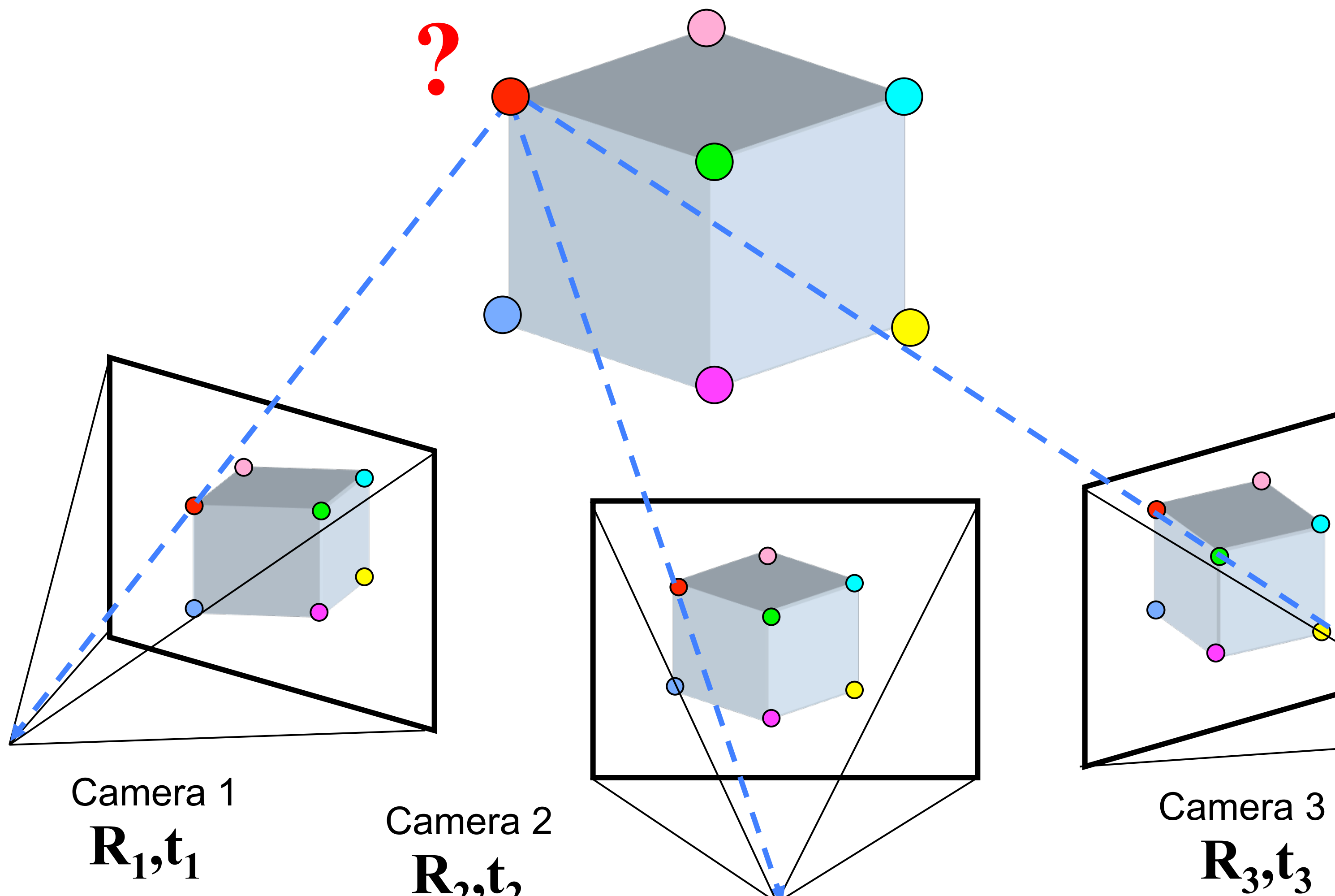
Multi-view geometry

Multi-view geometry problems

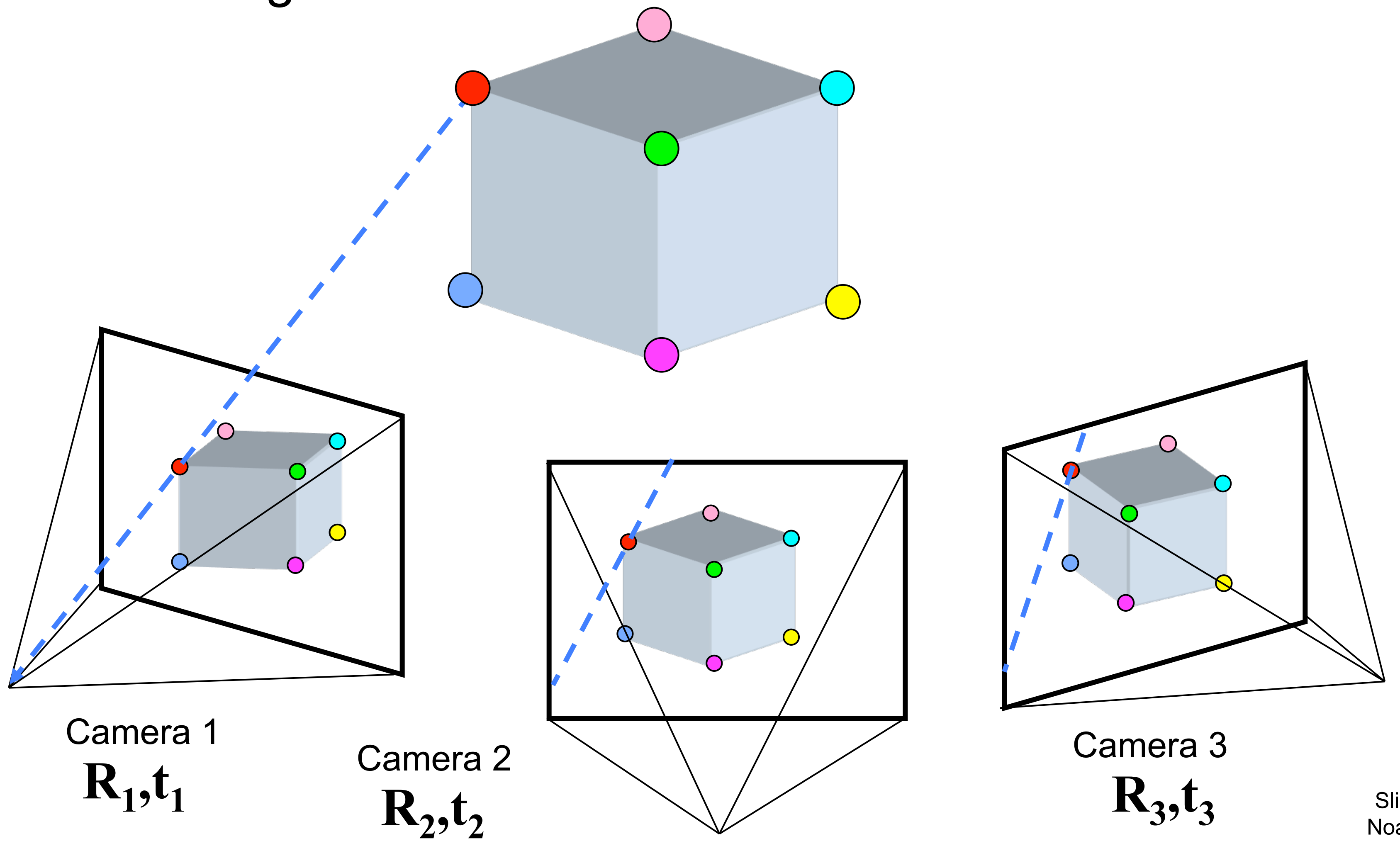
- **Structure:** Given projections of the same 3D point in two or more images, compute the 3D coordinates of that point



Slide credit: Noah Snavely

Multi-view geometry problems

- **Stereo correspondence:** Given a point in one of the images, where could its corresponding points be in the other images?



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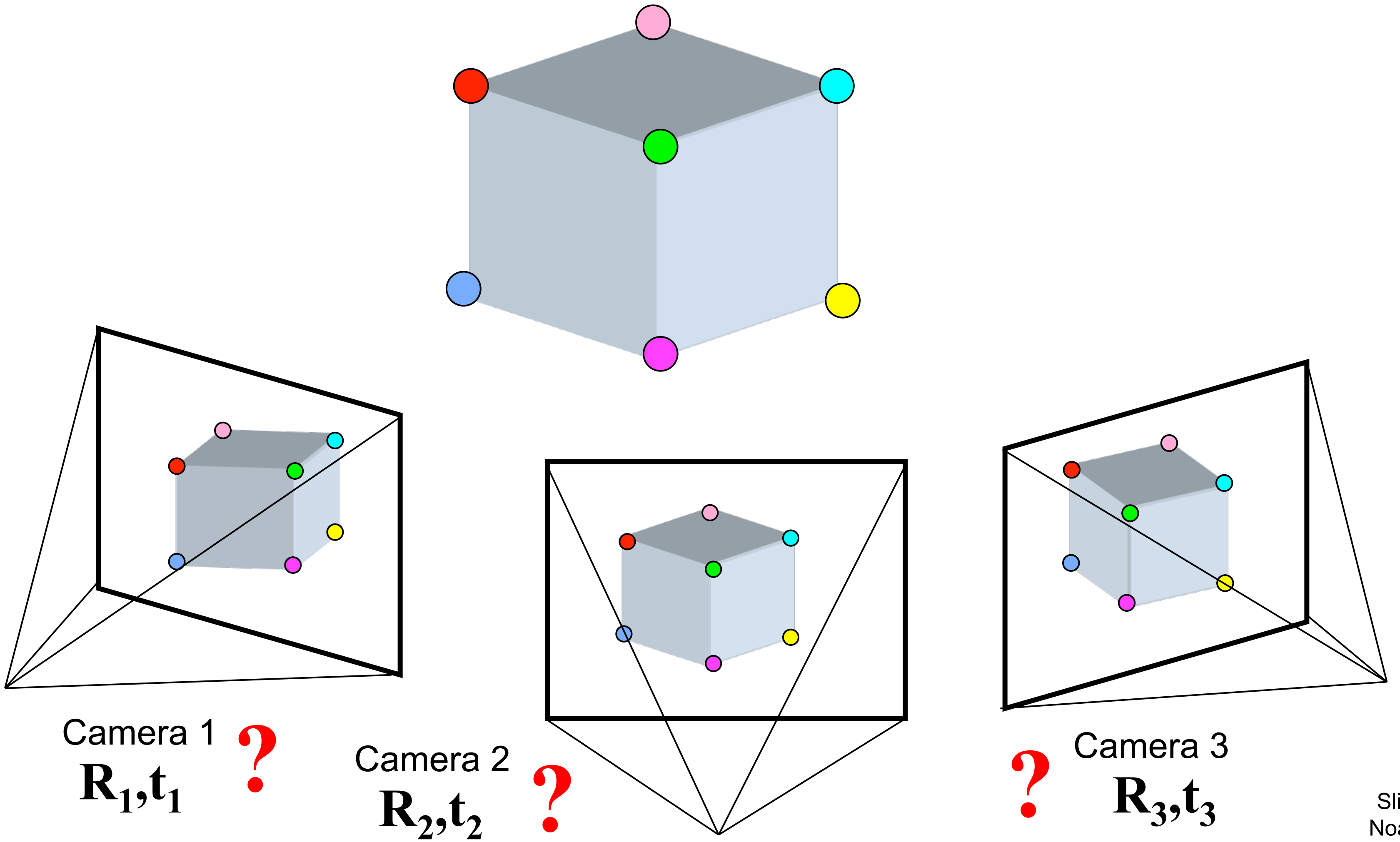
Multi-view geometry problems

