < 1$, let $b \coloneqq \log(1 + (a-1)) = \log(a)$, i.e., $a = \exp(b)$. Therefore, if $u \in \mathrm{GL}(A)$ and $\|v - u\| < \|u^{-1}\|^{-1}$, this means that $v = \exp(b)u$ for some b. From this treatment, if follows that $\exp(A)$ is a open and closed topological subspace of $\mathrm{GL}^{(0)}(A)$ which contains the unity, i.e., $\mathrm{GL}^{(0)}(A)$ coincides with $\exp(A)$. $\square$

***Remark* 2.3.2.** Let $M \in \mathrm{GL}_n(\mathbb{C})$. Since 0 cannot be an eigenvalue of M (which is a finite set), it's possible to find $\alpha \neq 0$ such that $[0,\infty)\cdot\alpha$ doesn't contains any of the eigenvalues of M or 1. Therefore, $1 - \alpha t \neq 0$ for all $t \geqslant 0$ and $M_t \coloneqq (1-\alpha t)^{-1}(M - t\alpha I_n)$ is a continuous path from M to the identity, i.e., $\mathrm{GL}_n(\mathbb{C})$ is connected.

In a not so long future, the following result will be important in the presence of an ideal $I \lhd A$, the considering of the projection $A \twoheadrightarrow A/I$.

Corollary 2.3.3. Any continuous surjection $A \longrightarrow B$ induces a lift from every element in $\mathrm{GL}_n^{(0)}(B)$ to one in $\mathrm{GL}_n^{(0)}(A)$.

Proof. Using 2.3.1, write $\prod_i \exp(b_i) \in \mathrm{GL}_n(B)^{(0)}$ for any desired element. Since there is a surjection, there exists lifts to each b_i, i.e., $a_i \in \mathrm{GL}_n(A)$ such that $\prod_i \exp(a_i) \in \mathrm{GL}_n(A)_0$. $\square$

Considering the homotopy equivalence relation, two elements in $\mathrm{GL}_\infty(A)$ are homotopical whenever they are in the same connected component in some $\mathrm{GL}_n(A)$. Denote the equivalence class by $[\,\cdot\,]_1$. Whence, the quotient

$\mathrm{GL}_\infty(A)/\mathrm{GL}_\infty^{(0)}(A)$ is an abelian group with the multiplication $[x]_1[y]_1 = [xy]_1$, which is commutative once you note it is possible to find a connected path[5] between $\mathrm{diag}(y,1)$ and $\mathrm{diag}(1,y)$, hence

$$[x]_1[y]_1 = [xy]_1 = \left[\begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix}\right]_1 = \left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right]_1 = \left[\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right]_1$$

and similarly, one shows that $[xy]_1 = [\mathrm{diag}(y,x)]_1 = [y]_1[x]_1$. There we have, our $K_1(A)$ group. Since $\mathrm{GL}_n(\mathbb{C})$ is connected, it follows immediately that $K_1(\mathbb{C}) = 0$ and, therefore, we can deal with units the way it is intended: for non necessarily unital algebras A, let $K_1(A) \coloneqq K_1(\widetilde{A})$.

Definition 2.3.4. The functor K_1 can be seen as the following:

$$\begin{array}{cccc} K_1\colon & \boldsymbol{\mathscr{B}}\textbf{-Alg} & \longrightarrow & \textbf{GrpAb} \\ & A & \longmapsto & \mathrm{GL}_\infty(\widetilde{A})\big/\mathrm{GL}_\infty^{(0)}(\widetilde{A}) \qquad [x]_1 \\ & {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\phi_1}\big\downarrow \qquad\qquad \big\downarrow \\ & B & \longmapsto & \mathrm{GL}_\infty(\widetilde{B})\big/\mathrm{GL}_\infty^{(0)}(\widetilde{B}) \quad [\phi(x)]_1 \end{array}$$

***Remark* 2.3.5.** It should be stated that if one is dealing with a C^*-algebra, then K_1 can be obtained by the set of unitary matrices $U_n(A)$, i.e., $u^* = u^{-1}$; Since $U_n(A)/U_n^{(0)}(A) \cong \mathrm{GL}_n(A)/\mathrm{GL}_n^{(0)}(A)$, one can obtain a deformation retraction from $\mathrm{GL}_n(A)$ to $U_n(A)$ by the polar decomposition, hence, $K_1(A)$ is isomorphic to $U_\infty(A)/U_\infty^{(0)}(A)$.

2.4 The index map

We are now ready to define the so called index map. This name comes from the Fredholm operator theory since what we are about to construct is a generalization of the index of those operators. Consider $\mathscr{B}(H)$ the C^*-algebra of bounded operators between a Hilbert space H, and $\mathscr{K}(H)$ the ideal of compact operators. The *Atikinson* theorem states precisely that the *Calkin* algebra $\mathscr{Q}(H) \coloneqq \mathscr{B}(H)/\mathscr{K}(H)$ is a classifying one: T is a Fredholm operator if, and only if, $(T \bmod \mathscr{K}(H)) \in \mathrm{GL}\,\mathscr{Q}(H)$.

Since $K_0\mathscr{K}(H) = \mathbb{Z}$ and $K_1\mathscr{Q}(H)$ can be seen the set of Fredholm operators up to homotopy[6], the index map $\textit{ind} : K_1\mathscr{Q}(H) \longrightarrow K_0\mathscr{K}(H)$ is well defined. Our index map ∂ will generalize this application.

[5] Let $z(\theta) = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ be the rotation matrix by some angle θ. Therefore, the continuous map $[0,\pi/2] \ni \theta \longrightarrow z(\theta)\,\mathrm{diag}(y,1)z(\theta)^{-1}$ is the desired path.

[6] Remind that two Fredholm operators in the same realm have the same index if, and only if they are homotopic.

Construction 2.4.1. Let $I \triangleleft A$ and consider the following short exact sequence:

$$0 \longrightarrow I \hookrightarrow A \twoheadrightarrow {}^{A}\!/_{I} \longrightarrow 0$$

We are in position to construct $\partial : K_1(A/I) \longrightarrow K_0(I)$. For $[x]_1 \in K_1(A/I)$, let n be such that $x \in \mathrm{GL}_n(\widetilde{A/I})$. It's about time to the corollary 2.3.3 to shine: Since the projection $A \twoheadrightarrow A/I$ is a continuous surjection, so it is the unifisation induced morphism between the algebras, hence, one can lift the element $\mathrm{diag}(x, x^{-1}) \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A/I})$ to some $w \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A})$.

If $\pi : \mathrm{GL}_\infty(\widetilde{A}) \twoheadrightarrow \mathrm{GL}_\infty(\widetilde{A/I})$ is the quotient projection, notice that $\pi(wp_nw^{-1}) = p_n$, so that $wp_nw^{-1} \in \widetilde{I}$. Since wp_nw^{-1} is also an idempotent, notice that $[wp_nw^{-1}]_0 - [p_n]_0 \in K_0(I)$. And this is the image of the index map ∂ of some element $[x]_1$. ☕

An anxious mind would immediately panic. We have a TO-DO list before calling it a day:

(i) Check that $[wp_nw^{-1}]_0 - [p_n]_0$ doesn't depend on the lift w choosen;

(ii) Check that $\partial([x]_1) = \partial([y]_1)$ for $x \equiv y$.

(iii) Check that ∂ is a group morphism.

Proof of TO-DO list items. If v is another lift of $\mathrm{diag}(x, x^{-1})$, notice that

$$vp_nv^{-1} = (vw^{-1})wp_nw^{-1}(vw^{-1})^{-1},$$

i.e., vp_nv^{-1} is similar to wp_nw^{-1}. This is enough to take care of (i).

In order to show that the index is well defined, suppose that $y \in \mathrm{GL}_n(\widetilde{A/I})$ is another representant of class $[x]_1$. Notice that

$$x^{-1}y \in \mathrm{GL}_n^{(0)}(\widetilde{A/I}) \quad \text{and} \quad \begin{pmatrix} x & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & y^{-1} \end{pmatrix} \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A/I})$$

so by the corollary 2.3.3 again, let $a \in \mathrm{GL}_n^{(0)}(\widetilde{A})$ and $b \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A})$ be the lifts respectively. But then $u := w\,\mathrm{diag}(a, b)$ is a lift of $\mathrm{diag}(y, y^{-1})$. From the fact that p_n commutes with $\mathrm{diag}(a, b)$, it is obtained that $up_nu^{-1} = wp_nw^{-1}$. Since we already show that the choice of lift doesn't matter, (ii) is checked.

For $x, y \in \mathrm{GL}_n(\widetilde{A/I})$, suppose that w is a lift of $\mathrm{diag}(x, x^{-1})$ and v is a lift of $\mathrm{diag}(y, y^{-1})$. Notice that $\varpi := \mathrm{diag}(w, v)$ is a lift of $\mathrm{diag}(x, y, x^{-1}, y^{-1})$, hence

$$\begin{aligned}
\partial([x]_1[y]_1) &= [\varpi p_{2n}\varpi^{-1}]_0 - [p_{2n}]_0 \\
&= \left[\begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}\begin{pmatrix} p_n & 0 \\ 0 & p_n \end{pmatrix}\begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}^{-1}\right]_0 - \left[\begin{pmatrix} p_n & 0 \\ 0 & p_n \end{pmatrix}\right]_0 \\
&= \left[\begin{pmatrix} wp_nw^{-1} & 0 \\ 0 & vp_nv^{-1} \end{pmatrix}\right]_0 - \left[\begin{pmatrix} p_n & 0 \\ 0 & p_n \end{pmatrix}\right]_0 \\
&= [wp_nw^{-1}]_0 - [p_n]_0 + [vp_nv^{-1}]_0 - [p_n]_0 = \partial[x]_1 + \partial[y]_1
\end{aligned}$$

Therefore, it is a group morphism as our final item *(iii)* assures. □

Definition 2.4.2 (Index map in K-theory)**.** Using construction 2.4.1, the group morphism so called *index* map is given by

$$\begin{aligned}
\partial: \quad K_1(A/I) &\longrightarrow K_0(I) \\
[x]_1 &\longmapsto [wp_nw^{-1}]_0 - [p_n]_0
\end{aligned}$$

whenever $x \in \mathrm{GL}_n(\widetilde{A/I})$ and $w \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A})$ is a lift of $\mathrm{diag}(x, x^{-1})$.

Example 2.4.3. In a unital C^*-algebra A, if a unitary[7] idempotent element u in $\mathrm{GL}_n(A/I)$ lifts to $v \in \mathbb{M}_n(A)$, the element

$$w := \begin{pmatrix} v & I_n - v^*v \\ I_n - vv^* & v^* \end{pmatrix}$$

is a lift for $\mathrm{diag}(u, u^{-1})$. Therefore:

$$\begin{aligned}
\partial[u]_1 &= \left[wp_nw^{-1}\right]_0 - [p_n]_0 \\
&= \left[\left(\begin{smallmatrix} v & I_n - v^*v \\ I_n - vv^* & v^* \end{smallmatrix}\right)\left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} v^* & I_n - vv^* \\ I_n - v^*v & v \end{smallmatrix}\right)\right]_0 - \left[\left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]_0 \\
&= \left[\left(\begin{smallmatrix} vv* & 0 \\ 0 & I_n - vv^* \end{smallmatrix}\right)\right]_0 - \left[\left(\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]_0 \\
&= [I_n - v^*v]_0 - [I_n - vv^*]_0
\end{aligned}$$

Notice that when $A = \mathscr{B}(H)$ and $I = \mathscr{K}(H)$, we dealing again with Fredholm operators living in the Calking algebra $\mathscr{Q}(H)$ and ∂ coincides with the Fredholm index.

[7] I.e., $u^* = u^{-1}$.

Chapter 3

Hilbert C^*-modules

Hilbert modules first appear in the work of *I. Kaplaski* [13] and *W. Paschke* [19] later. There are three main areas where Hilbert C^*-modules are heavily used to formulate mathematical concepts envolving:

(i) Induced representations of Morita equivalence [5], [23], [22];

(ii) Kasparov's KK-theory [15];

(iii) C^*-algebraic quantum groups.

In what is tangible to this work, we address the Morita equivalence target by building a Fredholm operator approach between Hilbert modules, introduced by Ruy Exel [10]. Hence, this chapter is responsible for defining and studying those objects.

The material source contains for this chapter contains the well written textbooks like [16], [12], [17].

3.1 The interest object

Definition 3.1.1 (Inner product Module)**.** A right module E over a C^*-algebra (non-necessarily unital) blessed with an generalized inner product $\langle \cdot, \cdot \rangle : E \times E \longrightarrow A$ will be said to be a *Inner product module* when $\langle \cdot, \cdot \rangle$ attends the following properties:

(i) **Twisted A-sesquilinear**: The first coordinate are involuted-linear and the second one linear, i.e.,

$$\begin{cases} \langle x + ya, z \rangle = \langle x, z \rangle + a^* \langle y, z \rangle \\ \langle z, x + ya \rangle = \langle z, x \rangle + \langle z, y \rangle a \end{cases} \qquad \begin{pmatrix} x, y, z \in E \\ a \in A \end{pmatrix}$$

(ii) **A-Hermitian symmetry**: $\langle x, y \rangle = \langle y, x \rangle^*$ whenever $x, y \in E$.

(*iii*) **Positive definite**: For any $x \in E$, $\langle x, x\rangle = 0 \Leftrightarrow x = 0$. By (*ii*), we can say that $\langle x, x\rangle \geqslant 0$ since it is self-adjoint.

One could argue that we only need the inner product to be linear in the second coordinate and by the Hermitian symmetry conclude as a proposition that every inner product over Inner product modules is indeed twisted sesquilinear.

Proposition 3.1.2 (Cauchy-Schwartz inequality)**.** For any Inner product module E over A, the following inequality holds:

$$\|\langle x, y\rangle\|^2 \leqslant \|\langle x, x\rangle\| \cdot \|\langle y, y\rangle\|. \qquad (x, y \in E) \tag{3.1}$$

Proof. Given the fact that $0 \leqslant \langle a, a\rangle$ for $a \in A$, notice that with the acessory elements $a := \langle x, x\rangle$, $b := \langle y, y\rangle$ and $c := \langle x, y\rangle$,

$$\begin{aligned} 0 &\leqslant \langle x - ytc^*, x - ytc^*\rangle \\ &= \langle x, x - ytc^*\rangle - tc\langle y, x - ytc^*\rangle \\ &= \langle x, x\rangle - \langle x, y\rangle\, tc^* - tc\langle y, x\rangle + tc\langle y, y\rangle tc^* \\ &= a - 2tcc^* + t^2cbc^* \end{aligned} \qquad (t \in \mathbb{R})$$

Since $2tcc^*$ is self-adjoint, we can add in both sides and maintain the inequality in the C^*-realm. Using the A-norm and assumig $t \geqslant 0$, by A.4.7,

$$\begin{aligned} 2t\|cc^*\| &\leqslant \|a\| + t^2\|cbc^*\| \\ &\leqslant \|a\| + t^2\|c\|\|b\|\|c^*\| \\ \Rightarrow\ 2t\|c\|^2 &\leqslant \|a\| + t^2\|b\|\|c\|^2 \end{aligned} \tag{3.2}$$

With a fairly nice quadratic polynomial in $\mathbb{R}[t]$ calved by (3.2) in our hands witch is allways non negative, the discriminant must be non positive. Therefore:

$$\begin{aligned} (-2\|c\|^2)^2 - 4\|b\|\|c\|^2\|a\| &\leqslant 0 \\ \Rightarrow\ \|\langle x, y\rangle\|^4 - \|\langle y, y\rangle\|\|\langle x, y\rangle\|^2\|\langle x, x\rangle\| &\leqslant 0 \end{aligned} \tag{3.3}$$

Assuming $\|\langle x, y\rangle\|^2 \neq 0$ means that (3.3) can be simplified into Cauchy-Schwartz inequality (3.1) by cancelling $\|\langle x, y\rangle\|^2$. Otherwise[1], $\langle x, y\rangle = 0$ is a trivial case of the desired inequality. $\square$

For any A-valued inner product as above, we define a norm $\|x\| := \sqrt{\|\langle x, x\rangle\|_A}$ on a Inner product C^*-module. Which means that for arbitrary $x, y \in E$ and $a \in A$, the following holds:

[1] Note that $\|\langle x, y\rangle\|^2 = 0$ if and only if $\langle x, y\rangle = 0$.

(i) $\|x\| = 0 \;\Leftrightarrow\; x = 0$.

(ii) $\|xa\| = \|a\|_A \|x\|$.

(iii) $\|x + y\| \leqslant \|x\| + \|y\|$.

Notice that the triangle inequality (iii) is a direct consequence of 3.1.2:

$$\begin{aligned}
\|x+y\|^2 &= \|\langle x+y, x+y\rangle\|_A \\
&= \|\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\|_A \\
&\leqslant \|x\|^2 + \|\langle x,y\rangle\|_A + \|\langle x,y\rangle^*\|_A + \|y\|^2 \\
&= \|x\|^2 + 2\|\langle x,y\rangle\|_A + \|y\|^2 \\
&\overset{3.1.2}{\leqslant} \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2
\end{aligned}$$

as in the good old days. One identity that still remais is the polarization one: For every sesquilinear form $\varsigma : E \times E \longrightarrow A$

$$4\varsigma(y,x) = \sum_{n=0}^{3} i^n \varsigma(x + i^n y, x + i^n y). \qquad (x, y \in E) \tag{3.4}$$

Since it should be a normed space, hence a complex vector space, one may be concerned about the fact that A doesn't necessarily have a unit and therefore, zx for $z \in \mathbb{C}$ should be an worry.

Proposition 3.1.3. All Inner product modules are naturally complex vector spaces, even the ones over non necessarily unital C^*-algebras.

Proof. Any Inner product module E is a $\mathbb{Z}$-module naturally because it is an abelian group with respect to the addition, and so is that $-\langle x,y\rangle = \langle x, -y\rangle$. Therefore, since the proof of Cauchy-Schwartz inequality 3.1.2 doesn't depend on the unity of A, we safe unitl now. For any approximate unit $(u_\lambda)_\lambda \subset A$, $(xu_\lambda)_\lambda \subset E$ converges to x, whence, for $z \in \mathbb{C}$, let $zx := \lim_\lambda x(zu_\lambda)$. Since A is a vector space, all properties are guaranteed and we are done. □

Definition 3.1.4. Inner product modules are called *Hilbert C^*-modules* when the induced norm is complete in the Cauchy sense.

Proposition 3.1.5. For a Hilbert C^*-module E over A, let EA denote the linear span of elements given by xa, for $x \in E$ and $a \in A$. Therefore $\overline{EA} = E$.

Proof. If $(u_\lambda)_\lambda \subset A$ is a approximate unit for A, then for all $x \in E$:

$$\begin{aligned}
\lim_\lambda \langle x - xu_\lambda, x - xu_\lambda\rangle &= \lim_\lambda \left(\langle x,x\rangle - u_\lambda\langle x,x\rangle\right) \\
&\quad - \lim_\lambda \left(\langle x,x\rangle u_\lambda - u_\lambda \langle x,x\rangle u_\lambda\right) = 0.
\end{aligned}$$

Hence the elements of the form xu_λ are dense in E. □

***Remark* 3.1.6.** Let A and B be C^*-algebras. If E is a Hilbert B-module and the ideal I of the clousure of the elements spanned by $\langle x, y\rangle$ is contained in A, then there is a way to make E into a Hilbert A-module without changing the inner product. Namely, let $(u_\lambda)_\lambda$ be an approximate unit for I. Then the identity

$$\begin{aligned}\langle xu_\eta a - xu_\lambda a, xu_\eta a - xu_\lambda a\rangle &= a^*u_\eta\langle x,x\rangle u_\eta a + a^*u_\lambda\langle x,x\rangle u_\lambda a \\ &\quad - a^*u_\eta\langle x,x\rangle u_\lambda a - a^*u_\lambda\langle x,x\rangle u_\eta a,\end{aligned}$$

holds for all $x \in E$ and $a \in A$, showing that $(xu_\lambda a)_\lambda$ converges in E. We can define $xa = \lim xu_\lambda a$, and it is straightforward to check that this makes E into a Hilbert A-module. This is particularly when dealing with non unital C^*-algebras A, and we might have a look into the same module over $\widetilde{A}$. ☕

Examples 3.1.7.

(i) Any traditional complex Hilbert space is a Hilbert $\mathbb{C}$-module.

(ii) Let $(E_i)_{i\in I}$ be a family of Hilbert C^*-modules over A. The direct sum will be:

$$\bigoplus_{i\in I} E_i \coloneqq \Big\{x \in \prod_{i\in I} E_i \mid \sum_{i\in I}\langle x_i, x_i\rangle \in A\Big\}$$

It should be noticed that the convergence of $\sum_i\langle x_i, x_i\rangle$ is a weaker condition than requiring that the series of norms $\sum_i \|\langle x_i, x_i\rangle\|$ should converge. With the addition inner product $\langle x, y\rangle = \sum_i \langle x_i, y_i\rangle_{E_i}$, $\bigoplus_i E_i$ is a Hilbert C^*-module it self.

(iii) Subexamples of (ii) are: A it-self endowed with $\langle a, b\rangle \coloneqq a^*b$; $A^n = \bigoplus_{i=1}^n A$ for any natural number n.

(iv) **The standard Hilbert A-module $\mathscr{H}_A$**: A more especif subexample of (ii) can be given by $\mathscr{H}_A \coloneqq \bigoplus_{n\in\mathbb{N}} A$, consisting of all sequences $(a_n)_n \subset A$ which $\sum_n a_n^*a_n$ converges.

(v) Given a Hilbert space H, the algebraic tensor product of H by A can be seeing as a Inner product C^*-module, with the bond:

$$\langle x\otimes a, y\otimes b\rangle \coloneqq \langle x, y\rangle_H a^*b$$

$H \otimes A$ stands for its completion.

(vi) Let $X \in$ **CHaus** and $E \longrightarrow X$ a complex vector bundle. As we mention, $C(X)$ is a unital C^*-algebra. Whenever $d : E\times E \longrightarrow [0,\infty)$ is an Hermitian metric over E, the set $\Gamma(E)$ of continuous sections over E holds the title of Hilbert module over $C(X)$ when endowed with

$$\begin{aligned}\langle\cdot,\cdot\rangle : \quad \Gamma(E)\times\Gamma(E) &\longrightarrow C(X)\\ (a,b) &\longmapsto d(a(\cdot), b(\cdot))\end{aligned}$$

as an inner product.

Lemma 3.1.8. Given two nets $(x_\lambda)_\lambda$ and $(y_\lambda)_\lambda$ and x, y in a Hilbert module E over a C^*-algebra A such that $x_\lambda \to x$ and $y_\lambda \to y$, $\lim_\lambda \langle x_\lambda, y_\lambda \rangle = \langle x, y \rangle$ holds.

Proof. From the Cauchy-Schartz inequallity 3.1.2, is easy to obtain that

$$\|\langle x_\lambda - x, z\rangle\|_A \overset{(3.1)}{\leqslant} \|x_\lambda - x\|_E \|z\|_E \qquad (z \in E, \lambda \in \mathbb{A})$$

Analogously, $\|\langle z, y_\lambda - y\rangle\|_A \leqslant \|y_\lambda - y\|_E \|z\|_E$. For each and every index λ, it is possible to obtain the following inequality:

$$\begin{aligned} \|\langle x_\lambda, y_\lambda\rangle - \langle x, y\rangle\| &= \|\langle x_\lambda, y_\lambda\rangle - \langle x_\lambda, y\rangle + \langle x_\lambda, y\rangle - \langle x, y\rangle\| \\ &\leqslant \|\langle x_\lambda, y_\lambda - y\rangle\| + \|\langle x_\lambda - x, y\rangle\| \\ &\leqslant \|y_\lambda - y\|\|x_\lambda\| + \|y\|\|x_\lambda - x\| \end{aligned}$$

Let $\varepsilon > 0$. Notice that $x_\lambda \to x$, means that $\|x_\lambda\| \to \|x\|$. By $\|y_\lambda - y\| \to 0$, there exists λ_0 in which $\|y_\lambda - y\| \, \|x_\lambda\| < \varepsilon/2$. Similarly, there allways exists λ_1 such that $\|x_\lambda - x\|_E < \varepsilon/2(\|y\| + 1)$ for $\lambda \succcurlyeq \lambda_1$. Since it exists λ_2 such that $\lambda_2 \succcurlyeq \lambda_0$ and $\lambda_2 \succcurlyeq \lambda_1$, we conclude that $\|\langle x_\lambda, y_\lambda\rangle - \langle x, y\rangle\| < \varepsilon$ for all $\lambda \succcurlyeq \lambda_2$. □

3.2 Adjointable operators

Definition 3.2.1 (Adjoint)**.** Let E, F be Hilbert modules over a C^*-algebra A. A function $T : E \longrightarrow F$ is said to be *adjointable* if there exists a function $T^* : F \longrightarrow E$ which satisfies the following relation:

$$\langle Tx, y\rangle_F = \langle x, T^*y\rangle_E \qquad ((x, y) \in E \times F)$$

Besides talking about Hilbert modules, we had defined the adjoint concept for any function between Hilbert modules. That's because inner product relations naturally require these functions to be linear operators, and if they exist, the adjoint is unique. For an adjointable T, T^* is unique, adjointable and $T^{**} = T$. Moreover, $(ST)^* = T^*S^*$ for adjointable operators T and S.

Proposition 3.2.2. Every adjointable operator $T : E \longrightarrow F$, between Hilbert A-modules is bounded and continuous.

Proof. Since the set $\{\langle Tx, y\rangle_F = \langle x, T^*y\rangle_E \mid \|x\| \leqslant 1\} \subset A$ is bounded for all $y \in F$, Banach-Steinhaus theorem 3.2.3 implies that T is bounded.

