# Blur Kernel

Last digit of matriculation number: $6 \Rightarrow 6 \; mod \; 4 = 2$
$\Rightarrow \; k_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

# 1.1 Discretization of E

*Finite difference approximation of the objective function E: Choose forward differences for the discretization. In particular, use eq. (6) for the Gaussian prior implementation and derive the corresponding discretization for the anisotropic prior. Write the main steps of your calculations in the report.*

## 1.1.1 Discretization of anisotropic total variation regularization term

The second term is defined as:

$$R[u] \;=\; |\nabla u|_1 \;=\; \sum_{i=0}^{m} \sum_{j=0}^{n} |\nabla u[i,j]|_1 \tag{5}$$

The analytic derivative with forward differences is defined as follows:

$$\nabla u[x] \;=\; \lim_{\epsilon \to 0} \frac{u[x+\epsilon] - u[x]}{\epsilon}$$

The grid step $\epsilon$ is set to 1, because it is the smalles possible grid step:

$$\nabla u[i,j] \;\simeq\; \begin{bmatrix} u[i+1,j] - u[i,j] \\ u[i,j+1] - u[i,j] \end{bmatrix}$$

Because we are applying the L1-norm on this derivative we can write:

$$|\nabla u[i,j]|_1 \;\simeq\; |u[i+1,j] - u[i,j]| + |u[i,j+1] - u[i,j]|$$

Now we need to make sure that the formula doesn't reach "out-of-bounds":

**1. Every pixel except the ones with $u[i,j] = u[m-1,j]$ and $u[i,j] = u[j,n-1]$**

$$\Rightarrow \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} |u[i+1,j] - u[i,j]| + |u[i,j+1] - u[i,j]|$$

**2. Every pixel with $u[i,j] = u[m,j]$ (lower edge)**

$$\Rightarrow \sum_{j=0}^{n-1} |u[m,j+1] - u[m,j]|$$

**3. Every pixel with $u[i,j] = u[j,n]$ (right edge)**

$$\Rightarrow \sum_{i=0}^{m-1} |u[i+1,n] - u[i,n]|$$

**Joined together this will give us the discretized regularization term**

$$R[u] = |\nabla u|_1 = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |u[i+1,j]-u[i,j]|+|u[i,j+1]-u[i,j]|+\sum_{j=0}^{n-1}|u[m,j+1]-u[m,j]|+\sum_{i=0}^{m-1}|u[i+1,n]-u[i,n]| \tag{7}$$

Simplifying this:

$$\begin{aligned}
R_{TV}[u] &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |u[i+1,j]-u[i,j]| + |u[i,j+1]-u[i,j]| \\
&\quad + \sum_{j=0}^{n-1}|u[m,j+1]-u[m,j]| + \sum_{i=0}^{m-1}|u[i+1,n]-u[i,n]| \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |u[i+1,j]-u[i,j]| + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|u[i,j+1]-u[i,j]| \\
&\quad + \sum_{j=0}^{n-1}|u[m,j+1]-u[m,j]| + \sum_{i=0}^{m-1}|u[i+1,n]-u[i,n]| \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |u[i+1,j]-u[i,j]| + \sum_{i=0}^{m-1}|u[i+1,n]-u[i,n]| \\
&\quad + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}|u[i,j+1]-u[i,j] + \sum_{j=0}^{n-1}|u[m,j+1]-u[m,j]| \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n} |u[i+1,j]-u[i,j]| + \sum_{i=0}^{m}\sum_{j=0}^{n-1}|u[i,j+1]-u[i,j]| \qquad (7a)
\end{aligned}$$

### 1.1.2 Simplification of term (3) $|u * k_2 - g|^2$

Taking the term (3) and simplifying it by appliying the kernel $k_2$ to it:

$$\begin{aligned}
|u * k_2 - g|^2 &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - \sum_{p=0}^{1}\sum_{q=0}^{1} k(p,q)\, u[i-p+1, j-q+1]|_2^2 \qquad (3) \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - (\frac{1}{2}\, u[i+1,j+1] + 0 + 0 + \frac{1}{2}\, u[i,j])|_2^2 \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - \frac{1}{2}\, u[i+1,j+1] - \frac{1}{2}\, u[i,j]|_2^2 \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - \frac{1}{2}\, (u[i+1,j+1] + u[i,j])|_2^2 \qquad (3a)
\end{aligned}$$

