# Fredholm Operators over Hilbert $C^*$-Modules

Mezzovilla

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# Contents

# Chapter 1

# $K$-theory of Banach Algebras

## 1.1 General portrait of homological theories

A homological theory for a category $\mathcal{C}$ consist in a sequence of covariant functors $H_n : \mathcal{C} \longrightarrow \mathbf{GrpAb}$ for each $n \in \mathbb{N}$ which satisfies some set of axioms, which depends on what theory one is interested. For example, if $\mathcal{C}$ contains a nice homotopical concept, its rather common to ask for homotopical invariance. If exact sequences naturally pops in the domain encoding a lot of information, some other axioms are required to obtain long exact sequences. The usual notation is:

$$\begin{aligned} H_n\colon\ \mathcal{C} &\longrightarrow \mathbf{GrpAb} \\ A &\longmapsto H_n(A) \\ \phi\Big\downarrow\ & \qquad\ \ \Big\downarrow\phi_n \\ B &\longmapsto H_n(B) \end{aligned}$$

We also need a way to translate short exact sequences of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

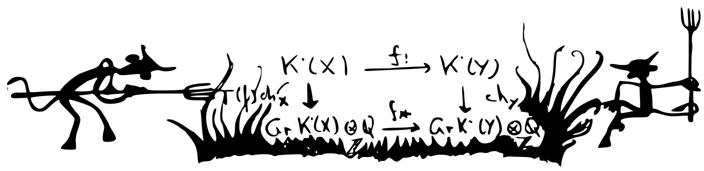
from the original category to higher counterparts obtained by $H_n$, hence, every homology theory seeks to define a connecting morphism $\delta_n : H_n(C) \longrightarrow H_{n+1}(A)$ into a long exact sequence:

$$\begin{tikzcd} H_0(A) \arrow[r] & H_0(B) \arrow[r] & H_0(C) \arrow[dll, rounded corners, "\delta_0"'] \\ H_1(A) \arrow[r] & H_1(C) \arrow[r] & H_1(B) \arrow[dll, rounded corners, "\delta_1"'] \\ H_2(A) \arrow[r] & H_2(B) \arrow[r] & \cdots \end{tikzcd}$$

In the other hand, as everything containing the prefix “co”, cohomology theories are consisted of contra-variant functors $(H^n)_n$ with the same pay-off. The position of the index on the notation usually indicates what sort of theory one is dealing with.

Here we are concerned with a homology theory for complex Banach Algebras $\mathscr{B}$-**Alg** or, more popularly, for $C^*$-algebras $\boldsymbol{C^*}$- **Alg**, a.k.a., $K$-theory for Operator Algebras. It is the mirror image of Topological $K$-theory, in light of *Gelfand Duality* connecting the category of Locally Compact Hausdorff spaces and complex abelian $C^*$-algebras, but not restricted to commutative spaces, which is often referred to as the "Non-Commutative Topology".

In his work to reformulate Riemman-Roch theorem [3], *A. Grothendieck* introduces the group $K(A)$ associated to a subcategory of an abelian category, which nowadays, it is the so called “Grothendieck’s group”. That’s where is from the letter $K$, which he had chosen for “Klassen”. His reformulation famously contains his legendary drawing:




> *Riemann-Roch Theorem: the new black: The diagram*
>
> [. . .] *is commutative!*
>
> *I would need to misuse about* $2h$ *of my listener’s time in order to impart only an approximal understanding of this statement for* $f : X \longrightarrow Y$.
>
> *In cold print (as in Springer’s lecture notes) this would take around* 400-500 *pages.*
>
> *A thrilling example of how our urge for knowledge and discoveries decays into a lifeless and ideological delirium while life itself goes thousandfold to the devil - and is threatened with final destruction.*
>
> *It’s high time to change our course!*
>
> (16.12.1971) Alexander Grothendieck

With his work settled in what is about to be algebraic $K$-theory, topological $K$-theory would be a product of *M. Atiyah* and *F. Hirzebruch* replicating Grothendieck’s construction for topological vector bundles on compact Hausdorff spaces.

We'll construct functors $K_n : \mathscr{B}\text{-}\mathbf{Alg} \longrightarrow \mathbf{GrpAb}$ and the connecting maps will be call *index map*, denoted by $\partial$. A remarkable aspect of operator $K$-theory is the *Bott periodicity*: $K_n \simeq K_{n+2}$, which then describes for any short exact sequence $0 \longrightarrow I \longrightarrow A \longrightarrow A/I \longrightarrow 0$, where $I \triangleleft A$, a six-term exact sequence:

$$\begin{tikzcd}
K_0(I) \arrow[r] & K_0(A) \arrow[r] & K_0(A/I) \arrow[d, "\partial"] \\
K_1(A/I) \arrow[u, "\partial"] & K_1(A) \arrow[l] & K_1(I) \arrow[l]
\end{tikzcd}$$

The details will be spared in what is outside of our scope, which will include the definition of the groups $K_0$ and $K_1$ for complex Banach algebras, and the index map mentioned. The connecting map will be used in the classfication of finite rank modules, and later on, the index of our Fredholm operators. Hence it is important to define it in a helpfull way.

## 1.2 The $K_0$-group

Our object is to deal with Hilbert $C^*$-modules, witch are right $A$-modules with a generalized $A$-valued inner product for a given $C^*$-algebra $A$ (plus some other details), generalizing the concept of Hilbert space. Therefore, it is reasonable to understand some $K$-theory for $C^*$-algebras, as they are our underlying space. Unfortunately, as we'll see later on, there is no *Riesz representation lemma* and, there exists bounded linear operators that aren't adjointable between Hilbert modules. Hence, dealing with self-adjoint operators is a restriction for sure. Thankfully, the $K$-theory for Banach algebras is good enougth in order to fill our needs.

In topological $K$-theory, in order to define the $0^{\underline{\text{th}}}$ $K$ group, one would consider a complex vector bundle $E$ over $X \in \mathbf{CHaus}$ and take the right $C(X)$-module $\Gamma(X, E)$ of continuous sections $s : X \longrightarrow E$ with pointwise scalar multiplication[1]. Compactness of $X$ implies that $\Gamma(X, E)$ is a projective $C(X)$-module, and *Serre-Swan* theorem [10, Thr. 6.18] states that $E \longmapsto \Gamma(X, E)$ induces an equivalence between the category of complex vector bundles and finitely generated projective $C(X)$-modules. Hence, $K^0(X)$ is the *Grothendieck group* of the set of equivalence classes of isomorphisms between vector bundles over $X$.

For a given Banach Algebra $A$, the following definitions and constructions mimics the above paragraph, by replacing vector bundles by finitely generated projective $A$-modules.

**Definition 1.2.1.** In any given Banach algebra $A$, for two idempotent elements $x$ and $y$, define the following notions of equivalence:

[1] That is to say, for $f \in C(X)$ and $s \in \Gamma(X, E)$, let $x \longmapsto s(x)f(x)$.

$(i)$ **Murray-von-Neuman equivalent**: There are elements $p, q \in A$ such that $x = pq$ and $y = qp$.

$(ii)$ **Similarly equivalent**: There exists an invertible[2] element $u \in \mathrm{GL}(A)$ such that $x = u^{-1}yu$.

$(iii)$ **Homotopic**: There are a continuous path $\gamma \in C([0,1], A)$ of idempotents between $x$ and $y$, i.e.,

$$\gamma(0) = x, \gamma(1) = y \quad \text{and} \quad \forall\, t \in [0,1], \gamma(t)^2 = \gamma(t).$$

If $A$ is assured to be a $C^*$-algebra, those definitions are concerned with self-adjoint idempotent elements, a.k.a., projections. Two projections $x, y$ are equivalent if there exists a *partial isometry* $u$ such that $x = u^*u$ and $y = uu^*$.

For the canonical embedding $x \longmapsto \mathrm{diag}(x, 0)$ over matrices, consider the inductive limit $\mathbb{M}_\infty(A) := \varinjlim_{n\in\mathbb{N}} \mathbb{M}_n(A)$, which can be seen as the set of infinite matrices over $A$ but only finitely many of the entries are non-zero.

***Remark* 1.2.2.** Note that $\mathbb{M}_\infty(A)$ contains no unity, but that doesn't stop us to declaring two elements $x, y$ to be similar when they are similar in some square matrix space $\mathbb{M}_n(A)$. Therefore, all equivalence relations listed in the definition 1.2.1 coincide in $\mathbb{M}_\infty(A)$.

Simply shouting "Let $A$ be an $C^*$-algebra" in the crowd is a powerfull classification tool, whenever is a mathematicians crowd[3].

$(i)$ If you hear in response "unital or not?", you know that there is some $C^*$-algebraic fellow around you.

$(ii)$ If the crowd contains mathematicians and no-one ask wetter $A$ contains a unity or not, no $C^*$-algebraist is contained in the crowd. They are instantly assuming the unity is there.

This is because dealing without unital rings outside $C^*$-theories are usually simple. Just unitize and go on. However, the presence of unity in $C^*$-algebras is crucial to determine theyr underlying hidden topology, as explicitly is made in *Gelfand's duality* theorem.

The next definition is in charge to define the functor $K_0$ for both cases, but some intermediate steps are required from one to another.

**Definition 1.2.3.** Let $A$ be a Banach algebra. The set of equivalence classes over $\mathbb{M}_\infty(A)$ considering any relation $\sim$ contained in 1.2.1 is an abelian semi-group with $[x] + [y] := [\mathrm{diag}(x, y)]$. Before defining $K_0$, in order to include

[2] Assuming that $A$ is unital.

[3] Otherwise, you are just playing creepy at dinner table again.

the non necessarily unital algebras, it is needed to be considered an auxiliar functor $K_{00}$ much closer to the topological counterpart $K^0$. This is necessary in order to obtain the Bott periodicity result for Banach algebras, and other good functorial properties.

(*i*) $\boldsymbol{K_{00}}$: It is the Grothendieck group construction associated with the semi-group $V(A) := \mathbb{M}_\infty(A)/\!\sim$ where addition is given by $[x]+[y] := [\mathrm{diag}(x,y)]$, generalising the construction of $\mathbb{Z}$ from $\mathbb{N}$ considering formal differences. In lighter sheets, for pairs $(a,b)$ and $(c,d)$ of elements in $V(A)$, let $(a,b) \equiv (c,d)$ whenever there exists[4] $z \in V(A)$ such that $a+d+z = c+b+z$. This is an equivalence relation over the pais, and $[\,\cdot\,]_{00}$ will denote the related equivalence class.

We are mimicking the formal differences construction, so it's natural to define the addition operation coordinate-wise and let $x - y := [(x,y)]_{00}$. Therefore, it is well defined the folowing covariant functor:

$$
\begin{array}{rcccc}
K_{00}\colon & \mathscr{B}\textbf{-Alg} & \longrightarrow & \textbf{GrpAb} & \\
& A & \longmapsto & V(A)\times V(A)/\!\equiv & x-y \\
& {\scriptstyle\phi}\big\downarrow & & {\scriptstyle\phi_{00}}\big\downarrow & \big\downarrow \\
& B & \longmapsto & V(B)\times V(B)/\!\equiv & \phi(x)-\phi(y)
\end{array}
$$

Since every element in $V(A)$ is the class of some idempotent matrix $p$, we can state that every element in $K_{00}(A)$ is on the form $[p]_{00} - [q]_{00}$. Two formal differences $[p]_{00} - [q]_{00}$ and $[x]_{00} - [y]_{00}$ coincide in $K_{00}(A)$ precisely when the operators $\mathrm{diag}(p,y)$ and $\mathrm{diag}(x,q)$ are *stably* homotopic.

(*ii*) $\boldsymbol{K_0}$: In our next step, it's crucial to know exactly who $K_{00}(\mathbb{C})$ is. Hence, remeber that two idempotents in $\mathbb{M}_n(\mathbb{C})$ are similar if, and only if, their images has the same dimension. Therefore $V(\mathbb{C}) \simeq \mathbb{N}$, and by historical nightmares with Analysis I exercises constructing the integer numbers, it is easy to infer that $K_{00}(\mathbb{C}) = \mathbb{Z}$.

For non necessarily unital $A$, consider $\widetilde{A} := A \oplus \mathbb{C}$ the *unifization* of $A$ and the complex projection $\varepsilon : \widetilde{A} \twoheadrightarrow \mathbb{C}$, which induces the short exact sequence:

$$0 \longrightarrow A \hookrightarrow \widetilde{A} \xrightarrow{\ \varepsilon\ } \mathbb{C} \longrightarrow 0$$

The urge to obtain Bott periodicity theorem for Banach algebras, which is a relation between $K_0$ and $K_1$ in the presence of short exact

[4]Since it is only a semi-group, the cancelation property do not hold necessarily over $V(A)$. One might check that this is the case if, and only if, the inclusion of $V(A)$ at the Grothendieck's associated group is injective.

sequences, will obligate the exactness of the following:

$$0 \longrightarrow K_0(A) \hookrightarrow K_0(\widetilde{A}) \xrightarrow{\varepsilon_0} K_0(\mathbb{C}) \longrightarrow 0$$

Since it is a morphism between unital Banach algebras, the induced map $\varepsilon_{00} : K_{00}(\widetilde{A}) \longrightarrow \mathbb{Z}$ is a well defined morphism, hence, it is possible to define the following:

$$\begin{tikzcd} K_0\colon & \mathscr{B}\textbf{-Alg} \arrow[r] & \textbf{GrpAb} & \\ & A \arrow[r, mapsto] \arrow[d, "\phi"'] & \ker(K_{00}(\widetilde{A}) \to \mathbb{Z}) \arrow[d, "\phi_0"'] & a+z \arrow[d, mapsto] \\ & B \arrow[r, mapsto] & \ker(K_{00}(\widetilde{B}) \to \mathbb{Z}) & \phi(a)+z \end{tikzcd}$$

Notice that $K_0(A)$ is precisely the set of elements $[p]_0 - [q]_0 \in K_{00}(\widetilde{A})$ such that $\varepsilon(p) \sim \varepsilon(q)$. If $A$ is allready unital, it is possible to show that $K_0(A) \simeq K_{00}(A)$.

***Remark* 1.2.4.** The argument to show that $V(\mathbb{C}) \simeq \mathbb{N}$ is equivalent for compact operators in a infinite-dimensional Hilbert space $H$, i.e., $V(\mathscr{K}(H)) \simeq \mathbb{N}$, hence $K_0\mathscr{K}(H) = \mathbb{Z}$. On the other hand, any two infinite rank projections in $\mathscr{B}(H)$ are equivalent, hence $V\mathscr{B}(H) \simeq \mathbb{N} \cup \{\infty\}$, which is a semi-group without the cancellation property. Since everyone is equivalent to $\infty$, it is obtained that $K_{00}\mathscr{B}(H) \simeq 0$. The semi-group $V(A)$ posses the cancellation property if, and only if, the inclusion $V(A) \hookrightarrow K_{00}(A)$ is injective.