> **Summoning 3.2.3** (Banach-Steinhaus "Uniform Boundness Principle" - [27])**.** Let $\mathcal{F}$ be a family of bounded linear operators from a Banach space X to a normed linear space Y. If $\mathcal{F}$ is

pointwise bounded, then $\mathcal{F}$ is norm-bounded, i.e.,

$$\forall\, x \in X, \sup_{T\in\mathcal{F}} \|Tx\| < \infty \quad\Rightarrow\quad \sup_{T\in\mathcal{F}} \|T\| < \infty.$$

A sketch of Sokal's really simple proof. For any operator $T \in \mathcal{F}$, one obtain that[a] $\sup_{z\in B(x,r)} \|Tz\| \geqslant \|T\|r$ for any $x \in X$ and $r > 0$. Hence, if $\sup \|\mathcal{F}\| = \infty$, there exists a sequence $(T_n)_n \subset \mathcal{F}$ such that $\|T_n\| \geqslant 4^n$. By the first claim, one can build (inductively) $(x_n)_n \subset X$, with $x_0 = 0$, $\|x_{n+1} - x_n\| \leqslant 1/3^{n+1}$ and $\|T_n x\| \geqslant \|T\|2/3^{n+1}$, which happens to be a Cauchy. Since X is complete, there exists x such that $\|x - x_n\| \leqslant 3^n/2$, hence $\|T_n x\| \geqslant \|T_n\|/(2.3^n) \geqslant (4/3)^n/2$, contradicting pointwise convergence of $\mathcal{F}$. ☕

[a] Just observe that

$$\max(\|T(x+\xi)\|, \|T(x-\xi)\|) \geqslant \frac{1}{2}(\|T(x+\xi)\| + \|T(x-\xi)\|) \geqslant \|T\xi\|$$

foll all $\xi \in X$. Hence, take the supremum over $\xi \in B(0,r)$.

By 3.1.8, for a convergent net $x_\lambda \to x$, the following holds for all $y \in E$:

$$\begin{aligned} 0 &= \lim_\lambda \left[\langle T^*y, x_\lambda\rangle - \langle T^*y, x_\lambda\rangle\right] \\ &= \lim_\lambda \langle y, Tx_\lambda\rangle - \lim_\lambda \langle T^*y, x_\lambda\rangle \\ &= \langle y, \lim_\lambda Tx_\lambda\rangle - \langle T^*y, \lim_\lambda x_\lambda\rangle \\ &= \langle y, \lim_\lambda Tx_\lambda\rangle - \langle T^*y, x\rangle \\ &= \langle y, \lim_\lambda Tx_\lambda\rangle - \langle y, Tx\rangle = \langle y, \lim_\lambda Tx_\lambda - Tx\rangle \end{aligned}$$

Especially when $y := \lim_\lambda Tx_\lambda - Tx$, so that $\lim_\lambda Tx_\lambda = Tx$. □

Example 3.2.4. Given $x, y \in E$, the maps $y\langle x, \cdot\,\rangle$ and $x\langle y, \cdot\,\rangle$ are adjoints of each other. The linear span of those operator are what we will call the *finite rank* operators. ☕

In traditional Hilbert spaces, every bounded operator is adjointable, thanks to the Riesz Lemma ([21]). But when talking about Hilbert modules, that can't be the case anymore:

Counterexample 3.2.5 (*Non-adjointable bounded operator* - [19])**.** Suppose that J is a closed right ideal of a unital C^*-algebra A such that no element of J^* acts as a left multiplicative identity on J[2]. Consider the right module $J \times A$ with inner product defined by

$$\langle (a_1, b_1), (a_2, b_2)\rangle = a_2^* a_1 + b_2^* b_1$$

[2] For instance, the algebra of complex valued continuous functions on the unit interval $C[0,1]$, and the ideal $C_0[0,1]$ of functions which vanish at 0.

for $a_1, a_2 \in J$ and $b_1, b_2 \in A$. In this new space we have $\|(a,b)\|^2_{J\times A} = \|a^*a + b^*b\| \leqslant \|a\|^2 + \|b\|^2$, hence $\|\cdot\|_{J\times A}$ is complete, i.e., $J \times A$ is a Hilbert module.

The operator $T(a,b) := (0,a)$ for each $(a,b) \in J \times A$ is obviously a bounded one, but if we suppose that there exists T^* and let $(x,y) := T^*(0,1)$, notice that

$$a = \langle T(a,b), (0,1)\rangle = \langle (a,b), T^*(0,1)\rangle = x^*a + y^*b$$

for all (a,b). Necessarilly, it is the case that $y = 0$ and $x^*a = a$ for all $a \in J$, hence x^* acts as a left multiplicative identity on J. But $x^* \in J^*$, and this contradicts our assumption of J. ☕

Counterexample 3.2.6. Let $X \in$ **CHaus** and $Y \subset X$ a closed non-empty subset with dense complement. Let $E := \{f \in C(X) \mid f(Y) = \{0\}\}$ and $\iota : E \hookrightarrow C(X)$ the bounded inclusion map. If ι were adjointable, $E \ni \iota^*(I_{C(X)}) = I_{C(X)} \notin E$, i.e., the inclusion is a non-adjointable bounded operator. ☕

Proposition 3.2.7. With the operator norm $\|T\| := \sup_{\|x\|=1} \|Tx\|$, the adjointable operators $\mathscr{L}(E,F)$ is C^*-algebra.

Proof. It is straightfoward checking that $\mathscr{L}(E,F)$ is an involution Banach algebra. To check the C^*-norm property, for each adjointable T,

$$\begin{aligned} \|Tx\|^2 &= \|\langle Tx, Tx\rangle\| = \|\langle T^*Tx, x\rangle\| \overset{3.1.2}{\leqslant} \|T^*Tx\|\|x\| \\ &\leqslant \|T^*T\|\|x\|^2 \leqslant \|T^*\|\|T\|\|x\|^2 \end{aligned} \tag{3.5}$$

for all $x \in E$. A direct calculation using (3.5), shows that

$$\|T\|^2 = \sup_{\|x\|=1} \|Tx\|^2 \leqslant \sup_{\|x\|=1} \|T^*\|\|T\|\|x\|^2 = \|T^*\|\|T\|,$$

which means: $\|T\| \leqslant \|T^*\|$ and by extension, $\|T\| = \|T^*\|$. This automatically garantee the C^*-norm property. □

Proposition 3.2.8 (Equivallence of positivity)**.** Beeing positive some element T in the C^*-algebra $\mathscr{L}(E)$ of adjointable automorphisms of a Hilbert C^*-module is equivalent to beeing positive in the inner algebra: $\langle Tx, x\rangle \geqslant 0$ for all $x \in E$.

Proof. Assume that T is a positive element in the C^*-algebra of operators. From A.4.4, let S be such that $T = S^*S$. Therefore:

$$\langle Tx, x\rangle = \langle S^*Sx, x\rangle = \langle Sx, Sx\rangle \overset{3.1.1(iii)}{\geqslant} 0. \qquad (x \in E) \tag{3.6}$$

Conversely, positive elements are self-adjoint, i.e., $\langle Tx, x\rangle = \langle x, Tx\rangle$. From the polarization identity 3.4, one can see that $\langle Tx, y\rangle = \langle x, Ty\rangle$ for all $x, y \in$

E, showing that T is self-adjoint. By the Hahn decomposition A.4.2(i), there exists two positive elements T_+ and T_- such that $T = T_+ - T_-$ and $T_+T_- = T_+T_- = 0$. Then $\langle T_-y, y\rangle \leqslant \langle T_+y, y\rangle$ for any $y \in E$. In particular,

$$\langle T_-^3x, x\rangle = \langle T_-^2x, T_-x\rangle \leqslant \langle T_+T_-x, T_-x\rangle = 0.$$

On the other hand, $T_- \geqslant 0$ and $T_-^3 \geqslant 0$, hence $\langle T_-^3x, x\rangle \geqslant 0$ (because the statement in this direction is already proved). So the only possibility left is $\langle T_-^3x, x\rangle = 0$ for any x. By the polarization equality 3.4, this implies $\langle T_-^3x, y\rangle = 0$ for all $x, y \in \mathcal{M}$, hence $T_-^3 = 0$, $T_- = 0$. Thus, $T = T_+ \geqslant 0$. □

Proposition 3.2.9. If $T \in \mathscr{L}(E, F)$ and $x \in E$, then $\langle Tx, Tx\rangle \leqslant \|T\|^2\langle x, x\rangle$.

Proof. Let ρ be a state of A. By repeated application of the Cauchy-Schwartz inequality to $\rho(\langle\,\cdot\,,\,\cdot\,\rangle)$:

$$\begin{aligned}
\rho(\langle T^*Tx, x\rangle) &\leqslant \rho\big(\langle T^*Tx, T^*Tx\rangle\big)^{\frac{1}{2}}\rho\big(\langle x, x\rangle\big)^{\frac{1}{2}}\\
&= \rho\big(\langle (T^*T)^2x, x\rangle\big)^{\frac{1}{2}}\rho\big(\langle x, x\rangle\big)^{\frac{1}{2}}\\
&\leqslant \rho\big(\langle (T^*T)^4x, x\rangle\big)^{\frac{1}{4}}\rho\big(\langle x, x\rangle\big)^{\frac{1}{2}+\frac{1}{4}}\\
&\ \vdots\\
&\leqslant \rho\big(\langle (T^*T)^{2^n}x, x\rangle\big)^{\frac{1}{2^n}}\rho\big(\langle x, x\rangle\big)^{\frac{1}{2}+\frac{1}{4}+\dots+\frac{1}{2^n}}\\
&\leqslant \|x\|^{2^{1-n}}\|T^*T\|\rho\big(\langle x, x\rangle\big)^{1-\frac{1}{2^n}}
\end{aligned}$$

As $n \to \infty$, one deduces that $\rho(\langle Tx, Tx\rangle) \leqslant \|T\|^2\rho(\langle x, x\rangle)$. Since this is true for all states ρ, the desired inequality holds. □

3.3 Finite rank operators

We are heading towards the definition of generalized Fredholm operators between Hilbert modules, and for that, we need a replacement for the finite dimensional condition. Henceforth, we shall explore the example 3.2.4.

Let M be a Hilbert A-module. Consider the following operator:

$$\begin{aligned}
\Omega :\quad M^n &\longrightarrow \mathscr{L}(A^n, M)\\
(x_i)_i &\longmapsto \Big((a_i)_i \xmapsto{\Omega_x} \sum_{i=1}^n x_ia_i\Big)
\end{aligned} \tag{3.7}$$

In order the obtain the adjoint operator, as far as algebraic manipulation goes, $\Omega_x^* : M \longrightarrow A^n$ has no other option else besides beeing the coordinate inner decomposition $(\langle x_i, \cdot \rangle)_i$: For $a \in A^n$ and $\xi \in M$,

$$\begin{aligned}
\langle \Omega_x a, \xi \rangle_M &= \Big\langle \sum_{i=1}^{n} x_i a_i, \xi \Big\rangle_M \\
&= \sum_{i=1}^{n} a_i^* \langle x_i, \xi \rangle_M \\
&= \begin{bmatrix} a_1^* & \cdots & a_n^* \end{bmatrix} \begin{bmatrix} \langle x_1, \xi \rangle_M \\ \vdots \\ \langle x_n, \xi \rangle_M \end{bmatrix} \\
&= \Big\langle a, (\langle x_1, \xi \rangle_M, \ldots, \langle x_n, \xi \rangle_M) \Big\rangle_{A^n} = \langle a, \Omega_x^* \xi \rangle_{A^n}
\end{aligned}$$

One should note that for $x \in M^n$ and $y \in N^n$, $\Omega_y \Omega_x^*$ rises a fair notion of *finite rank*, since it's image elements are given by

$$\Omega_y \Omega_x^* \xi = \Omega_y^n \left(\langle x_1, \xi \rangle_M, \ldots, \langle x_n, \xi \rangle_M \right) = \sum_{i=1}^{n} y_i \langle x_i, \xi \rangle_M. \tag{3.8}$$

Definition 3.3.1 (Compact and finite-rank operators)**.** Every operator of the form $\Omega_y \Omega_x^* : M \longrightarrow N$, where $(x, y) \in M^n \times N^n$ will be said to be a *A-finite rank operator*. The set of finite-rank operators will be denoted by $\mathrm{FR}_A(M, N)$. The set of *A-compact operators* between M and N are defined as the topological closure of $\mathrm{FR}(M, N)$ and it's denoted as $\mathscr{K}_A(M, N)$.

Unfortunately, A-compact operators need not to be compact in the sense of Banach spaces:

Counterexample 3.3.2. In a unital C^*-algebra A, the identity can be viewed as $\Omega_1 \Omega_1^* = I_A$ on the Hilbert module A. Hence $I_A \in \mathscr{K}(A)$, but it is a compact operator on the Banach space A if and only if A is finite-dimensional, since it is a invertible compact[3].

Proposition 3.3.3. In the standard Hilbert module $\mathscr{H}_A$ over a unital C^*-algebra A, the classical compact notion of compact operator is well rescued: If $L_n(A) \subset \mathscr{H}_A$ denotes the free submodule generated by the first n canonical elements e_i ($i \leqslant n$), the following are equivallent:

(i) $K \in \mathscr{K}_A(A)$.

(ii) The norms of restrictions of K onto the orthogonal complements $L_n(A)^\perp$ of the submodules $L_n(A)$ vanish as $n \longrightarrow \infty$.

Proof.

[3]If $T \in \mathrm{Hom}_{\mathscr{B}\mathbf{an}}(X, Y)$ is a invertible compact operator between Banach spaces, the boundedness of T^{-1} rises a constant C such that $\|T^{-1} y\| \leqslant C\|y\|$, and by invertibility, $\|x\| \leqslant C\|Tx\|$. Thus, the image by T of the unit ball in X contains an open ball in Y. Since T is compact, Y is finite-dimensional, and so do X.

$(i) \Rightarrow (ii)$ Let $p_n : \mathscr{H}_A \twoheadrightarrow L_n(A)^\perp$ by the orthogonal projection. Then, for any $z \perp L_n(A)$, one has

$$\begin{aligned} \|\Omega_x\Omega_y^* z\|^2 &= \|\langle \Omega_x\Omega_y^* z, \Omega_x\Omega_y^* z\rangle\| \\ &= \|\langle x\langle y,z\rangle, x\langle y,z\rangle\rangle\| \\ &= \|\langle y,z\rangle^*\langle x,x\rangle\langle y,z\rangle\| \\ &\leqslant \|x\|^2\|\langle y,z\rangle\|^2 \\ &= \|x\|^2\,\|\langle p_n y, z\rangle\|^2 \\ &\leqslant \|x\|^2\,\|p_n y\|^2\,\|z\|^2. \end{aligned}$$

Since $\|p_n y\|$ tends to zero, the same is true for the norm of the restriction of the operator $\Omega_x\Omega_y^*$ to the submodule $L_n(A)^\perp$, hence, for the norm of any compact operator K.

$(i) \Leftarrow (ii)$ For a operator $K \in \mathscr{L}(\mathscr{H}_A)$, suppose that $\|K\restriction_{L_n(A)^\perp}\|$ vanishes when $n \to \infty$. If z is an orthogonal element with respect to $L_n(A)$, i.e., $\langle e_i, z\rangle = 0$ for any $i \leqslant n$, notice that if $e^n := (e_1, \ldots, e_n)$, then $\Omega_{Ke^n}\Omega_{e^n}^*(z) = 0$. Therefore:

$$\begin{aligned} &\lim_{n\to\infty} \sup_{\substack{z\in L_n(A)^\perp \\ \|z\|\leqslant 1}} \left\| Kz - \sum_{i=1}^n Ke_i\langle e_i, z\rangle \right\| \\ = &\lim_{n\to\infty} \sup_{\substack{z\in L_n(A)^\perp \\ \|z\|\leqslant 1}} \|Kz\| = \lim_{n\to\infty} \|K\restriction_{L_n(A)^\perp}\| = 0. \end{aligned}$$

If $z \in L_n(A)$, one can see that $Kz = \Omega_{Ke^n}\Omega_{e^n}^*(z)$, so that the supreme can be taken in the hole $\mathscr{H}_A$. Therefore:

$$K = \lim_{n\to\infty} \sum_{i=1}^n \Omega_{Ke_i}\Omega_{e_i}^* \in \mathscr{K}_A(A). \qquad \square$$

Examples 3.3.4.

(i) Let A be your favorite C^*-algebra. We shall see that $\mathscr{K}(A) \simeq A$. Notice that the map

$$\begin{aligned} L:\ \ A &\longrightarrow \mathscr{L}(A) \\ a &\longmapsto (b \xmapsto{L_a} ab) \end{aligned}$$

is well defined since L_{a^*} is the adjoint of L_a. It is no mistery that $\|L_a\| \leqslant \|a\|$, but notice that $\|L_a(a^*)\| = \|a\|\,\|a^*\|$, hence L is an isomorphism of A onto a closed $*$-subalgebra of $\mathscr{L}(A)$.

Since $a\langle b, \cdot\rangle = L_{ab^*}$, it follows that $\mathscr{K}(A)$ is the clousure of (the image under L of) the linear span of products in A. But every C^*-algebra contains an approximate identity, such products are dense, thus L is an isomorphism between A and $\mathscr{K}(A)$.

(ii) If A is a unital algebra, $\mathscr{K}(A) = \mathscr{L}(A)$ since any adjointable operator T consists of left multiplication by $T(1)$.

(iii) Reviewing 3.1.7(v), one can obtain that $\mathscr{K}(H \otimes A) \simeq \mathscr{K}(H) \otimes A$.

(iv) $\mathscr{K}(E^m, F^n) \simeq \mathbb{M}_{m\times n}(\mathscr{K}(E, F))$ and $\mathscr{L}(E^m, F^n) \simeq \mathbb{M}_{m\times n}(\mathscr{L}(E, F))$.

☕

Proposition 3.3.5. For each $x \in M^n$, Ω_x is a A-compact operator.

Proof. Let $(u_\lambda)_\lambda \subset A$ be an approximate unit. Considering A it-self as a Hilbert module, $\Omega^*_{u_\lambda} : A^1 \longrightarrow A$ is the standard multiplication by u_λ. Therefore, for each $a \in A$:

$$\begin{aligned} \Omega_x a &= xa = x \lim\nolimits_\lambda u_\lambda a \\ &\overset{u_\lambda = u_\lambda^*}{=} \lim\nolimits_\lambda x\langle u_\lambda, a\rangle_A \overset{(3.8)}{=} \lim\nolimits_\lambda \Omega_x \Omega_{u_\lambda}^* a \end{aligned} \tag{3.9}$$

So every finite 1-rank operators is indeed compact. For a general n, we arrive at a sum of finite 1-rank operators. Given $x \in M^n$ and $a \in A^n$, for each λ, we wish to obtain $y \in M^n$ and $b \in (A^n)^n$ such that,

$$\sum_{i=1}^n \Omega_{x_i} \Omega^*_{u_\lambda} a_i \overset{(*)}{=} \Omega_y \Omega^*_b a$$

in order to write a general Ω_x as a compact operator. To addres $(*)$, let $y := x$ and $b := \mathrm{Diag}(u_\lambda) \in \mathbb{M}_{n\times n}(A) \simeq (A^n)^n$ the diagonal matrix whose non zero entries are u_λ. Notice that

$$\begin{aligned} \Omega_y \Omega^*_b a &= \Omega_x \Omega^*_{\mathrm{Diag}(u_\lambda)} a \\ &= \sum_{i=1}^n x_i \underbrace{\langle b_i, a\rangle_{A^n}}_{\sum_{j=1}^n \langle b_{ij}, a_j\rangle_A} \\ &= \sum_{i=1}^n x_i \langle b_{ii}, a_i\rangle_A \\ &\overset{u_\lambda = b_{ii}}{=} \sum_{i=1}^n \Omega_{x_i} \Omega^*_{u_\lambda} a_i \end{aligned} \qquad (a \in A^n) \tag{3.10}$$

Therefore, for any $a \in A^n$, the following holds and the claim is proved.

$$\Omega_x a = \sum_{i=1}^n x_i a_i \overset{(3.9)}{=} \lim_\lambda \sum_{i=1}^n \Omega_{x_i} \Omega^*_{u_\lambda} a_i \overset{(3.10)}{=} \lim_\lambda \Omega_x \Omega^*_{\mathrm{Diag}(u_\lambda)} a. \qquad \square$$

Definition 3.3.6 (Finite-rank Hilbert Module)**.** A Hilbert Module M over an C^*-algebra A whose identity operator I_M is A-finite rank will be said to be an A-*finite rank* module.

Proposition 3.3.7. For an A-finite rank Hilbert module M over A, the family of finite rank automorphisms $\mathrm{FR}(M)$ over A is a two-sided ideal of the adjointable operators $\mathscr{L}(M)$.

Proof. Extending coordinatewise, let $Ty \coloneqq (Ty_1, \dots, Ty_n)$ and notice that, for $T, S \in \mathscr{L}(M)$,

$$\begin{aligned} T\Omega_y\Omega_x^* + \Omega_w\Omega_z^* S &= \sum_{i=1}^{n} Ty_i\langle x_i, \cdot\rangle + \sum_{i=1}^{n} w_i\langle S^* z_i, \cdot\rangle \\ &= \Omega_{(Ty,w)}\Omega_{(x,S^*z)}^*. \end{aligned}$$

Therefore, any $\mathscr{L}(M)$-linear combination of finite rank operators is itself a finite rank one, as showed above, i.e., $\mathrm{FR}(M) \triangleleft \mathscr{L}(M)$. ☐

Proposition 3.3.8. If the identity I_E in a Hilbert module E is a compact operator, then E has finite rank, i.e., $I_E \in \mathscr{K}(E) \Rightarrow I_E \in \mathrm{FR}(E)$.

Proof. Suppose that I_M is a compact operator in a given Hilbert module E. By construction, $\mathrm{FR}(E)$ is a dense subset of compact operators, so every open non empty set $U \subset \mathscr{K}(E)$ obeys

$$U \cap \mathrm{FR}(E) \neq \varnothing \qquad (U \subset \mathscr{K}(E))$$

Since $\mathscr{K}(E)$ is a unital C^*-algebra, the invertible operators $\mathrm{GL}(\mathscr{K}(E))$ constitute a non empty open set by A.1.11, hence there is a finite-rank invertible operator $F \in \mathrm{GL}(\mathscr{K}(E)) \cap \mathrm{FR}(E)$. Since $\mathrm{FR}(E)$ contains an invertible and is an ideal, it follows that $I_E = FF^{-1} \in \mathrm{FR}(E)$, i.e., the identity has finite rank. ☐

In order to fully characterize finite rank Hilbert Modules over a given C^*-algebra, the same K-theoretic bias is seen right here through the representation by idempotents.

Theorem 3.3.9. A Hilbert module M has finite rank if, and only if, there exists an idempotent matrix $p \in \mathbb{M}_{n\times n}(A)$ such that M is isomorphic, as Hilbert A-modules, to pA^n.

Proof. Assume that M has finite rank, i.e., $I = \Omega_y\Omega_x^*$ for some $x, y \in M^n$. As presented in (3.11), $\Omega_x^*\Omega_y \in \mathscr{L}(A^n)$ is an idempotent operator, which corresponds to left multiplication by the idempotent matrix $p \coloneqq (\langle x_i, y_j\rangle)_{i,j} \in \mathbb{M}_{n\times n}(A)$.

$$I = \Omega_y\Omega_x^* \;\Rightarrow\; \Omega_x^* = (\Omega_x^*\Omega_y)\Omega_x^* \;\Rightarrow\; \Omega_x^*\Omega_y = (\Omega_x^*\Omega_y)^2 \tag{3.11}$$

$\Omega_x^* : M \longrightarrow pA^n$ is invertible: The middle term of (3.11) tells us that $\Omega_x^* = p\Omega_x^*$. Therefore, consider the following:

$$\begin{aligned} T: \quad pA^n &\longrightarrow M \\ pa &\longmapsto \Omega_y a \end{aligned}$$

That operator show us that Ω_x^* is an invertible operator: Given $a \in A^n$, $\xi \in M$, one obtains that $\Omega_x^* T(pa) = \Omega_x^* \Omega_y a = pa$ and $T\Omega_x^* \xi = T(p\Omega_x^* \xi) = \Omega_y \Omega_x^* \xi = \xi$, i.e., $T = \Omega_x^{*-1}$.

Since it is an adjointable, functional continuous calculus allow us to extract the square root $|\Omega_x^*| := (\Omega_x \Omega_x^*)^{1/2}$, which is self-adjoint. Besides beeing a linear bijection, Ω_x^* doesn't preserves inner products. But notice that $U := \Omega_x^* |\Omega_x^*|^{-1}$ does: For $\xi, \zeta \in M$,

$$\begin{aligned} \langle U\xi, U\zeta \rangle_{A^n} &= \langle \Omega_x^* |\Omega_x^*|^{-1} \xi, \Omega_x^* |\Omega_x^*|^{-1} \zeta \rangle_{A^n} \\ &= \langle \xi, |\Omega_x^*|^{-1} \Omega_x \Omega_x^* |\Omega_x^*|^{-1} \zeta \rangle_M \\ &= \langle \xi, |\Omega_x^*|^{-1} |\Omega_x^*|^2 |\Omega_x^*|^{-1} \zeta \rangle_M = \langle \xi, \zeta \rangle_M . \end{aligned}$$

Hence U is a Hilbert isomorphism between M and pA^n. □

3.4 Kasparov Stabilization Theorem

We follow [18].

Definition 3.4.1. If a is a positive element in a C^*-algebra A and $\phi(a) \neq 0$ for all *states*[4] ϕ on A, then a is said to be *strictly positive*.

Proposition 3.4.2. A positive element $a \geqslant 0$ in a unitary C^*-algebra is strictly positive if and only if it is a invertible element.

Proof. We follow [11]. Suppose $a \in A$ is strictly positive. Since A is unital, the state space of A is weak*-compact, and it follows that $\varepsilon := \inf\{\phi(a) \mid \phi \text{ is a state on } A\} > 0$. It then follows that $a - \varepsilon$ is positive, from which it follows that the spectrum of a is contained in $[\varepsilon, \infty)$, i.e., 0 is not in the spectrum of a, hence a is invertible.

If $a \in A$ is positive and invertible, its spectrum is a compact subset of $(0, \infty)$, and thus $a - \varepsilon$ is positive for some $\varepsilon > 0$. Thus if ϕ is a non-zero positive linear functional on A, we have

$$\phi(a) = \varepsilon\phi(1) + \phi(a - \varepsilon) \geqslant \varepsilon\|\phi\| + 0 > 0. \qquad \square$$

Lemma 3.4.3. Let $a \in A$ be a positive element. Therefore a is strictly positive if, and only if, aA is dense in A.

Proof. Conjure the following statement:

[4] Norm 1 positive linear functional $\phi : A \longrightarrow \mathbb{C}$.

