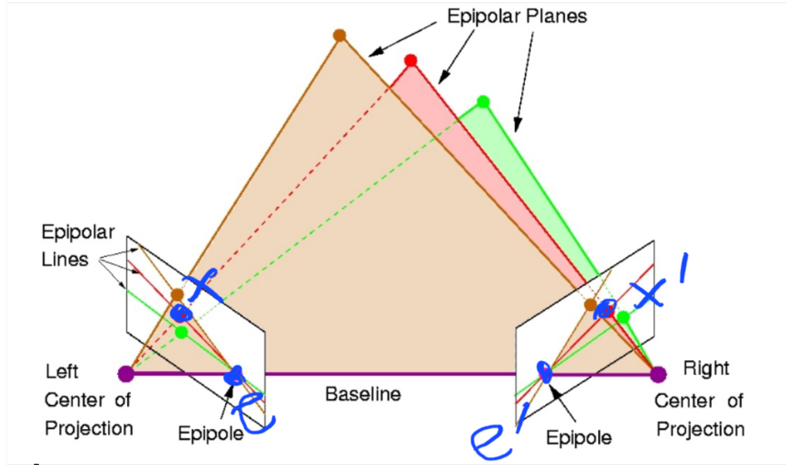


Problem Set 7 Solutions

Computer Vision 2020
University of Bern

1 Epipolar Geometry

1. In this exercise we will derive the essential and fundamental matrices for a pair of cameras related by a rigid transformation $[\mathbf{R}|\mathbf{t}]$, by looking at the common 3D point $\mathcal{P}$.

   Let $\mathbf{X}_1 = [X_1\ Y_1\ Z_1]^T \in R^3$ be the coordinates of $\mathcal{P}$ defined in the first camera coordinate system with origin in the first camera center C_1. Let $\mathbf{R}$ be the rotation matrix and $\mathbf{t}$ be the translation vector that transform the first camera coordinate system into the second camera coordinate system with origin C_2. Remember that $\mathbf{R}$ is a 3×3 orthogonal matrix with $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ and $\mathbf{t} \in R^3$. Finally, let the coordinates of the 3D point $\mathcal{P}$ in the second camera coordinate system be $\mathbf{X}_2 = [X_2\ Y_2\ Z_2]^T \in R^3$.

   (a) How do you compute the projection $\mathbf{m}_1 = [x_1\ y_1\ 1]^T$ of $\mathcal{P}$ in the first coordinate system using the coordinate of $\mathbf{X}_1$ and how do you compute the projection $\mathbf{m}_2 = [x_2\ y_2\ 1]^T$ of $\mathcal{P}$ in the second coordinate system using the coordinate of $\mathbf{X}_2$?

   Solution $\mathbf{m}_1 = \mathbf{X}_1/Z_1$ and $\mathbf{m}_2 = \mathbf{X}_2/Z_2$.

   (b) Give the relationship between $\mathbf{X}_1$ and $\mathbf{X}_2$ using $\mathbf{R}$ and $\mathbf{t}$.

   Solution $\mathbf{X}_2 = \mathbf{R}\mathbf{X}_1 + \mathbf{t}$.

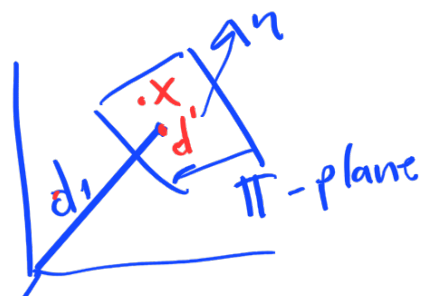
   (c) How are the projections $\mathbf{m}_1$ and $\mathbf{m}_2$ related (always taking the first image plane as coordinate system)?

   $x_1 = z_1 m_1 \quad x_2 = R x_1 + t$
   $x_2 = z_2 m_2$

   Solution $Z_2\mathbf{m}_2 = Z_1\mathbf{R}\mathbf{m}_1 + \mathbf{t}$.

   (d) Suppose that $\mathcal{P}$ lies on a plane $\mathcal{P} \in \pi$, where π has a normal vector $\mathbf{n}_1$ (expressed in the first camera coordinate system) and its distance to C_1 is d_1. Assuming that $\mathbf{n}_1$ verifies $\|\mathbf{n}_1\| = \sqrt{\mathbf{n}_1^T\mathbf{n}_1} = 1$, give the

$n_1^T (X - d') = 0$
$n_1^T X = n_1^T d'$



$n_1^T d' = |n| \cdot |d'| \cdot \cos\varphi$

$= 1 \cdot |d'| \cdot \cos 0 = d_1 \Rightarrow n_1^T X_1 = d_1$

equation of the plane π.

Solution We can use the Hessian normal form of the equation of a plane, that is $\mathbf{n}_1^T\mathbf{X} = d_1$.

