Koszul Pairs

Mario Román

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2.1 Abelian categories

2.1.1 Additive category

Definition 1. $\mathcal{C}$ is an **additive category** if:

- $Hom(A, B)$ is an *abelian group*.
- *Distributivity* holds: $b \circ (f + g) = b \circ f + b \circ g$ and $(f + g) \circ a = f \circ a + g \circ a$.
- Has a *zero object*.
- Has finite *products* and *coproducts*.

A functor T between two additive categories is and **additive functor** if $T(f + g) = Tf + Tg$. [1]

2.1.2 Abelian category

Definition 2. An abelian category is an *additive category* such that

- every morphism has a kernel and cokernel.
- every monomorphism is a kernel.
- every epimorphism is a cokernel.

The category of R-modules is an abelian category, but also the category of chain complexes of an arbitrary abelian category, $\mathtt{Ch}(\mathcal{A})$, is an abelian category.

2.2 Chain complexes and homology

2.2.1 Chain complex

Definition 3. A **chain complex** is a family of R-modules $\{C_n\}$ and homomorphisms $d_n \colon C_n \to C_{n-1}$ called *differentials*, such that each composite of consecutive differentials is zero, i.e. $d_{n-1} \circ d_n = 0$. [2]

Theorem 1. *Given an abelian category $\mathcal{A}$, the category $\mathtt{Ch}(\mathcal{A})$ is an abelian category.*

2.2.2 Exact sequences

Theorem 2.

2.2.3 Homology

3 Derived functors

3.1 Projective, injective and flat resolutions

3.1.1 Definitions

Definition 4. An R-module D is:

1. **Projective** if $Hom(D, -)$ is exact.
2. **Injective** if $Hom(-, D)$ is exact.
3. **Flat** if $D \otimes -$ is exact.

We know that $Hom(D, -)$ and $Hom(-, D)$ are left-exact and that $D \otimes -$ is right-exact; so for them to be exact, we only need:

- A module D is **projective** when $B \longrightarrow C$ surjective induces $Hom(D, B) \longrightarrow Hom(D, C)$ surjective.

$$\begin{tikzcd}
 & B \arrow[d, two heads] \\
D \arrow[r] \arrow[ur, dashed, "\exists"] & C
\end{tikzcd}$$

- A module D is **injective** when $A \longrightarrow B$ surjective induces $Hom(B, D) \longrightarrow Hom(A, D)$ surjective.

$$\begin{tikzcd}
 & A \arrow[d, two heads] \arrow[dl] \\
D & B \arrow[l, dashed, "\exists"]
\end{tikzcd}$$

- A module D is **flat** when $A \longrightarrow B$ injective induces $D \otimes A \longrightarrow D \otimes B$ injective.

3.1.2 Resolutions

Definition 5. A **projective resolution** is a resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

where every P_i is projective.

Definition 6. An **injective resolution** is a resolution

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \dots$$

where every E_i is injective.

Definition 7. A **flat resolution** is a resolution

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

where F_i is flat.

Explicit construction Notice that, given a module M, we can always find a surjection from a proyective module (if we have *enough projectives*). So we can construct a projective resolution as follows:

$$\begin{tikzcd}
 & \ker f_2 \arrow[dr] & & & & \ker \pi \arrow[dr] & & & \\
\dots & & P_2 \arrow[dr, two heads, "f_2"] & & P_1 \arrow[ur, two heads, "f_1"] & & P_0 \arrow[r, two heads] & M \arrow[r] & 0 \\
 & & & \ker f_1 \arrow[ur] & & & & &
\end{tikzcd}$$

We can also reverse the arrows to obtain an injective resolution.

3.1.3 Derived functors

Definition 8.

Construction of the right derived functor Let F be additive, covariant and left-exact. Let $0 \longrightarrow M \longrightarrow E^\bullet$ be an injective resolution with M deleted; then $F(E^\bullet)$ is a complex, and we define:

$$R^i F(M) = H^i(F(E^\bullet)) = \frac{\ker\{F(E_i) \longrightarrow F(E_{i+1})\}}{\mathrm{Im}\{F(E_{i-1}) \longrightarrow F(E_i)\}}$$

That is, if we take the *injective resolution*

$$0 \longrightarrow M \longrightarrow E_0 \longrightarrow E_1 \longrightarrow \dots$$

we can delete M and apply F to get a (non neccesarily exact) complex where we can compute the homology

$$0 \longrightarrow F(E_0) \longrightarrow F(E_1) \longrightarrow F(E_2) \longrightarrow \dots.$$

Construction of the left derived functor Let F be additive, contravariant and left-exact. Let $P^\bullet \longrightarrow M \longrightarrow 0$ be a projective resolution with M deleted; then $F(P^\bullet)$ is a complex, and we define:

$$R^i F(M) = H^i(F(P^\bullet)) = \frac{\ker\{F(P_i) \longrightarrow F(P_{i+1})\}}{\mathrm{Im}\{F(P_{i-1}) \longrightarrow F(P_i)\}}$$

That is, if we take the *projective resolution*:

$$\dots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Delete M and apply F to get a (non neccesarily exact) complex where we can compute the homology:

$$0 \longrightarrow F(P_0) \longrightarrow F(P_1) \longrightarrow F(P_2) \longrightarrow \dots$$