> **Summoning 3.4.4** ([7] - Lemma 2.9.4)**.** Let A be a C^*-algebra and L, L' two closed left ideals of A such that $L \subseteq L'$. Suppose every positive form on A that vanishes on L also vanishes on L'. Then $L = L'$.

Suppose that $L := aA \subset A =: L'$ isn't dense, i.e., $L \neq L'$. By the summoning, there is a state of A vanishing on aA. Such a state must vanish on a, so a is not strictly positive. Conversely, if ϕ is a state which $\phi(a) = 0$. Then, by Cauchy-Schwarz inequality for states:

$$|\phi(ab)|^2 \leqslant \phi(b^*b)\phi(a^*a) = 0 \qquad (b \in A)$$

i.e., ϕ vanishes on aA, hence it is not dense. $\square$

Notice that if $a \geqslant 0$ and is strictly positive, obviously, $a \neq 0$ since $\phi(0) = 0$ for all states ϕ. Conversely, we wish to show that $a > 0$ is in fact, strictly positive.

Proposition 3.4.5. Let E be a Hilbert A-module and T a positive element in the C^*-algebra $\mathscr{K}(E)$. Then T is strictly positive if and only if T has dense range.

Proof. If T is strictly positive, by 3.4.3 then $\overline{T\mathscr{K}(E)} = \mathscr{K}(E)$. Since $\overline{\mathscr{K}(E)E} = E$, we have that $\overline{\operatorname{Im} T} = \overline{T\mathscr{K}(E)E} = \overline{\mathscr{K}(E)E} = E$, i.e., T has dense image. Conversely, suppose that T has dense range. Therefore, for any $x, y \in E$, choose a sequence $(z_n)_n \subset E$ with $Tz_n \longrightarrow x$. Therefore,

$$\Omega_x\Omega_y^* = \lim_{n\to\infty} T\Omega_{z_n}\Omega_y^* \in \overline{T\mathscr{K}(E)}.$$

So $T\mathscr{K}(E)$ is dense and T is strictly positive. $\square$

A Hilbert A-module M is countably generated if there is a sequence $(x_n)_n \subset X$ such that every x is the limit A-linear combinations of $(x_n)_n$.

Theorem 3.4.6 (Kasparov Stabilization Theorem)**.** If M is a countably generated Hilbert A-module, then there exists a isomorphism $U : \mathscr{H}_A \longrightarrow M \oplus \mathscr{H}_A$. Whenever A is unital, U is compact.

Proof. Consider the case only when M is Hilbert $\widetilde{A}$-module, which is sufficient since $\overline{MA} = M$ and $\overline{\mathscr{H}_A A} = \mathscr{H}_A$. Therefore, assume that A is a unital C^*-algebra.

Let $(\eta_n)_n$ be a bounded sequence of generators for M, with each generator repeated infinitely often. Let $(e_n)_n$ be the canonical orthonormal basis for $\mathscr{H}_A$, i.e., only the n-th coordinate of e_n is 1 and 0 elsewhere. Define $T : \mathscr{H}_A \longrightarrow M \oplus \mathscr{H}_A$ linearly by $Te_n := 2^{-n}\eta_n \oplus 4^{-n}e_n$. Notice that

$$T = \sum_{n=1}^{\infty} 2^{-n}\Omega_{\zeta_n}\Omega_{e_n}^* = \sum_{n=1}^{\infty} 2^{-n}(\eta_n + 2^{-n}e_n)\langle e_n, \cdot\rangle \qquad (\zeta_n := \eta_n + 2^{-n}e_n)$$

Therefore, T is a compact bijection. Since each η_n is repeated infinitely often, it is true that $\eta_n \oplus 2^{-m} e_m = T(2^m e_m) \in \operatorname{Im} T$ for infinitely many m which $\eta_n = \eta_m$. Going through the limit when $m \longrightarrow \infty$, we see that

$$\eta_n \oplus 0 = \lim_{m\to\infty} \eta_n \oplus 2^{-m} e_m = \lim_{m\to\infty} T(2^m e_m) \in \overline{\operatorname{Im} T}$$

and $0 \oplus e_n = 4^n(Te_n - 2^{-n}(\eta_n \oplus 0))$. Therefore, both $\eta_n \oplus 0$ and $0 \oplus e_n$ are in the closure $\overline{\operatorname{Im} T}$.

Since $\{\eta_n \oplus 0, 0 \oplus e_n\}_n$ generates a dense submodule of $M \oplus \mathscr{H}_A$, T has dense range. Define new operators S and R given by $Se_n \coloneqq 0 \oplus 4^{-n} e_n$ and $Re_n \coloneqq 2^{-n}\eta_n \oplus 0$, in order that

$$\begin{aligned} T^*T &= S^*S + R^*R \\ &= \begin{pmatrix} 4^{-4} & 0 & 0 & \cdots \\ 0 & 4^{-8} & 0 & \cdots \\ 0 & 0 & 4^{-12} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} + \begin{pmatrix} 4^{-2}\langle\eta_1,\eta_1\rangle & 4^{-3}\langle\eta_1,\eta_2\rangle & 4^{-4}\langle\eta_1,\eta_4\rangle & \cdots \\ 4^{-3}\langle\eta_2,\eta_1\rangle & 4^{-4}\langle\eta_2,\eta_2\rangle & 4^{-5}\langle\eta_2,\eta_3\rangle & \cdots \\ 4^{-4}\langle\eta_3,\eta_1\rangle & 4^{-5}\langle\eta_3,\eta_2\rangle & 4^{-6}\langle\eta_3,\eta_1\rangle & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{aligned}$$

Hence $T^*T \geqslant S^*S$. Notice that S^*S has dense range, and by 3.4.5, it is strictly positive and so do T^*T. Therefore, one can define $U \coloneqq T|T|^{-1}$ which is isomorphism between Hilbert modules, and since T is a compact operator, so is U. □

3.5 Rank definition of Finite Rank Hilbert Modules

In order to define the actual *rank* of a finite rank module M, we wish to deal with the K_0-group as the K-algebraic theoreticals like, so we shall visualise our A-modules over the unifization $\widetilde{A}$ of the underlying C^*-algebra A, regardless if it already posseses a unity or not. This process naturally enrich our module, as showed in 3.5.2.

Definition 3.5.1 (Projective module)**.** Let M be an A-module. M is *projective* whenever there exists a map h such that it is commutative the following diagram:

$$\begin{tikzcd} & M \arrow[dl, dashed, "h"'] \arrow[d] \\ E \arrow[r, two heads] & F \end{tikzcd} \qquad (E, F \in \boldsymbol{A}\text{-}\mathbf{Mod})$$

That is to say, for every epimorphism[5] $g : E \twoheadrightarrow F$ and $f : M \longrightarrow F$, there allways exists a map h such that $g \circ h = f$.

Proposition 3.5.2. Every finite rank Hilbert A-module is a finitely generated projective Hilbert $\widetilde{A}$-module.

[5]For our pourpose and needs, surjective morphism

Proof. Let M be a finite rank Hilbert A-module. If $(a, \lambda) \in \widetilde{A}$, letting $(a, \lambda) \cdot \xi := a\xi + \lambda\xi$ turns M into an Hilbert $\widetilde{A}$-module. If $p \in \mathscr{L}(\widetilde{A}^n)$ is the idempotent such that $M \simeq p\widetilde{A}^n$ given by 3.3.9, notice that:

(*i*) $\boldsymbol{\widetilde{A}^n \simeq M \oplus \ker p}$: We shall see that it is a direct sum:

$$\widetilde{A}^n = p\widetilde{A}^n \oplus (I_{\widetilde{A}^n} - p)\widetilde{A}^n \tag{3.12}$$

Indeed, if $x \in p\widetilde{A}^n \cap (I_{\widetilde{A}^n} - p)\widetilde{A}^n$, than $x = pa = (I_{\widetilde{A}^n} - p)b$ for some tuples a, b. Therefore

$$x = pa = p(pa) = p(I_{\widetilde{A}^n} - p)b = (p - p^2)b = 0,$$

and the sum in (3.12) is fact direct. Since $\ker p = (I_{\widetilde{A}^n} - p)\widetilde{A}^n$, the desired isomorphism holds by the givenness of p.

(*ii*) **If there exists $\boldsymbol{Q}$ such that $\boldsymbol{M \oplus Q}$ is free, then $\boldsymbol{M}$ is projective**: Let E, F be $\widetilde{A}$-modules, $g : E \twoheadrightarrow F$ a surjective map and $f : M \longrightarrow F$. Let $(b_i)_i$ be a basis of $M \oplus Q$. By surjectivity, for all i, there is allways $x_i \in M$ such that $g(x_i) = f(b_i)$.

Define $\widetilde{h} : M \oplus Q \longrightarrow E$ extending linearly $\widetilde{h}(b_i) := x_i$, in order that $g \circ \widetilde{h} = f$. Therefore, $h := \widetilde{h}\restriction_M$ is the one necessary so that M is projective.

(*iii*) **If $\boldsymbol{M}$ is a direct summand of a free rank module, then it is finitely generated**: Let $M \oplus Q \simeq \widetilde{A}^n$ for some $\widetilde{A}$-module Q. That way, Q can be both projected and embbeded in $\widetilde{A}^n$ by morphisms. Let $\psi : \widetilde{A}^n \longrightarrow \widetilde{A}^n$ be the compositions of those. Then

$$\operatorname{Im} \psi = \{(0, q) \in M \oplus Q \mid q \in Q\}$$

is the kernel of the canonical projection $\Pi_M : \widetilde{A}^n \twoheadrightarrow M$. Therefore, the composition $\Pi_M \circ \psi : \widetilde{A}^n \twoheadrightarrow M$ is a surjection, telling the world that M must be finitely generated.

Since $\widetilde{A}^n$ is free module, setting $Q := \ker p$, one can see that M is projective by (*ii*) and finitely generated by (*iii*). □

Lemma 3.5.3. Let p, q be idempotent square matrices with entries living in A. The following are equivalent:

(*i*) As Hilbert modules, $pA^m \simeq qA^n$.

(*ii*) p and q are Murray von-Neumann equivalent: There are r, s such that $p = rs$ and $q = sr$.

Proof.

$(i) \Rightarrow (ii)$ Let $T : pA^m \longrightarrow qA^n$ be a isomorphism. There exists unique matrices $r \in \mathbb{M}_{n\times m}(A)$ and $s \in \mathbb{M}_{m\times n}(A)$ which corresponds $spa = T(pa)$ and $rqb = T^{-1}(qb)$ for every $a \in A^m$ and $b \in A^n$. Notice that

$$rs(pa) = T^{-1}T(pa) = pa = p^2a \quad \text{and} \quad sr(qb) = TT^{-1}(qb) = qb = q^2b$$

Therefore $p = rs$ and $q = sr$.

$(i) \Leftarrow (ii)$ The left multiplications maps $s : pA^m \longrightarrow qA^n$ and $r : qA^n \longrightarrow pA^m$ are mutual inverses of each otter:

$$\begin{cases} s(r(qb)) = (sr)^2b = q^2b = qb, & (b \in A^n). \\ r(s(pa)) = (rs)^2a = p^2a = pa, & (a \in A^m). \end{cases}$$

Therefore $pA^m \simeq qA^n$. $\square$

For a A-finite rank module M, M can be viewed as an finitely gerated projective $\widetilde{A}$-module, hence let p be as in 3.3.9 embeded in $\mathbb{M}_\infty(\widetilde{A})$. As an element of $V(\widetilde{A})$, the equivalence class of idempotents which represents M, is the set:

$$[M] \coloneqq \{q \in \mathbb{M}_\infty(\widetilde{A}) \mid q^2 = q \in \mathbb{M}_n(\widetilde{A}), M \simeq qA^n\} = [p] \in V(\widetilde{A})$$

which by 3.5.3 is well defined.

Letting $[\,\cdot\,]_0 : V(\widetilde{A}) \hookrightarrow K_0(\widetilde{A})$ be the natural inclusion, $[q]_0 \coloneqq [q] - [s(q)]$, one may see $[M]_0 \in K_0(\widetilde{A})$ as an element of $K_0(A)$. In fact, if $\varepsilon : \widetilde{A} \twoheadrightarrow \mathbb{C}$ is the projection of the complex component, $\varepsilon(p) = 0$ since M is originally a Hilbert A-module. Therefore,

$$\varepsilon^*\left([M]_0\right) = \varepsilon^*([p]-[s(p)]) = [\varepsilon(p)]-[\varepsilon(s(p))] = 0 \quad \Rightarrow \quad [M]_0 \in \ker \varepsilon^* = K_0(A).$$

Proposition 3.5.4. Let $P \in \mathscr{K}(E)$ be a compact self-adjoint idempotent operator over a Hilbert A-module E. Therefore, $\operatorname{Im} P$ is an A-finite rank one.

Proof. Notice that $I_{\operatorname{Im} P} = P$. Since it is compact in M, there are nets $(y_\lambda)_\lambda, (x_\lambda)_\lambda \subset E^\infty$ such that $P = \lim_\lambda \Omega_{y_\lambda}\Omega^*_{x_\lambda}$. Therefore:

$$\begin{aligned} I_{\operatorname{Im} P} &= P = P^3 = P\Big(\lim_\lambda \Omega_{y_\lambda}\Omega^*_{x_\lambda}\Big)P \\ &= \lim_\lambda P\Omega_{y_\lambda}\Omega^*_{x_\lambda}P = \lim_\lambda \Omega_{Py_\lambda}\Omega^*_{P^*x_\lambda} \overset{P^*=P}{=} \lim_\lambda \Omega_{Py_\lambda}\Omega^*_{Px_\lambda}. \end{aligned}$$

In light of the 3.3.8, $\operatorname{Im} P$ is indeed a finite-rank module. $\square$

***Remark* 3.5.5.** The range of an idempotent operator coincide with the range of some projection, i.e., self-adjoint idempotent operator. To see this, suppose that $a \in A$ is an idempotent in a unital C^*-algebra A. Let

$$h \coloneqq 1 + (a - a^*)(a^* - a) = 1 + aa^* - a^* - a + a^*a$$

With h in hands, one can draw the following conclusions:

(i) $h^* = h$.

(ii) Notice that $(a-a^*)(a^*-a) \geqslant 0$, hence $\operatorname{Spec}((a-a^*)(a^*-a)) \subset [0,\infty)$. By spectral mapping theorem A.1.18,

$$\operatorname{Spec}(h) = 1 + \operatorname{Spec}((a-a^*)(a^*-a)) \subset [1,\infty).$$

Since $0 \notin \operatorname{Spec}(h)$, h is invertible.

(iii) $ah = aa^*a = ha$ and $a^*h = a^*aa^* = ha^*$.

(iv) $p := aa^*h^{-1} = h^{-1}aa^*$. Indeed:

$$hp = haa^*h^{-1} = aa^*hh^{-1} = aa^*.$$

(v) p is self-adjoint:

$$p^* = (h^{-1})^*aa^* = (h^*)^{-1}aa^* = h^{-1}aa^* = p$$

(vi) p is idempotent:

$$p^2 = h^{-1}\underbrace{(aa^*a)}_{ha}a^*h^{-1} = h^{-1}h(aa^*h^{-1}) = p.$$

(vii) $pa = a$ and $ap = p$.

In particular, if $a := Q \in \mathscr{L}(E)$ is an idempotent operator, $\operatorname{Im} Q$ coincides with the range of p by (vii), which is a self-adjoint idempotent operator. In particular $E = \operatorname{Im} Q \oplus \operatorname{Im} Q^{\perp}$.

Definition 3.5.6. The *rank* of a finite rank Hilbert A-module will be defined as the class ***rank*** $(M) := [p]_0 \in K_0(A)$. By proposition 3.5.4 and the remark 3.5.5, we shall also define the rank of a given compact idempotent operator P as the rank of $\operatorname{Im} P$.

Lemma 3.5.7. Let E be a Hilbert module and $P, Q \in \mathscr{L}(E)$ be two compact idempotent operators, and let p and q be the idempotent square matrices given in the proof of Theorem 3.3.9. If P and Q are similar, i.e., there exists $u \in \operatorname{GL}\mathscr{L}(E)$ such that $P = uQu^{-1}$, the matrices p and q are Murray-von Neumann equivalent.

Proof. We have that $P = uQu^{-1}$, $\operatorname{Im} P \simeq pA^n$ and $\operatorname{Im} Q \simeq qA^n$. Since their ranges are finite-rank modules (3.5.5 and 3.5.4), there are $x, y \in (\operatorname{Im} P)^n$ and $z, w \in (\operatorname{Im} Q)^m$ such that

$$P = I_{\operatorname{Im} P} = \Omega_y \Omega_x^* \quad \text{and} \quad Q = I_{\operatorname{Im} Q} = \Omega_w \Omega_z^*.$$

We know that $p = (\langle x_i, y_j\rangle)_{i,j}$ and $q = (\langle z_i, w_j\rangle)_{i,j}$ by the argument used in . Therefore:

$$\Omega_y\Omega_x^* = P = uQu^{-1} = u\Omega_w\Omega_z^*u^{-1} = \Omega_{u(w)}\Omega^*_{(u^{-1})^*(z)},$$

which means that

$$\sum_{i=1}^{n} y_i\langle x_i, \cdot\rangle = \sum_{i=1}^{m} u(w_i)\langle (u^{-1})^*(z_i), \cdot\rangle$$

Notice that the matrix $\widehat{q} \coloneqq (\langle (u^{-1})^*(z_i), u(y_j)\rangle)_{i,j}$ obeys $\operatorname{Im} uQu^{-1} \simeq \widehat{q}A^s$ by the construction of 3.3.9. But since $\widehat{q} = q$, we obtain two isomorphisms such that the following diagram commutes.

$$\begin{tikzcd}
\operatorname{Im} P \arrow[r, equal] \arrow[d] & \operatorname{Im} uQu^{-1} \arrow[r, dashed] \arrow[d] & \operatorname{Im} Q \arrow[d] \\
pA^n \arrow[r, dashed] & qA^m \arrow[r, equal] & qA^m
\end{tikzcd}$$

Since $pA^n \simeq qA^m$ if, and only if p and q are Murray-von Neumann equivalent by 3.5.3, the result follows. □

Corollary 3.5.8. Let P and Q be compact idempotent operators in $\mathscr{L}(E)$. If $[P]_0 = [Q]_0$, then $\textbf{rank}\ (P) = \textbf{rank}\ (Q)$.

3.6 Quasi-stably-isomorphic Hilbert modules

If X, Y, Z and W are Hilbert A-modules and T is in $\mathscr{L}(X \oplus Y, Z \oplus W)$, then T can be represented by a matrix

$$T = \begin{pmatrix} T_{ZX} & T_{ZY} \\ T_{WX} & T_{WY} \end{pmatrix}$$

where T_{ZX} is in $\mathscr{L}(X, Z)$ and similarly for the other matrix entries. Matrix notation is used to define our next important concept.

We are in touch with pretty algebraic properties of Hilbert modules, and in our case of finite rank ones. Since those modules can be seen as projective finitely generated, an algebraist might convince you that two generators $[M]_0, [N]_0 \in K_0(A)$ are equal if, and only if they are *stably-isomorphic*, i.e., there exists n such that $M \oplus A^n \simeq N \oplus A^n$. We will present a generalization of this concept in terms of the **rank** , darkly hidden:

Definition 3.6.1 (Quasi-stably-isomorphic finite rank)**.** Two Hilbert modules E and F are said to be *quasi-stably-isomorphic* if there exists a Hilbert module X and an invertible operator $T \in \mathrm{GL}\,\mathscr{L}(E \oplus X, F \oplus X)$ such that $I_X - T_{XX}$ is compact.

Before lighting that relationship, first we enrich the definition on the special case of finite-rank modules.

Lemma 3.6.2. Assume M and N are A-finite rank modules. If M and N are quasi-stably-isomorphic then the module X referred to in 3.6.1 can be taken to be countably generated.

Proof. Let X and T as in 3.6.1. Using matrix notation, we have:

$$\begin{aligned} T: \quad M \oplus X &\longrightarrow N \oplus X \\ (\xi, \eta) &\longmapsto \begin{pmatrix} T_{NM} & T_{NX} \\ T_{XM} & T_{XX} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \end{aligned}$$

Since it is an invertible operator, consider the inverse given in matrix notation as well:

$$T^{-1} = \begin{pmatrix} S_{MN} & S_{MX} \\ S_{XN} & S_{XX} \end{pmatrix}$$

We shall construct inductively a enumerable collection $\mathscr{C} \subset X$ in order to generate a specific submodule. Such a construction will be given by a collection $(\mathscr{C}_n)_{n\in\mathbb{N}} \subset \wp(X)$, in order that each $\mathscr{C}_n \subset X$ satisfies:

(*i*) The images of the operators T_{XM}, S_{XN}, T^*_{NX} and S^*_{XM} are contained in the submodule of X generated by $\mathscr{C}_n$.

(*ii*) $I_X - T_{XX}$ can be approximated by finite rank operators of the form $\Omega_y\Omega^*_x$, where the components of x and y belong to $\mathscr{C}_n$.

For a given $i \leqslant n$, let π_i be the projection in the i-th coordinate of a tuple. In order to choose wisely, we write a technical issue:

> **Choice of generators**: Let $r \in \mathscr{L}(P, X)$ where P is a finite rank Hilbert module. By assumption, there exists a natural number $n \in \mathbb{N}$ and tuples $x, y \in P^n$ such that $I_P = \Omega_y\Omega^*_x$. Notice that their existence depends on the domain of r. Therefore,
>
> $$r = r\Omega_y\Omega^*_x = \sum_{i=1}^{n} r\pi_i(y)\langle \pi_i(x), \cdot \rangle$$
>
> Hence, $\operatorname{Im} r$ is contained in the submodule generated by $\big(r\pi_i(y)\big)_{i\leqslant n}$.

In order to obey (*i*), our initial collection $\mathscr{C}_0$ must contain all elements of the form $r\pi_i(y(r))$ for r varying over the desired operators. Since $I_X - T_{XX}$ is a compact set, there exists tuple sequences $(\boldsymbol{\xi}_n)_{n\in\mathbb{N}}, (\boldsymbol{\zeta}_n)_{n\in\mathbb{N}}$ such that

$$I_X - T_{XX} = \lim_{n\to\infty} \Omega_{\boldsymbol{\zeta}_n}\Omega^*_{\boldsymbol{\xi}_n}.$$

Let $\mathscr{C}_0$ be given by:

$$\begin{aligned}\mathscr{C}_0 \quad :=& \quad \{r\pi_i(y) \mid r \in \{T_{XM}, S_{XN}, T^*_{NX}, S^*_{XM}\}, I_{\mathrm{dom}(r)} = \Omega_y \Omega^*_x\}_{i\in\mathbb{N}} \\ & \cup \bigcup_{n\in\mathbb{N}} \{\pi_i(\boldsymbol{\xi_n}), \pi_i(\boldsymbol{\zeta_n})\}_{i\in\mathbb{N}}\end{aligned}$$

By construction, those properties are obeyed. Inductively, we set new collections in order to obey to above properties in terms of the operator T_{XX} and S_{XX}:

$$\begin{aligned}\mathscr{C}_{n+1} \quad :=& \quad \mathscr{C}_n \cup T_{XX}(\mathscr{C}_n) \cup S_{XX}(\mathscr{C}_n) \\ & \cup T^*_{XX}(\mathscr{C}_n) \cup S^*_{XX}(\mathscr{C}_n)\end{aligned} \qquad (n \in \mathbb{N})$$

in order that each $\mathscr{C}_n$ satisfies *(i)* and *(ii)*. Therefore, the union $\mathscr{C} := \bigcup_n \mathscr{C}_n$ is then obviously countable, and also obeys those same properties above. In addition, the following one belongs to package:

(iii) $\mathscr{C}$ **is invariant under $\boldsymbol{T_{XX}, T^{-1}_{XX}, T^*_{XX}}$ and $\boldsymbol{(T^{-1}_{XX})^*}$.** Let $r \in \mathscr{L}(X)$ be one the operators. If it was the case that $\mathscr{C}$ wasn't invariant over r, necessarily it would exists $w \in r(\mathscr{C})\backslash\mathscr{C}$, hence, $r^{-1}(w) \in \mathscr{C}_n$ for some n. However,

$$w = r(r^{-1}(w)) \in \mathscr{C}_n \cup r(\mathscr{C}_n) \subset \mathscr{C}_{n+1} \subset \mathscr{C}$$

i.e., it cant be the case.

Let $X' := \overline{\langle\mathscr{C}\rangle}$ be the Hilbert submodule of X generated by $\mathscr{C}$. Because of *(i)* and *(iii)* we see that

$$T(M \oplus X') \subseteq N \oplus X' \quad \text{and} \quad T^*(N \oplus X') \subseteq M \oplus X'.$$

The restriction of T then gives an operator T' in $\mathscr{L}(M \oplus X', N \oplus X')$. The same reasoning applies to T^{-1} providing $(T^{-1})'$ in $\mathscr{L}(N \oplus X', M \oplus X')$ which is obviously the inverse of T'. In virtue of *(ii)* it is clear that T' satisfies the conditions of definition 3.6.1. □

In terms of the Morita equivalence, this is a big deal since we are not restricting ourselves into C^*-algebras with countably approximate identities. After studying Fredholm operators between Hilbert modules, one shall construct group morphisms $K_*(A) \longrightarrow K_*(B)$ for generic C^*-algebras A and B. Those maps turn out to be isomorphisms when A and B are strongly Morita equivalent.

Theorem 3.6.3. Let M and N be quasi-stably-isomorphic finite rank Hilbert modules over a C^*-algebra A. Therefore, $\boldsymbol{\mathit{rank}}\,(M) = \boldsymbol{\mathit{rank}}\,(N)$.

Proof. Let $T \in \mathrm{GL}(\mathscr{L}(M \oplus X, N \oplus X))$ with X beeing a countably generated Hilbert module (3.6.2) and $I_X - T_{XX} \in \mathscr{K}(X)$. By the countability condition, we can apply Kasparov's Stabilization Theorem 3.4.6 in order to obtain that $X \oplus \mathscr{H}_A \simeq \mathscr{H}_A$. Without loss of generality, we can assume that $X = \mathscr{H}_A$.

