Koszul Pairs

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1 Introduction

Koszul algebras have numerous applications in diverse fields of Mathematics such as Algebraic Topology, Combinatorics, Representation Theory or Algebraic Geometry, as it is showed in [1], and many of its fundamental properties still hold in **Koszul rings**, a particular case of graded rings.

Almost-Koszul pairs are a tool for the study of Koszul rings; to every strongly graded ring corresponds a canonical almost-Koszul pair. Every almost-Koszul pair has three associate chain complexes and three cochain complexes. If any of this six complexes is exact, all the others are exact too; in this case, we call the pair a **Koszul pair**.

2 Area of interest

In order to define Koszul pairs, we need to introduce some homological algebra prerequsites. We will define abelian categories in general, even if we are going to use later only the particular case of module categories, where we will define projective, injective and flat modules.

In particular, we will need to define **Hoschschild comohology**.

2.1 Abelian categories

The theory of abelian categories was introduced by Buchsbaum and Grothendieck in [2] to unify the multiple cohomology theories at the time.

Additive category

The original motivation for additive categories is the category of abelian groups, and, more generally, the category of momdules over a fixed ring R. In these categories, morphisms between two objects form an abelian group; and we can define functors preserving this group structure.

Definition 1. $\mathcal{C}$ is an **additive category** if:

- $\mathrm{Hom}(A, B)$ is an *abelian group*.
- *Distributivity* holds: $b \circ (f + g) = b \circ f + b \circ g$ and $(f + g) \circ a = f \circ a + g \circ a$.
- Has a *zero object*.
- Has finite *products* and *coproducts*.

A functor T between two additive categories is and **additive functor** if $T(f + g) = Tf + Tg$. [3]

Abelian category

Our interest is specifically on abelian categories. We will need to assume that every morphism has a kernel and a cokernel in order to prove results on homological algebra.

Definition 2. An **abelian category** is an *additive category* such that

- every morphism has a kernel and cokernel.
- every monomorphism is a kernel.
- every epimorphism is a cokernel.

The category of R-modules is an abelian category, but also the category of chain complexes of an arbitrary abelian category, $\mathtt{Ch}(\mathcal{A})$, is an abelian category.

2.2 Chain complexes and homology

Homology was originally defined in algebraic topology as a rigorous method allowing the topological distinction of manifolds with arbitrary dimensional holes. The same construction can be translated into multiple different homology theories.

In abelian categories, the homology provides a formal description of the failure of a functor to be exact.

Chain complexes

Chain complexes were initially a representation the relationships between cycles and boundaries on a topological space; we will study chain complexes in the abstract setting of module categories, devoided of any explicit relation to its motivating example.

Definition 3. A **chain complex** is a family of R-modules $\{C_n\}$ and homomorphisms $d_n\colon C_n \to C_{n-1}$ called *differentials*, such that each composite of consecutive differentials is zero, i.e. $d_{n-1} \circ d_n = 0$.

Theorem 1. *Given an abelian category $\mathcal{A}$, the category $\mathtt{Ch}(\mathcal{A})$ is an abelian category. [4]*

Exact sequences

Exact sequences provide a convenient framework for homological questions such as the completion of the middle term of a particular sequence in such a way that the homology groups are exactly zero. This kind of problems, in particular in the category of groups, have been proven to be useful to the resolution of problems such as the classification of finite simple groups.

Definition 4. A pair of composable morphisms is **exact** in the object where they can be composed when $\mathrm{img}(f) = \ker(g)$. Equivalently, when $\mathrm{coker}(f) = \mathrm{coimg}(g)$.

Definition 5. A **short exact sequence** is a diagram

$$0 \longrightarrow a \xrightarrow{f} b \xrightarrow{g} c \longrightarrow 0$$

exact on a,b and c.

Definition 6. A **morphism of short exact sequences** is defined by three morphisms f, g, h making the following diagram commute

$$\begin{tikzcd}
0 \arrow[r] & \cdot \arrow[r, "m"] \arrow[d, "f"] & \cdot \arrow[r, "e"] \arrow[d, "g"] & \cdot \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & \cdot \arrow[r, "m'"] & \cdot \arrow[r, "e'"] & \cdot \arrow[r] & 0
\end{tikzcd} .$$

The short exact sequences of an abelian category A define a category with these morphisms called $\mathtt{Ses}(A)$, which is preadditive with the component by component sum.

