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Modal Logic
Exercise 01

Problem 1.1

1.1.1 $M, w_1 \Vdash q$

$M, w \Vdash q$ iff $w \in V(q)$ but $w_1 \notin V(q)$, so it is False.

1.1.2 $M, w_3 \Vdash \neg q$

$M, w \Vdash \neg q$ iff $w \in V(\neg q)$. $w_3 \in V(\neg q)$, so it is True.

1.1.3 $M, w_1 \Vdash p \lor q$

$M, w \Vdash p \lor q$ iff $w \in V(p)$ or $w \in V(q)$. $w_1 \in V(p)$, so it is True.

1.1.4 $M, w_1 \Vdash \Box(p \lor q)$

$$\begin{aligned} M, w_1 \Vdash \Box(p \lor q) &\Rightarrow \forall v \in V : Rw_1v \Rightarrow M, v \Vdash (p \lor q) \\ &\Rightarrow \forall v \in V : Rw_1v \Rightarrow v \in V(p) \text{ OR } v \in V(q) \\ &\text{BECAUSE } w_3 \in V \textit{ with } Rw_1w_3, \textit{ but } w_3 \notin V(p) \text{ AND } w_3 \notin V(q) \\ &\Rightarrow \text{FALSE} \end{aligned}$$

1.1.5 $M, w_3 \Vdash \Box q$

$$\begin{aligned} M, w_3 \Vdash \Box q &\Rightarrow \underbrace{\forall v \in V : Rw_1v}_{\text{FALSE}} \Rightarrow M, v \Vdash q \\ &\text{BECAUSE } \textit{the left side of the statement is } \text{FALSE } \textit{everything can follow} \\ &\Rightarrow \text{TRUE} \end{aligned}$$

1.1.6 $M, w_3 \Vdash \Box\bot$

$$\begin{aligned} M, w_3 \Vdash \Box\bot &\Rightarrow \underbrace{\forall v \in V : Rw_1v}_{\text{FALSE}} \Rightarrow M, v \Vdash \bot \\ &\text{BECAUSE } \textit{the left side of the statement is } \text{FALSE } \textit{everything can follow} \\ &\Rightarrow \text{TRUE} \end{aligned}$$

1.1.7 $M, w_1 \Vdash \Diamond q$

$$\begin{aligned} M, w_1 \Vdash \Diamond q &\Rightarrow \exists v \in V : Rw_1v \Rightarrow M, v \Vdash q \\ &\Rightarrow \exists v \in V : Rw_1v \Rightarrow v \in V(q) \\ &\text{BECAUSE } w_2 \in V : Rw_1w_2 \textit{ and } w_2 \in V(q) \\ &\Rightarrow \text{TRUE} \end{aligned}$$

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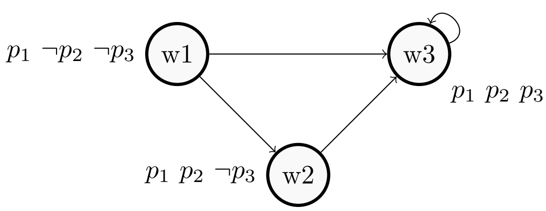
1.1.8 $M, w_1 \Vdash \Box q$

$$\begin{aligned}
M, w_1 \Vdash \Box q &\Rightarrow \forall v \in V : Rw_1v \Rightarrow M, v \Vdash q \\
&\Rightarrow \forall v \in V : Rw_1v \Rightarrow v \in V(q) \\
&\text{BECAUSE } w_3 \in V : Rw_1w_3,\ \textit{but } w_3 \notin V(q) \\
&\Rightarrow \text{FALSE}
\end{aligned}$$