### 1.1.3 Simplification of Gaussian Prior Regularization Term (6)

$$
\begin{aligned}
R_{GP}[u] &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (u[i+1,j]-u[i,j])^2 + (u[i,j+1]-u[i,j])^2 \\
&\quad + \sum_{j=0}^{n-1}(u[m,j+1]-u[m,j])^2 + \sum_{i=0}^{m-1}(u[i+1,n]-u[i,n])^2 \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (u[i+1,j]-u[i,j])^2 + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(u[i,j+1]-u[i,j])^2 \\
&\quad + \sum_{j=0}^{n-1}(u[m,j+1]-u[m,j])^2 + \sum_{i=0}^{m-1}(u[i+1,n]-u[i,n])^2 \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} (u[i+1,j]-u[i,j])^2 + \sum_{i=0}^{m-1}(u[i+1,n]-u[i,n])^2 \\
&\quad + \sum_{i=0}^{m-1}\sum_{j=0}^{n-1}(u[i,j+1]-u[i,j])^2 + \sum_{j=0}^{n-1}(u[m,j+1]-u[m,j])^2 \\
&= \sum_{i=0}^{m-1}\sum_{j=0}^{n} (u[i+1,j]-u[i,j])^2 + \sum_{i=0}^{m}\sum_{j=0}^{n-1}(u[i,j+1]-u[i,j])^2 \qquad (6a)
\end{aligned}
$$

### 1.1.4 Full Energy Terms

**No Regularization $\lambda = 0$**

$$E_{NR}[u] = \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - \tfrac{1}{2}(u[i+1,j+1] + u[i,j])|_2^2 \qquad (I)$$

**Gaussian Prior Regularization Term**

$$
\begin{aligned}
E_{GP}[u] &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - \frac{1}{2}(u[i+1,j+1] + u[i,j])|_2^2 \\
&\quad + \sum_{i=0}^{m-1}\sum_{j=0}^{n}(u[i+1,j]-u[i,j])^2 + \sum_{i=0}^{m}\sum_{j=0}^{n-1}(u[i,j+1]-u[i,j])^2 \qquad (II)
\end{aligned}
$$

**Anisotropic Total Variation Regularization Term**

$$
\begin{aligned}
E_{TV}[u] &= \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - \frac{1}{2}(u[i+1,j+1] + u[i,j])|_2^2 \\
&\quad + \sum_{i=0}^{m-1}\sum_{j=0}^{n}|u[i+1,j]-u[i,j]| + \sum_{i=0}^{m}\sum_{j=0}^{n-1}|u[i,j+1]-u[i,j]| \qquad (III)
\end{aligned}
$$

## 1.2 Gradient Calculations

### 1.2.a Compute the gradient $\nabla_u E$ at pixels inside the image, i.e. $1 \leq i \leq m-1$ and $1 \leq j \leq n-1$

**No Regularization $\lambda = 0$**

For simplification and because this term's derivative is used at the later points of this exercise, I will call the particular term:

$$E_{NR}(u) = |u * k_2 - g|^2 \;=\; \sum_{i=0}^{m-1}\sum_{j=0}^{n-1} |g[i,j] - \frac{1}{2}\,(u[i+1,j+1] + \;u[i,j])|_2^2 \;:= D(u)$$

**Derivative of $E_{NR}(u) \;=\; D(u)$**

The terms that contain the particular $u[i,j]$ are filtered out to create a new term $D^*(u)$ with the rest of the function which will count as a constant denoted as $K$. This new function will then be derived to get the solution:

$$\begin{aligned}
D^*(u) &= (g[i,j] - \frac{1}{2}(u[i+1,j+1] + u[i,j]))^2 + (g[i-1,j-1] - \frac{1}{2}(u[i,j] + u[i-1,j-1]))^2 + K \\
\frac{\partial D^*(u)}{\partial u[i,j]} &= -(g[i,j] - \frac{1}{2}(u[i+1,j+1] + u[i,j])) - (g[i-1,j-1] - \frac{1}{2}(u[i,j] + u[i-1,j-1])) \\
&= -g[i,j] + \frac{1}{2}u[i+1,j+1] + \frac{1}{2}u[i,j] - g[i-1,j-1] + \frac{1}{2}u[i,j] + \frac{1}{2}u[i-1,j-1] \\
&= u[i,j] + \frac{1}{2}u[i+1,j+1] + \frac{1}{2}u[i-1,j-1] - g[i,j] - g[i-1,j-1] \qquad (Da)
\end{aligned}$$

**Gaussian Prior Regularization Term**

First I will derive the Gaussian prior regularization term and then add this to the derivative from the no-regularization term, in order to get the whole derivative.