- **Motion:** Given a set of corresponding points in two or more images, compute the camera parameters



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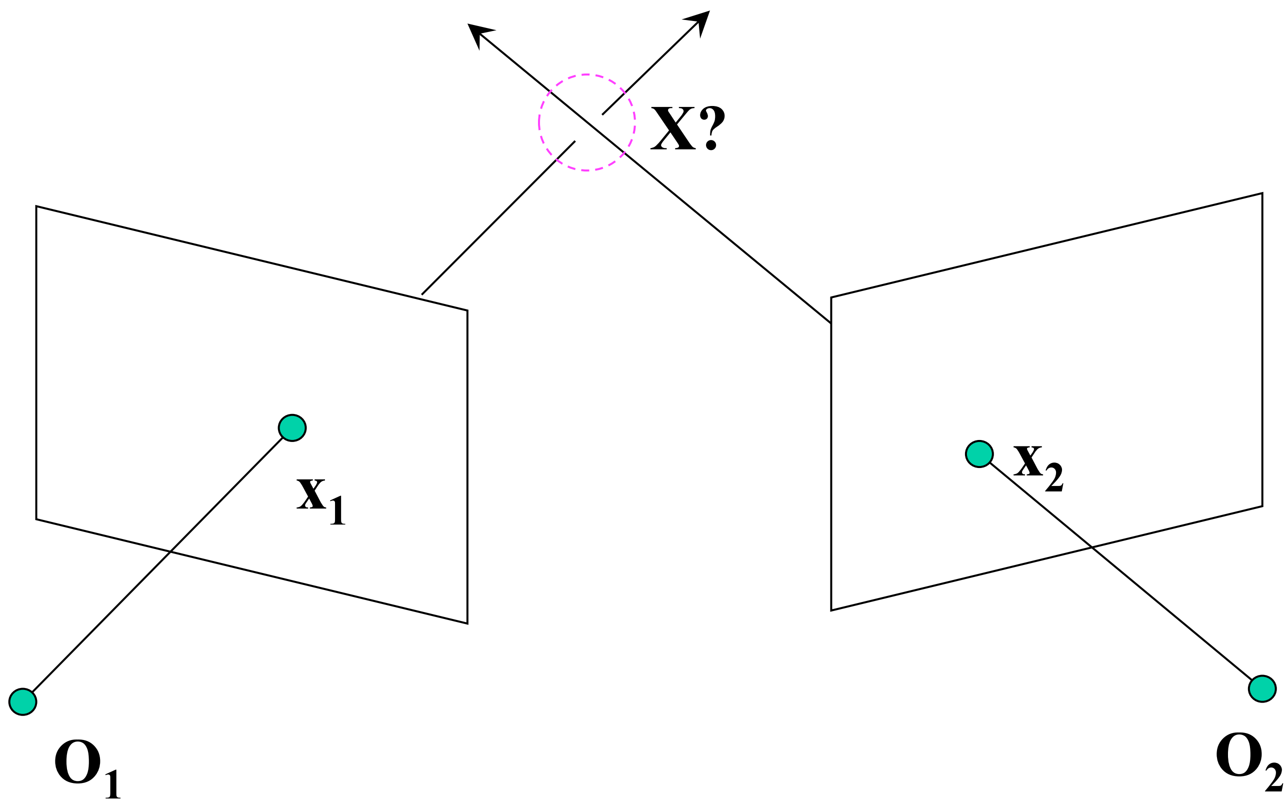
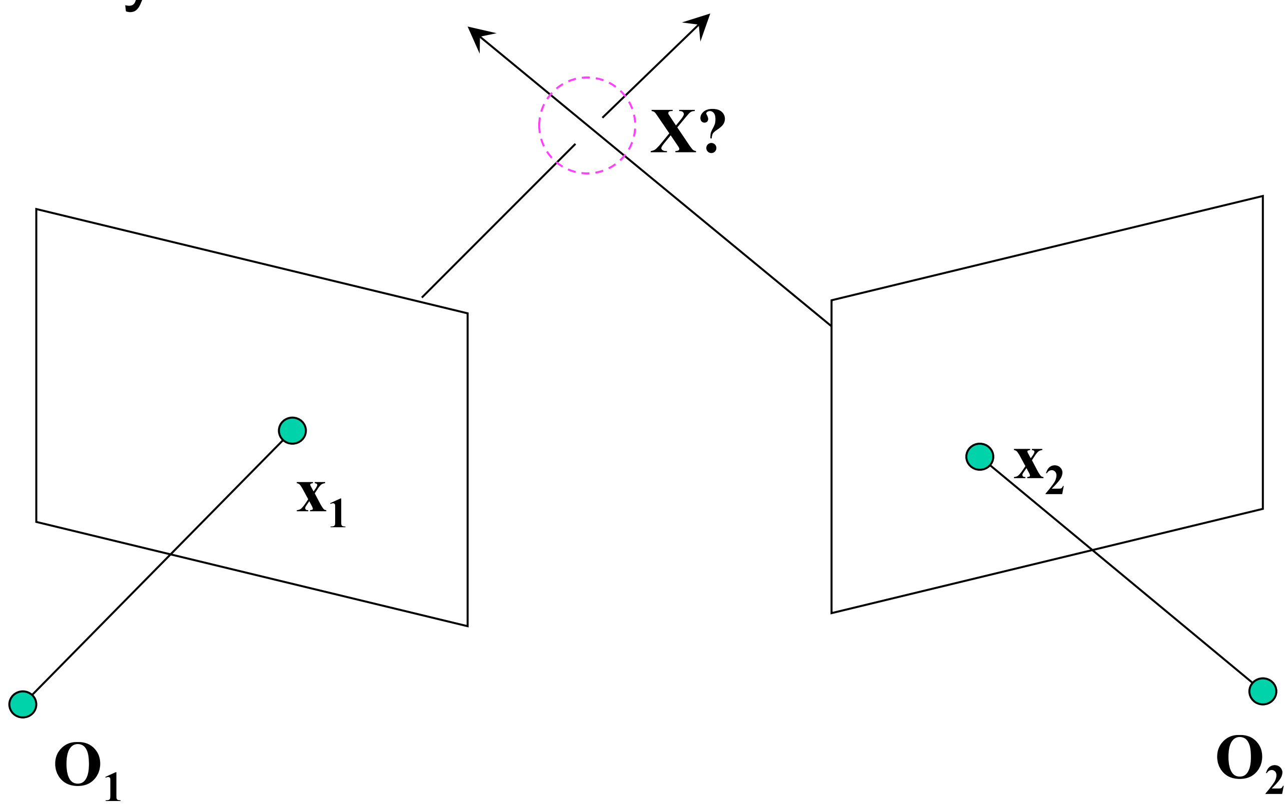
Structure: Triangulation

- Given projections of a 3D point in two or more images (with known camera matrices), find the coordinates of the point



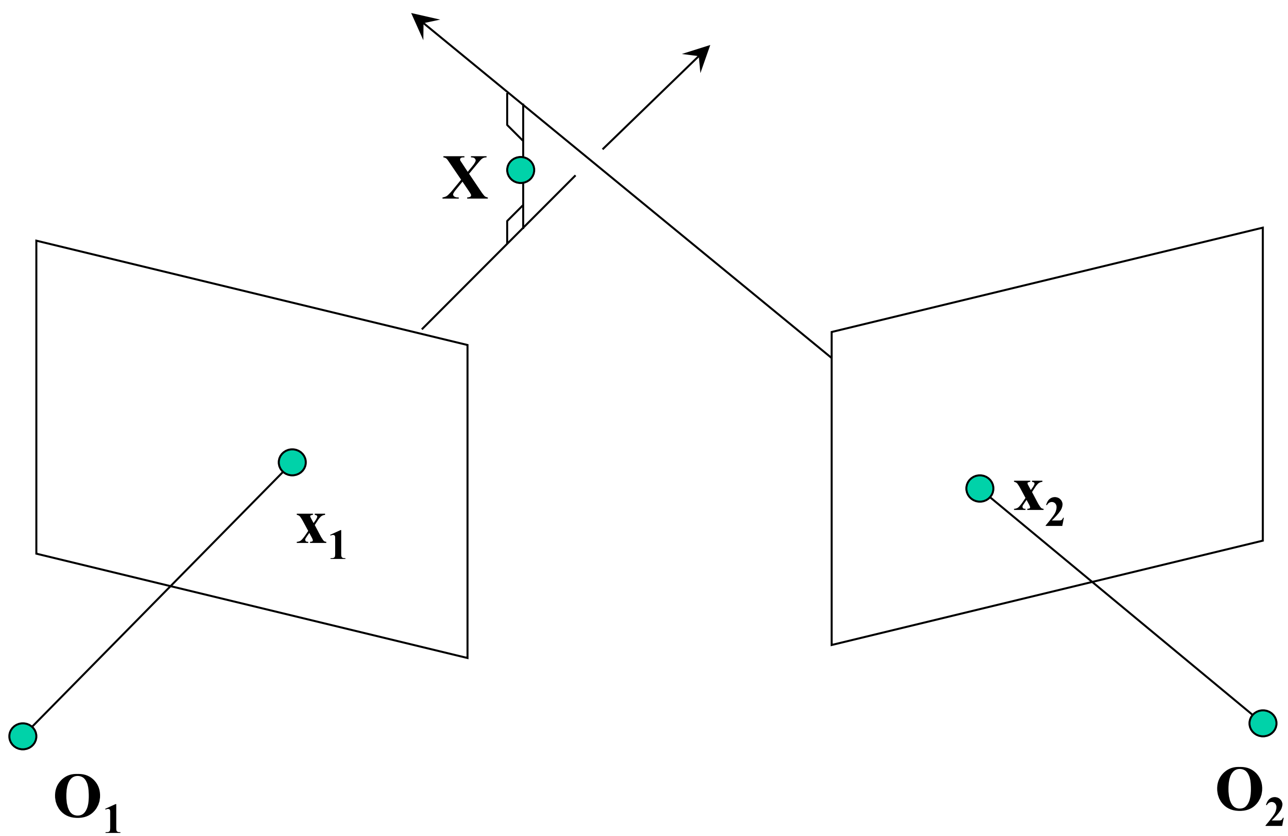
Structure: Triangulation

- We want to intersect the two visual rays corresponding to $\mathbf{x_1}$ and $\mathbf{x_2}$, but because of noise and numerical errors, they don't meet exactly