1.1.9 $M, w_1 \Vdash \neg\Box\Box\neg q$

$$\begin{aligned}
M, w_1 \Vdash \neg\Box\Box\neg q &\Rightarrow \nexists v_1 \in V : Rw_1v_1 \Rightarrow M, v_1 \Vdash \Box\neg q \\
&\Rightarrow \nexists v_1 \in V : Rw_1v_1 \Rightarrow \underbrace{\forall v_2 \in V : Rv_1v_2}_{\text{FALSE}} \Rightarrow M, v_2 \Vdash \neg q \\
&\Rightarrow \nexists v_1 \in V : Rw_1v_1 \Rightarrow \text{ TRUE} \\
&\Rightarrow \text{FALSE}
\end{aligned}$$

Problem 1.5



1.5.1 $p \to \Box p$

For w_1:

$M, w_1 \Vdash p_1 \to \Diamond p_1$ holds, because w_2, w_3 have p_1.
$M, w_1 \Vdash p_2 \to \Diamond p_2$ holds, because $M, w_1 \nVdash p_2$.
$M, w_1 \Vdash p_3 \to \Diamond p_3$ holds, because $M, w_1 \nVdash p_3$.

For w_2:

$M, w_2 \Vdash p_1 \to \Diamond p_1$ holds, because w_3 has p_1.
$M, w_2 \Vdash p_2 \to \Diamond p_2$ holds, because w_3 has p_2.
$M, w_21 \Vdash p_3 \to \Diamond p_3$ holds, because $M, w_2 \nVdash p_3$.

For w_3:

$M, w_3 \Vdash p_1 \to \Diamond p_1$ holds, because we have Rw_3w_3 with $M, w_3 \Vdash p_1$.
Same for p_2 and p_3.

$\Rightarrow p \to \Diamond p$ holds for all $w \in W$.

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1.5.2 $A \to \Diamond A$

$A = ((p_2 \to \bot) \land (p_3 \to \bot))$

Then $M, w_1 \Vdash A$ holds
but $M, w_2 \nVdash A$ and $M, w_3 \nVdash A$.

Therefore $A \to \Box A$ is not True.

1.5.3 $\Box p \to p$

does not hold because:
$M, w_1 \Vdash \Box p_2$ holds but $M, w_1 \nVdash p_2$

1.5.4 $\neg p \to \Diamond\Box p$

For w_1:

$M, w_1 \Vdash \neg p_1 \to \Diamond\Box p_1$ holds, because $M, w_1 \nVdash \neg p_1$.

$M, w_1 \Vdash \neg p_2 \to \Diamond\Box p_2$ holds, because we have Rw_1w_2 with $M, w_2 \Vdash \Box p_2$, because we have Rw_2w_3 with $M, w_3 \Vdash p_2$.

$M, w_1 \Vdash \neg p_3 \to \Diamond\Box p_3$ holds, because we have Rw_1w_2 with $M, w_2 \Vdash \Box p_3$, because we have Rw_2w_3 with $M, w_3 \Vdash p_3$.

For w_2:

$M, w_2 \Vdash \neg p_1 \to \Diamond\Box p_1$ holds, because $M, w_2 \nVdash \neg p_1$.

$M, w_2 \Vdash \neg p_2 \to \Diamond\Box p_2$ holds, because $M, w_2 \nVdash \neg p_2$.

$M, w_2 \Vdash \neg p_3 \to \Diamond\Box p_3$ holds, because we have Rw_2w_3 with $M, w_3 \Vdash \Box p_3$, because we have Rw_3w_3 with $M, w_3 \Vdash p_3$.

For w_3:

$M, w_3 \Vdash \neg p_1 \to \Diamond\Box p_1$ holds, because $M, w_3 \nVdash \neg p_1$.

$M, w_3 \Vdash \neg p_2 \to \Diamond\Box p_2$ holds, because $M, w_3 \nVdash \neg p_2$.

$M, w_3 \Vdash \neg p_3 \to \Diamond\Box p_3$ holds, because $M, w_3 \nVdash \neg p_3$.

$\Rightarrow \neg p \to \Diamond\Box p$ holds for all $w \in W$.