Snake lemma

The *snake lemma* will provide us with a tool to construct long exact sequences, which will be used in the definition of derived functors. This is a first result on algebraic homology theory.

Theorem 2. *Given a morphism of short exact sequences f, g, h; there exists a morphism $\delta\colon \ker h \to \mathrm{coker}\, f$ such that the following sequence is exact*

$$\begin{tikzcd}
0 \arrow[r] & \ker(f) \arrow[r, "m"] & \ker(g) \arrow[r, "e"] & \ker(h) \arrow[dll, "\delta"] & \\
& \mathrm{coker}(f) \arrow[r, "m'"] & \mathrm{coker}(g) \arrow[r, "e'"] & \mathrm{coker}(h) \arrow[r] & 0
\end{tikzcd}$$

Proof. In this extended diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & ker(f) \arrow[r] \arrow[d] & ker(g) \arrow[r] \arrow[d] & ker(h) \arrow[d] \arrow[dddll, "\delta"] & \\
0 \arrow[r] & a \arrow[r, "m"] \arrow[d, "f"] & b \arrow[r, "e"] \arrow[d, "g"] & c \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & a' \arrow[r, "m'"] \arrow[d] & b' \arrow[r, "e'"] \arrow[d] & c' \arrow[r] \arrow[d] & 0 \\
 & coker(f) \arrow[r] \arrow[d] & coker(g) \arrow[r] \arrow[d] & coker(h) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

we can first define the morphism δ using the properties of abelian categories to prove that it exists and then prove that it is exact using again the properties of monomorphisms and epimorphisms. □

Homology

The definition of *homology* tries to capture the failure of the complex to be exact as the quotient of the kernel and the image of the sucessive differential maps.

Definition 7. Given a chain complex $\{C_n\}$ with differentials d_n, we define the nth homology group as

$$H_n(C) \cong \ker(d_n)/\mathrm{im}(d_{n+1}).$$

It is important to notice that a sequence will be exact if and only if all their homology groups are zero.

2.3 Projective, injective and flat resolutions

Resolutions of injective modules are needed to define Hochschild homology.

Definitions

Definition 8. An R-module D is:

1. **Projective** if $\mathrm{Hom}(D, -)$ is exact.
2. **Injective** if $\mathrm{Hom}(-, D)$ is exact.
3. **Flat** if $D \otimes -$ is exact.

We know that $\mathrm{Hom}(D, -)$ and $\mathrm{Hom}(-, D)$ are left-exact and that $D \otimes -$ is right-exact; so for them to be exact, we only need:

- A module D is **projective** when $B \longrightarrow C$ epimorphism induces $\mathrm{Hom}(D, B) \longrightarrow \mathrm{Hom}(D, C)$ epimorphism.

$$\begin{tikzcd}
& B \arrow[d, two heads] \\
D \arrow[r] \arrow[ur, dashed, "\exists"] & C
\end{tikzcd}$$

- A module D is **injective** when $A \longrightarrow B$ epimorphism induces $\operatorname{Hom}(B, D) \longrightarrow \operatorname{Hom}(A, D)$ epimorphism.

$$\begin{tikzcd}
& A \arrow[d, two heads] \arrow[dl] \\
D & B \arrow[l, dashed, "\exists"]
\end{tikzcd}$$

- A module D is **flat** when $A \longrightarrow B$ monomorphism induces $D \otimes A \longrightarrow D \otimes B$ monomorphism.

Resolutions

Definition 9. A **projective resolution** is a resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where every P_i is projective.

Definition 10. An **injective resolution** is a resolution

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow E^2 \longrightarrow \dots$$

where every E^i is injective.

Definition 11. A **flat resolution** is a resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_i is flat.

Explicit construction Notice that, given a module M, we can always find a surjection from a proyective module (if we have *enough projectives*). So we can construct a projective resolution as

$$\begin{tikzcd}
& \ker f_2 \arrow[dr] & & & & \ker \pi \arrow[dr] & & & \\
\dots & & P_2 \arrow[dr, two heads, "f_2"] & & P_1 \arrow[ur, two heads, "f_1"] & & P_0 \arrow[r, two heads] & M \arrow[r] & 0 \\
& & & \ker f_1 \arrow[ur] & & & & &
\end{tikzcd}$$

We can also reverse the arrows to obtain an injective resolution.

