UNIVERSITÄT BERN

Applied Optimization
Exercise 1 - Convex Sets and Convex Functions

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Hand-in instructions:

Please hand-in **only one** compressed file named after the following convention: `Exercise`*n*`-GroupMemberNames.zip`, where n is the number of the current exercise sheet. This file should contain:

- **Only** the files you changed (headers and source). It is up to you to make sure that all files that you have changed are in the zip.
- A `readme.txt` file containing a description on how you solved each exercise (use the same numbers and titles) and the encountered problems.
- Other files that are required by your `readme.txt` file. For example, if you mention some screenshot images in `readme.txt`, these images need to be submitted too.
- Submit your solutions to ILIAS before the submission deadline.

Convex Sets (5 pts)

Example sets (1 pt)

Sketch the following sets in $\mathbb{R}^2$

1. $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -0.5 \\ -0.5 \end{pmatrix}\right\}$
2. $\operatorname{span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ -0.5 \end{pmatrix}\right\}$
3. $\operatorname{aff}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$
4. $\operatorname{conv}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \end{pmatrix}\right\}$

Convexity (1 pt)

Let $C \in \mathbb{R}^n$ be a convex set, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbb{R}$ satisfy $\theta_i \geq 0, \theta_1 + \ldots + \theta_k = 1$. Show that $\theta_1 x_1 + \ldots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) Hint. Use induction on k.

Linear Equations (1 pt)

Show that the solution set of linear equations $\{x | Ax = b\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ is an affine set.

Linear Inequations (1 pt)

1. Show that the solution set of linear inequations $\{x | Ax \preceq b, Cx = d\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{k \times n}$ and $d \in \mathbb{R}^k$ is a convex set. Here $\preceq$ means componentwise less or equal.
2. Is it an affine set?

Voronoi description of halfspace (1 pt)

Let a and b be distinct points in $\mathbb{R}^n$. Show that the set of all points that are closer (in Euclidean norm) to a than b, i.e., $x | \|x - a\|^2 \leq \|x - b\|^2$, is a halfspace. Describe it explicitly as an inequality of the form $c^T x \leq d$. Draw a picture.

Convex Functions (5 pts)

Convexity Test (5 pts)

By definition, a function $f : \mathbb{R}^n \rightarrow R$ is convex if $\mathbf{dom} f$ is a convex set and if for all $x, y \in \mathbf{dom} f$, and θ with $0 \leq \theta \leq 1$, we have

$$f(\theta x + (1 - \theta) y) \leq \theta f(x) + (1 - \theta) f(y).$$

One way to check is to prove theoretically, the other straight forward way is to run a brutal-force search until you find a counter example. In this task, you are requested to implement such a search algorithm to check if the following functions are convex:

1. $h(x, y) = (y - x^2)^2 + cos^2(4 * y) * (1 - x)^2 + x^2 + y^2$, with $x, y \in \mathbb{R}$.

2. $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T A\mathbf{x} + \mathbf{b}^T\mathbf{x} + c$, with $A \in \mathbb{R}^{n\times n}$ being positive semi-definite and $\mathbf{b}, \mathbf{x} \in \mathbb{R}^n$.

Function 1 and 2 are already implemented in `FunctionNonConvex2D.hh` and `FunctionQuadraticND.hh` in the last exercise. The idea of the search algorithm is to randomly sample pairs of points on the domain and check if the convex condition as described is satisfied. You need to implement the function called `isConvex(...)` in the file `ConvexFunctionTest.hh`. The algorithm should return a pair of points that violates the condition or keep searching. For the function that the algorithm cannot find an answer, you need to prove if it is convex or not.

Convex Illumination Problem (3 pts (bonus))

Show that the solution $p^* = (p_1^*, p_2^*, \ldots, p_n^*)^T \in \mathbb{R}^n$ of the non-convex illumination problem from the lecture

$$\begin{array}{ll} \text{minimize} & \max\limits_{k=1\ldots m} |\log I_k - log I_{des}| \\ \text{subject to} & 0 \le p_j \le p_{max}, \quad j = 1 \ldots n \end{array}$$

with $I_k = \sum_{j=1}^m a_{kj} p_j$ for geometric constants $a_{jk} \in \mathbb{R}$, a constant desired illumination $I_{des} \in \mathbb{R}$ and an upper bound $p_{max} \in \mathbb{R}$ on the lamp power, is identical to the solution of the following equivalent (convex) problem

$$\begin{array}{ll} \text{minimize} & \max\limits_{k=1\ldots m} h(I_k / I_{des}) \\ \text{subject to} & 0 \le p_j \le p_{max}, \quad j = 1 \ldots n \end{array}$$

with $h(u) = \max\{u, 1/u\}$.