Triangulation: Geometric approach

- Find shortest segment connecting the two viewing rays and let **X** be the midpoint of that segment



Triangulation: Linear approach

$$\lambda_1 \mathbf{x_1} = \mathbf{P_1 X} \qquad \mathbf{x_1} \times \mathbf{P_1 X} = \mathbf{0} \qquad [\mathbf{x_{1\times}}]\mathbf{P_1 X} = \mathbf{0}$$

$$\lambda_2 \mathbf{x_2} = \mathbf{P_2 X} \qquad \mathbf{x_2} \times \mathbf{P_2 X} = \mathbf{0} \qquad [\mathbf{x_{2\times}}]\mathbf{P_2 X} = \mathbf{0}$$

Cross product as matrix multiplication:

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix} = [\mathbf{a}_\times]\mathbf{b}$$

Triangulation: Linear approach

$$\lambda_1 \mathbf{x_1} = \mathbf{P_1 X} \qquad \mathbf{x_1} \times \mathbf{P_1 X} = \mathbf{0} \qquad [\mathbf{x}_{\mathbf{1}\times}]\mathbf{P_1 X} = \mathbf{0}$$

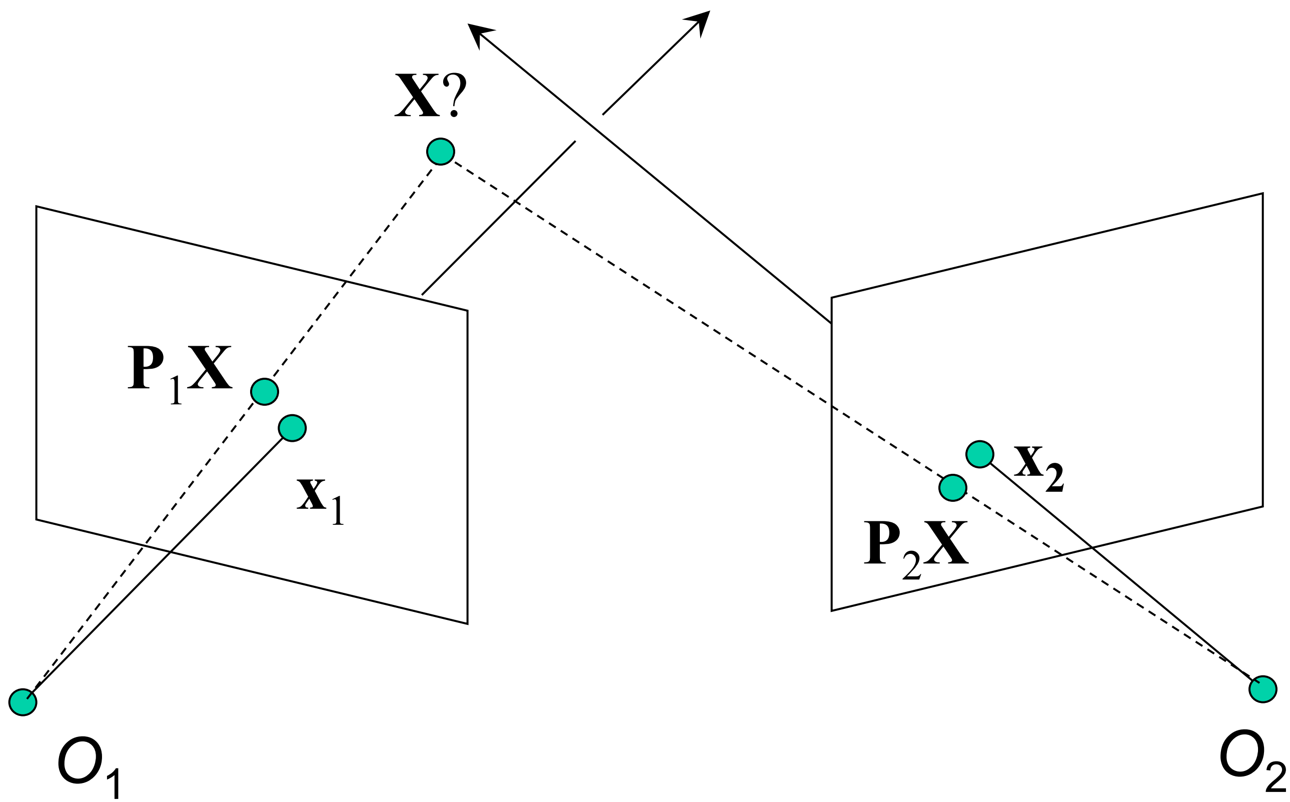
$$\lambda_2 \mathbf{x_2} = \mathbf{P_2 X} \qquad \mathbf{x_2} \times \mathbf{P_2 X} = \mathbf{0} \qquad [\mathbf{x}_{\mathbf{2}\times}]\mathbf{P_2 X} = \mathbf{0}$$

