Modal Logic

Important Information:

Exercises: nenad.savic@inf.unibe.ch
Exam: Tuesday, December 17, 2019
Lecture starts 9:20
Exercises start in two weeks
Next week no lecture
mcs.unibnf.ch request for academia access

Chapter 1

Syntacs and Semantics of Normal Modal Logics

1.1 Introduction

New Connectives: $\Box$ and $\Diamond$
$\Box$A means A is necessary
$\Diamond$A means A is possible

$\Box$A holds if A is true in all possible worlds
$\Diamond$A holds if A is true in some possible worlds

$\Box$ and $\Diamond$ are dual operators
$\Diamond$A holds if $\neg\Box\neg$A

$\Box$ and $\Diamond$ are intensional operators (not possible to calculate if $\Box$A truth-value from the truth value of A)... in contrast to extensional operators
A $\wedge$ B $\rightarrow$ A and B

Epistic:
$\Box$A means A **is known** or A **is believed**

Temporal:
$\Box$A means **always A**
$\Diamond$A means **eventually** A
$\circ$A means **in the next world is** A

Deontic:
$\Box$A means A **is obligatory**
$\Diamond$A means A **is permitted**

Proof Theoretic:
$\Box$A means A **is provable**

Basic principles in modal logic:
$\Box A \land \Box(A \to B) \to \Box B$
$\Box(A \to B) \land \Box A \to \Box B$
$\left.\begin{smallmatrix}\Box A \to A \\ \Box A \to \Box\Box A\end{smallmatrix}\right\}$ *depends on the definition of* $\Box$
If A is provable, so is $\Box$A (**Neccessitation**, you can proof A in every world, so $\Box$A holds)

1.2 Lecture notes: Boxes and Diamonds

1.2.1 Relations

Special properties (pages 178ff.):

Reflexivity:
A relation R $\subseteq X^2$ is *reflexive* iff, for every x $\in$ X, R_{xx}.

Transitivity:
A relation R $\subseteq X^2$ is *transitive* iff, whenever R_{xy} and R_{yz}, then also R_{xz}.

Symmetry:
A relation R $\subseteq X^2$ is *symmetric* iff, whenever R_{xy}, then also R_{yx}.

Anti-Symmetry:
A relation R $\subseteq X^2$ is *antisymmetric* iff, whenever both R_{xy} and R_{yx}, then x = y (or, in other words: if x $\neq$ y then either $\neg R_{xy}$ or $\neg R_{yx}$)

Connectivity:
A relation R $\subseteq X^2$ is *connected* if for all $x, y \in X$, if x $\neq$ y, then either R_{xy} or R_{yx}.

Partial order:
A relation R $\subseteq X^2$ that is reflexive, transitive, and anti-symmetric is called a partial order.

Linear order:
A partial order that is also connected is called a *linear order*.

Equivalence relation:
A relation R $\subseteq X^2$ that is reflexive, symmetric, and transitive is called an *equivalence relation*. x and y are called *R-equivalent* if R_{xy}.

Equivalence class:
The R-equivalence class containing x, or $[x]_R$, or $[x]$ if R is clear, is defined to be the set $\{y : R_x y\}$. x is said to be the *representative* of this R-equivalence class when we write $[x]_R$.

Orders (pages 180 ff.):

Preorder:
A relation which is both reflexive and transitive is called a *preorder*.

Partial order:
A preorder which is also antisymmetric is called a *partial order*.

Linear order:
A partial order which is also connected is called a *total order* or *linear order*.

Irreflexivity:
A relation R on X is called *irreflexive* if, for all x ∈ X, $\neg R_{xx}$.

Asymmetry:
A relation R on X is called *asymmetric* if for no pair x,y ∈ X we have R_{xy} and R_{yx}.

Strict order:
A *strict order* is a relation which is irreflexive, asymmetric, and transitive.

Strict linear order:
A strict order whichh is also connected is called a *strict linear order*.

Proposition:

1. If R is a strict (linear) order on X, then $R^+ = R \cup Id_X$ is a partial order (linear order).
2. If R is a partial order (linear order) on X, then $R^- = R \setminus Id_X$ is a strict linear order.

1.2.2 Syntacs and Semantics

Formulas (pages 189 ff.)

1. A countable infinite set At_0 of propositional variables p_0, p_1 ...
2. The propositional constant for falsity $\perp$.
3. The logical connectives: $\neg$ (negation), $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (conditional)
4. Punctuation marks: (,), and the comma.

1.2.3 Axiomatic Dervations

Axioms for the Propositional Connectives (page 203)

The set of Ax_0 of *axioms* for the propositional connectives comprises all formulas of the following forms:

(D.1) $(A \wedge B) \rightarrow A$
(D.2) $(A \wedge B) \rightarrow B$
(D.3) $A \rightarrow (B \rightarrow (A \wedge B))$
(D.4) $A \rightarrow (A \vee B)$
(D.5) $A \rightarrow B \vee A$
(D.6) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
(D.7) $A \rightarrow (B \rightarrow A)$
(D.8) $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$
(D.9) $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$
(D.10) $\neg A \rightarrow (A \rightarrow B)$
(D.11) $\top$
(D.12) $\bot \rightarrow A$
(D.13) $(A \rightarrow \bot) \rightarrow \neg A$
(D.14) $\neg\neg A \rightarrow A$

Modus ponens

If B and B $\rightarrow$ A already occur in a derivation, then A is a correct inference step.

Deduction Theorem

$\Gamma \wedge \{$A$\} \vdash$ B iff $\Gamma \vdash$ A $\rightarrow$ B
$\{$D$\} \vdash$ D iff $\vdash$ D $\rightarrow$ D

Soundness

IF $\Gamma \vdash$ A then $\Gamma \models$ A

1.2.4 Tableaux

1.2.5 The Completeness Theorem

IF $\Gamma \vdash$ A then $\Gamma \models$ A is given
The proof of the other side round