**Proposition 1.2.5** (Standard portrait of $K_0$)**.** Every element of $K_0(A)$ can be written as $[x + p_n]_0 - [p_n]_0$.

*Proof.* Let $p, q \in \mathbb{M}_\infty(\widetilde{A})$ be some idempotent square matrices with order no longer than $n$, such that $\varepsilon_0\left([p]_0 - [q]_0\right) = 0$, i.e., $[p]_0 - [q]_0 \in K_0(A)$. Matrices $p \in \mathbb{M}_n(\widetilde{A})$ can be written as $(p_A, p_\mathbb{C}) \in \mathbb{M}_n(A) \oplus \mathbb{M}_n(\mathbb{C})$, i.e., an algebraic part $p_A$ and a scalar part $p_\mathbb{C}$. Stating that $\varepsilon_0\left([p]_0 - [q]_0\right) = 0$ means that the scalar parts of $p$ and $q$ coincide.

The identity $I_n \in \mathbb{M}_\infty(\widetilde{A})$ can be seen as the projection operator of the first $n$-th coordinates, by filling it with 0's, but to avoid confusions, let it be $p_n$. With $y \leqslant x$ be given by $xy = yx = y$, one may see that $p < p_n$ and $y < p_n$. Notice that $\operatorname{diag}(0, p) \in \mathbb{M}_{2n}(\widetilde{A})$ is similar to $p$ and orthogonal to $I_n$, i.e.,

$$\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix} = 0.$$

Hence, $x \coloneqq \operatorname{diag}(-q, p)$ is such that $x + p_n$ is an idempotent operator and:

$$\begin{aligned} [x + p_n]_0 - [p_n]_0 &= [\operatorname{diag}(0, p)]_0 + [p_n - q]_0 - \left([p_n - q]_0 + [q]_0\right) \\ &= [p]_0 - [q]_0 \,. \end{aligned}$$

□

## 1.3 The $K_1$-group

While $K_0$ is build upon equivallence classes of idempotent, $K_1$ uses invertible elements, but simpler. Therefore, let $\mathrm{GL}_\infty(A) := \varinjlim_{n\in\mathbb{N}} \mathrm{GL}_n(A)$ considering the embedding $x \longmapsto \mathrm{diag}(x, 1)$. Calculus is back, and we shall consider exponentials inside a unital algebra $A$:

$$\exp(a) := \sum_{n=0}^{\infty} \frac{a^n}{n!} \quad \text{and} \quad \log(1+a) := \sum_{n=1}^{\infty} -\frac{a^n}{n} \qquad (a \in A)$$

where the log is defined whenever $\|a\| < 1$. This is the case since elements of the form $z - a$ for complex $z$ are invertible if $\|a\| \leqslant |z|$. If $a$ and $b$ doesn't commute, $\exp(a)\exp(b) \neq \exp(a+b)$, which means that the set of exponentials isn't closed by multiplications.

**Lemma 1.3.1.** For a unital Banach algebra $A$, the component of the unity is the group generated by $\{\exp(a) \mid a \in A\} \subset \mathrm{GL}(A)$, denoted by $\exp(A)$.

*Proof.* Let $\mathrm{GL}^{(0)}(A)$ be the refered set of connected components of 1. Notice that $t \longmapsto \exp(tb)$ for $t \in [0,1]$ is a continuous path of invertible elements between 1 and $\exp(b)$ for any $b$, hence $\exp(A) \subset \mathrm{GL}^{(0)}(A)$. It remais only to show the converse inclusion.

For some $a$ with $\|1 - a\| < 1$, let $b := \log(1 + (a-1)) = \log(a)$, i.e., $a = \exp(b)$. Therefore, if $u \in \mathrm{GL}(A)$ and $\|v - u\| < \|u^{-1}\|^{-1}$, this means that $v = \exp(b)u$ for some $b$. From this treatment, if follows that $\exp(A)$ is a open and closed topological subspace of $\mathrm{GL}^{(0)}(A)$ which contains the unity, i.e., $\mathrm{GL}^{(0)}(A)$ coincides with $\exp(A)$. $\square$

***Remark* 1.3.2.** Let $M \in \mathrm{GL}_n(\mathbb{C})$. Since 0 cannot be an eigenvalue of $M$ (which is a finite set), it's possible to find $\alpha \neq 0$ such that $[0,\infty)\cdot\alpha$ doesn't contains any of the eigenvalues of $M$ or 1. Therefore, $1 - \alpha t \neq 0$ for all $t \geqslant 0$ and $M_t := (1-\alpha t)^{-1}(M - t\alpha I_n)$ is a continuous path from $M$ to the identity, i.e., $\mathrm{GL}_n(\mathbb{C})$ is connected. ☕

In a not so long future, the following result will be important in the presence of an ideal $I \triangleleft A$, the considering of the projection $A \twoheadrightarrow A/I$.

**Corollary 1.3.3.** Any continuous surjection $A \longrightarrow B$ induces a lift from every element in $\mathrm{GL}_n^{(0)}(B)$ to one in $\mathrm{GL}_n^{(0)}(A)$.

*Proof.* Using 1.3.1, write $\prod_i \exp(b_i) \in \mathrm{GL}_n(B)^{(0)}$ for any desired element. Since there is a surjection, there exists lifts to each $b_i$, i.e., $a_i \in \mathrm{GL}_n(A)$ such that $\prod_i \exp(a_i) \in \mathrm{GL}_n(A)_0$. $\square$

Considering the homotopy equivalence relation, two elements in $\mathrm{GL}_\infty(A)$ are homotopical whenever they are in the same connected component in some $\mathrm{GL}_n(A)$. Denote the equivalence class by $[\,\cdot\,]_1$. Whence, the quotient

$\mathrm{GL}_\infty(A)/\mathrm{GL}_\infty^{(0)}(A)$ is an abelian group with the multiplication $[x]_1[y]_1 = [xy]_1$, which is commutative once you note it is possible to find a connected path[5] between $\mathrm{diag}(y,1)$ and $\mathrm{diag}(1,y)$, hence

$$[x]_1[y]_1 = [xy]_1 = \left[\begin{pmatrix} xy & 0 \\ 0 & 1 \end{pmatrix}\right]_1 = \left[\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}\right]_1 = \left[\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}\right]_1$$

and similarly, one shows that $[xy]_1 = [\mathrm{diag}(y,x)]_1 = [y]_1[x]_1$. There we have, our $K_1(A)$ group. Since $\mathrm{GL}_n(\mathbb{C})$ is connected, it follows immediately that $K_1(\mathbb{C}) = 0$ and, therefore, we can deal with units the way it is intended: for non necessarily unital algebras $A$, let $K_1(A) \coloneqq K_1(\widetilde{A})$.

**Definition 1.3.4.** The functor $K_1$ can be seen as the following:

$$\begin{array}{cccccc} K_1\colon & \boldsymbol{\mathscr{B}}\textbf{-Alg} & \longrightarrow & \textbf{GrpAb} & & \\ & A & \longmapsto & \mathrm{GL}_\infty(\widetilde{A})/\mathrm{GL}_\infty^{(0)}(\widetilde{A}) & & [x]_1 \\ & {\scriptstyle\phi}\downarrow & & {\scriptstyle\phi_1}\downarrow & & \downarrow \\ & B & \longmapsto & \mathrm{GL}_\infty(\widetilde{B})/\mathrm{GL}_\infty^{(0)}(\widetilde{B}) & & [\phi(x)]_1 \end{array}$$

## 1.4 The index map

We are now ready to define the so called index map. This name comes from the Fredholm operator theory since what we are about to construct is a generalization of the index of those operators. Consider $\mathscr{B}(H)$ the $C^*$-algebra of bounded operators between a Hilbert space $H$, and $\mathscr{K}(H)$ the ideal of compact operators. The *Atikinson* theorem states precisely that the *Calkin* algebra $\mathscr{Q}(H) \coloneqq \mathscr{B}(H)/\mathscr{K}(H)$ is a classifying one: $T$ is a Fredholm operator if, and only if, $(T \bmod \mathscr{K}(H)) \in \mathrm{GL}\,\mathscr{Q}(H)$.

Since $K_0\mathscr{K}(H) = \mathbb{Z}$ and $K_1\mathscr{Q}(H)$ can be seen the set of Fredholm operators up to homotopy[6], the index map $\mathit{ind} : K_1\mathscr{Q}(H) \longrightarrow K_0\mathscr{K}(H)$ is well defined. Our index map $\partial$ will generalize this application.

**Construction 1.4.1.** Let $I \vartriangleleft A$ and consider the following short exact sequence:

$$0 \longrightarrow I \hookrightarrow A \twoheadrightarrow {}^{A}\!/_{I} \longrightarrow 0$$

We are in position to construct $\partial : K_1(A/I) \longrightarrow K_0(I)$. For $[x]_1 \in K_1(A/I)$, let $n$ be such that $x \in \mathrm{GL}_n(\widetilde{A/I})$. It's about time to the corollary 1.3.3 to shine: Since the projection $A \twoheadrightarrow A/I$ is a continuous surjection, so it is the

[5]Let $z(\theta) = \left(\begin{smallmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{smallmatrix}\right)$ be the rotation matrix by some angle $\theta$. Therefore, the continuous map $[0,\pi/2] \ni \theta \longrightarrow z(\theta)\,\mathrm{diag}(y,1)z(\theta)^{-1}$ is the desired path.

[6]Remind that two Fredholm operators in the same realm have the same index if, and only if they are homotopic.

unifisation induced morphism between the algebras, hence, one can lift the element $\operatorname{diag}(x, x^{-1}) \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A/I})$ to some $w \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A})$.

If $\pi : \mathrm{GL}_\infty(\widetilde{A}) \twoheadrightarrow \mathrm{GL}_\infty(\widetilde{A/I})$ is the quotient projection, notice that $\pi(wp_nw^{-1}) = p_n$, so that $wp_nw^{-1} \in \widetilde{I}$. Since $wp_nw^{-1}$ is also an idempotent, notice that $[wp_nw^{-1}]_0 - [p_n]_0 \in K_0(I)$. And this is the image of the index map $\partial$ of some element $[x]_1$.

An anxious mind would immediately panic. We have a TO-DO list before calling it a day:

$(i)$ Check that $[wp_nw^{-1}]_0 - [p_n]_0$ doesn't depend on the lift $w$ choosen;

$(ii)$ Check that $\partial([x]_1) = \partial([y]_1)$ for $x \equiv y$.

$(iii)$ Check that $\partial$ is a group morphism.

*Proof of TO-DO list items.* If $v$ is another lift of $\operatorname{diag}(x, x^{-1})$, notice that

$$vp_nv^{-1} = (vw^{-1})wp_nw^{-1}(vw^{-1})^{-1},$$

i.e., $vp_nv^{-1}$ is similar to $wp_nw^{-1}$. This is enough to take care of $(i)$.

In order to show that the index is well defined, suppose that $y \in \mathrm{GL}_n(\widetilde{A/I})$ is another representant of class $[x]_1$. Notice that

$$x^{-1}y \in \mathrm{GL}_n^{(0)}(\widetilde{A/I}) \quad \text{and} \quad \begin{pmatrix} x & 0 \\ 0 & I_n \end{pmatrix} \begin{pmatrix} I_n & 0 \\ 0 & y^{-1} \end{pmatrix} \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A/I})$$

so by the corollary 1.3.3 again, let $a \in \mathrm{GL}_n^{(0)}(\widetilde{A})$ and $b \in \mathrm{GL}_{2n}^{(0)}(\widetilde{A})$ be the lifts respectively. But then $u := w \operatorname{diag}(a, b)$ is a lift of $\operatorname{diag}(y, y^{-1})$. From the fact that $p_n$ commutes with $\operatorname{diag}(a, b)$, it is obtained that $up_nu^{-1} = wp_nw^{-1}$. Since we already show that the choice of lift doesn't matter, $(ii)$ is checked.

For $x, y \in \mathrm{GL}_n(\widetilde{A/I})$, suppose that $w$ is a lift of $\operatorname{diag}(x, x^{-1})$ and $v$ is a lift of $\operatorname{diag}(y, y^{-1})$. Notice that $\varpi := \operatorname{diag}(w, v)$ is a lift of $\operatorname{diag}(x, y, x^{-1}, y^{-1})$, hence

$$\begin{aligned}
\partial([x]_1[y]_1) = {} & \left[\varpi p_{2n} \varpi^{-1}\right]_0 - [p_{2n}]_0 \\
= {} & \left[\begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix} \begin{pmatrix} p_n & 0 \\ 0 & p_n \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & v \end{pmatrix}^{-1}\right]_0 - \left[\begin{pmatrix} p_n & 0 \\ 0 & p_n \end{pmatrix}\right]_0 \\
= {} & \left[\begin{pmatrix} wp_nw^{-1} & 0 \\ 0 & vp_nv^{-1} \end{pmatrix}\right]_0 - \left[\begin{pmatrix} p_n & 0 \\ 0 & p_n \end{pmatrix}\right]_0 \\
= {} & [wp_nw^{-1}]_0 - [p_n]_0 + [vp_nv^{-1}]_0 - [p_n]_0 = \partial[x]_1 + \partial[y]_1
\end{aligned}$$

Therefore, it is a group morphism as our final item $(iii)$ assures. $\square$

**Definition 1.4.2.** Using construction 1.4.1, the application

$$\begin{aligned} \partial : \quad K_1(A/I) &\longrightarrow K_0(I) \\ [x]_1 &\longmapsto [wp_nw^{-1}]_0 - [p_n]_0 \end{aligned}$$

whenever $x \in \mathrm{GL}_n(\widetilde{A/I})$ and $w$ is a lift of $\mathrm{diag}(x, x^{-1})$ is the group morphism so called *index* map.

**Example 1.4.3.** ☕

# Chapter 2

# Hilbert $C^*$-modules

Hilbert modules first appear in the work of *I. Kaplaski* [9] and *W. Paschke* [14] later. There are three main areas where Hilbert $C^*$-modules are heavily used to formulate mathematical concepts envolving:

$(i)$ Induced representations of Morita equivalence [4], [16], [15];

$(ii)$ Kasparov's $KK$-theory [11];

$(iii)$ $C^*$-algebraic quantum groups.

In what is tangible to this work, we address the Morita equivalence target by building a Fredholm operator approach between Hilbert modules, introduced by Ruy Exel [7]. Hence, this chapter is responsible for defining and studying those objects.

The material source contains for this chapter contains the well written textbooks like [12], [8], [13].

