Lecture 1

Sec 1.1. Absolute Values & Triangle Inequalities

What's math?
↳ A Language : Describe the world.

Notation Shorthand:

$\forall$ for all	$\exists$ there exist	$\ni$ Such that
$\therefore$ Therefore	wlog : without loss of generality	
$\in$ is an element of	WSIC why should I care?	

Absolute value:

Mathematical Definition:

$$|x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases}$$

Note: $|x| = |-x|$

Verbal sense definition:

$|x|$ gives distance of x from 0.
↳ aka is magnitude / size.

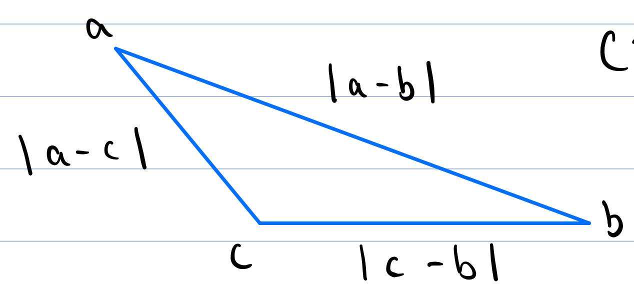
From this thought process, we could define the distance from two arbitrary points, using abs values.

So in general, $\forall\ a,b \in \mathbb{R}$, the dist between a and b is given by

$$|a-b| = |b-a| = \text{distance}.$$

One fundamental Inequality in math: the triangle inequality.

"The sum of lengths of any two sides of the triangle exceeds the length of the third."



(2D Case)

The statements & pic lead us to:

Theorem I: The Δ Inequality

For $a, b, c \in \mathbb{R}$ we have:

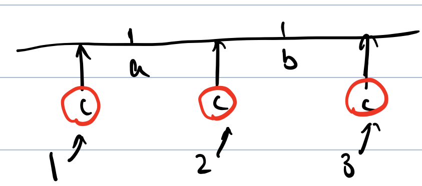
$$|a-b| \leq |a-c| + |c-b|$$

Proof for the 1D Case:

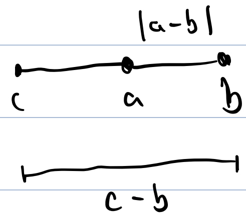
反过来也一样

$\because |a-b| = |b-a|$, assume wlog that $a<b$.

Then, there're 3 possible cases for the value of c.



Case 1: $c < a$



From the diagram: $|a-b| < |c-b|$

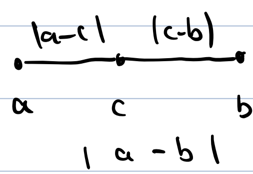
Then, since $|a-c| > 0$

Thus, $|a-b| < |c-b| + |a-c|$

Therefore, $|a-b| \leq |c-b| + |a-c|$

?: $\leq$ → less than OR equals to, 满足一项即 True.

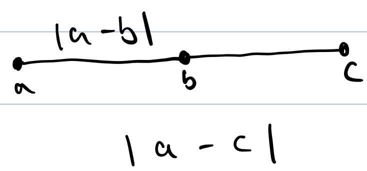
Case 2: $a \leq c \leq b$



we can see: $|a-c| + |c-b| = |a-b|$

and therefore: $|a-c| + |c-b| \geq |a-b|$

Case 3: $c > b$



we see that $|a-c| > |a-b|$

since $|c-b| > 0$

we have $|a-c| + |c-b| > |a-b|$

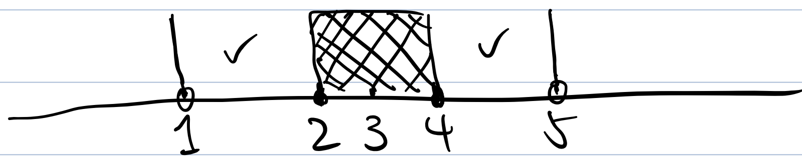
Therefore $|a-c| + |c-b| \geq |a-b|$

Important Inequalities:

Let $\delta > 0$.

1. $|x-a| \leq \delta$ if and only if $x \in [a-\delta, a+\delta]$
2. $|x-a| < \delta$ if and only if $x \in (a-\delta, a+\delta)$
3. $0 < |x-a| < \delta$ if and only if $x \in (a-\delta, a+\delta) \setminus \{a\}$.

Example. $1 \leq |x-3| < 2$



$\therefore\ x \in (1,2] \cup [4,5) = (1,5) \setminus (2,4)$

Practice Problems:

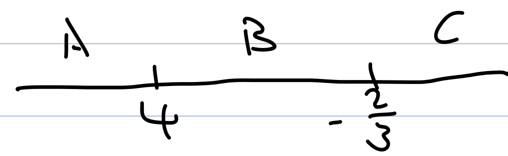
Q 1. (a) $|x-2| = 5$

To the right of the 2, if the distance between x and 2 has to be exactly 5, then x must be $x = 2+5 = 7$ such that $|7-2| = 5$.

To the left of the 2, if the distance between x and 2 has to be exactly 5, then x must be $x = 2-5 = -3$ such that $|-3-2| = 5$.

$\therefore x \in \{-3, 7\}$

b) $|x-4| = |3x+2|$



Case 1: $x-4 = (3x+2)$

$x = 3x+6$

$-2x = 6$

$x = -3$

Case 2: $x-4 = -(3x+2)$

$x-4 = -3x-2$

$4x = 2$

$x = \frac{1}{2}$

$\therefore x \in \{-3, \frac{1}{2}\}$

c) $|x+1| > 1$

Case 1: $x + 1 > 1$

$x > 0$

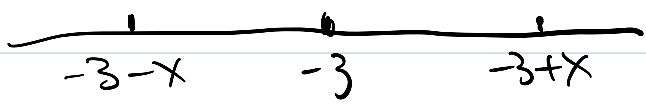
Case 2: $x + 1 < -1$

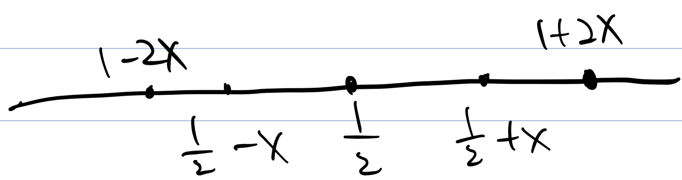
$x < -2$

$\therefore (-\infty, -2) \cup (0, \infty)$ is the solution.

d)

$2(\frac{1}{2} - x)$





Case 1. $(x+3) - 5 \le 1 - 2x$

$x - 2 - 1 \le -2x$

$3x \le 3$

$x \le 1$

Case 2. $(x+3)-5 \geq -1+2x$

$$x-2+1 \geq 2x$$

$$-1 \geq x$$

$$x \leq -1$$

$\because$ $x \leq 1$

$\therefore$ ans : $(-\infty, 1]$

e) $|x-4|\,|x+2| > 7$

Case 1 $(x-4)(x+2) > 7$

$$x^2+2x-4x-8 > 7$$

$$x^2-2x-15 > 0$$

solve for $x^2-2x-15 = 0$

$$(x+3)(x-5) > 0$$

$\therefore$ $x < -3$, $x > 5$

Case ② $(x-4)(-x-2) < 7$

$$-x^2 + 2x + 8 < 7$$

$$-x^2 + 2x + 1 < 0$$

we know root formula for quadratic equations:

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x_1 = \frac{-2 + \sqrt{4 - 4\times(-1)\times 1}}{-2}$$

$$x_1 = \frac{-2 + 2\sqrt{2}}{-2}$$

$$x_1 = 1 + \sqrt{2}$$

$$x_2 = \frac{-2 - \sqrt{4 - 4\times(-1)\times 1}}{-2}$$

$$x_2 = 1 - \sqrt{2}$$

$\because$ x_1 and x_2 are the point by which

$|x-4|\,|x+2| = 7$

$\therefore$ $x > 1-\sqrt{2}$ and $x < 1+\sqrt{2}$

Now we know the domains of x are:

$x < -3$, $x > 5$, $x > 1-\sqrt{2}$, $x < 1+\sqrt{2}$

also written as:

$x \in (-\infty, -3) \cup (1-\sqrt{2}, 1+\sqrt{2}) \cup (5, \infty)$

Q2. a) Let x, y, z be a, 0, and $-b$.

according to the triangle inequality I, we know

$\forall x, y, z \in \mathbb{R}$, $|x-y| \leq |x-z| + |z-y|$

$\therefore |a-0| \leq |a-(-b)| + |-b-0|$

$|a| \leq |a-(-b)| + |b|$

we have $|a| - |b| \leq |a+b|$

b) Let x, y, z be a, 0, and b.

according to the triangle inequality I, we know

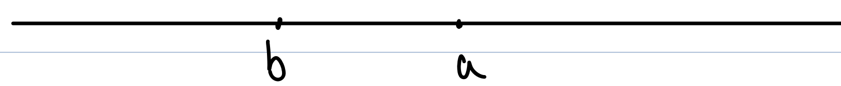
$\forall x, y, z \in \mathbb{R}$, $|x-y| \leq |x-z| + |z-y|$

$$\therefore \quad |a - 0| \leq |a - b| + |0 - b|$$

$$|a| \leq |a - b| + |b|$$

we have $|a| - |b| \leq |a - b|$

Q3. Case 1: $a \geq b$



In this case, $|a-b| = (a-b)$

$\therefore$ piecewise function for this case is :

$$\frac{1}{2}(a+b+a-b)$$

$$= \frac{1}{2} \times 2a = a$$

Case 2: $a < b$

In this case, $|a-b| = -(a-b) = b-a$

$\therefore$ piecewise function for this case is :

$$\frac{1}{2}(a+b+b-a)$$

$$= \frac{1}{2} \times 2b = b$$

$\therefore$ The piecewise function of this expression is :

$$f(x) = \begin{cases} a, & a \geq b \\ b, & a < b \end{cases}$$

Q4. a) F

b) T

$$|x-0| \leq |x-y| + |y-0|$$

$$|x| \leq |x-y| + |y|$$

$$|x| - |y| \leq |x-y|$$

By triangle inequality:

$$|x-0| \leq |y-0| + |x-y|$$

Case 1: if $|x| \geq |y|$

$$|x| - |y| \leq |x-y|$$

$$\big||x| - |y|\big| \leq |x-y|$$

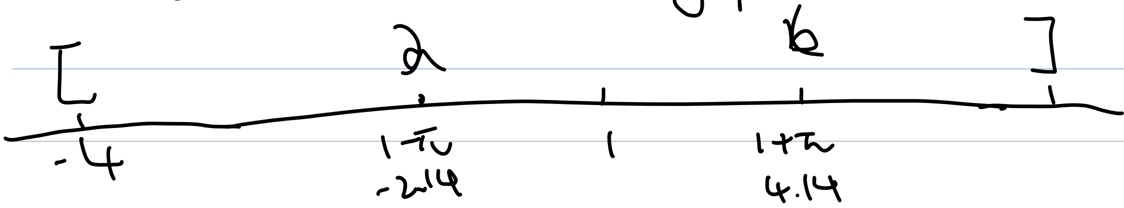
Case 2: if $|x| < |y|$

$$|y| - |x| \leq |x-y|$$

Since $\big||y| - |x|\big|$ is equivalent as $\big||x| - |y|\big|$

therefore $\big||x| - |y|\big| \leq |x-y|$

Consider the follow line graph:



As segment a, if $x < 1$,

$\pi < 1-x \leq 5$

$[-4, 1-\pi) \cup (1+\pi, 6]$

$\pi - 1 < -x \leq 4$

$1-\pi > x \geq -4$

As segment b, if $x \geq 1$,

$\pi < x-1 \leq 5$

$\pi+1 < x \leq 6$

-2.14

(c) (d) (f)

Lecture 2

Sec 1.1. Absolute Values & Triangle Inequalities

Theorem 2: Triangle Inequality 2.0

For $a, b \in \mathbb{R}$, we have

$$|a+b| \leq |a| + |b|$$

Proof:

$$|a+b| = |a-(-b)| \leq |a-0| + |0-(-b)|$$

$$\leq |a| + |b|$$

WSIC?

We'll need to deal with ineqs involving abs values as part of fundamental definitions and for examining errors on approximation throughout this course.

A sidebar on notation: (Interval vs. set notation)

The largest set of numbers in this course is $\mathbb{R}$.
When dealing with solving ineqs, our answers will be some subset of $(\mathbb{R})$.

$\hookrightarrow 1 < x < 2 \longrightarrow x \in (1,2)$

↳ $1 < x \leq 2 \longrightarrow x \in (1,2]$

↳ $4 \leq x \leq -2 \longrightarrow x \in [-4,-2] \cup [3,\infty)$

Common equalities → 3 forms:

$x \in R$, fixed point $a \in R$, tolerant $0 < \delta \in R$.

1) $|x-a| < \delta$

"Think, all real numbers x, whose distance from a is less than δ."

↳ $x \in (a-\delta, a+\delta)$ ↳ $a-\delta < x < a+\delta$

$-\delta < x-a < \delta$

2) $|x-a| \leq \delta$

↳ $x \in [a-\delta, a+\delta]$

3) $0 < |x-a| < \delta$

↳ $x \in (a-\delta, a+\delta) \setminus \{a\}$

↳ or $x \in (a-\delta, a) \cup (a, a+\delta)$

Ex. 1

$$|-2x+6| < 5$$

$$\hookrightarrow -5 < -2x+6 < 5$$

$$-11 < -2x < -1$$

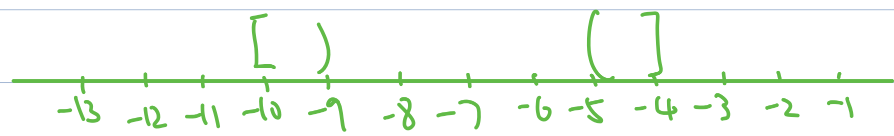
$$\frac{11}{2} > x > \frac{1}{2} \quad \text{or} \quad \boxed{x \in \left(\tfrac{1}{2}, \tfrac{11}{2}\right)}$$

Ex. 2

$$2 < |x+7| \leq 3$$

Distance : $2 < |x-(-7)| \leq 3$

Translate: All x whose distance from -7 is at most equal to 3, and at least 2.



$\therefore \quad x \in [10,-9) \cup (-5,-4]$

Purely algebraicly :

$2 < |x+7|$ AND $|x+7| \leq 3$

Case 1: $x \geq -7$

$$2 < x+7$$

$$x > -5$$

$$-3 \leq x+7 \leq 3$$

$$-10 \leq x \leq 4$$

Case 2: $x < -7$

$$2 < -x-7$$

Putting together on a num

$9 < -x$

$-9 > x$

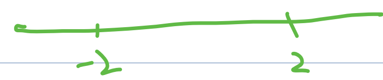
$x < -9$

line, we recover our previous result:

$x \in [-10, -9) \cup (-5, -4]$

$x > -5$ or $x < -9$

Ex. 3 $\dfrac{|x+2|}{|x-2|} > 5$



$|x+2| > 5|x-2| \quad (x \neq 2)$

Case 1: $x < -2$

$-x-2 > 5(-x+2)$

$-x-2 > -5x+10$

$4x > 12$

$\Rightarrow \quad x > 3$

No solution for $x < -2$, $x > 3$

Case 2: $-2 \leq x < 2$

$x+2 > 5(-(x-2))$

$\Rightarrow x > 4/3$

$\Rightarrow \frac{4}{3} < x < 2$

Case 3: $x > 2$

$x + 2 > 5(x - 2)$

$\Rightarrow \quad x < 3$

$\Rightarrow \quad 2 < x < 3$

$\therefore \; x \in \left(\frac{4}{3}, 2\right) \left(2, 3\right)$

LECTURE 3

1.2 Sequence & Their Limits

A sequence is an ordered list.
↳ discrete data: (a list of data points)

↳ Approximation → approx sqr root → p14
→ approx solns we can't obtain explicitly. (Newton's Method)

We can express infinite sequences explicitly as:

set → $\{a_1, a_2, \dots, a_n\}$ ①

terms — Index $\in \mathbb{N}$

OR $\{a_n\}_{n=1}^{\infty}$ ② ∞ ↔ Infinite for us

OR $\{a_n\}$ ③

E.x. $\{\frac{1}{n}\}_1^{\infty} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$

$\{n^2\}_{n=0}^{\infty} = \{0, 1, 4, \dots\}$

$$\{(-1)^n\}_{n=1}^{\infty} = \{-1, 1, -1, \ldots\}$$

We can also do it recursively:

- $a_1 = 1 \quad a_2 = 1, \quad a_n = a_{n-1} + a_{n+2} \; (n \geq 3)$
- $a_1 = 16, \; a_{n+1} = \frac{1}{2}\left(a_n + \frac{260}{a_n}\right)$ [Heron's Algo to approx $\sqrt{260}$]

KEY Question: What's happening to our sequence at large n?

↳ $\frac{1}{n}$ goes infinitely approach to 0.

↳ (-1^n) oscillates between -1 and 1 forever.

What does our sequence approach?

Terminology:

We can extract infinitely many terms of a given sequence to create any sequence. (preserve order)

E.X.

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \rightarrow \text{take every odd term} \rightarrow \left\{1, \frac{1}{3}, \frac{1}{5}, \ldots\right\}$$

$$\rightarrow \{a_{2k+1}\} = \left\{\frac{1}{2k+1}\right\}_{k=0}^{\infty}$$

We've extracted a sub-sequence

Definition: Subsequence

Let $\{a_n\}$ be a seq. Let $\{n_1, n_2, \dots n_k, \dots\}$ be $\mathbb{N}$ seq.

Then $\{a_{n_k}\} = \{a_{n_1}, a_{n_2}, \dots, a_{n_k} \dots\}$ is a subsequence of $\{a_n\}$.

Given that we're curious about our sequence at large n, we are then interested in the subsequence which is all terms of $\{a_n\}$ beyond some cutoff a_k.

This is the tail.

Defn: Tail

Let a_n be a sequence. Let $k \in \mathbb{N}$, then the $\boxed{\text{subseq}}$ $\{a_k, a_{k+1}, \dots\}$ is called the tail of $\{a_n\}$ with cut off term a_k. (with cutoff k).

Including a_k.

We need to be careful with our language, it's ~~good enough~~ to say a limit is a value the seq approaches as n gets larger.

E.x. The terms of $\{\frac{1}{n}\}$ also approach -1. X

N = index ε = bound line

The correct way to think about it:

"L is the limit of $\{a_n\}$ if as n gets larger, a_n gets infinitely closer to L"

That is for any positive tolerance $\varepsilon > 0$ given, we can find an $N \in \mathbb{N}$ such that a_n approximates L with an error less than ε for $n \geq N$.

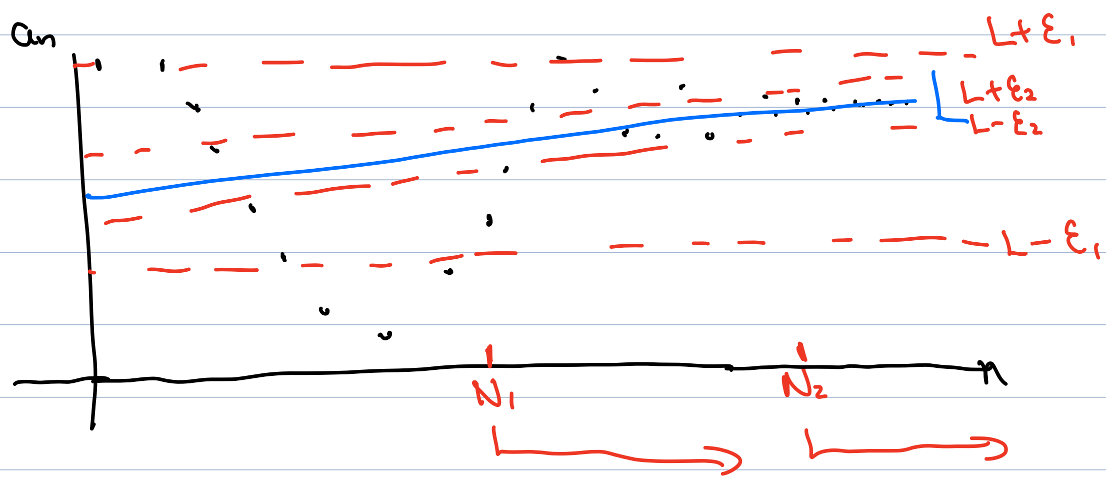
<u>Defn: Formal defn of limit of a sequence.</u>

L is the limit of $\{a_n\}$ (as $n \to \infty$) if for every $\varepsilon > 0$ $\exists N \in \mathbb{N} \ni$ if $n \geq N$ then $|a_n - L| < \varepsilon$

If such L exist, then $\{a_n\}$ is convergent
$\left[\lim_{n \to \infty} a_n = L \text{ or } a_n \to L \right]$

If no such L exists, then $\{a_n\}$ is divergent.

Picture:



For smaller ε, we'll need to go to larger n to be written error bounds.

This is becoming a very boring game:

Evil Mathematician vs You

give you an $\varepsilon > 0$

find an $N \in \mathbb{N}$, such that for $n \geq N$, $|a_n - L| < \varepsilon$

This repeat over and over with the evil giving you smaller $\varepsilon > 0$.