**Derivative of $R_{GP}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}(u[i+1,j] - u[i,j])^2 + \sum_{i=0}^{m}\sum_{j=0}^{n-1}(u[i,j+1] - u[i,j])^2$**

In order to derive this term, the terms that matter for a particular $u[i,j]$ are "filtered" out and will create a new function $R^*_{GP}(u)$ (the $K$ which will occur is always the rest of the function which does not matter for the particular case), which is then derived to get the solution:

$$\begin{aligned}
R^*_{GP}(u) &= (u[i,j] - u[i-1,j])^2 + (u[i+1,j] - u[i,j])^2 + (u[i,j+1] - u[i,j])^2 + (u[i,j] - u[i,j-1])^2 + K \\
\frac{\partial R^*_{GP}(u)}{\partial u[i,j]} &= 2 * (u[i,j] - u[i-1,j]) - 2(u[i+1,j] - u[i,j]) - 2(u[i,j+1] - u[i,j]) + 2(u[i,j] - u[i,j-1]) \\
&= 2 * (4 * u[i,j] - u[i-1,j] - u[i+1,j] - u[i,j-1] - u[i,j+1])
\end{aligned}$$

**Derivative of $\nabla E_{GP}(u) \;=\; \nabla D(u) + \lambda \times \nabla R_{GP}(u)$**

Adding both derivatives together we will get:

$$\begin{aligned}
\nabla E_{GP}(u) &= u[i,j] \;+ \frac{1}{2}u[i+1,j+1] + \frac{1}{2}u[i-1,j-1] - g[i,j] - g[i-1,j-1] \\
&\quad + \lambda \times 2 * (4 * u[i,j] - u[i-1,j] - u[i+1,j] - u[i,j-1] - u[i,j+1]) \\
&= (8\lambda + 1) * \; u[i,j] + \frac{1}{2} * (u[i+1,j+1] + u[i-1,j-1]) \\
&\quad - 2\lambda * (u[i-1,j] + u[i+1,j] + u[i,j-1] + u[i,j+1]) - g[i,j] - g[i-1,j-1]
\end{aligned}$$

**Anisotropic Total Variation Regularization Term**

First I will derive the anisotropic total variation regularization term and then add the result to the derivative of the image data term calculated before.

**Derivative of** $R_{ATV}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}|u[i+1,j]-u[i,j]| + \sum_{i=0}^{m}\sum_{j=0}^{n-1}|u[i,j+1]-u[i,j]|$

In order to derive this term, the terms that matter for a particular $u[i,j]$ are "filtered" out and will create a new function $R^*_{ATV}(u)$ (the $K$ which will occur is always the rest of the function which does not matter for the particular case), which is then derived to get the solution:

$$\begin{aligned}
R^*_{ATV}(u) &= |u[i,j]-u[i-1,j]| + |u[i+1,j]-u[i,j]| + |u[i,j+1]-u[i,j]| + |u[i,j]-u[i,j-1]| + K \\
\frac{\partial R^*_{ATV}(u)}{\partial u[i,j]} &= sign(u[i,j]-u[i-1,j]) - sign(u[i+1,j]-u[i,j]) - sign(u[i,j+1]-u[i,j]) + sign(u[i,j]-u[i,j-1])
\end{aligned}$$

**Derivative of** $\nabla E_{ATV}(u) = \nabla D(u) + \lambda \times \nabla R_{ATV}(u)$

Adding both derivatives together we will get:

$$\begin{aligned}
\nabla E_{ATV}(u) = u[i,j] &+ \frac{1}{2}u[i+1,j+1] + \frac{1}{2}u[i-1,j-1] - g[i,j] - g[i-1,j-1] \\
&+ \lambda \times (sign(u[i,j]-u[i-1,j]) - sign(u[i+1,j]-u[i,j]) - sign(u[i,j+1]-u[i,j]) + sign(u[i,j]-u[i,j-1]))
\end{aligned}$$

### 1.2.b Compute the gradient $\nabla_u E$ at the four corners, i.e. $(i,j) \in \{(0,0);(0,n);(m,0);(m,n))\}$

**No Regularization** $\lambda = 0$

I will follow the same process as before.

**Derivative of** $E_{NR}(u) = D(u)$

The naming convention is the same:

(i) **For** $u[0,0]$**:**

$$\begin{aligned}
D^*(u) &= (g[0,0] - \frac{1}{2}(u[1,1]+u[0,0]))^2) + K \\
\frac{\partial D^*(u)}{\partial u[0,0]} &= -(g[0,0] - \frac{1}{2}(u[1,1]+u[0,0])) \\
&= \frac{1}{2}u[0,0] + \frac{1}{2}u[1,1] - g[0,0] && (D_i(u))
\end{aligned}$$

(ii) **For** $u[0,n]$**:**

$$\begin{aligned}
D^*(u) &= K \\
\frac{\partial D^*(u)}{\partial u[0,n]} &= 0 && (D_{ii}(u))
\end{aligned}$$

(iii) **For** $u[m,0]$**:**

$$\begin{aligned}
D^*(u) &= K \\
\frac{\partial D^*(u)}{\partial u[m,0]} &= 0 && (D_{iii}(u))
\end{aligned}$$

(iv) **For** $u[m,n]$**:**

$$\begin{aligned}
D^*(u) &= (g[m-1,n-1] - \frac{1}{2}(u[m,n]+u[m-1,n-1]))^2) + K \\
\frac{\partial D^*(u)}{\partial u[m,n]} &= -(g[m-1,n-1] - \frac{1}{2}(u[m,n]+u[m-1,n-1])) \\
&= \frac{1}{2}u[m,n] + \frac{1}{2}u[m-1,n-1] - g[m-1,n-1] && (D_{iv}(u))
\end{aligned}$$

**Gaussian Prior Regularization Term**

I will follow the same process as before.