## 2.1 The interest object

**Definition 2.1.1** (Inner product Module)**.** A right module $E$ over a $C^*$-algebra (non-necessarily unital) blessed with an generalized inner product $\langle \cdot , \cdot \rangle : E \times E \longrightarrow A$ will be said to be a *Inner product module* when $\langle \cdot , \cdot \rangle$ attends the following properties:

$(i)$ **Twisted $A$-sesquilinear**: The first coordinate are involuted-linear and the second one linear, i.e.,

$$\begin{cases} \langle x + ya, z \rangle = \langle x, z \rangle + a^* \langle y, z \rangle \\ \langle z, x + ya \rangle = \langle z, x \rangle + \langle z, y \rangle a \end{cases} \qquad \begin{pmatrix} x, y, z \in E \\ a \in A \end{pmatrix}$$

$(ii)$ **$A$-Hermitian symmetry**: $\langle x, y \rangle = \langle y, x \rangle^*$ whenever $x, y \in E$.

(*iii*) **Positive definite**: For any $x \in E$, $\langle x, x\rangle = 0 \Leftrightarrow x = 0$. By (*ii*), we can say that $\langle x, x\rangle \geqslant 0$ since it is self-adjoint.

One could argue that we only need the inner product to be linear in the second coordinate and by the Hermitian symmetry conclude as a proposition that every inner product over Inner product modules is indeed twisted sesquilinear.

**Proposition 2.1.2** (Cauchy-Schwartz inequality)**.** For any Inner product module $E$ over $A$, the following inequality holds:

$$\|\langle x, y\rangle\|^2 \leqslant \|\langle x, x\rangle\| \cdot \|\langle y, y\rangle\|. \qquad (x, y \in E) \tag{2.1}$$

*Proof.* Given the fact that $0 \leqslant \langle a, a\rangle$ for $a \in A$, notice that with the acessory elements $a := \langle x, x\rangle$, $b := \langle y, y\rangle$ and $c := \langle x, y\rangle$,

$$\begin{aligned} 0 &\leqslant \langle x - ytc^*, x - ytc^*\rangle \\ &= \langle x, x - ytc^*\rangle - tc\langle y, x - ytc^*\rangle \\ &= \langle x, x\rangle - \langle x, y\rangle\, tc^* - tc\langle y, x\rangle + tc\langle y, y\rangle tc^* \qquad (t \in \mathbb{R}) \\ &= a - 2tcc^* + t^2 cbc^* \end{aligned}$$

Since $2tcc^*$ is self-adjoint, we can add in both sides and maintain the inequality in the $C^*$-realm. Using the $A$-norm and assumig $t \geqslant 0$, by A.4.7,

$$\begin{aligned} 2t\|cc^*\| &\leqslant \|a\| + t^2\|cbc^*\| \\ &\leqslant \|a\| + t^2\|c\|\|b\|\|c^*\| \\ \Rightarrow \quad 2t\|c\|^2 &\leqslant \|a\| + t^2\|b\|\|c\|^2 \end{aligned} \tag{2.2}$$

With a fairly nice quadratic polynomial in $\mathbb{R}[t]$ calved by (2.2) in our hands witch is allways non negative, the discriminant must be non positive. Therefore:

$$\begin{aligned} (-2\|c\|^2)^2 - 4\|b\|\|c\|^2\|a\| &\leqslant 0 \\ \Rightarrow \quad \|\langle x, y\rangle\|^4 - \|\langle y, y\rangle\|\|\langle x, y\rangle\|^2\|\langle x, x\rangle\| &\leqslant 0 \end{aligned} \tag{2.3}$$

Assuming $\|\langle x, y\rangle\|^2 \neq 0$ means that (2.3) can be simplified into Cauchy-Schwartz inequality (2.1) by cancelling $\|\langle x, y\rangle\|^2$. Otherwise[1], $\langle x, y\rangle = 0$ is a trivial case of the desired inequality. $\square$

For any $A$-valued inner product as above, we define a norm $\|x\| := \sqrt{\|\langle x, x\rangle\|_A}$ on a Inner product $C^*$-module. Which means that for arbitrary $x, y \in E$ and $a \in A$, the following holds:

[1]Note that $\|\langle x, y\rangle\|^2 = 0$ if and only if $\langle x, y\rangle = 0$.

$(i)$ $\|x\| = 0 \quad \Leftrightarrow \quad x = 0.$

$(ii)$ $\|xa\| = \|a\|_A \|x\|.$

$(iii)$ $\|x + y\| \leqslant \|x\| + \|y\|.$

Notice that the triangle inequality $(iii)$ is a direct consequence of 2.1.2:

$$\begin{aligned}
\|x+y\|^2 &= \|\langle x+y, x+y\rangle\|_A \\
&= \|\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\|_A \\
&\leqslant \|x\|^2 + \|\langle x,y\rangle\|_A + \|\langle x,y\rangle^*\|_A + \|y\|^2 \\
&= \|x\|^2 + 2\|\langle x,y\rangle\|_A + \|y\|^2 \\
&\overset{2.1.2}{\leqslant} \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2
\end{aligned}$$

as in the good old days. One identity that still remais is the polarization one: For every sesquilinear form $\varsigma : E \times E \longrightarrow A$

$$4\varsigma(y,x) = \sum_{n=0}^{3} i^n \varsigma(x + i^n y, x + i^n y). \qquad (x, y \in E) \tag{2.4}$$

Since it should be a normed space, hence a complex vector space, one may be concerned about the fact that $A$ doesn't necessarily have a unit and therefore, $zx$ for $z \in \mathbb{C}$ should be an worry.

**Proposition 2.1.3.** All Inner product modules are naturally complex vector spaces, even the ones over non necessarily unital $C^*$-algebras.

*Proof.* Any Inner product module $E$ is a $\mathbb{Z}$-module naturally because it is an abelian group with respect to the addition, and so is that $-\langle x,y\rangle = \langle x, -y\rangle$. Therefore, since the proof of Cauchy-Schwartz inequality 2.1.2 doesn't depend on the unity of $A$, we safe unitl now. For any approximate unit $(u_\lambda)_\lambda \subset A$, $(xu_\lambda)_\lambda \subset E$ converges to $x$, whence, for $z \in \mathbb{C}$, let $zx := \lim_\lambda x(zu_\lambda)$. Since $A$ is a vector space, all properties are guaranteed and we are done. □

**Definition 2.1.4.** Inner product modules are called *Hilbert $C^*$-modules* when the induced norm is complete in the Cauchy sense.

**Proposition 2.1.5.** For a Hilbert $C^*$-module $E$ over $A$, $\overline{\operatorname{Span} EA} = E$.

*Proof.* If $(u_\lambda)_\lambda \subset A$ is a approximate unit for $A$, then for all $x \in E$:

$$\begin{aligned}
\lim_\lambda \langle x - xu_\lambda, x - xu_\lambda\rangle &= \lim_\lambda \left(\langle x,x\rangle - u_\lambda\langle x,x\rangle\right) \\
&\quad - \lim_\lambda \left(\langle x,x\rangle u_\lambda - u_\lambda\langle x,x\rangle u_\lambda\right) = 0.
\end{aligned}$$

Hence the elements of the form $xu_\lambda$ are dense in $E$. □

***Remark* 2.1.6.** Let $A$ and $B$ be $C^*$-algebras. If $E$ is a Hilbert $B$-module and the ideal $I$ of the clousure of the elements spanned by $\langle x, y\rangle$ is contained in $A$, then there is a way to make $E$ into a Hilbert $A$-module without changing the inner product. Namely, let $(u_\lambda)_\lambda$ be an approximate unit for $I$. Then the identity

$$\begin{aligned}\langle xu_\eta a - xu_\lambda a, xu_\eta a - xu_\lambda a\rangle &= a^*u_\eta\langle x,x\rangle u_\eta a + a^*u_\lambda\langle x,x\rangle u_\lambda a\\ &\quad - a^*u_\eta\langle x,x\rangle u_\lambda a - a^*u_\lambda\langle x,x\rangle u_\eta a,\end{aligned}$$

holds for all $x \in E$ and $a \in A$, showing that $(xu_\lambda a)_\lambda$ converges in $E$. We can define $xa = \lim xu_\lambda a$, and it is straightforward to check that this makes $E$ into a Hilbert $A$-module. This is particularly when dealing with non unital $C^*$-algebras $A$, and we might have a look into the same module over $\widetilde{A}$. ☕

**Examples 2.1.7.**

(*i*) Any traditional complex Hilbert space is a Hilbert $\mathbb{C}$-module.

(*ii*) Let $(E_i)_{i\in I}$ be a family of Hilbert $C^*$-modules over $A$. The direct sum will be:

$$\bigoplus_{i\in I} E_i := \Big\{x \in \prod_{i\in I} E_i \mid \sum_{i\in I}\langle x_i, x_i\rangle \in A\Big\}$$

It should be noticed that the convergence of $\sum_i\langle x_i, x_i\rangle$ is a weaker condition than requiring that the series of norms $\sum_i \|\langle x_i, x_i\rangle\|$ should converge. With the addition inner product $\langle x, y\rangle = \sum_i \langle x_i, y_i\rangle_{E_i}$, $\bigoplus_i E_i$ is a Hilbert $C^*$-module it self.

(*iii*) Subexamples of (*ii*) are: $A$ it-self endowed with $\langle a, b\rangle := a^*b$; $A^n = \bigoplus_{i=1}^n A$ for any natural number $n$.

(*iv*) **The standard Hilbert *A*-module $\mathscr{H}_A$**: A more especif subexample of (*ii*) can be given by $\mathscr{H}_A := \bigoplus_{n\in\mathbb{N}} A$, consisting of all sequences $(a_n)_n \subset A$ which $\sum_n a_n^* a_n$ converges.

(*v*) Given a Hilbert space $H$, the algebraic tensor product of $H$ by $A$ can be seeing as a Inner product $C^*$-module, with the bond:

$$\langle x \otimes a, y \otimes b\rangle := \langle x, y\rangle_H a^*b$$

$H \otimes A$ stands for its completion.

(*vi*) Let $X \in$ **CHaus** and $E \longrightarrow X$ a complex vector bundle. As we mention, $C(X)$ is a unital $C^*$-algebra. Whenever $d : E \times E \longrightarrow [0, \infty)$ is an Hermitian metric over $E$, the set $\Gamma(E)$ of continuous sections over $E$ holds the title of Hilbert module over $C(X)$ when endowed with

$$\begin{aligned}\langle\,\cdot\,,\,\cdot\,\rangle :\ \Gamma(E) \times \Gamma(E) &\longrightarrow C(X)\\ (a, b) &\longmapsto d(a(\,\cdot\,), b(\,\cdot\,))\end{aligned}$$

as an inner product.

**Lemma 2.1.8.** Given two nets $(x_\lambda)_\lambda$ and $(y_\lambda)_\lambda$ and $x, y$ in a Hilbert module $E$ over a $C^*$-algebra $A$ such that $x_\lambda \to x$ and $y_\lambda \to y$, $\lim_\lambda \langle x_\lambda, y_\lambda \rangle = \langle x, y \rangle$ holds.

*Proof.* From the Cauchy-Schartz inequallity 2.1.2, is easy to obtain that

$$\|\langle x_\lambda - x, z\rangle\|_A \overset{(2.1)}{\leqslant} \|x_\lambda - x\|_E \|z\|_E \qquad (z \in E, \lambda \in \mathbb{A})$$

Analogously, $\|\langle z, y_\lambda - y\rangle\|_A \leqslant \|y_\lambda - y\|_E \|z\|_E$. For each and every index $\lambda$, it is possible to obtain the following inequality:

$$\begin{aligned} \|\langle x_\lambda, y_\lambda\rangle - \langle x, y\rangle\| &= \|\langle x_\lambda, y_\lambda\rangle - \langle x_\lambda, y\rangle + \langle x_\lambda, y\rangle - \langle x, y\rangle\| \\ &\leqslant \|\langle x_\lambda, y_\lambda - y\rangle\| + \|\langle x_\lambda - x, y\rangle\| \\ &\leqslant \|y_\lambda - y\|\|x_\lambda\| + \|y\|\|x_\lambda - x\| \end{aligned}$$

Let $\varepsilon > 0$. Notice that $x_\lambda \to x$, means that $\|x_\lambda\| \to \|x\|$. By $\|y_\lambda - y\| \to 0$, there exists $\lambda_0$ in which $\|y_\lambda - y\| \, \|x_\lambda\| < \varepsilon/2$. Similarly, there allways exists $\lambda_1$ such that $\|x_\lambda - x\|_E < \varepsilon/2(\|y\| + 1)$ for $\lambda \succcurlyeq \lambda_1$. Since it exists $\lambda_2$ such that $\lambda_2 \succcurlyeq \lambda_0$ and $\lambda_2 \succcurlyeq \lambda_1$, we conclude that $\|\langle x_\lambda, y_\lambda\rangle - \langle x, y\rangle\| < \varepsilon$ for all $\lambda \succcurlyeq \lambda_2$. ☐

**Proposition 2.1.9.** If $E$ is Hilbert $A$-module, $\|x\| = \sup\{\|\langle x, y\rangle\| \mid \|y\| \leqslant 1\}$.

## 2.2 Adjointable operators

- no riesz lemma :/ - examples of non-adjointables - the (unital) C*-algebra of adjointable operators - equivalence of positivity

# Chapter 3

# Fredholm Operators

## 3.1 Definitions and comparisons

- comments about how the proof of ?? induces a Fredholm operator theory idea - talk about a trivialization of atikinson theorem - continuous family of classical fredholm operators - no topological clousure of the image - comment about moore-penrose idea

## 3.2 Regular Fredholm Operators and their indexes

- moore penrose idea - definition - kernel is finite rank module - index definition - ind(T) in K0(A) - index properties - (I-T) compact implies ind(T) = 0 - T1-T2 compact implies ind(T1)=ind(T2) - families index

## 3.3 Regularization of General Fredholm Operators

- construction of regular - ind(T) in K0(A)

## 3.4 Toolkit of theorems

- index properties - index locally constant - ind(T2T1) = ind(T1)+ind(T2)

## 3.5 The Fredholm Picture of $K_0$

- surjectivity of the index - classification of ind(T)=0 - equivalence relation - Atiyah-Janich theorem

# Appendix A

# Usefull and handy theorems in $C^*$-algebras

## A.1 $C^*$-algebras 101

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**Definition A.1.1.** In a (complex) algebra $A$, it is called an *involution* an operation $a \longmapsto a^*$ attending: conjugate-linearity, order 2 and anti-commutativivite. That is:

$$(a+\lambda b)^* = a^* + \overline{\lambda} b^*; \qquad a^{**} = a; \qquad (ab)^* = b^* a^* \qquad (a, b \in A, \lambda \in \mathbb{C})$$

If $(A, \|\cdot\|)$ is a Banach algebra[1] and the involution satisfies $\|a^*a\| = \|a\|^2$ for all $a$ ($C^*$-property), then $A$ its said to be an $C^*$*-algebra*. Setting $*$-morphisms to be the algebraic morphisms that respect involution[2], we construct the category of $C^*$-algebras $\boldsymbol{C^*}\text{-}\mathbf{Alg} := \boldsymbol{C^*}\text{-}\mathbf{Alg}(\mathbb{C})$.

