13-123-922
Elias Wipfli
13-933-262
Lorenzo Wipfli
16-124-836
Marcel Zauder

Applied Optimization
Exercise 03

3.1 Vector Composition

First we can compute the second derivation of this function:

$$\begin{aligned} f(x) &= h(g_1(x), ..., g_k(x)) = h(g(x)) \\ f''(x) &= g'(x)^T \bigtriangledown^2 h(g(x)) g'(x) + \bigtriangledown h(g(x))^T g''(x) \end{aligned}$$

For convexity the second derivation must be $f''(x) \geq 0$.

1. *h is convex and nondecreasing in the i-t h argument, and g_i is convex:*
   Therefore we have for the second derivation:

   $$f''(x) = \underbrace{g'(x)^T \underbrace{\bigtriangledown^2 h(g(x))}_{(1)} g'(x)}_{(2)} + \underbrace{\underbrace{\bigtriangledown h(g(x))}_{(3)}^T \underbrace{g''(x)}_{(4)}}_{(5)}$$

   (1) $\bigtriangledown^2 h(g(x)) \geq 0$, because h is convex
   (2) Therefore $g'(x)^T \bigtriangledown^2 h(g(x)) g'(x) \geq 0$
   (3) $\bigtriangledown h(g(x)) \geq 0$, because h is nondecreasing
   (4) $g''(x) \geq 0$, because all g_i are convex
   (5) $\bigtriangledown h(g(x))^T g''(x) \geq 0$, because of (3) and (4).

   Therefore $f''(x) \geq 0$.

2. *h is convex and nonincreasing in the i-t h argument, and g_i is concave:*
   Therefore we have for the second derivation:

   $$f''(x) = \underbrace{g'(x)^T \underbrace{\bigtriangledown^2 h(g(x))}_{(1)} g'(x)}_{(2)} + \underbrace{\underbrace{\bigtriangledown h(g(x))}_{(3)}^T \underbrace{g''(x)}_{(4)}}_{(5)}$$

   (1) $\bigtriangledown^2 h(g(x)) \geq 0$, because h is convex
   (2) Therefore $g'(x)^T \bigtriangledown^2 h(g(x)) g'(x) \geq 0$
   (3) $\bigtriangledown h(g(x)) \leq 0$, because h is nonincreasing
   (4) $g''(x) \leq 0$, because all g_i are concave
   (5) $\bigtriangledown h(g(x))^T g''(x) \geq 0$, because of (3) and (4).

   Therefore $f''(x) \geq 0$.

13-123-922
Elias Wipfli
13-933-262
Lorenzo Wipfli
16-124-836
Marcel Zauder

3. *All g_i are affine*
   Therefore $g_i(x)$ is of the form $Ax + b$. For $x, y \in \mathbb{R}^n$ and $\theta \in [0, 1]$, we have:
   Note that: $g(\theta x + (1-\theta)y) \;=\; \theta g(x) + (1-\theta)g(y)$

$$\begin{aligned} f(\theta x + (1-\theta)y) &= h(g(\theta x + (1-\theta)y)) \\ &= h(\theta g(x) + (1-\theta)g(y)) \\ h \textit{ convex} \Rightarrow \quad &\leq \theta h(g(x)) + (1-\theta)h(g(y)) \\ &= \theta f(x) + (1-\theta)f(y) \end{aligned}$$

   Therefore f is convex.
   Also we can compute the second derivative of f:

$$\bigtriangledown^2 f(x) = A^T \bigtriangledown^2 h(g(x))g'(x) = A^T \bigtriangledown^2 h(g(x))A$$

   , which is clearly always positive which implies that f is convex.