Derived functors

Derived functors provide a canonical way to extend the image of an exact sequence by a non two-sided exact functor. We need an abelian category A with enough injectives to construct left-derived functors; and we need enough projectives to construct right-derived functors.

Construction of the right derived functor Let F be additive, covariant and left-exact. Let $0 \longrightarrow M \longrightarrow E^\bullet$ be an injective resolution with M deleted; then $F(E^\bullet)$ is a complex, and we define:

$$R^iF(M) = H^i(F(E^\bullet)) = \frac{\ker\{F(E^i) \longrightarrow F(E^{i+1})\}}{\mathrm{Im}\{F(E^{i-1}) \longrightarrow F(E^i)\}}$$

That is, if we take the *injective resolution*

$$0 \longrightarrow M \longrightarrow E^0 \longrightarrow E^1 \longrightarrow \dots$$

we can delete M and apply F to get a (non neccesarily exact) complex where we can compute the homology

$$0 \longrightarrow F(E^0) \longrightarrow F(E^1) \longrightarrow F(E^2) \longrightarrow \dots.$$

Construction of the left derived functor Let F be additive, contravariant and left-exact. Let $P^\bullet \longrightarrow M \longrightarrow 0$ be a projective resolution with M deleted; then $F(P^\bullet)$ is a complex, and we define

$$R^iF(M) = H^i(F(P^\bullet)) = \frac{\ker\{F(P_i) \longrightarrow F(P_{i+1})\}}{\mathrm{Im}\{F(P_{i-1}) \longrightarrow F(P_i)\}}$$

That is, if we take the *projective resolution*

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Delete M and apply F to get a (non neccesarily exact) complex where we can compute the homology:

$$0 \longrightarrow F(P_0) \longrightarrow F(P_1) \longrightarrow F(P_2) \longrightarrow \dots$$

2.4 Hochschild homology

Preliminaries

Our main interest will be on a particular kind of homology and comology called *Hochschild homology*. This section defines the preliminary necessary concepts to develop the notion of Hochschild homology.

Opposite algebra. The definition of an *opposite algebra* is a trvial notion which will be crucial to create the definition of a standard resolution of an algebra over a field.

Definition 12. If A is an R-algebra, the **opposite algebra** of A, with a multiplication given by yuxtaposition is A^*; an algebra with the same set of elements and where the multiplication $\circ$ is defined as $x \circ y = yx$.

Enveloping algebra.

Definition 13. Let R be a k-algebra, the **enveloping algebra** of R is the tensor product $R^e = R \otimes R^{op}$, where the product is defined as

$$(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_2 s_1).$$

Theorem 3. *Given any k-algebra, R, the following categories are isomorphic:*

- $(R; R)$*-Mod*
- R^e*-Mod*
- *Mod-*R^e

Proof. If M is an $(R; R)$-module, we can provide it with R^e-module structure by defining $(r \otimes s)m = rms$. It is trivial to check that this structure is compatible with our previously defined product, as

$$(a \otimes b)(c \otimes d)m = acmdb = (ac \otimes bd)m.$$

If M is an R^e-module, we can provide it with $R; R$-module structure taking $rms = (r \otimes s)m$. Compatibility relation can be checked by the same reasoning. □

Standard resolution.

Definition 14. Given R, a k-algebra we define the **standard resolution** $(P_\bullet, d_\bullet)$ of R in $(R; R)-$Mod as

- $P_n = R \otimes (R^{\otimes n}) \otimes R$
- $d_n = \sum_{i=0}^{n} (-1)^i d_n^i$

where

$$d_n^i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

Hochschild homology

Definition 15. Given R, a K-algebra, and M, an $(R; R)$-module, we define:

- The **Hochschild cohomology** of R in M as $HH^\bullet(R, M) = \mathrm{Ext}^\bullet_{R^e}(R, M)$.
- The **Hochschild homology** of R in M as $HH_\bullet(R, M) = \mathrm{Tor}^{R^e}_\bullet(R, M)$.