To ensure victory regardless of given $\varepsilon > 0$:

$\boxed{\text{return an } N \text{ dependent on } \varepsilon}$

Lecture 4

Ex. $\lim_{n \to \infty} \frac{1}{\sqrt[3]{n}} = 0$

a) $\varepsilon = \frac{1}{1000}$

b) $\varepsilon > 0$

a) <u>Rough work:</u>

We must find an $N \in \mathbb{N} \ni$ for $n \geq N$

$$|a_n - 0| < \frac{1}{1000}$$

$$\left|\frac{1}{\sqrt[3]{n}} - 0\right| < \frac{1}{1000}$$

$$\frac{1}{\sqrt[3]{n}} < \frac{1}{1000}$$

$$1000 < \sqrt[3]{n}$$

$$1000{,}000{,}000 < n$$

Proof: Given $\varepsilon = \frac{1}{1000}$, let $N = 1000{,}000{,}001$

Then, for $n \geq N$, we have

$$|a_n - 0| = \left|\frac{1}{\sqrt[3]{n}}\right| = \frac{1}{\sqrt[3]{n}} \leq \frac{1}{\sqrt[3]{N}} < \frac{1}{\sqrt[3]{1000000000}}$$
$$= \frac{1}{1000}$$

$$\therefore |a_n - 0| < \frac{1}{1000}$$

b) <u>Rough work:</u>

We must find $N \in \mathbb{N} \ni$ for $n \geq N$

$$|a_n - 0| < \varepsilon$$
$$\left|\frac{1}{\sqrt[3]{n}}\right| < \varepsilon$$
$$\frac{1}{\sqrt[3]{n}} < \varepsilon$$
$$\frac{1}{\varepsilon^3} < n$$

Proof / Solution:

1. Given $\varepsilon > 0$, let $N > \frac{1}{\varepsilon^3}$, $N \in \mathbb{N}$
2. Then, for $n \geq N$, we have

③ $\left| a_n - 0 \right| = \dfrac{1}{\sqrt[3]{n}} \leq \dfrac{1}{\sqrt[3]{N}} < \dfrac{1}{\sqrt[3]{\left(\frac{1}{\varepsilon}\right)^3}}$

④ $\Rightarrow \left| a_n - 0 \right| < \varepsilon, \ \forall \varepsilon > 0$

⑤ $\therefore \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[3]{n}} = 0$

E.x. 2

Show $\lim\limits_{n\to\infty} \dfrac{3n^2+2n}{4n^2+n+1} = \dfrac{3}{4}$ using the formal defn.

<u>Rough work:</u>

We need $N \in \mathbb{N} \ni$ for $n \geq N$.

$$\left| a_n - \frac{3}{4} \right| < \frac{3}{4}$$

$$\left| \frac{3n^2+2n}{4n^2+n+1} - \frac{3}{4} \right| < \frac{3}{4}$$

$$\left| \frac{4(3n^2+2n) - 3(4n^2+n+1)}{4(4n^2+n+1)} \right| < \varepsilon$$

$$\left| \frac{5n-3}{16n^2+4n+4} \right| < \varepsilon$$

Now, note:

$$\frac{5n-3}{16n^2+4n+4} < \frac{5n}{16n^2+4n+4} < \frac{5n}{16n^2+4n} < \frac{5n}{16n^2} = \frac{5}{16n}$$

We'd like $\frac{5}{16n} < \varepsilon$

$$\Longrightarrow \frac{5}{16\varepsilon} < n$$

Proof / Solution:

① Give $\varepsilon > 0$, let $N > \frac{5}{16\varepsilon}$, $N \in \mathbb{N}$

② Then for $n \in \mathbb{N}$, $n \geq N$, we've:

③ $$\left|a_n - \frac{3}{4}\right| = \left|\frac{5n-3}{16n^2+4n+4}\right| \leq \left|\frac{5n}{16n^2}\right| = \frac{5}{16n} \leq \frac{5}{16N} < \frac{5}{16\left(\frac{5}{16\varepsilon}\right)} = \varepsilon$$

④ $\Longrightarrow \left|a_n - \frac{3}{4}\right| < \varepsilon, \ \forall \varepsilon > 0$

⑤ Thus, $\lim\limits_{n\to\infty} \frac{3n^2+2n}{16n^2+4n+1} = \frac{3}{4}$

Formal Defn 2.0.

$\lim_{n\to\infty} a_n = L$ if $\forall\, \varepsilon > 0$ the interval

$(L-\varepsilon, L+\varepsilon)$ contains a tail of $\{a_n\}$

Further, for any interval (a,b) containing L,

we can find a small enough $\varepsilon > 0 \ni$

$(L-\varepsilon, L+\varepsilon) \subseteq (a,b)$

So any interval (a,b) containing L has a tail of

$\{a_n\}$.

Theorem 3.

These are equivalent.

1) $\lim_{n\to\infty} a_n = L$

2) Every interval $(L-\varepsilon, L+\varepsilon)$ contains a tail of $\{a_n\}$

3) Every interval $(L-\varepsilon, L+\varepsilon)$ contains all but finitely many terms of $\{a_n\}$.

4) Every interval (a,b) containing L contains a tail of $\{a_n\}$

5) Every interval (a,b) containing L contains all but finitely many terms of $\{a_n\}$

* Changing $\boxed{\text{finitely}}$ many terms of a sequence does not affect convergence.

Ex.

$\{(-1)^n\} \longrightarrow$ Let's show there's no limit L. (divergent)

Assume $L=-1$.

This would mean for example $(-1.5, -0.5)$ must contain a tail

However 1 is not in this interval, so no interval.

$\longrightarrow$ Contradiction

Similar argument for $L=1$.

Assume L is another value.

Then it must be that if we're given $\varepsilon = \frac{3}{4}$, $(L-\frac{3}{4}, L+\frac{3}{4})$ must contain a tail.

This interval is 1.5 wide. But ± 1 are 2 apart!

$\therefore$ No tail in $(L-\frac{3}{4}, L+\frac{3}{4})$. Contradiction

$\Rightarrow$ No L, divergent

Theorem: Uniqueness of Limits.

If the sequence $\{a_n\}$ has a limit L, it is unique

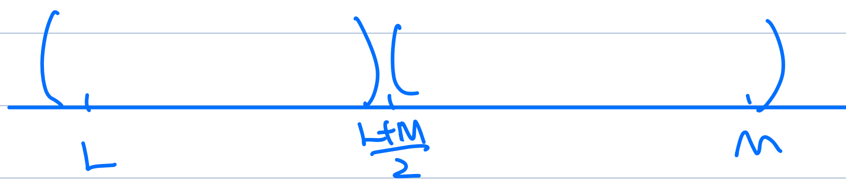
Proof:
Suppose $\{a_n\}$ has two different limits, L & M.

wlog, $L < M$

Consider that $L \in \left(L-1, \frac{M+L}{2}\right)$ and

$M \in \left(\frac{M+L}{2}, L-1\right)$

with these two intervals being disjoint.



Note that each interval contain infinitely many terms.

Since they have infinitely many terms, eventually beyond some cutoff, at least one term in the tail must live in both interval.

But this is a contradiction as the interval are disjoint.

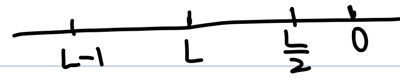
Proposition 5:

If $a_n \geq 0$, $\forall n \in \mathbb{N}$ in $\{a_n\}$, then $\lim_{n\to\infty} a_n \geq 0$

Proof.

Assume $L < 0$

Consider: $L \in (L-1, \frac{L}{2})$



This interval is strictly negative.

So no terms of $\{a_n\}$ reside in it $\Rightarrow$ no tail.

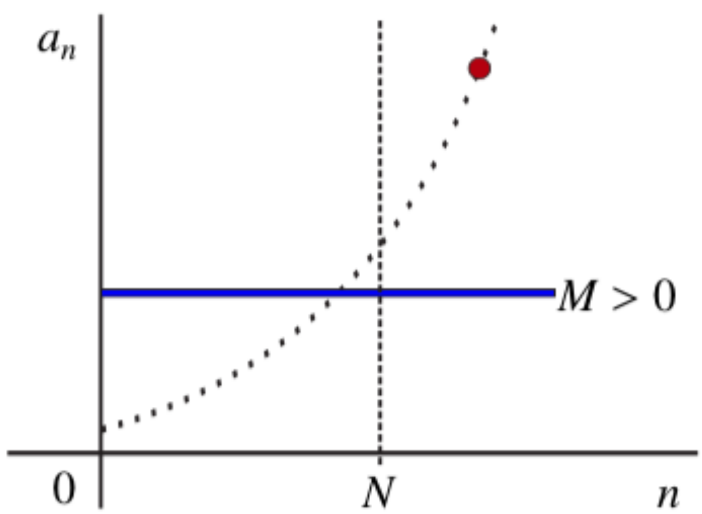
A contradiction. Thus, $L<0$ isn't a limit of $\{a_n\}$.

$\{n^2\} \to$ growth without bounds

$\hookrightarrow \lim_{n\to\infty} \underbrace{a_n = \infty}$

* Not a #

* Diverges



Lecture 5 1.2.5. Divergence to Infinity

Defn: Divergence to ∞

For every $M>0$ we can find a cutoff $N \in \mathbb{N}$ so that if $n \geq N$, then

$$a_n > M.$$

In this case, we write.

$$\lim_{n\to\infty} a_n = \infty.$$

We also have $\lim_{n\to\infty} a_n = \infty$ if every interval of the form (M, ∞) contains a tail of the sequence.

Defn: Divergence to $-\infty$

For every $M<0$ we can find a cutoff $N \in \mathbb{N}$ so that if $n \geq N$, then

$$a_n < M.$$

In this case, we write.

$$\lim_{n\to\infty} a_n = -\infty.$$

We also have $\lim_{n\to\infty} a_n = -\infty$ if every interval of the form $(-\infty, M)$ contains a tail of the

Sequence.

E.x.
Show $\lim_{n\to\infty} n^3 = \infty$

Pf: Given $M>0$, let $N > \sqrt[3]{M}$, $N \in \mathbb{N}$

Then, for $n \geq N$, we have
$a_n = n^3 \geq N^3 > (\sqrt[3]{M})^3 = M$

Aside:
For a given $M>0$.
we want $N \in \mathbb{N} \ni$
$a_n > M$
$n^3 > M$
$n > \sqrt[3]{M}$

$\therefore a_n > M \ \forall M>0$, when $n \geq N$. So $\lim_{n\to\infty} n^3 = \infty$.

THEOREM 6 :

(i) If $\alpha > 0$, then:
$\lim_{n\to\infty} n^{\alpha} = \infty$. Divergent to Infinity.

(ii) If $\alpha < 0$, then:
$\lim_{n\to\infty} n^{\alpha} = 0$ Convergent to zero

1.2.6 Arithmetic for Limits of Sequences.

THEOREM 7: Arithmetic Rules for Limits of Sequences.

Let $\{a_n\}$ and $\{b_n\}$ be sequences. Assume that $\lim_{n\to\infty} a_n = L \in \mathbb{R}$ and $\lim_{n\to\infty} b_n = M \in \mathbb{R}$, where L and M are Real numbers. Then

* Note: We CANNOT use any of these rules unless we know all limits exist.

i) For any $c \in \mathbb{R}$, if $a_n = c$ for every n, then $c = L$.

ii) For any $c \in \mathbb{R}$, $\lim_{n\to\infty} c a_n = cL$.

iii) $\lim_{n\to\infty} (a_n + b_n) = L + M$

iv) $\lim_{n\to\infty} a_n b_n = LM$.

v) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{M}$ if $M \neq 0$

vi) If $a_n \geq 0$ for all n and if $a > 0$, then $\lim_{n\to\infty} a_n^a = L^a$.

vii) For any $k \in \mathbb{N}$, $\lim_{n\to\infty} a_{n+k} = L$.

Proof of #3:

Let $\varepsilon > 0$ be given. Since $\lim_{n\to\infty} a_n = L$, $\lim_{n\to\infty} b_n = M$,

we know by defn, that

$\forall \varepsilon_1, \varepsilon_2 > 0$, $|a_n - L| < \varepsilon_1$ and $|b_n - M| < \varepsilon_2$

Aside:

We want $|(a_n + b_n) - (L+M)| < \varepsilon$

use Δ inequality $|(a_n - L) + (b_n - M)| < \varepsilon$

$|(a_n - L) + (b_n - M)| \leq |a_n - L| + |b_n - M|$

We've freedom to choose $\varepsilon_1, \varepsilon_2$, let
$\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$

Let $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, for cutoff $N_1, N_2 \in \mathbb{N}$,

Take $N = \max\{N_1, N_2\}$.

Now, examine $|(a_n + b_n) - (L+M)| \leq |a_n - L| + |b_n - M|$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Thus, for $N = \max\{N_1, N_2\}$, we've for $n \geq N$

$$|(a_n + b_n) - (L+M)| < \varepsilon$$

Thus, $\lim\limits_{n\to\infty} (a_n + b_n) = L + M$

THEOREM (8):

Assume $\{a_n\}$ & $\{b_m\}$ are sequenes and that $\lim\limits_{n\to\infty} b_n = 0$

Further, assume $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n}$ exist. Then $\lim\limits_{n\to\infty} a_n = 0$

Proof:

Let $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = k \in \mathbb{R}$. Let $\lim\limits_{n\to\infty} b_n = 0$

Notice $a_n = \dfrac{a_n}{b_n} \cdot b_n$

Then, by limit law # 4 we get:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{a_n}{b_n} \cdot b_n\right)$$

$$= \lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) \cdot \lim_{n\to\infty} (b_n)$$

$$= (k)(0) = 0$$

Highest power

Ex. 1 $\lim\limits_{n\to\infty} \dfrac{3n^2+2n \cdot \frac{1}{n^2}}{4n^2+n=1 \cdot \frac{1}{n^2}} = \lim\limits_{n\to\infty} \dfrac{3+\frac{1}{n}}{4+\frac{1}{n}+\frac{1}{n^2}}$

Lim Laws $\dfrac{3+0}{\quad} - \dfrac{3}{\quad}$

$= \quad 4+0+0 \quad = \quad 4$

Ex 13 on P. 36 of the CN gives a short cut.

Ex. 2

$$\lim_{n\to\infty} \sqrt{n^2+n} - n \longrightarrow \text{looks like } "\infty - \infty"$$

KEY: Multiply by the conjugate.

$$= \lim_{n\to\infty} \left(\sqrt{n^2+n} - n\right) \cdot \frac{\sqrt{n^2+n} + n}{\sqrt{n^2+n} + n}$$

$$= \lim_{n\to\infty} \frac{n^2+n-n^2}{\sqrt{n^2+n} + n} = \lim_{n\to\infty} \frac{n \cdot \frac{1}{n}}{\sqrt{n^2+n} + n \cdot \frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{\frac{n}{n}}{\sqrt{\frac{n^2}{n^2} + \frac{n}{n^2}} + \frac{n}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}} + 1}$$

$$\underset{\text{Lim Laws}}{=} \frac{1}{\sqrt{1+0} + 1} = \frac{1}{2}$$

Ex. 3

$$a_1 = 16, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{260}{a_n}\right).$$

Say that we know $\lim_{n\to\infty} a_n = L$.

Then using limit law #7, $\lim_{n\to\infty} a_{n+1} = L$

$$\text{So, } \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{1}{2}\left(a_n + \frac{260}{n}\right)$$

$$L = \frac{1}{2} L + \frac{260}{L}$$

$$L^2 = \frac{1}{2}(L^2 + 260)$$

$$\frac{1}{2} L^2 = 130$$

$$L^2 = 260$$

$$L = \pm\sqrt{260}$$ but note $a_n \geq 0 \ \forall n$

Then by proposition 5 from class, $L \geq 0$

$$\text{So, } L = 260$$

Week 3

1.4 Squeeze Theorem

Collorary to Thm 8

Assume $\{a_n\}$ & $\{b_n\}$ are sequences.

If $\lim_{n\to\infty} b_n = 0$ and $\lim_{n\to\infty} a_n \neq 0$ then

$$\lim_{n\to\infty} \frac{a_n}{b_n} \text{ DNE}$$

We've a few more tools to looking at convergence of sequences:

Let's look at $\left\{ \frac{\sin(n)}{n} \right\}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\sin(n)}{n} \stackrel{?}{=} \lim_{n\to\infty} \sin(n) \cdot \lim_{n\to\infty} \frac{1}{n} \to 0$$

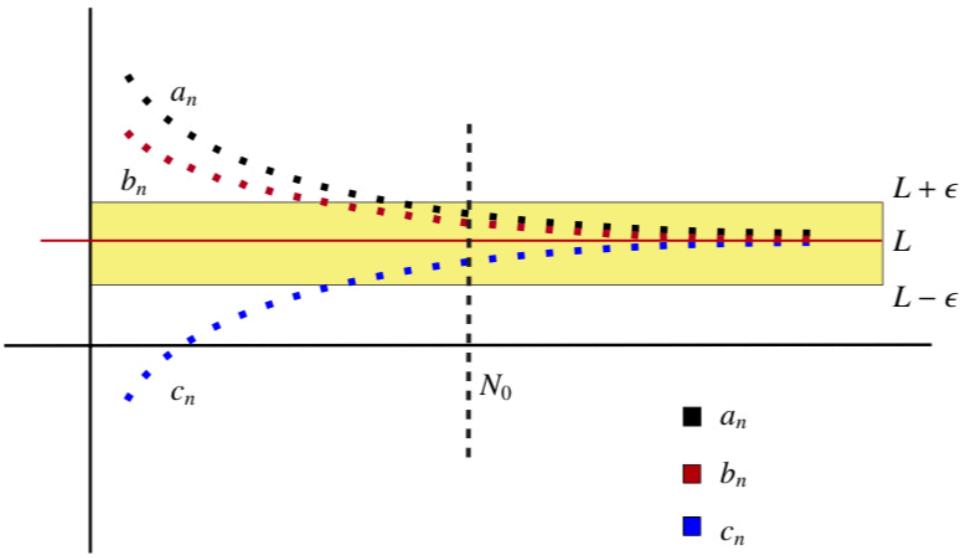
No!

Temping to say $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$ by this multiplication.

But, we can't use limit laws because $\{\sin(n)\}$ diverges.

Assume $a_n \leq b_n \leq c_n$ $\forall n \in \mathbb{N}$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$

Then $\{b_n\}$ converges and $\lim_{n\to\infty} b_n = L$.



Proof:

Since $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then for $\varepsilon > 0$ given, we can find $N \in \mathbb{N} \ni$ for $n \geq N$, we can have $a_n \in (L-\varepsilon, L+\varepsilon)$ and $c_n \in (L-\varepsilon, L+\varepsilon)$.

That is, for $n \geq N$, $L-\varepsilon < a_n \leq b_n \leq c_n < L+\varepsilon$

This means that for $n \geq N$, $b_n \in (L-\varepsilon, L+\varepsilon)$ $\forall \varepsilon > 0$. Thus $\{b_n\}$ cvgs & $\lim_{n\to\infty} b_n = L$. $\square$

Ex. 1

$$\lim_{n\to\infty} \frac{\sin(n)}{n}$$

Note $\frac{-1}{n} \leq \frac{\sin(n)}{n} \leq \frac{1}{n}$

and $\lim_{n\to\infty} \pm \frac{1}{n} = 0$

thus by Squeeze Theorem, $\lim_{n\to\infty} \frac{\sin(n)}{n} = 0$

E.X.2.

$\lim_{n\to\infty} \frac{4+(-1)^n}{n^3+n^2-1}$ Note: $\underbrace{\frac{3}{n^3+n^2-1}}_{a_n} \leq \underbrace{\frac{4+(-1)^n}{n^3+n^2-1}}_{b_n} \leq \underbrace{\frac{5}{n^3+n^2-1}}_{c_n}$

and $\lim_{n\to\infty} a_n = 0 = \lim_{n\to\infty} c_n$

thus, by Squeeze Theorem, $\lim_{n\to\infty} b_n = 0$.

There is a special class of sequences for which we can state a key stuff.

Defns:

We say that a sequence $\{a_n\}$ is $(\forall n \in \mathbb{N})$.

- Increasing if $a_n < a_{n+1}$.
- Non-decreasing if $a_n \leq a_{n+1}$
- Decreasing if $a_n > a_{n+1}$
- Non-Increasing if $a_n \geq a_{n+1}$
- Monotonic if you non-inc or non-dec

More Defns:

Let $S \in \mathbb{R}$. We say α is an upperbound of S if $x \leq \alpha$ $\forall x \in S$.

If $\exists$ an upper bound, S is bounded above.

We say β is a lower bound of S if $x > \beta$ $\forall x \in S$.

If $\exists$ an lower bound, S is bounded below.

We say S is <u>bounded</u> if bounded <u>above</u> and <u>below</u>.

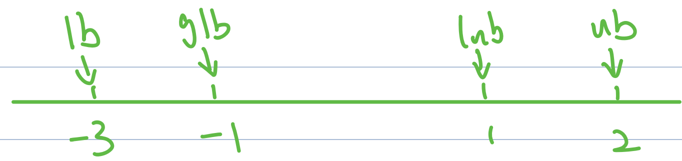
That is, $\exists M \in \mathbb{R} \ni S \subseteq [-M, M]$.

Even more defns.

Let $S \subseteq \mathbb{R}$. We say α is the least upper bound (lub or sup) of S if α is the smallest upper bd of S. That is if $x \leq \gamma$ $\forall x \in S$, then $x \leq \gamma$.

Let $S \subseteq \mathbb{R}$. We say β is the greatest lower bound (glb or inf) of S if β is the largest lower bd of S. That is if $x \geq \gamma$ $\forall x \in S$, then $x \geqq \gamma$.

Ex. $[-1,1)$



Now an axiom:

Axiom:

Let $S \subset \mathbb{R}$ be non-empty & bounded abv / blw.
Then S has a lub / glb.

THEOREM 11 Monotone Convergence T

Let $\{a_n\}$ be a non-decreasing sequence.

1) If $\{a_n\}$ is bounded above, then $\{a_n\}$ converges to $L = \text{lub}(\{a_n\})$.

2) If $\{a_n\}$ is not bounded above, then $\{a_n\}$ diverges to ∞.

In particular, $\{a_n\}$ converges $\Longleftrightarrow$ it is bounded above.

Note: Non-increasing : glb $\Rightarrow$ lub, $\infty \Rightarrow -\infty$.

Proof: 1) let $\{a_n\}$ be non-decr. let $\mathrm{lub}(\{a_n\}) = L$.

