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Applied Optimization
Exercise 05

5.1 Exact line search for the convex quadratic function

Given properties:

(i) **objective function:** $f(x) \;=\; \frac{1}{2} \cdot x^T Q x + q^T x + c$
(ii) **search direction:** $\Delta x \;=\; -\bigtriangledown f(x) = -(Qx + q)$

With this we can calculate the following:

$$\begin{aligned} f(x + s\Delta x) &= \frac{1}{2}((x + s\Delta x)^T Q (x + s\Delta x)) + q^T(x + s\Delta x) + c \\ &= \frac{1}{2}(\Delta x^T Q \Delta x) \cdot s^2 + \underbrace{(x^T Q + q^T)\Delta x}_{=-\Delta x^T \Delta x} \cdot s + x^T Q x + q^T x + c \end{aligned}$$

This equation is of the form: $g(s) = as^2 + bs + c$. The solution for $arg\,\min_{s\geq 0}(g(s))$ is $\frac{-b}{2a}$. Therefore we have:

$$\begin{aligned} t &= arg\min_{s\geq 0}(f(x + s\Delta x)) \\ &= \frac{\Delta x^T \cdot \Delta x}{\Delta x^T Q \Delta x} \end{aligned}$$

5.2 Gradient descent with exact line search

Given properties:

(i) **objective function:** $f(x) \;=\; \frac{1}{4} \cdot x_1^2 + x_2^2$
(ii) **starting point:** $x^{(0)} \;=\; (2, 1)$

With this we can calculate the following:

$$\begin{aligned} & \bigtriangledown f(x) &&= \begin{pmatrix} \frac{1}{2}x_1 \\ 2x_2 \end{pmatrix} \\ \Rightarrow\quad & \bigtriangledown f(x^{(0)}) &&= \begin{pmatrix} \frac{1}{2} \cdot 2 \\ 2 \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} =: \; -\Delta x \\ \Rightarrow\quad & x^{(1)} &&:= x^{(0)} + t\Delta x \\ & &&= \begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ & &&= \begin{pmatrix} 2 - t \\ 1 - 2t \end{pmatrix} \end{aligned}$$

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Now we will use the exact line search algorithm to find t:

$$
\begin{aligned}
\textbf{minimize} \qquad & \Phi(t) = f(x + t\Delta x) \\
\Rightarrow \qquad & \Phi(t) = f\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix} - t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right) \\
& = \frac{1}{4} \cdot (2-t)^2 + (1-2t)^2 \\
& = 1 - t + \frac{1}{4}t^2 + 1 - 4t + 4t^2 \\
& = \frac{17}{4}t^2 - 5t + 2 \\
\Rightarrow \qquad & \nabla\Phi(t) = \frac{17}{2}t - 5 \\
\Rightarrow \qquad & t = \frac{10}{17}
\end{aligned}
$$

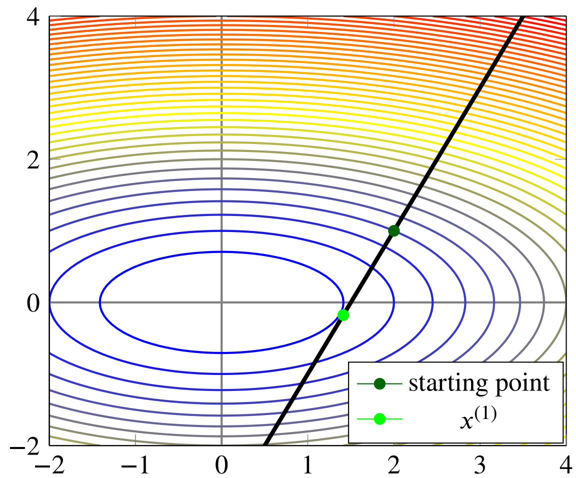
For our $x^{(1)}$ we will get:

$$
\begin{aligned}
x^{(1)} &= x^{(0)} + \frac{10}{17}\Delta x \\
&= \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{10}{17} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 2 - {}^{10}/_{17} \\ 1 - 2 \cdot ({}^{10}/_{17}) \end{pmatrix} \\
&= \begin{pmatrix} {}^{24}/_{17} \\ -({}^{3}/_{17}) \end{pmatrix}
\end{aligned}
$$

The last step is to calculate $\|\nabla f(x^{(1)}\|_2$:

$$
\begin{aligned}
\|\nabla f(x^{(1)}\|_2 &= \| \begin{pmatrix} ({}^{1}/_{2}) \cdot ({}^{24}/_{17}) \\ 2 \cdot (-{}^{3}/_{17}) \end{pmatrix} \|_2 \\
&= \sqrt{(\frac{12}{17})^2 + (-\frac{6}{17})^2} \\
&= \sqrt{\frac{144}{289} + \frac{36}{289}} \\
&= \sqrt{\frac{180}{289}} = \frac{6 \cdot \sqrt{5}}{17}
\end{aligned}
$$

Graphical Plot



5.3 Programming Exercise: Constrained Mass Spring System

Filling in the functions f, $grad\_f$, $eval\_f$, $eval\_gradient$, $solve$ and $backtracking\_line\_search$ was done as stated in the exercise. Solving the $setup\_problem$ was too much effort to do because of the lack of documentation of this code. We just did not know what each part of this code does and therefore we did not do this.
More explaining or comments of the code will be helpful.