**Examples A.1.2.** Some of the following examples are relatively bizarre for an initial view of the theory, but these represent important applications of $C^*$-algebras.

($i$) The complex conjugation in $\mathbb{C}$ naturally fits into the definition of an involution. Since $\mathbb{C}$ is complete, it is a $C^*$-algebra.

($ii$) The square matrices algebra $\mathbb{M}_{n\times n}(\mathbb{C})$ with $M \longmapsto \overline{M}^t$ as the given involution is fated to also be an $C^*$-algebra.

($iii$) For any $x \in \mathbf{Top}$ and $A \in \boldsymbol{C^*}\text{-}\mathbf{Alg}$, the algebras $C_b(x, A)$ and $C_0(x, A)$ of bounded and infinity-vanishing functions respectively, endowed with the point-wise involution[3] also becomes $C^*$-algebras.

[1] Norm Cauchy-complete and submultiplicative $\|ab\| \leqslant \|a\| \cdot \|b\|$.
[2] That is to say, linear maps $\phi : A \longrightarrow B$ such that $\phi(a^*) = \phi(a)^*$ for every $a \in A$.
[3] $f^* : x \longmapsto f(x)^*$ for any $f : x \longrightarrow A$.

(*iv*) **Concrete $C^*$-algebras**: Every $*$-closed subalgebra[4] of bounded operators $T$ over a Hilbert space fits the above definition, since $\langle Tx, y\rangle = \langle x, T^*y\rangle$. In fact, the first $C^*$-algebras in town where precisely these, called the *concrete* $C^*$-algebras. [17] introduced the term to describe precisely these mathematical beings, to which he refers to as "uniformly closed, self-adjoint algebra of bounded operators on a Hilbert space".

(*v*) **Irrational rotation algebra**: Let $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number. For $S^1 := \{z \in \mathbb{C} \mid |z| = 1\}$, consider $L^2(S^1)$ the Hilbert space of square-integral functions[5] and the operators $U$ and $V$ given by:

$$Uf(z) := z \cdot f(z) \quad \text{and} \quad Vf(z) := f(e^{2\pi i\theta} z) \qquad (z \in S^1)$$

One should check that $VU = e^{2\pi i\theta} UV$. The $C^*$-algebra generated by $\{U, V\}$ is denoted by $A_\theta$. It is known that $A_\theta$ is a simple algebra (contains no bilateral ideals) and also that for $0 < \theta_1 < \theta_2 < 1/2$, the algebras $A_{\theta_1}$ and $A_{\theta_2}$ are not isomorphic ([15], Theorem 2). The proofs of these two facts are closely linked to the great advance of the theory of $C^*$-algebras in the last twenty-five years.

(*vi*) **Toeplitz algebras and the Hardy Space**: Consider the *shift* operator $T : H \longrightarrow H$ in separable Hilbert space $H$, given by $Te_n := e_{n+1}$ where $(e_n)_{n\in\mathbb{N}}$ is an orthonormal basis. The *Toeplitz algebra* is the smallest $C^*$-subalgebra $\mathscr{T}(H) \subset \mathscr{B}(H)$ containing $T$.

Is a prototypical example of a Toeplitz algebra the *Hardy space* $\mathscr{H}^2(S^1) \subset L^2(S^1)$ whose elements are precisely those $f \in L^2(S^1)$ in which all the negative Fourier coefficients are zero.

One of the most surprising observations about Toeplitz algebras is the fact that $\mathscr{T}(H)$ does not depend on $H$ nor on the orthonormal basis $(e_n)_n$ (up to isomorphism).

> **Summoning A.1.3** ([6], Theorem 7.23)**.** Let $H$ be a Hilbert space and $\mathscr{K}(H) \subset \mathscr{T}(H)$ be the ideal of compact operators over $H$. There exists a $*$-isomorphism
>
> $$\mathscr{T}(H)\big/\mathscr{K}(H) \longrightarrow C(S^1)$$
>
> mapping the class of the shift operator into the identity $I_{C(S^1)}$.

In fact, $\mathscr{T}(H)$ is the universal unital $C^*$-algebra generated by an isometry, providing a more categorical definition of an Toeplitz algebra:

[4] A subalgebra $A$ for which $T \in A \Rightarrow T^* \in A$.
[5] $f : S^1 \longrightarrow \mathbb{C}$ whose $\int |f(z)|^2 \mathrm{d}z < \infty$.

> **Summoning A.1.4** (Coburn)**.** For any unital $A \in \boldsymbol{C^*\text{-}\mathbf{Alg}^u}$ and any isometry $w \in A$, there is a unique unital $*$-morphism $\mathscr{T}(H) \longrightarrow A$ mapping the shift operator into $w$.

To list one important aplication, [5] provide a simple proof of Bott periodicity for $C^*$-algebras, in which only the functorial properties of $K_0$ and $K_1$ are required. The heart of the proof consists in the usage of the short exact sequence:

$$0 \longrightarrow \mathscr{K}(\ell^2(\mathbb{N})) \longrightarrow \mathscr{T}(\ell^2(\mathbb{N})) \longrightarrow C(S^1) \longrightarrow 0.$$

**Definition A.1.5** (Invertible elements)**.** An element $a \in A$ in a unital ring, is said to be *invertible* when there exists $b \in A$ such that $ab = ba = 1$. The set of invertibles are denoted by $\mathrm{GL}(A)$.

**Proposition A.1.6.** Let $A$ be a unital algebra and let $a, b, c \in A$.

$(i)$ If $a$ is invertible, then there is a unique $b$ such that $ab = ba = 1$, and therefore, it can be denoted by $a^{-1}$.

$(ii)$ If $ab = ca = 1$, then $b = c$ and hence $a$ is invertible.

$(iii)$ If $ab = ba$, then $ab$ is invertible if and only if both $a$ and $b$ are invertible.

$(iv)$ If $A$ is an involution algebra, $a \in \mathrm{GL}(A) \Rightarrow (a^*)^{-1} = (a^{-1})^*$.

*Proof.*

$(i)$ Whenever $b$ and $b'$ are the interesting ones, $b(ab) = b(ab') = (ba)b' = b'$.

$(ii)$ Notice that $c = c(ab) = (ca)b = b$.

$(iii)$ If $a$ and $b$ are invertible, therefore

$$ab(a^{-1}b^{-1}) = baa^{-1}b^{-1} = 1 = a^{-1}b^{-1}ba = (a^{-1}b^{-1})ab.$$

Now, suppose that $ab$ is invertible and let $c \coloneqq (ab)^{-1}$.

$(iv)$ Using the anti-commutativity of the involution,

$$a^*(a^{-1})^* = (a^{-1}a)^* = 1 = (aa^{-1})^* = (a^{-1})^*a^*,$$

that is to say, $(a^*)^{-1} = (a^{-1})^*$. $\square$

Proposition A.1.6 shows that $\mathrm{GL}(A)$ is a group under multiplication.

**Definition A.1.7** (Spectrum)**.** For a unital Banach algebra $A$, the *spectrum* of an element $a \in A$ is the set of scalar $\lambda$ such the elements $a - \lambda := a - \lambda 1$ aren't invertible, i.e.,

$$\operatorname{Spec}(a) := \{\lambda \in \mathbb{C} \mid a - \lambda \notin \operatorname{GL}(A)\}.$$

For non-unital Banach algebras, since $A$ can be isometrically embeded into a unital algebra $A^u$, $\operatorname{Spec}(a)$ stand for the spectrum over its identification.

**Examples A.1.8.**

(*i*) Let $x$ be a compact Hausdorff topological space, we'll show that $\operatorname{Spec}(f) = \operatorname{Im} f$ for every continuous function $f \in C(x)$. Given $\lambda \in \operatorname{Spec}(f)$, notice that $f - \lambda \mathit{1}$ is non-invertible if and only if, there exists $x \in x$ such that $(f - \lambda \mathit{1})(x) = 0$, which means $f(x) = \lambda$.

(*ii*) Whenever $T \in \mathbb{M}_n(\mathbb{C})$, $\operatorname{Spec} T$ stands for the set of eigenvalues of $T$, due to the fact that those values $\lambda$ are the ones that $\det(T - \lambda I_n) = 0$; precisely the operators which $T - \lambda I_n$ aren't invertible.

(*iii*) Let $A$ be a unital $C^*$-algebra and $a \in A$. Let $\lambda$ be a complex number. By proposition A.1.6(*iv*) it is safe to say that $a - \overline{\lambda} = (a^* - \lambda)^*$ isn't invertible if and only if $a^* - \lambda$ isn't as well. Therefore, $\operatorname{Spec}(a^*) = \overline{\operatorname{Spec}(a)}$.

**Lemma A.1.9.** For elements $a, b$ in a $C^*$-algebra $A$, $\operatorname{Spec}(ab)\backslash\{0\} = \operatorname{Spec}(ba)\backslash\{0\}$.

*Proof.* Assume that $A$ is unital. It suffices to prove that $1 - ab \in \operatorname{GL}(A)$ implies $1 - ba \in \operatorname{GL}(A)$ Let $u := (1 - ab)^{-1}$. Then

$$\begin{aligned}(1 + bua)(1 - ba) &= 1 - ba + bua - buaba \\ &= 1 - b(1 - u + uab)a \\ &= 1 - b(1 - u(1 - ab))a = 1\end{aligned}$$

Similarly $(1 - ba)(1 + bua) = 1$. Hence $1 - ba \in \operatorname{GL}(A)$. $\square$

**Lemma A.1.10.** In a unitary Banach algebra $A$, if $\lambda \in \mathbb{C}$, elements of the form $\lambda - a$ with $a \in A$, are invertible if $\|a\| < |\lambda|$. Furthermore, any algebraic morphism of the form $\phi : A \longrightarrow \mathbb{C}$ is continuous and $\|\phi\| \leqslant 1$.

*Proof.* Consider the sequence of partial sums of the form $1 + a/\lambda + a^2/\lambda^2 + \cdots + a^n/\lambda^n$, and notice that they forms a Cauchy sequence, so it converges to a element $b$, given that $\|a/\lambda\| < 1$. Notice that

$$\lambda\Big(1 - \frac{a}{\lambda}\Big) \underbrace{\lim_{n\to\infty} \sum_{j=0}^{n} \Big(\frac{a}{\lambda}\Big)^j}_{b} = \lim_{n\to\infty} \lambda\Big(1 - \frac{a^{n+1}}{\lambda^{n+1}}\Big) = \lambda$$

So $(\lambda - a)^{-1} = b$. Without loss of generality, we can assume that $\phi$ is non-null and $\phi(1) = 1$. As we have seen,

$$(\lambda - a)\, b = 1 \quad \Rightarrow \quad (\lambda - \phi(a))\phi(b) = 1,$$

meaning $\phi(a) \neq \lambda$. Since this holds for all $\lambda$ such that $|\lambda| > \|a\|$, it follows that $|\phi(a)| \leqslant \|a\|$, which proves our claim that $\|\phi\| \leqslant 1$. □

**Proposition A.1.11.** The group $\mathrm{GL}(A)$ of invertible elements in a unitary Banach algebra $A$ constitute an open set.

*Proof.* Let $x$ be an invertible. Given $y$ such that $\|x - y\| < \|x^{-1}\|^{-1}$, let's see that $y$ will be an invertible as well. From the asssumption and submultiplicativity of the norm, its easy to see that

$$\|1 - x^{-1}y\| = \|x^{-1}(x - y)\| \leqslant \|x^{-1}\|\|x - y\| < \|x^{-1}\|\|x^{-1}\|^{-1} = 1,$$

hence $1 - (1 - x^{-1}y) = x^{-1}y$ is invertible by A.1.10. But since $\mathrm{GL}(A)$ is a group, $y$ must also be invertible, i.e., the open ball around $x$ of radious $\|x^{-1}\|^{-1}$ is fully contained in $\mathrm{GL}(A)$, so it is open. □

**Proposition A.1.12.** Over a unitary Banach algebra $A$, the invertion map $a \longmapsto a^{-1}$ is a continuous function.

*Proof.* Choose $x, y \in \mathrm{GL}(A)$ such that $\|x - y\| < \|x^{-1}\|^{-1}$. As showed in the proof of A.1.11, $\|1 - x^{-1}y\| < 1$ and therefore,

$$\begin{aligned} & y^{-1}x = 1 - (1 - x^{-1}y)^{-1} = \sum_{n=0}^{\infty}(1 - x^{-1}y)^n \\ \Rightarrow \quad & y^{-1} = \sum_{n=0}^{\infty}(1 - x^{-1}y)^n x^{-1} \end{aligned} \tag{A.1}$$

If we want to estimate how close $x$ and $y$ need to be in order that $\|y^{-1} - x^{-1}\|$ can be arbitrally small, (A.1) might help to found an upper bound:

$$\|y^{-1} - x^{-1}\| = \Big\| \sum_{n=1}^{\infty}(1 - x^{-1}y)^n x^{-1} \Big\| \leqslant \|x^{-1}\| \sum_{n=1}^{\infty} \|1 - x^{-1}y\|^n = \|x^{-1}\| \cdot \frac{\|1 - x^{-1}y\|}{1 - \|1 - x^{-1}y\|}$$

For a given $\varepsilon > 0$, since $t \longmapsto t/(1 - t)$ is a increasing continuous function for $t < 1$, there allways exists $\delta_{x,\varepsilon} > 0$ right for the job:

$$t < \delta_{x,\varepsilon} := \frac{\frac{\varepsilon}{\|x^{-1}\|}}{\frac{\varepsilon}{\|x^{-1}\|} + 1} \quad \Rightarrow \quad \frac{t}{1 - t} < \frac{\varepsilon}{\|x^{-1}\|},$$

so whenever $\|x - y\| < \delta_{x,\varepsilon}$, $\|y^{-1} - x^{-1}\| < \varepsilon$. □

**Proposition A.1.13.** The *resolvent function* (A.2) is holomorphic for each and every unital Banach algebra $A$.
(A.2)

$$\begin{aligned} R_a : \quad \mathbb{C} \setminus \operatorname{Spec}(a) &\longrightarrow A \\ \lambda \longmapsto (\lambda - a)^{-1} &= \sum_{n=0}^{\infty} \left(\frac{a}{\lambda}\right)^n \end{aligned} \qquad (a \in A)$$

*Proof.* Let $z, w \in \mathbb{C} \setminus \operatorname{Spec}(a)$. Let $\alpha := z - a$ and $\beta := w - a$. Notice that

$$\begin{aligned} R_a(z) - R_a(w) &= \alpha^{-1} - \beta^{-1} \\ &= \alpha^{-1}(1 - \alpha\beta^{-1}) \\ &= \alpha^{-1}(\beta - \alpha)\beta^{-1} \\ &= R_a(z)(w - z)R_a(w) \end{aligned}$$

Therefore:

$$\lim_{w \to z} \frac{R_a(z) - R_a(w)}{z - w} = \lim_{w \to z} -R_a(z)R_a(w) = -R_a(z)^2 \qquad (z \in \mathbb{C} \setminus \operatorname{Spec}(a))$$

So the limit exists, $\partial R_a = -R_a^2$ and it is analytic over the domain. □

**Theorem A.1.14.** The spectrum of any element $a$ in a Banach algebra is a compact non-empty set, and satisfies:

$$\operatorname{Spec}(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leqslant \|a\|\}.$$

*Proof.* Since life is really good and Heine-Borel's theorem holds for complex numbers, we need only to show that $\operatorname{Spec}(a)$ is closed and bounded.