Let us be given $\varepsilon > 0$. Then $L - \varepsilon < L$.

Thus $L-\varepsilon$ cannot be an upperbound of $\{a_n\}$

Thus $\exists N \in \mathbb{N} \ni a_N > L - \varepsilon$.

Then, for $n \geq N$ we have $a_n \underset{\substack{\uparrow \\ \text{Non-decreasing}}}{\geq} a_N > L - \varepsilon$.

Also, note that since $L = \mathrm{lub}(\{a_n\})$,

we have $L \geq a_n \geq a_N > L - \varepsilon$

Further, $L + \varepsilon > L \geq a_n > L - \varepsilon \quad \forall \varepsilon > 0$

That is, for $N \in \mathbb{N}$, $a_n \in (L-\varepsilon, L+\varepsilon) \ \forall \varepsilon > 0$.

Thus, $\lim_{n\to\infty} a_n = L$

2) Let a_n be non-decreasing. Let $\{a_n\}$ be not bounded above. Let us be given $M \in \mathbb{R}^+$.

Note that M is not an upper bound of $\{a_n\}$.

Thus, $\exists N \in \mathbb{N} \ni a_N > M$.

Thus, for $n \geq N$ we've $a_n \geq a_N > M$, $\forall M > 0$

Thus, $\lim_{n\to\infty} a_n = \infty$.

To use MCT we need a new proof technique:

Induction: use for recursive sequences.

1) Prove a base case is true ($n=1$).

2) Make an inductive hypothesis (IH) ($n=k$ for some $k \geq 1$)

3) Use the IH to show the next step is true ($n=k+1$)

Then, we have shown the claim to be true $\forall n \in \mathbb{N}$.

To use MCT on recursive sequences, we'll:

1) Pv the seq is monotonic.

2) Pv the seq is bounded above or needed as needed] Induction

3) Conclude convergence by MCT.

4) Use the limit laws (#7) $\lim_{n\to\infty} a_{n+1} = L$, to find L.

Ex. 1

Let $a_1 = 1$ and $a_{n+1} = \sqrt{3+2a_n}$. Show cvg & find L.

Pv. monotonicity. We guess that we're non decreasing.

Base Case: $a_1 = 1$, $a_2 = \sqrt{3+2\cdot 1} = \sqrt{5} \geq 1$.

$\therefore a_2 \geq a_1$

Inductive Hypo:

Assume $a_k \leq a_{k+1}$ for some $k \geq 1$.

Then $2a_k \leq 2a_{k+1}$

Then $3+2a_k \leq 3+2a_{k+1}$

$\sqrt{3+2a_k} \leq \sqrt{3+2a_{k+1}}$

That is, $a_{k+1} \leq a_{k+2}$

Thus, non-dec. Thus monotonic by induction.

Next, pv bound above. I guess bounded above by 7.

BC

$a_1 = 1 \leq 7$. $\therefore a_1 \leq 7$.

IH

Assume $a_k \leq 7$ for some $k \geq 1$.

Then $2a_k \leq 14$

Then, $3+2a_k \leq 17$

Then, $\sqrt{3+2a_k} \leq \sqrt{17}$.

That is, $a_{k+1} \leq \sqrt{17} \leq 7$.

Thus, I am bounded above by 7 by Induction.

Then, by MCT, the sequence converges.

Then, we have that $\lim_{n\to\infty} a_n = L$.

By lim laws, $\lim_{n\to\infty} a_{n+1} = L$.

So, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{3+2a_n} = L$

$\sqrt{3+2L} = L$

$3+2L = L^2$

$$L^2 - 2L - 3 = 0$$

$$(L-3)(L+1) = 0 \Rightarrow L = -1, 3.$$

But $a_1 = 1$, we pv non-dec, so $L \neq -1$.

$\therefore L = 3$.

Ex. 2.

$a_1 = 4$, $a_{n+1} = \dfrac{7+a_n}{22}$. Show cvg & find L.

First pv monotonicity. We are non-increasing.

<u>BC</u>

$a_1 = 4$, $a_2 = \dfrac{7+4}{22} = \dfrac{1}{2} \leq 4 = a_1$

$\therefore a_2 \leq a_1$.

<u>IH</u>:

Assume $a_k \geq a_{k+1}$ for some $k \geq 1$.

Then $7 + a_k \geq 7 + a_{k+1}$

Then, $\dfrac{7+a_k}{22} \geq \dfrac{7+a_{k+1}}{22}$

$a_{k+1} \geq a_{k+2}$

Thus, non-inc. Thus monotonic by induction.

Alert:
Be careful with manipulations
For ex, dividing by a_k, you may justify bounded below by 0 (>0)

Next, pv bounded below. Guess bounded below by -10.

BC

$a_1 = 4 \geq -10 \quad \therefore a_1 \geq -10$

IH

Assume $a_k \geq -10$ for some $k \geq 1$.

Then, $7 + a_k \geq -3$

Then, $\frac{7+a_k}{22} \geq -\frac{3}{22}$

$$a_{k+1} \geq \frac{-3}{22} \geq -10$$

Thus, bounded below by -10 by induction.

Then, by MCT the sequence cvgs.

Now $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} a_{n+1}$ (by lim Laws).

Then, $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{7+a_n}{22} = L$

$$\frac{7+L}{22} = L$$

$$7 + L = L \cdot 22$$

$$L = \frac{7}{21} = \frac{1}{3}$$

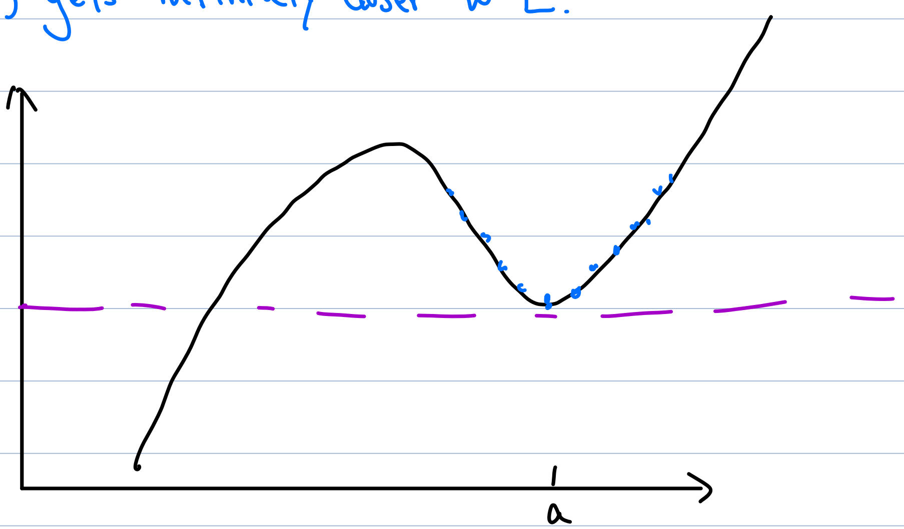
Lecture 9

4.1 Limits of Function.

$$\lim_{x \to a} f(x) = L.$$

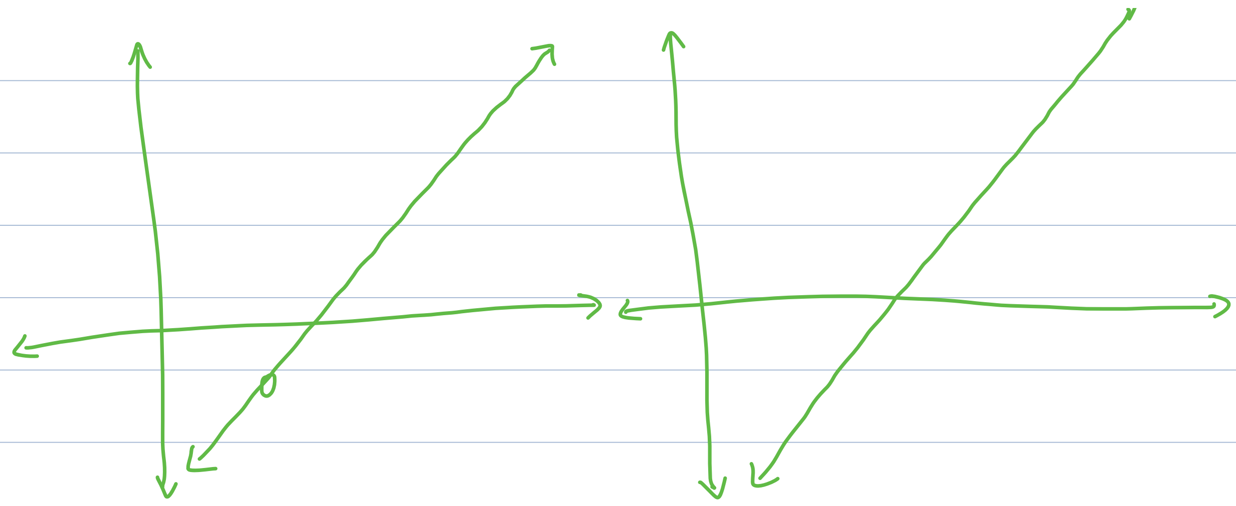
In words:

As x gets infinitely closer to a, without reaching a, $f(x)$ gets infinitely closer to L.



A cautionary tale:

Ex. $f(x) = \dfrac{x^2-3x+2}{x-1}$. We would like if we could say this is the same function as $g(x) = x-2$. but it is now.



However, $\lim_{x\to 1} f(x) = \lim_{x\to 1} g(x)$. We approach but never reach 1.

Formal Definition of the Limit of a function

Let $f: \mathbb{R} \to \mathbb{R}$ be a function & Let $a \in \mathbb{R}$.

Then, $\lim_{x\to a} f(x) = L$

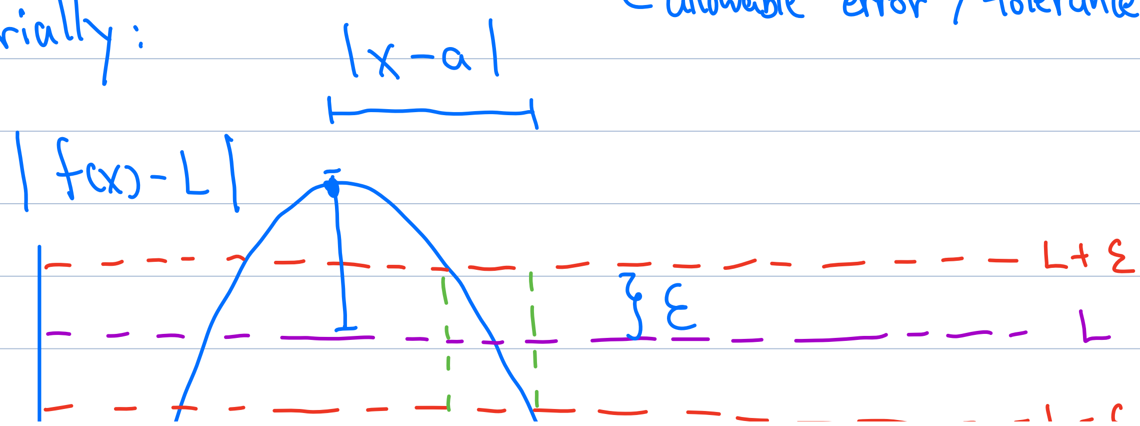
if $\forall \varepsilon > 0 \ \exists \delta > 0 \ni$ if $0 < |x-a| < \delta$

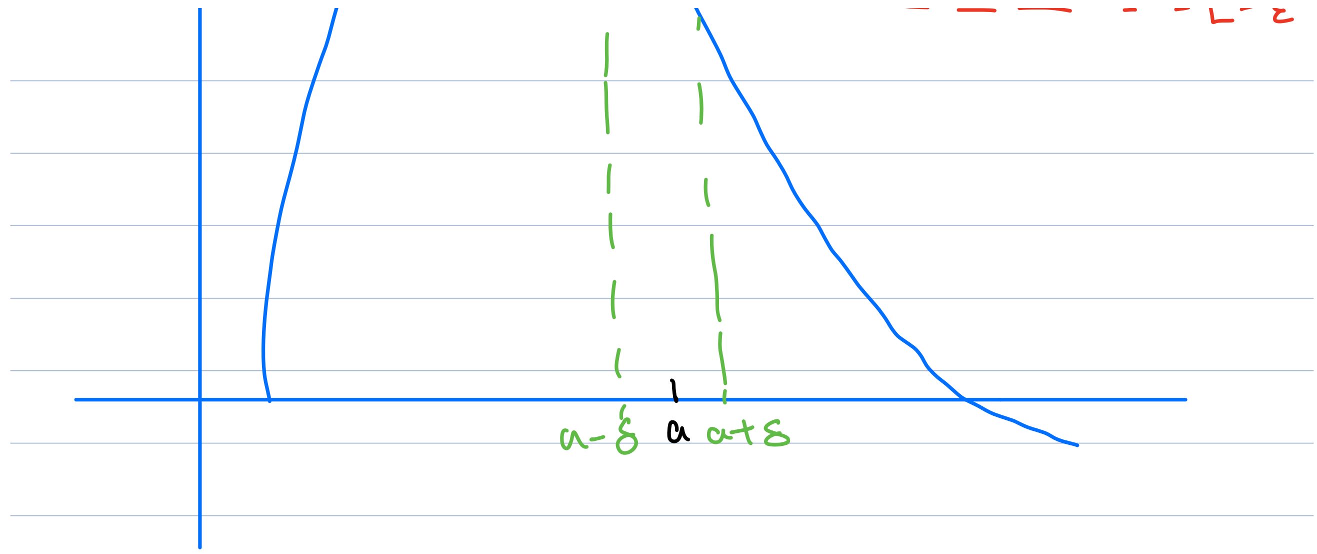
Delta Cutoff

then $|f(x) - L| < \varepsilon$

allowable error / tolerance.

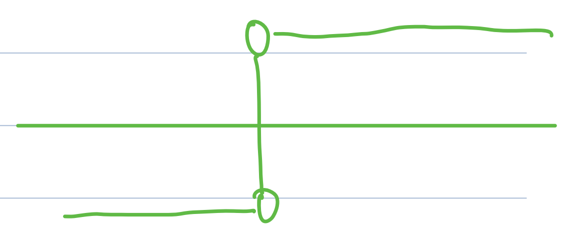
Pictorially:





Ex 1)

$$f(x) = \begin{cases} 3 & \text{if } x > 0 \\ -2 & \text{if } x < 0 \end{cases}$$



Exame $\lim_{x \to 0} f(x)$. Approach -2 from left & 3 from right.
Try to show $\lim_{x \to 0} f(x)$ D.N.E.

Proof:

Suppose $\lim_{x \to 0} f(x) = L$. Say we are given $\varepsilon = 1$.
From the formal definition, we should be able to find
$\delta > 0 \ni$ for $0 < |x - 0| < \delta$ we get $|f(x) - L| < 1$.

$$x \in (-\delta, 0) \cup (0, \delta)$$

First look at $x \in (-\delta, 0)$:

First, look at $x \in (-\delta, 0)$.

There, $f(x) = -2$. Then $|-2-L| < 1$.
That is, $L \in (-3,-1)$

Now, look at $x \in (0, \delta)$.
Here $f(x) = 3$. Then, $|3-L| < 1$. That is $L \in (2,4)$.

These intervals are disjoint. Contradiction
$\therefore \lim_{x \to 0} f(x) = $ D.N.E.

E.X
Show $\lim_{x \to 7} 8x-3 = 53$.

Rough Work:
(If $0 < |x-1| < \delta$)
Want: $|(8x-3)-53| < \varepsilon$
$|8x-56| < \varepsilon$
$8|x-7| < \varepsilon$
$|x-7| < \frac{\varepsilon}{8}$

Proof: Let $\varepsilon > 0$ be given. Let $\delta = \varepsilon/8$.
Then, if $0 < |x-7| < \delta$, we have

$$0 < |x-7| < \varepsilon/8$$
$$0 < 8|x-7| < \varepsilon$$
$$0 < |8x-56| < \varepsilon$$
$$0 < |(8x-3)-53| < \varepsilon$$

That is, $|f(x)-L| < \varepsilon \quad \forall \varepsilon > 0$
Thus, $\lim_{x \to 7} 8x-53 = 53$.

Ex.

Show $\lim_{x\to 1} x^2+3x+4=8$

Rough Work:

(If $0<|x-1|<\delta$)

Want: $|(x^2+3x+4)-8|<\varepsilon$

$|x^2+3x-4|<\varepsilon$

$|\underbrace{(x-1)}(x+4)|<\varepsilon$

depends on x. (δ only contains constants).

Assume $\delta\le 1$.

$|x-1|<1$

$\hookrightarrow x\in(0,2)\Rightarrow |x+4|<6.$

Then, returning to , we have

$|x+4||x-1|<6|x-1|<\varepsilon$

$|x-1|<\varepsilon/6.$

We will ask $\delta=\min\{1,\varepsilon/6\}$, since this gives that

- $\delta\le 1\to$ this gives $|x+4|<6$
- $\delta\le\varepsilon/6\to$ this helps us say $|x-1|<\varepsilon/6.$

Proof:

Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/6\}$.

Then, if $0 < |x-1| < \delta$, we have following:

$$|(x^2+3x+4)-8| = |x+4|\,|x-1|$$

Then, since $\delta \leq 1$, we have $|x-1| < 1 \Rightarrow x \in (0,2)$

So, $|x+4| < 6$.

Thus, $|(x^2+3x+4)-8| < 6|x-1|$

$$< 6\delta < 6(\varepsilon/6) = \varepsilon.$$

That is, $|(x^2+3x+4)-8| < \varepsilon \quad \forall \varepsilon > 0$

Thus, $\lim_{x\to 1} x^2+3x+4 = 8$.

Another example : p64, Ex. 4.

Week 4

Notes:

1) For $\lim_{x \to a} f(x)$ to be existed, f must be defined on an open intervals (α, β) containing $x=a$, exept possibly at $x=a$.

2) The value of $f(a)$, if defined at all, does not affect the existence of the limit of its value.

3) If two funs are equal, except possibly at $x=a$, then their limiting behaving at a is identical.

WSIC?

We must understand limits of funcs in order to examine the instantaneous role of change of funcs. That is we need limits to define the derivative. The basic building blocks of calculus!

How can sequences be useful when examing funcs limits.

Thm 1: Sequential characterization of limits.

Let f be defined on an open interval containing $x=a$, except possibly at $x=a$.

Then $\lim_{x \to a} f(x) = L \xLeftrightarrow{\text{iff}} \{x_n\}$ is a sequence with $x_n \neq a$ and $x_n \to a$ then $\lim_{n\to\infty} f(x_n) = L$.

Pf for $\Rightarrow$ direction:

Let $\varepsilon > 0$ be given.

Since $\lim_{x\to a} f(x) = L$, we can find $\delta > 0 \ni$ if $0 < |x-a| < \delta$, then $|f(x) - L| < \varepsilon$

And since $x_n \to a$, ($\lim_{n\to\infty} x_n = a$), we can find a cut off $N \in \mathbb{N} \ni$ for $n \geq N$, we have $|x_n - a| < \delta$.

($\forall \delta > 0$, that is, including the one we found.

Thus, we have $|f(x_n) - L| < \varepsilon$ for $n \geq N$

$\therefore \lim_{n\to\infty} f(x_n) = L$

Since we know sequence limits are unique, we get:

Thm: Uniqueness of lim of funcs.

Assume $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} f(x) = M$, then $L = M$.

That is, fcn lims are unique.

Sequential Characterization can help show that limit DNE:

1) Find $\{x_n\}$ with $x_n \to a$ and $x_n \neq a$ for which $\lim_{n\to\infty} f(x_n) = \text{DNE}$

2) Find $\{x_n\}$ and $\{y_n\}$ with $x_n y_n \to a$ and $x_n \neq a, y_n \neq a$.

for which $\lim_{n\to\infty} f(x_n) = L$ and $\lim_{n\to\infty} f(y_n) = M$, but $L \neq M$.

E.X.

$\lim_{x\to 0} \cos\left(\frac{1}{x}\right)$

Let $x_n = \frac{1}{2\pi n}$, $y_n = \frac{1}{\pi + 2\pi n}$.

Note $x_n y_n \to 0$ and $x_n \neq 0$ & $y_n \neq 0$.

Hower, $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \cos\left(\frac{1}{\frac{1}{2\pi n}}\right) = \lim_{n\to\infty} \cos(2\pi n)$.

$= 1$ for $n \in \mathbb{N}$.

$= 1$

And, $\lim_{n\to\infty} f(y_n) = \lim_{n\to\infty} \cos\left(\frac{1}{\frac{1}{\pi+2\pi n}}\right)$

$= \lim_{n\to\infty} \cos(\pi + 2\pi n)$

$= -1$ for $n \in \mathbb{N}$.

$= -1$.

Thus, $\lim_{n\to\infty} f(x_n) \neq \lim_{n\to\infty} f(y_n)$, so $\lim_{n\to\infty} \cos\left(\frac{1}{n}\right)$ DNE.

Just as with sequences, we don't tend to use formal defns in practice.

Theorem 3: Arithmetic Rules for Lims of funcs:

Let f & g be fcns with $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

Let $a \in \mathbb{R}$.

1) If $f(x) = c$ $\forall x \in \mathbb{R}$, then $\lim_{x \to a} f(x) = c$
2) For any $c \in \mathbb{R}$, $\lim_{x \to a} [c f(x)] = cL$.
3) $\lim_{x \to a} [f(x) + g(x)] = L + M$.
4) $\lim_{x \to a} [f(x) g(x)] = LM$
5) $\lim_{x \to a} \left[\frac{f(x)}{g(x)} \right] = \frac{L}{M}$ if $M \neq 0$.
6) $\lim_{x \to a} [f(x)]^a = L^a$ $\forall a > 0, L > 0$

Another carryover:

Theorem 4:

Assume $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists and $\lim_{x \to a} g(x) = 0$.

Then $\lim_{x \to a} f(x) = 0$.

