

# Modal Logic

## 1. Syntax and Semantics of Normal Modal Logics

### 1.3 Simultaneous Substitution

$$\Box p_1 \wedge \neg p_2$$

$$\begin{aligned}\Box p \wedge q &= (\Box p) \wedge q \\ \neg p &\Rightarrow (p \rightarrow \perp)\end{aligned}$$

$$\begin{aligned}p_1 &= \Diamond p_2 \vee \Diamond \neg p_2 \\ \Box(\Diamond p_2 \vee \Diamond \neg p_2) &\wedge \neg p_2\end{aligned}$$

$$\begin{aligned}p_1 \rightarrow \Box(p_1 \wedge p_2) &[\Diamond(p_2 \rightarrow p_2) / p_1 \\ &\neg \Box p_1 / p_2] \\ &\equiv \Diamond(p_2 \rightarrow p_1) \rightarrow \Box(\Diamond(p_2 \rightarrow p_1) \wedge \neg \Box p_1)\end{aligned}$$

The substitution is simultaneous.

### 1.4 Relational Models

$M, w \models A$ , means  $A$  is true in the world  $w$  in the model  $M$  (Truth relation)

TO 7.  $A \equiv \Box B$  means iff  $B$  is TRUE in every accessible world

TO 8.  $A \equiv \Diamond B$  means iff  $B$  is TRUE in at least one accessible world

### 1.5 Truth at a World

$\Box$  and  $\Diamond$  are dual operators.

### 1.7 Validity

$$\models A \Rightarrow \models \Box A$$

BUT: this does not imply  $\models A \rightarrow \Box A$

## 1.8 Tautological Instances

$$\begin{array}{l} p \vee \neg p \quad [\Box A / p] \\ \Rightarrow \Box A \vee \neg \Box A \end{array}$$

## 1.9 Schemas and Validity

$$\begin{array}{l} p \vee \neg p \quad [\Box A / p] \quad (\text{characteristic formula}) \\ \Box A \vee \neg \Box A \quad (\text{instance}) \end{array}$$

Instances have the "same shape" as the characteristic formula.

$$\begin{array}{l} \Box \Box p \rightarrow \Box p \\ w_1 \longrightarrow w_2 \longrightarrow w_3 \\ \Box \Box p \quad \Box p \quad p \\ \neg \Box p \quad p \end{array}$$

$$\begin{array}{l} \Box \diamond p \rightarrow \diamond \Box p \\ w_1 \longrightarrow w_2 \longrightarrow w_3 \\ \Box \diamond p \quad \diamond p \quad p \\ \neg \diamond \Box p \neg \Box p \\ \hookrightarrow w_4 \\ \neg p \end{array}$$

## 1.10 Entailment

$$\begin{array}{l} \models A \quad A \text{ is valid} \\ C \models A \quad A \text{ is valid in } C \\ \Gamma \models A \quad \Gamma \text{ entails } A \end{array}$$