$(i)$ **Spec($a$) is a closed set**. Since $a$ is fixed element, notice the projection $a + \lambda \longmapsto \lambda$ is a continuous function by A.1.10 and so is their inverse $\lambda \longmapsto a - \lambda$. Since $\operatorname{Spec}(a)$ is the set-theoretic complement of the pre-image of $\mathrm{GL}(A)$ (an open set - A.1.11) by a continuous function, it must be closed.

$(ii)$ **Spec($a$) is bounded**. From A.1.10, spectral elements of the form $\lambda - a$ must necessarily obey $|\lambda| \leqslant \|a\|$, i.e., $\operatorname{Spec}(a) \subset \{\lambda \in \mathbb{C} \mid |\lambda| \leqslant \|a\|\}$.

Therefore, $\operatorname{Spec}(a)$ is compact set. In order to give a little bit of flavor, we present the proof of [19] the non-emptiness of the spectrum, with a clever tricky about derivation under the integral sign.

$(iii)$ **Spec($a$) is non-empty**. Suppose $a \neq 0$, since otherwise the result follows immediately. Assume that $\operatorname{Spec}(a)$ is empty, so that $R_a$ exists in all complex plane. In particular, $a \in \mathrm{GL}(A)$. For any bounded linear functional $f : A \longrightarrow \mathbb{C}$, the function

$$\begin{aligned} g_f : \quad \mathbb{R}^2 &\longrightarrow \mathbb{C} \\ (r,\theta) &\longmapsto f(R_a(re^{i\theta})) \end{aligned}$$

is continuously differentiable with respect to both $r$ and $\theta$:

(A.3)
$$\partial_\theta g_f(r,\theta) = f\big(-R_a(re^{i\theta})^2\big)ire^{i\theta} \quad \text{and} \quad \partial_r g_f(r,\theta) = f\big(-R_a(re^{i\theta})^2\big)e^{i\theta}.$$

Hence $\partial_\theta g_f = ir\partial_r g$. Now let $F(r) \coloneqq \int_0^{2\pi} g_f(r,\theta)\,\mathrm{d}\theta$. By differentiating under the integral sign[6] and using $\partial_r g_f$, we obtain

$$\begin{aligned} F'(r) &= \int_0^{2\pi} \partial_r g_f(r,\theta)\,\mathrm{d}\theta = \int_0^{2\pi} f\big(-R_a(re^{i\theta})^2\big)e^{i\theta}\,\mathrm{d}\theta \\ \Rightarrow \qquad irF'(r) &= \int_0^{2\pi} f\big(-R_a(re^{i\theta})^2\big)ire^{i\theta}\,\mathrm{d}\theta = \int_0^{2\pi} \partial_\theta g_f(r,\theta)\,\mathrm{d}\theta \\ &= f(R_a(re^{i2\pi})) - f(R_a(re^{i0})) = 0 \\ \Rightarrow \qquad F(r) &= F(0) = 2\pi f(a^{-1}) \end{aligned}$$

We now choose $f$ so that $f\left(a^{-1}\right) \neq 0$. Let $h : \mathrm{Span}(a^{-1}) \longrightarrow \mathbb{C}$, $h(za^{-1}) = z\|a^{-1}\|$. Since $\|\cdot\|$ is a positive sublinear functional, we are able to conjure:

> **Summoning A.1.15** (Complex Hahn-Banach)**.** Let $X$ be a $\mathbb{K}$-vector space with $\mathbb{K} \in \{\mathbb{R},\mathbb{C}\}$ and $p$ a positive sublinear functional[a]. If $Y \subset X$ is a subspace and $h : Y \longrightarrow \mathbb{K}$ a linear functional dominated by $p$ (i.e., $|h(y)| \leqslant p(y)$ for $y \in Y$), the there exists a *Hahn-Banach extension* $f : X \longrightarrow \mathbb{K}$ such that $f$ is also dominated by $p$. ☕
>
> [a] I.e., for all vectors $x, y$ and scalar $\alpha$, $p(x) \geqslant 0$, $p(x+y) \leqslant p(x) + p(y)$ and $p(\alpha x) \leqslant |\alpha| p(x)$.

Therefore, choose a Hahn-Banach extension $f$ of $h$ such that $f(a^{-1}) = \|a^{-1}\| \neq 0$. Since

$$f(R_a(re^{i\theta})) = f\big((re^{i\theta}1 - a)^{-1}\big) = r^{-1}e^{-i\theta}f\big((1 - ar^{-1}e^{-i\theta})^{-1}\big)$$

it follows that $\left|f\big(R_a(re^{i\theta})\big)\right|$ can be made as small as we like, independently of $\theta$, by choosing $r$ sufficiently large, by the continuity of the invertion

[6] Such a differentiation is possible, since $g$ is clearly a $C^1$-function by (A.3).

map $x \longmapsto x^{-1}$ (A.1.12). We fix $r$ such that $|f(R_a(re^{i\theta}))| < |f(a^{-1})|/2$. For this $r$ we have

$$2\pi|f(a^{-1})| = |F(r)| \leqslant \int_0^{2\pi} |f(a(re^{i\theta}))|\,\mathrm{d}\theta \leqslant \pi|f(a^{-1})|$$

which is not possible. Therefore, $\operatorname{Spec}(a)$ must in fact be non-empty.

□

Typically, in order to prove A.1.14(*iii*), one would argue that if otherwise, $R_a$ would be *entire* function and conclude that it must constant by Liouville's theorem. The advantage of the argument presented, as showed in [19], is a more elementary proof, and both the fundamental theorem of Algebra and Liouville's theorem as corollary.

**Corollary A.1.16** (Fundamental Theorem of Algebra)**.** If $p \in \mathbb{C}[z]$, there exists a complex root $w \in \mathbb{C}$ such that $p(w) = 0$.

*Proof.* In the algebra of complex square matrices $\mathbb{M}_n(\mathbb{C})$, set $T$ as the operator in which $p$ is the characteristic polynomial. If $p(z) = z^n + \alpha_{n-1}z^{n-1} + \cdots + \alpha_1 z + \alpha_0$, one may let

$$T := \begin{bmatrix} 0 & 0 & \cdots & 0 & -\alpha_0 \\ 1 & 0 & \cdots & 0 & -\alpha_1 \\ 0 & 1 & \cdots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\alpha_{n-1} \end{bmatrix}$$

By A.1.14(*iii*), there exists an eigenvalue $w \in \operatorname{Spec} T$ thus $p(w) = 0$. □

**Corollary A.1.17** (Liouville's Theorem)**.** Let $A \in \boldsymbol{\mathscr{B}}\textbf{-Alg}^{\boldsymbol{u}}$ be a unitary Banach algebra. If $f : \mathbb{C} \longrightarrow A$ is a bounded function which is holomorphic everywhere (i.e., *entire*), then it must be constant.

*Proof.* Let $f$ be a bounded entire function. Fix an arbitrary $w \in \mathbb{C}$. Define $g$ on $\mathbb{C}$ as follows:

$$g(z) := \begin{cases} (f(z+w) - f(w))/z & \text{if } z \neq 0. \\ f'(w) & \text{if } z = 0. \end{cases}$$

Then $g$ is certainly continuous in $\mathbb{C}$ and analytic in $\mathbb{C}_{\neq 0}$. Define $F(r) = \int_0^{2\pi} g(re^{i\theta})\,\mathrm{d}\theta$. By the same idea used in the argument of A.1.14(*iii*), $F$ must be constant. However, from the boundedness of $f$ it follows that $g$ tends to 0 uniformly in $\theta$ as $r$ tends to infinity, so $F$ tends to 0. Therefore, $F(r) = F(0) = 0 = 2\pi f'(w)$. As $w$ was an arbitrary point of $\mathbb{C}$, we conclude that $f'$ is identically zero, therefore $f$ is constant. □

**Theorem A.1.18** (Spectral Mapping Theorem)**.** Given an unital Banach algebra $A$ and $a \in A$, the following holds for each complex-polynomial:

$$\mathrm{Spec}(p(a)) = \{p(\lambda) \mid \lambda \in \mathrm{Spec}(a)\} = p(\mathrm{Spec}(a)). \qquad (p \in \mathbb{C}[x])$$

*Proof.* Fix $p \in \mathbb{C}[x]$ and by the Fundamental Theorem of Algebra A.1.16, for each $\lambda \in \mathbb{C}$, it is possible to decompose $\lambda - p(z)$ into linear factors. By the Cayley-Hamilton theorem, it is safe to say that there is $\lambda_, \ldots, \lambda_n, \alpha \in \mathbb{C}$ such:

$$\lambda - p(a) = \alpha \prod_{i=1}^{n} (\lambda_i 1 - a). \qquad (a \in A)$$

If $\lambda_i \notin \mathrm{Spec}(a)$ for each index $i$, every factor $\lambda_i 1 - a$ is invertible, hence $\lambda - p(a)$ are as well. In order to $\lambda \in \mathrm{Spec}(p(a))$, there must exist a root $\lambda_j$ such that $\lambda_j \in \mathrm{Spec}(a)$. In that case, $\lambda = p(\lambda_j) \in p(\mathrm{Spec}(a))$, and we shall conclude that $\mathrm{Spec}(p(a)) \subset p(\mathrm{Spec}(a))$.

To prove the other inclusion, let $\lambda \in p(\mathrm{Spec}(a))$, meaning that there is $\eta \in \mathbb{C}$ such that $\eta 1 - a$ is non-invertible and $p(\eta) = \lambda$. But because decomposition is unique, there must be $j$ such that $\eta = \lambda_j$ and by the same argument, one concludes that $\mathrm{Spec}(p(a)) \supset p(\mathrm{Spec}(a))$, obtaining the equallity. □

***Remark* A.1.19.** In quantum mechanics, physical observables are represented mathematically by linear operators on Hilbert spaces. One remarkable corollary of the non-emptiness of the spectrum imply that, it is not possible to describe the position and momentum operators as elements of a Banach algebra.

> **Proposition**. The *Weyl algebra* $\mathbb{C}[x, \partial_x] \simeq \mathbb{C}[x,y] / (yx - xy - 1)$ does not embed into a Banach algebra.
>
> *Proof.* Suppose to the contrary that $x, y$ are elements of a Banach algebra satisfying $yx - xy = 1$. From the spectral mapping theorem A.1.18,
>
> $$\mathrm{Spec}(yx) = \mathrm{Spec}(xy + 1) = \mathrm{Spec}(xy) + 1.$$
>
> But we know that $\mathrm{Spec}(yx) = \mathrm{Spec}(xy)$, so the spectrum of the product $xy$ must be invariant under addition by 1. By non-emptyness, it must contain arbitrarily large elements, and this contradicts compactness. □

Hence, it is necessary to look at the unitary groups of strongly continuous functions that the position and momentum operators generate, through the Stone-von Neumann theorem, stating that, the canonical commutation relations on two generators (canonical coordinate $q$ and canonical momentum

$p$) in the form $[q,p] = i\hbar$ may be represented as unbounded operators on the Hilbert space of square integrable functions $L^2(\mathbb{R})$ on the real line by defining them on the dense subspace of smooth functions $\psi : \mathbb{R} \longrightarrow \mathbb{C}$ as

$$(q\psi)(x) \coloneqq x\psi(x) \quad \text{and} \quad (p\psi)(x) \coloneqq i\hbar\partial_x\psi(x),$$

where on the right we have the derivative along the canonical coordinate function on real numbers axis.

**Corollary A.1.20** (Complex Gelfand-Mazur)**.** Any complex unital Banach division algebra $A$ (every non-zero element is invertible) is isometrically isomorphic to $\mathbb{C}$.

*Proof.* Let $a$ be a non-zero element, i.e., invertible. Since $\operatorname{Spec}(a)$ is non-empty A.1.14(*iii*), there must exist $\lambda_a$ such that $a - \lambda_a 1$ isn't invertible. The only non-invertible elemen't in $A$ is the zero element, which means that $a = \lambda_a 1$. □

**Theorem A.1.21** (Beurling-Gelfand formula)**.** The *spectral radius* $r$ of an element $a \in A$ can be directly calculated by the following formula:

$$r(a) \coloneqq \sup|\operatorname{Spec}(a)| = \lim_{n\to\infty} \sqrt[n]{\|a^n\|}$$

for *any Banach norm* $\|\cdot\|$.

*Proof.* By the Spectral Mapping theorem A.1.18 and the continuity of the inversion map A.1.12(*ii*), is easy to estabilish that $r(a)^n = r(a^n) \leqslant \|a^n\|$ for any given natural $n$, hence $r(a) \leqslant \inf \|a^n\|^{1/n}$. Notice that if $|\lambda| > r(a)$, then $\lambda - a$ is invertible by A.1.10. For any bounded linear $\phi : A \longrightarrow \mathbb{C}$, the operator

$$(\phi \circ R_a)\lambda = \phi((\lambda - a)^{-1}) = \sum_{n=0}^{\infty} \frac{1}{\lambda^n}\phi(a^n) \qquad (|\lambda| \geqslant r(a))$$

is analytic, and hence, converges absolutely. That can only happend if the general term of the series $\phi(a^n/\lambda^n)$ converges to 0 for any bounded linear $\phi$. Since every weakly convergent sequence is bounded (**??**), for each $\lambda$, there is $M_\lambda > 0$ such that $\|a^n\| \leqslant M_\lambda |\lambda^n|$. Therefore,

$$\limsup_{n\to\infty} \sqrt[n]{\|a^n\|} \leqslant \limsup_{n\to\infty} \sqrt[n]{M_\lambda}|\lambda| = |\lambda|$$

Since this holds for every $|\lambda| > r(a)$ we have

$$r(a) \leqslant \inf_{n\in\mathbb{N}} \sqrt[n]{\|a^n\|} \leqslant \liminf_{n\to\infty} \sqrt[n]{\|a^n\|} \leqslant \limsup_{n\to\infty} \sqrt[n]{\|a^n\|} \leqslant r(a),$$

which finishes the proof. □

**Example A.1.22.** In A.1.12(*ii*), $\sup|\operatorname{Spec}(a)| \leqslant \|a\|$, but this inequallity can be strict: Over $\mathbb{M}_{2\times 2}(\mathbb{C})$, notice that

$$\sup\left|\operatorname{Spec}\begin{pmatrix} 0 & 1/t \\ t & 0 \end{pmatrix}\right| = \sup|\{1,-1\}| = 1 \qquad (t>0)$$

but $\left\|\begin{smallmatrix} 0 & 1/t \\ t & 0 \end{smallmatrix}\right\| = t^2 + 1/t^2$.