A few common functions limits:

1) If $P(x)$ is a polynomial, then $\lim_{x \to a} P(x) = P(a)$.
2) If $r(x) = \frac{p(x)}{q(x)}$ is a rational func (poly/poly).

   a) if $q(x) \neq 0$, then $\lim_{x \to a} r(x) = r(a)$.

   b) if $q(a) = 0$, and $p(a) \neq 0$ then $\lim_{x \to a} r(x)$ DNE.

   c) if $q(a) = 0 = P(a)$, then $(x-a)$ must be a factor of both. Factor out, cancel, look at new Function.

   ↳ Long Division / Synthetic.

Lecture 10

We have a visual / conceptual understanding that we approach different y-values as we approach 0 from the left / right (Formerly proved $\lim_{x \to 0} f(x)$ DNE)

Defn:

Right-Sided Limits: $\lim_{x \to a^+} f(x) = L$

↳ f has limit L approaching from the right.

↳ $\forall \varepsilon > 0 \; \exists \delta > 0 \ni$ if $0 < |x-a| < \delta$ and $x > a$ then $|f(x) - L| < \varepsilon$.

Left-Sided Limits: $\lim_{x \to a^-} f(x) = L$

↳ f has limit approaching from left.

↳ $\forall \varepsilon > 0, \exists \delta > 0 \ni$ if $0 < |x-a| < \delta$ and $x < a$, then $|f(x) - L| < \varepsilon$

To relate one-sided limits to their two sideded counter-parts:

Theorem 6:

$$\lim_{x\to a} f(x) = L \overset{(\text{iff})}{\iff} \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

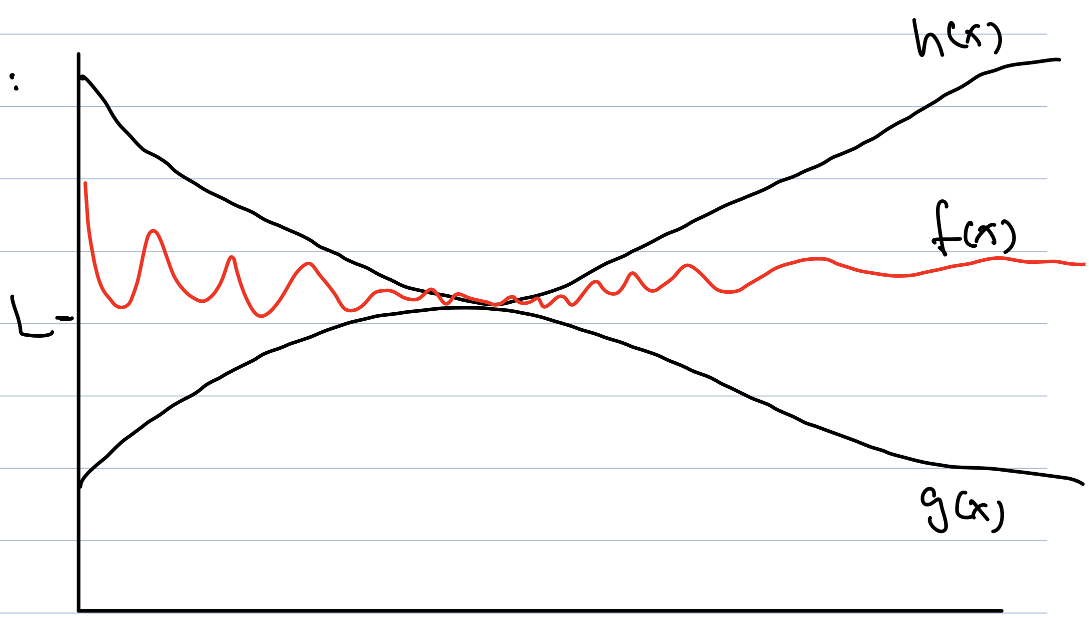
* All the arithmetic rules and sequential characterization holds for one-sided limits.

Another carry over from the sequence limits:

Thm 7: Squeeze

Assume f, g, h are defined on an open interval I, containing $x = a$, except possibly at $x = a$. Assume $\forall x \in I$, except possibly at $x = a$ that $g(x) \le f(x) \le h(x)$, and that $\lim_{x\to a} g(x) = L = \lim_{x\to a} h(x)$. Then, $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} f(x) = L$.

Pictorially:



Ex. $\lim_{x\to 0} x^3 \cos\left(\frac{1}{x}\right)$

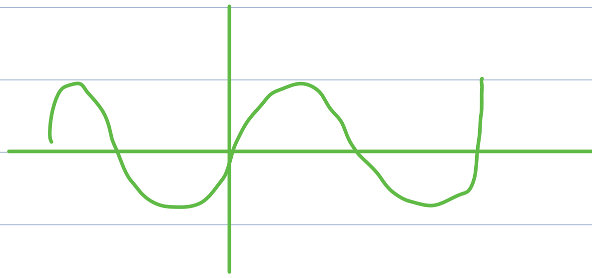
Note: $-1 \le \cos\left(\frac{1}{x}\right) \le 1 \quad \forall x \in \mathbb{R} \setminus \{0\}$

$$-x^8 \le x^8\cos\left(\tfrac{1}{x}\right) \le x^8$$

Then, $\lim_{x\to 0} \pm x^8 = 0$, so by Sqz Theorem $\lim_{x\to 0} x^8\cos\left(\frac{1}{x}\right) = 0$.

Ex.

$\lim_{x\to 0} \sin(x)$



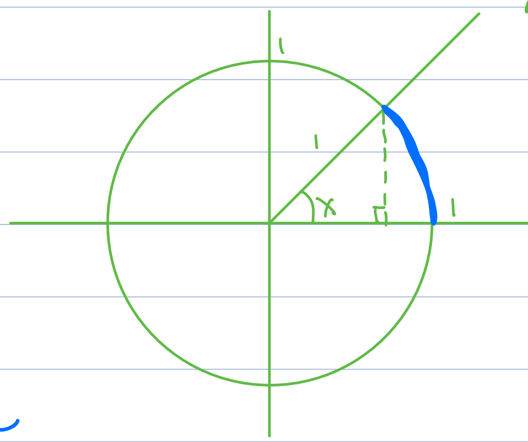
We see $\sin(x)$ goes to zero as x goes to zero.
We can properly show $\lim_{x\to 0} \sin(x) = 0$ using Squeeze Theorem.

Examine the unit circle for $0 < x < \frac{\pi}{2}$ (looking for $\lim_{x\to a^+}$)

$\sin(x) = \frac{O}{H}$ $\qquad \cos(x) = \frac{A}{H}$

$\sin(x) = \frac{O}{1}$ $\qquad \cos(x) = \frac{A}{1}$

$\sin(x) = O$ $\qquad \cos(x) = A$



$P = 2\pi r = 2\pi(1) = 2\pi$

$\text{Arclength} = P \times \frac{x}{2\pi} = 2\pi \cdot \frac{x}{2\pi} = x$

We can see from the diagram that for $x \in (0, \pi/2)$

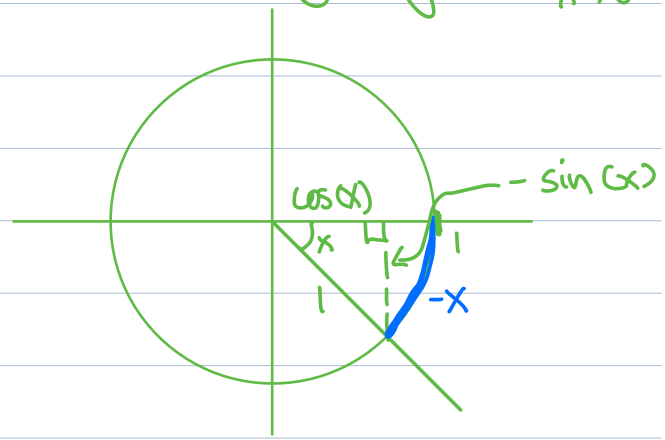
that $\qquad 0 < \sin(x) < x$

We know $\lim_{x\to 0^+} 0 = 0 = \lim_{x\to 0^+} x$

Thus, by Sqz Thm, $\lim_{x\to 0^+} \sin x = 0$.

Now, let's look at the unit circle for $-\frac{\pi}{2} < x < 0$

(looking @ $\lim_{x \to 0^-}$)



So, we see that for $x \in \left(-\frac{\pi}{2}, 0\right)$ that

$$0 < -\sin(x) < -x$$

Which means $\quad 0 > \sin(x) > x.$

We know $\lim_{x \to 0^-} 0 = 0 = \lim_{x \to 0^-} x$

Thus, by Sqz Thm, $\lim_{x \to 0^-} \sin(x) = 0.$

Thus, $\lim_{x \to 0^-} \sin(x) = 0.$

Then, we can look at the following:

We know that for $x \in \left(-\pi/2, \pi/2\right)$ that:

$$\cos(x) = \sqrt{1 - \sin^2 x}$$

$$\text{Thus } \lim_{x \to 0} \cos(x) = \lim_{x \to 0} \sqrt{1 - \sin^2 x}$$

$$= \sqrt{\lim_{x \to 0} 1 - \lim_{x \to 0} \sin^2 x}$$

$$= \sqrt{1 - 0}$$

$\lim_{x \to 0} \cos(x) = 1$ (as expected)

We also know that $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

So, $\lim\limits_{x\to 0} \tan(x) = \lim\limits_{x\to 0} \dfrac{\sin(x)}{\cos(x)}$

$$= \frac{\lim\limits_{x\to 0} \sin(x)}{\lim\limits_{x\to 0} \cos(x)}$$

$$\lim_{x\to 0} \tan(x) = \frac{0}{1} = 0$$

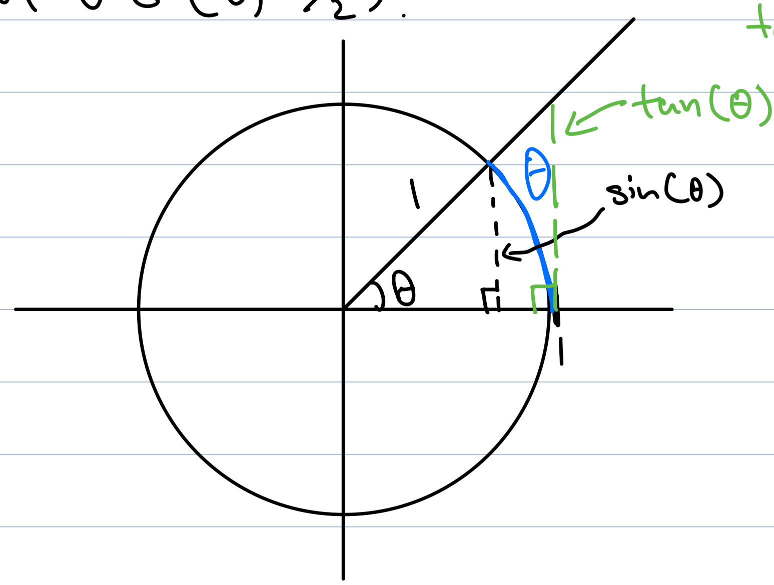
We can use squeeze theorem in a similar manner to find:

$\lim\limits_{\theta\to 0} \dfrac{\sin(\theta)}{\theta}$ ('Fundamental Trig Limits)

Let's revisit the idea of a unit circle.

We'll just look at the $\lim\limits_{\theta\to 0^+} \dfrac{\sin(\theta)}{\theta}$, $\lim\limits_{\theta\to 0^-} \dfrac{\sin(\theta)}{\theta}$ is very similar.

Let's look at $\theta \in (0, \pi/2)$.



$\tan(\theta) = \dfrac{O}{A}$

$= \dfrac{O}{1}$

$= O$

So here, we have three areas of interest:

1) Smaller triangle ($\cos\theta$ / $\sin\theta$ / 1)

2) Sector of circle (1 / 1 / θ)

3) Larger Triangle (1 / $\tan\theta$ / ?)

Notice that 1) < 2) < 3)

We can now get the formulas for these areas.

1) $\frac{\sin\theta \cdot \cos\theta}{2} = \frac{1}{2}\cos\theta\sin\theta$

2) $(\pi(1^2)) \cdot \left(\frac{\theta}{2\pi}\right) = \frac{1}{2}\theta$

3) $\frac{(1)(\tan\theta)}{2} = \frac{1}{2}\tan\theta$

So, we have:

$$\frac{1}{2}\cos\theta\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

It is very important to note that for $\theta \in \left(0, \frac{\pi}{2}\right)$ we have:

$\sin\theta > 0$ $\cos\theta > 0$

$\tan\theta > 0$ $\theta > 0$

Then, multiplying by $\frac{2}{\sin\theta}$

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}$$

$$\frac{1}{\cos\theta} > \frac{\sin\theta}{\theta} > \cos\theta$$

Then, $\lim_{x\to\theta^+} \cos\theta = 1 = \lim_{x\to\theta^+} \frac{1}{\cos\theta}$

So by squeeze theorem,

$$\lim_{\theta\to 0^+} \frac{\sin(\theta)}{\theta} = 1$$

The L-H Limit follows similarily, so we get

$$\lim_{x\to 0} \frac{\sin(\theta)}{\theta} = 1$$

Ex. 1.

$$\lim_{x\to 0} \frac{\sin(72x)}{\tan(9x)} = \lim_{x\to 0} \sin(72x) \cdot \frac{\cos(9x)}{\sin(9x)}$$

$$= \lim_{x\to 0} \sin(72x) \cdot \frac{72x}{72x} \cdot \frac{\cos(9x)}{\sin(9x)} \cdot \frac{9x}{9x}$$

$$= \lim_{x\to 0} \frac{\sin(72x)}{72x} \cdot \frac{9x}{\sin(9x)} \cdot \cos(9x) \cdot \frac{72x}{9x}$$

Side bar:

Note that $\lim_{\theta\to 0} \frac{\theta}{\sin\theta} = \lim_{\theta\to 0} 1 \Big/ \frac{\sin\theta}{\theta}$

$$= \frac{\lim_{\theta\to 0} 1}{\lim_{\theta\to 0} \frac{\sin\theta}{\theta}} = \frac{1}{1} = 1$$

$$= 1 \cdot 1 \cdot 1 \cdot 8 = 8$$

Ex. 2.

$$\lim_{x \to 1} \frac{\sin(x^2-1)}{\sin(x-1)} = \lim_{x \to 1} \frac{\sin(x^2-1)}{\sin(x-1)} \cdot \frac{x^2-1}{x^2-1}$$

$$= \lim_{x \to 1} \frac{\sin(x^2-1)}{\sin(x-1)} \cdot \frac{(x-1)(x+1)}{x^2-1}$$

$$= \lim_{x \to 1} \frac{\sin(x^2-1)}{x^2-1} \cdot \frac{x-1}{\sin(x-1)} \cdot (x+1)$$

$$= 1 \cdot 1 \cdot 2 = 2$$

$$\frac{\cancel{\cos x}}{x} \cdot \frac{\sin x}{\cancel{\cos x}} \cdot \frac{\cos x}{\sin x}$$

$$\frac{\sin x}{x} \cdot \frac{\cos x}{\sin x}$$

$$(2x+5)(x-2)$$

$$2x^2 + 5x - 4x - 10$$

E.x.

Show $\lim_{x \to 1} 2x^2 + x - 3 = 7$

Rough Work:

(If $0 < |x-2| < \delta$)

Want: $|(2x^2+x-3)-7| < \varepsilon$

$|2x^2+x-10| < \varepsilon$

$|\underbrace{(2x+5)}(x-2)| < \varepsilon$

depends on x. (δ only contains constants).

Assume $\delta \leq 1$.

$|x-2| < 1$

$\hookrightarrow x \in (1,3) \Rightarrow |2x+5| < 11.$

Then, returning to , we have

$|2x+5||x-2| < 11|x-2| < \varepsilon$

$|x-2| < \varepsilon/11.$

We will ask $\delta = \min\{1, \varepsilon/11\}$, since this gives that

- $\delta \leq 1 \Rightarrow$ this gives $|2x+5| < 11$
- $\delta \leq \varepsilon/11 \rightarrow$ this helps us say $|x-2| < \varepsilon/11$

Proof:

Given $\varepsilon > 0$, let $\delta = \min\{1, \varepsilon/11\}$.

Then, if $0 < |x-2| < \delta$, we have following:

$$|(2x^2+x-3)-7| = |2x+5|\,|x-2|$$

Then, since $\delta \leq 1$, we have $|x-2| < \delta \leq 1 \Rightarrow x \in (1,3)$

So, $|2x+5| < 11$.

Also, since $\delta \leq \varepsilon/11$, we have $|x-2| < \delta < \varepsilon/11$

Thus, $|(2x^2+x-3)-7| < 11|x-2|$

$$< 11\delta < 11(\varepsilon/11) = \varepsilon.$$

That is, $|(2x^2+x-3)-7| < \varepsilon \quad \forall \varepsilon > 0$

Thus, $\lim_{x\to 1} 2x^2+x-3 = 7$

Week 5 Infinity limits.

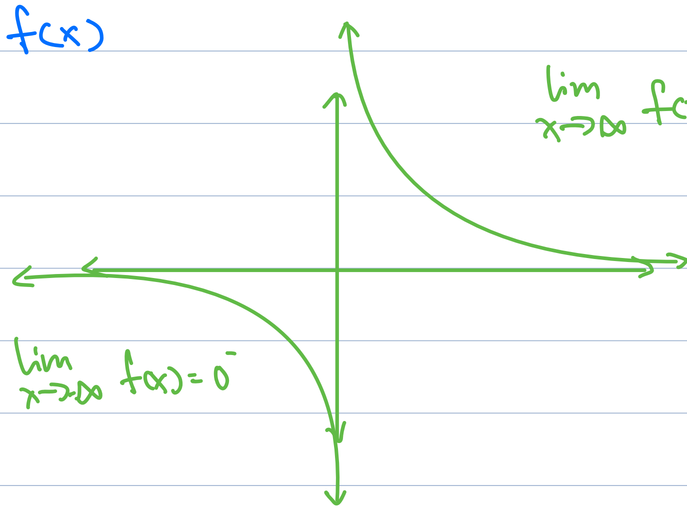
↳ End behaviour

↳ $\lim_{x \to \pm\infty} f(x)$

For e.x.:

$f(x) = \frac{1}{x}$

$\lim_{x \to \infty} f(x) = 0^+$



$\lim_{x \to -\infty} f(x) = 0^-$

Defn: limits at infinity

f is a fcn, $L \in \mathbb{R}$.

- $\lim_{x \to +\infty} f(x) = L$: $\forall \varepsilon > 0 \ \exists N > 0 \in \mathbb{R} \ni$
  if $x > N$, then $|f(x) - L| < \varepsilon$

- $\lim_{x \to -\infty} f(x) = L$: $\forall \varepsilon > 0 \ \exists N < 0 \in \mathbb{R} \ni$
  if $x < N$, then $|f(x) - L| < \varepsilon$

We can also define :

Defn: Horizontal Asymptote:

Assume $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$. Then, the line $y = L$ is a horizontal asymptote of $f(x)$.

Ex.

For $f(x)=\frac{1}{x}$, $y=0$ is a horizontal asymptote.

Some function may shoot off without bounds as $x\to\pm\infty$ rather than approach a value.

Defn: limits at $\pm\infty$.

- $\lim_{x\to\infty} f(x)=+\infty$: $\forall M>0\ \exists N>0 \ni$ if $x>N$ then $f(x)>M$.
- $\lim_{x\to\infty} f(x)=-\infty$: $\forall M<0\ \exists N>0 \ni$ if $x>N$ then $f(x)<M$.
- $\lim_{x\to-\infty} f(x)=+\infty$: $\forall M>0\ \exists N<0 \ni$ if $x<N$ then $f(x)>M$.

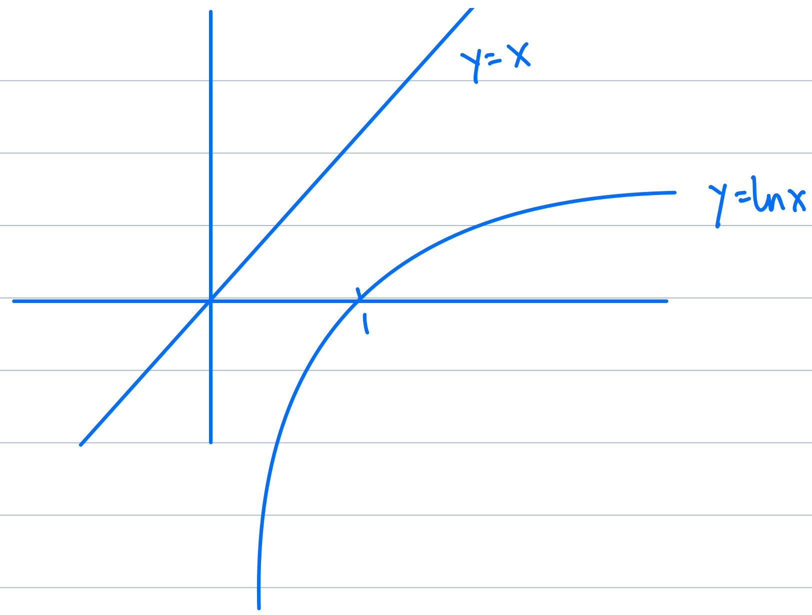
Squeeze Theorem still holds way out at $\pm\infty$:

$\boxed{\text{Thm 9: Sqz Thm at } \pm\infty.}$

- Assume $g(x)\le f(x)\le h(x)$ $\forall x\ge N$. If $\lim_{x\to\infty} g(x)=L=\lim_{x\to\infty} h(x)$ then $\lim_{x\to+\infty} f(x)=L$.
- Assume $g(x)\le f(x)\le h(x)$ $\forall x\le N$. If $\lim_{x\to\infty} g(x)=L=\lim_{x\to\infty} h(x)$ then $\lim_{x\to-\infty} f(x)=L$.