Two independent equations each in terms of three unknown entries of **X**

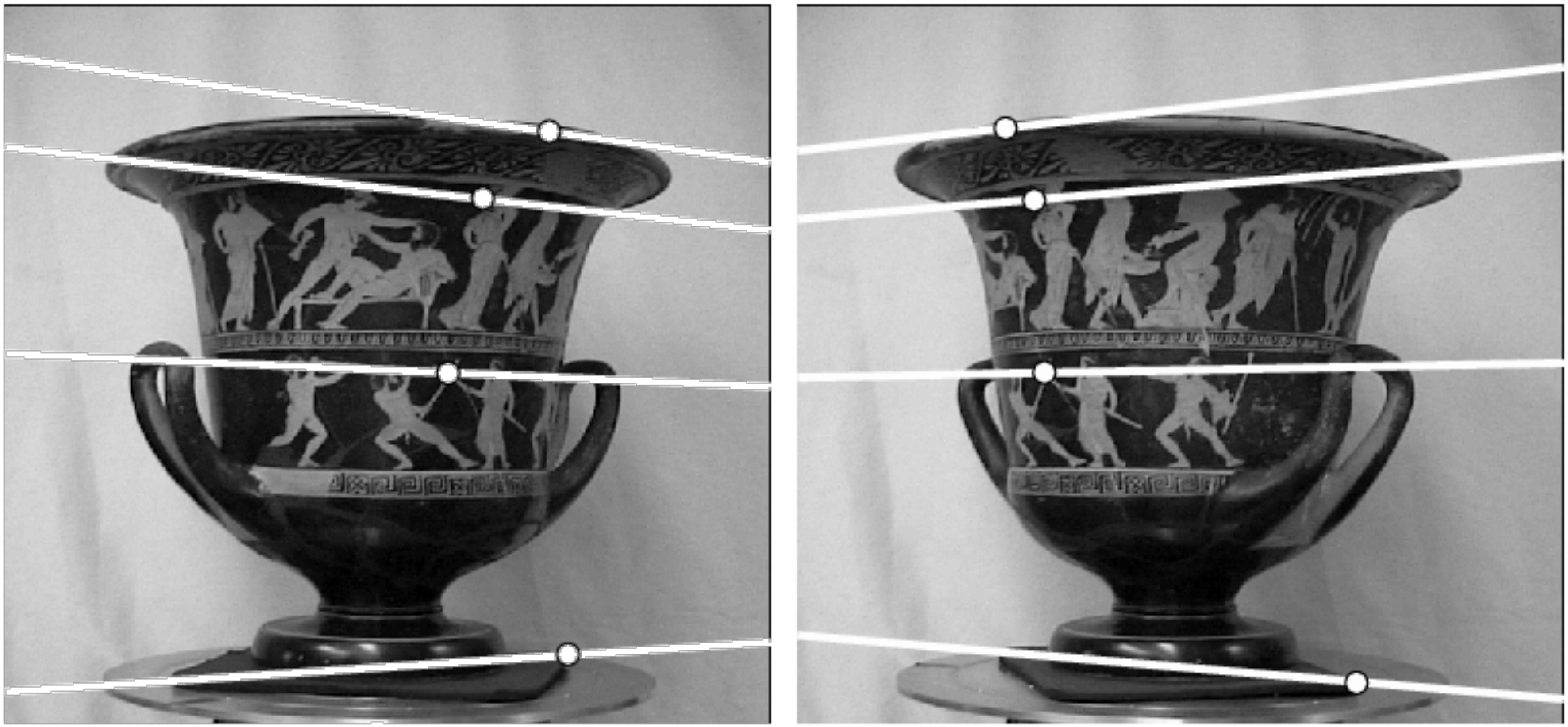
Triangulation: Nonlinear approach

Find X that minimizes

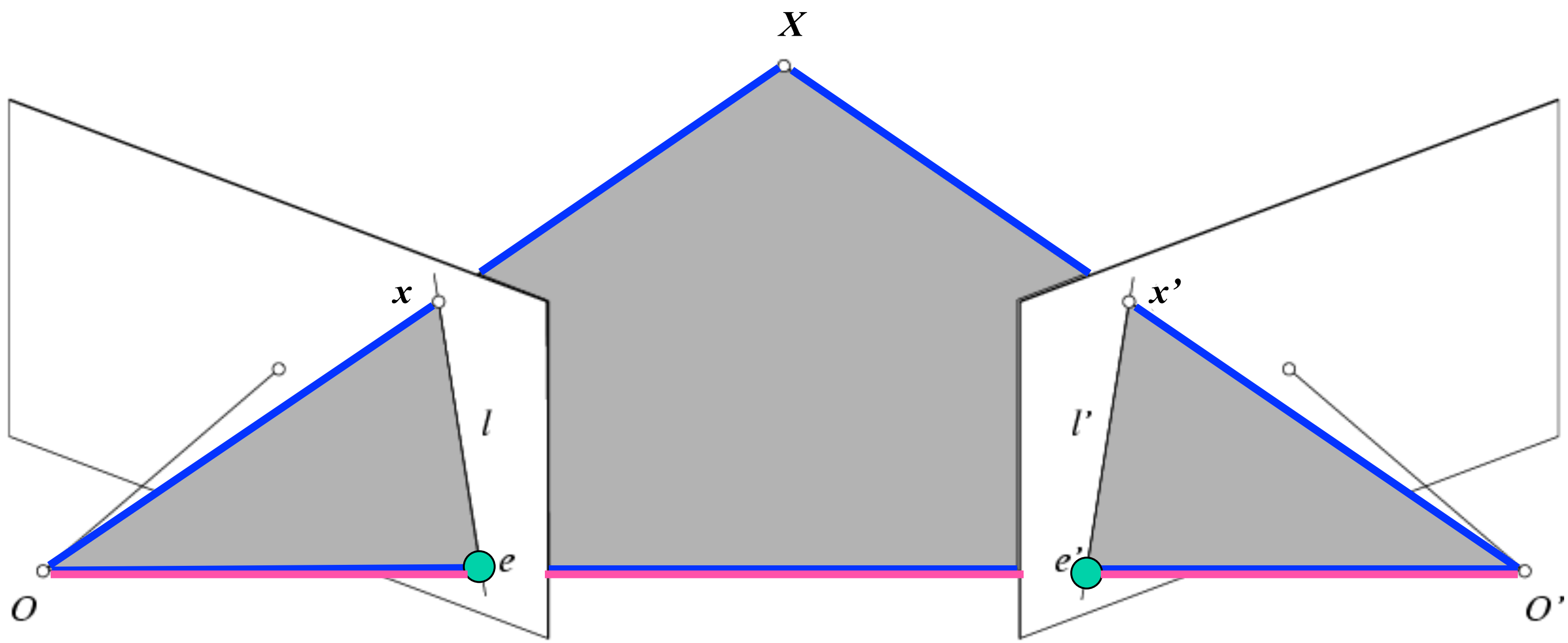
$$d^2(\mathbf{x_1}, \mathbf{P_1 X}) + d^2(\mathbf{x_2}, \mathbf{P_2 X})$$



Two-view geometry



Epipolar geometry



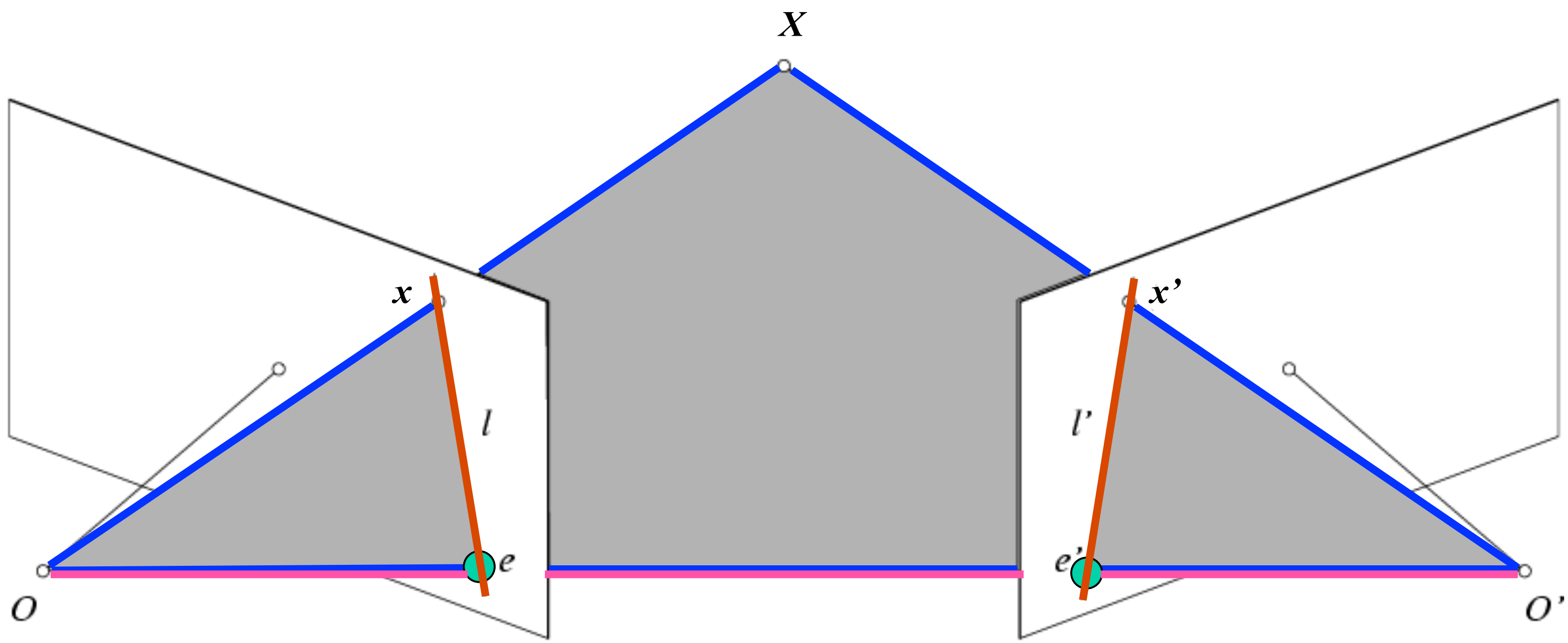
- **Baseline** – line connecting the two camera centers
- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipoles**

= intersections of baseline with image planes
= projections of the other camera center
= vanishing points of the baseline (motion direction)

The Epipole

Photo by Frank Dellaert

Epipolar geometry



- **Baseline** – line connecting the two camera centers
- **Epipolar Plane** – plane containing baseline (1D family)
- **Epipoles**

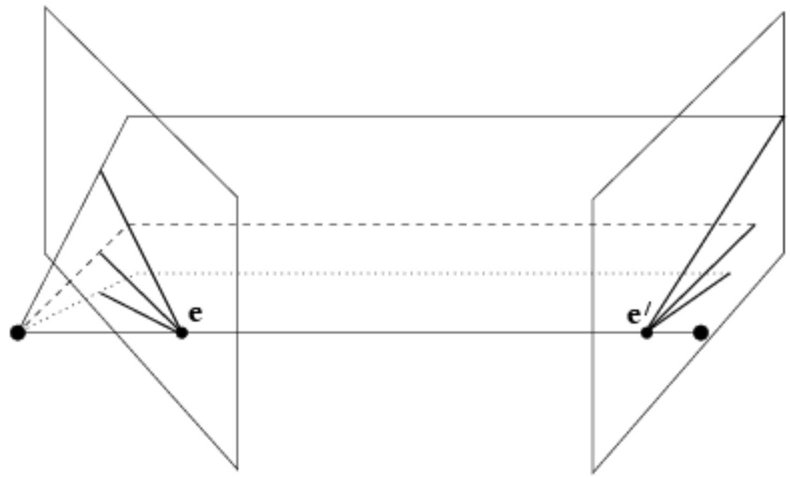
= intersections of baseline with image planes

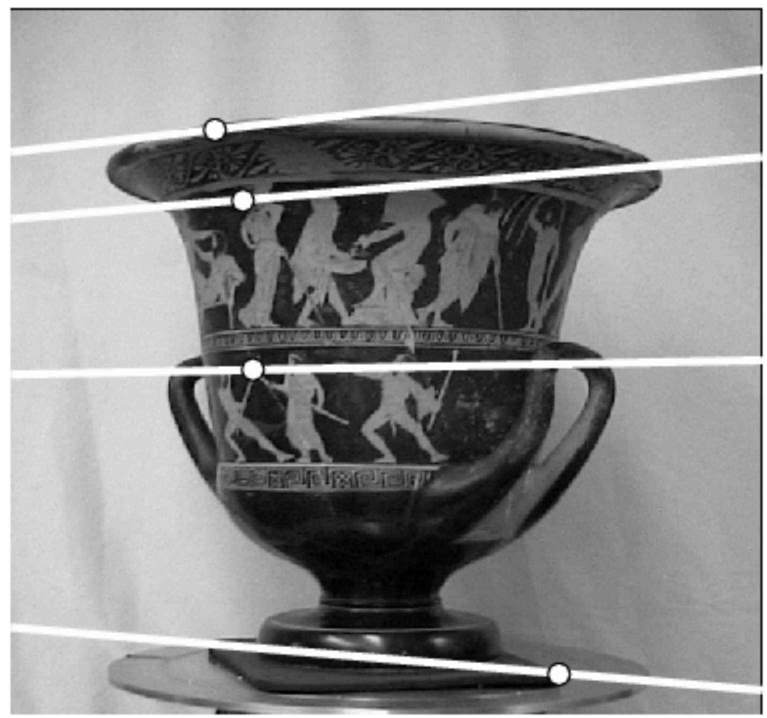
= projections of the other camera center

= vanishing points of the baseline (motion direction)

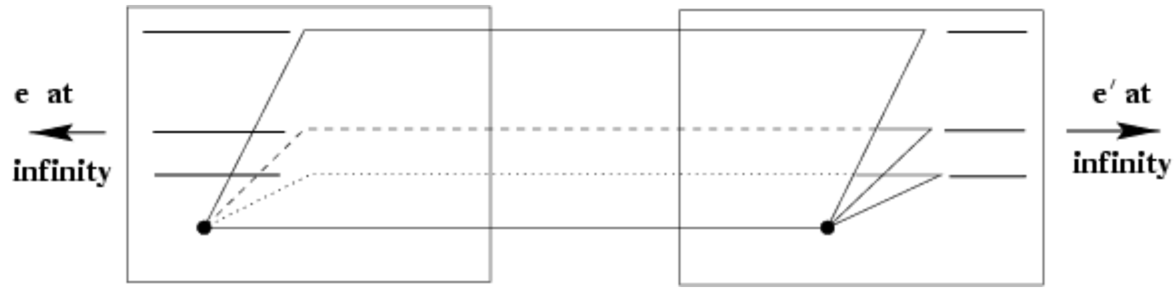
- **Epipolar Lines** - intersections of epipolar plane with image planes (always come in corresponding pairs)