**Derivative of** $R_{GP}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}(u[i+1,j]-u[i,j])^2 + \sum_{i=0}^{m}\sum_{j=0}^{n-1}(u[i,j+1]-u[i,j])^2$

The naming convention is the same:

(i) **For** $u[0,0]$**:**

$$\begin{aligned} R^*_{GP}(u) &= (u[1,0]-u[0,0])^2 + (u[0,1]-u[0,0])^2 + K \\ \frac{\partial R^*_{GP}(u)}{\partial u[0,0]} &= -2 * (u[1,0]-u[0,0]) - 2(u[0,1]-u[0,0]) \\ &= 2 * (2 * u[0,0] - u[1,0] - u[0,1]) \end{aligned}$$

**Derivative of** $\nabla E_{GP}(u) = \nabla D_i(u) + \lambda \times \nabla R_{GP}(u)$

Adding both derivatives together we will get:

$$\begin{aligned} \nabla E_{GP}(u) &= \frac{1}{2}u[0,0] + \frac{1}{2}u[1,1] - g[0,0] + \lambda \times 2 * (2 * u[0,0] - u[1,0] - u[0,1]) \\ &= (4\lambda + \frac{1}{2}) * u[0,0] + \frac{1}{2}u[1,1] - 2\lambda * (u[1,0] + u[0,1]) - g[0,0] \end{aligned}$$

(ii) **For** $u[0,n]$**:**

$$\begin{aligned} R^*_{GP}(u) &= (u[0,n]-u[0,n-1])^2 + (u[1,n]-u[0,n])^2 + K \\ \frac{\partial R^*_{GP}(u)}{\partial u[0,n]} &= 2 * (u[0,n]-u[0,n-1]) - 2(u[1,n]-u[0,n]) \\ &= 2 * (2 * u[0,n] - u[1,n] - u[0,n-1]) \end{aligned}$$

**Derivative of** $\nabla E_{GP}(u) = \nabla D_{ii}(u) + \lambda \times \nabla R_{GP}(u)$

Adding both derivatives together we will get:

$$\begin{aligned} \nabla E_{GP}(u) &= 0 + \lambda \times 2 * (2 * u[0,n] - u[1,n] - u[0,n-1]) \\ &= 2\lambda * (2 * u[0,n] - u[1,n] - u[0,n-1]) \end{aligned}$$

(iii) **For** $u[m,0]$**:**

$$\begin{aligned} R^*_{GP}(u) &= (u[m,0]-u[m-1,0])^2 + (u[m,1]-u[m,0])^2 + K \\ \frac{\partial R^*_{GP}(u)}{\partial u[m,0]} &= 2 * (u[m,0]-u[m-1,0]) - 2(u[m,1]-u[m,0]) \\ &= 2 * (2 * u[m,0] - u[m-1,0] - u[m,1]) \end{aligned}$$

**Derivative of** $\nabla E_{GP}(u) = \nabla D_{iii}(u) + \lambda \times \nabla R_{GP}(u)$

Adding both derivatives together we will get:

$$\begin{aligned} \nabla E_{GP}(u) &= 0 + \lambda \times 2 * (2 * u[m,0] - u[m-1,0] - u[m,1]) \\ &= 2\lambda * (2 * u[m,0] - u[m-1,0] - u[m,1]) \end{aligned}$$

(iv) **For** $u[m,n]$**:**

$$\begin{aligned} R^*_{GP}(u) &= (u[m,n]-u[m,n-1])^2 + (u[m,n]-u[m-1,n])^2 + K \\ \frac{\partial R^*_{GP}(u)}{\partial u[m,n]} &= 2 * (u[m,n]-u[m,n-1]) + 2(u[m,n]-u[m-1,n]) \\ &= 2 * (2 * u[m,n] - u[m,n-1] - u[m-1,n]) \end{aligned}$$

**Derivative of** $\nabla E_{GP}(u) \;=\; \nabla D_{iv}(u) + \lambda \times \nabla R_{GP}(u)$
Adding both derivatives together we will get:

$$\begin{aligned}\nabla E_{GP}(u) &= \frac{1}{2}u[m,n] + \frac{1}{2}u[m-1,n-1] - g[m-1,n-1] + \lambda \times 2 * (2 * u[m,n] - u[m,n-1] - u[m-1,n]) \\ &= (4\lambda + \frac{1}{2}) * u[m,n] + \frac{1}{2}u[m-1,n-1] - 2\lambda * (u[m,n-1] + u[m-1,n]) - g[m-1,n-1]\end{aligned}$$

**Anisotropic Total Variation Regularization Term**

I will follow the same process as before.