1.5.5 $\Diamond\Box A$

$A = \neg p_1$

$M, w_1 \Vdash \Diamond\Box\neg p_1$ does not hold because this implies Rw_1w_2 with $M, w_2 \Vdash \Box\neg p_1$ but we have Rw_2w_3 with $M, w_3 \nVdash \neg p_1$.

1.5.6 $\Box \Diamond p$

For w_1:

$M, w_1 \Vdash \Box \Diamond p_1$ holds because:
For Rw_1w_2 with $M, w_2 \Vdash \Diamond p_1$ we have Rw_2w_3 with $M, w_3 \Vdash p_1$
For Rw_1w_3 with $M, w_3 \Vdash \Diamond p_1$ we have Rw_3w_3 with $M, w_3 \Vdash p_1$
$M, w_1 \Vdash \Box \Diamond p_2$ holds because:
For Rw_1w_2 with $M, w_2 \Vdash \Diamond p_2$ we have Rw_2w_3 with $M, w_3 \Vdash p_2$
For Rw_1w_3 with $M, w_3 \Vdash \Diamond p_2$ we have Rw_3w_3 with $M, w_3 \Vdash p_2$
$M, w_1 \Vdash \Box \Diamond p_3$ holds because:
For Rw_1w_2 with $M, w_2 \Vdash \Diamond p_3$ we have Rw_2w_3 with $M, w_3 \Vdash p_3$
For Rw_1w_3 with $M, w_3 \Vdash \Diamond p_3$ we have Rw_3w_3 with $M, w_3 \Vdash p_3$

For w_2:

$M, w_2 \Vdash \Box \Diamond p_1$ holds because:
For Rw_2w_3 with $M, w_3 \Vdash \Diamond p_1$ we have Rw_3w_3 with $M, w_3 \Vdash p_1$
$M, w_2 \Vdash \Box \Diamond p_2$ holds because:
For Rw_2w_3 with $M, w_3 \Vdash \Diamond p_2$ we have Rw_3w_3 with $M, w_3 \Vdash p_2$
$M, w_2 \Vdash \Box \Diamond p_3$ holds because:
For Rw_2w_3 with $M, w_3 \Vdash \Diamond p_3$ we have Rw_3w_3 with $M, w_3 \Vdash p_3$

For w_3:

$M, w_3 \Vdash \Box \Diamond p_1$ holds because:
For Rw_3w_3 with $M, w_3 \Vdash \Diamond p_1$ we have Rw_3w_3 with $M, w_3 \Vdash p_1$
$M, w_3 \Vdash \Box \Diamond p_2$ holds because:
For Rw_3w_3 with $M, w_3 \Vdash \Diamond p_2$ we have Rw_3w_3 with $M, w_3 \Vdash p_2$
$M, w_3 \Vdash \Box \Diamond p_3$ holds because:
For Rw_3w_3 with $M, w_3 \Vdash \Diamond p_3$ we have Rw_3w_3 with $M, w_3 \Vdash p_3$

$\Rightarrow \Box \Diamond p$ holds for all $w \in W$.

Problem 1.6

Show that the following are valid:

1.6.1 $\models \Box p \to \Box(q \to p)$

We will show that the negation of this statement is always False:

$$\begin{array}{c}
\neg(\Box p \to \Box(q \to p)) \\
\hline
\begin{array}{c|c}
\begin{array}{c} \Box p \\ \hline p \end{array} &
\begin{array}{c} \neg(\Box(q \to p)) \\ \hline \neg(q \to p) \\ \hline \begin{array}{c|c} q & \neg p \end{array} \end{array}
\end{array}
\end{array}$$

Because we get p and $\neg p$ to be True we can conclude that our assumption must be False.

1.6.2 $\models \Box\neg\bot$

We will show that the negation of this statement is always False:

$$\begin{array}{c} \neg(\Box\neg\bot) \\ \hline \neg\neg\bot \\ \hline \bot \end{array}$$

It is obvious that this statement is always False.