Since M is finitely generated as an A-module, by Kasparov's theorem again, there exists a isomorphism $\varphi : \mathscr{H}_A \longrightarrow M \oplus \mathscr{H}_A$. Now, we construct operators F and G given by the compositions:

$$F : \mathscr{H}_A \xrightarrow{\varphi} M \oplus \mathscr{H}_A \xrightarrow{T} N \oplus \mathscr{H}_A \twoheadrightarrow \mathscr{H}_A \hookrightarrow M \oplus \mathscr{H}_A \xrightarrow{\varphi^{-1}} \mathscr{H}_A$$

$$G : \mathscr{H}_A \xrightarrow{\varphi} M \oplus \mathscr{H}_A \twoheadrightarrow \mathscr{H}_A \hookrightarrow N \oplus \mathscr{H}_A \xrightarrow{T^{-1}} M \oplus \mathscr{H}_A \xrightarrow{\varphi^{-1}} \mathscr{H}_A$$

We state that

(*i*) **Both $\boldsymbol{I - FG}$ and $\boldsymbol{I - GF}$ are compact**: Let $\Pi_M = I_M \oplus 0$ and $\Pi_{\mathscr{H}_A} = 0 \oplus I$ be the coordinate projections. When composing, one can simplify:

$$FG : \mathscr{H}_A \xrightarrow{\varphi} M \oplus \mathscr{H}_A \xrightarrow{\Pi_{\mathscr{H}_A}} M \oplus \mathscr{H}_A \xrightarrow{\varphi^{-1}} \mathscr{H}_A$$

$$GF : \mathscr{H}_A \xrightarrow{\varphi} M \oplus \mathscr{H}_A \xrightarrow{T} N \oplus \mathscr{H}_A \xrightarrow{\Pi_{\mathscr{H}_A}} N \oplus \mathscr{H}_A \xrightarrow{T^{-1}} M \oplus \mathscr{H}_A \xrightarrow{\varphi^{-1}} \mathscr{H}_A$$

Meaning that

$$\begin{aligned} I - FG &= I - \varphi^{-1}\Pi_{\mathscr{H}_A}\varphi \\ &= \varphi^{-1}\Pi_M\varphi \end{aligned} \quad \text{and} \quad \begin{aligned} I - GF &= I - \varphi^{-1}T^{-1}\Pi_{\mathscr{H}_A}T\varphi \\ &= (T\varphi)^{-1}\Pi_N(T\varphi) \end{aligned}$$

Therefore, $I - FG$ and $I - GF$ are unitarily equivalent to Π_M and Π_N, which are compact operators.

(*ii*) **$\boldsymbol{I - FG}$ and $\boldsymbol{I - GF}$ are idempotents**: Notice that $F = \varphi^{-1}(0_M \oplus \Pi_{\mathscr{H}_A}T\varphi(\,\cdot\,))$. Therefore:

$$\begin{aligned} FGF &= \varphi^{-1}(0_M \oplus \Pi_{\mathscr{H}_A}T\varphi GF(\,\cdot\,)) \\ &= \varphi^{-1}\big(0_M \oplus \Pi_{\mathscr{H}_A}T\varphi\varphi^{-1}T^{-1}\Pi_{\mathscr{H}_A}T\varphi(\,\cdot\,)\big) \\ &= \varphi^{-1}(0_M \oplus \Pi_{\mathscr{H}_A}T\varphi(\,\cdot\,)) = F. \end{aligned}$$

Similarly, one can check that $GFG = G$. Hence, $I - FG$ and $I - GF$ are idempotents.

For sake of notation, let $\mathscr{Q}(E) := \mathscr{L}(E)/\mathscr{K}(E)$ for a Hilbert A-module E. Since the compact set is an ideal of the adjointable maps, one can consider the following exact sequence of C^*-algebras:

$$0 \longrightarrow \mathscr{K}(\mathscr{H}_A) \hookrightarrow \mathscr{L}(\mathscr{H}_A) \xrightarrow{\pi} \mathscr{Q}(\mathscr{H}_A) \longrightarrow 0 \tag{3.13}$$

Notice that we can consider the index map ∂ induced by (3.13). Since $T_{\mathscr{H}_A\mathscr{H}_A}$ is a compact perturbation of the identity and both M and N are finite rank modules, one concludes that F is also a compact perturbation of the identity. Therefore, $\pi(F) = 1_{\mathscr{Q}(\mathscr{H}_A)}$. Since I_2 is a lift of $\mathrm{diag}(1_{\mathscr{Q}(\mathscr{H}_A)}, 1_{\mathscr{Q}(\mathscr{H}_A)}{}^{-1})$, one can obtain that $\partial([1_{\mathscr{Q}(\mathscr{H}_A)}]_1) = [I_2 p_2 I_2^{-1}]_0 - [p_2]_0 = 0$.

We can extract more information by writing down who is the index in terms of F and G. By (*i*), is easy to see that $\pi(F)$ and $\pi(G)$ are each others inverse in $\mathscr{Q}(\mathscr{H}_A)$. In order to compute the index, the element

$$w := \begin{pmatrix} F & I - FG \\ I - GF & G \end{pmatrix} \in \mathrm{GL}_2^{(0)}(\mathscr{Q}(\mathscr{H}_A))$$

is a lift of $\mathrm{diag}(\pi(F), \pi(F)^{-1})$, and its inverse just swaps F and G places. Therefore:

$$\begin{aligned} 0 &= \partial\left([\pi(F)]_1\right) \\ &= \left[wp_2w^{-1}\right]_0 - [p_2]_0 \\ &= \left[\left(\begin{smallmatrix} F & I-FG \\ I-GF & G \end{smallmatrix}\right)\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} G & I-GF \\ I-FG & F \end{smallmatrix}\right)\right]_0 - \left[\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]_0 \\ &= \left[\left(\begin{smallmatrix} FG & 0 \\ 0 & I-GF \end{smallmatrix}\right)\right]_0 - \left[\left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)\right]_0 \\ &= [I - GF]_0 - [I - FG]_0 \end{aligned}$$

Hence $[I - FG]_0 = [I - GF]_0$ and by the corollary 3.5.8, one obtains that $\boldsymbol{rank}\ (I - FG) = \boldsymbol{rank}\ (I - GF)$. We had seen that $I - FG = \varphi^{-1}\Pi_M\varphi$, i.e., $\mathrm{Im}(I - FG) \simeq M$. Therefore:

$$\boldsymbol{rank}\ (M) = [M]_0 = [\mathrm{Im}(I - FG)]_0 = \boldsymbol{rank}\ (I - FG).$$

Similarly, one obtains that $\boldsymbol{rank}\ (N) = \boldsymbol{rank}\ (I - GF)$ and consequentially, $\boldsymbol{rank}\ (M) = \boldsymbol{rank}\ (N)$ as we were looking. $\square$

Chapter 4

Fredholm Operators

In regular Fredholm theory, one would define Fredholm operators as those with both kernel and cokernel were finite dimensional. Here, our concepts involving "finiteness" depends on a fixed C^*-algebra A, so, talking about dimension is kinda tricky. But classical Atikinson's theorem characterizes Fredholm operators in terms only of compactness, which we fully characterize in Theorem 3.6.3. That points towards a generalization.

Definition 4.0.1 (A-Fredholm)**.** A given $T \in \mathscr{L}(E, F)$ is said to be A-Fredholm if it is invertible modulo $\mathrm{FR}(E, F)$, i.e., exists $S \in \mathscr{L}(F, E)$ such that both $I_E - ST$ and $I_F - TS$ are A-finite rank operators.

If adjointable operators S, S' and T are such that both TS and $S'T$ are Fredholm, one can obtain operators R and R' such that $I - T(SR)$ and $I - (R'S')T$ are finite-rank ones. Hence T is Fredholm.

Proposition 4.0.2. If T is invertible modulo $\mathscr{K}(E, F)$, then it is A-Fredholm. Hence, definition 4.0.1 really fits the sentence "Atikinson's theorem trivialization".

Proof. Let us digress a little bit:

> ***Remark* 4.0.3.** Let $J \triangleleft A$ be a non-closed ideal of a unital C^*-algebra, and suppose that $a \in A$ is invertible modulo $\overline{J}$ (the topological closure), i.e., there exists $b \in A$ such that $(1 - ab) \in \overline{J}$. Therefore, there exists a sequence $(j_n)_n \subset J$ such that $j_n \to (1 - ab)$, which means that at least one of then, say j, obeys $\|1 - ab - j\| < 1$. Since $ab + j = 1 - (1 - ab - j)$, $ab + j$ is invertible by A.1.10. Therefore, some algebraic manipulation shows that
>
> $$a\underbrace{b(ab + j)^{-1}}_{x_1} - 1 = -j(ab + j)^{-1} \in J$$
>
> i.e., a is right invertible modulo J. But since b also ensure that $(1 - ba) \in \overline{J}$, hence it exist x_2 such that $(x_2 a - 1) \in J$. Notice

that
$$x_2(ax_1 - 1) = \big((x_2 a)x_1 - x_2\big) = (x_1 - x_2) \in J,$$
i.e., $[x_1] = [x_2]$ in the quotient algebra A/J. Therefore, there is a representative of the class of x_1 and x_2, say x, such that both $ax - 1$ and $xa - 1$ belong to J, i.e., a is invertible modulo J. ☕

With this digression in mind, notice that if $(I_E - TS) \in \mathscr{K}(E, F)$, there exist a element S such that both $I_E - TS$ and $I_F - ST$ are finite-rank operators, i.e., T is Fredholm. □

***Remark* 4.0.4.** Let H an W be complex Hilbert spaces. The set of classical Fredholm operators coincide with the $\mathbb{C}$-Fredholm ones. ☕

Example 4.0.5 ([28])**.** Let $X \in$ **CHaus** and $\mathscr{F}(\ell^2(\mathbb{N}))$ the set of $\mathbb{C}$-Fredholm classical operators over the separable Hilbert space $\ell^2(\mathbb{N})$. We avoid writing $\ell^2(\mathbb{N})$ everywhere, so $\mathscr{B}$, $\mathscr{K}$ and $\mathscr{F}$ will respectively denote the bounded, compact and Fredholm operators over $\ell^2(\mathbb{N})$.

For any continuous family of operators[1] $T : X \longrightarrow \mathscr{B}$, is possible to see T as a $C(X)$-endomorphism over the standard Hilbert $C(X)$-module $\mathscr{H}_{C(X)}$:

$$\begin{aligned} \widehat{T}: \quad \mathscr{H}_{C(X)} &\longrightarrow \mathscr{H}_{C(X)} \\ \xi &\longmapsto (x \mapsto T_x \xi(x)) \end{aligned}$$

Each $T_x := T(x)$ is a classical operator in a Hilbert space, so $T^* \in C(X, \mathscr{B})$. Hence $\widehat{T}$ is adjointable. In order to show that continuous families of Fredholm operators extent to $C(X)$-Fredholm ones, we need the following claim:

(i) **$\widehat{T}$ is $C(X)$-compact whenever T is continuous family of compact operators**: Let $\varepsilon > 0$ and $T \in C(X, \mathscr{K})$. For each $x \in X$, pick a finite rank operator R_x so that $\|R_x - T_x\| \leqslant \varepsilon/3$, and pick a neighborhood U_x such that $\|T_x - T_y\| \leqslant \varepsilon/3$ for $y \in U_x$. Since X is compact, extract a finite subcover $U_1, \ldots, U_n$ of $(U_x)_{x \in X}$, and a partition of unity $\lambda_1, \ldots, \lambda_n$. Then
$$\|T_x - \sum_{i=1}^{n} \lambda_j(x) R_{x_j}\| \leqslant \varepsilon\,.$$
Therefore, is a finite rank the operator $\sum_i \lambda_i(\,\cdot\,) R_{x_i}$ and, $\widehat{T}$ is compact.

Suppose that the range of T is constituted only by Fredholm operators, i.e., $\mathscr{F}$. In order to see the extension of T to the standard Hilbert $C(X)$-module, $\widehat{T} \in \mathscr{L}_{C(X)}(\mathscr{H}_{C(X)})$ is a $C(X)$-Fredholm operator, we must conjure the following:

[1] For example, $T : [0,1] \longrightarrow \mathscr{F}$, $T_x(\xi) := (x\xi_n)_n$.

> **Summoning 4.0.6** (Classical Bartle-Graves theorem - Corollary 17.67 [2])**.** Every surjective continuous linear operator between Banach spaces[a] admits continuous right inverse, but not necessarily a linear one.
>
> [a] More generally, completely metrizable locally convex spaces, i.e., Fréchet spaces.

Bartle-Graves theorem has significant improvements and different versions (e.g., [8]), but for our needs, we are fine with the above.

Consider the Calkin algebra given by the quotient of compact operators: $\mathscr{Q} := \mathscr{B}/\mathscr{K}$ and π be the quotient map. Bartle-Graves theorem offers a continuous section $\sigma : \mathscr{Q} \longrightarrow \mathscr{B}$ such that $\pi\sigma A = A$. Let S be given by the composition:

$$\begin{tikzcd}
X \arrow[rr, dashed, "S"] \arrow[d, "T"'] & & \mathscr{F} \\
\mathscr{F} \arrow[r, two heads, "\pi"'] & \mathrm{GL}\,\mathscr{Q} \arrow[r, "(\,\cdot\,)^{-1}"'] & \mathrm{GL}\,\mathscr{Q} \arrow[u, "\sigma"']
\end{tikzcd}$$

Since $T_x \in \mathscr{F}$ if and only if $\pi(T_x)$ is invertible in $\mathscr{Q}$ (by Atikinson's theorem), S is well defined and continuous (each arrow above is), hence defines $\widehat{S} : C(X, \ell^2(\mathbb{N})) \longrightarrow C(X, \ell^2(\mathbb{N}))$. We are left to show that both $I - \widehat{S}\widehat{T}$ and $I - \widehat{T}\widehat{S}$ are $C(X)$-compact operators in $\mathscr{H}_{C(X)}$.

Since $1_{C(X)} - T \cdot S$ and $1_{C(X)} - S \cdot T$ are continuous families with compact range, $\widehat{T}$ is indeed $C(X)$-Fredholm by *(i)*. ☕

In the classical Fredhom theory between Hilbert spaces, one only requires that $\ker T$ and $\operatorname{coker} T$ are finite dimensional. Those assumptions are sufficient to guarantee that every classical Fredholm operator has closed range, hence orthogonal decompositions are abundant in the proofs. Only if we could extend it so naturally, maybe we woudn't be here with a slightly different definition.

Example 4.0.7 (Non closed range A-Fredholm operator)**.** $C[0,1]$ is a unital C^*-algebra, that we shall consider as a Hilbert C^*-module. Choose T to be

$$\begin{aligned} T : \quad C[0,1] &\longrightarrow C[0,1] \\ f &\longmapsto (x \mapsto xf(x)) \end{aligned}$$

Since the algebra is unital, any adjointable operator is $C[0,1]$-compact: $\mathscr{L}(C[0,1]) \simeq \mathscr{K}(C[0,1])$, hence must be $C[0,1]$-Fredholm as well. Unfortunately, the square root $\sqrt{\cdot}$ can be approximated by functions in the range $\operatorname{Im} T$, but it doesn't belong to it. ☕

Therefore, in order to develop the theory of Fredholm operator between Hilbert modules and, in some extent, try to obtain a correspondence with the classical theory, we shall dodge that closure of the Fredholm range. Hence, we will focus on a smaller class of operators: those which admit pseudo-inverse, henceforth, the *regular* ones. Later, we shall extent our results to general Fredholm operators, showing that each and everyone is, in some extent, regularizable.

4.1 Regular Fredholm operators

Definition 4.1.1 (Regular operators)**.** It is said to be *regular* any operator $T \in \mathscr{L}(E, F)$ that admits a *pseudo-inverse*, i.e., there exists $S \in \mathscr{L}(F, E)$ such that $TST = T$ and $STS = S$. When S is such that TS and ST are idempotents, it is called a *Moore-Penrose* inverse.

For a regular Fredholm operator T, such a pseudo-inverse S fits the Fredholm criteria of T: If S' is such that $I_E - S'T$ and $I_F - TS'$ are finite-rank operators,

$$\begin{aligned}(I_E - S'T)(I_E - ST) &= (I_E - S'T) - (I_E - S'T)ST \\ &= I_E - S'T - ST + \underbrace{S'TST}_{S'T} = I_E - ST\end{aligned}$$

Since $\mathrm{FR}(E, F)$ is an ideal, the above manipulation shows that $I_E - ST$ is indeed a finite-rank operator (and similarly for $I_F - TS$).

Example 4.1.2. To motivate the study of regular Fredholm operators as some way to deal with a weaker version of "the range is closed", we exhibit the following theorem:

> **Theorem**. For a Hilbert space H, a bounded operator $T \in \mathscr{B}(H)$ admits a Moore-Penrose pseudo-inverse S if, and only if, $\operatorname{Im} T$ is closed.

One way is disrespectfully trivial: If there exists a Moore-Penrose pseudo-inverse S, $\operatorname{Im} T = \operatorname{Im} TS$. Since TS is a orthogonal projection by hypothesis, $\operatorname{Im} T$ is closed.

Conversely, consider the following decompositions

$$H = \ker T \oplus \operatorname{Im} T^* = \ker T^* \oplus \operatorname{Im} T.$$

Therefore, $T\restriction_{\operatorname{Im} T^*}$ is an injective bounded operator, which posses a bounded inverse S. Similarly, $T^*\restriction_{\operatorname{Im} T}$ contains a bounded inverse R. Those inverses can be extended to all the space, by setting it to zero (which is fine, since the kernels are all there is left in each case). One can verify that $R = S^*$ and that S induces a Moore-Penrose inverse. ☕

Proposition 4.1.3. Let $T \in \mathscr{L}(E, F)$ be a A-Fredholm operator. If T admits a pseudo-inverse S, then:

(i) $I_E - ST$ and TS are idempotents with ranges $\ker T$ and $\operatorname{Im} T$.

(ii) $\ker T$ and $\ker T^*$ are finite rank modules.

Proof. Notice that $(ST)^2 = S(TST) = ST$ and similarly for TS, i.e., they are idempotents. It is easy to see that $I_E - ST$ also has the idempotent badge, $\operatorname{Im}(I_E - ST) = \ker T$ and $\operatorname{Im} TS = \operatorname{Im} T$.

Is easy to see that $I_{\ker T} = (I_E - ST)\restriction_{\ker T}$. When supposing that T is A-Fredholm, let $x, y \in E^n$ be such that $I_E - ST = \Omega_y \Omega_x^*$. Remind that idempotent operators share their range with some projection by the remark 3.5.5. Since $I_E - ST$ is an idempotent, there exists a self-adjoint idempotent operator P such that $\operatorname{Im}(I_E - St) = \operatorname{Im} P = \ker T$. Therefore, with $a = \Omega_y \Omega_x^*$ and $p = P$, 3.5.5.(vii) guarantee that

$$\Omega_y \Omega_x^* \restriction_{\ker T} = P \Omega_y \Omega_x^* P \overset{P^* = P}{=} \Omega_{Py} \Omega_{Px}^*.$$

Since $Py, Px \in (\ker T)^n$, it follows that $I_{\ker T} = \Omega_{Py} \Omega_{Px}^*$ is a finite-rank operator over $\ker T$, i.e., $\ker T$ is a finite-rank module. Very much the same is sufficient to obtain that $\ker T^*$ also is a finite-rank module. $\square$

The rank of a finite rank module is well defined as seen before. Hence, the above proposition enable us to define the Index of regular Fredholm operators.

Definition 4.1.4. If T is a A-Fredholm operator who admits a pseudo-inverse (i.e., regular), set their *index* to be the $K_0(A)$ element given by

$$\boldsymbol{ind}\, T := \boldsymbol{rank}\,(\ker T) - \boldsymbol{rank}\,(\ker T^*).$$

Proposition 4.1.5. Our elementary propositions of classical Fredholm operators stands nicely. If $T \in \mathscr{L}(E, F)$ is a regular Fredholm operator, then:

(i) $\boldsymbol{ind}\, T^* = -\boldsymbol{ind}\, T$.

(ii) For any pseudo-inverse S, $\boldsymbol{rank}\,(\ker T^*) = \boldsymbol{rank}\,(\ker S)$ and $\boldsymbol{ind}\, S = -\boldsymbol{ind}\, T$.

(iii) If there are invertible operators U and V between Hilbert modules such that:

$$\begin{tikzcd} X \arrow[r, "\simeq", "U"'] \arrow[rrr, bend left, "VTU"] & E \arrow[r, "T"'] & F \arrow[r, "\simeq", "V"'] & Y \end{tikzcd}$$

Therefore VTU is Fredholm and $\boldsymbol{ind}\,(VTU) = \boldsymbol{ind}\, T$.

(iv) If $T_i \in \mathscr{L}(E_i, F_i)$ is a regular Fredholm operator for $i \in \{1, 2\}$, the direct sum $T_1 \oplus T_2$ is also regular Fredholm and $\boldsymbol{ind}\,(T_1 \oplus T_2) = \boldsymbol{ind}\, T_1 + \boldsymbol{ind}\, T_2$.

Proof.

(i) Clear.

(ii) Since S and T^* are Fredholm operators, $\ker T^*$ and $\ker S$ are finite rank modules (4.1.3). In what comes next, keep in mind that $(\operatorname{Im} T)^\perp = \ker T^*$. Visiting again the remark 3.5.5, one can conclude that for any idempotent $Q : F \longrightarrow E$, $F = \operatorname{Im} Q \oplus (\operatorname{Im} Q)^\perp$. Since TS is an idempotent, we obtain the following diagram of equality's:

$$\begin{tikzcd} \operatorname{Im}(I_F - TS) \oplus \operatorname{Im} TS \arrow[r, equal] \arrow[d, equal] & F \arrow[r, equal, "3.5.5"] & (\operatorname{Im} TS)^\perp \oplus \operatorname{Im} TS \arrow[d, equal, "(\operatorname{Im} TS)^\perp = (\operatorname{Im} T)^\perp = \ker T^*"] \\ \ker S \oplus \operatorname{Im} TS & & \ker T^* \oplus \operatorname{Im} TS \end{tikzcd}$$

Therefore, $\ker S$ and $\ker T^*$ are quasi-stably-isomorphic. Therefore, 3.6.3 guarantee that $\boldsymbol{rank}\,(\ker S) = \boldsymbol{rank}\,(\ker T^*)$. Consequentially, $\boldsymbol{ind}\, S = -\boldsymbol{ind}\, T$.

(iii) Notice that $\ker VT = \ker T$ since V is an invertible one, hence $\boldsymbol{rank}\,(\ker VT) = \boldsymbol{rank}\,(\ker T)$. Analysing $U\restriction_{\ker TU}$, one obtains that $\ker TU \simeq \ker T$, thus $\boldsymbol{rank}\,(\ker TU) = \boldsymbol{rank}\,(\ker T)$. The exact same roll goes for the adjoints. Therefore, the indexes coincide.

(iv) It is the case that $\Omega_{\xi_1 \oplus \xi_2} = \Omega_{\xi_1} \oplus \Omega_{\xi_2}$ for any $\xi_1 \oplus \xi_2 \in E_1 \oplus E_2$, which is sufficient to infer that $T_1 \oplus T_2$ is a Fredholm operator.

Since $\ker(T_1 \oplus T_2) = \ker T_1 \oplus \ker T_2$, $\ker(T_1 \oplus T_2)$ is a finite rank module. If $\boldsymbol{rank}\, T_i = [p_i]_0$, it is clear that

$$\begin{aligned} \boldsymbol{rank}\,(\ker(T_1 \oplus T_2)) &= [\operatorname{diag}(p_1, p_2)]_0 \\ &= [p_1]_0 + [p_2]_0 \\ &= \boldsymbol{rank}\,(\ker T_1) + \boldsymbol{rank}\,(\ker T_2). \end{aligned}$$

Therefore, the desired index relation follows. ☐

Since our compact operators aren't necessarily the same as in Hilbert space case, the index invariance under compact perturbations needs to be handed carefully.

Proposition 4.1.6. If $T \in \mathscr{L}(E)$ is a regular Fredholm operator such that $(I - T) \in \mathscr{K}(E)$, then $\boldsymbol{ind}\, T = 0$.

Proof. Let S be a pseudo-inverse of T. Since $I - T$ is a compact operator and the compact operators is an ideal, notice that S is a compact perturbation of the identity:

$$S = I + S(I - T) - (I - ST)$$

Considering the isomorphism map $U : \ker T \oplus \operatorname{Im} S \longrightarrow \ker S \oplus \operatorname{Im} S$ given by

$$U := \begin{pmatrix} I - TS & I - TS \\ S & S \end{pmatrix} \qquad U^{-1} = \begin{pmatrix} I - ST & (I - ST)T \\ ST & STT \end{pmatrix}$$

with the fact that $I - S = I_{\operatorname{Im} S} - U_{\operatorname{Im} S \operatorname{Im} S}$ is compact, the modules $\ker T$ and $\ker S$ are quasi-stably-isomorphic. By 3.6.3, $\boldsymbol{rank}\,(\ker T) = \boldsymbol{rank}\,(\ker S)$, hence

$$\begin{aligned} \boldsymbol{ind}\, T &= \boldsymbol{rank}\,(\ker T) - \boldsymbol{rank}\,(\ker T^*) \\ &\overset{4.1.5(ii)}{=} \boldsymbol{rank}\,(\ker T) - \boldsymbol{rank}\,(\ker S) = 0. \quad \square \end{aligned}$$

Theorem 4.1.7. If $T_1, T_2 \in \mathscr{L}(E, F)$ are regular Fredholm operators such that $T_1 - T_2$ is compact, then $\boldsymbol{ind}\, T_1 = \boldsymbol{ind}\, T_2$.

Proof. The action plan for the proof will be as follows: Build accessory operators U and R in function of the given maps, such that U is invertible and $\boldsymbol{ind}\, R = \boldsymbol{ind}\, T_2 - \boldsymbol{ind}\, T_1$. Hence, $\boldsymbol{ind}\,(UR)$ will be a compact perturbation of the identity, so we can use the previous theorem and obtain that $\boldsymbol{ind}\, R = \boldsymbol{ind}\,(UR) = 0$.