**Lemma A.1.23.** Let $a \in A$ be an element of a $C^*$-algebra. If $a \in A$ is a self-adjoint, then $\operatorname{Spec}(a) \subset \mathbb{R}$.

*Proof.* Let $\lambda \in \operatorname{Spec}(a)$. For each integer $n$, let $b_n := a + (in \operatorname{Im}\lambda - \operatorname{Re}\lambda)$. Considering the rational function $f$ given by $f(z) = z - \operatorname{Re}\lambda + in\operatorname{Im} y$ we conclude that $f(\lambda) \in \operatorname{Spec} f(a)$ from the Spectral Mapping theorem A.1.18. Hence, $i(n+1)\operatorname{Im}\lambda \in \operatorname{Spec}(b_n)$.

From A.1.10, one obtains that $|i(n+1)\operatorname{Im}\lambda| \leqslant \|b_n\|$. Therefore

$$\begin{aligned}
(n^2+2n+1)(\operatorname{Im}\lambda)^2 &= |i(n+1)\operatorname{Im}\lambda|^2 \\
&\leqslant \|b_n\|^2 \\
&= \|b_n^* b_n\| \\
&\overset{a^*=a}{=} \|(a - \operatorname{Re}\lambda - in\operatorname{Im}\lambda)(a - \operatorname{Re}\lambda + in\operatorname{Im}\lambda)\| \\
&= \|(a-\operatorname{Re}\lambda)^2 + n^2(\operatorname{Im}\lambda)^2\| \leqslant \|a - \operatorname{Re}\lambda\|^2 + n^2(\operatorname{Im}\lambda)^2
\end{aligned}$$

which implies that $(2n+1)(\operatorname{Im}\lambda)^2 \leqslant \|a-\operatorname{Re}\lambda\|^2$. Since $n$ is arbitrary, it follows that $(\operatorname{Im}\lambda)^2$ is arbitrarily small, i.e., $\operatorname{Im}\lambda = 0$, hence $\operatorname{Spec}(a) \subset \mathbb{R}$. □

**Lemma A.1.24.** If $a \in A$ is a self-adjoint element of a unitary $C^*$-algebra, then $r(a) = \|a\|$.

*Proof.* We have $\|a^2\| = \|a^*a\| = \|a\|^2$. We appeal to induction: Suppose $n$ is such that $\|a^{2^n}\|^{2^{-n}} = \|a\|$. Therefore:

$$\left\|a^{2^{n+1}}\right\|^{2^{-n-1}} = \sqrt{\left\|(a^{2^n})^2\right\|^{2^{-n}}} = \sqrt{\|a^{2^n}\|^{2^{-n}}} \overset{\text{I.H.}}{=} \sqrt{\|a\|^2} = \|a\|.$$

By induction, the above assumption is valid for all natural $n$. Therefore, the sequence $\left(\|a^n\|^{1/n}\right)_{n\in\mathbb{N}}$, which converges to $r(a)$ by A.1.21, has a constant subsequence equal to $\|a\|$. Then $r(a) = \|a\|$. □

## A.2 Gelfand-Naimark representation theorem

To explore (*iii*), both bounded and infinity-vanishing functions can be seeing as a contra-variant functor from topological spaces to abelian $C^*$-algebras.

We will focus on the infinity-vanishing ones:

$$C_0\colon\quad \mathbf{Top} \longrightarrow \boldsymbol{C^*}\text{-}\,\mathbf{Alg}^{com}$$

$$\begin{array}{ccccc} X & \longmapsto & C_0(X) & & \phi\circ f \\ {\scriptstyle f}\big\downarrow & & \big\uparrow & & \big\uparrow \\ Y & \longmapsto & C_0(Y) & & \phi \end{array}$$

**Definition A.2.1** (Gelfand transformation)**.** For a given $A \in \mathscr{B}$-**Alg**, the *character space* of $A$ is the set of non-zero morphisms $\Gamma A \coloneqq \mathrm{Hom}_{\mathscr{B}\text{-}\mathbf{Alg}}(A,\mathbb{C})_{\neq 0}$. The *Gelfand transformation* is the evaluation functional $\kappa \coloneqq \mathbf{ev}_{(\,\cdot\,)}$ given by:

$$\text{(A.4)}\qquad \begin{aligned} \kappa :\quad A &\longrightarrow C_0(\Gamma A) \\ a &\longmapsto (\phi \overset{\mathbf{ev}_a}{\mapsto} \phi(a)) \end{aligned}$$

Later, we will endow a topological flavor onto $\Gamma A$. Before, we gave a look into a alternative definition of the character space.

**Proposition A.2.2.** If $A$ is a unitary commutative Banach algebra, let $\mathscr{I}_m(A)$ be the set of maximal ideals[7] of $A$. So the function $\ker(\,\cdot\,) : \Gamma A \longrightarrow \mathscr{I}_m(A)$ is a bijection. Since in algebraic lands $\mathscr{I}_m(A)$ is often called *spectrum of* $A$, the character space may also be called the *spectrum*.

*Proof.* Of course $\ker(\,\cdot\,)$ is well defined, i.e. $\ker\phi$ is always a maximal ideal of $A$ for any character $\phi$. That's because $\phi$ is necessarily surjective ($\phi(z1_A) = z\phi(1_A) = z$ for any $z \in \mathbb{C}$) and by the First Isomorphism Theorem, $A/\ker\phi \simeq \mathbb{C}$ is a field and therefore, $\ker\phi$ is maximal (since $A$ is a commutative ring).

In order to $\ker(\,\cdot\,)$ be a bijection, we verify:

(*i*) **ker( · ) is an injection**: Let $\phi, \psi \in \widehat{A}$ characters such that $\ker\phi = \ker\psi$. Notice that

$$\phi(a)1_A - a \in \ker\phi = \ker\psi \qquad (a \in A)$$

That is, $\psi(\phi(a)1 - a) = 0$ from which it follows that $\phi = \psi$.

(*ii*) **ker( · ) is a surjection**: Let $I \triangleleft_m A$ be a maximal ideal and consider the projection $\pi_I : x \longmapsto x + I$. So $\ker\pi_I = I$. It remains to verify that $\pi_I$ is a character but this will follow from the fact that $A/I$ is a Banach algebra where all its elements are invertible.

[7] We say that $I \subset A$ is an ideal if $I$ is closed by $A$-linear combinations, i.e., given $x, y \in I$ and $a, b \in A$, then $ax + by \in A$. We denote $I \triangleleft A$. We say that $I \triangleleft_m A$ is a maximal ideal if it is not trivial (like $\{0\}$ or $A$ ) and if any other ideal $J \triangleleft A$ such that $I \subset J \subset A$ collapses with either $I$ or $A$.

Since $I$ is maximal it follows that $A/I$ is a field and in particular an algebra. Using the Gelfand-Mazur theorem A.1.20, all non-zero elements of $A/I$ are all invertible and therefore we guarantee that $A/I \simeq \mathbb{C}$.

This concludes the proof. □

**Lemma A.2.3.** Endowed with the weak* topology, $\Gamma A$ is a locally compact Hausdorff topological space. Whenever $A$ is unitary, then $\Gamma A$ is compact.

*Proof.* The weak* topology on $\Gamma A \subset \mathscr{B}(A, \mathbb{C})$ is induced by the *separant*[8] familly of seminorms $(p_a)_{a \in A}$ given by $p_a : \phi \longmapsto |\mathbf{ev}_a(\phi)| = |\phi(a)|$, in the sense that it is the smallest topological space such that each $p_a$ be continuous. Therefore, $\Gamma A$ is a Hausdorff topological vector space.

Let $S := \mathrm{Hom}_{\boldsymbol{C^*}\textbf{-}\mathbf{Alg}}(A, \mathbb{C})$ for convention. We afirm that:

($i$) ***S* is a closed subspace of the unity ball over $\mathscr{B}(\boldsymbol{A}, \mathbb{C})$**: Let $B$ be the unit ball on the dual space and choose $\phi \in B$. For any net $(\phi_\lambda)_\lambda \subset S$ whoose $\lim_\lambda \phi_\lambda = \phi$, one have

$$\begin{aligned} \phi(ab) &= \lim_\lambda \phi_\lambda(ab) \\ &\overset{\phi_\lambda \in S}{=} \lim_\lambda \phi_\lambda(a)\phi_\lambda(b) \qquad (a, b \in A) \\ &= \lim_\lambda \phi_\lambda(a) \lim_\lambda \phi_\lambda(b) = \phi(a)\phi(b) \end{aligned}$$

i.e., $\phi \in S$, hence it is a closed subspace.

($ii$) **Whenever *A* is unitary, $\boldsymbol{0 : x \longmapsto 0}$ is a isolated point of *S***: For any net $(\phi_\lambda)_\lambda \subset S \backslash \{0\}$ converging to a given $\phi \in S$, notice that $\phi(1_A) = \lim_\lambda \phi_\lambda(1_A) = 1$. Hence $\phi \neq 0$, i.e., $S \backslash \{0\}$ is a closed set.

To conclude, we shall invoque:

> **Summoning A.2.4** (Banach-Alaoglu - [1], Theorem 1.3)**.** The closed unit ball $\{f \in \mathscr{B}(X, \mathbb{C}) \mid \|f\|_\infty \leqslant 1\}$ over the dual of a complex normed space $X$ is a compact Hausdorff space. ☕

Banach-Alaoglu theorem A.2.4 in contrast with ($i$), guarantee us that $\Gamma A = S \backslash \{0\}$ is locally compact Hausdorff. By ($ii$), $\Gamma A$ must be compact when $A$ contais a unity. □

**Example A.2.5.** Suppose that for a Banach algebra $A$, the character space separates points, i.e., for any $a, b \in A$, there exists $\phi \in \Gamma A$ such that $\phi(a) \neq \phi(b)$. In this configuration, $A$, must necessarily be abelian:

$$\phi(ab - ba) = \phi(a)\phi(b) - \phi(b)\phi(a) = 0 \overset{\phi \neq 0}{\Rightarrow} ab = ba.$$

[8] I.e., given an arbitrary $a$, there exists $\phi \in \Gamma A$ such that $|\phi(a)| \neq 0$

**Example A.2.6.** Given $A \in \mathscr{B}$-**Alg**, let $[A, A]$ be the closed linear span of the commutators $[a_1, a_2] = a_1a_2 - a_2a_1$. This is a closed two-sided ideal of $A$, and the quotient $A/[A, A]$ is the *abelianization* of $A$: the universal commutative Banach algebra to which $A$ maps.

Any character of $A$ must be a character of its abelianization, so, in order to determinate if $\Gamma A$ is a empty set for a arbitrary $A$ reduces immediately to the commutative case.

**Example A.2.7.** Give to $(\mathbb{C}^2, \|\cdot\|_{\max})$ the product $(z_1, z_2) \cdot (w_1, w_2) := (z_1w_1, 0)$, turning $\mathbb{C}^2$ into a commutative non-unital Banach algebra. The character space of this algebra is a singleton.

**Example A.2.8.** $\mathscr{B}(\mathbb{C}^n)$ with $n \geqslant 2$ has only trivial two-sided ideals. Hence it has no characters.

**Theorem A.2.9** (Gelfand theorem for $\mathscr{B}$-$\mathbf{Alg}_u^{com}$)**.** Let $A$ be a commutative unital Banach algebra. Therefore,

$$a \in \mathrm{GL}(A) \quad \Leftrightarrow \quad \mathbf{ev}_a \in \mathrm{GL}(C_0(\Gamma A)) \quad \Leftrightarrow \quad \forall\, \phi \in \Gamma A, \phi(a) \neq 0.$$

*Proof.* Since $A$ is unital, $1 = \phi(a)\phi(a^{-1})$ for each $a \in \mathrm{GL}(A)$, i.e., $\phi(a) \neq 0$ for every $\phi \in \Gamma A$. Now for the hard part, suppose that $\phi(a) \neq 0$ and we shall consider the set of every proper ideal $J \triangleleft A$ which contains $aA$. A technical issue is needed:

($i$) **Every maximal ideal $J \triangleleft_m A$ is closed**: Let $J \triangleleft_m A$ and notice that $\overline{J} \supset J$ is also a maximal ideal. Suppose that $\overline{J} = A$, hence $J$ is a dense subset. If $B$ is the open ball of radius 1 centered at the unit, by density of $J$, there exists $x \in J \cap B$. Notice that by A.1.10, $x$ is an invertible element, hence $1 = xx^{-1} \in J$, i.e., $J$ isn't proper. Therefore, it must be the case that $J = \overline{J}$.

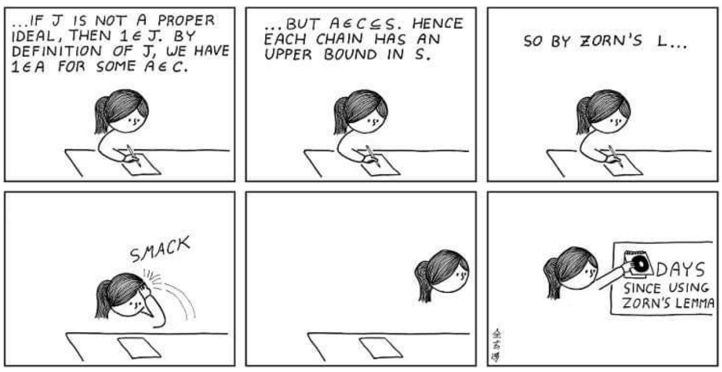


Figure A.1: You reading this proof

Back to business: Suppose $a \notin \mathrm{GL}(A)$. As strongly suggested by Figure A.1, we shall apply the Zorn Lemma to the set $\mathscr{C} := \{J \lhd A \mid aA \subseteq J \subsetneq A\}$ of proper ideals that contain the one generated by $a$, obtaining a maximal ideal $\mathcal{J} \lhd A$, closed by $(i)$.

> **Summoning A.2.10** (Zorn's Lemma)**.** If, in a non-empty and partially ordered set $\mathscr{C}$, every fully ordered subset has an upper quota, then $\mathscr{C}$ has a maximal element.