(e) Use the results above, including $\mathcal{P} \in \pi$, to find a transformation of the form $\mathbf{m}_2 = T\mathbf{m}_1$.

dimensions

$$\underbrace{[3\times1]}_{m_2} = \left(\underbrace{[3\times3]}_{R} - \underbrace{[3\times1]}_{t}\underbrace{[1\times3]}_{n_1^T}\right) \cdot \underbrace{[3\times1]}_{m_1}$$

$n_1^T X_1 = d_1$

$\Rightarrow \frac{n_1^T X_1}{d_1} = 1$

Solution

$$Z_2\mathbf{m}_2 = Z_1\mathbf{R}\mathbf{m}_1 + \mathbf{t}$$

$$Z_2\mathbf{m}_2 = Z_1\mathbf{R}\mathbf{m}_1 + \mathbf{t}\frac{\mathbf{n}_1^T\mathbf{X}_1}{d_1}$$

$$Z_2\mathbf{m}_2 = Z_1\mathbf{R}\mathbf{m}_1 + Z_1\mathbf{t}\frac{\mathbf{n}_1^T\mathbf{m}_1}{d_1}$$

$$\mathbf{m}_2 = \frac{Z_1}{Z_2}\left(\mathbf{R} + \mathbf{t}\frac{\mathbf{n}_1^T}{d_1}\right)\mathbf{m}_1$$

Given two (perspective) images of a world plane, the transformation relating points in two images which correspond to the same point $\mathbf{X}$ on the work plane is a homography.

(f) Given the normalized image coordinates $\mathbf{m}_1$ and $\mathbf{m}_2$ express the transformation of two points $\mathbf{p}_1$ and $\mathbf{p}_2$ in pixel coordinates with the help of the intrinsic matrix $\mathbf{K}$.

Solution We know that $\mathbf{p}_1 = \mathbf{K}_1\mathbf{m}_1$ and $\mathbf{p}_2 = \mathbf{K}_2\mathbf{m}_2$. Substituting this information we finally get a homograph mapping pixels to pixels:

$p_1 = K_1 m_1 \Rightarrow m_1 = K_1^{-1} m_1$

$p_2 = K_2 m_2 \Rightarrow m_2 = K_2^{-1} m_2$

$$\mathbf{p}_2 = \mathbf{K}_2\frac{Z_1}{Z_2}\left(\mathbf{R} + \mathbf{t}\frac{\mathbf{n}_1^T}{d_1}\right)\mathbf{K}_1^{-1}\mathbf{p}_1 \tag{1}$$

(g) Try to find a relationship between $\mathbf{m}_1$ and $\mathbf{m}_2$ without knowing the normal of the plane passing from $\mathcal{P}$.

Solution The normal vector of the plane can be derived from the cross product $\mathbf{t} \times \mathbf{X}_2 = [\mathbf{t}]_\times\mathbf{X}_2$, therefore we have

$$\mathbf{X}_2 = \mathbf{R}\mathbf{X}_1 + \mathbf{t}$$

$$[\mathbf{t}]_\times\mathbf{X}_2 = [\mathbf{t}]_\times\mathbf{R}\mathbf{X}_1 + \underbrace{[\mathbf{t}]_\times\mathbf{t}}_{=0}$$

$$[\mathbf{t}]_\times\mathbf{X}_2 = [\mathbf{t}]_\times\mathbf{R}\mathbf{X}_1$$

Since $\mathbf{X}_2$ lies on the plane, we have $\mathbf{X}_2^T[\mathbf{t}]_\times\mathbf{X}_2 = 0$, Replacing with the normalized image coordinates we have

$$0 = Z_1Z_2\mathbf{m}_2^T[\mathbf{t}]_\times\mathbf{R}\mathbf{m}_1$$

$$0 = \mathbf{m}_2^T\mathbf{E}\mathbf{m}_1$$

$\Rightarrow X_2^T \cdot [t]_x X_2 = X_2^T [t]_x R X_1$

$X_2^T [t]_x R X_1 = 0$

$X_1 = Z_1 m_1$

$X_2 = Z_2 m_2$

$z_2 m_2^T [t]_\times R\, z_1 m_1 = z_1 z_2 \, m_2^T \underbrace{[t]_\times R}_{\text{Essential mat.}} m_1 = 0$

$E \in \mathbb{R}^{3\times 3}$

where **E** is the essential matrix.

(h) Give the relation between the points $\mathbf{p}_1$ and $\mathbf{p}_2$ in pixel coordinates.

Solution Using the intrinsic matrices **K** we have

$$\begin{aligned}
\mathbf{m}_2^T\mathbf{E}\mathbf{m}_1 &= 0 \\
(\mathbf{K}_2^{-1}\mathbf{p}_2)^T\mathbf{E}(\mathbf{K}_1^{-1}\mathbf{p}_1) &= 0 \\
\mathbf{p}_2^T(\mathbf{K}_2^{-T}\mathbf{E})\mathbf{K}_1^{-1}\mathbf{p}_1 &= 0 \\
\mathbf{p}_2^T\mathbf{F}\mathbf{p}_1 &= 0
\end{aligned}$$

$F \in \mathbb{R}^3$

where **F** is the fundamental matrix.