Let S_1 and S_2 be pseudo inverses for T_1 and T_2. Define operators U and R in $\mathscr{L}(E \oplus F)$ by

$$U := \begin{pmatrix} I_E - S_1 T_1 & S_1 \\ T_1 & I_F - T_1 S_1 \end{pmatrix} \quad \text{and} \quad R := \begin{pmatrix} 0 & S_1 \\ T_2 & 0 \end{pmatrix}.$$

(*i*) $\boldsymbol{ind}\, \boldsymbol{R} = \boldsymbol{ind}\, \boldsymbol{T_2} - \boldsymbol{ind}\, \boldsymbol{T_1}$: Using the coordinate switch operator (which has index zero since it is a invertible one), one obtains that:

$$\begin{tikzcd} E \oplus F \arrow[rr, "R"] \arrow[dr, "(x,y)\mapsto(y,x)"'] & & E \oplus F \\ & F \oplus E \arrow[ur, "S_1 \oplus T_2"'] & \end{tikzcd}$$

$$\begin{aligned} \boldsymbol{ind}\, R &\overset{4.1.5(iii)}{=} \boldsymbol{ind}\,(S_1 \oplus T_2) \\ &\overset{4.1.5(iv)}{=} \boldsymbol{ind}\, S_1 + \boldsymbol{ind}\, T_2 \\ &\overset{4.1.5(ii)}{=} -\boldsymbol{ind}\, T_1 + \boldsymbol{ind}\, T_2. \end{aligned}$$

(*ii*) $\boldsymbol{U}$ **is invertible**: We'll show even more: it is a order 2 nilpotent element. Since $I_E - S_1 T_1$ and $I_F - T_1 S_1$ are idempotents,

$$\begin{aligned} U^2 &= \begin{pmatrix} I_E - S_1 T_1 & S_1 \\ T_1 & I_F - T_1 S_1 \end{pmatrix}^2 \\ &= \begin{pmatrix} (I_E - S_1 T_1)^2 + S_1 T_1 & (I_E - S_1 T_1) S_1 + S_1 (I_F - T_1 S_1) \\ T_1 (I_E - S_1 T_1) + (I_F - T_1 S_1) T_1 & T_1 S_1 + (I_F - T_1 S_1)^2 \end{pmatrix} \\ &= \begin{pmatrix} I_E & 0 \\ 0 & I_F \end{pmatrix} = I_{E \oplus F}. \end{aligned}$$

Therefore U is invertible.

(*iii*) ***UR*** **is a compact perturbation of identity**: First, we obtain UR:

$$UR = \begin{pmatrix} I_E - S_1T_1 & S_1 \\ T_1 & I_F - T_1S_1 \end{pmatrix} \begin{pmatrix} 0 & S_1 \\ T_2 & 0 \end{pmatrix} = \begin{pmatrix} S_1T_2 & 0 \\ (I_F - T_1S_1)T_2 & T_1S_1 \end{pmatrix}$$

Bravely evaluating the difference, we must determine if is compact the following operator:

$$I_{E\oplus F} - UR = \begin{pmatrix} I_E - S_1T_2 & 0 \\ (T_1S_1 - I_F)T_2 & I_F - T_1S_1 \end{pmatrix}$$

Notice that all operators in the second row are compact since T_1 is Fredholm. From the hypothesis, $T_1 - T_2$ is compact, hence:

$$\begin{aligned} I_E - S_1T_2 &= I_E - S_1T_1 + S_1T_1 - S_1T_2 \\ &= (I_E - S_1T_1) + S_1(T_1 - T_2) \in \mathscr{K}(E). \end{aligned}$$

Since each entry of $I_{E\oplus F} - UR$ is a compact operator, the claim is proved.

Using 4.1.5(*iii*) again, we have that $\mathbf{ind}\,(UR) = \mathbf{ind}\,R$. Since UR is a compact perturbation of the identity, it follows that $\mathbf{ind}\,(UR) = 0$ by Proposition 4.1.6. □

4.2 Regularization of Fredholm operators

Time to extend our concepts to general Fredholm operators. A change in algebras will be necessary, so we write our next lemma with new a C^*-algebra notation.

Lemma 4.2.1. Let B be a unital C^*-algebra and $T \in \mathscr{L}_B(E, F)$ a B-Fredholm but not necessarily regular. There exists a natural n and some $x \in E^n$ such that

$$\begin{pmatrix} T & 0 \\ \Omega_x^* & 0 \end{pmatrix} : E \oplus B^n \longrightarrow F \oplus B^n$$

is a regular B-Fredholm operator.

Proof. Let S be a pseudo-inverse of T such that both $I_E - ST$ and $I_F - TS$ are finite rank operators, and $I_E - ST = \Omega_y\Omega_x^*$ for some $y \in F^n$, $x \in E^n$. We will construct operators $\widetilde{T}$ and $\widetilde{S}$ that are regular Fredholm. Define the following operators:

$$\widetilde{T} := \begin{pmatrix} T & 0 \\ \Omega_x^* & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{S} := \begin{pmatrix} S & \Omega_y \\ 0 & 0 \end{pmatrix}.$$

(*i*) $\boldsymbol{\widetilde{T}}$ **and** $\boldsymbol{\widetilde{S}}$ **are pseudo-inverses of each other, hence regular**: In what follows, we need the expressions:

(a) $T\Omega_y\Omega_x^* = T(I_E - ST) = T - TST = 0.$

(b) $\Omega_x^*(ST + \Omega_y\Omega_x^*) = \Omega_x^*(ST + I_E - ST) = \Omega_x^*.$

Notice that

$$\begin{aligned}\widetilde{T}\widetilde{S}\widetilde{T} &= \begin{pmatrix} T & 0 \\ \Omega_x^* & 0 \end{pmatrix}\begin{pmatrix} S & \Omega_y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & 0 \\ \Omega_x^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} TS & T\Omega_y \\ \Omega_x^* S & \Omega_x^*\Omega_y \end{pmatrix}\begin{pmatrix} T & 0 \\ \Omega_x^* & 0 \end{pmatrix} \\ &= \begin{pmatrix} TST+T\Omega_y\Omega_x^* & 0 \\ \Omega_x^*(ST+\Omega_y\Omega_x^*) & 0 \end{pmatrix} \overset{(a)+(b)}{=} \begin{pmatrix} T & 0 \\ \Omega_x^* & 0 \end{pmatrix} = \widetilde{T}\end{aligned}$$

Similarly, one can obtain that $\widetilde{S}\widetilde{T}\widetilde{S} = \widetilde{S}$, hence $\widetilde{T}$ and $\widetilde{S}$ are regular due to the fact that they are each others pseudo-inverses.

(ii) **$\widetilde{T}$ and $\widetilde{S}$ are Fredholm operators**: Notice that:

$$I_{E\oplus B^n} - \widetilde{S}\widetilde{T} = \begin{pmatrix} I_E & 0 \\ 0 & I_{B^n} \end{pmatrix} - \begin{pmatrix} ST+\Omega_y\Omega_x^* & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & I_{B^n} \end{pmatrix}$$

(4.1)

$$I_{F\oplus B^n} - \widetilde{T}\widetilde{S} = \begin{pmatrix} I_F & 0 \\ 0 & I_{B^n} \end{pmatrix} - \begin{pmatrix} TS & T\Omega_y \\ \Omega_x^* S & \Omega_x^*\Omega_y \end{pmatrix} = \begin{pmatrix} I_F - TS & -T\Omega_y \\ -\Omega_x^* S & I_{B^n} - \Omega_x^*\Omega_y \end{pmatrix}$$

Lets check that every entry in those matrices are compact:

(a) **I_E is finite rank**: Since B is unital, $I_{B^n} = \Omega_{(1_E,\dots,1_E)}\Omega^*_{(1_E,\dots,1_E)}$.

(b) **$I_F - TS$ is finite rank**: By assumption.

(c) **$-\Omega_x^* S, -T\Omega_y$ and $I_{B^n} - \Omega_x^*\Omega_y$ are compact**: This is due to the fact that Ω_y and Ω_x^* are compact (proposition 3.3.5) and the set of compact operators is an ideal.

Therefore, $I_{E\oplus B^n} - \widetilde{S}\widetilde{T}$ and $I_{F\oplus B^n} - \widetilde{T}\widetilde{S}$ are compact operators. Finally, proposition 4.0.2 guarantee that both $\widetilde{T}$ and $\widetilde{S}$ are regular Fredholm operators. ☐

Definition 4.2.2 (Regularization of a Fredholm operator)**.** Given a A-Fredholm $T \in \mathscr{L}_A(E, F)$, the *regularization of* T is the $\widetilde{A}$-Fredholm $\widetilde{T} \in \mathscr{L}_{\widetilde{A}}(E \oplus \widetilde{A}^n, F \oplus \widetilde{A}^n)$ constructed using lemma 4.2.1 for $B = \widetilde{A} := A \oplus \mathbb{C}$ being the unitization of A.

Proposition 4.2.3. For any A-Fredholm operator T, despite the fact that the regularization $\widetilde{T}$ is a $\widetilde{A}$-Fredholm operator, the index of $\widetilde{T}$ lies in $K_0(A)$.

Proof. Let $\varepsilon : \widetilde{A} \twoheadrightarrow \mathbb{C}$ the complex projection. Since $K_0(A)$ is the kernel of ε_0, we seek to obtain that $\varepsilon_0(\mathtt{ind}\,\widetilde{T}) = 0$. We borrow notations and results from the proof of 4.2.1, i.e., $I_E - ST = \Omega_y\Omega_x^*$ and

$$\widetilde{T} := \begin{pmatrix} T & 0 \\ \Omega_x^* & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{S} := \begin{pmatrix} S & \Omega_y \\ 0 & 0 \end{pmatrix}$$

are regular $\widetilde{A}$-Fredholm operators and pseudo-inverses of each otter. To compute the index of $\widetilde{T}$, first we obtain that $\boldsymbol{rank}\ (\ker \widetilde{T}) = n{\cdot}1_{K_0(A)}$; in order to obtain $\boldsymbol{rank}\ (\ker \widetilde{T}^*) = \boldsymbol{rank}\ (\ker \widetilde{S})$, we will introduce two new operators P and Q, such that the rank of $\ker \widetilde{S}$ will coincide with the embedding of trace of $\varepsilon(Q)$, which will be equal to n.

Now, we look to verify those claims:

(*i*) $\boldsymbol{rank}\ (\ker \widetilde{\boldsymbol{T}}) = \boldsymbol{n \cdot 1_{K_0(A)}}$: In the proof of Lemma 4.2.1 (4.1), we saw that $I_{E\oplus\widetilde{A}^n} - \widetilde{S}\widetilde{T} = 0 \oplus I_{\widetilde{A}^n}$. Since $\ker \widetilde{T} = \operatorname{Im}(I_{E\oplus\widetilde{A}^n} - \widetilde{S}\widetilde{T})$, it follows that

$$\boldsymbol{rank}\ (\ker \widetilde{T}) = [0 \oplus I_{\widetilde{A}^n}]_0 = n \cdot [1_{\widetilde{A}}]_0 = n \cdot 1_{K_0(A)}.$$

For notation sake, let:

$$P := I_{F\oplus\widetilde{A}^n} - \widetilde{T}\widetilde{S} \overset{(4.1)}{=} \begin{pmatrix} I_F - TS & -T\Omega_y \\ -\Omega_x^* S & I_{B^n} - \Omega_x^*\Omega_y \end{pmatrix} \tag{4.2}$$

(*ii*) $\mathbf{ker}\ \widetilde{\boldsymbol{S}} = \mathbf{Im}\ \boldsymbol{P}$. Lets check that the two sets coincide: In one direction, $\widetilde{S} - \widetilde{S}\widetilde{T}\widetilde{S} = 0$ since $\widetilde{S}$ and $\widetilde{T}$ are pseudo-inverses of each other. Hence $\ker \widetilde{S} \supset \operatorname{Im} P$. Conversely, the elements of the range of P can be written as:

$$\begin{aligned} P(\zeta + a) &= \begin{pmatrix} I_F - TS & -T\Omega_y \\ -\Omega_x^* S & I_{B^n} - \Omega_x^*\Omega_y \end{pmatrix} \begin{pmatrix} \zeta \\ a \end{pmatrix} \\ &= \begin{pmatrix} \zeta - T(S\zeta + \Omega_y a) \\ a - \Omega_x^*(\Omega_y a + S\zeta) \end{pmatrix} \end{aligned}$$

whenever $\zeta \in F$ and $a \in \widetilde{A}^n$. If $(\zeta + a) \in \ker \widetilde{S}$, then $P(\zeta + a) = \zeta + a$, hence $\zeta \oplus a$ is in the range of P, i.e., $\ker \widetilde{S} \subset \operatorname{Im} P$.

Hence, we shall compute $\boldsymbol{rank}\ (\operatorname{Im} P)$. Since $\widetilde{S}$ is a regular Fredholm operator, $\operatorname{Im} P = \ker \widetilde{S}$ is a finite-rank module (4.1.3), i.e., $I_{\operatorname{Im} P}$ can be written as $\Omega_\phi \Omega_\psi^*$ for some $m \in \mathbb{N}$ and a pair of tuples $\phi, \psi \in (F \oplus B^n)^m$, hence

$$I_{\operatorname{Im} P} = \Omega_\phi\Omega_\psi^* \quad \Rightarrow \quad P = \Omega_\phi\Omega_\psi^* P$$

Replacing if necessary each coordinate ϕ_i with $P\phi_i$ if necessary, we can assume that $P\Omega_\phi = \Omega_\phi$. This will lead us to the next claim:

(*iii*) $\boldsymbol{Q := \Omega_\psi^*\Omega_\phi \in \mathscr{L}(\widetilde{A}^n)}$ **is an idempotent operator**: Indeed:

$$Q^2 \overset{P\Omega_\phi = \Omega_\phi}{=} (\Omega_\psi^* P\Omega_\phi)^2 = \Omega_\psi^* P \underbrace{(\Omega_\phi\Omega_\psi^*)}_{P} P\Omega_\phi = \Omega_\psi^*\Omega_\phi = Q.$$

Therefore, Q is an idempotent operator in $\mathscr{L}(\widetilde{A}^n)$ which corresponds to left multiplication by the matrix $(\langle \phi_i, \psi_j \rangle)_{i,j}$, and $\operatorname{Im} Q \simeq \operatorname{Im} P$ as $\widetilde{A}$-modules.

(*iv*) $\boldsymbol{Tr}\, \boldsymbol{\varepsilon(Q) = n}$: Let $(e_r)_r$ be the canonical basis of $\widetilde{A}^n$. We shall write the coordinates of ϕ and ψ as:

$$\phi_i = \zeta_i + a_i \quad \text{and} \quad \psi_i = \xi_i + b_i$$

for $\zeta_i, \xi_i \in F$ and $a_i, b_i \in \widetilde{A}^n$. Hence $\varepsilon(\langle\psi_i, \phi_i\rangle) = \varepsilon(\langle b_i, a_i\rangle)$ which enables us to expand in the following way:

$$\begin{aligned} \boldsymbol{Tr}\, \varepsilon(Q) &= \sum_{i=1}^{m} \varepsilon\left(\langle\psi_i, \phi_i\rangle\right) \\ &= \sum_{i=1}^{m} \varepsilon\left(\langle b_i, a_i\rangle\right) \\ &= \varepsilon\left(\sum_{i=1}^{m}\sum_{r=1}^{n} \langle b_i, e_r\rangle \langle e_r, a_i\rangle\right) \\ &= \varepsilon\left(\sum_{i=1}^{m}\sum_{r=1}^{n} \langle e_r, a_i \langle b_i, e_r\rangle\rangle\right) \\ &= \varepsilon\left(\sum_{r=1}^{n} \langle (0, e_r), P(0, e_r)\rangle\right) \end{aligned}$$

Using the definition of P in (4.2), the term $\sum_{r=1}^{n} \langle (0, e_r), P(0, e_r)\rangle$ can be expressed as

$$\sum_{r=1}^{n} \langle e_r, (I_{B^n} - \Omega_x^* \Omega_y) e_r\rangle = n \cdot 1_A - \sum_{r=1}^{n} \langle x_r, y_r\rangle$$

hence $\boldsymbol{Tr}\, \varepsilon(Q) = n$.

With all these steps, we conclude that

$$\boldsymbol{rank}\, (\ker \widetilde{S}) = \boldsymbol{rank}\, (\operatorname{Im} P) = \boldsymbol{rank}\, (\operatorname{Im} Q) = \boldsymbol{Tr}\, \varepsilon(Q) \cdot 1_{K_0(A)} = n \cdot 1_{K_0(A)}$$

and finally that $\varepsilon_0(\boldsymbol{ind}\, \widetilde{T}) = 0$. $\square$

The statement of 4.2.3 is meant to refer to the specific construction of $\widetilde{T}$ obtained in 4.2.2. But note that any regular Fredholm operator in $\mathscr{L}_{A^u}(E \oplus \widetilde{A}^n, F \oplus \widetilde{A}^n)$, which has T in the upper left corner, will differ from the $\widetilde{T}$ above, by an $\widetilde{A}$-compact operator. Therefore its index will coincide with that of $\widetilde{T}$ by 4.2.1, and so will be in $K_0(A)$ as well.

Definition 4.2.4. If T is a Fredholm operator in $\mathscr{L}(E, F)$, then the Fredholm index of T, denoted $\boldsymbol{ind}\, T$, is defined to be the index of the regular Fredholm operator $\widetilde{T}$ constructed in proposition 4.2.3.

It is clear that all properties listed in 4.1.5 are naturally extended to general Fredholm operators;

As consequence of the Atikinson's theorem in the classical theory, one can obtain that the original index is locally constant. Since it is now our definition, we can extract the same proof.

Proposition 4.2.5. Let $(E,F)Q := \mathscr{L}(E,F)/\mathscr{K}(E,F)$ be the Calkin algebra and $\pi : \mathscr{L}(E,F) \longrightarrow \mathscr{Q}(E,F)$ be the quotient projection. The set of Fredholm operators $\mathscr{F}(E,F) := \pi^{-1}\operatorname{GL}\mathscr{Q}(E,F) \subset \mathscr{L}(E,F)$ is an open subset and $\boldsymbol{ind} : \mathscr{F}(E,F) \longrightarrow K_0(A)$ is locally constant.

Proof. The fact that $\mathscr{F}(E,F)$ is an open set follows from the continuity of π on the the invertible elements of a unital C^*-algebra A.1.11.

To check the continuity of the index, let T be a Fredholm operator and S be one of its pseudo-inverses. If R is a Fredholm operator in the open ball around T of radius $\|S\|^{-1}$,

$$\|TS - RS\| \leqslant \|T - R\|\|S\| \leqslant 1.$$

Hence $I - (TS - RS)$ is a invertible Fredholm operator, which means that it has index 0. Notice that $(I - (TS - RS))T = RST$. Therefore:

$$\begin{aligned} \boldsymbol{ind}\, T &= \boldsymbol{ind}\,((I - (TS - RS))T) \\ &= \boldsymbol{ind}\,(RST) = \boldsymbol{ind}\, R + \boldsymbol{ind}\, S + \boldsymbol{ind}\, T \end{aligned}$$

hence $\boldsymbol{ind}\, R = -\boldsymbol{ind}\, S$. Since R was an arbitrary element of the open ball, it follows necessarily that $\boldsymbol{ind} \restriction_{B(T,\|S\|^{-1})}(R) = -\boldsymbol{ind}\, S$, i.e., the index is locally constant. $\square$

Proposition 4.2.6. Choose A-Fredholm operators such that

$$\begin{tikzcd} E \arrow[r, "T_1"'] \arrow[rr, bend left, "T_2T_1"] & F \arrow[r, "T_2"'] & G \end{tikzcd}$$

Therefore, T_2T_1 is a Fredholm operator and $\boldsymbol{ind}\,(T_2T_1) = \boldsymbol{ind}\, T_1 + \boldsymbol{ind}\, T2$.

Proof. Consider $H_t : F \oplus E \longrightarrow G \oplus E$ given by

$$H_t := \begin{pmatrix} T_2 & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & T_1 \end{pmatrix} = \begin{pmatrix} \cos(t)T_2 & -\sin(t)T_2T_1 \\ \sin(t) & \cos(t)T_1 \end{pmatrix}$$

for $t \in [0,\pi]$. Since $\boldsymbol{ind}\,(H_0) = \boldsymbol{ind}\, T_2 + \boldsymbol{ind}\, T_1$ and $\boldsymbol{ind}\,(H_\pi) = \boldsymbol{ind}\,(T_2T_1)$ $\square$

***Remark* 4.2.7.** In light of example 4.0.5, let $T : X \longrightarrow \mathscr{F}(\ell^2(\mathbb{N}))$ be a continuous family of classical Fredholm operators. We saw that $\widehat{T} : \xi \longmapsto T_{(\,\cdot\,)}\xi(\,\cdot\,)$ for $\xi \in \mathscr{H}_{C(X)}$ was a $C(X)$-Fredholm operator.

However, since $\ell^2(\mathbb{N})$ is a separable Hilbert space, one can consider an orthogonal basis $(e_n)_n \subset \ell^2(\mathbb{N})$ and let H_n be the closed subspace spanned by $e_0, e_1, \ldots, e_{n-1}$, and let p_n be the associated projection. Let $\operatorname{Vect}(X)$ denote the abelian semi-group of isomorphism classes of all (complex) vector bundles over X.

Using topological K-theory, one can define a analytical index given by

$$\boldsymbol{ind}_{\,\mathrm{an}} T \coloneqq [\ker p_n \circ T]_0 - [X \times H_n^{\perp}]_0$$

for a sufficiently large n, where $[\,\cdot\,]_0$ is the isomorphism class in $\mathrm{Vect}(X) \hookrightarrow K(X)$.

Proposition 4.2.8. For any $\alpha \in K_0(A)$, there exists a Fredholm operator T with $\boldsymbol{ind}\,(T) = \alpha$.

Proof. Write $\alpha = [p]_0 - [q]_0$ with $\varepsilon_0([p]_0 - [q]_0) = 0$, for self-adjoint idempotent matrices. Hence $\varepsilon(p)$ and $\varepsilon(q)$ are similar matrix. After performing a conjugation of, say q, by a complex unitary matrix, we may assume that $\varepsilon(p)$ and $\varepsilon(q)$ are in fact equal, hence $(p - q) \in \mathbb{M}_n(A)$. With bricks in hands, choose

$$\begin{aligned} T: \quad pA^n &\longrightarrow qA^n \\ x &\longmapsto qx \end{aligned}$$

which is

(*i*) ***T* is a Fredholm operator**: Let $S : qA^n \longrightarrow pA^n$ be the similar operator given by $Sy = py$. Let $(u_\lambda)_\lambda$ be an approximate identity for A. Therefore, consider the tuples ξ and η^λ, where their coordinates are given by:

$$\xi_i = p(p-q)p_i \quad \text{and} \quad \eta_i^\lambda = p_i u_\lambda \qquad (1 \leqslant i \leqslant n)$$

With the tuples defined, remember that $\langle a, b\rangle_A = a^* b$, hence:

$$\begin{aligned} \Omega_\xi \Omega^*_{\eta^\lambda} x &= \sum_{i=1}^n \xi_i \langle \eta_i^\lambda, x\rangle_A \\ &= \sum_{i=1}^n \xi_i\big((p_i u_\lambda)^* x\big) \\ &= \sum_{i=1}^n p(p-q)p_i u_\lambda p_i^* x \end{aligned}$$

for each $x \in pA^n$. Therefore, the following converges uniformly:

$$\begin{aligned} \lim_\lambda \Omega_\xi \Omega^*_{\eta^\lambda} x &= \sum_{i=1}^n p(p-q)p_i p_i^* x \\ &= p(p-q)px \qquad\qquad (\|x\| \leqslant 1) \\ &= x - pqx = (I - ST)x \end{aligned}$$

The above shows that $I_{pA^n} - ST$ is compact, and the same conclusions can be drawn for $I_{qA^n} - TS$ also. By applying 4.0.2, the claim is proved.

(*ii*) $\boldsymbol{ind}\, \boldsymbol{T} = \boldsymbol{\alpha}$: In order to compute the index of T, consider the operators

$$T' = \begin{pmatrix} qp & q(I-p) \\ (I-q)p & (I-q)(I-p) \end{pmatrix} \quad \text{and} \quad S' = \begin{pmatrix} pq & p(I-q) \\ (I-p)q & (I-p)(I-q) \end{pmatrix}$$

Direct computation shows that

$$S'T' = \begin{pmatrix} I_{pA^n} & 0 \\ 0 & I_{\widetilde{A}^n}-p \end{pmatrix} \quad \text{and} \quad T'S' = \begin{pmatrix} I_{qA^n} & 0 \\ 0 & I_{\widetilde{A}^n}-q \end{pmatrix},$$

from which it follows that S' is a pseudo-inverse for T' and hence that T' is a regular A^u-Fredholm operator.

By construction, $I - S'T'$ and $I - T'S'$ are compact self-adjoint idempotents, hence their ranges are finite rank modules (3.5.4). Moreover, we already know that

$$\operatorname{Im}(I - S'T') = \ker T' \quad \text{and} \quad \operatorname{Im}(I - T'S') = \ker S'$$

which turns possible the index calculation:

$$\begin{aligned} \boldsymbol{ind}\, T &= \boldsymbol{ind}\, T' \\ &= \boldsymbol{rank}\, (\ker T') - \boldsymbol{rank}\, (\ker S') \\ &= \boldsymbol{rank}\, \operatorname{Im}(I - S'T') - \boldsymbol{rank}\, \operatorname{Im}(I - T'S') \\ &= [p]_0 - [q]_0 = \alpha \end{aligned}$$

as desired. □

4.3 Fredholm Picture of $K_0(A)$

We already saw that every element of $K_0(A)$ is the index of some Fredholm operator.

In order to avoid pissing any set-theoretical reader, choose ω to be one of your favorite cardinal numbers, as long as it is greater than the cardinality of each and every A^n for every integer n. Denote by $F_0(A)$ the family of all A-Fredholm operators whose domain and codomain are Hilbert modules with cardinality no larger than ω. We will introduce a equivalence relation in $F_0(A)$, by characterizing whenever two operators T_1 and T_2 contains the same index.

Notice that whenever $T \oplus I_{A^n}$ is a compact perturbation of an invertible operator, one can conclude by 4.1.7 that $\boldsymbol{ind}\, T = 0$. This is a good indicator for an equivalence relation:

Proposition 4.3.1. Let $T \in \mathscr{L}(E, F)$ be Fredholm operator with $\boldsymbol{ind}\, T = 0$. Therefore, there exists some integer n such that $T \oplus I_{A^n}$ is a compact perturbation of an invertible operator.

Proof. Let $\widetilde{T}$ be the regularization of T and, as always, there is an integer n and a operator S such that $I - \widetilde{S}\widetilde{T} = 0 \oplus I_{\widetilde{A}^n}$ described in the proof of Lemma 4.2.1 (4.1).