**Derivative of** $R_{ATV}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n} |u[i+1,j] - u[i,j]| + \sum_{i=0}^{m}\sum_{j=0}^{n-1} |u[i,j+1] - u[i,j]|$

The naming convention is the same:

(i) **For** $u[0,0]$**:**

$$\begin{aligned}R^*_{ATV}(u) &= |u[1,0] - u[0,0]| + |u[0,1] - u[0,0]| + K \\ \frac{\partial R^*_{ATV}(u)}{\partial u[0,0]} &= -sign(u[1,0] - u[0,0]) - sign(u[0,1] - u[0,0])\end{aligned}$$

**Derivative of** $\nabla E_{ATV}(u) \;=\; \nabla D_{i}(u) + \lambda \times \nabla R_{ATV}(u)$
Adding both derivatives together we will get:

$$\begin{aligned}\nabla E_{ATV}(u) &= \frac{1}{2}u[0,0] + \frac{1}{2}u[1,1] - g[0,0] + \lambda \times (-sign(u[1,0] - u[0,0]) - sign(u[0,1] - u[0,0])) \\ &= \frac{1}{2}u[0,0] + \frac{1}{2}u[1,1] - g[0,0] - \lambda \times (sign(u[1,0] - u[0,0]) + sign(u[0,1] - u[0,0]))\end{aligned}$$

(ii) **For** $u[0,n]$**:**

$$\begin{aligned}R^*_{ATV}(u) &= |u[0,n] - u[0,n-1]| + |u[1,n] - u[0,n]| + K \\ \frac{\partial R^*_{ATV}(u)}{\partial u[0,n]} &= sign(u[0,n] - u[0,n-1]) - sign(u[1,n] - u[0,n])\end{aligned}$$

**Derivative of** $\nabla E_{ATV}(u) \;=\; \nabla D_{ii}(u) + \lambda \times \nabla R_{ATV}(u)$
Adding both derivatives together we will get:

$$\begin{aligned}\nabla E_{ATV}(u) &= 0 + \lambda \times (sign(u[0,n] - u[0,n-1]) - sign(u[1,n] - u[0,n])) \\ &= \lambda \times (sign(u[0,n] - u[0,n-1]) - sign(u[1,n] - u[0,n]))\end{aligned}$$

(iii) **For** $u[m,0]$**:**

$$\begin{aligned}R^*_{ATV}(u) &= |u[m,0] - u[m-1,0]| + |u[m,1] - u[m,0]| + K \\ \frac{\partial R^*_{ATV}(u)}{\partial u[m,0]} &= sign(u[m,0] - u[m-1,0]) - sign(u[m,1] - u[m,0])\end{aligned}$$

**Derivative of** $\nabla E_{ATV}(u) \;=\; \nabla D_{iii}(u) + \lambda \times \nabla R_{ATV}(u)$
Adding both derivatives together we will get:

$$\begin{aligned}\nabla E_{ATV}(u) &= 0 + \lambda \times (sign(u[m,0] - u[m-1,0]) - sign(u[m,1] - u[m,0])) \\ &= \lambda \times (sign(u[m,0] - u[m-1,0]) - sign(u[m,1] - u[m,0]))\end{aligned}$$

(iv) **For** $u[m,n]$**:**

$$\begin{aligned}R^*_{ATV}(u) &= |u[m,n] - u[m,n-1]| + |u[m,n] - u[m-1,n]| + K \\ \frac{\partial R^*_{ATV}(u)}{\partial u[m,n]} &= sign(u[m,n] - u[m,n-1]) + sign(u[m,n] - u[m-1,n])\end{aligned}$$

**Derivative of** $\nabla E_{ATV}(u) \;=\; \nabla D_{iv}(u) + \lambda \times \nabla R_{ATV}(u)$
Adding both derivatives together we will get:

$$\nabla E_{ATV}(u) \;=\; \frac{1}{2}u[m,n] + \frac{1}{2}u[m-1,n-1] - g[m-1,n-1] + \lambda \times (sign(u[m,n]-u[m,n-1]) + sign(u[m,n]-u[m-1,n]))$$

### 1.2.c Compute the gradient $\nabla_u E$ at pixels on the upper edge, i.e. $i = 0$ and $1 \leq j \leq n-1$

**No Regularization** $\lambda = 0$

I will follow the same process as before.

**Derivative of** $E_{NR}(u) \;=\; D(u)$

The naming convention is the same:

$$\begin{aligned} D^*(u) \;&=\; (g[0,j] - \frac{1}{2}(u[1,j+1]+u[0,j]))^2 + K \\ \frac{\partial D^*(u)}{\partial u[0,j]} \;&=\; -(g[0,j] - \frac{1}{2}(u[1,j+1]+u[0,j])) \\ &=\; \frac{1}{2}u[0,j] + \frac{1}{2}u[1,j+1] - g[0,j] \qquad (Dc) \end{aligned}$$

**Gaussian Prior Regularization Term**

I will follow the same process as before.