In order to compute the cohomology, we can take the following cochain complex

$$\mathrm{Hom}_K(K, M) \xrightarrow{b^0} \mathrm{Hom}_K(R, M) \xrightarrow{b^1} \mathrm{Hom}_K(R^{\otimes 2}, M) \xrightarrow{b^2} \cdots.$$

where the b^n are defined as

- $b^0(m)(a) = am - ma$
- $b^n = \sum_{i=0}^{n+1} (-1)^i b_i^n$

and the auxiliary morphisms b_i^n are defined as

$$b_i^n(f)(a_1 \otimes \cdots \otimes a_{n+1}) = \begin{cases} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) & \text{if } i = 0 \\ f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} & \text{if } i = 1, \ldots, n \\ f(a_1 \otimes \cdots \otimes a_n) a_{n+1} & \text{if } i = n+1 \end{cases}.$$

3 Methods

Our methodology is based on the review of the basic bibliography of the subject. This article is an attempt to collect all the needed prerequisites to work with homology and cohomology theory and ultimately to work with Hochschild homology and Koszul pairs.
Apart from the bibliographic review, we have developed materials for future students interested in the subject.

Wikipedia articles

In order to achieve a higher level of understanding of the topic and to provide future students with accesible resources, we have written the following articles for the Spanish wikipedia; they are published using a Creative Commons license.

- Compleción (álgebra)
- Lema de escisión
- Lema de la serpiente
- Funtor Tor
- Homología de Hochschild
- Categoría coma

Student-organized seminars

Two student-organized seminars have been held in the university. In those seminars, the author has lectured about category theory; giving a basic introduction to mathematics and computer science students.

4 Discussion

4.1 Koszul algebra

Koszul rings will be a generalization of *Koszul algebras*. A pair of a ring and its categorical dual, a coring; toghether with certain coherence relations, will constitute our definition of a Koszul pair.

Definition 16. A graded algebra A is a **Koszul algebra** over a field k if every graded module has a graded projective resolution $P_\bullet$ where the projective module P_j is generated by homogeneous elements of degree j.

Quadratic algebras

Koszul algebras are a particular case of quadratic algebras. We can describe them in full generality with the following definition.

Definition 17. A graded algebra A is a **quadratic algebra** if the natural application from its tensor algebra $T(A) \to A$ is surjective and its kernel J_A is generated from $J_A \cap (A^1 \otimes A^1)$. [1]

In other words, a graded quadratic algebra is determined as the quotient of a vector space A_1 by a subspace of homogeneous quadratic relations $S \subset V \otimes V$ as

$$A = T(V)/\langle S\rangle .$$

Theorem 4. *Every Koszul R-ring is a quadratic algebra.*

4.2 Almost-koszul pairs

Previous to the definition of Koszul pairs, we are going to define *almost-koszul pairs*. Those will provide us with a weaker set of requirements and we will be able to obtain a definition of Koszul pairs suitable to every almost-koszul pair.

Graded rings

Definition 18. A **graded ring** is a ring that can be written as a direct sum of abelian groups

$$A = \bigoplus_{n\in\mathbb{N}} A_n$$

such that $A_i A_j \subset A_{i+j}$.

A **homogeneous element** is an element of any submodule A_i of the decomposition.

Koszul rings

Definition 19. A graded ring A is a **Koszul ring** if A^0 is a semisimple ring and it has a resolution P_* by projective graded left A-modules such that each P_n is generated by homogeneous elements of degree n. [5]

R-rings

Definition 20. An R-ring is an associative and unital algebra. It is an associative and unital ring A together with a morphism $u : R \longrightarrow A$.

A R-ring is **graded** if it is equipped with a decomposition

$$A = \bigoplus_{n\in\mathbb{N}} A^n$$

such that multiplicaton $m^{p,q}$ maps $A^p \otimes A^q$ into A^{p+q}. It is **connected** when $A_0 = R$. It is **strongly graded** when $m^{1,p}$ is surjective. We call π^n_A to the projection of A onto A^n.

R-coring

Corings will be the categorical dual of rings. In order to define them, we proceed by giving defitions of coalgebra and certain properties of this kind of algebras. Those are also the dual notions to the preliminary definitions we described at the start of this chapter.

Definition 21. A **coalgebra** over a field K is a **vector space** V together with linear maps $\Delta : V \longrightarrow V \otimes V$ and $\varepsilon : V \longrightarrow K$ such that:

1. $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$
2. $(id \otimes \varepsilon) \circ \Delta = id = (\varepsilon \otimes id) \circ \Delta$

When writting in coalgebras, we will follow the **Sweedler notation**. [6]

Definition 22. An **R**-coring is a *coassociative* and *counital coalgebra*. It is an R-bimodule with a *comultiplication* $\Delta : C \longrightarrow C \otimes C$ and a *counit* $\epsilon : C \longrightarrow R$.