We will now apply sqz Thm at ∞ to examine:

$$\lim_{x\to\infty}\frac{\ln(x)}{x}$$ "Fundamental Log Limit"



Note that for $x \geq 1$ we have $x > 0$ & $\ln(x) \geq 0$.

As well, note that for $x \geq 0$, $\ln(x) \leq x$.

We are looking at $x \to \infty$, in this regime both observations hold true.

Then for $x \to \infty$ we have $0 \leq \frac{\ln(x)}{x}$

By being lil sneaky, we can note:

$$\frac{\ln x}{x} = \frac{\ln\left(x^{\frac{1}{3}}\right)^3}{x^{\frac{2}{3}}\, x^{\frac{1}{3}}} = \frac{3}{x^{\frac{2}{3}}}\left(\frac{\ln x^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right) \leq 1$$

That is, we have:

$$0 \leq \frac{\ln(x)}{x} = \frac{3}{x^{\frac{2}{3}}}\left(\frac{\ln x^{\frac{1}{3}}}{x^{\frac{1}{3}}}\right) \leq \frac{3}{x^{\frac{2}{3}}}$$

$$0 \leq \frac{\ln(x)}{x} \leq \frac{3}{x^{\frac{2}{3}}}$$

Now, $\lim_{x \to \infty} 0 = 0 = \lim_{x \to \infty} \frac{3}{x^{\frac{2}{3}}}$

Then, by Sqz Thm,

$$\boxed{\lim_{x\to\infty} \frac{\ln x}{x} = 0}$$

E.x(1)

$$\lim_{x\to\infty} \frac{\ln(x)}{x^p} = \lim_{x\to\infty} \frac{\ln(x^p)^{\frac{1}{p}}}{x^p} = \lim_{x\to\infty} \frac{1}{p} \cdot \frac{\ln(x^p)}{x^p}$$

$$= \frac{1}{p} \cdot (0) = 0$$

E.x.2

$$\lim_{x\to\infty} \frac{\ln(x^p)}{x} = \lim_{x\to\infty} p \cdot \frac{\ln(x)}{x}$$

$$= p \cdot 0 = 0$$

E.x.3 $(p>0)$

$$\lim_{x\to\infty} \frac{x^p}{e^x}$$ let $u = e^x$ note, for $x\to\infty$
$u\to\infty$

$$= \lim_{u\to\infty} \frac{\ln u^p}{u}$$ $\Rightarrow x = \ln u$

$$= \lim_{u\to\infty} \left[\frac{\ln u}{u^{\frac{1}{p}}}\right]^p = 0^p = 0$$

E.x.4 $(p>0)$

$\lim x^p \ln(x)$ let $u = \frac{1}{}$ note for $x\to 0^+$

$x \to 0^+$ ... let $u = \frac{1}{x}$, note for $x \to 0^+$

$x = \frac{1}{u}$ $\qquad u \to +\infty$

$= \lim_{x \to 0^+} \left(\frac{1}{u}\right)^p \ln\left(\frac{1}{u}\right)$

$= \lim_{x \to 0^+} \left(\frac{1}{u^p}\right) \ln(u^{-1})$

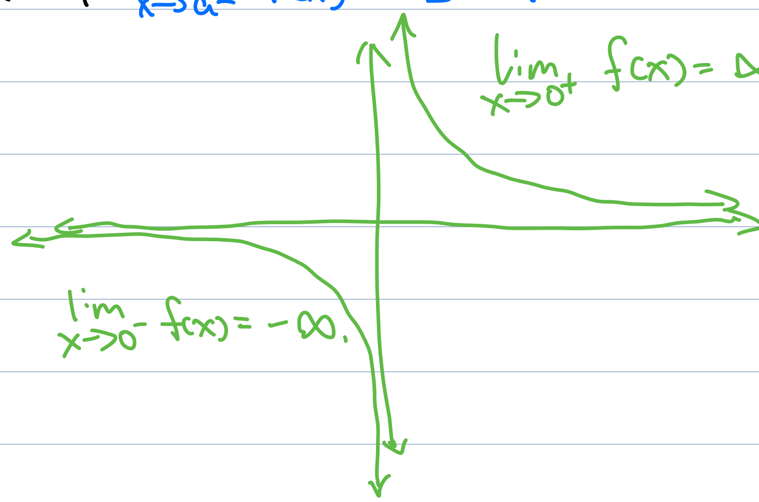
$= \lim_{x \to \infty} - \frac{\ln(u)}{u^p} = 0^p = 0$

Observation: $(p > 0)$

$$[\ln(x)]^p \ll x^p \ll p^x \ll x^x \text{ as } x \to \infty$$

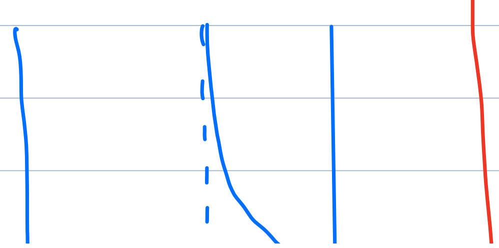
We saw previously that we could have $\lim_{x \to \pm\infty} f(x) = \pm\infty$.
Now, we look at $\lim_{x \to a^\pm} f(x) = \pm\infty$.

Ex.
$f(x) = \frac{1}{x}$

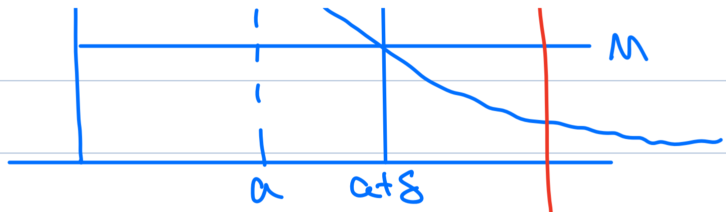


Defn: Infinite limits.

- Right side $\Rightarrow$ $\lim_{x \to a^+} f(x) = +\infty$:



$\forall M > 0 \ \exists \delta > 0 \ni$ if $|x - a| < \delta$ and $x > a$, that is $a < x < a + \delta$, then $f(x) > M$.



$\lim_{x \to a^+} f(x) = +\infty$

$\forall M < 0 \ \exists \delta > 0 \ni$ if
$|x - a| < \delta$ and $x > a$,
that is $a < x < a + \delta$, then
$f(x) < M$.

- Left-side
  - $\lim_{x \to a^-} f(x) = +\infty$
  - $\lim_{x \to a^-} f(x) = -\infty$

From our work with two vs one-sided limits,
we say $\lim_{x \to a} f(x) = \pm\infty$ if $\lim_{x \to a^-} f(x) = \pm\infty = \lim_{x \to a^+} f(x)$

Note: $\pm\infty$ are not values, limit doesn't exist.

Defn: Vertical Asymptotes
If any of $\lim_{x \to a^\pm} f(x) = \pm\infty$ then the line
$x = a$ is a vertical asymptote of $f(x)$.

Ex.
$f(x) = \frac{1}{x}$ has a vertical asymptote at $x = 0$.

Defn: Continuity (In 3 flavours)

- We say a fnc is continuous at a point $x=a$ if
  a) $\lim_{x\to a} f(x)$ exists, and
  b) $\lim_{x\to a} f(x) = f(a)$

  Otherwise we say the function is discontinuous at $x=a$, which is a point of discontinuity.

- We say a function is continuous at $x=a$ if $\forall \varepsilon>0\ \exists \delta>0 \ni$ if $|x-a|<\delta$ then $|f(x)-f(a)|<\varepsilon$.

- A function f is continuous at $x=a$

  $\overset{\text{iff}}{\Longleftrightarrow}$

  $\{x_n\}$ is a sequence with $\lim_{n\to\infty} x_n = a$ then $\lim_{x\to a} f(x_n) = f(a_n)$

We can rewrite our continuity statement:

Notice that for $x\neq a$ we have $x=a+h$, $h\neq 0$.

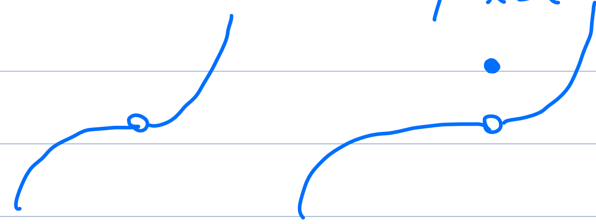
Then, $\lim_{x\to a} f(x) = \lim_{h\to 0} f(a+h)$

So a function is continuous at $x=a$

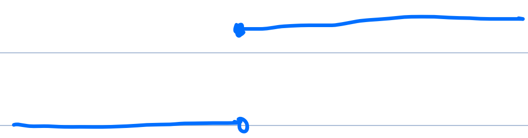
$\overset{\text{iff}}{\Longleftrightarrow}$ $\lim_{h\to 0} f(a+h) = f(a)$

There are a few different kinds of discontinuities:

- holes / removable discontinuity: $\lim_{x\to a}$ exist but $\neq f(a)$



- Jumps: finite / infinite: $\lim_{x\to a^-} \neq \lim_{x\to a^+}$



Essentials discty

$\lim_{x\to a}$ DNE

- Oscillatory: $\lim_{x\to a}$ DNE due to infinite oscillation

  ↳ $f(x) = \cos\left(\frac{1}{x}\right)$

Types of discontinuities

We can examine the continuity of common fcns:

Polynomials: $\lim_{x\to a} p(x) = p(a)$

↳ Definition of continuity.

$\therefore$ Polynomials are continuous $\forall\, a \in \mathbb{R}$.

Sin(x) / cos(x): We showed $\lim_{x\to 0} \sin(x) = 0 = \sin(0)$ using squeeze thm. We use that result to also show $\lim_{x\to 0} \cos(x) = 1 = \cos(0)$

Then, $\lim_{x\to a} \sin(x) = \lim_{h\to 0} \sin(a+h)$

Trig

$$= \lim_{h\to 0} \left(\sin(a)\cos(h) + \sin(h)\cos(a)\right)$$

$$= \sin(a)(1) + (0)\cos(a)$$

$$= \sin(a)$$

$\therefore$ sin(x) is continuous $\forall\, a \in \mathbb{R}$

Try cos(x) is cts $\forall\, a \in \mathbb{R}$.

e^x / ln(x): These are not simple to show.

"Power series" (M138).

But if we assume e^x is cts at $x = 0$, we can show it is cts $\forall a \in \mathbb{R}$.

So: take as fact: $\lim_{x\to 0} e^x = 1 = e^0$

Then, $\lim_{x\to a} e^x = \lim_{h\to 0} e^{a+h} = \lim_{h\to 0} e^a e^h = e^a(1) = e^a$

$\therefore$ e^x is cts $\forall a \in \mathbb{R}$.

We make a geometric argument for $\ln(x)$.

Note $\ln(x)$, being inverse of e^x, is a reflection of e^x across $y = x$.

Then, as there are no brakes in e^x. there will be no breaks in the reflection, $\ln(x)$.

$\therefore$ $\ln(x)$ is cts on its domain.

Theorem 12: Continuity of Inverses.

If $y = f(x)$ is invertible with inverse $f^{-1}(y) = x$, and $f(a) = b$ and $f(x)$ is continuous at $x = a$, then $f^{-1}(x)$ is cts at $x = b$.

We can expand our base tool kit of cts fcns:

Thm 13: Continuity of Sums & Products

Let f & g be cts at $x=a$. Then,

1) $f+g$ is cts at $x=a$
2) fg is cts at $x=a$

Thm 14: Continuity of Quotients

Let f and g be continuous at $x=a$. If $g(a) \neq 0$ then f/g is cts at $x=a$.

Thm 15: Continuity of Compositions

Let f be continuous at $x=a$ and let g be cts at $x=f(a)$. Then $h(x) = g(f(x))$ is continuous at $x=a$

Proof:

Let f be cts at $x=a$, let g be cts at $x=f(a)$, let $h=g(f(x))$.

Let $\{x_n\}$ be a sequence $\ni x_n \to a$

Then, since f is cts at $x=a$, by the sequential characterization defn of cty, $\lim_{n\to\infty} f(x_n) = f(a)$.

That is, $\{f(x_n)\}$ is a sequence $\ni f(x_n) \to f(a)$.

Then, since g is cts at $x=f(a)$, by the sequential characterization defn of continuity, $\lim_{n\to\infty} g(f(x_n)) = g(f(a))$.

That is, we have $\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} g(f(x_n)) = g(f(a)) = h(a)$.

Then by sequential characterization defn of cty, $h(x)$ is cts at $x=a$, $\forall a \in \mathbb{R}$. $\square$

E.x.

$f(x) = 2^{2x\sin(e^x)}$

This fcn is cts $\forall a \in \mathbb{R}$ cuz:

- e^x cts $\forall a \in \mathbb{R}$
- $\sin(x)$ cts $\forall a \in \mathbb{R}$
- $2 \cdot x$ cts $\forall a \in \mathbb{R}$
- 2^x cts $\forall a \in \mathbb{R}$
- applications of cty of product & compositions.

There is some nuance to deal with:

Defn: Continuity on (a,b) [or $\mathbb{R}$]

We say f is cts on the open interval (a,b) [or $\mathbb{R}$] if it is continuous at each $x \in (a,b)$ [or $x \in \mathbb{R}$].

Defn: Continuity on $[a,b]$

We say f is cts on closed interval $[a,b]$ if:

1) it is cts $\forall x \in (a,b)$.
2) $\lim_{x \to a^+} f(x) = f(a)$.
3) $\lim_{x \to b^-} f(x) = f(b)$.

Ex.

$f(x) = x^{\frac{1}{4}} = \sqrt[4]{x}$

This is cts on $x \in [0, \infty)$ since it is cts $\forall x \in (0, \infty)$ and $\lim_{x \to 0^+} x^{\frac{1}{4}} = 0 = 0^{\frac{1}{4}}$.

WSIC?

Continuity of fcns will be crucial requirements for many major calculus theorems. We have to be mindful of our fcns before applying these theorems.

Week 6.

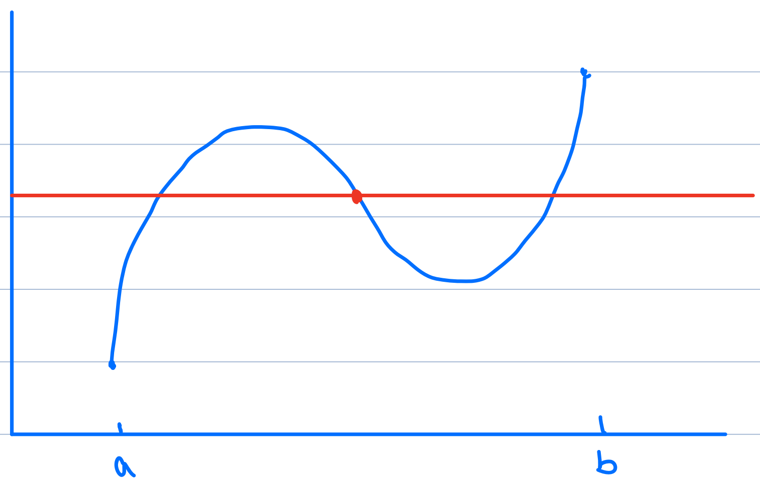
We now state a key theorem:

Thm 16: Intermediate Value Theorem (IVT)

Assume that f is cts on closed interval $[a,b]$ and either $f(a) < \alpha < f(b)$ or $f(a) > \alpha > f(b)$.

α : some intermediate value

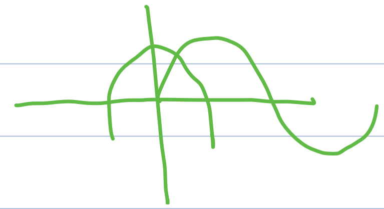
Then there exist a $c \in (a,b)$ such that $f(c) = \alpha$.



This makes intuition sense. A cts fcn which starts above / below α and ends below / above α at two point must hit α at some point in between.

E.x.1.

Prove that $\sin(x)$ & $1-x^2$ intersects for some $c \in (0,1)$.

1) Graph 

We are trying to show $\sin(x) = 1-x^2$ for some $c \in (0,1)$.

This is the same as showing fcn

$$f(x) = \sin(x) - (1-x^2) = 0$$

for some $c \in (0,1)$.

First & foremost, $f(x)$ is cts on $[0,1]$.
↳ We have the sum of two cts fcns.

Now, note $f(0) = \sin(0) - (1-0^2) = -1 < 0$
And, note $f(1) = \sin(1) - (1-1^2) = \sin(1) > 0$
Then, by IVT, $f(x) = 0$ for some $c \in (0,1)$.
That is, $\sin(x) = 1-x^2$ for some $c \in (0,1)$.

E.X. 2

Prove that x^3+3x^2-x-3 has a root on $[-5,-2]$.

First note $f(x) = x^3+3x^2-x-3$ is a polynomial so it is cts on $[-5,-2]$.
Now we find $f(-5) = -48 > 0$
and we find $f(-2) = 3 > 0$
Then, by IVT, $f(x) = 0$ for some $c \in (-5,-2)$.

Now, we can make a good apprxmtn for solns to $f(x)=0$ for cts fcns $f(x)$.

We saw for $f(x) = x^3 + 3x^2 - x - 3$ that $f(-5) < 0$ and $f(-2) > 0$

So $f(x) = 0$ btw $x = -5$ & $x = -2$.

Let's check the midpoint of the interval $x = -\frac{7}{2}$.

We'd find $f(-\frac{7}{2}) = -\frac{45}{8} < 0$

Then since $f(-2) > 0$, by IVT we know $f(x) = 0$ between $x = -\frac{7}{2}$ & $x = -2$.

We check the new midpoint & find $f(-\frac{7}{2}) = \frac{105}{64} > 0$

Then, since $f(-\frac{7}{2}) < 0$, by IVT we know $f(x) = 0$ btw $x = -\frac{7}{2}$ & $x = -\frac{11}{4}$.

We will get closer to the exact solution $(x = -3)$ every iteration.

Every iteration cuts our interval in half.

Our newest interval had length $|-3.5 - (-2.75)| = 0.75$ whereas the original interval has length $|-5 - (-2)| = 3$.

So, our final interval had length

$$\frac{1}{2^2}(3)$$

Our final guess to the solution would be the midpoint of the final interval.

In the worst case the final guess is half the final interval away from the exact soln.

That is $x = -\frac{25}{8} = -3.125$ is at most $\frac{1}{2^3}(3)$ away from the exact soln.

(The actual error is 0.125).

Note :

- $\frac{1}{2^4} = \frac{1}{16} < \frac{1}{10} \Rightarrow$ 4 iterations $\Rightarrow$ acc improve 1 Decimal points.

- $\frac{1}{2^{10}} = \frac{1}{1024} < \frac{1}{1000} \Rightarrow$ 10 iterations $\Rightarrow$ acc improve 3 Decimal points.

Bisection Methods

Suppose we want to approximate the soln to $F(x) = 0$.
(So for $f(x) = g(x)$ we set $F(x) = f(x) - g(x)$).
Here F is cts on the relevant interval, and we have some error to be less than, ε.

Step 1: Find two points $a_0 < b_0 \ni F(a_0)$ and $F(b_0)$ are on opposite sides of 0.
Then IVT guarantees $\exists\, c \in (a_0, b_0) \ni F(c) = 0$.

Step 2: Find the midpoint of interval $[a_0, b_0]$,

$d = \frac{a_0 + b_0}{2}$, and evaluate $F(d)$.

Step 3: If $F(a_0)$ and $F(d)$ have the same sign, let $a_1 = d$ and $b_1 = d$. Otherwise, let $a_1 = a_0$ and $b_1 = d$. This gives new interval $[a_1, b_1]$ of length $\frac{1}{2}(b_0 - a_0)$ containing a soln to $F(x) = 0$.

Step 4: Repeat Step 2 & 3 to obtain new soln, containing a soln to $F(x) = 0$.

The kth interval has length $\frac{1}{2^k}(b_0 - a_0)$.

Step 5: Stop when $\frac{1}{2^{k+1}}(b_0 - a_0) < \varepsilon$. Take the final approx $d = \frac{a_k + b_k}{2}$

Then, we know $\exists c \ni F(c) = 0$ where $|d - c| < \varepsilon$.

Next up: another theorem, terminologies

Defn: Global Maxima & Global Minima (or Absolute)

Suppose $f: I \to \mathbb{R}$ where I is some interval.

- We say c is a global maximum for f on I if $c \in I$ and $f(x) \leq f(c)$ $\forall x \in I$.

- We say c is a global minimum for f on I if $c \in I$ and $f(x) \geq f(c)$ $\forall x \in I$.

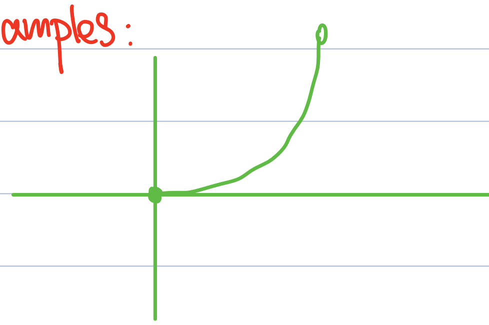
- We say c is global extremum for f on I if either a global max or global min.

Can we say if f is defined on a non-empty interval I, then f achieves a global max & global min on I?

No! Consider two counter-examples:

C-E.x.1)

Take $f(x) = x^2$ on $[0, 1)$

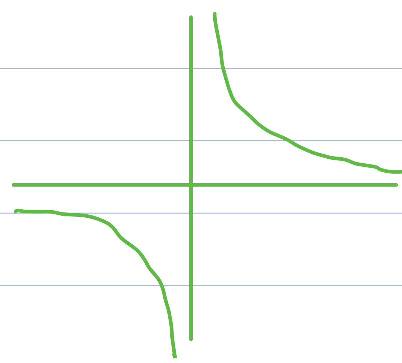


Here $f(x)$ has a global min of 0 at $x=0$, but no global max.