4 Hochschild homology

4.1 Preliminaries

4.1.1 Enveloping algebra

Definition 9. Let R be a k-algebra, the **enveloping algebra** of R is $R^e = R \otimes R^{op}$, where the product is defined as

$$(r_1 \otimes s_1)(r_2 \otimes s_2) = (r_1 r_2) \otimes (s_2 s_1).$$

Theorem 3. *Given any k-algebra, R, the following categories are isomorphic:*

- *$(R; R)$-Mod*
- *R^e-Mod*
- *Mod-R^e*

Proof. If M is an $(R;R)$-module, we can provide it with R^e-module structure by defining $(r \otimes s)m = rms$. It is trivial to check that this structure is compatible with our previously defined product, as

$$(a \otimes b)(c \otimes d)m = acmdb = (ac \otimes bd)m.$$

If M is an R^e-module, we can provide it with $R;R$-module structure taking $rms = (r \otimes s)m$. Compatibility relation can be checked by the same reasoning. $\square$

4.1.2 Standard resolution

Given R, a k-algebra we define the **standard resolution** $(P_\bullet, d_\bullet)$ of R in $(R;R)-$mod as

- $P_n = R \otimes (R^{\otimes n}) \otimes R$
- $d_n = \sum_{i=0}^{n} (-1)^i d_n^i$

where

$$d_n^i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

4.2 Hochschild homology

Definition 10. Given R, a K-algebra, and M, an $(R;R)$-module, we define:

- The **Hochschild cohomology** of R in M as $HH^\bullet(R,M) = \mathrm{Ext}^\bullet_{R^e}(R,M)$.
- The **Hochschild homology** of R in M as $HH_\bullet(R,M) = \mathrm{Tor}^{R^e}_\bullet(R,M)$.

In order to compute the cohomology, we can take the following cochain complex

$$\mathrm{Hom}_K(K,M) \xrightarrow{b^0} \mathrm{Hom}_K(R,M) \xrightarrow{b^1} \mathrm{Hom}_K(R^{\otimes 2},M) \xrightarrow{b^2} \ldots.$$

where the b^n are defined as

- $b^0(m)(a) = am - ma$
- $b^n = \sum_{i=0}^{n+1} (-1)^i b_i^n$

and the auxiliary morphisms b_i^n are defined as

$$b_i^n(f)(a_1 \otimes \cdots \otimes a_{n+1}) = \begin{cases} a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) & \text{if } i = 0 \\ f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} & \text{if } i = 1, \ldots, n \\ f(a_1 \otimes \cdots \otimes a_n) a_{n+1} & \text{if } i = n+1 \end{cases}.$$

5 Koszul pairs

5.1 Preliminary definitions

5.1.1 Graded rings

Definition 11. A **graded ring** is a ring that can be written as a direct sum of abelian groups

$$A = \bigoplus_{n \in \mathbb{N}} A_n$$

such that $A_i A_j \subset A_{i+j}$.

A **homogeneous element** is an element of any factor A_i of the decomposition.

5.1.2 Koszul rings

Definition 12. A graded ring A is a **koszul ring** if A^0 is a semisimple ring and it has a resolution P_* by projective graded left A-modules such that each P_n is generated by homogeneous elements of degree n. [3]

5.1.3 R-rings

Definition 13. An R-ring is an associative and unital algebra. It is an associative and unital ring A together with a morphism $u : R \longrightarrow A$.

A R-ring is **graded** if it is equipped with a decomposition:

$$A = \bigoplus_{n \in \mathbb{N}} A^n$$

such that multiplicaton $m^{p,q}$ maps $A^p \otimes A^q$ into A^{p+q}. It is **connected** when $A_0 = R$. It is **strongly graded** when $m^{1,p}$ is surjective. We call π_A^n to the projection of A onto A^n.

5.1.4 R-coring

Definition 14. A **coalgebra** over a field K is a **vector space** V together with linear maps $\Delta : V \longrightarrow V \otimes V$ and $\varepsilon : V \longrightarrow K$ such that:

1. $(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta$
2. $(id \otimes \varepsilon) \circ \Delta = id = (\varepsilon \otimes id) \circ \Delta$

When writting in coalgebras, we will follow the **Sweedler notation**. [4]

Definition 15. An **R**-coring is a *coassociative* and *counital coalgebra*. It is an R-bimodule with a *comultiplication* $\Delta : C \longrightarrow C \otimes C$ and a *counit* $\epsilon : C \longrightarrow R$.

A R-coring is **graded** if it is equipped with a decomposition $C = \bigoplus_{n \in \mathbb{N}} C_n$, such that

$$\Delta(C_n) \subset \bigoplus_{p=0}^{n} C_p \otimes C_{n-p}.$$

5.1.5 Almost-koszul pair

Definition 16. An **almost-Koszul pair** is a connected R-ring and R-coring (A, C) with an isomorphism $\theta_{C,A} : C_1 \longrightarrow A^1$ that satisfies the relation

$$m^{1,1} \circ (\theta_{C,A} \otimes \theta_{C,A}) \circ \Delta_{1,1} = 0.$$

Using Sweedler notation we can rewrite the condition as follows: for any $c \in C_2$,

$$\sum \theta_{C,A}(c_{(1,1)}) \theta_{C,A}(c_{(2,1)}) = 0.$$

6 References

References

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[3] P. Jara Martínez, J. López Peña, and D. Ştefan. Koszul pairs and applications. *ArXiv e-prints*, November 2010.

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