**Derivative of** $R_{GP}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}(u[i+1,j]-u[i,j])^2 + \sum_{i=0}^{m}\sum_{j=0}^{n-1}(u[i,j+1]-u[i,j])^2$

The naming convention is the same:

$$\begin{aligned} R^*_{GP}(u) \;&=\; (u[1,j]-u[0,j])^2 + (u[0,j+1]-u[0,j])^2 + (u[0,j]-u[0,j-1])^2 + K \\ \frac{\partial R^*_{GP}(u)}{\partial u[0,j]} \;&=\; -2 * (u[1,j]-u[0,j]) - 2(u[0,j+1]-u[0,j]) + 2(u[0,j]-u[0,j-1]) \\ &=\; 2 * (3 * u[0,j] - u[1,j] - u[0,j-1] - u[0,j+1]) \end{aligned}$$

**Derivative of** $\nabla E_{GP}(u) \;=\; \nabla D(u) + \lambda \times \nabla R_{GP}(u)$
Adding both derivatives together we will get:

$$\begin{aligned} \nabla E_{GP}(u) \;&=\; \frac{1}{2}u[0,j] + \frac{1}{2}u[1,j+1] - g[0,j] + \lambda \times 2 * (3 * u[0,j] - u[1,j] - u[0,j-1] - u[0,j+1]) \\ &=\; (6\lambda + \frac{1}{2}) * u[0,j] + \frac{1}{2}u[1,j+1] - 2\lambda * (u[1,j] + u[0,j-1] + u[0,j+1]) - g[0,j] \end{aligned}$$

**Anisotropic Total Variation Regularization Term**

I will follow the same process as before.

**Derivative of** $R_{ATV}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}|u[i+1,j]-u[i,j]| + \sum_{i=0}^{m}\sum_{j=0}^{n-1}|u[i,j+1]-u[i,j]|$

The naming convention is the same:

$$\begin{aligned} R^*_{ATV}(u) \;&=\; |u[1,j]-u[0,j]| + |u[0,j+1]-u[0,j]| + |u[0,j]-u[0,j-1]| + K \\ \frac{\partial R^*_{ATV}(u)}{\partial u[0,j]} \;&=\; -sign(u[1,j]-u[0,j]) - sign(u[0,j+1]-u[0,j]) + sign(u[0,j]-u[0,j-1]) \end{aligned}$$

**Derivative of** $\nabla E_{ATV}(u) \;=\; \nabla D(u) + \lambda \times \nabla R_{ATV}(u)$
Adding both derivatives together we will get:

$$\nabla E_{ATV}(u) \;=\; \frac{1}{2}u[0,j] + \frac{1}{2}u[1,j+1] - g[0,j] + \lambda \times (sign(u[0,j]-u[0,j-1]) - sign(u[1,j]-u[0,j]) - sign(u[0,j+1]-u[0,j]))$$

### 1.2.d Compute the gradient $\nabla_u E$ at pixels on the left edge, i.e. $j = 0$ and $1 \leq i \leq m-1$

**No Regularization $\lambda = 0$**

I will follow the same process as before.

**Derivative of $E_{NR}(u) = D(u)$**

The naming convention is the same:

$$\begin{aligned} D^*(u) &= (g[i,0] - \frac{1}{2}(u[i+1,1] + u[i,0]))^2 + K \\ \frac{\partial D^*(u)}{\partial u[i,0]} &= -(g[i,0] - \frac{1}{2}(u[i+1,1] + u[i,0])) \\ &= \frac{1}{2}u[i,0] + \frac{1}{2}u[i+1,1] - g[i,0] \end{aligned} \tag{Dd}$$

**Gaussian Prior Regularization Term**

I will follow the same process as before.

**Derivative of $R_{GP}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}(u[i+1,j] - u[i,j])^2 + \sum_{i=0}^{m}\sum_{j=0}^{n-1}(u[i,j+1] - u[i,j])^2$**

The naming convention is the same:

$$\begin{aligned} R^*_{GP}(u) &= (u[i+1,0] - u[i,0])^2 + (u[i,0] - u[i-1,0])^2 + (u[i,1] - u[i,0])^2 + K \\ \frac{\partial R^*_{GP}(u)}{\partial u[i,0]} &= -2 * (u[i+1,0] - u[i,0]) + 2(u[i,0] - u[i-1,0]) - 2(u[i,1] - u[i,0]) \\ &= 2 * (3 * u[i,0] - u[i+1,0] - u[i-1,0] - u[i,1]) \end{aligned}$$