Given $x \in A\backslash\mathcal{J}$, notice that $\widetilde{\mathcal{J}} := \{bx + j \mid b \in A, j \in \mathcal{J}\}$ contains properlly $\mathcal{J}$. Since it is maximal, $\widetilde{\mathcal{J}} = A$ resulting in $1_A - xy \in \mathcal{J}$ for some $y \in A$. Hence, every non-zero element of $A/\mathcal{J}$ is invertible.

$$\begin{tikzcd} A \arrow[d] \arrow[dr, dashed, "\phi"] & \\ A/\mathcal{J} \arrow[r, "\simeq"'] & \mathbb{C} \end{tikzcd}$$

By Gelfand-Mazur theorem A.1.20, $A/\mathcal{J} \simeq \mathbb{C}$. Considering the canonical projection, let $\phi$ be the one who commutes the diagram at the right side. Necessarily, $\phi(1_A) = 1$ and $\phi(a) = 0$ since $a \in \mathcal{J}$. $\square$

**Corollary A.2.11.** For a unital Banach algebra $A$, the spectrum of each element $a \in A$ coincides with the image of Gelfand transformation (A.4), i.e., $\operatorname{Spec} a = \{\phi(a) \mid \phi \in \Gamma A\} = \operatorname{Im} \mathbf{ev}_a$.

**Theorem A.2.12** (Gelfand-Naimark representation theorem (1943))**.** For any commutative $C^*$-algebra $A$, Gelfand transformation (A.4) is an isometric isomorphism between involution algebras.

*Proof.* Whe shall break in cases since the presence of a unity in $A$ changes compactness local to total.

(*i*) **The unital case**: By lemma A.2.3, $\Gamma A$ is a compact space and $C_0(\Gamma A) = C(\Gamma A)$ is a $C^*$-algebra. By means of spectral theory, we shall prove that $\|\kappa(a)\|_\infty = \|a\|$ for all $a \in A$. For an arbitrary $\phi \in \Gamma A$,

$$\{\phi(x) \mid \phi \in \Gamma A\} \overset{A.2.11}{=} \operatorname{Spec} x \overset{A.1.23}{\subset} \mathbb{R} \qquad (x = x^* \in A)$$

whenever $x^* = x \in A$. Therefore $\phi(x) = \overline{\phi(x)}$ and hence,

$$\begin{aligned} \phi(a^*) &= \phi\Big(\frac{a^*+a}{2} - i\frac{a-a^*}{2i}\Big) \\ &= \phi\Big(\frac{a^*+a}{2}\Big) - i\phi\Big(\frac{a-a^*}{2i}\Big) \\ &= \overline{\phi\Big(\frac{a^*+a}{2}\Big)} - i\,\overline{\phi\Big(\frac{a-a^*}{2i}\Big)} \\ &= \overline{\phi\Big(\frac{a^*+a}{2} + i\frac{a-a^*}{2i}\Big)} = \overline{\phi(a)}. \end{aligned}$$

Since $\phi$ is arbitrary, $\kappa(a^*) = \mathbf{ev}_{a^*} = \overline{\mathbf{ev}_a} = \overline{\kappa(a)}$, thus the Gelfand transformation is an $*$-morphism. Therefore, one may notice that $\|\kappa(a)\|_\infty = \sup_{\phi\in\Gamma A} |\phi(a)| = \sup \operatorname{Spec} a = r(a)$ from A.2.11. Therefore, it follows that:

$$\|\kappa(a)\|_\infty^2 = \|\overline{\kappa(a)}\kappa(a)\|_\infty = \|\kappa(a^*a)\|_\infty = r(a^*a) = \|a^*a\| = \|a\|^2 \qquad (a \in A$$

i.e., $\kappa$ is an isometry, hence injective. Now it only rests to show that $\kappa$ is surjective.

> **Summoning A.2.13** (Complex Stone-Weierstraß- [18] - Theorem 36.B)**.** Let $X \in \mathbf{CHaus}$ and $A \subset C(X)$ a closed $*$-subalgebra containing all the constant functions. Therefore $A$ is dense if and only if separates points[a].
>
> [a]For any $x, y \in A$ distinct, there exists $g \in A$ such that $g(x) \neq g(y)$.

Since each isometry with Banach domain has closed image, $\operatorname{Im}\kappa$ is a closed dense $*$-subalgebra of $C(\Gamma A)$ containg all constant functions that separates poits, hence by Stone-Weierstraß A.2.13, $\operatorname{Im}\kappa = \overline{\operatorname{Im}\kappa} = C(\Gamma A)$, ensuring surjectivity.

$(ii)$ **The non-unital case**: The proof uses the fact that both One-point compactification and unitalization constructions are pairwised: $\widetilde{C_0(X)} \simeq C_0(X \sqcup \{\infty\})$ and $\Gamma\widetilde{A} \simeq \Gamma A \sqcup \{\infty\}$. The proof comes from guaranteeing that those functors are natural transformations. Since we allready proved the "compact-unital" version, the "local compact-non unital" follows.

$\square$

## A.3 Continuos Functional Calculus

**Lemma A.3.1.** Let $X \in \mathbf{CHaus}$ and $f \in C(X)$. If $f(x_0) = 0$ for some $x_0 \in X$, then for every $\varepsilon > 0$ there is $g \in C(X)$ such that $\|g\|_\infty = 1$ and $\|gf\|_\infty < \varepsilon$.

*Proof.* The set $V := \{x \in X \mid |f(x)| < \varepsilon\}$ is an open one containing $x_0$, because it is the pre-image of $|f(\cdot)|$ at the $\varepsilon$ ball centered at origin. Time to use some big guns from topology, in order to establish the existence of continuous extensions.

**Summoning A.3.2** (Urysohn's Lemma - [18], Theorem 28.A)**.** Let $X$ be a topological *normal space*[a], and let $A$ and $B$ be disjoint closed subspaces of $X$. Then there exists a continuous real function $f$ defined on $X$, all of whose values lie in the closed unit interval $[0, 1]$, such that $f(A) = 0$ and $f(B) = 1$.

[a] A $T_1$-space (all singletons $\{x\}$ are closed) in which each pair of disjoint closed sets can be separated by open sets, in the sense that they have disjoint neighborhoods.

As stated in [18], Theorem 27.A, every compact Hausdorff space is topologically normal, thus Urysohn's lemma A.3.2 apply: There exists $g : X \longrightarrow [0, 1]$ continuous such that $g(x_0) = 1$ and $g(x) = 0$ if $x \in V_{\neq x_0}$. So, $\|g\|_\infty = 1$ and $\|gf\|_\infty < \varepsilon$ □

For a unital $A \in \boldsymbol{C^*}\text{-}\mathbf{Alg}_{\boldsymbol{u}}$, suppose that an element $a \in A$ is *normal*, i.e., $a^*a = aa^*$. For each polynomial $p \in \mathbb{C}[z, \overline{z}]$, we extend it to $p \in \mathbb{C}[a, a^*]$ in the natural way. Since $a$ is normal, $\mathbb{C}[a, a^*]$ is a commutative $*$-subalgebra of $A$ containing $a$ and 1. Its closure is denoted by $C^*(1, a)$ and it is the smallest $C^*$-subalgebra of $A$ containing 1 and $a$.

**Proposition A.3.3.** Let $A$ be a $C^*$ algebra with unity and $a \in A$ normal.

$(i)$ If $b \in C^*(1, a)$ has an inverse $b^{-1} \in A$, then $b^{-1} \in C^*(1, a)$, i.e.,

$$\mathrm{GL}(A) \cap C^*(1, a) = \mathrm{GL}(C^*(1, a)).$$

$(ii)$ Let $B \subset A$ be a $C^*$-subalgebra containing the unity. The same deal applies:

$$\mathrm{GL}(A) \cap B = \mathrm{GL}(B). \qquad (1 \in B \subset A)$$

*Proof.*

$(i)$ Suppose $b \in \mathrm{GL}(A)$. Since it is a commutative unital $C^*$-algebra, $b \notin \mathrm{GL}(C^*(1, a))$ if and only if $\kappa(b)$ vanishes at some point in $\Gamma C^*(1, a)$ by Gel'fand theorem A.2.9. In the case which $b$ isn't invertible in the generated $C^*$-algebra, it is posssible to obtain $g \in C(\Gamma C^*(1, a))$ such that

$$\|g\|_\infty = 1 \quad \text{and} \quad \|g\kappa(b)\|_\infty < \|b^{-1}\|^{-1}$$

by lemma A.3.1. By the Gel'fand-Naimark representation A.2.12, $\kappa^{-1}(g) \in \Gamma C^*(1, a)$ obeys $\|\kappa^{-1}(g)\| = \|g\|_\infty = 1$ and

$$\|\kappa^{-1}(g)b\| = \|\kappa(\kappa^{-1}(g)b)\|_\infty = \|g\kappa(b)\|_\infty < \|b^{-1}\|^{-1}.$$

That inequality tells us that

$$1 = \|\kappa^{-1}(g)\| = \|b^{-1}(b\kappa^{-1}(g))\| \leqslant \|b^{-1}\|\|b\kappa^{-1}(g)\| < 1,$$

which is a contradiction. Therefore, $b \in \mathrm{GL}(C^*(1,a))$. $\square$

$(ii)$ Notice that $b^{-1} = (b^*b)^{-1}b^*$ for $b \in \mathrm{GL}(A)$. Since $b^*b$ is a self-adjoint element, hence normal, $(b^*b)^{-1} \in C^*(1, b^*b) \subset B$ by $(i)$. Therefore $b^{-1} \in B$. $\square$

**Corollary A.3.4** (Spectral Invariance)**.** Let $B \subset A$ be a $C^*$-subalgebra containg the unity of $A$. Therefore,

$$\mathrm{Spec}_B(b) = \mathrm{Spec}_A(b). \qquad (b \in B)$$

**Counterexample A.3.5.** Let $D \subset \mathbb{C}$ be the unitary open disk of complex numbers such that $|z| < 1$, such that $S^1 = \partial D$. The algebra of restrictions of holomorphic functions can be given by the set:

$$E := \{f \in C(S^1) \mid f = g\restriction_{S^1}, g \in C(\overline{D}), g\restriction_D \in \mathrm{Hol}(D)\}$$

Equipped with natural complex-conjugation $f \longmapsto \overline{f}$, $C(S^1)$ is a $C^*$-algebra, but $E$ isn't invariant by the induced involution. Therefore, the corollary A.3.4 doesn't apply.

**Proposition A.3.6.** The evaluation at a normal element $a \in A$ of a unital $C^*$-algebra

$$\begin{aligned} \mathbf{ev}_a : \ \Gamma C^*(1,a) &\longrightarrow \mathrm{Spec}\, a \\ \phi &\longmapsto \phi(a) \end{aligned}$$

is a homeomorphism.

*Proof.* In order to see that the image of the evaluation is in fact the spectrum, notice that $\mathrm{Spec}_{\Gamma C^*(1,a)}(a) = \{\mathbf{ev}_a(\phi) \mid \phi \in \Gamma C^*(1,a)\}$ by A.2.11 and $\mathrm{Spec}_{\Gamma C^*(1,a)}(a) = \mathrm{Spec}\, a$ by A.3.4. Therefore, the evaluation is well defined and it is surjective.

Suppose that for $\phi, \psi \in \Gamma C^*(1,a)$, one has that $\phi(a) = \psi(a)$. Notice that for every complex polynomial $p \in \mathbb{C}[z, \overline{z}]$, $\phi(p(a,a^*)) = \psi(p(a,a^*))$ since $\phi$ and $\psi$ are $*$-morphisms. The fact that $\mathbb{C}[a,a^*]$ is dense in $C^*(1,a)$ shows that necessarily, $\phi = \psi$, i.e., $\mathbf{ev}_a$ is injective.

The weak* topology is the smallest topology over $\Gamma C^*(1,a)$ such that each $p_b : \phi \longmapsto |\mathbf{ev}_b(\phi)|$ $(b \in C^*(1,a))$ be continuous. Therefore, for a converging net $(\phi_\alpha)_\alpha \subset \Gamma C^*(1,a)$, $\phi_\alpha \longrightarrow \phi$, one can see that $\mathbf{ev}_a$ is in fact continuous:

$$\begin{aligned} & \lim_\alpha |\mathbf{ev}_a(\phi_\alpha)| = \lim_\alpha p_a(\phi_\alpha) = p_a\Big(\lim_\alpha \phi_\alpha\Big) = p_a(\phi) = |\mathbf{ev}_a(\phi)| \\ \Leftrightarrow \quad & \lim_\alpha \mathbf{ev}_a(\phi_\alpha) = \mathbf{ev}_a(\phi). \end{aligned}$$

Notice that $\Gamma C^*(1,a) \in$ **CHaus** by lemma A.2.3 since the inner algebra is unital. In the other direction, $\operatorname{Spec} a$ is compact by A.1.14 and is Hausdorff because is a subset of the complex numbers $\mathbb{C}$. Since $\mathbf{ev}_a$ is a continuous bijection between compact Hausdorff spaces, **??** guarantee us that $\mathbf{ev}_a$ is an homeomorphism. □

**Theorem A.3.7** (The Continous Functional Calculus)**.** Let $a \in A$ be a normal element of a unital $C^*$-algebra. There exists a isometric $*$-morphism $\mathfrak{C}_a : C(\operatorname{Spec} a) \longrightarrow C^*(1,a)$ such that, for every $p \in \mathbb{C}[a,a^*]$, with $f(z) := p(z,\overline{z}) = \sum_{n,m} b_{n,m} z^n \overline{z}^m$, one does have

$$\mathfrak{C}_a(f) = \sum_{n,m} b_{n,m} a^n (a^*)^m = f(a). \tag{A.5}$$

*Proof.* You better like composition, because this is the one! The Gel'fand transform is given by $\kappa = \mathbf{ev}_{(\,\cdot\,)} : C^*(1,a) \longrightarrow C(\Gamma C^*(1,a))$ and it is an $*$-isometric isomorphism (A.2.12). By A.3.6, the composition function

$$\begin{aligned} \mathbf{ev}_a^* : \quad C(\operatorname{Spec} a) &\longrightarrow C(\Gamma C^*(1,a)) \\ f &\longmapsto f \circ \mathbf{ev}_a \end{aligned}$$

also become an $*$-isometric isomorphism. Therefore the composition $\mathfrak{C}_a := \kappa^{-1} \circ \mathbf{ev}_a^*$ holds the same title. In particular,

$$\mathfrak{C}_a(1_{C(\operatorname{Spec} a)}) = \kappa^{-1}(\mathbf{ev}_a^*(1_{C(\operatorname{Spec} a)})) = \kappa^{-1}(1_{C(\operatorname{Spec} a)}(\mathbf{ev}_a)).$$

Notice that $1_{C(\operatorname{Spec} a)} : \operatorname{Spec} a \longrightarrow \{1\}$. Therefore,

$$\begin{aligned} \kappa(\mathfrak{C}_a(1_{C(\operatorname{Spec} a)}))(\phi) &= 1_{C(\operatorname{Spec} a)}(\mathbf{ev}_a(\phi)) \\ &= 1 \qquad\qquad (\phi \in C(\Gamma C^*(1,a))) \\ &= \kappa(1_A)(\phi). \end{aligned}$$

Hence $\kappa(\mathfrak{C}_a(1_{C(\operatorname{Spec} a)})) = \kappa(1_A)$ which imply by injectivity that $\mathfrak{C}_a(1_{C(\operatorname{Spec} a)}) = 1_A$. One can also verify that the statement $I_{C(\operatorname{Spec} a)} \circ \mathbf{ev}_a = \kappa(a)$, implies that $\mathfrak{C}_a(I_{C(\operatorname{Spec} a)}) = a$. Thus, (A.5) holds. □

In summary, for a normal element $a \in A$, notice $f(a) := \kappa^{-1}(f(\mathbf{ev}_a))$ makes totally sense for $f \in C(\operatorname{Spec} a)$, extending $f : A \longrightarrow A$. Moreover, if $g \in C(f(\operatorname{Spec} a))$, then $g(f(a)) = (g \circ f)(a)$. Unfortunately, the continuous functional calculus does not work for non normal elements, since the generated $C^*$-algebra woudn't necessarily be commutative.

