Categories for the working mathematician

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1. Categories, functors and natural transformations

1.3. Functors

Exercise 1

Show how each of the following constructions can be regarded as a functor:

- *the field of quotients of an integral domain.*
- *the Lie algebra of a Lie group.*

Field of quotients Given two integral domains and a ring homomorphism $f : R \to S$, we define an homomorphism between its fields of quotients as:

$$\widetilde{f}\left(\frac{a}{b}\right) = \frac{f(a)}{f(b)}$$

We can prove it is well-defined using that $ab = cd$ implies $f(a)f(b) = f(c)f(d)$, and then:

$$\widetilde{f}\left(\frac{a}{b}\right) = \frac{f(a)}{f(b)} = \frac{f(c)}{f(d)} = \widetilde{f}\left(\frac{c}{d}\right)$$

And it respects sums and products:

$$\widetilde{f}\left(\frac{a}{b} + \frac{c}{d}\right) = \widetilde{f}\left(\frac{ad+cb}{bd}\right) = \frac{f(a)f(d) + f(c)f(d)}{f(b)f(d)} = \frac{f(a)}{f(b)} + \frac{f(c)}{f(d)}$$

$$\widetilde{f}\left(\frac{ac}{bd}\right) = \frac{f(a)f(c)}{f(b)f(d)}$$

So it is a field homomorphism.

TODO Lie algebra Given $\phi : G \to H$, a Lie group homomorphism, we can compute its first derivative at the identity ϕ^*.

Exercise 2

Show that functors $1 \to C$, $2 \to C$, and $3 \to C$ correspond respectively to objects, arrows, and composable pairs of arrows in C.

A functor $F\colon 1 \to C$ is determined by $F1$. A functor $F\colon 2 \to C$ is determined by $F(1 \leq 2)\colon F1 \to F2$. A functor $F\colon 3 \to C$ is determined by $F(1 \leq 2)$ and $F(2 \leq 3)$, which must be composable in $F2$.

Exercise 3

Interpret "functor" in the following special types of categories:

1. *A functor between two preorders is a function T which is monotonic (i.e. $p \leq p'$ implies $Tp \leq Tp'$).*
2. *A functor between two groups (one-object categories) is a morphism of groups.*
3. *If G is a group, a functor $G \to \mathtt{Set}$ is a permutation representation of G, while $G \to Matr_K$ is a matrix representation of G.*

First statement A functor must be a monotonic function, as it has to send $(p \leq p')$ into a morphism between Tp and Tp'. This morphism exists if and only if $Tp \leq Tp'$.

Second statement It respects the identity and the group operation, as functors respect the identity and the composition.

Third statement A functor $F\colon G \to \mathtt{Set}$ is determined by FG and the assignment of every element of G to a set automorphism, that is, an element of the permutation group of the set. The functor $F\colon G \to \mathtt{Matr}_K$ selects a dimension n, and sends every element of the group to an invertible matrix $M_{n\times n}$.

Exercise 4

Prove that there is no functor $\mathtt{Grp} \to \mathtt{Ab}$ sending each group G to its center. (Consider $S_2 \to S_3 \to S_2$, the symmetric groups).

A functor must preserve identities and composition. We have the following diagram in $\mathtt{Grp}$,

$$\begin{tikzcd} S_2 \arrow[r, hook] \arrow[rr, bend left, "id"] & S_3 \arrow[r] & S_2 \end{tikzcd}$$

that cannot be translated into $\mathtt{Ab}$ by this functor

$$S_2 \longrightarrow \{id\} \longrightarrow S_2$$

as we know that the identity is not the zero morphism.

Exercise 5

Find two different functors $T : \texttt{Grp} \to \texttt{Grp}$ with object function $T(G) = G$ the identity for every group G.

The identity functor and a functor sending every morphism to the zero morphism.