Example: Converging cameras





Example: Motion parallel to image plane

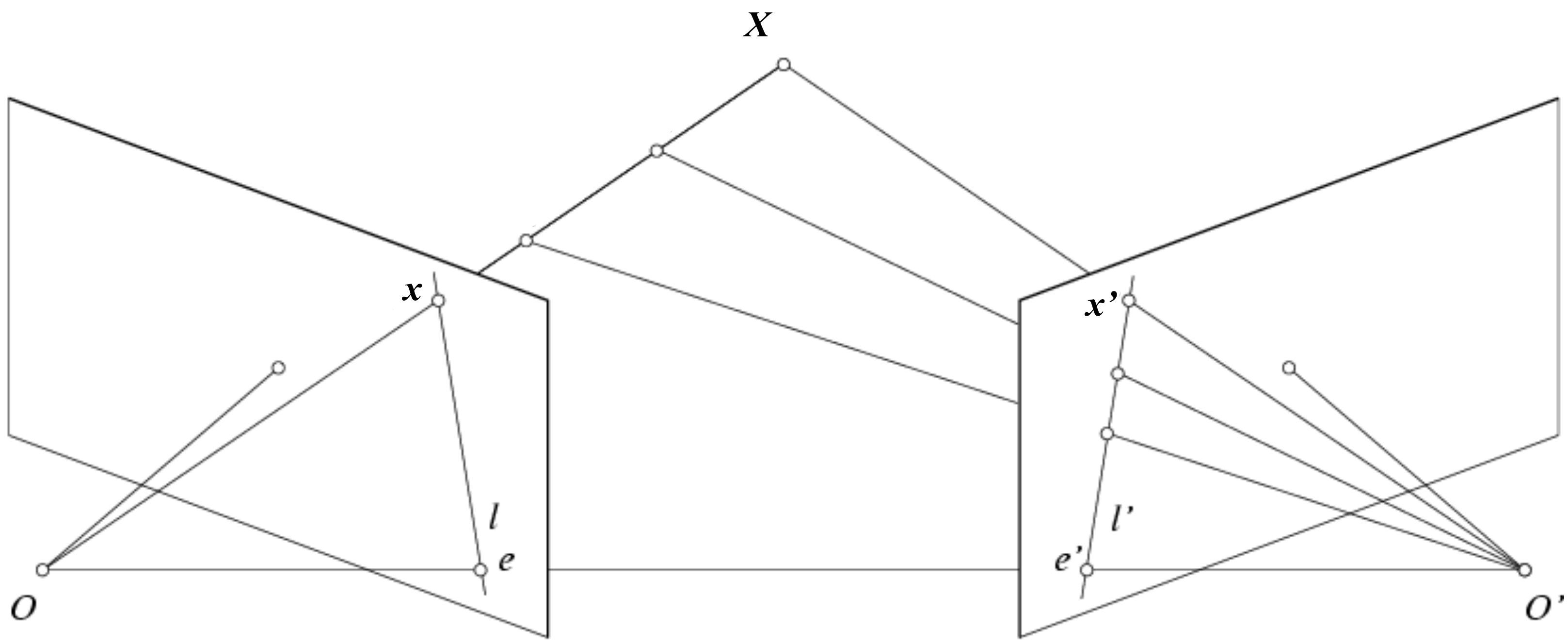


Example: Motion perpendicular to image plane

Example: Motion perpendicular to image plane

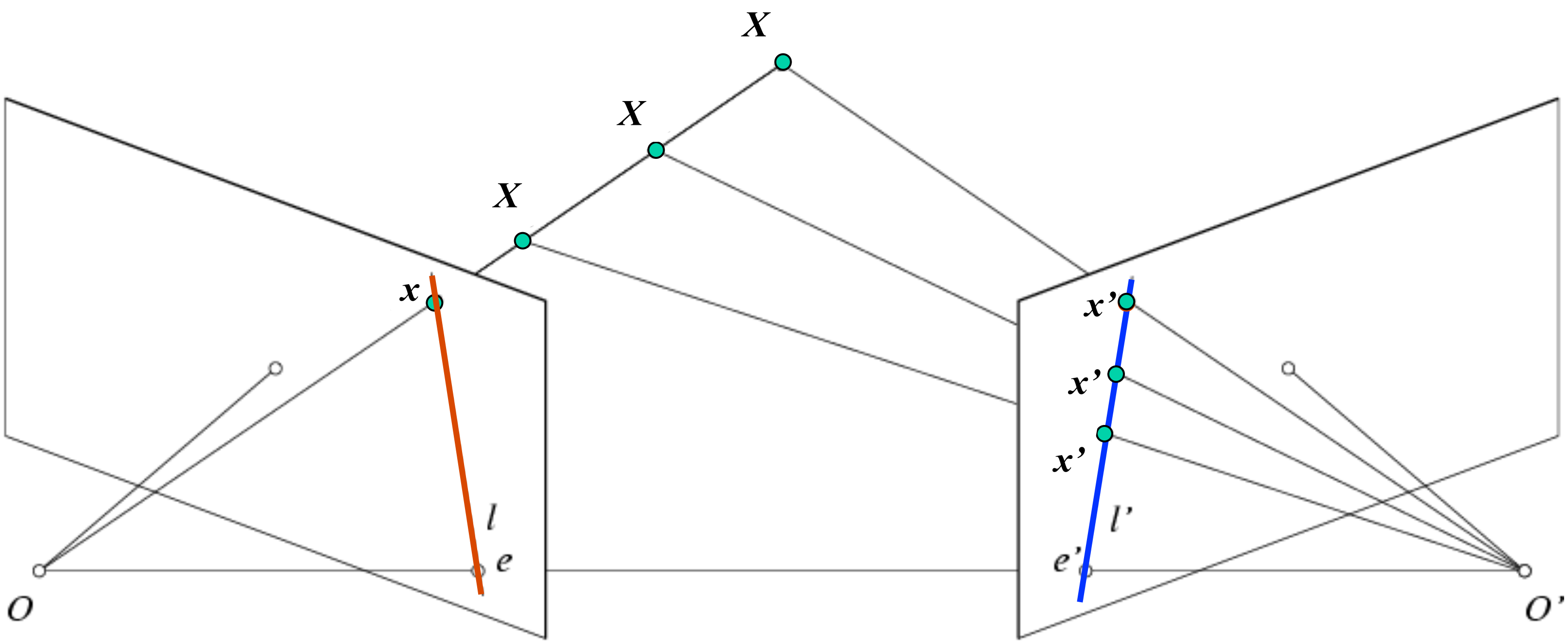
- Points move along lines radiating from the epipole: “focus of expansion”
- Epipole is the principal point

Epipolar constraint



- If we observe a point ***x*** in one image, where can the corresponding point ***x'*** be in the other image?