2. The epipolar geometry is the intrinsic projective geometry between two views I and I'. It depends only on the cameras' intrinsic parameters and their relative pose (rotation and translation between the camera centers). The **Fundamental Matrix F** is a 3×3 matrix with rank$(\mathbf{F}) = 2$.

dimensions

(a) How is the fundamental matrix **F** related to pairs of corresponding points x, x' in the two images?

$(1\times 3)(3\times 3)(3,1) = (1\times 1)$

transpose property

$(ABC)^T = C^T B^T A^T$

Solution $\mathbf{x'}_i^T\mathbf{F}\mathbf{x}_i = 0$.

(b) How are the fundamental matrices **F**, going from I and I', and **F'**, going from I' and I, related?

Solution $\mathbf{F'} = \mathbf{F}^T$.

$(x_i'^T \cdot F \cdot x_i)^T = x_i^T F^T \cdot x_i' \Rightarrow F^T = F'$

(c) What is the geometric meaning of the epipoles **e** and **e'**? How are they related to the fundamental matrix (algebraically)?

Solution Geometrically the epipoles are the points of the intersection of the line joining the camera centers (the baseline) with the image planes. Equivalently, an epipole is is the image in one view of the camera center of the other view. Algebraically, **e** is the right null space of **F**, i.e. $\mathbf{Fe} = 0$. Similarly, **e'** is the left null space of **F**, i.e. $\mathbf{e'}^T\mathbf{F} = 0$.

(d) Are the epipoles always visible in the two views?

Solution No, the intersection can happen outside the image or even at infinity, when the two image planes are parallel and the translation between their centers is perpendicular to the image planes.

(e) How can one determine the epipolar line **l'** passing trough a given point **x'**?

Solution By finding the epipole and using $\mathbf{l'} = \mathbf{e'} \times \mathbf{x'}$.

$l^T p = [a \;\; b \;\; c]\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = ax + by + c = 0$

if line l is passing through 2 points p_1, p_2 (cross product

$\Rightarrow l^T p_1 = 0 \wedge l^T p_2 = 0 \Rightarrow l \propto p_1 \times p_2$

p_1, p_2 are in homogeneous coordinates

(f) What is the effect of applying the fundamental matrix $\mathbf{F}$ to a point $\mathbf{x}$?

Solution The fundamental matrix maps a point in one image to a line in the second image. $\mathbf{l'} = \mathbf{F}\mathbf{x}$ and $\mathbf{l} = \mathbf{F}^T\mathbf{x'}$.

$x'^T F x = 0 \Rightarrow x'^T l' = 0 \Rightarrow l' = Fx$; $(x'^T F x)^T = x^T F^T x' = 0$

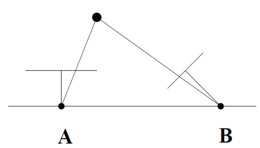
$\Rightarrow \underline{l = F^T x'}$

2 Calibrated Reconstruction

For a given essential matrix E with singular value decomposition $E = U\text{diag}(1,1,0)V^T$ and first camera matrix $P = [I \mid 0]$, there are four possible choices for the second camera matrix P', namely

$$P' = [UWV^T \mid u_3] \quad \text{or} \quad [UWV^T \mid -u_3] \quad \text{or} \quad [UW^TV^T \mid u_3] \quad \text{or} \quad [UW^TV^T \mid -u_3],$$

where W is a specific orthogonal matrix and u_3 is the third column of U. The Figure below illustrates the geometric interpretation of the first solution. Draw the three missing configurations.



(a)

(b)

(c)

(d)

Solution
See page 260 of Hartley and Zisserman, Multiple View Geometry in Computer Vision, Second Edition, 2004.
`https://www.robots.ox.ac.uk/~vgg/hzbook/hzbook2/HZepipolar.pdf`
(a) points in front of both views; (b) points behind both A and B; (c) points in front of B but behind A; (d) points in front of A but behind B.
Author: Adrian Wälchli

2.c) the first camera's pinhole is at the origin $O = \begin{pmatrix}0\\0\\0\end{pmatrix}$

It's image in image 2 is at

$X_2 = R X_1 + t = R \cdot O + t = t$

$\boxed{e_2^T \ell_2 = 0}$ where $\ell_2 = E m_1$; we are claiming that for all ℓ_2, $e_2^T \ell_2 = 0$

$e_2^T E m_1 = 0$ for all m_1

$$e_2^T E m_1 = t^T E m_1 = t^T [t]_\times R m_1$$
$$= t^T (t \times (R m_1)) = 0$$

thus e_2 lies on every epipolar line in image 2.

$x'^T E x = 0$

$x'^T \cdot \ell' = 0$ where ℓ' is epipolar line

we also know that $e'^T \cdot \ell' = 0$

$\Rightarrow \underbrace{e'^T \cdot E}\, x = 0$

$e'^T \cdot E = 0$