1.4. Natural transformations

Exercise 1

Let S be a fixed set, and X^S the set of all functions $h : S \longrightarrow X$. Show that $X \mapsto X^S$ is the object function of a functor $\texttt{Set} \longrightarrow \texttt{Set}$, and that evaluation $e_X : X^S \times S \longrightarrow X$ defined by $e(h, s) = h(s)$, the value of the function h at $s \in S$ is a natural transformation.

We define the functor $\_^S$ on arrows as follows. Given a $f : X \to Y$ and a $g : S \longrightarrow X$:

$$f^S(g) = f \circ g$$

And it follows the functor laws.
We can see that evaluation is a natural transformation with the naturality square:

$$\begin{array}{ccc} X^S \times S & \xrightarrow{f^S, id} & Y^S \times S \\ \downarrow{\scriptstyle e_X} & & \downarrow{\scriptstyle e_Y} \\ X & \xrightarrow{\quad f \quad} & Y \end{array}$$

Which commutes on its elements:

$$\begin{array}{ccc} (f, s) & \xrightarrow{g^S, id} & (g \circ f, s) \\ \downarrow{\scriptstyle e_X} & & \downarrow{\scriptstyle e_Y} \\ f(s) & \xrightarrow{\quad f \quad} & g(f(s)) \end{array}$$

Exercise 2

If H is a fixed group, show that $G \mapsto H \times G$ defines a functor $H \times -\colon \texttt{Grp} \to \texttt{Grp}$ and that each morphism $f\colon H \to K$ of groups defines a natural transformation $H \times - \stackrel{\cdot}{\to} K \times -$.

The functor will send a morphism $f\colon G \to G'$ to $\mathrm{id} \times f\colon H \times G \to H \times G'$. The naturality condition is satisfied if the following diagram commutes

$$\begin{array}{ccc} H \times G & \xrightarrow{g \times \mathrm{id}} & K \times G \\ \downarrow{\scriptstyle \mathrm{id} \times f} & & \downarrow{\scriptstyle \mathrm{id} \times f} \\ H \times G' & \xrightarrow{g \times \mathrm{id}} & K \times G' \end{array}$$

but it is trivial to check commutativity.

Exercise 3

If B and C are groups (regarded as categories with one object each) and $S, T\colon B \to C$ are functors (homomorphisms of groups), show that there is a natural transformation $S \dot{\to} T$ if and only if S and T are conjugate; i.e. if and only if there is an element $h \in C$ with $Tg = h(Sg)h^{-1}$ for all $g \in B$.

If the only object in B is b, then $Sb = Tb = c$ must be the only object in c. Naturality gives us

$$\begin{tikzcd} c \arrow[r, "\varphi"] \arrow[d, "Sf"'] & c \arrow[d, "Tf"] \\ c \arrow[r, "\varphi"] & c \end{tikzcd}$$

so we know that $Sf \circ \varphi = \varphi \circ Tf$, and this is the conjugate condition.

Exercise 4

For functors $S, T\colon C \to P$ where C is a category and P a preorder, show that there is a natural transformation $S \dot{\to} T$ (which is then unique) if and only if $Sc \leq Tc$ for every object $c \in C$.

To define a natural transformation, we must have a familiy of morphisms

$$Sc \to Tc \quad \forall c \in C,$$

but this morphism exists if and only if $Sc \leq Tc$ for every object. If the morphisms exist, the naturality condition is trivial, as there will be an unique morphism between two objects and all squares will commute.

Exercise 5

Show that every natural transformation $\tau\colon S \dot{\to} T$ defines a function (also called τ) which sends each arrow $f\colon c \to c'$ of C to an arrow $\tau f\colon Sc \to Tc'$ of B in such a way that $Tg \circ \tau f = \tau(gf) = \tau g \circ Sf$ for each composable pair (g, f). Conversely, show that every such function τ comes from a unique natural transformation with $\tau_c = \tau(1_c)$. (This gives an arrows only description of a natural transformation.)