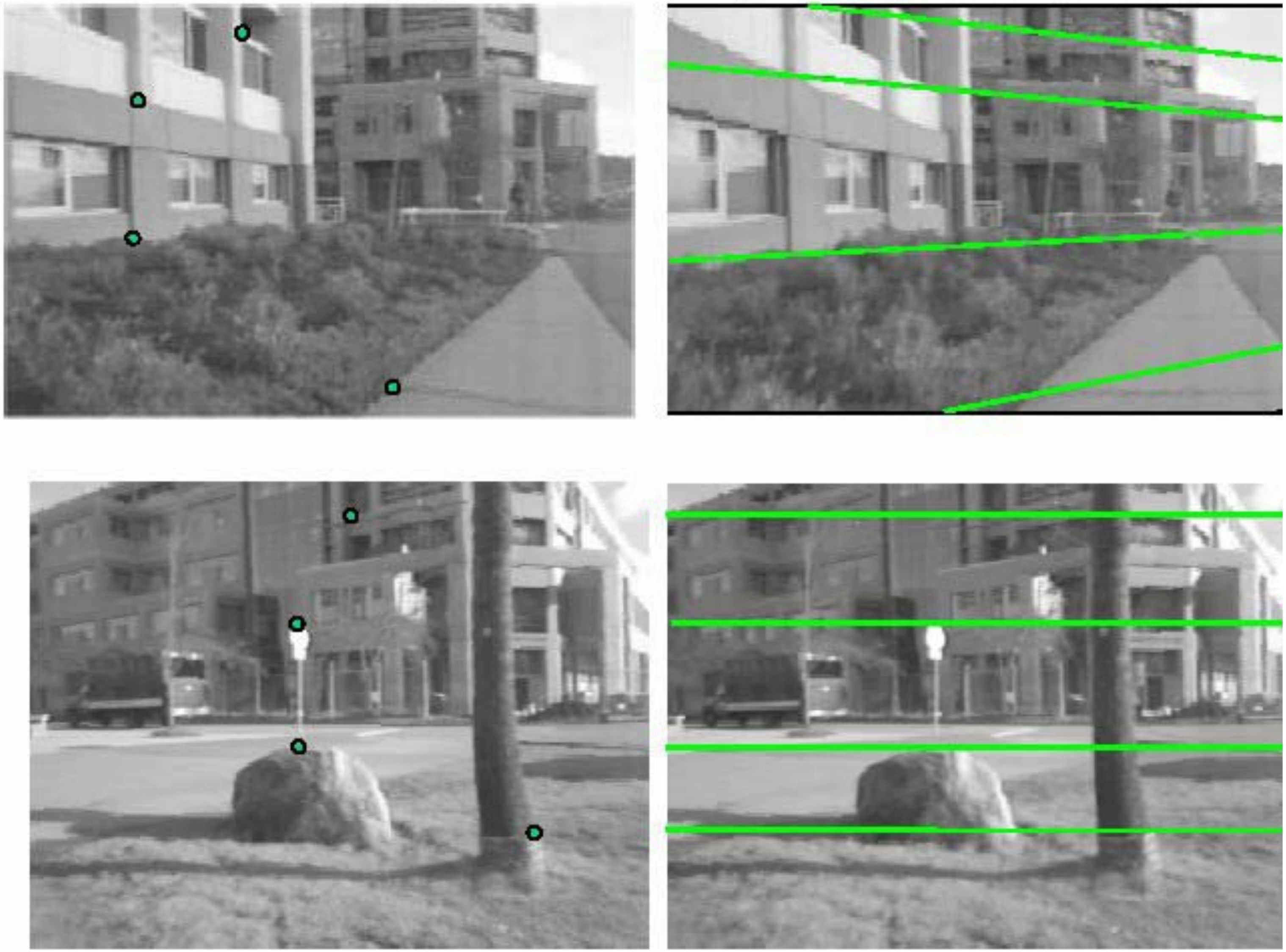
Epipolar constraint



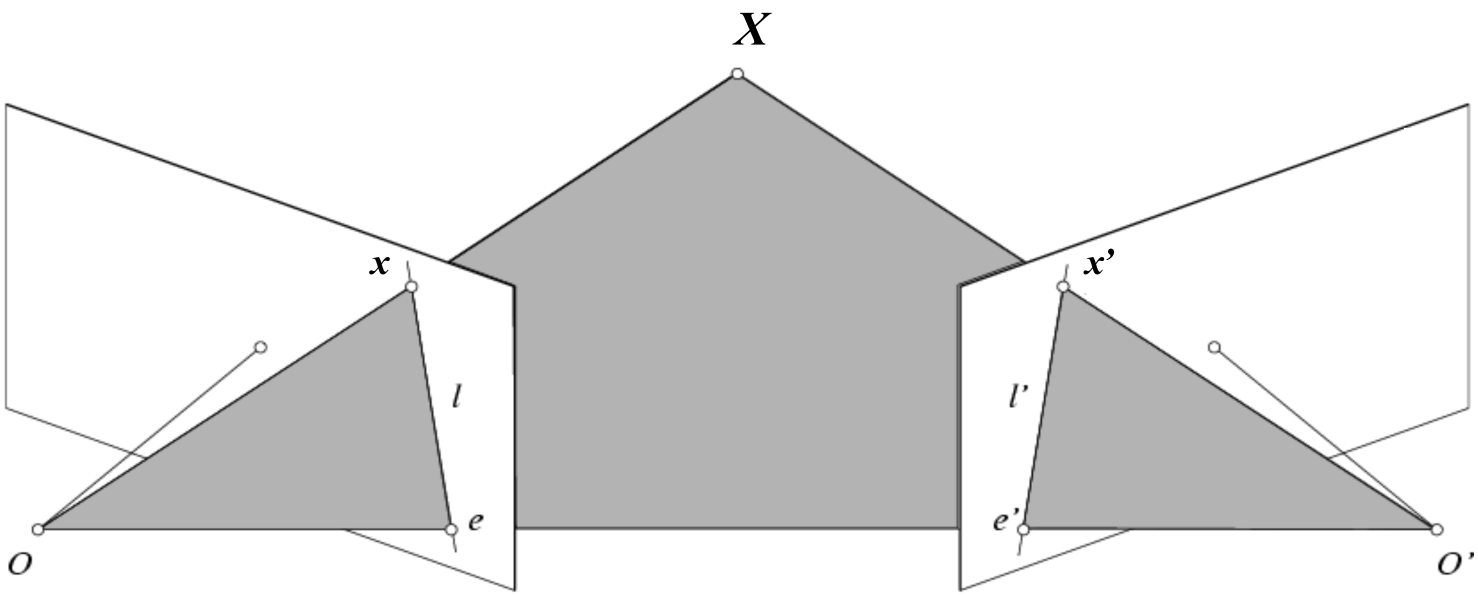
- Potential matches for ***x*** have to lie on the corresponding epipolar line ***l'***.

- Potential matches for ***x'*** have to lie on the corresponding epipolar line ***l***.

Epipolar constraint example

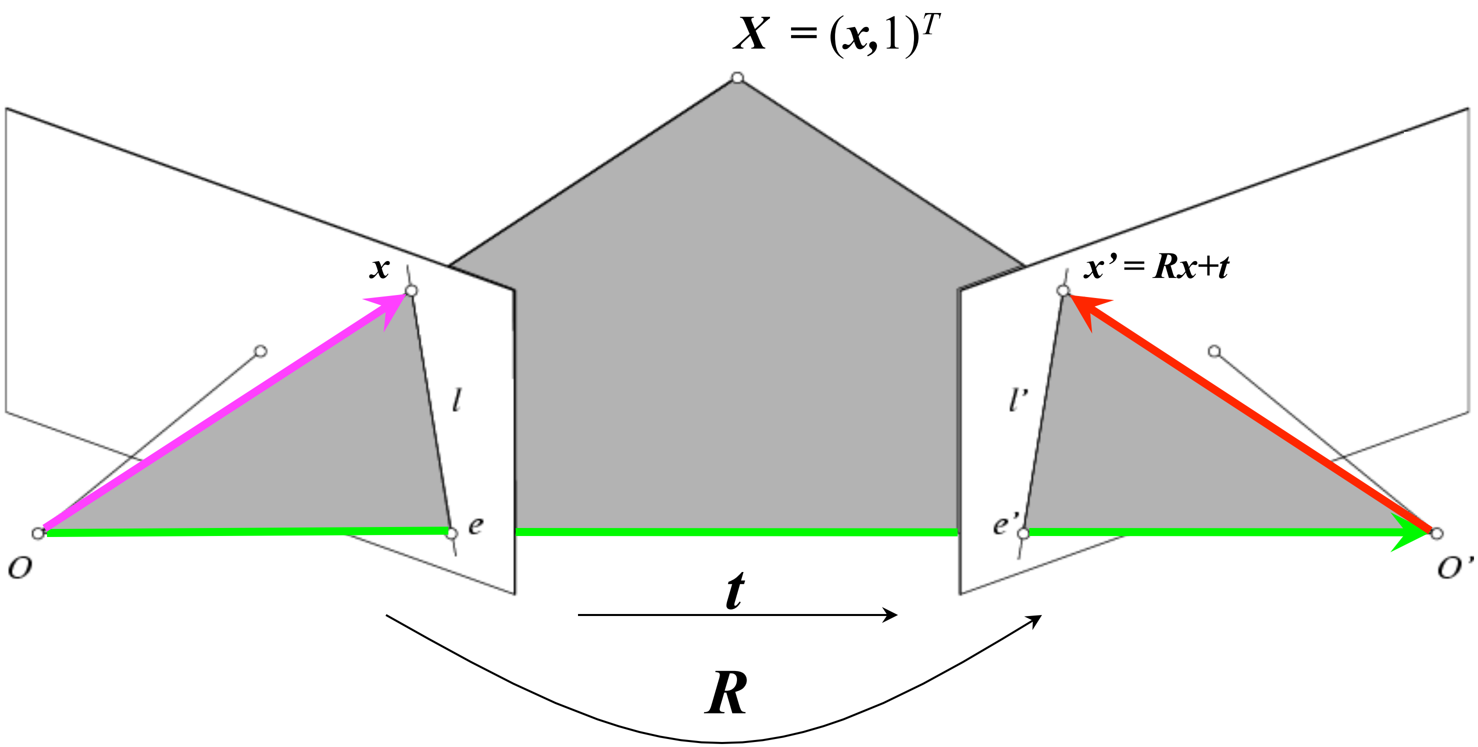


Epipolar constraint: Calibrated case



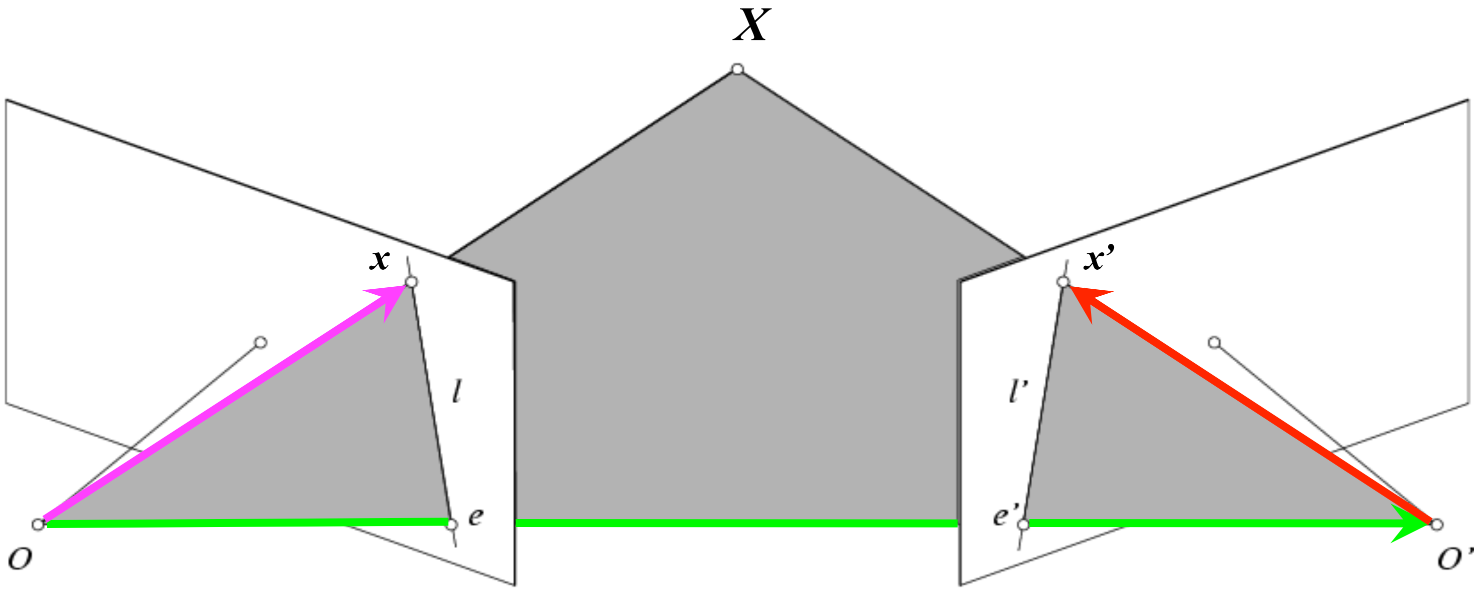
- Assume that the intrinsic and extrinsic parameters of the cameras are known
- We can multiply the projection matrix of each camera (and the image points) by the inverse of the calibration matrix to get *normalized* image coordinates
- We can also set the global coordinate system to the coordinate system of the first camera. Then the projection matrices of the two cameras can be written as $[\mathbf{I} \mid \mathbf{0}]$ and $[\mathbf{R} \mid \mathbf{t}]$