(*i*) **There exists $\boldsymbol{n}$ such that $\mathbf{Im}(\boldsymbol{I}-\widetilde{\boldsymbol{T}}\widetilde{\boldsymbol{S}})\cong\widetilde{\boldsymbol{A}}^{\boldsymbol{n}}$**: The hypothesis of null index is equivalent to $\boldsymbol{rank}\ \mathrm{Im}(I-\widetilde{S}\widetilde{T})=\boldsymbol{rank}\ \mathrm{Im}(I-\widetilde{T}\widetilde{S})$, hence

$$\boldsymbol{rank}\ \mathrm{Im}(I-\widetilde{T}\widetilde{S})=\boldsymbol{rank}\ \mathrm{Im}(0\oplus I_{\widetilde{A}^n})=n\cdot 1_{K_0(A)}$$

i.e., $\mathrm{Im}(I-\widetilde{T}\widetilde{S})$ is stably isomorphic to $\widetilde{A}^n$ as $\widetilde{A}$-modules, meaning that for some integer r, $\mathrm{Im}(I-\widetilde{T}\widetilde{S})\oplus\widetilde{A}^r\simeq\widetilde{A}^{n+r}$. Remember that $\Omega_x^*=(\langle x_i,\cdot\rangle)_{i\leqslant n}$, hence, for $0\in E^m$, $\Omega^*_{(x,0)}(\,\cdot\,)=(\Omega^*_x(\,\cdot\,),0)$ and the regularization $\widetilde{T}$ can be updated to

$$\widetilde{T}=\begin{pmatrix}(T,0) & 0\\ \Omega^*_{(x,0)} & 0\end{pmatrix}$$

As seen, n can be increased without essentially changing $\widetilde{T}$. Therefore, there is no danger in assuming that $\mathrm{Im}(I-\widetilde{T}\widetilde{S})\simeq\widetilde{A}^n$.

(*ii*) **There is a orthonormal generating set $((\boldsymbol{\zeta_i}+\boldsymbol{a_i}))_{\boldsymbol{i}\leqslant\boldsymbol{n}}\subset\mathbf{Im}(\boldsymbol{I}-\widetilde{\boldsymbol{T}}\widetilde{\boldsymbol{S}})$ such that $\boldsymbol{\varepsilon}((\boldsymbol{a_1},\ldots,\boldsymbol{a_n}))=I_{\mathbb{M}_n(\widetilde{A})}$**: Let $(p_i)_{i\leqslant n}\subset\mathrm{Im}(I-\widetilde{T}\widetilde{S})$ with $p_i=(\zeta_i+a_i)\in F\oplus\widetilde{A}^n$. The elements p_i can be choosen so that they generate the module and $\langle p_i,p_j\rangle=\delta_{i,j}$, i.e., they are orthonormal. Since each $a_i\in\widetilde{A}^n$, one can write $a_i=(a_{i,r})_{r\leqslant n}$. Hence, orthonormality can be written as:

$$\begin{aligned}\delta_{i,j} &= \langle p_i,p_j\rangle_{F\oplus\widetilde{A}^n}\\ &= \langle\zeta_i,\zeta_j\rangle_F+\langle a_i,a_j\rangle_{\widetilde{A}^n}\\ &= \langle\zeta_i,\zeta_j\rangle_F+\sum_{r=1}^n\langle a_{i,r},a_{j,r}\rangle_{\widetilde{A}}\\ &= \langle\zeta_i,\zeta_j\rangle_F+\sum_{r=1}^n a^*_{i,r}a_{j,r}\end{aligned}$$

The projected matrix $u:=\varepsilon((a_{i,r})_{i,r})$ is unitary, i.e., $uu^*=u^*u=I_{\mathbb{M}_n(\widetilde{A})}$. Whence, setting $q_i:=\sum_j u^*_{ij}p_j$, we obtain $q_i=\xi_i+b_i$ in which $\varepsilon(b_{i,j})=\delta_{i,j}$, i.e., $\varepsilon((b_1,\ldots,b_n))=I_{\mathbb{M}_n(\widetilde{A})}$. At the end of the day, one can suppose that $(p_i)_{i\leqslant n}$ attends the required condition, otherwise, replace p by q.

(*iii*) $\boldsymbol{U}:=\left(\begin{smallmatrix}\boldsymbol{T} & \boldsymbol{\Omega_\zeta}\\ \boldsymbol{\Omega^*_x} & \boldsymbol{\Omega_a}\end{smallmatrix}\right)$ **is invertible**: As in the proof of 4.2.1, we suppose that $I-ST=\Omega_y\Omega^*_x$. By the features ensured about the elements $a=(a_1,\ldots,a_n)$, notice that $\Omega_a:\widetilde{A}^n\longrightarrow\mathrm{Im}(I-\widetilde{T}\widetilde{S})$ is an isomorphism.

Composing with $\widetilde{T}$, produces an operator

$$\begin{aligned}U:\ \ E\oplus\widetilde{A}^n &\longrightarrow F\oplus\widetilde{A}^n\\ \xi+b &\longmapsto \left(\begin{smallmatrix}T & \Omega_\zeta\\ \Omega^*_x & \Omega_a\end{smallmatrix}\right)\left(\begin{smallmatrix}\xi\\ b\end{smallmatrix}\right)\end{aligned}$$

The complex component can be forget, so that $\overline{(E \oplus \widetilde{A}^n) \cdot A} = A^n$ for any Hilbert A-module E. Thus, U can be restricted to an element in $\mathrm{GL}\,\mathscr{L}(E \oplus A^n, F \oplus A^n)$. With this in mind, notice that the difference operator

$$U - T \oplus I_{A^n} = \begin{pmatrix} 0 & \Omega_\zeta \\ \Omega_x^* & ((a_{ij} - \delta_{ij}))_{i,j} \end{pmatrix}$$

is compact, since the right lower entry $((a_{ij} - \delta_{ij}))_{i,j}$ is compact. But this matrix was seen to be in $\mathbb{M}_n(A)$, since its projection by ε is zero. $\square$

As consequence, proposition 4.3.1 immediately characterizes whenever two Fredholm operators between different Hilbert modules have the same index.

Corollary 4.3.2. Whenever two Fredholm operators $T_i \in \mathscr{L}(E_i, F_i)$ $(i \in \{1, 2\})$ share the same index $\textit{ind}\, T_1 = \textit{ind}\, T_2$, there exists a integer n such that

$$T_1 \oplus T_2^* \oplus I_{A^n} : E_1 \oplus F_2 \oplus A^n \longrightarrow E_2 \oplus F_1 \oplus A^n$$

is an A-compact perturbation of a invertible operator.

Declare two operators in $T_1, T_2 \in F_0(A)$ to be equivalent whenever $T_1 \oplus T_2^* \oplus I_{A^n}$ is a compact perturbation of an invertible operator, i.e., $\textit{ind}\, T_1 = \textit{ind}\, T_2$. Denote $F(A)$ the be the induced set of equivalence classes, which is an abelian group when equipped with the direct sum operation $\oplus$, where $(\,\cdot\,)^{-1} : T \longmapsto T^*$.

Considering the index map between $F(A)$ and $K_0(A)$ is the most natural think up to this point, since 4.2.8 already shows that it is a surjective map, and the equivalence relation ensures the injectivity. More over, since $[\mathrm{diag}(x, y)]_0 = [x]_0 + [y]_0$ in $K_0(A)$, we had produced the following Atiyah-Jänich analogue:

Corollary 4.3.3. The index map

$$\textit{ind} \; : F(A) \longrightarrow K_0(A)$$

is a group isomorphism.

Not only analogue, equivalent: Let $K^0(X)$ be the K-group of a compact Hausdorff space $X \in \mathbf{CHaus}$, in the realm of topological K-theory.

Appendix A

Usefull and handy theorems in C^*-algebras

A.1 C^*-algebras 101

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Definition A.1.1. In a (complex) algebra A, it is called an *involution* an operation $a \longmapsto a^*$ attending: conjugate-linearity, order 2 and anti-commutativite. That is:

$$(a+\lambda b)^* = a^*+\overline{\lambda}b^*; \qquad a^{**} = a; \qquad (ab)^* = b^*a^* \qquad (a, b \in A, \lambda \in \mathbb{C})$$

If $(A, \|\cdot\|)$ is a Banach algebra[1] and the involution satisfies $\|a^*a\| = \|a\|^2$ for all a (C^*-property), then A its said to be an C^**-algebra*. Setting $*$-morphisms to be the algebraic morphisms that respect involution[2], we construct the category of C^*-algebras $\boldsymbol{C^*}\text{-}\mathbf{Alg} := \boldsymbol{C^*}\text{-}\mathbf{Alg}(\mathbb{C})$.

Examples A.1.2. Some of the following examples are relatively bizarre for an initial view of the theory, but these represent important applications of C^*-algebras.

(i) The complex conjugation in $\mathbb{C}$ naturally fits into the definition of an involution. Since $\mathbb{C}$ is complete, it is a C^*-algebra.

(ii) The square matrices algebra $\mathbb{M}_{n\times n}(\mathbb{C})$ with $M \longmapsto \overline{M}^t$ as the given involution is fated to also be an C^*-algebra.

(iii) For any $x \in \mathbf{Top}$ and $A \in \boldsymbol{C^*}\text{-}\mathbf{Alg}$, the algebras $C_b(x, A)$ and $C_0(x, A)$ of bounded and infinity-vanishing functions respectively, endowed with the point-wise involution[3] also becomes C^*-algebras.

[1] Norm Cauchy-complete and submultiplicative $\|ab\| \leqslant \|a\| \cdot \|b\|$.

[2] That is to say, linear maps $\phi : A \longrightarrow B$ such that $\phi(a^*) = \phi(a)^*$ for every $a \in A$.

[3] $f^* : x \longmapsto f(x)^*$ for any $f : x \longrightarrow A$.

(*iv*) **Concrete C^*-algebras**: Every $*$-closed subalgebra[4] of bounded operators T over a Hilbert space fits the above definition, since $\langle Tx, y\rangle = \langle x, T^*y\rangle$. In fact, the first C^*-algebras in town where precisely these, called the *concrete* C^*-algebras. [24] introduced the term to describe precisely these mathematical beings, to which he refers to as "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space".

(*v*) **Irrational rotation algebra**: Let $\theta \in \mathbb{R}\setminus\mathbb{Q}$ be an irrational number. For $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, consider $L^2(S^1)$ the Hilbert space of square-integral functions[5] and the operators U and V given by:

$$Uf(z) := z{\cdot}f(z) \quad \text{and} \quad Vf(z) := f(e^{2\pi i\theta}z) \qquad (z \in S^1)$$

One should check that $VU = e^{2\pi i\theta}UV$. The C^*-algebra generated by $\{U, V\}$ is denoted by A_θ. It is known that A_θ is a simple algebra (contains no bilateral ideals) and also that for $0 < \theta_1 < \theta_2 < 1/2$, the algebras A_{θ_1} and A_{θ_2} are not isomorphic ([22], Theorem 2). The proofs of these two facts are closely linked to the great advance of the theory of C^*-algebras in the last twenty-five years.

(*vi*) **Toeplitz algebras and the Hardy Space**: Consider the *shift* operator $T : H \longrightarrow H$ in separable Hilbert space H, given by $Te_n := e_{n+1}$ where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis. The *Toeplitz algebra* is the smallest C^*-subalgebra $\mathscr{T}(H) \subset \mathscr{B}(H)$ containing T.

Is a prototypical example of a Toeplitz algebra the *Hardy space* $\mathscr{H}^2(S^1) \subset L^2(S^1)$ whose elements are precisely those $f \in L^2(S^1)$ in which all the negative Fourier coefficients are zero.

One of the most surprising observations about Toeplitz algebras is the fact that $\mathscr{T}(H)$ does not depend on H nor on the orthonormal basis $(e_n)_n$ (up to isomorphism).

> **Summoning A.1.3** ([9], Theorem 7.23)**.** Let H be a Hilbert space and $\mathscr{K}(H) \subset \mathscr{T}(H)$ be the ideal of compact operators over H. There exists a $*$-isomorphism
>
> $$\mathscr{T}(H)\big/\mathscr{K}(H) \longrightarrow C(S^1)$$
>
> mapping the class of the shift operator into the identity $I_{C(S^1)}$. ☕

In fact, $\mathscr{T}(H)$ is the universal unital C^*-algebra generated by an isometry, providing a more categorical definition of an Toeplitz algebra:

[4] A subalgebra A for which $T \in A \Rightarrow T^* \in A$.

[5] $f : S^1 \longrightarrow \mathbb{C}$ whose $\int |f(z)|^2 \mathrm{d}z < \infty$.

> **Summoning A.1.4** (Coburn)**.** For any unital $A \in \boldsymbol{C^*}\text{-}\,\mathbf{Alg}^{\boldsymbol{u}}$ and any isometry $w \in A$, there is a unique unital $*$-morphism $\mathscr{T}(H) \longrightarrow A$ mapping the shift operator into w.

To list one important aplication, [6] provide a simple proof of Bott periodicity for C^*-algebras, in which only the functorial properties of K_0 and K_1 are required. The heart of the proof consists in the usage of the short exact sequence:

$$0 \longrightarrow \mathscr{K}(\ell^2(\mathbb{N})) \longrightarrow \mathscr{T}(\ell^2(\mathbb{N})) \longrightarrow C(S^1) \longrightarrow 0.$$

Definition A.1.5 (Invertible elements)**.** An element $a \in A$ in a unital ring, is said to be *invertible* when there exists $b \in A$ such that $ab = ba = 1$. The set of invertibles are denoted by $\mathrm{GL}(A)$.

Proposition A.1.6. Let A be a unital algebra and let $a, b, c \in A$.

(*i*) If a is invertible, then there is a unique b such that $ab = ba = 1$, and therefore, it can be denoted by a^{-1}.

(*ii*) If $ab = ca = 1$, then $b = c$ and hence a is invertible.

(*iii*) If $ab = ba$, then ab is invertible if and only if both a and b are invertible.

(*iv*) If A is an involution algebra, $a \in \mathrm{GL}(A) \Rightarrow (a^*)^{-1} = (a^{-1})^*$.

Proof.

(*i*) Whenever b and b' are the interesting ones, $b(ab) = b(ab') = (ba)b' = b'$.

(*ii*) Notice that $c = c(ab) = (ca)b = b$.

(*iii*) If a and b are invertible, therefore

$$ab(a^{-1}b^{-1}) = baa^{-1}b^{-1} = 1 = a^{-1}b^{-1}ba = (a^{-1}b^{-1})ab.$$

Now, suppose that ab is invertible and let $c \coloneqq (ab)^{-1}$.

(*iv*) Using the anti-commutativity of the involution,

$$a^*(a^{-1})^* = (a^{-1}a)^* = 1 = (aa^{-1})^* = (a^{-1})^*a^*,$$

that is to say, $(a^*)^{-1} = (a^{-1})^*$. □

Proposition A.1.6 shows that $\mathrm{GL}(A)$ is a group under multiplication.

Definition A.1.7 (Spectrum)**.** For a unital Banach algebra A, the *spectrum* of an element $a \in A$ is the set of scalar λ such the elements $a - \lambda := a - \lambda 1$ aren't invertible, i.e.,

$$\operatorname{Spec}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda \notin \operatorname{GL}(A)\}.$$

For non-unital Banach algebras, since A can be isometrically embeded into a unital algebra A^u, $\operatorname{Spec}(a)$ stand for the spectrum over its identification.

Examples A.1.8.

(*i*) Let x be a compact Hausdorff topological space, we'll show that $\operatorname{Spec}(f) = \operatorname{Im} f$ for every continuous function $f \in C(x)$. Given $\lambda \in \operatorname{Spec}(f)$, notice that $f - \lambda \mathit{1}$ is non-invertible if and only if, there exists $x \in x$ such that $(f - \lambda \mathit{1})(x) = 0$, which means $f(x) = \lambda$.

(*ii*) Whenever $T \in \mathbb{M}_n(\mathbb{C})$, $\operatorname{Spec} T$ stands for the set of eigenvalues of T, due to the fact that those values λ are the ones that $\det(T - \lambda I_n) = 0$; precisely the operators which $T - \lambda I_n$ aren't invertible.

(*iii*) Let A be a unital C^*-algebra and $a \in A$. Let λ be a complex number. By proposition A.1.6(*iv*) it is safe to say that $a - \overline{\lambda} = (a^* - \lambda)^*$ isn't invertible if and only if $a^* - \lambda$ isn't as well. Therefore, $\operatorname{Spec}(a^*) = \overline{\operatorname{Spec}(a)}$.

Lemma A.1.9. For elements a, b in a C^*-algebra A, $\operatorname{Spec}(ab)\backslash\{0\} = \operatorname{Spec}(ba)\backslash\{0\}$.

Proof. Assume that A is unital. It suffices to prove that $1 - ab \in \operatorname{GL}(A)$ implies $1 - ba \in \operatorname{GL}(A)$ Let $u := (1 - ab)^{-1}$. Then

$$\begin{aligned}(1 + bua)(1 - ba) &= 1 - ba + bua - buaba \\ &= 1 - b(1 - u + uab)a \\ &= 1 - b(1 - u(1 - ab))a = 1\end{aligned}$$

Similarly $(1 - ba)(1 + bua) = 1$. Hence $1 - ba \in \operatorname{GL}(A)$. □

Lemma A.1.10. In a unitary Banach algebra A, if $\lambda \in \mathbb{C}$, elements of the form $\lambda - a$ with $a \in A$, are invertible if $\|a\| < |\lambda|$. Furthermore, any algebraic morphism of the form $\phi : A \longrightarrow \mathbb{C}$ is continuous and $\|\phi\| \leqslant 1$.

Proof. Consider the sequence of partial sums of the form $1 + a/\lambda + a^2/\lambda^2 + \cdots + a^n/\lambda^n$, and notice that they forms a Cauchy sequence, so it converges to a element b, given that $\|a/\lambda\| < 1$. Notice that

$$\lambda\Big(1 - \frac{a}{\lambda}\Big) \underbrace{\lim_{n\to\infty} \sum_{j=0}^{n} \Big(\frac{a}{\lambda}\Big)^j}_{b} = \lim_{n\to\infty} \lambda\Big(1 - \frac{a^{n+1}}{\lambda^{n+1}}\Big) = \lambda$$

So $(\lambda - a)^{-1} = b$. Without loss of generality, we can assume that ϕ is non-null and $\phi(1) = 1$. As we have seen,

$$(\lambda - a)\, b = 1 \quad \Rightarrow \quad (\lambda - \phi(a))\phi(b) = 1,$$

meaning $\phi(a) \neq \lambda$. Since this holds for all λ such that $|\lambda| > \|a\|$, it follows that $|\phi(a)| \leqslant \|a\|$, which proves our claim that $\|\phi\| \leqslant 1$. □

Proposition A.1.11. The group $\mathrm{GL}(A)$ of invertible elements in a unitary Banach algebra A constitute an open set.

Proof. Let x be an invertible. Given y such that $\|x - y\| < \|x^{-1}\|^{-1}$, let's see that y will be an invertible as well. From the asssumption and submultiplicativity of the norm, its easy to see that

$$\|1 - x^{-1}y\| = \|x^{-1}(x - y)\| \leqslant \|x^{-1}\|\|x - y\| < \|x^{-1}\|\|x^{-1}\|^{-1} = 1,$$

hence $1 - (1 - x^{-1}y) = x^{-1}y$ is invertible by A.1.10. But since $\mathrm{GL}(A)$ is a group, y must also be invertible, i.e., the open ball around x of radious $\|x^{-1}\|^{-1}$ is fully contained in $\mathrm{GL}(A)$, so it is open. □

Proposition A.1.12. Over a unitary Banach algebra A, the invertion map $a \longmapsto a^{-1}$ is a continuous function.

Proof. Choose $x, y \in \mathrm{GL}(A)$ such that $\|x - y\| < \|x^{-1}\|^{-1}$. As showed in the proof of A.1.11, $\|1 - x^{-1}y\| < 1$ and therefore,

$$\begin{aligned} & y^{-1}x = 1 - (1 - x^{-1}y)^{-1} = \sum_{n=0}^{\infty}(1 - x^{-1}y)^n \\ \Rightarrow \quad & y^{-1} = \sum_{n=0}^{\infty}(1 - x^{-1}y)^n x^{-1} \end{aligned} \tag{A.1}$$

If we want to estimate how close x and y need to be in order that $\|y^{-1} - x^{-1}\|$ can be arbitrally small, (A.1) might help to found an upper bound:

$$\|y^{-1} - x^{-1}\| = \Big\| \sum_{n=1}^{\infty}(1 - x^{-1}y)^n x^{-1} \Big\| \leqslant \|x^{-1}\| \sum_{n=1}^{\infty} \|1 - x^{-1}y\|^n = \|x^{-1}\| \cdot \frac{\|1 - x^{-1}y\|}{1 - \|1 - x^{-1}y\|}$$

For a given $\varepsilon > 0$, since $t \longmapsto t/(1 - t)$ is a increasing continuous function for $t < 1$, there allways exists $\delta_{x,\varepsilon} > 0$ right for the job:

$$t < \delta_{x,\varepsilon} := \frac{\frac{\varepsilon}{\|x^{-1}\|}}{\frac{\varepsilon}{\|x^{-1}\|} + 1} \quad \Rightarrow \quad \frac{t}{1 - t} < \frac{\varepsilon}{\|x^{-1}\|},$$

so whenever $\|x - y\| < \delta_{x,\varepsilon}$, $\|y^{-1} - x^{-1}\| < \varepsilon$. □

Proposition A.1.13. The *resolvent function* (A.2) is holomorphic for each and every unital Banach algebra A.
(A.2)

$$\begin{aligned} R_a : \quad \mathbb{C}\setminus \mathrm{Spec}(a) &\longrightarrow A \\ \lambda &\longmapsto (\lambda - a)^{-1} = \sum_{n=0}^{\infty} \left(\frac{a}{\lambda}\right)^n \end{aligned} \qquad (a \in A)$$

Proof. Let $z, w \in \mathbb{C}\setminus \mathrm{Spec}(a)$. Let $\alpha \coloneqq z - a$ and $\beta \coloneqq w - a$. Notice that

$$\begin{aligned} R_a(z) - R_a(w) &= \alpha^{-1} - \beta^{-1} \\ &= \alpha^{-1}(1 - \alpha\beta^{-1}) \\ &= \alpha^{-1}(\beta - \alpha)\beta^{-1} \\ &= R_a(z)(w - z)R_a(w) \end{aligned}$$

Therefore:

$$\lim_{w \to z} \frac{R_a(z) - R_a(w)}{z - w} = \lim_{w \to z} -R_a(z)R_a(w) = -R_a(z)^2 \qquad (z \in \mathbb{C}\setminus \mathrm{Spec}(a))$$

So the limit exists, $\partial R_a = -R_a^2$ and it is analytic over the domain. $\square$

Theorem A.1.14. The spectrum of any element a in a Banach algebra is a compact non-empty set, and satisfies:

$$\mathrm{Spec}(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leqslant \|a\|\}.$$

Proof. Since life is really good and Heine-Borel's theorem holds for complex numbers, we need only to show that $\mathrm{Spec}(a)$ is closed and bounded.

(*i*) **Spec(a) is a closed set**. Since a is fixed element, notice the projection $a + \lambda \longmapsto \lambda$ is a continuous function by A.1.10 and so is their inverse $\lambda \longmapsto a - \lambda$. Since $\mathrm{Spec}(a)$ is the set-theoretic complement of the pre-image of $\mathrm{GL}(A)$ (an open set - A.1.11) by a continuous function, it must be closed.

(*ii*) **Spec(a) is bounded**. From A.1.10, spectral elements of the form $\lambda - a$ must necessarily obey $|\lambda| \leqslant \|a\|$, i.e., $\mathrm{Spec}(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leqslant \|a\|\}$.

Therefore, $\mathrm{Spec}(a)$ is compact set. In order to give a little bit of flavor, we present the proof of [26] the non-emptiness of the spectrum, with a clever tricky about derivation under the integral sign.

(*iii*) **Spec(a) is non-empty**. Suppose $a \neq 0$, since otherwise the result follows immediately. Assume that $\mathrm{Spec}(a)$ is empty, so that R_a exists in all complex plane. In particular, $a \in \mathrm{GL}(A)$. For any bounded linear functional $f : A \longrightarrow \mathbb{C}$, the function

$$\begin{aligned} g_f : \quad \mathbb{R}^2 &\longrightarrow \mathbb{C} \\ (r, \theta) &\longmapsto f(R_a(re^{i\theta})) \end{aligned}$$

is continuously differentiable with respect to both r and θ:

$$\partial_\theta g_f(r,\theta) = f\big(-R_a(re^{i\theta})^2\big)ire^{i\theta} \quad \text{and} \quad \partial_r g_f(r,\theta) = f\big(-R_a(re^{i\theta})^2\big)e^{i\theta}. \tag{A.3}$$

Hence $\partial_\theta g_f = ir\partial_r g$. Now let $F(r) := \int_0^{2\pi} g_f(r,\theta)\,\mathrm{d}\theta$. By differentiating under the integral sign[6] and using $\partial_r g_f$, we obtain

$$\begin{aligned} F'(r) &= \int_0^{2\pi} \partial_r g_f(r,\theta)\,\mathrm{d}\theta = \int_0^{2\pi} f\big(-R_a(re^{i\theta})^2\big)e^{i\theta}\,\mathrm{d}\theta \\ \Rightarrow \qquad irF'(r) &= \int_0^{2\pi} f\big(-R_a(re^{i\theta})^2\big)ire^{i\theta}\,\mathrm{d}\theta = \int_0^{2\pi} \partial_\theta g_f(r,\theta)\,\mathrm{d}\theta \\ &= f(R_a(re^{i2\pi})) - f(R_a(re^{i0})) = 0 \\ \Rightarrow \qquad F(r) &= F(0) = 2\pi f(a^{-1}) \end{aligned}$$

We now choose f so that $f\left(a^{-1}\right) \neq 0$. Let $h : \mathrm{Span}(a^{-1}) \longrightarrow \mathbb{C}$, $h(za^{-1}) = z\|a^{-1}\|$. Since $\|\cdot\|$ is a positive sublinear functional, we are able to conjure:

> **Summoning A.1.15** (Complex Hahn-Banach)**.** Let X be a $\mathbb{K}$-vector space with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and p a positive sublinear functional[a]. If $Y \subset X$ is a subspace and $h : Y \longrightarrow \mathbb{K}$ a linear functional dominated by p (i.e., $|h(y)| \leqslant p(y)$ for $y \in Y$), the there exists a *Hahn-Banach extension* $f : X \longrightarrow \mathbb{K}$ such that f is also dominated by p. ☕
>
> [a] I.e., for all vectors x, y and scalar α, $p(x) \geqslant 0$, $p(x+y) \leqslant p(x) + p(y)$ and $p(\alpha x) \leqslant |\alpha| p(x)$.

