Exercise II.6.1

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April 30, 2017

The group of invertible $n \times n$ matrices with entries in $\mathbb{R}$ is denoted $GL_n(\mathbb{R})$. Similarly, $GL_n(\mathbb{C})$ denotes the group of $n \times n$ invertible matrices with complex entries. Consider the following sets of matrices:

- $SL_n(\mathbb{R}) = \{M \in GL_n(\mathbb{R}) \mid \det(M) = 1\}$;
- $SL_n(\mathbb{C}) = \{M \in GL_n(\mathbb{C}) \mid \det(M) = 1\}$;
- $O_n(\mathbb{R}) = \left\{M \in GL_n(\mathbb{R}) \mid MM^t = M^tM = I_n\right\}$;
- $SO_n(\mathbb{R}) = \{M \in O_n(\mathbb{R}) \mid \det(M) = 1\}$;
- $U(n) = \left\{M \in GL_n(\mathbb{C}) \mid MM^\dagger = M^\dagger M = I_n\right\}$;
- $SU(n) = \{M \in U_n(\mathbb{C}) \mid \det(M) = 1\}$;

Here I_n stands for the $n \times n$ identity matrix, M^t is the transpose *of M, $M^\dagger$ is the* conjugate transpose *of M, and $\det(M)$ denotes the* determinant *of M. Find all possible inclusions among these sets, and prove that in every case the smaller set is a subgroup of the larger one.*

We are dealing with three different properties:

1. The matrix has entries in $\mathbb{C}$.
2. The matrix has its conjugate transpose as its inverse, $MM^\dagger = I_n$.
3. The matrix has determinant 1, $\det(M) = 1$.

None of them implies the others. The three properties give rise to this three-dimensional cube

$$\begin{tikzcd}
 & GL(\mathbb{C}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
U(n) \arrow[d, dash] \arrow[dr, dash] & SL_n(\mathbb{C}) \arrow[dl, dash] \arrow[dr, dash] & GL(\mathbb{R}) \arrow[dl, dash] \arrow[d, dash] \\
SU(n) \arrow[dr, dash] & O_n(\mathbb{R}) \arrow[d, dash] & SL_n(\mathbb{R}) \arrow[dl, dash] \\
 & SO_n(\mathbb{R}) &
\end{tikzcd} .$$