Epipolar constraint: Calibrated case



The vectors ***Rx***, ***t***, and ***x'*** are coplanar

Epipolar constraint: Calibrated case

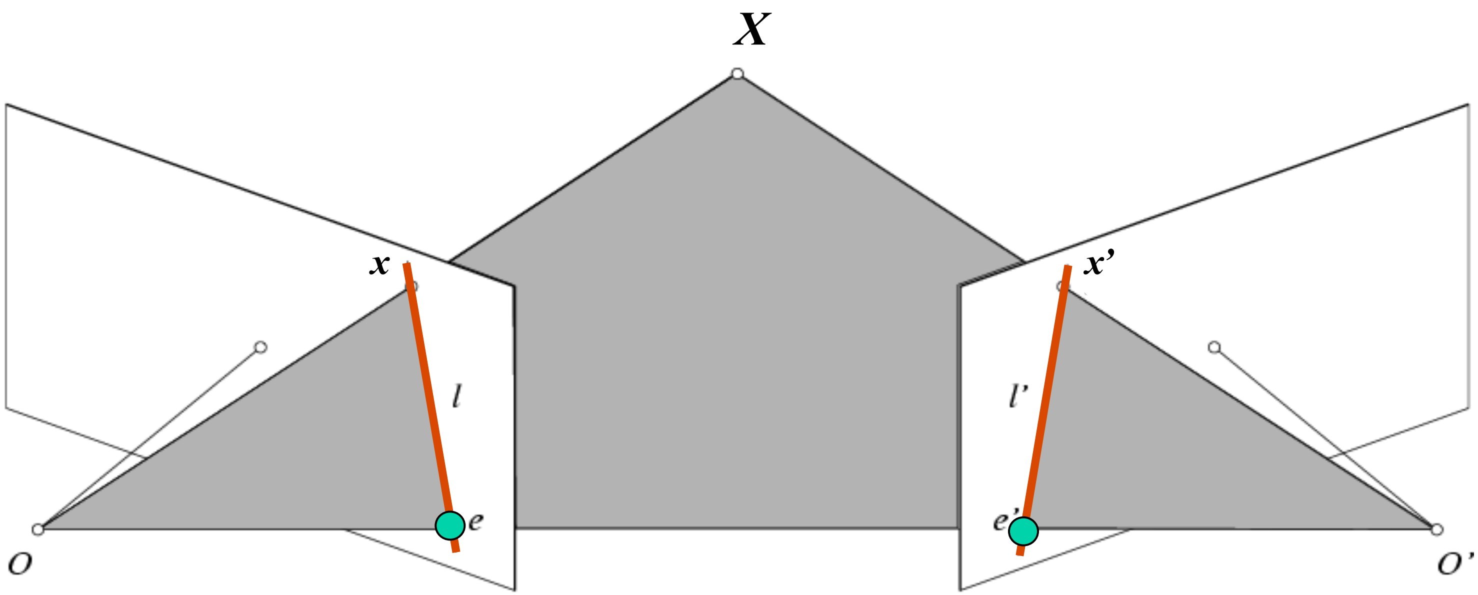


$$\boldsymbol{x}' \cdot [\boldsymbol{t} \times (\boldsymbol{R}\boldsymbol{x})] = 0 \quad \Rightarrow \quad \boldsymbol{x}'^T \boldsymbol{E} \boldsymbol{x} = 0 \quad \text{with} \quad \boldsymbol{E} = [\boldsymbol{t}_\times]\boldsymbol{R}$$

Essential Matrix
(Longuet-Higgins, 1981)

The vectors $\boldsymbol{Rx}$, $\boldsymbol{t}$, and $\boldsymbol{x}'$ are coplanar

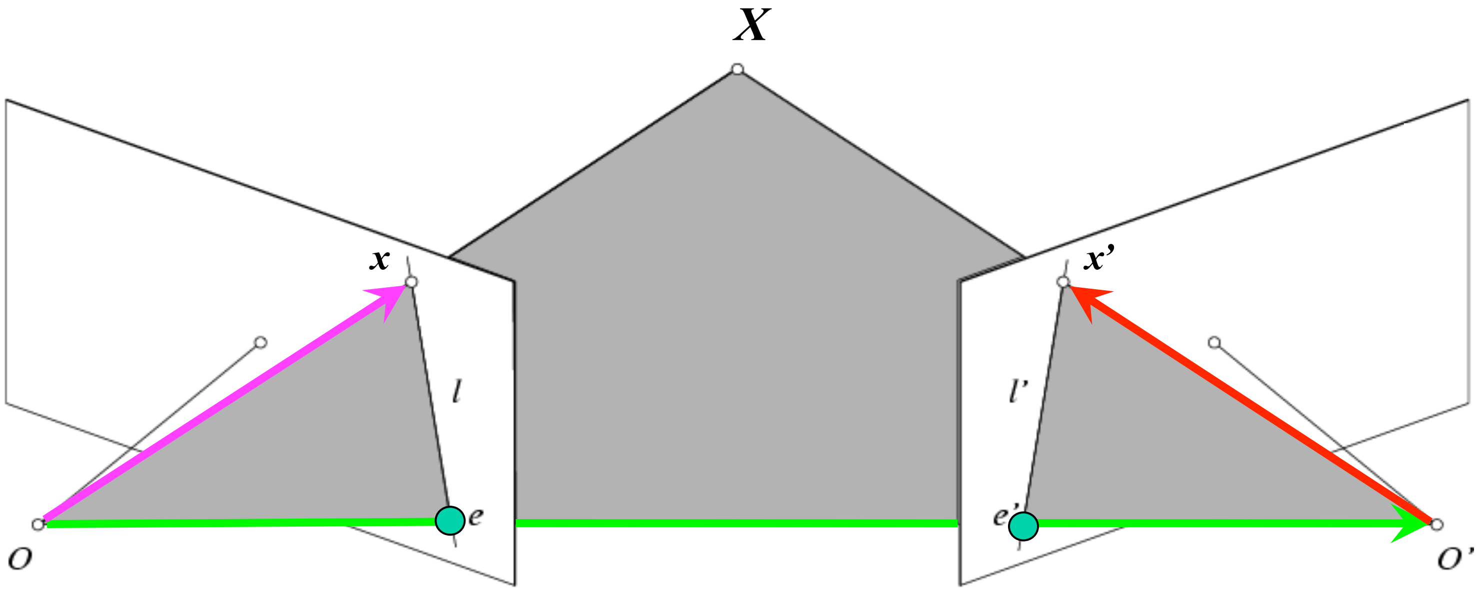
Epipolar constraint: Calibrated case



$$\boldsymbol{x}' \cdot [\boldsymbol{t} \times (\boldsymbol{R}\boldsymbol{x})] = 0 \quad \Rightarrow \quad \boldsymbol{x}'^T \boldsymbol{E}\, \boldsymbol{x} = 0 \quad \text{with} \quad \boldsymbol{E} = [\boldsymbol{t}_\times]\boldsymbol{R}$$

- $\boldsymbol{E}\,\boldsymbol{x}$ is the epipolar line associated with $\boldsymbol{x}$ ($\boldsymbol{l}' = \boldsymbol{E}\,\boldsymbol{x}$)
- $\boldsymbol{E}^T\boldsymbol{x}'$ is the epipolar line associated with $\boldsymbol{x}'$ ($\boldsymbol{l} = \boldsymbol{E}^T\boldsymbol{x}'$)
- $\boldsymbol{E}\,\boldsymbol{e} = 0$ and $\boldsymbol{E}^T\boldsymbol{e}' = 0$
- $\boldsymbol{E}$ is singular (rank two)
- $\boldsymbol{E}$ has five degrees of freedom

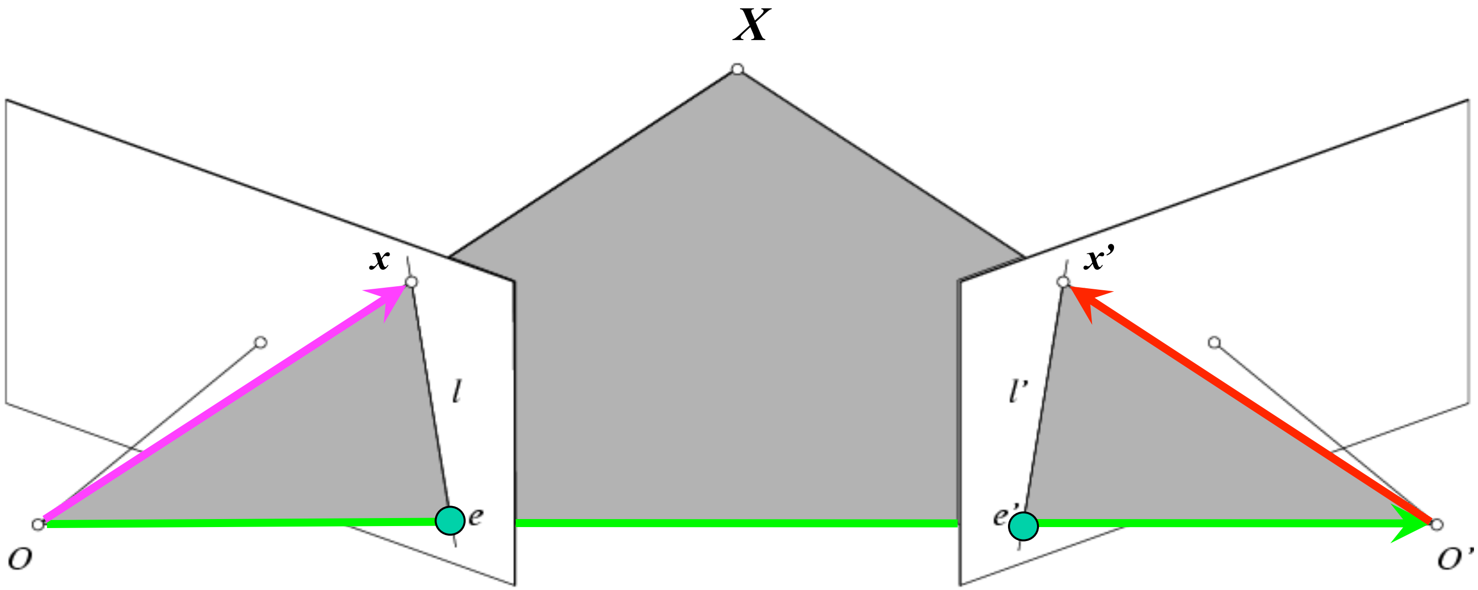
Epipolar constraint: Uncalibrated case



- The calibration matrices ***K*** and ***K'*** of the two cameras are unknown
- We can write the epipolar constraint in terms of *unknown* normalized coordinates:

$$\hat{\mathbf{x}}'^T \mathbf{E} \hat{\mathbf{x}} = 0 \qquad\qquad \hat{\mathbf{x}} = \mathbf{K}^{-1}\mathbf{x}, \quad \hat{\mathbf{x}}' = \mathbf{K}'^{-1}\hat{\mathbf{x}}'$$