Given $f\colon c \to c'$, we apply the naturality condition and take τ to be the diagonal

$$\begin{tikzcd} Sc \arrow[r, "\tau"] \arrow[d, "Sf"'] \arrow[dr, "\tau f"] & Tc \arrow[d, "Tf"] \\ Sc' \arrow[r, "\tau"] & Tc' \end{tikzcd}$$

i.e. we have defined $\tau f = Tf \circ \tau_c = \tau_{c'} \circ Sf$. The condition holds now trivially, as we know that

$$Tg \circ \tau f = Tg \circ Tf \circ \tau_c = \tau(gf) = \tau_{c'} \circ Sg \circ Sf = \tau g \circ Sf.$$

TODO Exercise 6

Let F be a field. Show that te category of all finite-dimensional vector spaces over F (with morphisms all lineal transformations) is equivalent to the category `Matr`.

1.5. Monics, epis and zeros

Exercise 1

Find a category with an arrow which is both epi and monic, but not invertible (e.g., dense subset of a topological space).

In the `Top` category of topological spaces with continuous functions, we can include a dense subset in its base space. This inclusion will be a monomorphism (as it is injective) and an epimorphism as we know that, if $i\colon U \subset V$ is our inclusion,

$$f \circ i = i \circ g \implies f|_U = g|_U,$$

and because it is a dense subset, by continuity, $f = g$.
But it has not to be an isomorphism. In fact, it won't be if U is a proper subset.

Exercise 2

Prove that the composite of monics is monic, and likewise for epis.

If f, g are monics, we can apply the definition twice to get

$$f \circ g \circ a = f \circ g \circ b \implies g \circ a = g \circ b \implies a = b.$$

The same proof can be applied in reverse.

Exercise 3

If a composite $g \circ f$ is monic, so is f. Is this true of g?

No, f could be a zero morphism and g could still give $g \circ h = g \circ h'$ for two $h \neq h'$.

Exercise 4

Show that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is epi in the category `Rng`.

If $f \circ i = g \circ i$, then $f|_{\mathbb{Z}} = g|_{\mathbb{Z}}$, and we can extend the morphisms uniquely to the ring $\mathbb{Q}$, as the ring morphisms have to preserve inverses.

TODO Exercise 5

In `Grp` *prove that every epi is surjective (Hint. If* $\varphi\colon G \to H$ *has image* M *not* H*, use the factor group* H/M *if* M *has index 2. Otherwise, let* $\mathrm{Perm}\ H$ *be the group of all permutations of the set* H*, choose three different cosets* M, Mu *and* Mv *of* M*, define* $\sigma \in \mathrm{Perm}\ H$ *by* $\sigma(xu) = xv$*,* $\sigma(xv) = xu$ *for* $x \in M$*, and* σ *otherwise the identity. Let* $\psi\colon H \to \mathrm{Perm}\ H$ *send each* h *to left multiplication* ψ_h *by* h*, while* $\psi'_h = \sigma^{-1}\psi_h\sigma$*. Then* $\psi\varphi = \psi'\varphi$*, but* $\psi \neq \psi'$*).*

TODO Exercise 6

In `Set`*, show that all idempotents split.*

Given $f\colon A \to A$ idempotent, we can define the set $\mathrm{img}\ f$, and two functions $g\colon A \to \mathrm{img}\ f$, $h\colon \mathrm{img}\ f \to A$ defined naturally and satisfiying the conditions. Notice that g is an epimorphism and h a monomorphism.

2. Constructions on categories

2.6. Comma categories

Exercise 1

If K *is a commutative ring, show that the comma category* $(K \downarrow \mathtt{CRng})$ *is the usual category of all small commutative* K*-algebras.*

A K-algebra can be defined as an inclusion from K on a ring, morphisms of algebras must preserve this inclusion.

Exercise 2

If t *is a terminal object in* C*, prove that* $(C \downarrow t)$ *is isomorphic to* C*.*

By definition of terminal object, there will be only an arrow $*\colon u \to t$ for any $u \in C$. Every morphism will create a commutative diagram because of the unicity of the morphisms.