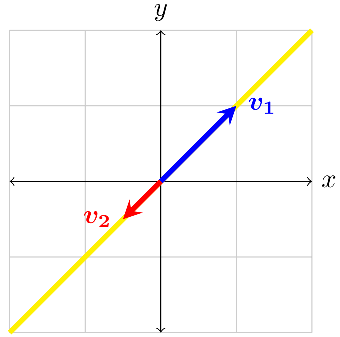
13-123-922
Elias Wipfli
13-933-262
Lorenzo *Wipfli*
16-124-836
Marcel Zauder

Applied Optimization
Exercise 01

1.1 Convex Sets

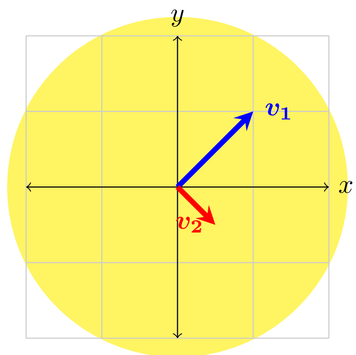
1.1.1 Example Sets

1. span$\{\binom{1}{1}, \binom{-0.5}{-0.5}\}$



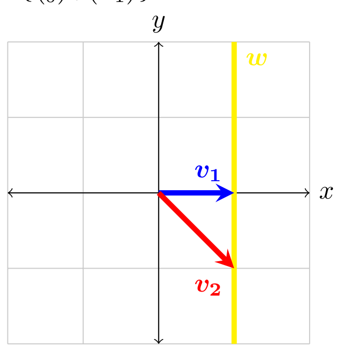
span$\{\binom{1}{1}, \binom{-0.5}{-0.5}\}$ is a line through the origin because the vectors are linear dependent.

2. span$\{\binom{1}{1}, \binom{0.5}{-0.5}\}$



span$\{\binom{1}{1}, \binom{0.5}{-0.5}\}$ is the whole $\mathbb{R}^2$ plane because the vectors are linear independent.

3. aff$\{\binom{1}{0}, \binom{1}{-1}\}$



aff$\{\binom{1}{0}, \binom{1}{-1}\}$ creates a line w with:
$w = v_1 + \beta(v_2 - v_1) = \binom{1}{0} + \beta\binom{0}{-1}$
which is a line parallel to the y-axis.

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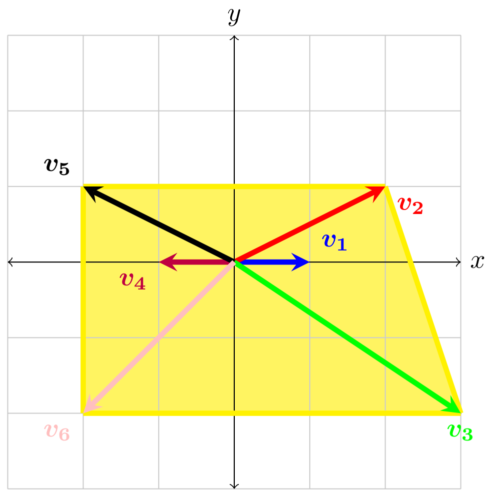
4. $\operatorname{conv}\{\binom{1}{0}, \binom{2}{1}, \binom{3}{-2}, \binom{-1}{0}, \binom{-2}{1}, \binom{-2}{-2}\}$



$\operatorname{conv}\{\binom{1}{0}, \binom{2}{1}, \binom{3}{-2}, \binom{-1}{0}, \binom{-2}{1}, \binom{-2}{-2}\}$ creates a quadrilateral with the vertices v_2, v_3, v_5 and v_6. v_1 and v_4 are inside the shape so there wouldn't be a difference in the shape if they were missing.

1.1.2 Convexity

$C \in \mathbb{R}^n$, $x_1, x_2, ..., x_k \in C$, $\theta_1, ..., \theta_k \in \mathbb{R}$, $\theta_i \geq 0$, $\theta_1 + ... + \theta_k = 1$
Show: $\theta_1 x_1 + ... + \theta_k x_k \in C$

$$\underline{\boldsymbol{k=2:}} \qquad \begin{aligned} & \theta_1 x_1 + \theta_2 x_2 \overset{?}{\in} C \\ & \theta_1 + \theta_2 = 1 \Rightarrow \theta_2 = 1 - \theta_1 \\ & \Rightarrow \theta_1 x_1 + (1-\theta_1) x_2 \\ & = \theta_1 x_1 + x_2 - theta_1 x_2 \\ & = \theta_1 (x_1 - x_2) + x_2 \in C \end{aligned}$$

$$\underline{\boldsymbol{k>2:}} \qquad u = \sum_{i=1}^{k-1} \theta_i = 1 - \theta_k$$

$$\textit{if } u = 0, \textit{ then } \sum_{i=1}^{k} \theta_i x_i = x_k \in C$$

$$\textit{else let } a_i = \frac{\theta_i}{u}, w := \sum_{i=1}^{k-1} a_i x_i \in C, \textit{ because } \sum_{i=1}^{k-1} a_i = 1 \textit{ and all } a_i \geq 0$$

$$\textit{Then } \sum_{i=1}^{k} \theta_i x_i = x_k + u \cdot (w - x_k) \textit{ is a point in the line segment } x_k \textit{ to } w, \textit{ hence in } C.$$

$\square$

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1.1.3 Linear Equations

Show that the solution set of linear equations $\{x \mid Ax = b\} \in C$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ is an affine set.

