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Modal Logic
Exercise 01

Problem 2.1

Complete the Proof of Theorem 2.1.

Model $M = \langle W, R, V\rangle$. If R has the property on the left side, every instance of the formula on the right side is true in M.

D: $\Box p \to \Diamond p$
$\Box A$ holds for every $w \in W$ ($M, w \Vdash A,\ \forall w \in W$)
, because of serial hypothesis we know $\forall w \in W$ there has to be a Rww' with $w' \in W$ with $M, w' \Vdash A$ (because of $\Box A$) so $M, w \Vdash \Diamond A$ holds.

$\Box$

T: $\Box p \to p$
$\Box A$ golds for every world $w \in W$ ($M, w \Vdash A$)
, because of reflexivity hypothesis we know for every $w \in W$ there is Rww, so $M, w \Vdash A$ because of the antecedent $\Box A$.

$\Box$

4: $\Box p \to \Box\Box p$
Suppose $\Box A \to \Box \Diamond A$ is an instance of (4). Let a world be an arbitrary $w \in W$ with $M, w \Vdash \Box A$. We need to show that for every $w' \in W$ such that Rww' and every $w'' \in W$ such that $Rw'w''$ we have $M, w'' \Vdash A$, i.e. $M, w \Vdash \Box\Box A$. But if Rww' and $Rw'w''$ then Rww'' since transitivity, and together with $M, w \Vdash \Box A$ we have then $M, w'' \Vdash A$. Hence $M, w \Vdash \Box\Box A$.

$\Box$

5: $\Box p \to \Box \Diamond p$
Relation R is euclidean. Let $w \in W$ be arbitrary such that $M, w \Vdash \Diamond A$. Then ther is a $w'' \in W$ with Rww'' and $M, w'' \Vdash A$. Now suppose $w' \in W$ with Rww', then $Rw'w''$ since R is euclidean. But then we have $M, w' \Vdash \Diamond A$, and hence $M, w \Vdash \Box \Diamond A$, i.e. $M, w \Vdash \Diamond A \to \Box \Diamond A$.

$\Box$

Problem 2.3

Let $M = \langle W, R, V\rangle$ be a model. Show that if R satisfies the left-hand properties of table 2.2, every instance of the corresponding right-hand formula is true in M.

Partially functional:
Relation R with $\forall w \forall u \forall v((Rwu \land Rwv) \to u = v)$: $\Diamond p \to \Box p,\ w, u, v \in W$, with an arbitrary world $w \in W$ with $M, w \Vdash \Diamond A$. Now suppose we have Rww' with $w' \in W$ and Rww'' with $w'' \in W$ where $M, w'' \Vdash A$ because of $M, w \Vdash \Diamond A$. But with the properties $w'' = w'$, hence $M, w' \Vdash A$, $M, w \Vdash \Box A$.

$\Box$

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Functional:
Relation R with $\forall w \exists v \forall u((Rwu \leftrightarrow u = v) : \ \diamond p \leftrightarrow \Box p, \ w, u, v \in W$, suppose arbitrary world $w \in W$ with $M, w \Vdash \Box A$. Suppose we have Rww' with $M, w' \Vdash A$ and Rww'' with $M, w'' \Vdash A$. So we have $M, w \Vdash \diamond A$, but $Rww' \rightarrow w' = w''$ (sp $\diamond p \rightarrow \Box p$ proof as above).

$\Box$