1.6.3 $\models \Box p \to (\Box q \to \Box p)$

We will show that the negation of this statement is always False:

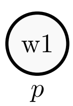
$$\begin{array}{ccc} \multicolumn{3}{c}{\neg(\Box p \to (\Box q \to \Box p))} \\ \hline \Box p & \mid & \neg(\Box q \to \Box p) \\ \hline p & & \begin{array}{ccc} \Box q & \mid & \neg\Box p \\ \hline q & & \neg p \end{array} \end{array}$$

Because We get p and $\neg p$ to be True we can conclude that our assumption must be False.

Problem 1.10

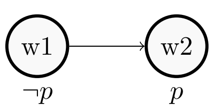
Show that none of the following formulas are valid:

1.10.1 $\Box p \to \Diamond p$



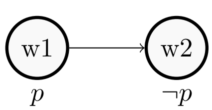
In this model $\Box p$ is clearly True. But because there is no world that can be reached from w_1 $\Diamond p$ is False and therefore $\Box p \to \Diamond p$ is also False.

1.10.2 $\Box p \to p$



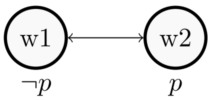
In this model $\Box p$ is clearly True in all worlds. But p is not True in all worlds. Therefore $\Box p \to p$ is False.

1.10.3 $p \to \Box\Diamond p$



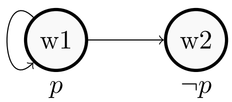
In this model is p is clearly True. But world w_2 has no following world and therefore $\Diamond p$ is False and therefore the statement is not valid.

1.10.4 $\Box p \rightarrow \Box\Box p$



In this model $\Box p$ is True but we cannot reach a world from w_2 where p is True and therefore $\Box\Box p$ is False.

1.10.5 $\Diamond p \rightarrow \Box \Diamond p$

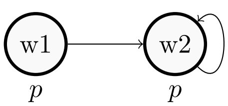


In this model $\Diamond p$ is True because we can reach w_1 from w_1. But there is no following world for w_2 and therefore $\Box \Diamond p$ is False.

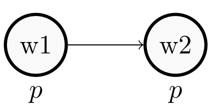
Problem 1.13

For each of the following schemes find a model M such that every instance of the formula is true in M:

1.13.1 $p \rightarrow \Diamond \Diamond p$



1.13.2 $\Diamond p \rightarrow \Box p$



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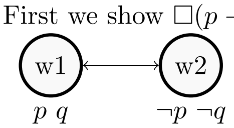
Problem 1.14

Show that $\Box(A \wedge B) \models \Box A$.
Let $\mathrm{M} = \langle W, R, V \rangle$ be a model and $w \in W$. Then we have:

$$\begin{aligned}
\Box(A \wedge B) \models \Box A \;&\Rightarrow\; \forall w \in W : M, w \Vdash \Box(A \wedge B) \\
&\Rightarrow\; \forall w' \in W : Rww' \Rightarrow M, w' \Vdash (A \wedge B) \\
&\Rightarrow\; w' \in V(A \wedge B) \\
&\Rightarrow\; w' \in V(A) \text{ AND } w' \in V(B) \\
&\Rightarrow\; M, w' \Vdash A \\
&\text{BECAUSE } w' \textit{ is arbitrary}: \\
&\Rightarrow\; \models \Box A \qquad\qquad \Box
\end{aligned}$$

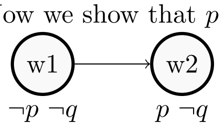
Problem 1.15

Show that $\Box(p \to q) \not\models p \to \Box q$ and $p \to \Box q \not\models \Box(p \to g)$.
First we show $\Box(p \to g) \not\models p \to \Box q$:



$\Box(p \to q)$ is True for w_1 and w_2 but $p \to \Box q$ is False for w_1.

Now we show that $p \to \Box q \not\models \Box(p \to g)$:



$p \to \Box q$ is True for w_1 and w_2 but $\Box(p \to q)$ is False for w_1.