Therefore, choose a Hahn-Banach extension f of h such that $f(a^{-1}) = \|a^{-1}\| \neq 0$. Since

$$f(R_a(re^{i\theta})) = f\big((re^{i\theta}1 - a)^{-1}\big) = r^{-1}e^{-i\theta}f\big((1 - ar^{-1}e^{-i\theta})^{-1}\big)$$

it follows that $\big|f\big(R_a(re^{i\theta})\big)\big|$ can be made as small as we like, independently of θ, by choosing r sufficiently large, by the continuity of the invertion

[6] Such a differentiation is possible, since g is clearly a C^1-function by (A.3).

map $x \longmapsto x^{-1}$ (A.1.12). We fix r such that $|f(R_a(re^{i\theta}))| < |f(a^{-1})|/2$. For this r we have

$$2\pi|f(a^{-1})| = |F(r)| \leqslant \int_0^{2\pi} |f(a(re^{i\theta}))| \,\mathrm{d}\theta \leqslant \pi|f(a^{-1})|$$

which is not possible. Therefore, $\mathrm{Spec}(a)$ must in fact be non-empty.

□

Typically, in order to prove A.1.14(*iii*), one would argue that if otherwise, R_a would be *entire* function and conclude that it must constant by Liouville's theorem. The advantage of the argument presented, as showed in [26], is a more elementary proof, and both the fundamental theorem of Algebra and Liouville's theorem as corollary.

Corollary A.1.16 (Fundamental Theorem of Algebra)**.** If $p \in \mathbb{C}[z]$, there exists a complex root $w \in \mathbb{C}$ such that $p(w) = 0$.

Proof. In the algebra of complex square matrices $\mathbb{M}_n(\mathbb{C})$, set T as the operator in which p is the characteristic polynomial. If $p(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0$, one may let

$$T := \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}$$

By A.1.14(*iii*), there exists an eigenvalue $w \in \mathrm{Spec}\, T$ thus $p(w) = 0$. □

Corollary A.1.17 (Liouville's Theorem)**.** Let $A \in \boldsymbol{\mathscr{B}}\text{-}\mathbf{Alg}^{\boldsymbol{u}}$ be a unitary Banach algebra. If $f : \mathbb{C} \longrightarrow A$ is a bounded function which is holomorphic everywhere (i.e., *entire*), then it must be constant.

Proof. Let f be a bounded entire function. Fix an arbitrary $w \in \mathbb{C}$. Define g on $\mathbb{C}$ as follows:

$$g(z) := \begin{cases} (f(z+w) - f(w))/z & \text{if } z \neq 0. \\ f'(w) & \text{if } z = 0. \end{cases}$$

Then g is certainly continuous in $\mathbb{C}$ and analytic in $\mathbb{C}_{\neq 0}$. Define $F(r) = \int_0^{2\pi} g(re^{i\theta})\,\mathrm{d}\theta$. By the same idea used in the argument of A.1.14(*iii*), F must be constant. However, from the boundedness of f it follows that g tends to 0 uniformly in θ as r tends to infinity, so F tends to 0. Therefore, $F(r) = F(0) = 0 = 2\pi f'(w)$. As w was an arbitrary point of $\mathbb{C}$, we conclude that f' is identically zero, therefore f is constant. □

Theorem A.1.18 (Spectral Mapping Theorem)**.** Given an unital Banach algebra A and $a \in A$, the following holds for each complex-polynomial:

$$\mathrm{Spec}(p(a)) = \{p(\lambda) \mid \lambda \in \mathrm{Spec}(a)\} = p(\mathrm{Spec}(a)). \qquad (p \in \mathbb{C}[x])$$

Proof. Fix $p \in \mathbb{C}[x]$ and by the Fundamental Theorem of Algebra A.1.16, for each $\lambda \in \mathbb{C}$, it is possible to decompose $\lambda - p(z)$ into linear factors. By the Cayley-Hamilton theorem, it is safe to say that there is $\lambda_, \ldots, \lambda_n, \alpha \in \mathbb{C}$ such:

$$\lambda - p(a) = \alpha \prod_{i=1}^{n} (\lambda_i 1 - a). \qquad (a \in A)$$

If $\lambda_i \notin \mathrm{Spec}(a)$ for each index i, every factor $\lambda_i 1 - a$ is invertible, hence $\lambda - p(a)$ are as well. In order to $\lambda \in \mathrm{Spec}(p(a))$, there must exist a root λ_j such that $\lambda_j \in \mathrm{Spec}(a)$. In that case, $\lambda = p(\lambda_j) \in p(\mathrm{Spec}(a))$, and we shall conclude that $\mathrm{Spec}(p(a)) \subset p(\mathrm{Spec}(a))$.

To prove the other inclusion, let $\lambda \in p(\mathrm{Spec}(a))$, meaning that there is $\eta \in \mathbb{C}$ such that $\eta 1 - a$ is non-invertible and $p(\eta) = \lambda$. But because decomposition is unique, there must be j such that $\eta = \lambda_j$ and by the same argument, one concludes that $\mathrm{Spec}(p(a)) \supset p(\mathrm{Spec}(a))$, obtaining the equallity. □

***Remark* A.1.19.** In quantum mechanics, physical observables are represented mathematically by linear operators on Hilbert spaces. One remarkable corollary of the non-emptiness of the spectrum imply that, it is not possible to describe the position and momentum operators as elements of a Banach algebra.

> **Proposition**. The *Weyl algebra* $\mathbb{C}[x, \partial_x] \simeq \mathbb{C}[x,y] / (yx - xy - 1)$ does not embed into a Banach algebra.
>
> *Proof.* Suppose to the contrary that x, y are elements of a Banach algebra satisfying $yx - xy = 1$. From the spectral mapping theorem A.1.18,
>
> $$\mathrm{Spec}(yx) = \mathrm{Spec}(xy + 1) = \mathrm{Spec}(xy) + 1.$$
>
> But we know that $\mathrm{Spec}(yx) = \mathrm{Spec}(xy)$, so the spectrum of the product xy must be invariant under addition by 1. By non-emptyness, it must contain arbitrarily large elements, and this contradicts compactness. □

Hence, it is necessary to look at the unitary groups of strongly continuous functions that the position and momentum operators generate, through the Stone-von Neumann theorem, stating that, the canonical commutation relations on two generators (canonical coordinate q and canonical momentum

p) in the form $[q,p] = i\hbar$ may be represented as unbounded operators on the Hilbert space of square integrable functions $L^2(\mathbb{R})$ on the real line by defining them on the dense subspace of smooth functions $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ as

$$(q\psi)(x) \coloneqq x\psi(x) \quad \text{and} \quad (p\psi)(x) \coloneqq i\hbar\partial_x\psi(x),$$

where on the right we have the derivative along the canonical coordinate function on real numbers axis.

Corollary A.1.20 (Complex Gelfand-Mazur)**.** Any complex unital Banach division algebra A (every non-zero element is invertible) is isometrically isomorphic to $\mathbb{C}$.

Proof. Let a be a non-zero element, i.e., invertible. Since $\mathrm{Spec}(a)$ is non-empty A.1.14(*iii*), there must exist λ_a such that $a - \lambda_a 1$ isn't invertible. The only non-invertible elemen't in A is the zero element, which means that $a = \lambda_a 1$. □

Theorem A.1.21 (Beurling-Gelfand formula)**.** The *spectral radius* r of an element $a \in A$ can be directly calculated by the following formula:

$$r(a) \coloneqq \sup|\,\mathrm{Spec}(a)| = \lim_{n\to\infty} \sqrt[n]{\|a^n\|}$$

for *any Banach norm* $\|\cdot\|$.

Proof. By the Spectral Mapping theorem A.1.18 and the continuity of the inversion map A.1.12(*ii*), is easy to estabilish that $r(a)^n = r(a^n) \leqslant \|a^n\|$ for any given natural n, hence $r(a) \leqslant \inf \|a^n\|^{1/n}$. Notice that if $|\lambda| > r(a)$, then $\lambda - a$ is invertible by A.1.10. For any bounded linear $\phi : A \longrightarrow \mathbb{C}$, the operator

$$(\phi \circ R_a)\lambda = \phi((\lambda - a)^{-1}) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n}\phi(a^n) \qquad (|\lambda| \geqslant r(a))$$

is analytic, and hence, converges absolutely. That can only happend if the general term of the series $\phi(a^n/\lambda^n)$ converges to 0 for any bounded linear ϕ. Since every weakly convergent sequence is bounded (**??**), for each λ, there is $M_\lambda > 0$ such that $\|a^n\| \leqslant M_\lambda\,|\lambda^n|$. Therefore,

$$\limsup_{n\to\infty} \sqrt[n]{\|a^n\|} \leqslant \limsup_{n\to\infty} \sqrt[n]{M_\lambda}|\lambda| = |\lambda|$$

Since this holds for every $|\lambda| > r(a)$ we have

$$r(a) \leqslant \inf_{n\in\mathbb{N}} \sqrt[n]{\|a^n\|} \leqslant \liminf_{n\to\infty} \sqrt[n]{\|a^n\|} \leqslant \limsup_{n\to\infty} \sqrt[n]{\|a^n\|} \leqslant r(a),$$

which finishes the proof. □

Example A.1.22. In A.1.12(*ii*), $\sup|\operatorname{Spec}(a)| \leqslant \|a\|$, but this inequallity can be strict: Over $\mathbb{M}_{2\times 2}(\mathbb{C})$, notice that

$$\sup\left|\operatorname{Spec}\begin{pmatrix} 0 & 1/t \\ t & 0 \end{pmatrix}\right| = \sup|\{1,-1\}| = 1 \qquad (t>0)$$

but $\left\|\begin{smallmatrix} 0 & 1/t \\ t & 0 \end{smallmatrix}\right\| = t^2 + 1/t^2$.

Lemma A.1.23. Let $a \in A$ be an element of a C^*-algebra. If $a \in A$ is a self-adjoint, then $\operatorname{Spec}(a) \subset \mathbb{R}$.

Proof. Let $\lambda \in \operatorname{Spec}(a)$. For each integer n, let $b_n := a + (in\operatorname{Im}\lambda - \operatorname{Re}\lambda)$. Considering the rational function f given by $f(z) = z - \operatorname{Re}\lambda + in\operatorname{Im} y$ we conclude that $f(\lambda) \in \operatorname{Spec} f(a)$ from the Spectral Mapping theorem A.1.18. Hence, $i(n+1)\operatorname{Im}\lambda \in \operatorname{Spec}(b_n)$.

From A.1.10, one obtains that $|i(n+1)\operatorname{Im}\lambda| \leqslant \|b_n\|$. Therefore

$$\begin{aligned}
(n^2+2n+1)(\operatorname{Im}\lambda)^2 &= |i(n+1)\operatorname{Im}\lambda|^2 \\
&\leqslant \|b_n\|^2 \\
&= \|b_n^* b_n\| \\
&\overset{a^*=a}{=} \|(a-\operatorname{Re}\lambda - in\operatorname{Im}\lambda)(a-\operatorname{Re}\lambda + in\operatorname{Im}\lambda)\| \\
&= \|(a-\operatorname{Re}\lambda)^2 + n^2(\operatorname{Im}\lambda)^2\| \leqslant \|a-\operatorname{Re}\lambda\|^2 + n^2(\operatorname{Im}\lambda)^2
\end{aligned}$$

which implies that $(2n+1)(\operatorname{Im}\lambda)^2 \leqslant \|a-\operatorname{Re}\lambda\|^2$. Since n is arbitrary, it follows that $(\operatorname{Im}\lambda)^2$ is arbitrarily small, i.e., $\operatorname{Im}\lambda = 0$, hence $\operatorname{Spec}(a) \subset \mathbb{R}$. □

Lemma A.1.24. If $a \in A$ is a self-adjoint element of a unitary C^*-algebra, then $r(a) = \|a\|$.

Proof. We have $\|a^2\| = \|a^*a\| = \|a\|^2$. We appeal to induction: Suppose n is such that $\|a^{2^n}\|^{2^{-n}} = \|a\|$. Therefore:

$$\left\|a^{2^{n+1}}\right\|^{2^{-n-1}} = \sqrt{\left\|(a^{2^n})^2\right\|^{2^{-n}}} = \sqrt{\|a^{2^n}\|^{2^{-n}}} \overset{\text{I.H.}}{=} \sqrt{\|a\|^2} = \|a\|.$$

By induction, the above assumption is valid for all natural n. Therefore, the sequence $\left(\|a^n\|^{1/n}\right)_{n\in\mathbb{N}}$, which converges to $r(a)$ by A.1.21, has a constant subsequence equal to $\|a\|$. Then $r(a) = \|a\|$. □

A.2 Gelfand-Naimark representation theorem

To explore (*iii*), both bounded and infinity-vanishing functions can be seeing as a contra-variant functor from topological spaces to abelian C^*-algebras.

We will focus on the infinity-vanishing ones:

$$
C_0\colon \quad \mathbf{Top} \longrightarrow \boldsymbol{C^*}\text{- }\mathbf{Alg}^{\boldsymbol{com}}
$$

$$
\begin{array}{ccccc}
X & \longmapsto & C_0(X) & & \phi \circ f \\
{\scriptstyle f}\big\downarrow & & \big\uparrow & & \big\uparrow \\
Y & \longmapsto & C_0(Y) & & \phi
\end{array}
$$

Definition A.2.1 (Gelfand transformation)**.** For a given $A \in \mathscr{B}$-**Alg**, the *character space* of A is the set of non-zero morphisms $\Gamma A := \mathrm{Hom}_{\mathscr{B}\text{-}\mathbf{Alg}}(A, \mathbb{C})_{\neq 0}$. The *Gelfand transformation* is the evaluation functional $\kappa := \mathbf{ev}_{(\,\cdot\,)}$ given by:

$$
\begin{aligned}
\kappa :\quad A &\longrightarrow C_0(\Gamma A) \\
a &\longmapsto (\phi \overset{\mathbf{ev}_a}{\mapsto} \phi(a))
\end{aligned} \tag{A.4}
$$

Later, we will endow a topological flavor onto ΓA. Before, we gave a look into a alternative definition of the character space.

Proposition A.2.2. If A is a unitary commutative Banach algebra, let $\mathscr{I}_m(A)$ be the set of maximal ideals[7] of A. So the function $\ker(\,\cdot\,) : \Gamma A \longrightarrow \mathscr{I}_m(A)$ is a bijection. Since in algebraic lands $\mathscr{I}_m(A)$ is often called *spectrum of* A, the character space may also be called the *spectrum*.

Proof. Of course $\ker(\,\cdot\,)$ is well defined, i.e. $\ker\phi$ is always a maximal ideal of A for any character ϕ. That's because ϕ is necessarily surjective ($\phi\,(z1_A) = z\phi\,(1_A) = z$ for any $z \in \mathbb{C}$) and by the First Isomorphism Theorem, $A/\ker\phi \simeq \mathbb{C}$ is a field and therefore, $\ker\phi$ is maximal (since A is a commutative ring).

In order to $\ker(\,\cdot\,)$ be a bijection, we verify:

(*i*) **ker(·) is an injection**: Let $\phi, \psi \in \widehat{A}$ characters such that $\ker\phi = \ker\psi$. Notice that

$$
\phi(a)1_A - a \in \ker\phi = \ker\psi \qquad (a \in A)
$$

That is, $\psi\,(\phi(a)1 - a) = 0$ from which it follows that $\phi = \psi$.

(*ii*) **ker(·) is a surjection**: Let $I \triangleleft_m A$ be a maximal ideal and consider the projection $\pi_I : x \longmapsto x + I$. So $\ker\pi_I = I$. It remains to verify that π_I is a character but this will follow from the fact that A/I is a Banach algebra where all its elements are invertible.

[7] We say that $I \subset A$ is an ideal if I is closed by A-linear combinations, i.e., given $x, y \in I$ and $a, b \in A$, then $ax + by \in A$. We denote $I \triangleleft A$. We say that $I \triangleleft_m A$ is a maximal ideal if it is not trivial (like $\{0\}$ or A) and if any other ideal $J \triangleleft A$ such that $I \subset J \subset A$ collapses with either I or A.

Since I is maximal it follows that A/I is a field and in particular an algebra. Using the Gelfand-Mazur theorem A.1.20, all non-zero elements of A/I are all invertible and therefore we guarantee that $A/I \simeq \mathbb{C}$.

This concludes the proof. □

Lemma A.2.3. Endowed with the weak* topology, ΓA is a locally compact Hausdorff topological space. Whenever A is unitary, then ΓA is compact.

Proof. The weak* topology on $\Gamma A \subset \mathscr{B}(A, \mathbb{C})$ is induced by the *separant*[8] familly of seminorms $(p_a)_{a \in A}$ given by $p_a : \phi \longmapsto |\mathbf{ev}_a(\phi)| = |\phi(a)|$, in the sense that it is the smallest topological space such that each p_a be continuous. Therefore, ΓA is a Hausdorff topological vector space.

Let $S \coloneqq \mathrm{Hom}_{\boldsymbol{C^*}\text{-}\mathbf{Alg}}(A, \mathbb{C})$ for convention. We afirm that:

(i) ***S* is a closed subspace of the unity ball over $\mathscr{B}(A, \mathbb{C})$**: Let B be the unit ball on the dual space and choose $\phi \in B$. For any net $(\phi_\lambda)_\lambda \subset S$ whoose $\lim_\lambda \phi_\lambda = \phi$, one have

$$\begin{aligned} \phi(ab) &= \lim_\lambda \phi_\lambda(ab) \\ &\overset{\phi_\lambda \in S}{=} \lim_\lambda \phi_\lambda(a)\phi_\lambda(b) \qquad (a, b \in A) \\ &= \lim_\lambda \phi_\lambda(a) \lim_\lambda \phi_\lambda(b) = \phi(a)\phi(b) \end{aligned}$$

i.e., $\phi \in S$, hence it is a closed subspace.

(ii) **Whenever *A* is unitary, $\mathit{0} : x \longmapsto 0$ is a isolated point of *S***: For any net $(\phi_\lambda)_\lambda \subset S \backslash \{\mathit{0}\}$ converging to a given $\phi \in S$, notice that $\phi(1_A) = \lim_\lambda \phi_\lambda(1_A) = 1$. Hence $\phi \neq \mathit{0}$, i.e., $S \backslash \{\mathit{0}\}$ is a closed set.

To conclude, we shall invoque:

> **Summoning A.2.4** (Banach-Alaoglu - [1], Theorem 1.3)**.** The closed unit ball $\{f \in \mathscr{B}(X, \mathbb{C}) \mid \|f\|_\infty \leqslant 1\}$ over the dual of a complex normed space X is a compact Hausdorff space.

Banach-Alaoglu theorem A.2.4 in contrast with (i), guarantee us that $\Gamma A = S \backslash \{\mathit{0}\}$ is locally compact Hausdorff. By (ii), ΓA must be compact when A contais a unity. □

Example A.2.5. Suppose that for a Banach algebra A, the character space separates points, i.e., for any $a, b \in A$, there exists $\phi \in \Gamma A$ such that $\phi(a) \neq \phi(b)$. In this configuration, A, must necessarily be abelian:

$$\phi(ab - ba) = \phi(a)\phi(b) - \phi(b)\phi(a) = 0 \overset{\phi \neq \mathit{0}}{\Rightarrow} ab = ba.$$

[8] I.e., given an arbitrary a, there exists $\phi \in \Gamma A$ such that $|\phi(a)| \neq 0$

Example A.2.6. Given $A \in \mathscr{B}$**-Alg**, let $[A, A]$ be the closed linear span of the commutators $[a_1, a_2] = a_1 a_2 - a_2 a_1$. This is a closed two-sided ideal of A, and the quotient $A/[A, A]$ is the *abelianization* of A: the universal commutative Banach algebra to which A maps.

Any character of A must be a character of its abelianization, so, in order to determinate if ΓA is a empty set for a arbitrary A reduces immediately to the commutative case.

Example A.2.7. Give to $(\mathbb{C}^2, \|\cdot\|_{\max})$ the product $(z_1, z_2) \cdot (w_1, w_2) := (z_1 w_1, 0)$, turning $\mathbb{C}^2$ into a commutative non-unital Banach algebra. The character space of this algebra is a singleton.

Example A.2.8. $\mathscr{B}(\mathbb{C}^n)$ with $n \geqslant 2$ has only trivial two-sided ideals. Hence it has no characters.

Theorem A.2.9 (Gelfand theorem for $\mathscr{B}$-$\mathbf{Alg}_u^{com}$)**.** Let A be a commutative unital Banach algebra. Therefore,

$$a \in \mathrm{GL}(A) \quad \Leftrightarrow \quad \mathbf{ev}_a \in \mathrm{GL}(C_0(\Gamma A)) \quad \Leftrightarrow \quad \forall\, \phi \in \Gamma A, \phi(a) \neq 0.$$

Proof. Since A is unital, $1 = \phi(a)\phi(a^{-1})$ for each $a \in \mathrm{GL}(A)$, i.e., $\phi(a) \neq 0$ for every $\phi \in \Gamma A$. Now for the hard part, suppose that $\phi(a) \neq 0$ and we shall consider the set of every proper ideal $J \triangleleft A$ which contains aA. A technical issue is needed:

(*i*) **Every maximal ideal $J \triangleleft_m A$ is closed**: Let $J \triangleleft_m A$ and notice that $\overline{J} \supset J$ is also a maximal ideal. Suppose that $\overline{J} = A$, hence J is a dense subset. If B is the open ball of radius 1 centered at the unit, by density of J, there exists $x \in J \cap B$. Notice that by A.1.10, x is an invertible element, hence $1 = xx^{-1} \in J$, i.e., J isn't proper. Therefore, it must be the case that $J = \overline{J}$.

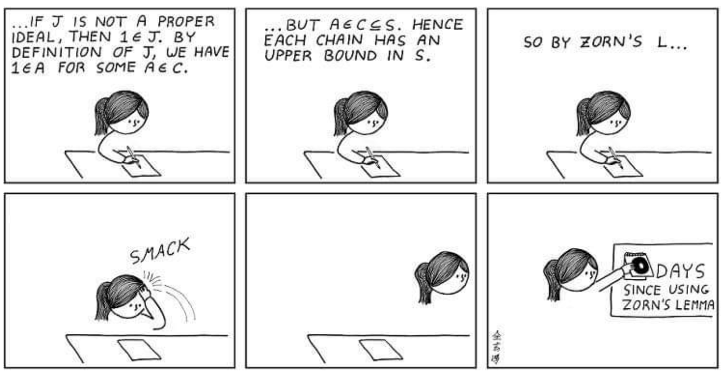

Figure A.1: You reading this proof

Back to business: Suppose $a \notin \mathrm{GL}(A)$. As strongly suggested by Figure A.1, we shall apply the Zorn Lemma to the set $\mathscr{C} := \{J \lhd A \mid aA \subseteq J \subsetneq A\}$ of proper ideals that contain the one generated by a, obtaining a maximal ideal $\mathcal{J} \lhd A$, closed by (i).

> **Summoning A.2.10** (Zorn's Lemma)**.** If, in a non-empty and partially ordered set $\mathscr{C}$, every fully ordered subset has an upper quota, then $\mathscr{C}$ has a maximal element.

Given $x \in A \backslash \mathcal{J}$, notice that $\widetilde{\mathcal{J}} := \{bx + j \mid b \in A, j \in \mathcal{J}\}$ contains properlly $\mathcal{J}$. Since it is maximal, $\widetilde{\mathcal{J}} = A$ resulting in $1_A - xy \in \mathcal{J}$ for some $y \in A$. Hence, every non-zero element of $A/\mathcal{J}$ is invertible.

$$\begin{tikzcd} A \arrow[d] \arrow[dr, dashed, "\phi"] & \\ A/\mathcal{J} \arrow[r, "\simeq"'] & \mathbb{C} \end{tikzcd}$$

By Gelfand-Mazur theorem A.1.20, $A/\mathcal{J} \simeq \mathbb{C}$. Considering the canonical projection, let ϕ be the one who commutes the diagram at the right side. Necessarily, $\phi(1_A) = 1$ and $\phi(a) = 0$ since $a \in \mathcal{J}$. $\square$

Corollary A.2.11. For a unital Banach algebra A, the spectrum of each element $a \in A$ coincides with the image of Gelfand transformation (A.4), i.e., $\operatorname{Spec} a = \{\phi(a) \mid \phi \in \Gamma A\} = \operatorname{Im} \mathbf{ev}_a$.

Theorem A.2.12 (Gelfand-Naimark representation theorem (1943))**.** For any commutative C^*-algebra A, Gelfand transformation (A.4) is an isometric isomorphism between involution algebras.

Proof. Whe shall break in cases since the presence of a unity in A changes compactness local to total.

(i) **The unital case**: By lemma A.2.3, ΓA is a compact space and $C_0(\Gamma A) = C(\Gamma A)$ is a C^*-algebra. By means of spectral theory, we shall prove that $\|\kappa(a)\|_\infty = \|a\|$ for all $a \in A$. For an arbitrary $\phi \in \Gamma A$,

$$\{\phi(x) \mid \phi \in \Gamma A\} \overset{A.2.11}{=} \operatorname{Spec} x \overset{A.1.23}{\subset} \mathbb{R} \qquad (x = x^* \in A)$$

whenever $x^* = x \in A$. Therefore $\phi(x) = \overline{\phi(x)}$ and hence,

$$\begin{aligned} \phi(a^*) &= \phi\Big(\frac{a^* + a}{2} - i\frac{a - a^*}{2i}\Big) \\ &= \phi\Big(\frac{a^* + a}{2}\Big) - i\phi\Big(\frac{a - a^*}{2i}\Big) \\ &= \overline{\phi\Big(\frac{a^* + a}{2}\Big)} - i\,\overline{\phi\Big(\frac{a - a^*}{2i}\Big)} \\ &= \overline{\phi\Big(\frac{a^* + a}{2} + i\frac{a - a^*}{2i}\Big)} = \overline{\phi(a)}. \end{aligned}$$

Since ϕ is arbitrary, $\kappa(a^*) = \mathbf{ev}_{a^*} = \overline{\mathbf{ev}_a} = \overline{\kappa(a)}$, thus the Gelfand transformation is an $*$-morphism. Therefore, one may notice that $\|\kappa(a)\|_\infty = \sup_{\phi\in\Gamma A} |\phi(a)| = \sup \operatorname{Spec} a = r(a)$ from A.2.11. Therefore, it follows that:

$$\|\kappa(a)\|_\infty^2 = \|\overline{\kappa(a)}\kappa(a)\|_\infty = \|\kappa(a^*a)\|_\infty = r(a^*a) = \|a^*a\| = \|a\|^2 \qquad (a \in A$$

i.e., κ is an isometry, hence injective. Now it only rests to show that κ is surjective.