In the end we can conclude that a composition of all of those different statements will lead to a convex function. □

3.2 Linear Programming

3.2.1 Transform

We have the following optimization problem:

$$\begin{aligned} &\textbf{\textit{minimize}} && \|(2x_1 \;+\; 3x_2,\; -3x_1)^T\|_\infty \\ &\textbf{\textit{subject to}} && |\; x_1 \;-\; 2x_2 \;| \leq \; 0 \end{aligned}$$

3.2.1.1 Transform to LP

1. First thing to do is to eliminate the L_∞-Norm:

$$\begin{aligned} &\textbf{\textit{minimize (over x, t)}} && t \\ &\textbf{\textit{subject to}} && t \;\geq\; 2x_1 + 3x_2 \\ & && t \;\geq\; -3x_1 \\ & && t \;\geq\; -2x_1 - 3x_2 \\ & && t \;\geq\; 3x_1 \\ & && |\; x_1 \;-\; 2x_2 \;| \;\leq\; 0 \end{aligned}$$

2. Next we eleminate the absolute value:

$$\begin{aligned} &\textbf{\textit{minimize (over x, t)}} && t \\ &\textbf{\textit{subject to}} && t \;\geq\; 2x_1 + 3x_2 \\ & && t \;\geq\; -3x_1 \\ & && t \;\geq\; -2x_1 - 3x_2 \\ & && t \;\geq\; 3x_1 \\ & && x_1 \;-\; 2x_2 \;\leq\; 0 \\ & && 2x_2 \;-\; x_1 \;\leq\; 0 \end{aligned}$$

13-123-922
Elias Wipfli
13-933-262
Lorenzo Wipfli
16-124-836
Marcel Zauder

3.2.1.2 Transform with variables in $\mathbb{R}^+$ and no inequalities

From the last two inequalities of the LP we can conclude the following:

$$\begin{aligned} & x_1 - 2x_2 \leq 0 \ \wedge \ 2x_2 - x_1 \leq 0 \\ \Rightarrow \ & x_1 - 2x_2 = 0 \ \Leftrightarrow \ x_1 = 2x_2 \end{aligned}$$

Therefore we can put these information into the other inequalities:

$$\begin{array}{lrcl} \textbf{\textit{minimize (over x, t)}} & t & & \\ \textbf{\textit{subject to}} & t & \geq & 4x_2 + 3x_2 \ = \ 7x_2 \\ & t & \geq & -6x_2 \\ & t & \geq & -4x_2 - 3x_2 \ = \ -7x_2 \\ & t & \geq & 6x_2 \\ & x_1 & = & 2x_2 \end{array}$$

Because one of both either $7x_2$ and $-7x_2$ (respectively $6x_2$ and $-6x_2$) is positive, we can write for any $x_2 \in \mathbb{R}^+$:

$$\begin{array}{lrcl} \textbf{\textit{minimize (over x, t)}} & t & & \\ \textbf{\textit{subject to}} & t & \geq & 7x_2 \\ & t & \geq & 6x_2 \\ & x_1 & = & 2x_2 \end{array}$$

Because $7x_2$ is always greater equal to $6x_2$, we have:

$$\begin{array}{lrcl} \textbf{\textit{minimize (over x, t)}} & t & & \\ \textbf{\textit{subject to}} & t & \geq & 7x_2 \\ & x_1 & = & 2x_2 \end{array}$$

Because we want to minimize t, we can also write:

$$\begin{array}{lrcl} \textbf{\textit{minimize (over x, t)}} & t & & \\ \textbf{\textit{subject to}} & t & = & 7x_2 \\ & x_1 & = & 2x_2 \end{array}$$

Therfore we have a LP with variables in $\mathbb{R}^+$ and no inequalities.

13-123-922
Elias Wipfli
13-933-262
Lorenzo Wipfli
16-124-836
Marcel Zauder

3.2.2 Differences between general LP and its standard form

The general LP represents a convex problem with affine objectives and constraints. The feasable set is a polyhedron, which is put up by the different constraints. The optimal solution $x^\star$ will be at one of the corners of the polyhedron.
In contradiction to the general form the standard form only contains non-negative constraints for all variables. Also all constrants are expressed as equality constraints. Its optimal solution $x^\star$ will be the minimum of the f_0 function within the feasable set of non-negative variables which fulfill all constraints.

3.3 Mass Spring System

First we have to say that it was very hard because of the lack of documentation of this code and not really knowing what we should do. Only a quick sentence does not explain anything.