Energy Minimization and an Introduction to the Bayesian Framework Problem Set 3

Computer Vision 2020
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1 Bayesian Framework

1. **Maximum a Posteriori** Suppose that g is a Gaussian random variable such that $g \sim \mathcal{N}(\mu, \Sigma)$ and that we observe m iid samples $g_1, \ldots, g_m$ of g. Let us assume that the parameters μ and Σ are independent random variables with Gaussian distribution (the prior distribution) with an extremely large covariance, so that we can claim that $p(\mu) \simeq \text{const}$ and $p(\Sigma) \simeq \text{const}$. Compute the Maximum a Posteriori solution for μ and Σ.
2. **Conditional mean (challenging)** Suppose that g is a Gaussian random variable such that $g \sim \mathcal{N}(\mu, \Sigma)$ and that we observe m iid samples $g_1, \ldots, g_m$ of g. Assume that Σ is a fixed parameter and that μ is a random variable with a Gaussian distribution (the prior distribution) with an extremely large covariance, so that we can claim that $p(\mu) \simeq \text{const}$. Compute the Conditional Mean estimate of the parameters μ.

2 Optimization and Regularization

1. **Energy minimization**
   Show that the solution to the following energy minimization problem
   $$\arg\min_u |Au - f|^2 \tag{1}$$
   where $A \in \mathbf{R}^{n\times n}$ and $u, f \in \mathbf{R}^n$ is
   $$u = \left(A^\top A\right)^{-1} A^\top f. \tag{2}$$
2. **Energy minimization with regularization**
   Show that the solution to the following energy minimization problem
   $$\arg\min_u |Au - f|^2 + \lambda |u|^2 \tag{3}$$
   where $A \in \mathbf{R}^{n\times n}$, $u, f \in \mathbf{R}^n$ is
   $$u = \left(A^\top A + \lambda I\right)^{-1} A^\top f. \tag{4}$$
3. **Energy minimization 2**
   Find the solution to the following energy minimization problem
   $$\arg\min_{u,\sigma} \frac{|Au - f|^2 + \epsilon}{\sigma} + \lambda \log \sigma. \tag{5}$$

4. **The role of regularization (challenging)**

   Suppose that the following energy is provided

$$\hat{x} = \arg\min_x |Ax - b|^2 + \lambda |x|^2 \tag{6}$$

   with a given matrix A, a vector b and regularization parameter λ. Analyze how the solution of this minimization problem varies with λ by using the singular value decomposition of A.

5. **Discretization of the energy [Python implementation available]**

   Consider the following discretized energy for a 1D denoising problem

$$\min_u \sum_{i=1}^{N-1} |u[i+1] - u[i]| + \frac{\lambda}{2} \sum_{i=1}^{N} (u[i] - f[i])^2 \tag{7}$$

   where f is a given discrete signal and $\lambda > 0$ is some regularization parameter. Compute the gradient and write the gradient descent algorithm. Implement it in Python and test it with some signals f.

6. **Gauss/Gauss-Seidel/SOR [Python implementation available]**

   Consider the linear system corresponding to the gradient of this energy

$$\min_u \sum_i \frac{1}{2} |\nabla u[i]|^2 + \frac{\lambda}{2} \sum_i (u[i] - f[i])^2 \tag{8}$$

   where f is a given 1D signal and $\lambda > 0$ is a regularization parameter. Compute the inverse of the linear system matrix by using a direct inverse, Gauss' algorithm, Gauss-Seidel's algorithm and the Successive Over-Relaxations algorithm. Implement these methods in Python and test with different f signals.

7. **Iterative linearization [Python implementation available]**

   Consider the energy

$$\min_u \sum_i \frac{1}{2} |\nabla u[i]|^2 + \frac{\lambda}{2} \sum_i \log \left[(u[i] - f[i])^2 + \epsilon \right] \tag{9}$$

   where f is a given 1D signal and $\lambda, \epsilon > 0$ are regularization parameters. Discretize the derivatives via forward differences, compute the gradient equations, linearize them around some solution u^t at iteration time t, and solve the corresponding linear system by using any of the solver seen in class (Gauss, Gauss-Seidel, or SOR). Write all as an iterative algorithm. Implement it in Python and test it with some signals f.

3 Shading

1. **Normal integration [Python implementation available]**

   Suppose that the normal map $\mathbf{n}$ of a depth map d is given. Write the equation that relates the depth map to the normal map and then write an algorithm to reconstruct the depth map from the normal map.

2. **Integrability and shading [Python implementation available]**

   Suppose that the image of a Lambertian object is made available. The illumination of the scene is due to a far point light source of known orientation $s = [s_1 \; s_2 \; s_3]^T \in S^2$ (a vector in the unit sphere) and intensity $L = 1$. We also assume that the albedo ρ of the object is constant and equal to 1. Given the shading model $I(x, y) =< n(x, y), s >$, where $n(x, y)$ is the normal vector at the pixel (x, y) on the surface, reconstruct the normal map n. Make use of the integrability constraint and use the simple parametrization in p and q. Write Python code to perform the reconstruction.

4 Photometric Stereo

1. Under certain simplifying assumptions, you can write that the intensity of a particular point P on an object illuminated by a single point light source as $\mathbf{I}_P = \rho_P \mathbf{N}_P \cdot \mathbf{S}$ where ρ_P is the albedo at a point P on the object surface, $\mathbf{N}_P$ is the normal vector at P and $\mathbf{S}$ is the light source direction expressed as a unit vector. We are interested in computing the albedo and the normal vector at P from the image intensities. You can change the direction of the light source and acquire multiple intensity measurements.
   (a) What is the minimum number of images you need to compute ρ and $\mathbf{N}$? Are there any restrictions that the measurement process must satisfy?
   (b) Show the steps involved in solving for ρ and $\mathbf{N}$ provided that you have exactly the required minimum number of images. Write down the equations involved.
   (c) Is it advantageous to have more than the minimum number of images? If yes, why? How would your solution change?
2. The relationship between a 3D point at world coordinates (X, Y, Z) and its corresponding 2D pixel at image coordinates (u, v) can be defined as a projective transformation, i.e. a 3×4 camera projection matrix P.
   - How many degrees of freedom does the projection matrix P have? Briefly justify your answer.
3. Consider a plane with a Lambertian surface defined by the equation $z = 0$. The surface has constant albedo $\rho = 1$ and is illuminated by a point light source. The plane is imaged by an orthographic camera. The light source is positioned at the 3D point $\begin{bmatrix} 5 & 5 & 10 \end{bmatrix}^\top$. Derive equations to prove that the brightest point on the surface is at the 3D point $\begin{bmatrix} 5 & 5 & 0 \end{bmatrix}^\top$.
   Hint: Use the Lambertian shading model. However, this case is different from the classic photometric stereo setting, where light sources are located at infinity. In this case the light source is close to the surface, therefore the light direction changes at each point on the plane.