We note that an end-point is then a key feature of obtaing a global max / min.

C. E.x.2

Take $f(x) = \frac{1}{x}$ on $[-1, 1]$



Here $f(x)$ has neither global max nor global min.
We note that discontinuities seem to matters.

Putting together:

Thm 17: Extreme Value Theorem (EVT)

Suppose f is cts on $[a,b]$. Then, there exist two numbers c_1 & c_2 $\in [a,b]$ s.t.

$$f(c_1) \le f(x) \le f(c_2)$$
$$\forall x \in [a,b].$$

That is, there exist a global max & a global min.

Note: This gives us existence, it doesn't tell us how to find a global extrema.

— END OF MIDTERM CONTENT

Derivative

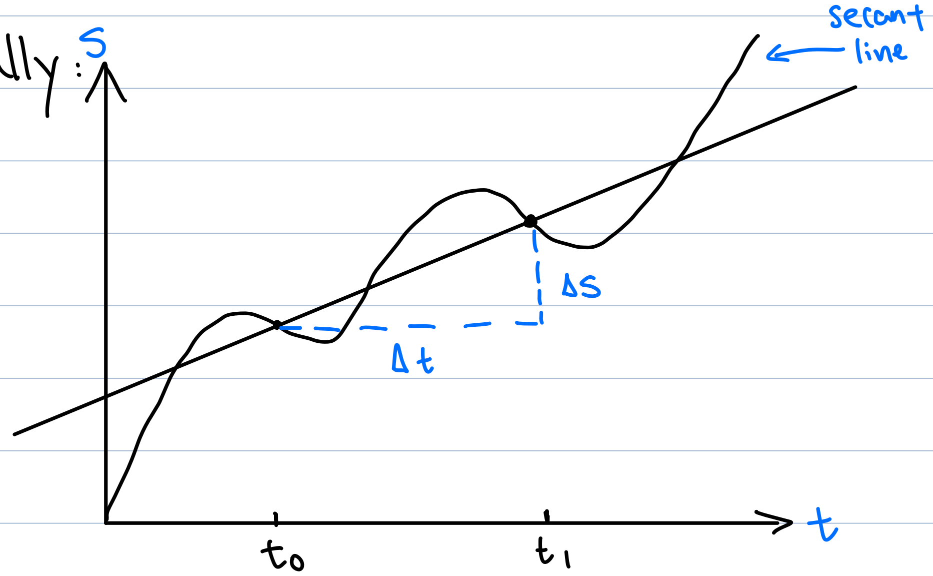
We are now interested in rate of change.

We start by looking at displacement and velocity.
The average velocity between two times t_0 & t_1, is given by:

$$V_{avg} = \frac{\Delta s}{\Delta t}$$

$$= \frac{s(t_1) - s(t_0)}{t_1 - t_0} = \frac{\text{change in displacement}}{\text{change in time}}$$

Graphically:



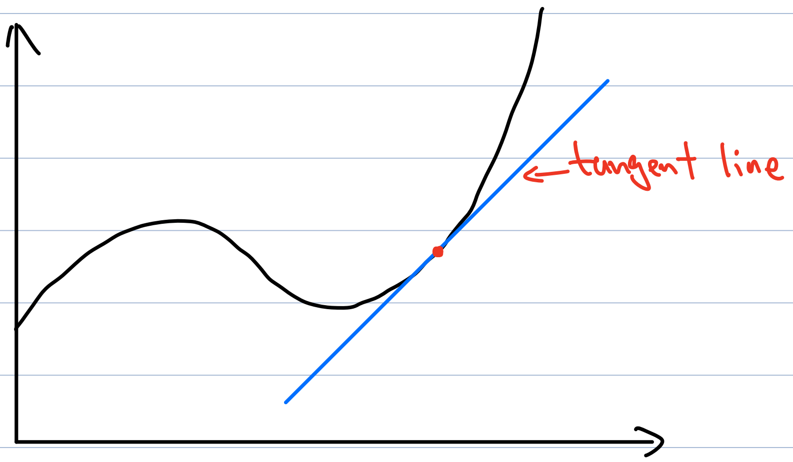
So, slope of the secant line is the average

We want instantaneous velocity.
Idea is to calculate slopes of secants as we bring t_1 closer to t_0.

To find velocity at time t_0, we examine the limit:

$$\lim_{t \to t_0} \frac{s(t)-s(t_0)}{t-t_0} = v(t_0)$$

Graphically:



The slope of the tangent line is the instantaneous r.o.c.

Reframe our limit like we did with cty:

Note that for $t \neq t_0$ we have $t = t_0 + h$.

$$V_{inst} = V(t_0) = \lim_{t \to t_0} \frac{s(t)-s(t_0)}{t-t_0}$$

$$= \lim_{h \to 0} \frac{s(t_0+h)-s(t_0)}{(t_0+h)-t_0} = \lim_{h \to 0} \frac{s(t_0+h)-s(t_0)}{h}$$

Defn: Average R.O.C.

The average r.o.c of a fcn f between $x=a$ & $x=b$.

is $f_{avg} = \dfrac{f(b)-f(a)}{b-a}$

Defn: Instantaneous R.O.C / The derivative at $x=a$.

The instantaneous r.o.c of f at $x=a$, or the derivative of x at $x=a$, denote as $f'(a)$, is:

Newton's Quotient

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a} = \boxed{\lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

If this limit exists, we say f is differentiable at $x=a$.

Point intercept of a line:

$$y = mx+b$$

Point-slope form:

$$y-y_1 = m(x-x_1) \Rightarrow y = m(x-x_1)+y_1$$

$m \Rightarrow$ slope

$(x_1, y_1) \Rightarrow$ point on the line.

Defn. Tangent Line

If f is differentiable at $x=a$, then the tangent line to f at $x=a$ is the line passing thru $(f, f(a))$ with slope $f'(a)$. Then, the equation of tangent line is:

$$y - f(a) = f'(a)(x-a)$$
$$y = f'(a)(x-a) + f(a)$$

Ex. 1

Find the instantaneous velocity of $s(t) = 5t^2 + 6t - 7$.

a) at $t = 7$

b) at $t = t_0$

a) $$s'(7) = \lim_{h \to 0} \frac{s(7+h) - s(7)}{h}$$

$$= \lim_{h \to 0} \frac{5(7+h)^2 + 6(7+h) - 7 + (5(7)^2 + 6 \cdot 7 - 7)}{h}$$

$$= \lim_{h \to 0} \frac{76h + 5h^2}{h}$$

$$= \lim_{h \to 0} 76 + 5h = 76.$$

b) $$s'(t_0) = \lim_{h \to 0} \frac{s(t_0+h) - s(t_0)}{h}$$

Note. Velocity is derivative of displacement.

$$= \lim_{h \to 0} \frac{5(t_0+h)^2 + 6(t_0+h) - 7 - 5t_0^2 - 6t_0 + 7}{h}$$

$$= \lim_{h \to 0} \frac{(10t_0 + 6)h + 5h^2}{h} = \lim_{h \to 0} 10t_0 + 6 + 5h$$

$$= 6 + 10t_0 = v(t_0)$$

Ex.2.

Find equation of tangent line to $f(x) = \frac{1}{x+5}$ at $x=3$.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(h)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{1}{8+h} - \frac{1}{3+5}}{h}$$

$$= \lim_{h\to 0} \left[\frac{8-(h+8)}{8(h+8)}\right]\left[\frac{1}{h}\right]$$

$$= \lim_{h\to 0} \frac{-1}{8(h+8)} = -\frac{1}{64}$$

We have $f'(3) = -\frac{1}{64}$, we have $x=3$,
$f(3) = \frac{1}{8}$.

Equation:

$$y = f'(a)(x-a) + f(a)$$
$$= -\frac{1}{64}(x-3) + \frac{1}{8}$$

Thm 1: Differentiability Implies Continuity.

If f is diff'ble at $x=a$, then it is cts at $x=a$.

[By contrapositive], If f is not cts at $x=a$, then it is not diff'ble at $x=a$.

Proof: Let f be diff'ble at $x=a$. Then,

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists.}$$

By thm earlier in the course, since $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists and $\lim_{x\to a} x-a=0$, then $\lim_{x\to a} [f(x)-f(a)] = 0$

$$\lim_{x\to a} [f(x)] - f(a) = 0$$

$$\lim_{x\to a} f(x) = f(a).$$

Thus f is cts at $x=a$.

Continuity $\Rightarrow$ Differentiability?

↳ No, Count-ex: Examine $f(x)=|x|$ at $x=0$.

We see $\lim_{x\to 0} |x| = 0 = |0|$, so $f(x)$ is cts at $x=0$.

But, $f'(0) = \lim_{h\to 0} \frac{|0+h|-|0|}{h} = \frac{|h|}{h}$

Now $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} -\frac{h}{h} = \lim_{h\to 0^-} -1 = -1$

$\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = \lim_{h\to 0^+} 1 = +1$

Then, $\lim_{h\to 0} \frac{|h|}{h}$ DNE, that is $f'(0)$ DNE.

$\rightarrow f$ isn't differentiable at $x=0$.

Defn: The Derivative Function

The derivative fcn f' is defined as.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

We say f is differentiable on interval I if $f'(a)$ exist $\forall a \in I$.

So, the derivative fcn is the derivative of f at each $x \in I$.

'Primed' Notation → Newton notation

There's also Leibniz Notation.

Given $y = f(x)$ the derivative is:

$$\frac{dy}{dx} = \frac{d}{dx}(y) = \frac{df}{dx} = \frac{d}{dx}(f)$$

Differential Operator

We write $f'(a)$ as $\left.\frac{dy}{dx}\right|_{x=a}$

Defn: N^{th} Derivative

If f is n times diff'ble, we denote the n^{th} derivative as: $f^{(n)} = \frac{d^n}{dx^n}(f) = \frac{d}{dx}(f^{(n-1)})$

Ex.

a) If f is 2 times diff'ble, the second derivative is:

$$f'' = \frac{d^2}{dx^2}(f) = \frac{d}{dx}(f')$$

b) If f is 17 times diff'ble, the 17^{th} derivative is:

$$f^{(17)} = \frac{d^{17}}{dx^{17}}(f) = \frac{d}{dx}(f^{16})$$

Derivative of a constant function.

Let $c \in \mathbb{R}$, $f(x) = c$. Then,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(h)}{h}$$

$$= \lim_{h \to 0} \frac{c-c}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

That is, for $f(x) = c$, we have $f'(x) = 0$.

Derivative of a Linear Fcn.

Let $f(x) = mx + b$, $m, b \in \mathbb{R}$. Then $f'(x) = m$.

Exercise: Show by defn.

Derivative of a Quadratic Fcn

Let $f(x) = ax^2 + bx + c$, $a, b, c \in \mathbb{R}$. Then $f'(x) = 2ax + b$.

Derivative of cos (x).

$$f'(x) = \lim_{h \to 0} \frac{\cos(x+h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$$

$$= \lim_{h \to 0} \left[\cos(x)\left(\frac{\cos(h) - 1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right]$$

$$\lim_{h \to 0} -\sin x\left(\frac{\sin(h)}{h}\right)$$

$$= (-\sin x)\ (1)$$

$$= -\sin(x)$$

Let's find:

$$\lim_{h \to 0} \frac{\cos(h) - 1}{h} \cdot \frac{\cos(h) + 1}{\cos(h) + 1}$$

$$= \lim_{h \to 0} \frac{\cos^2(h) - 1}{h(\cos(h) + 1)}$$

$$= \lim_{h \to 0} \frac{(1 - \sin^2(h)) - 1}{h(\cos(h) + 1)}$$

$$= \lim_{h \to 0} \frac{-\sin^2(h)}{h(\cos(h) + 1)}$$

$$= \lim_{h \to 0} \frac{\sin(h)}{h} \cdot \frac{-\sin(h)}{\cos(h) + 1} = 1\left(\frac{-0}{2}\right) = 0$$

Returning to the main problem:

$$f'(x) = \lim_{h\to 0}\left[\cos(x)\left(\frac{\cos h - 1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right]$$
$$= \cos(x)(0) - \sin(x)(1)$$
$$= -\sin(x)$$

That is, for $f(x) = \cos(x)$, we have $f'(x) = -\sin(x)$.

<u>Derivative of $\sin(x)$.</u>

For $f(x) = \sin(x)$, we have $f'(x) = \cos(x)$.

Derivative of e^x

There are several differen ways to define e:

Most Common:

- $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$
- $\lim_{n\to\infty}(1+n)^{\frac{1}{n}} = e$

For us today:

We consider e to be the unique value for which an exponential fcn of form $y = a^x$ has a tangent line with slope 1 thru $(0,1)$.

That is, we have defined that for $f(x) = e^x$, we have

$$f'(0) = 1 = \lim \frac{e^{0+h} - e^0}{}$$

$f'(0) = 1 = \lim_{h\to 0} \qquad h$

Now, for $f(x) = e^x$, we have:

$$f'(x) = \lim_{h\to 0} \frac{e^{x+h} - e^x}{h}$$

$$= \lim_{h\to 0} e^x \left(\frac{e^h - 1}{h} \right)$$

$$= e^x (1)$$

$$= e^x.$$

That is, for $f(x) = e^x$, we have $f'(x) = e^x$.

As with limits, we have arithmetic rules we use in practice.

Thm 7: Arithmetic Rules for Differentiation

Assume f & g are diff'ble at $x = a$.

1) Constant Multiple Rule: Let $h(x) = c\,f(x)$.
Then h is diff'ble at $x = a$, and:

$$h'(a) = c\,f'(a).$$

2) Sum Rule: Let $h(x) = f(x) + g(x)$.
Then, h is diff'ble at $x = a$, and:

$$h'(a) = f'(a) + g'(a).$$

3) Product Rule: Let $h(x) = f(x)\,g(x)$.

Then, h is diff'ble at $x=a$, and:

$uv: v\,du + u\,dv.$ $\quad h'(a) = f'(a)\,g(a) + f(a)\,g'(a).$

4) Reciprocal Rule: Let $h(x) = \frac{1}{g}(x)$. If $g(a) \neq 0$, then h is diff'ble at $x=a$, and:

$$h'(a) = \frac{-g'(a)}{[g(a)]^2}$$

5) Quotient Rule: Let $h(x) = f(x)/g(x)$. If $g(a) \neq 0$, then h is diff'ble at $x=a$, and:

$\frac{u}{v}: \frac{v\,du - u\,dv}{v^2}$ $\quad h'(a) = \dfrac{g(a)\,f'(a) - f(a)\,g'(a)}{[g(a)]^2}$

Pfs of Arithmetic Differentiation Rules

1) and 2) are literally limit laws.

3) [Product Rule]

By defn, $(fg)'(a) = \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a)g(a)}{h}$

$$= \lim_{h\to 0} \frac{f(a+h)g(a+h) - f(a+h)g(a+h) + f(a+h)g(a+h) - f(a)\,g(a)}{h}$$

$$= \lim_{h\to 0}\left[f(a+h)\left[\frac{g(a+h)-g(a)}{h}\right] + g(a)\left[\frac{f(a+h)-f(a)}{h}\right]\right]$$

$\circledast$ defn derivative constant defn derivative

$$= f(a)\, g'(a) + g(a)\, f'(a)$$

We can do step $\circledast$ because f is diff'ble which means it is continuous which by defn means $\lim_{x\to a} f(x) = f(a)$ or $\lim_{h\to 0} f(a+h) = f(a)$

4) [Reciprocal Rule]

By defn $\left(\frac{1}{g}\right)'(a) = \lim_{h\to 0} \dfrac{\frac{1}{g(a+h)} - \frac{1}{g(a)}}{h}$

$$= \lim_{h\to 0} \frac{g(a)-g(a+h)}{h[g(a)g(a+h)]}$$

$$= \lim_{h\to 0}\left[-\frac{g(a+h)-g(a)}{h}\cdot\frac{1}{g(a)\,g(a+h)}\right]$$

$$= -g'(a)\cdot\frac{1}{g(a)g(a)}$$

g is diff'ble $\Rightarrow$ cts so by defn.

$$= \frac{-g'(a)}{[g(a)]^2}$$

5) [Quotient Rule]

We can combine 3) & 4) have:

$$\left(\frac{f}{g}\right)'(a) = \left(f \cdot \frac{1}{g}\right)'(a)$$
$$= f'(a) \cdot \left(\frac{1}{g}\right)(a) + f(a)\left(\frac{1}{g}\right)'(a)$$
$$= \frac{f'(a)}{g(a)} + f(a)\left[\frac{-g'(a)}{[g(a)]^2}\right]$$
$$= \frac{f'(a)g(a) - f(a)g'(a)}{[g(a)]^2}$$

There r a few more rules we state without pf:

Thm 8: Power Rule

Assume $\alpha \in \mathbb{R}$, $\alpha \neq 0$, and $f(x) = x^\alpha$.
Then f is diff'ble and $f'(x) = \alpha \cdot x^{\alpha-1}$, wherever $x^{\alpha-1}$ is defined.

E.x.
$$f(x) = x^\pi \Rightarrow f'(x) = \pi \cdot x^{\pi-1}$$

Thm 9: The Chain Rule

Assume $y = f(x)$ is diff'ble at $x=a$ and $z = g(y)$ is diff'ble at $y = f(a)$. Then, $h(x) = g \circ f(x) = g(f(x))$ is diff'ble at $x = a$ and $h'(a) = g'(f(a)) f'(a)$.

Note: Leibniz Notation pays off here:

For $z = g(y)$ $\wedge$ $y = f(x)$ we get:

Then, for $z = g(y) = g(f(x))$ we have that

$$\frac{dz}{dy} = g'(f(x))\, f'(x) = \left.\frac{dz}{dy}\right|_{f(x)} \left.\frac{dy}{dx}\right|_{x}$$

That is $\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$

Ex.

$$f(x) = \sin^2(x) \Rightarrow f'(x) = \cos(x^2) \cdot \frac{d}{dx}(x^2)$$
$$= \cos(x^2)(2x)$$
$$= 2x \cos x^2$$

Ex.1

$$f(x) = \tan x = \frac{\sin(x)}{\cos(x)}$$

$$\frac{d}{dx}[\tan(x)] = \frac{d}{dx}\left[\frac{\sin(x)}{\cos(x)}\right]$$

$$= \frac{(\cos(x))\frac{d}{dx}(\sin(x)) - (\sin(x))\frac{d}{dx}(\cos(x))}{[\cos(x)]^2}$$

$$= \frac{(\cos(x))(\cos(x)) - (\sin(x))(-\sin(x))}{\cos^2(x)}$$

$$= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)}$$

$$= \frac{1}{\cos^2(x)} = \sec^2(x)$$

Ex. 2

$f(x) = \csc(x)$. Find $f'(x)$

$\csc(x) = \frac{1}{\sin(x)}$

$$\frac{d}{dx}[\csc(x)] = \frac{d}{dx}\left[\frac{1}{\sin(x)}\right]$$

$$= \frac{-\frac{d}{dx}(\sin(x))}{[\sin(x)]^2}$$

$$= \frac{-\cos(x)}{\sin^2(x)}$$

$$= -\frac{\cos(x)}{\sin(x)} \cdot \frac{1}{\sin(x)}$$

$$= -\cot(x)\cdot\csc(x)$$

$$\therefore \frac{d}{dx}[\csc(x)] = -\cot(x)\cdot\csc(x)$$

We find that $\frac{d}{dx}[\sec(x)] = \tan(x)\sec(x)$.

and $\frac{d}{dx}[\cot(x)] = -\csc^2(x)$ similarity.

Ex. 3

$f(x) = a^x$, $a > 0$. Find $f'(x)$

$$a^x = e^{\ln(a^x)} = e^{x\ln a}$$

$$\frac{d}{dx}(a^x) = \frac{d}{dx}\left(e^{x\ln(a)}\right)$$

$= e^{x \ln a} \cdot \ln(a)$

$= e^{\ln a^x} \cdot \ln(a)$

$= a^x \cdot \ln(a)$

We practice using derivative rule.

E.x

$f(x) = 2^x \sin(x)$. Find $f'(x)$.

$f'(x) = \frac{d}{dx}(2^x) \cdot \sin(x) + (2^x) \cdot \frac{d}{dx}(\sin(x))$

$= 2^x \cdot \ln 2 \cdot \sin(x) + 2^x \cos(x)$

E.x $f(x) = (\sqrt[3]{x^4} + 5x + 7)(\tan(x))$

$f'(x) = (\sqrt[3]{x^4} + 5x + 7)'(\tan(x)) +$
$(x^{\frac{4}{3}} + 5x + 7)(\tan(x))'$

$= \left(\frac{4}{3}\sqrt[3]{x} + 5\right)(\tan(x)) + (\sqrt[3]{x^4} + 5x + 7) \cdot \frac{1}{\cos^2 x}$

$f'(0) = 5(\tan 0) + (0 + 0 + 7) \cdot \frac{1}{\cos^2(0)}$
$= 5 \cdot 0 + 7 \cdot 1^2 = 7$

E.X.

We can extend product rules to more than 2 fcns.

$$(fgh)' = \frac{d}{dx}[(fg)(h)] = (fg)'h + (fg)h'$$

$$= (f'g + fg')h + fgh'$$

$$(fgh)' = f'gh + fg'h + fgh'$$
$$= 0 + 2\cdot 3\cdot 3e^7 + 2\cdot 4\cdot 3e^7$$
$$= 42e^7$$

- $f(7) = 2$, f is constant, $f'(7) = 0$
- $g'(7) = 3$, $g(-7) = 4$, $g(7) = 4$
- $h'(x) = h(x)$, $h(0) = 3$

  $\hookrightarrow h(x) = e^x$ $\quad\hookrightarrow e^0 = 1 \neq 3$

  $\hookrightarrow h(x) = 3e^x \Rightarrow h'(x) = 3e^x = h(x)$

  and $h(0) = 3e^0 = 3$

  $h(7) = 3e^7 = h'(7)$

Note: The point about $h(x)$ involved a "differential eqn."