**Derivative of $\nabla E_{GP}(u) = \nabla D(u) + \lambda \times \nabla R_{GP}(u)$**

Adding both derivatives together we will get:

$$\begin{aligned} \nabla E_{GP}(u) &= \frac{1}{2}u[i,0] + \frac{1}{2}u[i+1,1] - g[i,0] + \lambda \times 2 * (3 * u[i,0] - u[i+1,0] - u[i-1,0] - u[i,1]) \\ &= (6\lambda + \frac{1}{2}) * u[i,0] + \frac{1}{2}u[i+1,1] - 2\lambda * (u[i+1,0] + u[i-1,0] + u[i,1]) - g[i,0] \end{aligned}$$

**Anisotropic Total Variation Regularization Term**

I will follow the same process as before.

**Derivative of $R_{ATV}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}|u[i+1,j] - u[i,j]| + \sum_{i=0}^{m}\sum_{j=0}^{n-1}|u[i,j+1] - u[i,j]|$**

The naming convention is the same:

$$\begin{aligned} R^*_{ATV}(u) &= |u[i+1,0] - u[i,0]| + |u[i,0] - u[i-1,0]| + |u[i,1] - u[i,0]| + K \\ \frac{\partial R^*_{ATV}(u)}{\partial u[i,0]} &= -sign(u[i+1,0] - u[i,0]) + sign(u[i,0] - u[i-1,0]) - sign(u[i,1] - u[i,0]) \end{aligned}$$

**Derivative of $\nabla E_{ATV}(u) = \nabla D(u) + \lambda \times \nabla R_{ATV}(u)$**

Adding both derivatives together we will get:

$$\nabla E_{ATV}(u) = \frac{1}{2}u[i,0] + \frac{1}{2}u[i+1,1] - g[i,0] + \lambda \times (sign(u[i,0] - u[i-1,0]) - sign(u[i+1,0] - u[i,0]) - sign(u[i,1] - u[i,0]))$$

### 1.2.e Compute the gradient $\nabla_u E$ at pixels on the right edge, i.e. $j = n$ and $1 \leq i \leq m-1$

**No Regularization** $\lambda = 0$

I will follow the same process as before.

**Derivative of** $E_{NR}(u) = D(u)$

The naming convention is the same:

$$\begin{aligned} D^*(u) &= (g[i-1,n-1] - \frac{1}{2}(u[i,n] + u[i-1,n-1]))^2 + K \\ \frac{\partial D^*(u)}{\partial u[i,n]} &= -(g[i-1,n-1] - \frac{1}{2}(u[i,n] + u[i-1,n-1])) \\ &= \frac{1}{2}u[i,n] + \frac{1}{2}(u[i-1,n-1] - g[i-1,n-1] \qquad (De) \end{aligned}$$

**Gaussian Prior Regularization Term**

I will follow the same process as before.

**Derivative of** $R_{GP}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}(u[i+1,j] - u[i,j])^2 + \sum_{i=0}^{m}\sum_{j=0}^{n-1}(u[i,j+1] - u[i,j])^2$

The naming convention is the same:

$$\begin{aligned} R^*_{GP}(u) &= (u[i,n] - u[i,n-1])^2 + (u[i,n] - u[i-1,n])^2 + (u[i+1,n] - u[i,n])^2 + K \\ \frac{\partial R^*_{GP}(u)}{\partial u[i,n]} &= 2 * (u[i,n] - u[i,n-1]) + 2(u[i,n] - u[i-1,n]) - 2(u[i+1,n] - u[i,n]) \\ &= 2 * (3 * u[i,n] - u[i,n-1] - u[i-1,n] - u[i+1,n]) \end{aligned}$$

**Derivative of** $\nabla E_{GP}(u) = \nabla D(u) + \lambda \times \nabla R_{GP}(u)$

Adding both derivatives together we will get:

$$\begin{aligned} \nabla E_{GP}(u) &= \frac{1}{2}u[i,n] + \frac{1}{2}(u[i-1,n-1] - g[i-1,n-1] + \lambda \times 2 * (3 * u[i,n] - u[i,n-1] - u[i-1,n] - u[i+1,n]) \\ &= (6\lambda + \frac{1}{2}) * u[i,n] + \frac{1}{2}(u[i-1,n-1] - 2\lambda * (u[i,n-1] + u[i-1,n] + u[i+1,n]) - g[i-1,n-1] \end{aligned}$$

**Anisotropic Total Variation Regularization Term**

I will follow the same process as before.