> **Summoning A.2.13** (Complex Stone-Weierstraß- [25] - Theorem 36.B)**.** Let $X \in$ **CHaus** and $A \subset C(X)$ a closed $*$-subalgebra containing all the constant functions. Therefore A is dense if and only if separates points[a].
>
> [a] For any $x, y \in A$ distinct, there exists $g \in A$ such that $g(x) \neq g(y)$.

Since each isometry with Banach domain has closed image, $\operatorname{Im}\kappa$ is a closed dense $*$-subalgebra of $C(\Gamma A)$ containg all constant functions that separates poits, hence by Stone-Weierstraß A.2.13, $\operatorname{Im}\kappa = \overline{\operatorname{Im}\kappa} = C(\Gamma A)$, ensuring surjectivity.

(ii) **The non-unital case**: The proof uses the fact that both One-point compactification and unitalization constructions are pairwised: $\widetilde{C_0(X)} \simeq C_0(X \sqcup \{\infty\})$ and $\Gamma\widetilde{A} \simeq \Gamma A \sqcup \{\infty\}$. The proof comes from guaranteeing that those functors are natural transformations. Since we allready proved the "compact-unital" version, the "local compact-non unital" follows.

□

A.3 Continuos Functional Calculus

Lemma A.3.1. Let $X \in$ **CHaus** and $f \in C(X)$. If $f(x_0) = 0$ for some $x_0 \in X$, then for every $\varepsilon > 0$ there is $g \in C(X)$ such that $\|g\|_\infty = 1$ and $\|gf\|_\infty < \varepsilon$.

Proof. The set $V := \{x \in X \mid |f(x)| < \varepsilon\}$ is an open one containing x_0, because it is the pre-image of $|f(\,\cdot\,)|$ at the ε ball centered at origin. Time to use some big guns from topology, in order to establish the existence of continuous extensions.

> **Summoning A.3.2** (Urysohn's Lemma - [25], Theorem 28.A)**.** Let X be a topological *normal space*[a], and let A and B be disjoint closed subspaces of X. Then there exists a continuous real function f defined on X, all of whose values lie in the closed unit interval $[0, 1]$, such that $f(A) = 0$ and $f(B) = 1$.
>
> [a] A T_1-space (all singletons $\{x\}$ are closed) in which each pair of disjoint closed sets can be separated by open sets, in the sense that they have disjoint neighborhoods.

As stated in [25], Theorem 27.A, every compact Hausdorff space is topologically normal, thus Urysohn's lemma A.3.2 apply: There exists $g : X \longrightarrow [0, 1]$ continuous such that $g(x_0) = 1$ and $g(x) = 0$ if $x \in V_{\neq x_0}$. So, $\|g\|_\infty = 1$ and $\|gf\|_\infty < \varepsilon$ □

For a unital $A \in \boldsymbol{C^*}\text{-}\,\mathbf{Alg}_{\boldsymbol{u}}$, suppose that an element $a \in A$ is *normal*, i.e., $a^*a = aa^*$. For each polynomial $p \in \mathbb{C}[z, \overline{z}]$, we extend it to $p \in \mathbb{C}[a, a^*]$ in the natural way. Since a is normal, $\mathbb{C}[a, a^*]$ is a commutative $*$-subalgebra of A containing a and 1. Its closure is denoted by $C^*(1, a)$ and it is the smallest C^*-subalgebra of A containing 1 and a.

Proposition A.3.3. Let A be a C^* algebra with unity and $a \in A$ normal.

(i) If $b \in C^*(1, a)$ has an inverse $b^{-1} \in A$, then $b^{-1} \in C^*(1, a)$, i.e.,

$$\mathrm{GL}(A) \cap C^*(1, a) = \mathrm{GL}(C^*(1, a)).$$

(ii) Let $B \subset A$ be a C^*-subalgebra containing the unity. The same deal applies:

$$\mathrm{GL}(A) \cap B = \mathrm{GL}(B). \qquad (1 \in B \subset A)$$

Proof.

(i) Suppose $b \in \mathrm{GL}(A)$. Since it is a commutative unital C^*-algebra, $b \notin \mathrm{GL}(C^*(1, a))$ if and only if $\kappa(b)$ vanishes at some point in $\Gamma C^*(1, a)$ by Gel'fand theorem A.2.9. In the case which b isn't invertible in the generated C^*-algebra, it is posssible to obtain $g \in C(\Gamma C^*(1, a))$ such that

$$\|g\|_\infty = 1 \quad \text{and} \quad \|g\kappa(b)\|_\infty < \|b^{-1}\|^{-1}$$

by lemma A.3.1. By the Gel'fand-Naimark representation A.2.12, $\kappa^{-1}(g) \in \Gamma C^*(1, a)$ obeys $\|\kappa^{-1}(g)\| = \|g\|_\infty = 1$ and

$$\|\kappa^{-1}(g)b\| = \|\kappa(\kappa^{-1}(g)b)\|_\infty = \|g\kappa(b)\|_\infty < \|b^{-1}\|^{-1}.$$

That inequality tells us that

$$1 = \|\kappa^{-1}(g)\| = \|b^{-1}(b\kappa^{-1}(g))\| \leqslant \|b^{-1}\|\|b\kappa^{-1}(g)\| < 1,$$

which is a contradiction. Therefore, $b \in \mathrm{GL}(C^*(1,a))$. □

(ii) Notice that $b^{-1} = (b^*b)^{-1}b^*$ for $b \in \mathrm{GL}(A)$. Since b^*b is a self-adjoint element, hence normal, $(b^*b)^{-1} \in C^*(1, b^*b) \subset B$ by (i). Therefore $b^{-1} \in B$. □

Corollary A.3.4 (Spectral Invariance)**.** Let $B \subset A$ be a C^*-subalgebra containg the unity of A. Therefore,

$$\mathrm{Spec}_B(b) = \mathrm{Spec}_A(b). \qquad (b \in B)$$

Counterexample A.3.5. Let $D \subset \mathbb{C}$ be the unitary open disk of complex numbers such that $|z| < 1$, such that $S^1 = \partial D$. The algebra of restrictions of holomorphic functions can be given by the set:

$$E := \{f \in C(S^1) \mid f = g\restriction_{S^1}, g \in C(\overline{D}), g\restriction_D \in \mathrm{Hol}(D)\}$$

Equipped with natural complex-conjugation $f \longmapsto \overline{f}$, $C(S^1)$ is a C^*-algebra, but E isn't invariant by the induced involution. Therefore, the corollary A.3.4 doesn't apply. ☕

Proposition A.3.6. The evaluation at a normal element $a \in A$ of a unital C^*-algebra

$$\begin{aligned} \mathbf{ev}_a : \ \Gamma C^*(1,a) &\longrightarrow \mathrm{Spec}\, a \\ \phi &\longmapsto \phi(a) \end{aligned}$$

is a homeomorphism.

Proof. In order to see that the image of the evaluation is in fact the spectrum, notice that $\mathrm{Spec}_{\Gamma C^*(1,a)}(a) = \{\mathbf{ev}_a(\phi) \mid \phi \in \Gamma C^*(1,a)\}$ by A.2.11 and $\mathrm{Spec}_{\Gamma C^*(1,a)}(a) = \mathrm{Spec}\, a$ by A.3.4. Therefore, the evaluation is well defined and it is surjective.

Suppose that for $\phi, \psi \in \Gamma C^*(1,a)$, one has that $\phi(a) = \psi(a)$. Notice that for every complex polynomial $p \in \mathbb{C}[z, \overline{z}]$, $\phi(p(a,a^*)) = \psi(p(a,a^*))$ since ϕ and ψ are $*$-morphisms. The fact that $\mathbb{C}[a,a^*]$ is dense in $C^*(1,a)$ shows that necessarily, $\phi = \psi$, i.e., $\mathbf{ev}_a$ is injective.

The weak* topology is the smallest topology over $\Gamma C^*(1,a)$ such that each $p_b : \phi \longmapsto |\mathbf{ev}_b(\phi)|$ $(b \in C^*(1,a))$ be continuous. Therefore, for a converging net $(\phi_\alpha)_\alpha \subset \Gamma C^*(1,a)$, $\phi_\alpha \longrightarrow \phi$, one can see that $\mathbf{ev}_a$ is in fact continuous:

$$\begin{aligned} & \lim_\alpha |\mathbf{ev}_a(\phi_\alpha)| = \lim_\alpha p_a(\phi_\alpha) = p_a\Big(\lim_\alpha \phi_\alpha\Big) = p_a(\phi) = |\mathbf{ev}_a(\phi)| \\ \Leftrightarrow \quad & \lim_\alpha \mathbf{ev}_a(\phi_\alpha) = \mathbf{ev}_a(\phi). \end{aligned}$$

Notice that $\Gamma C^*(1,a) \in$ **CHaus** by lemma A.2.3 since the inner algebra is unital. In the other direction, $\operatorname{Spec} a$ is compact by A.1.14 and is Hausdorff because is a subset of the complex numbers $\mathbb{C}$. Since $\mathbf{ev}_a$ is a continuous bijection between compact Hausdorff spaces, **??** guarantee us that $\mathbf{ev}_a$ is an homeomorphism. ☐

Theorem A.3.7 (The Continous Functional Calculus)**.** Let $a \in A$ be a normal element of a unital C^*-algebra. There exists a isometric $*$-morphism $\mathfrak{C}_a : C(\operatorname{Spec} a) \longrightarrow C^*(1,a)$ such that, for every $p \in \mathbb{C}[a,a^*]$, with $f(z) := p(z,\overline{z}) = \sum_{n,m} b_{n,m} z^n \overline{z}^m$, one does have

$$\mathfrak{C}_a(f) = \sum_{n,m} b_{n,m} a^n (a^*)^m = f(a). \tag{A.5}$$

Proof. You better like composition, because this is the one! The Gel'fand transform is given by $\kappa = \mathbf{ev}_{(\,\cdot\,)} : C^*(1,a) \longrightarrow C(\Gamma C^*(1,a))$ and it is an $*$-isometric isomorphism (A.2.12). By A.3.6, the composition function

$$\begin{aligned} \mathbf{ev}_a^* : \quad C(\operatorname{Spec} a) &\longrightarrow C(\Gamma C^*(1,a)) \\ f &\longmapsto f \circ \mathbf{ev}_a \end{aligned}$$

also become an $*$-isometric isomorphism. Therefore the composition $\mathfrak{C}_a := \kappa^{-1} \circ \mathbf{ev}_a^*$ holds the same title. In particular,

$$\mathfrak{C}_a(1_{C(\operatorname{Spec} a)}) = \kappa^{-1}(\mathbf{ev}_a^*(1_{C(\operatorname{Spec} a)})) = \kappa^{-1}(1_{C(\operatorname{Spec} a)}(\mathbf{ev}_a)).$$

Notice that $1_{C(\operatorname{Spec} a)} : \operatorname{Spec} a \longrightarrow \{1\}$. Therefore,

$$\begin{aligned} \kappa(\mathfrak{C}_a(1_{C(\operatorname{Spec} a)}))(\phi) &= 1_{C(\operatorname{Spec} a)}(\mathbf{ev}_a(\phi)) \\ &= 1 \qquad\qquad (\phi \in C(\Gamma C^*(1,a))) \\ &= \kappa(1_A)(\phi). \end{aligned}$$

Hence $\kappa(\mathfrak{C}_a(1_{C(\operatorname{Spec} a)})) = \kappa(1_A)$ which imply by injectivity that $\mathfrak{C}_a(1_{C(\operatorname{Spec} a)}) = 1_A$. One can also verify that the statement $I_{C(\operatorname{Spec} a)} \circ \mathbf{ev}_a = \kappa(a)$, implies that $\mathfrak{C}_a(I_{C(\operatorname{Spec} a)}) = a$. Thus, (A.5) holds. ☐

In summary, for a normal element $a \in A$, notice $f(a) := \kappa^{-1}(f(\mathbf{ev}_a))$ makes totally sense for $f \in C(\operatorname{Spec} a)$, extending $f : A \longrightarrow A$. Moreover, if $g \in C(f(\operatorname{Spec} a))$, then $g(f(a)) = (g \circ f)(a)$. Unfortunately, the continuous functional calculus does not work for non normal elements, since the generated C^*-algebra woudn't necessarily be commutative.

Proposition A.3.8. Let A and B be two unital C^*-algebras. If $\varphi : A \longrightarrow B$ is a unital $*$-morphism, and if $a \in A$ is normal, then:

(*i*) $\operatorname{Spec} \varphi(a) \subseteq \operatorname{Spec} a$,

(*ii*) if $f \in C(\operatorname{Spec} a)$ then $\varphi(f(a)) = f(\varphi(a))$.

Proof.

(i) Suppose that $\lambda \notin \operatorname{Spec} a$, i.e., $\lambda 1 - a \in \mathrm{GL}(A)$. Therefore $\lambda 1 - \varphi(a) = \varphi(\lambda 1 - a) \in \mathrm{GL}(B)$, hence $\lambda \notin \operatorname{Spec} \varphi(a)$.

(ii) Let $\mathscr{P} = \{\mathbf{ev}_{(\,\cdot\,)}(p) : \operatorname{Spec} a \longrightarrow \mathbb{C} \mid p \in \mathbb{C}[z, \overline{z}]\} \subset C(\operatorname{Spec} a)$. With the complex conjugation induced in those function, this is a unital $*$-subalgebra that separates points of $\operatorname{Spec} a$. By the Stone-Weierstraß (A.2.13), $\mathscr{P}$ is dense. Therefore, there exists $(p_n)_n \subset \mathscr{P}$ such that $f = \lim_n p_n$.

Since $(p_n)_n$ converges uniformly to f on $\operatorname{Spec} \varphi(a) \subseteq \operatorname{Spec} a$, and by continuity of the functional calculus we conclude that:

$$\begin{aligned} \varphi(f(a)) &= \varphi\Big(\lim_{n\to\infty} p_n(a)\Big) \\ &= \lim_{n\to\infty} \varphi(p_n(a)) \\ &= \lim_{n\to\infty} p_n(\varphi(a)) = f(\varphi(a)). \quad \square \end{aligned}$$

A.4 Positive elements

Definition A.4.1. A element a of a C^*-algebra A is said to be *positive* and it can be written that $a \geqslant 0$, whenever it is self-adjoint $a = a^*$ and its spectrum is positive: $\operatorname{Spec} a \subset [0, \infty)$. Hence, a order relation pops out, stating that $a \geqslant b$ if $a - b \geqslant 0$.

Theorem A.4.2. Let $a \in A$ be a positive element.

(i) (The Hahn decomposition) There exists unique positive elements $a_+, a_- \in A$ such that $a = a_+ - a_-$ and $a_+ a_- = 0$.

(ii) When a and $-a$ are both positive, then $a = 0$.

(iii) For any $\lambda \geqslant \|a\|$, a is positive if, and only if, $\|\lambda - a\| \leqslant \lambda$.

(iv) If both a and b are positive, so it is their sum $a + b$.

Proof.

(i) Let $B := C^*(1, a)$ be the generated unital C^*-algebra containing a, a^* and 1. By Gel'fand-Naimark theorem A.2.12, $\kappa = \mathbf{ev}_{(\,\cdot\,)}$ is an isometric isomorphism between B and $C(\Gamma B)$, where ΓB is the set of non zero morphisms $\phi : B \longrightarrow \mathbb{C}$. Since a is self-adjoint, the expansion

$$\phi(a) = \mathbf{ev}_a \phi = \kappa(a)\phi = \kappa(a^*)\phi = \phi(a^*) = \overline{\phi(a)} \qquad (\phi \in \Gamma B)$$

holds and it shows that $\kappa(a)$ must be a real continuous function. Therefore, let

$$a_+ := \kappa^{-1}(\max\{\kappa(a), 0\}) \quad \text{and} \quad a_- := \kappa^{-1}(\min\{-\kappa(a), 0\})$$

Those elements are positive and they obey the following: $a = a_+ - a_-$ and $a_+ a_- = a_- a_+ = 0$.

(*ii*) By the Spectral Mapping theorem A.1.18, $\mathrm{Spec}(-a) = -\mathrm{Spec}\, a$. Hence, both a and $-a$ be positive means that $\mathrm{Spec}\, a = \{0\}$. Therefore, A.1.21 guarantee us that $\|a\| = r(a) = 0$, i.e., $a = 0$.

(*iii*) Let $\lambda \geqslant \|a\|$. Notice that by A.1.10 and A.1.23, $\mathrm{Spec}\, a \subset [-\|a\|, \|a\|] \subset [-\lambda, \lambda]$. Therefore,

$$\|\lambda - a\| = r(\lambda - a) = \sup \mathrm{Spec}(\lambda - a) = \sup_{\mu \in \mathrm{Spec}\, a} |\lambda - \mu|$$

Hence $\|\lambda - a\| \leqslant \lambda$ if and only if $\mathrm{Spec}\, a \subset [0, \infty)$.

(*iv*) By (*iii*), $\|\|x\| - x\| \leqslant \|x\|$ for $x \in \{a, b\}$. Therefore:

$$\|\|a\| + \|b\| - \|a + b\|\| \leqslant \|\|a\| - a\| + \|\|b\| - b\| \leqslant \|a\| + \|b\|$$

So (*iii*) again ensures that $a + b$ is positive.

□

A *square root* of an element $a \in A$ is a element $b \in A$ such that $b^2 = a$.

Theorem A.4.3. Each positive element $a \geqslant 0$ of a C^*-algebra A has a unique positive square root.

Proof. Since positive elements are normal, we are good to go. Pick the usual square root $\sqrt{\cdot}$ defined on the interval $[0, \|a\|] \supset \mathrm{Spec}\, a$. With the continuous functional calculus A.3.7, notice that $\sqrt{a} := \mathfrak{C}_a(\sqrt{\cdot}) = \kappa^{-1}(\sqrt{\mathbf{ev}_a})$ is a well defined element of A, self-adjoint since and the square root $\sqrt{\cdot}$ is a real-valued function. Moreover, $\mathrm{Spec}\, \sqrt{a} = \sqrt{\mathrm{Spec}\, a} \subset [0, \infty)$, i.e., $\sqrt{a} \geqslant 0$.

The notation wasn't choose randomly: For $p(x) := x^2$, notice that $p \in C(\sqrt{\mathrm{Spec}\, a})$, hence $\sqrt{a}^2 = p(\sqrt{a}) = (p \circ \sqrt{\cdot})(a) = a$. If b_1, b_2 were two positives square roots of a, $b_1^2 = a = b_2^2$, one can see that $b_1 = \sqrt{b_1^2} = \sqrt{a} = \sqrt{b_2^2} = b_2$, concluding uniqueness. □

Lemma A.4.4. Let A be a unital C^*-algebra and $a \in A$. The following are equivalent:

(*i*) a is positive.

(*ii*) There is a self-adjoint element $b \in A$ such that $b^2 = a$.

(*iii*) There is $b \in A$ such that $b^* b = a$.

Proof. The implication $(i) \Rightarrow (ii)$ is essentialy the theorem A.4.3 and $(ii) \Rightarrow (iii)$ is trivial.

$(iii) \Rightarrow (i)$ If $a = b^*b$, let a_+ and a_- be the Hahn decomposition such as in A.4.2(i). Notice that

$$-(ba_-)^*(ba_-) = -a_-b^*ba_- = -a_-(a_+ - a_-)a_- = (a_-)^3$$

Since it is a positive element, $\operatorname{Spec} a^3 = (\operatorname{Spec} a)^3$ CONTINUAR □

Lemma A.4.5. Whenever $0 \leqslant a \leqslant b$ are invertible elements in a unitary C^*-algebra A, then $b^{-1} \leqslant a^{-1}$.

Proof. Given two self-adjoint elements $x, y \in A$ such that $x \leqslant y$, notice that $z^*xz \leqslant z^*yz$ for all z. Indeed, since $x - y \geqslant 0$,

$$\begin{aligned} z^*yz - z^*xz &= z^*(x-y)z \\ &= z^*(\sqrt{x-y})^*(\sqrt{x-y})z \\ &= (\sqrt{x-y}\,z)^*(\sqrt{x-y}\,z) \overset{A.4.4}{\geqslant} 0 \qquad (z \in A) \\ \Rightarrow \quad z^*xz &\leqslant z^*yz \end{aligned}$$

Since $b - a \geqslant 0$, the above shows that

$$\begin{aligned} 0 &\leqslant \sqrt{b}^{-1}(b-a)\sqrt{b}^{-1} \\ &= \sqrt{b}^{-1}\,b\,\sqrt{b}^{-1} - \sqrt{b}^{-1}\,a\,\sqrt{b}^{-1} \\ &= 1 - \sqrt{b}^{-1}a\sqrt{b}^{-1} \end{aligned}$$

Thus, $(\sqrt{a}\sqrt{b}^{-1})^*(\sqrt{a}\sqrt{b}^{-1}) \leqslant 1$, implying that $\|\sqrt{a}\sqrt{b} - 1\| \leqslant 1$. Hence,

$$1 \geqslant (\sqrt{a}\sqrt{b}^{-1})(\sqrt{a}\sqrt{b}^{-1})^* = \sqrt{a}b^{-1}\sqrt{a}$$

Multiplying on both sides by $\sqrt{a}^{-1}$, we get

$$a^{-1} = \sqrt{a}^{-1}\,1\,\sqrt{a}^{-1} \geqslant \sqrt{a}^{-1}(\sqrt{a}\,b^{-1}\sqrt{a})\sqrt{a}^{-1} = b^{-1}.$$

□

Lemma A.4.6. If $a \in A$ is a positive element of a unital C^*-algebra, then

$$\|a\| = \inf\{\lambda \geqslant 0 \mid a \leqslant \lambda \cdot 1\}.$$

Proof. Notice that $\|a\| \in \{\lambda \geqslant 0 \mid a \leqslant \lambda \cdot 1\}$, because the function $t \longmapsto t - \|a\|$ is positive: Since $\|a\| \leqslant \|a\|$, $\|a\|1 - a$ isn't invertible by A.1.10, hence $\|a\| \in \operatorname{Spec} a$. □

Corollary A.4.7. Let A be a C^*-algebra and $a, b \in A$. Therefore $0 \leqslant a \leqslant b \Rightarrow \|a\| \leqslant \|b\|$.

A.5 Approximate Units

We can always trade unitary arguments with approximate units over Banach algebras, which is exactly what are we going to do. Traditionally in topology, nets can be much more useful to describe weird continuous functions spaces than sequences are, and, as long as Banach, C^* and Von-Neumann algebras are blazingly wild, we need to appeal to these objects.

Definition A.5.1 (Approximate Unit)**.** Let $(\mathbb{A}, \preccurlyeq)$ be an pre-ordered set[9]. The image of any function $\mathbb{A} \longrightarrow A$ will be said to be a *net*, and will be mentioned as $(u_\lambda)_{\lambda\in\mathbb{A}}$. An *approximate unit net* will allways denote a net $(u_\lambda)_{\lambda\in\mathbb{A}}$ such that $0 \leqslant \|u_\lambda\|_A \leqslant 1$ and

$$\lim_{\lambda\in\mathbb{A}} \|a - au_\lambda\| = 0 \quad \Big(= \lim_{\lambda\in\mathbb{A}} \|a - u_\lambda a\|\Big) \qquad (a \in A)$$

which means that: for every $\varepsilon > 0$, exists $\lambda_0 \in \mathbb{A}$ such that $\|a - au_\lambda\| < \varepsilon$ whenever $\lambda \succcurlyeq \lambda_0$. Therefore, we can stabilish that $\lim_\lambda au_\lambda = \lim_\lambda u_\lambda a = a$ for every element a.

Notice that for any complex z, $\delta \in [0, 1)$ can allways be chosen such that $|z - z\delta|$ is desirily small (with respect to the ordinarily euclidean norm), i.e., the non-negative numbers with norm less than one $[0, 1) = (\delta)_{\delta\in[0,1)}$ is a approximate unit over the complex numbers. We'll show that those in fact exists in each and every C^*-algebra.

Theorem A.5.2. The positive elements of any C^*-algebra A with norm less than one are a approximate unit.

Proof. Let $\mathbb{A} := \{a \in A \mid a \geqslant 0 \text{ and } \|a\| < 1\}$. To show that $\mathbb{A}$ is directed set, we invoque functional calculus by our side:

(i) $\mathbb{A}$ **is a pre-ordered set**: Consider the bijection:

$$\begin{aligned} g: \qquad [0,1) &\longrightarrow [0,\infty) \\ t &\longmapsto \frac{t}{1-t} \\ 1 - \frac{1}{1+t} &\longleftarrow t \end{aligned}$$

Since g and g^{-1} map 0 to 0, they will send A to A under the functional calculus, even if A is non-unital. Use the order $\geqslant$ in $\mathbb{A}$ induced by the positive elements in A. Now choose $a, a' \in \mathbb{A}$ and define:

$$b := g^{-1}(g(a) + g(a')) = 1 - (1 + g(a) + g(a'))^{-1}.$$

[9] For $\lambda, \lambda' \in \mathbb{A}$, there exists a μ which both $\lambda \preccurlyeq \mu$ and $\lambda' \preccurlyeq \mu$. Equivalently, any finite subset has an upper bound.

Then $\operatorname{Spec} b \subset [0,1)$ so $b \in \mathbb{A}$. Also, since $1+g(a)+g(a') \geqslant 1+g(a)$, lemma A.4.5 implies that

$$b = 1-(1+g(a)+g(a'))^{-1} \geqslant 1-(1-g(a))^{-1} = g^{-1}(g(a)) = a.$$

Likewise $b \geqslant a'$.

(ii) CONTINUAR $\square$

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