E.x

Let $f(x) = \frac{2x}{\sin(e^x)}$. Find $f'(x)$

We'll need quotient & chain rules.

$$f'(x) = \frac{\sin(e^x)(2x)' - \sin(e^x)'(2x)}{\sin^2(e^x)}$$

$$= \frac{2\sin e^x - \cos(e^x)\cdot e^x \cdot 2x}{\sin^2 e^x}$$

$$f(x) = \cot(\sin(x^{e^7}))$$

$$f'(x) = \cot(\sin(x^{e^7}))' \cdot [\sin(x^{e^7})]'$$

$$= -\csc^2(\sin(x^{e^7})) \cdot \sin(x^{e^7})' \cdot (x^{e^7})'$$

$$= -\csc^2(\sin(x^{e^7})) \cdot \cos(x^{e^7}) \cdot e^7 x^{e^7 - 1}$$

$$= -\csc^2(\sin(1^{e^7})) \cdot \cos(1^{e^7}) \cdot e^7 \cdot 1^{e^7 - 1}$$

$$= -\csc^2(\sin 1) \cdot \cos(1) \cdot e^7$$

$$= -e^7 \csc^2(\sin 1) \cdot \cos(1)$$ — end of W6.

Revisit definition of derivative.

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Then, for x values very close to $x = a$, we have

$$f'(a) \approx \frac{f(x) - f(a)}{x - a}$$

We can rearrange this formula to:

$$f(x) \approx f(a) + f'(a)(x - a) \quad *$$

for x-value close to $x = a$.

Note, We recognize ⊛ as the eqn of the tangent line to $f(x)$ at $x=a$.

Pictorially:

We can approaximate the fcn values close to $x=a$ by taking a linear factorization or a tangent line factorization.

Defn: Linear Approximation

Let $y=f(x)$ be diff'ble at $x=a$. Then, the linear approx to f at $x=a$ is the fcn:

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

Note: We just wrote $L_a(x)$ if f is clear from context.

E.x.

Use the linear approx to estimate $\sin(\sqrt{10})$.

Take $f(x)=\sin(\sqrt{x})$ and the approx is at $a=\pi^2$

$$L^{F}_{\pi^2} = f(\pi^2) + f'(\pi^2)(x-\pi^2)$$

Now, $f(\pi^2) = \sin\sqrt{\pi^2} = \sin(\pi) = 0$

$f'(x) = \cos(\sqrt{x})\left(\frac{1}{2\sqrt{x}}\right)$

$\Rightarrow f'(\pi^2) = \cos(\sqrt{\pi^2})\left(\frac{1}{2\sqrt{\pi^2}}\right)$

$= -\frac{1}{2\pi}$

$$\boxed{\begin{aligned}\frac{d}{dx}\sqrt{x} &= \frac{d}{dx}(x^{\frac{1}{2}}) \\ &= \frac{1}{2}x^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}}\end{aligned}}$$

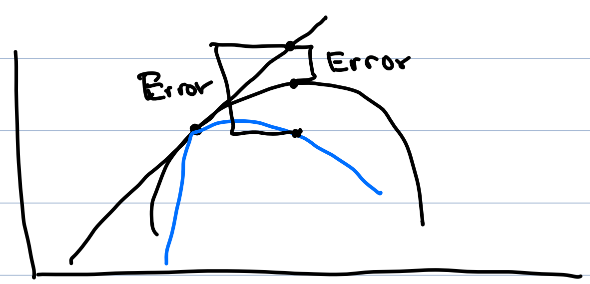
$\therefore L^{f}_{\pi^2}(x) = 0 + -\frac{1}{2\pi}(x-\pi^2)$

Then, $\sin(\sqrt{10}) \approx L^{f}_{\pi^2}(10) = -\frac{1}{2\pi}(10-\pi^2) = \frac{-5}{\pi} + \frac{\pi}{2}$

≈ -0.020753

(actual $\doteq$ 0.020683)

What affects how good our approx is?



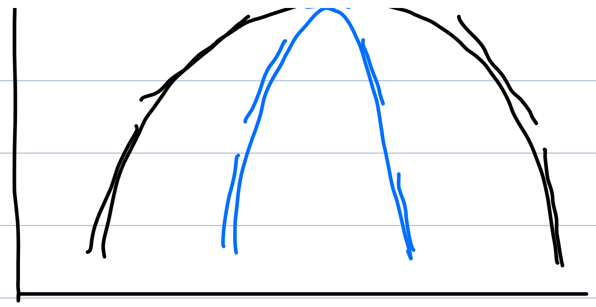
- how far away we are approaximating from $x=a$

↳ typically: further = worse

- how curved the fcn is near $x=a$.

↳ typically: more curved = worse.

How describe curvature?



We are interested in r.o.c. of r.o.c.

↳ 2nd derivative.

Thm 6: Error in Linear Approx

Assume that f is such that $|f''(x)| \leq M$ for each x in an interval I containing a point a.

Then, $\overbrace{|f(x) - L_a^f(x)|}^{\text{Error}} \leq \underbrace{\boxed{\frac{M}{2}(x-a)^2}}_{\text{Last Week}}$ Upper Bound. for each $x \in I$.

Ex.

Find an upper bound for the error on $L_{27}^{\sqrt[3]{x}}(x)$ on $[25, 30]$.

Let's deal with M first.

↳ We need $f''(x)$.

$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}} \Rightarrow f'(x) = \frac{x^{-\frac{2}{3}}}{3}$

$\Rightarrow f''(x) = -\frac{2x^{-\frac{5}{3}}}{9} = -\frac{2}{9x^{\frac{5}{3}}}$

Now $|f''(x)| = \left| -\frac{2}{9x^{\frac{5}{3}}} \right| = \frac{2}{9x^{\frac{5}{3}}}$ on $[25, 30]$

On $[25, 30]$, $|f''(x)| = \frac{2}{9x^{\frac{5}{3}}}$ is maximized at $x = 25$.

$\therefore M = \frac{2}{9(25)^{\frac{5}{3}}} \doteq 1.03965 \times 10^{-3}$

We have, on $[25, 30]$

$$|\sqrt[3]{x} - L_{27}(x)| \leq \frac{1.03965 \times 10^{-3}}{2} \underbrace{(x-27)^2}$$

this is maximized at $x = 30$ in $[25, 30]$.

$$\leq \frac{1.03965 \times 10^{-3}}{2} (3)^2$$

$$|\sqrt[3]{x} - L_{27}(x)| \leq 4.67842 \times 10^{-3}$$

WSIC?

The lin approx is useful for estimating fcn values for nasty function.

We care about errors to ensure our approx isn't useless.

Now, applications of lin approx.

Estimating Change

Assume we know $f(a)$.

How does $f(x)$ changed if we move to x_1 near $x=a$?

That is, what is: $\Delta y = f(x_1) - f(a)$ if $\Delta x = x_1 - a$.

Using $f(x_1) \approx L_a(x_1)$ (for Δx small):

$$\begin{aligned}
\Delta y &\approx L_a(x_1) - f(a) \\
&\approx (f(a) + f'(a)(x_1 - a)) - f(a) \\
&\approx f'(a)(x_1 - a) \\
\Delta y &\approx f'(a)\,\Delta x
\end{aligned}$$

Ex.

An ice cube of side length 3cm shrinks such that its side length reduces by 1mm.

Estimates the change in volume of the ice cube.

We have that $\Delta V \approx V'(30)(-1)$ where $V(s) = s^3$

$$V(s) = s^3 \Rightarrow V'(s) = 3 \cdot s^2 \Rightarrow V'(30) = 3(30^2) = 2700$$

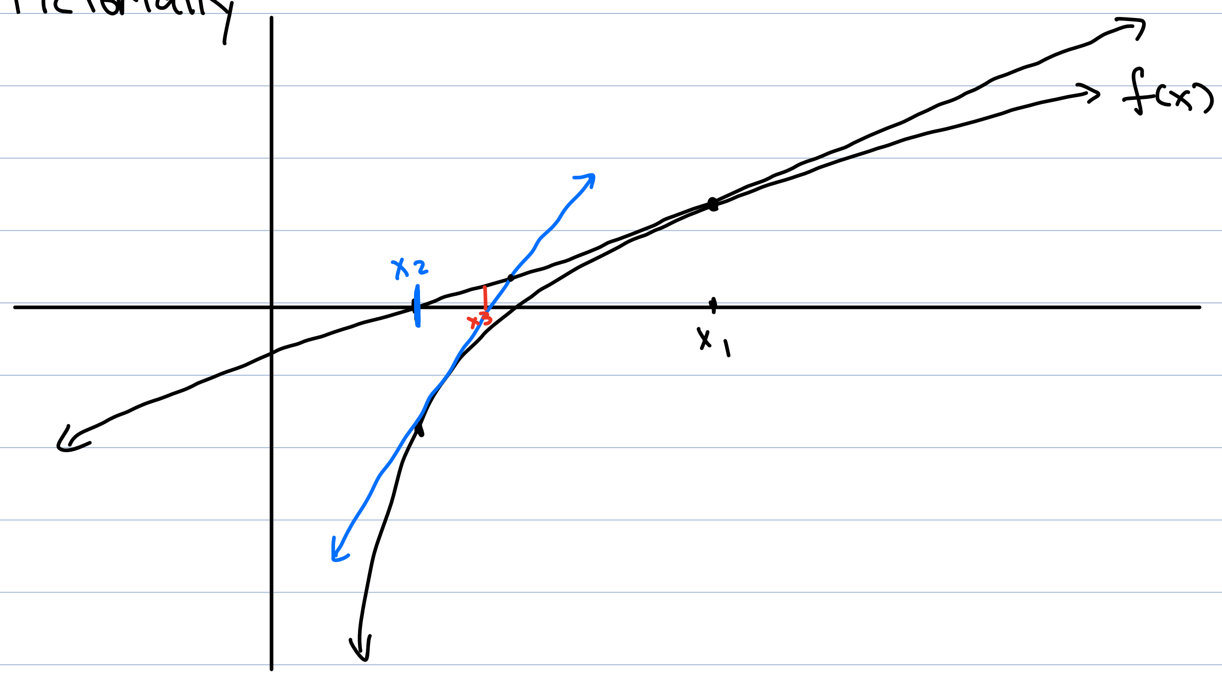
$$\therefore \Delta V \approx (2700)(-1) = -2700 \text{ mm}^3$$

(actual val, $V(29) - V(30) = -2611 \text{ mm}^3$)

IVT $\Rightarrow$ Bisection Method root-finding algorithm.

Newton's Method.

Pictorially:



Newton's Method

- Make a guess, x_1, of where the root is (IVT is useful).

(*) Take the linear appx, $L^f_{x_1}$, and find where it intersects the x-axis, call this value x_2.

- Repeat at x_2 to find x_3 and so on....

Examining (*), we are looking for $L^f_{x_1}(x_2) = 0$.

$$L^f_{x_1}(x_2) = f(x_1) + f'(x_1)(x_2 - x_1)$$

$$\Rightarrow f'(x_1)(x_2) = f'(x_1)x_1 - f(x_1)$$

$$\Rightarrow x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \quad \text{if } f'(x) \neq 0$$

$\therefore$ We can iteratively use:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Notes:

- Unlike Bisection Method, this doesn't always converge
  - ↳ fcn dependent
  - ↳ Lx_n is a horizontal tangent line.
  - ↳ pick a point too far from root, sequence might diverges / oscillates.

- Converges much faster than Bisection Method.

E.x.

Find the root of $x^3 + 5x^2 - 3x - 17$ on $[1, 3]$ accurate to 7 d.p..

Note $f(x)$ is continuous everywhere and $f(1) < 0$ and $f(3) > 0$. Then by IVT, $f(x)$ has a root on $(1, 3)$.

Let's take $x_1 = 2$. Now, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

We have $f(x) = x^3 + 5x^2 - 3x + 17$

$\Rightarrow f'(x) = 3x^2 + 10x - 3$

Then, $x_{n+1} = x_n - \dfrac{x_n^3 + 5x_n^2 - 3x_n + 17}{3x_n^2 + 10x_n - 3}$

$x_2 \doteq 2 - \frac{5}{29} = 1.82758621$

$x_3 \doteq 1.81486224$

$x_4 \doteq 1.81479452$

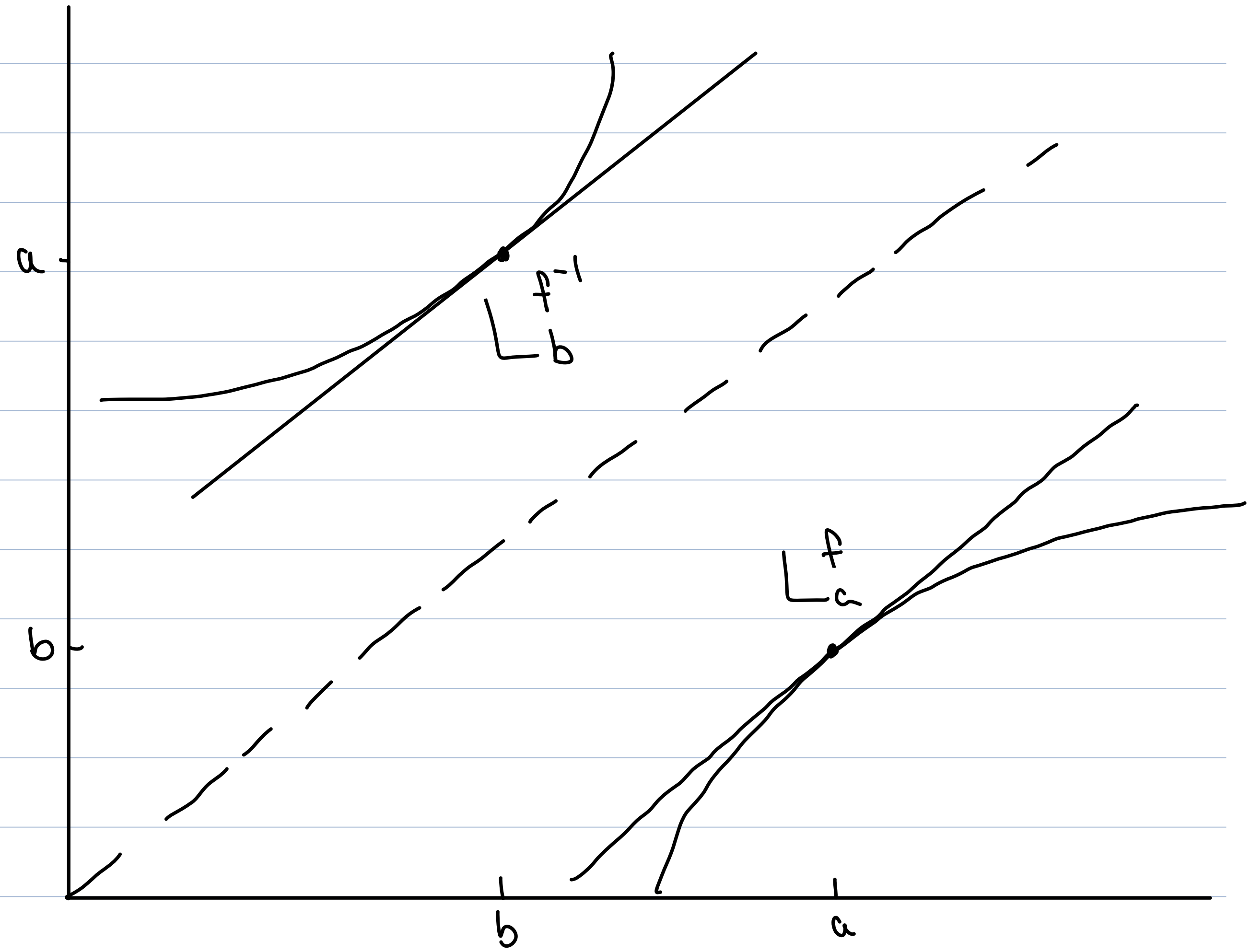
$x_5 \doteq 1.81479452$

$\therefore$ The root is $x \approx 1.8147945$

Return to land of derivative rules.

Lin approx can help us with the derivatives of inverse fcns.

Remember, geometrically, inversion is the reflection of a fcn across the line $y = x$.



We can formulate $L_b^{f^{-1}}$ in two ways:

1) Directly, by formula:

$$L_b^{f^{-1}} = f^{-1}(b) + (f^{-1})'(b)\,(x-b) \quad \circledast$$

2) Algebraically find the inverse of L_a^f:

$$L_a^f = y = f(a) + f'(a)(x-a)$$

Swap: $x = f(a) + f'(a)(y-a)$

solve: $y = \frac{1}{f'(a)}\big(x - f(a)\big) + a \quad (f'(a) \neq 0)$

Note: $f(a)=b$ and $f^{-1}(b)=a$

This gives $L_b^{f^{-1}} = \frac{1}{f'(a)}(x-b)+f^{-1}(b)$ (**)

Comparing terms, we see that $(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$

Thm 10: Inverse Fcn Theorem

Assume $y=f(x)$ is cts & invertible on $[c,d]$ with inverse $x=f^{-1}(y)$, and f is diff'ble at $a\in(c,d)$. If $f'(a)\neq 0$, then f^{-1} is diff'ble at $b=f(a)$, and

$$(f^{-1})'(b) = \frac{1}{f'(a)} = \frac{1}{f'(f^{-1}(b))}$$

Moreover, L_a^f is also invertible and $(L_a^f)^{-1}(x) = L_b^{f^{-1}}(x)$
$= L_{f(a)}^{f^{-1}}(x)$

(silly goofy) Example

Let $f(x)=5\sqrt{x}$. Find $(f^{-1})'(5)$ using IFT.

By IFT, $(f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))}$

there, $f(x)=5\sqrt{x} = 5x^{\frac{1}{2}} \Rightarrow f'(x) = \frac{5}{2}\frac{1}{\sqrt{x}} = \frac{5}{2\sqrt{x}}$

$f^{-1}(x)$: $y=5\sqrt{x} \to x=5\sqrt{y}$

$\to y = \frac{x^2}{25} = f^{-1}(x)$

Now, $f^{-1}(5) = \frac{5^2}{25} = 1$

Then, $f'(f^{-1}(5)) = f'(1) = \frac{5}{2\sqrt{1}} = \frac{5}{2}$

Then, by IFT, $(f^{-1})'(5) = \dfrac{1}{f'(f^{-1}(5))} = \dfrac{1}{\frac{5}{2}} = \frac{2}{5}$

Let's check: $f^{-1}(x) = \frac{x^2}{25} \Rightarrow (f^{-1})'(x) = \frac{2x}{25}$

$\Rightarrow (f^{-1})'(5) = \frac{2(5)}{25} = \frac{2}{5}$

We can also arrive via chain rule:

Assume f has inverse f', and both are diff'ble

By defn, $f(f^{-1}(x)) = x$

$$\frac{d}{dx} f(f^{-1}(x)) = \frac{d}{dx}[x]$$
$$= 1$$

$$f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$$
$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

Explicit functions: $y = f(x)$

Now implicit funcs:

Ex.

Find y' for $x^3y^5 + 2x = y^3 + 4$

We'll differentiate both sides:

$$\frac{d}{dx}\left[x^3y^5 + 2x\right] = \frac{d}{dx}\left[y^3 + 4\right]$$

$$3x^2y^5 + \underbrace{x^3 5y^4 \cdot \frac{dy}{dx}}_{\text{chain rule}} + 2 = \underbrace{3y^2 \cdot \frac{dy}{dx}}_{\text{chain rule}} + 0$$

$$x^3 5y^4 \cdot \frac{dy}{dx} - 3y^2 \cdot \frac{dy}{dx} = -2 - 3x^2y^5$$

$$\frac{dy}{dx}\left(x^3 5y^4 - 3y^2\right) = -2 - 3x^2y^5$$

$$\frac{dy}{dx} = \frac{-2 - 3x^2y^5}{x^3 5y^4 - 3y^2}$$

Note: ($x^2y^2 = -72$, we could proceed thru the above steps and find $y' = \frac{-y}{x}$.

There are no $(x,y) \ni x^2y^2 = -72$. $\therefore$ there's no curve,

$\therefore$ $y'(x)$ is meaningless.

Extend implicit diff'tion to discuss

"Logarithmic diff"

Standard motivator here is fcns of form

$$h(x) = g(x)^{f(x)} \quad (g(x) > 0)$$

E.x. 1.

$y = x^x \quad (x > 0)$. Find y'

Ln both sides:

$$\ln(y) = \ln(x)^x$$
$$\ln(y) = x\ln(x)$$

implicitly diff:

$$\frac{d}{dx}\ln(y) = \frac{d}{dx}(x \cdot \ln(x))$$
$$\frac{1}{y} \cdot y' = \ln x + x \cdot \left(\frac{1}{x}\right)$$
$$\frac{y'}{y} = \ln x + 1$$
$$y' = y(\ln x + 1)$$
$$y' = x^x(\ln x + 1)$$

Look at this:

E.x. 2

$$y = \frac{(x-3)^3(x+4)^2(y-1)}{(x+1)^2(x^2+x+1)^3}.\quad \text{Find } y'$$

Rather than use old tools

Log diff:

$$\ln y = \ln\left(\frac{(x-3)^3(x+4)^2(x-1)}{(x+1)^2(x^2+x+1)^3}\right)$$

$$\ln y = 3\ln(x-3) + 2\ln(x+4) + 2\ln(x-1)$$
$$- 2\ln(x+1) - 3\ln(x^2+x+1)$$

$$\frac{y'}{y} = \frac{3}{x-3}(1) + \frac{2}{x+4}(1) + \frac{2}{x-1}(1) - \frac{2}{x+1}(1)$$
$$- \frac{3}{x^2+x+1}(2x+1)$$

$$y' = \left(\frac{(x-3)^3(x+4)^2(y-1)}{(x+1)^2(x^2+x+1)^3}\right)\left(\frac{3}{x-3}(1) + \frac{2}{x+4}(1) + \frac{2}{x-1}(1)\right.$$
$$\left. - \frac{2}{x+1}(1) - \frac{3}{x^2+x+1}(2x+1)\right)$$

Now, revisit concept of extrema.