Epipolar constraint: Uncalibrated case



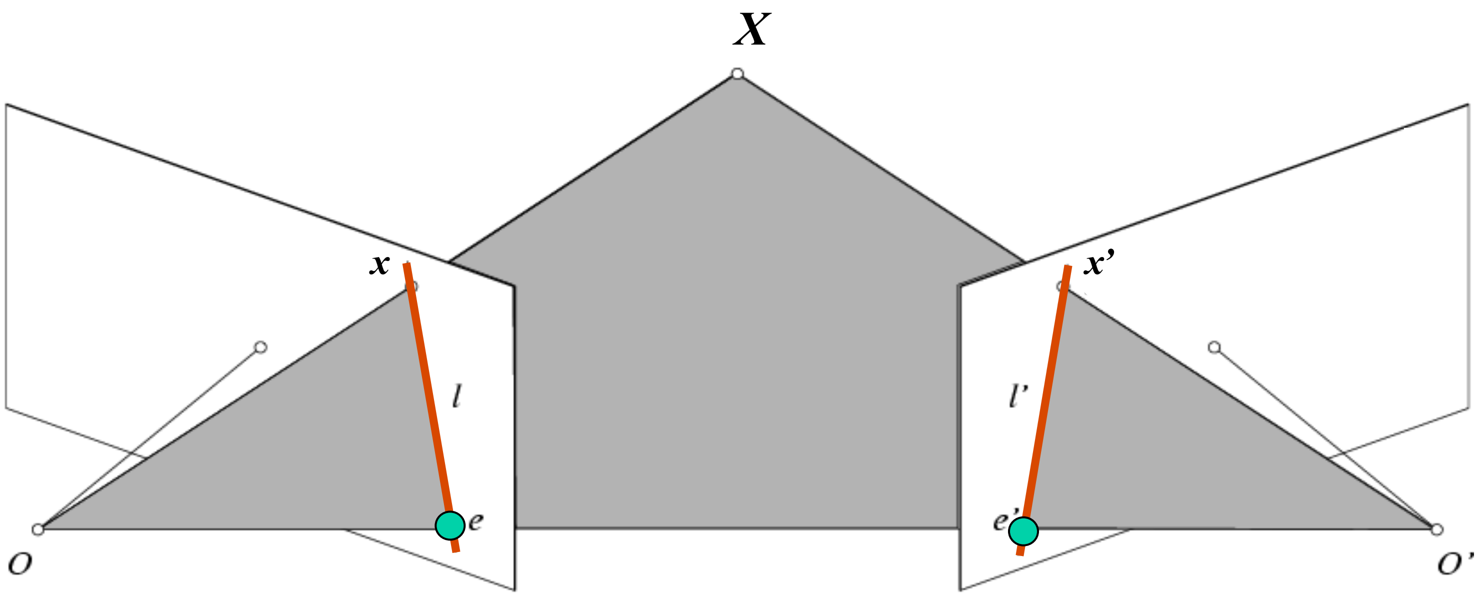
$$\hat{x}'^{T} E \hat{x} = 0 \Rightarrow x'^{T} F x = 0 \quad \text{with} \quad F = K'^{-T} E K^{-1}$$

$$\hat{x} = K^{-1} x$$

$$\hat{x}' = K'^{-1} x'$$

Fundamental Matrix
(Faugeras and Luong, 1992)

Epipolar constraint: Uncalibrated case



$$\hat{x}'^{T} E \hat{x} = 0 \quad \Rightarrow \quad x'^{T} F x = 0 \quad \text{with} \quad F = K'^{-T} E K^{-1}$$

- $\boldsymbol{F x}$ is the epipolar line associated with $\boldsymbol{x}$ ($\boldsymbol{l'} = \boldsymbol{F x}$)
- $\boldsymbol{F}^T\boldsymbol{x'}$ is the epipolar line associated with $\boldsymbol{x'}$ ($\boldsymbol{l'} = \boldsymbol{F}^T\boldsymbol{x'}$)
- $\boldsymbol{F e} = 0$ and $\boldsymbol{F}^T\boldsymbol{e'} = 0$
- $\boldsymbol{F}$ is singular (rank two)
- $\boldsymbol{F}$ has *seven* degrees of freedom

The eight-point algorithm

$$\boldsymbol{x} = (u, v, 1)^T, \quad \boldsymbol{x}' = (u', v', 1)$$

$$\begin{bmatrix} u' & v' & 1 \end{bmatrix} \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0 \quad \Rightarrow \quad \begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & u & v & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

Minimize:

$$\sum_{i=1}^{N} (\boldsymbol{x}_i'^T \boldsymbol{F} \boldsymbol{x}_i)^2$$

under the constraint

$||\boldsymbol{F}||^2 = 1$

The eight-point algorithm

- Meaning of error $\sum_{i=1}^{N} (\boldsymbol{x}_i'^T \boldsymbol{F} \boldsymbol{x}_i)^2$:

  sum of squared *algebraic* distances between points $\boldsymbol{x}'_i$ and epipolar lines $\boldsymbol{F}\boldsymbol{x}_i$ (or points $\boldsymbol{x}_i$ and epipolar lines $\boldsymbol{F}^T\boldsymbol{x}'_i$)
- Nonlinear approach: minimize sum of squared *geometric* distances

$$\sum_{i=1}^{N} \left[\mathrm{d}^2(\boldsymbol{x}_i', \boldsymbol{F}\boldsymbol{x}_i) + \mathrm{d}^2(\boldsymbol{x}_i, \boldsymbol{F}^T \boldsymbol{x}_i') \right]$$

Problem with eight-point algorithm

$$\begin{bmatrix} u'u & u'v & u' & v'u & v'v & v' & u & v \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -1$$

Problem with eight-point algorithm

250906.36	183269.57	921.81	200931.10	146766.13	738.21	272.19	198.81
2692.28	131633.03	176.27	6196.73	302975.59	405.71	15.27	746.79
416374.23	871684.30	935.47	408110.89	854384.92	916.90	445.10	931.81
191183.60	171759.40	410.27	416435.62	374125.90	893.65	465.99	418.65
48988.86	30401.76	57.89	298604.57	185309.58	352.87	846.22	525.15
164786.04	546559.67	813.17	1998.37	6628.15	9.86	202.65	672.14
116407.01	2727.75	138.89	169941.27	3982.21	202.77	838.12	19.64
135384.58	75411.13	198.72	411350.03	229127.78	603.79	681.28	379.48

$$\begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \end{bmatrix} = -1$$

Poor numerical conditioning

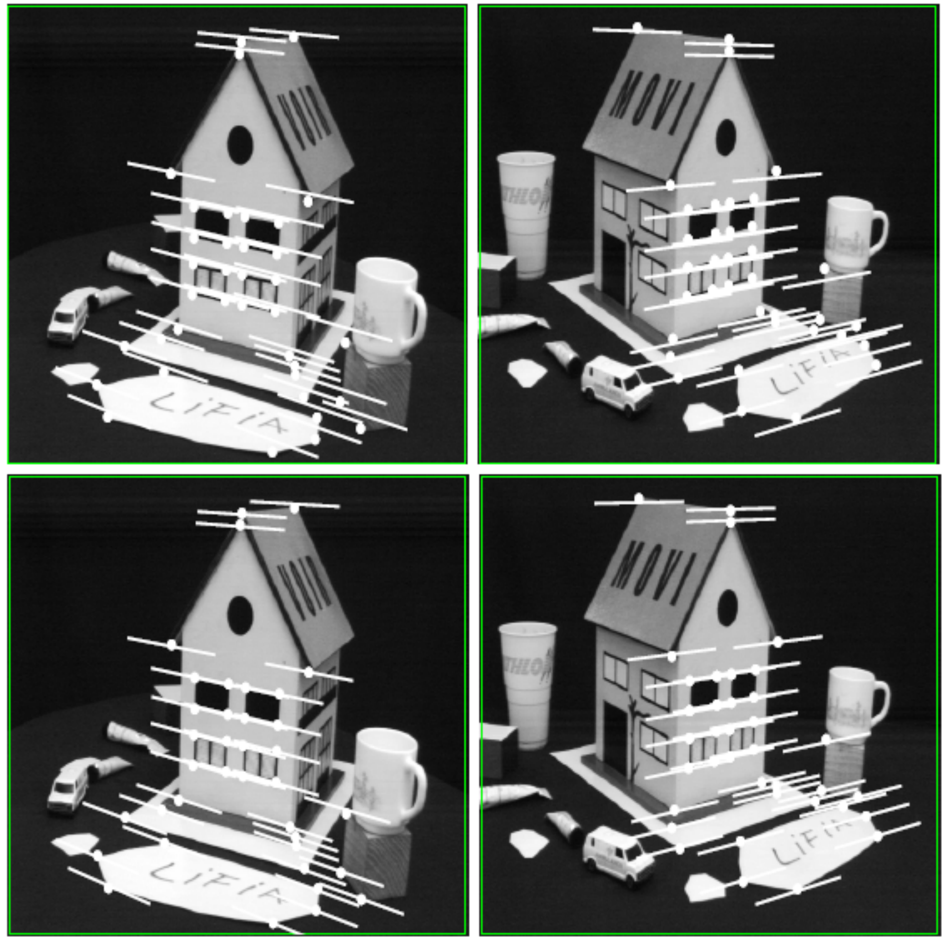
Can be fixed by rescaling the data

The normalized eight-point algorithm

(Hartley, 1995)

- Center the image data at the origin, and scale it so the mean squared distance between the origin and the data points is 2 pixels
- Use the eight-point algorithm to compute ***F*** from the normalized points
- Enforce the rank-2 constraint (for example, take SVD of ***F*** and throw out the smallest singular value)
- Transform fundamental matrix back to original units: if ***T*** and ***T'*** are the normalizing transformations in the two images, than the fundamental matrix in original coordinates is $T'^{T} F\, T$

Comparison of estimation algorithms



	8-point	Normalized 8-point	Nonlinear least squares
Av. Dist. 1	2.33 pixels	0.92 pixel	0.86 pixel
Av. Dist. 2	2.18 pixels	0.85 pixel	0.80 pixel

From epipolar geometry to camera calibration

- Estimating the fundamental matrix is known as "weak calibration"
- If we know the calibration matrices of the two cameras, we can estimate the essential matrix: $\boldsymbol{E} = \boldsymbol{K}'^{T}\boldsymbol{F}\boldsymbol{K}$
- The essential matrix gives us the relative rotation and translation between the cameras, or their extrinsic parameters