A R-coring is **graded** if it is equipped with a decomposition $C = \bigoplus_{n\in\mathbb{N}} C_n$, such that

$$\Delta(C_n) \subset \bigoplus_{p=0}^{n} C_p \otimes C_{n-p}.$$

Almost-koszul pair

Definition 23. An **almost-Koszul pair** is a connected R-ring and R-coring (A, C) with an isomorphism $\theta_{C,A} : C_1 \longrightarrow A^1$ that satisfies the relation

$$m^{1,1} \circ (\theta_{C,A} \otimes \theta_{C,A}) \circ \Delta_{1,1} = 0.$$

Using Sweedler notation we can rewrite the condition as follows

$$\sum \theta_{C,A}(c_{(1,1)})\theta_{C,A}(c_{(2,1)}) = 0,$$

for any $c \in C_2$.

4.3 Almost-koszul pair complexes

Six complexes will be associated to any given Koszul pair. Its exactness will be related, i.e., if any one of them is exact, the six complexes will be exact. This provides a definition of Koszul pairs relying only on this kind of complexes.

Definition 24. Let (A, C) be an almost-Koszul pair, if we define

$$K_l^{-1}(A, C) = R \quad \text{and} \quad K_l^n(A, C) = C \otimes A^n,$$

and the differential maps

$$d_l^n(c \otimes a) = \sum c_{(1,p-1)} \otimes \theta_{C,A}(c_{(2,1)})a,$$

with the exceptional case $n = -1$, where we take d_l^n to be the canonical bimodule morphisms $R \to C \otimes A^0$, mapping $1 \mapsto 1 \otimes 1 \in C_0 \otimes A^0$. We will also define the same notion on the opposite pair as

$$K_r^*(A, C) = K_l^*(A^{op}, C^{op}).$$

Definition 25. Let (A, C) be an almost-Koszul pair, we define

$$K^{-1}(A, C) = C \quad \text{and} \quad K^n(A, C) = C \otimes A^n \otimes C,$$

and the differential relations given by $d^{-1} = \Delta$ and

$$d^n = d_l^n \otimes I_C + (-1)^{n+1} I_C \otimes d_r^n.$$

Definition 26. Let (A, C) be an almost-Koszul pair, if we define

$$K^r_{-1}(A, C) = R \quad \text{and} \quad K^r_n(A, C) = C_n \otimes A,$$

and the differential maps

$$d^r_n(c \otimes a) = \sum c_{(1,n-1)} \otimes \theta_{C,A}(c_{(2,1)})a.$$

Appliying the same construction to the opposite almost-Koszul pair gives us the complex K^l_*.

Definition 27. Let (A, C) be an almost-Koszul pair, we define

$$K_{-1}(A, C) = A \quad \text{and} \quad K_n(A, C) = A \otimes C_n \otimes A,$$

and the differential relations given by d_0, induced by multiplication, and

$$d^l_n(a \otimes c) = \sum a\theta_{C,A}(c_{(1,1)}) \otimes c_{(2,n-1)};$$

defining $d_n = d^l_n \otimes I_A + (-1)^n I_A \otimes d^r_n$.

4.4 Koszul pairs

Theorem 5. *Given an almost-Koszul pair (A, C), if one of these six complexes is exact, all of them are exact, as it is showed in [5].*

- $K^l_*(A, C)$.
- $K^r_*(A, C)$.
- $K_*(A, C)$.
- $K^*_l(A, C)$.
- $K^*_r(A, C)$.
- $K^*(A, C)$.

Definition 28. An almost-Koszul pair (A, C) is said to be **Koszul** if and only if the previously discussed complexes are exact.

5 Conclusions

Starting from a very basic undergraduate mathematical level, all the necessary definitions of categories, chain complexes and homology have been developed in this work. This constitutes a reference for students interested on the specific field of Koszul pairs and sets the ground for future developments and undergraduate and graduate-level research.

In particular, future work should be able to find new characterizations of Koszul pairs in terms of homology and cohomology apart from the known characterizations found on [5].

6 References

References

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