$$\begin{aligned}
&\textit{If } Ax \neq b, \textit{ then } \{x \mid Ax = b\} \textit{ is empty and empty sets are affine.}\\
&\textit{If } Ax = b:\\
&\textit{suppose } x_1, x_2 \in C,\ Ax_1 = b,\ Ax_2 = b\\
&\textit{Then for any } \theta, \textit{ we have:}\\
&A(\theta x_1 + (1-\theta)x_2)\\
&= \theta Ax_1 + (1-\theta)Ax_2\\
&= \theta b + (1-\theta)b\\
&= b\\
&\textit{which shows that the affine combination } \theta x_1 + (1-\theta)x_2 \textit{ is also in } C.
\end{aligned}$$

1.1.4 Linear Inequations

1. Proof of a convex set

   Show that the solution set of linear inequalities $\{x \mid Ax \preceq b, Cx = d\}$ with $x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $C \in \mathbb{R}^{k \times n}$, $d \in \mathbb{R}^k$ is a convex set.

   Assumption:

   $$x_1, x_2 \in \{x \mid Ax \preceq b, Cx = d\}$$
   $$Ax_1 \preceq b,\ x_1 \geq 0,\ Ax_2 \preceq b,\ x_2 \geq 0 \textit{ with } \lambda \in [0,1]:$$

   $$\begin{aligned}
   \lambda A x_1 &\preceq \lambda b\\
   (1-\lambda)Ax_2 &\preceq (1-\lambda)b\\
   \Rightarrow A(\lambda x_1 + (1-\lambda)x_2) &\preceq b\\
   \Rightarrow \qquad \lambda x_1 + (1-\lambda)x_2 &\succeq 0
   \end{aligned}$$

   Vector a and scalar b, suppose x and y satisfy:

   $$a^\top x \geq b \textit{ and } a^\top y \geq b$$

   , therefore belong to the same halfspace.

   $$\textit{For a } \lambda \in [0,1]: \qquad a^\top(\lambda x + (1-\lambda)y) \geq \lambda b + (1-\lambda)b = b$$

   , which proves that $\lambda x + (1-\lambda)y$ also belongs to the halfspace.
   Therefore a halfspace is convex and since a polyhedron is the intersecion of a finite number of halfspaces, it is also convex. □

2. Is it an affine set?

   No, it is not halfspaces themself are not affine and by intersecting them you cannot get an affine set.

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1.1.5 Voronoi description of halfspaces

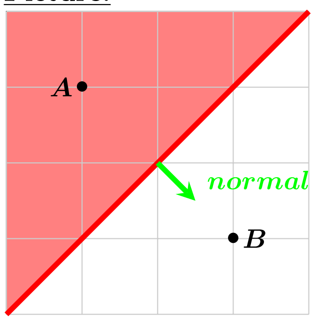
a,b distinct points in $\mathbb{R}^n$, with:

$$\{x \mid \|x-a\|_2 \leq \|x-b\|_2\}$$

Show that this is a halfspace:
A norm is alway nonnegative, so $\|x-a\|_2 \leq \|x-b\|_2$ holds if and only if $\|x-a\|_2^2 \leq \|x-b\|_2^2$, so:

$$\begin{aligned} \|x-a\|_2^2 &\leq \|x-b\|_2^2 \\ \Leftrightarrow \quad (x-a)^\top(x-a) &\leq (x-b)^\top(x-b) \\ \Leftrightarrow \quad x^\top x - 2a^\top x + a^\top a &\leq x^\top x - 2b^\top x + b^\top b \\ \Leftrightarrow \quad 2(b-a)^\top x &\leq b^\top b - a^\top a \end{aligned}$$

So c = 2(b-a) and d = $b^\top b - a^\top a$, points that are equidistant to A and B are given by a hyperplane with the normal $\frac{b-a}{|b-a|}$.
Picture:



1.2 Convex Functions

1.2.1 Convexity Test

The function isConvex() was implemented straight forward with the given convexity theorem. For the first function that is evaluated we get quite quickly the points that are not concave.
For the second function:

$$f(\theta x + (1-\theta)y) = 1/2((\theta x + (1-\theta)y)^\top A(\theta x + (1-\theta)y) + b^\top(\theta x + (1-\theta)y) + c =$$

$$1/2(\theta x^\top A\theta x + (1-\theta)y^\top A(1-\theta)y) + b^\top\theta x + b^\top(1-\theta)y + c =$$

$$1/2\ \theta^2(x^\top Ax) + 1/2\ (1-\theta)^2(y^\top Ay) + b^\top\theta x + b^\top(1-\theta)y + c$$

And:

$$\theta f(x) + (1-\theta)f(y) = \theta(1/2x^\top Ax + b^\top x + c) + (1-\theta)(1/2y^\top Ay + b^\top y + c) =$$

$$1/2\ \theta\ (x^\top Ax) + 1/2\ (1-\theta)\ (y^\top Ay) + b^\top\theta x + b^\top(1-\theta)y + \theta c + (1-\theta)c =$$

$$1/2\ \theta\ (x^\top Ax) + 1/2\ (1-\theta)\ (y^\top Ay) + b^\top\theta x + b^\top(1-\theta)y + c$$

Because we know that:

$$0 \leq \theta \leq 1$$

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We can see that:

$$1/2\ \theta^2\ (x^\top Ax) \leq 1/2\ \theta\ (x^\top Ax)$$

$$1/2\ (1-\theta)^2\ (y^\top Ay) \leq 1/2\ (1-\theta)\ (y^\top Ay)$$

The rest is the same in both formulas so:

$$f(\theta x + (1-\theta)y) \leq \theta f(x) + (1-\theta)f(y)\ \square$$

So the second function is convex.

1.3 Convex Illumination Problem

@ Holy moly, imma out of here!