**Derivative of** $R_{ATV}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}|u[i+1,j] - u[i,j]| + \sum_{i=0}^{m}\sum_{j=0}^{n-1}|u[i,j+1] - u[i,j]|$

The naming convention is the same:

$$\begin{aligned} R^*_{ATV}(u) &= |u[i,n] - u[i,n-1]| + |u[i,n] - u[i-1,n]| + |u[i+1,n] - u[i,n]| + K \\ \frac{\partial R^*_{ATV}(u)}{\partial u[i,n]} &= sign(u[i,n] - u[i,n-1]) + sign(u[i,n] - u[i-1,n]) - sign(u[i+1,n] - u[i,n]) \end{aligned}$$

**Derivative of** $\nabla E_{ATV}(u) = \nabla D(u) + \lambda \times \nabla R_{ATV}(u)$

Adding both derivatives together we will get:

$$\begin{aligned} \nabla E_{ATV}(u) = \frac{1}{2} \quad & u[i,n] + \frac{1}{2}(u[i-1,n-1] - g[i-1,n-1] \\ & + \lambda \times (sign(u[i,n] - u[i,n-1]) + sign(u[i,n] - u[i-1,n]) - sign(u[i+1,n] - u[i,n])) \end{aligned}$$

### 1.2.f Compute the gradient $\nabla_u E$ at pixels on the lower edge, i.e. $i = m$ and $1 \leq j \leq n-1$

**No Regularization** $\lambda = 0$

I will follow the same process as before.

**Derivative of** $E_{NR}(u) \;=\; D(u)$

The naming convention is the same:

$$\begin{aligned}
D^*(u) &= (g[m-1,j-1] - \frac{1}{2}(u[m,j] + u[m-1,j-1]))^2 + K \\
\frac{\partial D^*(u)}{\partial u[m,j]} &= -(g[m-1,j-1] - \frac{1}{2}(u[m,j] + u[m-1,j-1])) \\
&= \frac{1}{2}u[m,j] + \frac{1}{2}(u[m-1,j-1] - g[m-1,j-1] \qquad (Df)
\end{aligned}$$

**Gaussian Prior Regularization Term**

I will follow the same process as before.

**Derivative of** $R_{GP}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}(u[i+1,j] - u[i,j])^2 + \sum_{i=0}^{m}\sum_{j=0}^{n-1}(u[i,j+1] - u[i,j])^2$

The naming convention is the same:

$$\begin{aligned}
R^*_{GP}(u) &= (u[m,j] - u[m-1,j])^2 + (u[m,j] - u[m,j-1])^2 + (u[m,j+1] - u[m,j])^2 + K \\
\frac{\partial R^*_{GP}(u)}{\partial u[m,j]} &= 2 * (u[m,j] - u[m-1,j]) + 2(u[m,j] - u[m,j-1]) - 2(u[m,j+1] - u[m,j]) \\
&= 2 * (3 * u[m,j] - u[m-1,j] - u[m,j-1] - u[m,j+1])
\end{aligned}$$

**Derivative of** $\nabla E_{GP}(u) \;=\; \nabla D(u) + \lambda \times \nabla R_{GP}(u)$

Adding both derivatives together we will get:

$$\begin{aligned}
\nabla E_{GP}(u) &= \frac{1}{2}u[m,j] + \frac{1}{2}(u[m-1,j-1] - g[m-1,j-1] + \lambda \times 2 * (3 * u[m,j] - u[m-1,j] - u[m,j-1] - u[m,j+1]) \\
&= (6\lambda + \frac{1}{2}) * u[m,j] + \frac{1}{2}(u[m-1,j-1] - 2\lambda * (u[m-1,j] + u[m,j-1] + u[m,j+1]) - g[m-1,j-1]
\end{aligned}$$

**Anisotropic Total Variation Regularization Term**

I will follow the same process as before.

**Derivative of** $R_{ATV}(u) = \sum_{i=0}^{m-1}\sum_{j=0}^{n}|u[i+1,j] - u[i,j]| + \sum_{i=0}^{m}\sum_{j=0}^{n-1}|u[i,j+1] - u[i,j]|$

The naming convention is the same:

$$\begin{aligned}
R^*_{ATV}(u) &= |u[m,j] - u[m-1,j]| + |u[m,j] - u[m,j-1]| + |u[m,j+1] - u[m,j]| + K \\
\frac{\partial R^*_{ATV}(u)}{\partial u[m,j]} &= sign(u[m,j] - u[m-1,j]) + sign(u[m,j] - u[m,j-1]) - sign(u[m,j+1] - u[m,j])
\end{aligned}$$

**Derivative of** $\nabla E_{ATV}(u) \;=\; \nabla D(u) + \lambda \times \nabla R_{ATV}(u)$

Adding both derivatives together we will get:

$$\begin{aligned}
\nabla E_{ATV}(u) &= \frac{1}{2} \quad u[m,j] + \frac{1}{2}(u[m-1,j-1] - g[m-1,j-1] \\
&\quad + \lambda \times (sign(u[m,j] - u[m-1,j]) + sign(u[m,j] - u[m,j-1]) - sign(u[m,j+1] - u[m,j]))
\end{aligned}$$