**Proposition A.3.8.** Let $A$ and $B$ be two unital $C^*$-algebras. If $\varphi : A \longrightarrow B$ is a unital $*$-morphism, and if $a \in A$ is normal, then:

$(i)$ $\operatorname{Spec} \varphi(a) \subseteq \operatorname{Spec} a$,

$(ii)$ if $f \in C(\operatorname{Spec} a)$ then $\varphi(f(a)) = f(\varphi(a))$.

*Proof.*

(*i*) Suppose that $\lambda \notin \operatorname{Spec} a$, i.e., $\lambda 1 - a \in \mathrm{GL}(A)$. Therefore $\lambda 1 - \varphi(a) = \varphi(\lambda 1 - a) \in \mathrm{GL}(B)$, hence $\lambda \notin \operatorname{Spec} \varphi(a)$.

(*ii*) Let $\mathscr{P} = \{\mathbf{ev}_{(\,\cdot\,)}(p) : \operatorname{Spec} a \longrightarrow \mathbb{C} \mid p \in \mathbb{C}[z, \overline{z}]\} \subset C(\operatorname{Spec} a)$. With the complex conjugation induced in those function, this is a unital $*$-subalgebra that separates points of $\operatorname{Spec} a$. By the Stone-Weierstraß (A.2.13), $\mathscr{P}$ is dense. Therefore, there exists $(p_n)_n \subset \mathscr{P}$ such that $f = \lim_n p_n$.

Since $(p_n)_n$ converges uniformly to $f$ on $\operatorname{Spec} \varphi(a) \subseteq \operatorname{Spec} a$, and by continuity of the functional calculus we conclude that:

$$\begin{aligned} \varphi(f(a)) &= \varphi\Big(\lim_{n\to\infty} p_n(a)\Big) \\ &= \lim_{n\to\infty} \varphi(p_n(a)) \\ &= \lim_{n\to\infty} p_n(\varphi(a)) = f(\varphi(a)). \quad \square \end{aligned}$$

## A.4 Positive elements

**Definition A.4.1.** A element $a$ of a $C^*$-algebra $A$ is said to be *positive* and it can be written that $a \geqslant 0$, whenever it is self-adjoint $a = a^*$ and its spectrum is positive: $\operatorname{Spec} a \subset [0, \infty)$. Hence, a order relation pops out, stating that $a \geqslant b$ if $a - b \geqslant 0$.

**Theorem A.4.2.** Let $a \in A$ be a positive element.

(*i*) (The Hahn decomposition) There exists unique positive elements $a_+, a_- \in A$ such that $a = a_+ - a_-$ and $a_+ a_- = 0$.

(*ii*) When $a$ and $-a$ are both positive, then $a = 0$.

(*iii*) For any $\lambda \geqslant \|a\|$, $a$ is positive if, and only if, $\|\lambda - a\| \leqslant \lambda$.

(*iv*) If both $a$ and $b$ are positive, so it is their sum $a + b$.

*Proof.*

(*i*) Let $B := C^*(1, a)$ be the generated unital $C^*$-algebra containing $a$, $a^*$ and 1. By Gel'fand-Naimark theorem A.2.12, $\kappa = \mathbf{ev}_{(\,\cdot\,)}$ is an isometric isomorphism between $B$ and $C(\Gamma B)$, where $\Gamma B$ is the set of non zero morphisms $\phi : B \longrightarrow \mathbb{C}$. Since $a$ is self-adjoint, the expansion

$$\phi(a) = \mathbf{ev}_a \phi = \kappa(a)\phi = \kappa(a^*)\phi = \phi(a^*) = \overline{\phi(a)} \qquad (\phi \in \Gamma B)$$

holds and it shows that $\kappa(a)$ must be a real continuous function. Therefore, let

$$a_+ := \kappa^{-1}(\max\{\kappa(a), 0\}) \quad \text{and} \quad a_- := \kappa^{-1}(\min\{-\kappa(a), 0\})$$

Those elements are positive and they obey the following: $a = a_+ - a_-$ and $a_+ a_- = a_- a_+ = 0$.

(*ii*) By the Spectral Mapping theorem A.1.18, $\operatorname{Spec}(-a) = -\operatorname{Spec} a$. Hence, both $a$ and $-a$ be positive means that $\operatorname{Spec} a = \{0\}$. Therefore, A.1.21 guarantee us that $\|a\| = r(a) = 0$, i.e., $a = 0$.

(*iii*) Let $\lambda \geqslant \|a\|$. Notice that by A.1.10 and A.1.23, $\operatorname{Spec} a \subset [-\|a\|, \|a\|] \subset [-\lambda, \lambda]$. Therefore,

$$\|\lambda - a\| = r(\lambda - a) = \sup \operatorname{Spec}(\lambda - a) = \sup_{\mu \in \operatorname{Spec} a} |\lambda - \mu|$$

Hence $\|\lambda - a\| \leqslant \lambda$ if and only if $\operatorname{Spec} a \subset [0, \infty)$.

(*iv*) By (*iii*), $\|\|x\| - x\| \leqslant \|x\|$ for $x \in \{a, b\}$. Therefore:

$$\|\|a\| + \|b\| - \|a + b\|\| \leqslant \|\|a\| - a\| + \|\|b\| - b\| \leqslant \|a\| + \|b\|$$

So (*iii*) again ensures that $a + b$ is positive.

□

A *square root* of an element $a \in A$ is a element $b \in A$ such that $b^2 = a$.

**Theorem A.4.3.** Each positive element $a \geqslant 0$ of a $C^*$-algebra $A$ has a unique positive square root.

*Proof.* Since positive elements are normal, we are good to go. Pick the usual square root $\sqrt{\cdot}$ defined on the interval $[0, \|a\|] \supset \operatorname{Spec} a$. With the continuous functional calculus A.3.7, notice that $\sqrt{a} := \mathfrak{C}_a(\sqrt{\cdot}) = \kappa^{-1}(\sqrt{\mathbf{ev}_a})$ is a well defined element of $A$, self-adjoint since and the square root $\sqrt{\cdot}$ is a real-valued function. Moreover, $\operatorname{Spec} \sqrt{a} = \sqrt{\operatorname{Spec} a} \subset [0, \infty)$, i.e., $\sqrt{a} \geqslant 0$.

The notation wasn't choose randomly: For $p(x) := x^2$, notice that $p \in C(\sqrt{\operatorname{Spec} a})$, hence $\sqrt{a}^2 = p(\sqrt{a}) = (p \circ \sqrt{\cdot})(a) = a$. If $b_1, b_2$ were two positives square roots of $a$, $b_1^2 = a = b_2^2$, one can see that $b_1 = \sqrt{b_1^2} = \sqrt{a} = \sqrt{b_2^2} = b_2$, concluding uniqueness. □

**Lemma A.4.4.** Let $A$ be a unital $C^*$-algebra and $a \in A$. The following are equivalent:

(*i*) $a$ is positive.

(*ii*) There is a self-adjoint element $b \in A$ such that $b^2 = a$.

(*iii*) There is $b \in A$ such that $b^*b = a$.

*Proof.* The implication $(i) \Rightarrow (ii)$ is essentialy the theorem A.4.3 and $(ii) \Rightarrow (iii)$ is trivial.

$(iii) \Rightarrow (i)$ If $a = b^*b$, let $a_+$ and $a_-$ be the Hahn decomposition such as in A.4.2($i$). Notice that

$$-(ba_-)^*(ba_-) = -a_-b^*ba_- = -a_-(a_+ - a_-)a_- = (a_-)^3$$

Since it is a positive element, $\operatorname{Spec} a^3 = (\operatorname{Spec} a)^3$ CONTINUAR $\square$

**Lemma A.4.5.** Whenever $0 \leqslant a \leqslant b$ are invertible elements in a unitary $C^*$-algebra $A$, then $b^{-1} \leqslant a^{-1}$.

*Proof.* Given two self-adjoint elements $x, y \in A$ such that $x \leqslant y$, notice that $z^*xz \leqslant z^*yz$ for all $z$. Indeed, since $x - y \geqslant 0$,

$$\begin{aligned} z^*yz - z^*xz &= z^*(x-y)z \\ &= z^*(\sqrt{x-y})^*(\sqrt{x-y})z \\ &= (\sqrt{x-y}\,z)^*(\sqrt{x-y}\,z) \overset{A.4.4}{\geqslant} 0 \qquad (z \in A) \\ \Rightarrow \quad z^*xz &\leqslant z^*yz \end{aligned}$$

Since $b - a \geqslant 0$, the above shows that

$$\begin{aligned} 0 &\leqslant \sqrt{b}^{-1}(b-a)\sqrt{b}^{-1} \\ &= \sqrt{b}^{-1}b\sqrt{b}^{-1} - \sqrt{b}^{-1}a\sqrt{b}^{-1} \\ &= 1 - \sqrt{b}^{-1}a\sqrt{b}^{-1} \end{aligned}$$

Thus, $(\sqrt{a}\sqrt{b}^{-1})^*(\sqrt{a}\sqrt{b}^{-1}) \leqslant 1$, implying that $\|\sqrt{a}\sqrt{b} - 1\| \leqslant 1$. Hence,

$$1 \geqslant (\sqrt{a}\sqrt{b}^{-1})(\sqrt{a}\sqrt{b}^{-1})^* = \sqrt{a}b^{-1}\sqrt{a}$$

Multiplying on both sides by $\sqrt{a}^{-1}$, we get

$$a^{-1} = \sqrt{a}^{-1}\,1\,\sqrt{a}^{-1} \geqslant \sqrt{a}^{-1}(\sqrt{a}\,b^{-1}\sqrt{a})\sqrt{a}^{-1} = b^{-1}.$$

$\square$

**Lemma A.4.6.** If $a \in A$ is a positive element of a unital $C^*$-algebra, then

$$\|a\| = \inf\{\lambda \geqslant 0 \mid a \leqslant \lambda \cdot 1\}.$$

*Proof.* Notice that $\|a\| \in \{\lambda \geqslant 0 \mid a \leqslant \lambda \cdot 1\}$, because the function $t \longmapsto t - \|a\|$ is positive: Since $\|a\| \leqslant \|a\|$, $\|a\|1 - a$ isn't invertible by A.1.10, hence $\|a\| \in \operatorname{Spec} a$. $\square$

**Corollary A.4.7.** Let $A$ be a $C^*$-algebra and $a, b \in A$. Therefore $0 \leqslant a \leqslant b \Rightarrow \|a\| \leqslant \|b\|$.

## A.5 Approximate Units

We can always trade unitary arguments with approximate units over Banach algebras, which is exactly what are we going to do. Traditionally in topology, nets can be much more useful to describe weird continuous functions spaces than sequences are, and, as long as Banach, $C^*$ and Von-Neumann algebras are blazingly wild, we need to appeal to these objects.

**Definition A.5.1** (Approximate Unit)**.** Let $(\mathbb{A}, \preccurlyeq)$ be an pre-ordered set[9]. The image of any function $\mathbb{A} \longrightarrow A$ will be said to be a *net*, and will be mentioned as $(u_\lambda)_{\lambda \in \mathbb{A}}$. An *approximate unit net* will allways denote a net $(u_\lambda)_{\lambda \in \mathbb{A}}$ such that $0 \leqslant \|u_\lambda\|_A \leqslant 1$ and

$$\lim_{\lambda \in \mathbb{A}} \|a - au_\lambda\| = 0 \quad \Big( = \lim_{\lambda \in \mathbb{A}} \|a - u_\lambda a\| \Big) \qquad (a \in A)$$

which means that: for every $\varepsilon > 0$, exists $\lambda_0 \in \mathbb{A}$ such that $\|a - au_\lambda\| < \varepsilon$ whenever $\lambda \succcurlyeq \lambda_0$. Therefore, we can stabilish that $\lim_\lambda au_\lambda = \lim_\lambda u_\lambda a = a$ for every element $a$.

Notice that for any complex $z$, $\delta \in [0, 1)$ can allways be chosen such that $|z - z\delta|$ is desirily small (with respect to the ordinarily euclidean norm), i.e., the non-negative numbers with norm less than one $[0, 1) = (\delta)_{\delta \in [0,1)}$ is a approximate unit over the complex numbers. We'll show that those in fact exists in each and every $C^*$-algebra.

**Theorem A.5.2.** The positive elements of any $C^*$-algebra $A$ with norm less than one are a approximate unit.

*Proof.* Let $\mathbb{A} := \{a \in A \mid a \geqslant 0 \text{ and } \|a\| < 1\}$. To show that $\mathbb{A}$ is directed set, we invoque functional calculus by our side:

(*i*) $\mathbb{A}$ **is a pre-ordered set**: Consider the bijection:

$$\begin{aligned} g : \qquad [0, 1) &\longrightarrow [0, \infty) \\ t &\longmapsto \frac{t}{1 - t} \\ 1 - \frac{1}{1 + t} &\longleftarrow t \end{aligned}$$

Since $g$ and $g^{-1}$ map 0 to 0, they will send $A$ to $A$ under the functional calculus, even if $A$ is non-unital. Use the order $\geqslant$ in $\mathbb{A}$ induced by the positive elements in $A$. Now choose $a, a' \in \mathbb{A}$ and define:

$$b := g^{-1}(g(a) + g(a')) = 1 - (1 + g(a) + g(a'))^{-1}.$$

[9] For $\lambda, \lambda' \in \mathbb{A}$, there exists a $\mu$ which both $\lambda \preccurlyeq \mu$ and $\lambda' \preccurlyeq \mu$. Equivalently, any finite subset has an upper bound.

Then $\operatorname{Spec} b \subset [0,1)$ so $b \in \mathbb{A}$. Also, since $1 + g(a) + g(a') \geqslant 1 + g(a)$, lemma A.4.5 implies that

$$b = 1 - (1 + g(a) + g(a'))^{-1} \geqslant 1 - (1 - g(a))^{-1} = g^{-1}(g(a)) = a.$$

Likewise $b \geqslant a'$.

*(ii)* CONTINUAR $\square$

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