Recall, previously discussed global extrema & EVT.

Defn: Local Maxima / Minima

A point c is called a local max/min for a fcn f if $\exists$ an open interval (a,b) containing c s.t.

$$f(x) \le f(c) \;/\; f(x) \ge f(c) \quad \forall\, x \in (a,b)$$

Notes: • This means endpoints can't be local extrema.
• Global Extrema occur at endpoint are also local extrema.

☆ local / global Extrema.

Local Extrema Theorem (LFT)

If c is a local extremum for f and $f'(c)$ exists, then $f'(c) = 0$.

Proof:

Assume wlog we have c is a local min, and that $f'(c)$ exists.

Since $f'(c)$ exists, we have that

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

$$= \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h}$$

Since c is a local min, $\exists\, (a,b)$ s.t. $c \in (a,b)$

and $f(c) \le f(x) \ \forall x \in (a,b)$

Then, for $h > 0$ small enough that $c < c+h < b$, we have $f(c+h) \ge f(c)$

This means that

$$f'(c) = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h} \ge 0 \quad \left(\frac{\ge 0}{\ge 0} \right)$$

Similarily, for $h < 0$ small enough that $a < c+h < c$, we have $f(c+h) \ge f(c)$ as well.

This gives

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \le 0 \quad \left(\frac{\ge 0}{< 0} \right)$$

That is,

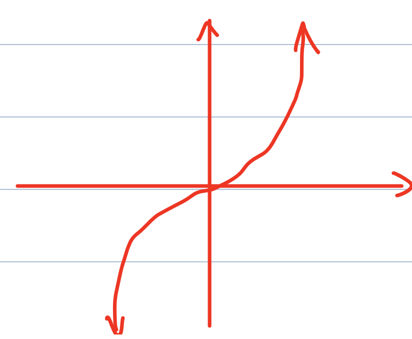
$$0 \le f'(c) \le 0$$
$$\Rightarrow f'(c) = 0$$

A similar proof follows for local maximum

Notes:

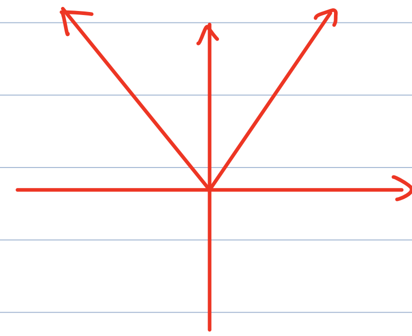
- $f'(c) = 0 \not\Rightarrow$ local max / min

  (e.x. $y = x^3$ @ $x = 0$:



· local max / min $\not\Rightarrow$ $f'(c) = 0$.

(e.x. $y = |x|$ @ $x = 0$:



We consider candidate points of interest :

Defn: Critical Points (cp's)

A point c in the domain of function f is called a critical point for f if either $f'(c) = 0$ or DNE.

We can put some pieces together.

· For a cts fcn on a closed interval, EVT guaranteed a global max / min, either on end points or on (a, b).

· If global extremum is on (a, b) it's also a local extremum.

· If local extremum, then also critical point.

"Closed Interval Method" for global extremum for cts f on $[a,b]$.

1. Calculate $f(a)$ & $f(b)$
2. Calculate $f'(x)$.
3. Find cp's (where $f'(x)=0$ or DNE)
4. Find func values at cp's.
5. Global max is the largest value from 1 & 4.
   Global min is the smallest value from 1 & 4.

E.X.

Find the global extrema of $f(x) = x\sqrt{4-x^2}$ on $[-1,2]$

We check endpoints: $\boxed{f(-1) = -\sqrt{3}}$, $\boxed{f(2) = 0}$

$$f'(x) = (1)(\sqrt{4-x^2}) + x\cdot\left(\tfrac{1}{2}(4-x^2)^{-\frac{1}{2}}\cdot(-2x)\right)$$

$$f'(x) = \sqrt{4-x^2} - \frac{x^2}{\sqrt{4-x^2}}$$

cp's: f' DNE $\longrightarrow$ div by zero when $x = \pm 2$

$\downarrow$ throw out -2 since out of interval.

$f' = 0$

$\hookrightarrow \sqrt{4-x^2} = \dfrac{x^2}{\sqrt{4-x^2}}$

$4 - x^2 = x^2 \Rightarrow x = \pm\sqrt{2}$

↳ threw out $-\sqrt{2}$ outside interval.

Calculate $f(2) = 0$, $f(\sqrt{2}) = 2$

$\therefore$ Global max at $(\sqrt{2}, 2)$

Global min at $(-1, -\sqrt{3})$.

Consider this scenario:

My drive into work is 5 kms. If I make it in 5 mins and the speed limits is 50 km/hr, can you guarantee that I sped at some point?

My avg velocity was: $\frac{\Delta s}{\Delta t} = \frac{5\text{ km}}{1/2\text{ hr}} = 60$ km/hr

2 choices:

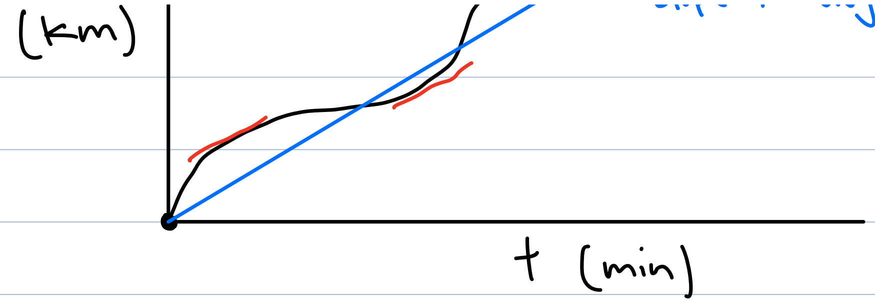
→ I constantly travel at 60 km/hr (sped)

→ sometimes, $v < 60$; sometimes $v > 60$.

↳ I had to be at exactly 60 km/hr at least once.

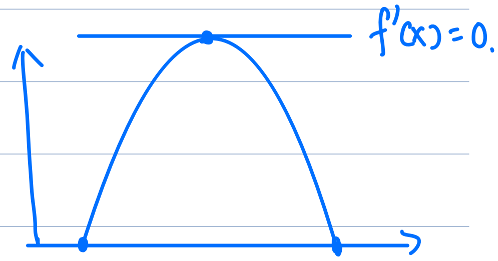
Take away, at some point, inst velo = avg velo.

Pictorially:



Motivator: Between zeroes of $f(x)$, there's a zero of $f'(x)$.

"what goes up must come down" theorem



<u>Thm 2: Rolle's Theorem</u>

Assume $f(x)$ is cts on $[a,b]$ and diff'ble on (a,b), and that $f(a) = f(b) = 0$.

<u>Then</u> $\exists\, c \in (a,b) \ni f'(c) = 0$.

<u>Proof:</u>

<u>Case 1:</u> $f(x) = 0$. Then $f'(x)=0$, so $f'(c) = 0 \ \forall c \in (a,b)$.

Case 2: $f(x_0) > 0$, for some $x_0 \in (a,b)$.

Then by EVT, since f is cts on $[a,b]$ $\exists$ a global max on $[a,b]$.

But since $f(a) = f(b) = 0 < f(x_0)$, the global max must be at some $c \in (a,b)$.

Then c is also a local max. Then since f is diff'ble on (a,b), by LET we must have $f'(c) = 0$.

Case 3: $f(x_0) < 0$ for some $x_0 \in (a,b)$.

Same as case 2, but with a global min argument.

<u>Thm 1: Mean Value Theorem</u>

Assume $f(x)$ is cts on $[a,b]$ and diff'ble on (a,b).
Then $\exists\ c \in (a,b) \ni$

$$\underbrace{f'(c)}_{\text{inst roc}} = \frac{f(b) - f(a)}{b - a} \quad \text{avg roc}$$

Proof:

Introduce fcn $h(x) = f(x) - \Big[\underbrace{f(a) + \frac{f(b) - f(a)}{b-a}(x-a)}_{\text{secant line between } a \,\&\, b.}\Big]$

$h(x)$ is the height of our fcn above the secant line.

Notice $h(x)$ is cts on $[a,b]$ & diff'ble on (a,b).

↳ cuz it's the difference of f (cts & diff'ble) and a line (cts / diff'ble everywhere)

Thus apply rolle's theorem to $h(x)$.

Thus, $\exists\, c \in (a,b) \ni h'(c) = 0$

Now, $h'(x) = f'(x) - \dfrac{f(b)-f(a)}{b-a}$

Then, $h'(c) = f'(c) - \dfrac{f(b)-f(a)}{b-a} = 0$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Note: Always check the hypothesis.

WSIC? MVT is a fundamental tool.

Defn: Antiderivative

Given a function f, an antiderivative is a fcn $F \ni$ $F'(x) = f(x)$

If $F'(x) = f(x)\ \forall x \in I$, we say F is an antiderivative for f or I.

Note: Unlike derivatives: Antiderivative aren't unique

E.x. $f(x) = 5$: $F(x) = 5x$, $F(x) = 5x+1$, $F(x) = 5x - \pi$

Thm 3: Constant Fcn thm

If $f'(x) = 0 \ \forall x \in I$, then $\exists \alpha \in \mathbb{R} \ni f(x) = \alpha \ \forall x \in I$.

Proof:

Choose $x_1, x_2 \in I$, $x_1 \neq x_2$. wlog let $x_2 > x_1$

Let $f(x_1) = \alpha$. Also, let $f'(x) = 0 \ \forall x \in I$.

Since f is diff'ble on I, it is also cts on I.

Thus, we can apply MVT on $[x_1, x_2]$

$\therefore \exists c \in (x_1, x_2) \ni$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$0 = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow f(x_2) = f(x_1)$$

$$\Rightarrow f(x_2) = \alpha$$

Since x_1 & x_2 are arbitrary, $f(x) = \alpha \ \forall x \in I$.

Now, we can prove:

Thm 4: Antiderivative Thm.

If $f'(x) = g'(x)\ \forall x \in I$, then $\exists\, \alpha \in \mathbb{R} \ni$

$$f(x) = g(x) + \alpha \quad \forall x \in I$$

Proof:

Consider $h(x) = f(x) - g(x)$

Notice h is diff'ble & cts on I since f & g are.

Also, notice $h'(x) = f'(x) - g'(x) = 0\ \forall x \in I$.

Then, by CFT, $h(x) = \alpha\ \forall x \in I$

$\Rightarrow f(x) - g(x) = \alpha$

$\Rightarrow f(x) = g(x) + \alpha \quad \forall \alpha \in I$.

Leibniz Notation for Antiderivatives:

$$\underbrace{\int f(x)\,dx}_{\text{Indefinite Integral of } x} \quad \text{denotes the family of antiderivatives of } f.$$

↳ $f(x)$ is the integrand.

Ex. $\int 5\,dx = 5x + C$

Thm 5: Power rule for antiderivative

If $\alpha \neq -1$, then

$$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + C$$

Pf: By differentiating

Find $\frac{d}{dx} \ln|x|$

for $x<0$: $\frac{d}{dx} \ln(-x)$

$= \frac{1}{-x} \cdot -1 = \frac{1}{x}$

domain matches $\frac{1}{x}$

- $\int 0 \, dx = C$
- $\int \frac{1}{x} \, dx = \ln|x| + C$
- $\int e^x \, dx = e^x + C$
- $\int a^x \, dx = \frac{a^x}{\ln(a)} + C$
- $\int \cos \, dx = \sin(x) + C$
- $\int \sin \, dx = -\cos(x) + C$
- $\int \sec^2 dx = \tan(x) + C$
- $\int \frac{dx}{1+x^2} dx = \arctan(x) + C$
- $\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) + C$
- $\int \frac{-dx}{\sqrt{1-x^2}} = \arccos(x) + C$
- $\int (\alpha_1 f(x) + \alpha_2 g(x) + \alpha_3 h(x) + \dots) \, dx$

$= \alpha_1 \int f(x) \, dx + \alpha_2 \int g(x) \, dx + \alpha_3 \int h(x) \, dx + \dots$

if the antiderivatives exists.

MVT to prove stuff.

Thm 6: Increasing / Decreasing Fcn Thm.

Let I be an interval and $x_1, x_2 \in I$, where $x_1 < x_2$.

1) If $f'(x) > 0 \ \forall x \in I$, then $f(x_1) < f(x_2)$

$\hookrightarrow$ that is f is increasing on I.

2) If $f'(x) \geq 0 \ \forall x \in I$, then $f(x_1) \leq f(x_2)$

$\hookrightarrow$ that is f is non-decreasing on I.

2) If $f'(x) < 0$ $\forall x \in I$, then $f(x_1) > f(x_2)$

↳ that is f is decreasing on I.

2) If $f'(x) \leq 0$ $\forall x \in I$, then $f(x_1) \geq f(x_2)$

↳ that is f is non-increasing on I.

Pf of 4)

Since $f'(x)$ exist on intrvl I, f is diff'ble on I and thus continuous on I.

Thus, f is cts on $[x_1, x_2]$ & diff'ble on (x_1, x_2)

So, we apply MVT to f on $[x_1, x_2]$

So, $\exists\ c \in (x_1, x_2) \ni$

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

But we have $f'(x) \leq 0$ $\forall x \in I$, so,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} \leq 0$$

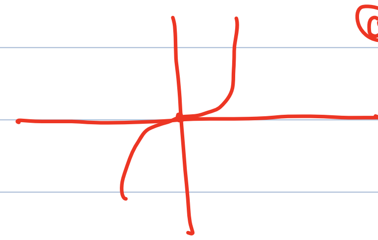
But since $x_2 > x_1$, we have $x_2 - x_1 > 0$, so:

$$f(x_2) - f(x_1) \leq 0$$

$$\Rightarrow f(x_2) \leq f(x_1), \text{ non-increasing.}$$

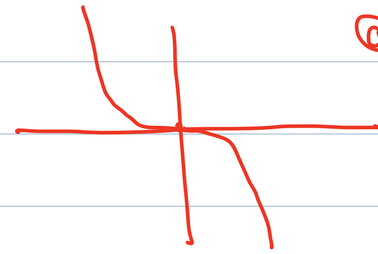
Note:

f' inc $\not\Rightarrow$ $f'>0$ e.g. x^3



@ $x=0$
$y'=0$

f' dec $\not\Rightarrow$ $f'<0$ e.g. x^3



@ $x=0$
$y'=0$

$\downarrow$

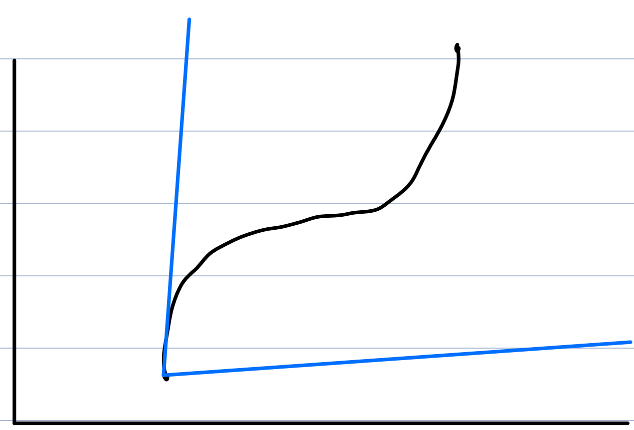
f' could be zero or DNE.

Thm. 7: Bounded Derivative Thm.

Assume f is cts on $[a,b]$ & diff'ble on (a,b),

$$m \le f'(x) \le M \quad \forall x \in (a,b)$$

Then, $f(a)+m(x-a) \le f(x) \le f(a)+M(x-a)$
$\forall x \in [a,b]$.



Proof:

By the hypothesis of the thm, we can apply MVT to f on $[a,b]$

Further, MVT would also apply on $[a, x_1]$ where $x_1 \in (a,b]$.

That is, $\exists\, c \in (a, x_1)$ s.t.

$$f'(c) = \frac{f(x_1) - f(a)}{x_1 - a}$$

Then, since $c \in (a,b)$, we have $m \le f'(c) \le M$

$$m \le \frac{f(x_1) - f(a)}{x_1 - a} \le M$$

$x_1 > a$
$\Rightarrow x_1 - a > 0$

$$(x_1 - a)\, m \le f(x_1) - f(a) \le M\,(x_1 - a)$$

$$f(a)\,(x_1 - a)\, m \le f(x_1) \le f(a) + M\,(x_1 - a)$$

x_1 is arbitrary, so holds $\forall\, x_1 \in (a,b)$. ■

E.x.

Prove that $\sqrt{50} \in \left[7 + \frac{1}{16}, 7 + \frac{1}{14}\right]$ using BDT.

Let $f(x) = \sqrt{x}$. We'll look at $f(x)$ on $[49, 64]$

We have $f'(x) = \frac{1}{2\sqrt{x}}$. We see that $f(x)$ is cts on $[49, 64]$ and diff'ble on $(49, 64)$.

Now, on $x \in (49, 64)$, we see that

$$\frac{1}{16} = \frac{1}{2\sqrt{64}} \le f'(x) \le \frac{1}{2\sqrt{x}} \le \frac{1}{2\sqrt{49}} = \frac{1}{14}$$

Then, by BDT, for $x \in [49, 64]$ we have

$$\sqrt{49} + \frac{1}{16}(x-49) \leq f(x) \leq \sqrt{49} + \frac{1}{14}(x-49)$$
$$7 + \frac{1}{16}(x-49) \leq \sqrt{x} \leq 7 + \frac{1}{14}(x-49)$$
$$7 + \frac{1}{16} \leq \sqrt{50} \leq 7 + \frac{1}{14}$$

Thm 8.

Assume that f and g are cts at $x=a$ with $f(a) = g(a)$.

1) If both f & g are diff'ble for $x > a$ and if $f'(x) \leq g'(x)$ for $x > a$, then
$$f(x) \leq g(x) \quad \text{for } x > a$$

2) If both f & g are diff'ble for $x < a$ and if $f'(x) \leq g'(x)$ for $x < a$, then
$$f(x) \geq g(x) \quad \text{for } x < a$$

Proof to 2:

Take all hypotheses. Define $h(x) = f(x) - g(x)$.

Then, h is cts at $x=a$ & diff'ble for $x < a$.
Now, $h'(x) = f'(x) - g'(x)$. Since we have

$f'(x) \le g'(x)$ for $x < a$, we have $h'(x) \le 0$ for $x<a$.
Note, we can apply MVT to $[x, a]$.
This says $\exists\, c \in (x, a)$ s.t.

$$h'(c) = \frac{h(a) - h(x)}{a - x} \le 0 \quad (\text{since } c < a)$$

Now, since $x<a$, $a-x>0$. We also have that
$h(a) = f(a) - g(a) = 0$. Thus, we have that

$$0 - h(x) \le 0$$
$$\Rightarrow \quad h(x) \ge 0$$
$$\Rightarrow f(x) - g(x) \ge 0$$
$$\Rightarrow f(x) \ge g(x) \quad \text{for } x < a. \ \square$$

Notes: If instead $f'(x) < g'(x)$ we get that $f(x) < g(x)$ for $x>a$
and $f(x) > g(x)$ for $x<a$

Ex Prove $x^2 > \ln(1+x^2)$ for $x < 0$.

Pf:
Let $f(x) = x^2$ and $g(x) = \ln(1+x^2)$
We know these are continuous $\Rightarrow$ poly $\Rightarrow$ composition.

Further, we see $f(0) = g(0) = 0$

We have $f'(x) = 2x$, $g'(x) = \dfrac{2x}{1+x^2}$

We see f and g are diff'ble everywhere.

We notice that for $x<0$, we have $1+x^2>1$.

Then, for $x<0$, we have

$$\frac{2x}{1+x^2} > \frac{2x}{1}$$

That is, for $x<0$, we have $g'(x) > f'(x)$.

Then, by Thm, for $x<0$, $f(x) > g(x)$ $\square$

Indeterminant form

- $\frac{0}{0}$ • $\pm\infty/\infty$ • $0\cdot\infty$ • $\infty-\infty$
- 1^{∞} • ∞^0 • 0^0

To deal with limits with ↑ forms, introduce new forms

Thm 14. L'Hôpital's Rule (L'HR)

Assume $f'(x)$ & $g'(x)$ exist near $x=a$, $g'(x)\neq 0$ near $x=a$ except possibly at $x=a$, and that $\lim\limits_{x\to a}\frac{f(x)}{g(x)}$ is of form $\frac{0}{0}$ and $\pm\frac{\infty}{\infty}$.

Then, $$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

if the latter limit exists or is $\pm\frac{\infty}{\infty}$.

Notes:

- L'HR applies to $a \in \mathbb{R}$, a "=" $\pm\infty$, and to one-sided limits
- May be used multiple times.

Type 0/0:

Ex. 1

$$\lim_{x\to 0} \frac{e^x - x - 1}{5\cos(x) + x^2 - 5} \overset{L'HR}{=} \lim_{x\to 0} \frac{e^x - 1 - 0}{-5\sin x + 2x - 0}$$

$$\overset{L'HR}{=} \lim_{x\to 0} \frac{e^x}{-5\cos x + 2} = \frac{1}{3}$$

Ex 2

$$\lim_{x\to 0} \frac{\sin x}{x} \overset{L'HR}{=} \lim_{x\to 0} \frac{\cos x}{1} = 1$$

Ex 3

$$\lim_{x\to 1} \frac{\ln(x)}{1 - \cos(x-1)} \overset{L'HR}{=} \frac{\frac{1}{x}}{\sin(x-1)} \overset{L'HR}{=} \frac{\frac{1}{-x^2}}{\cos(x-1)} = -1$$

X

it's not indefinite form. We get $\frac{1}{0^2} = \pm\infty$

$\hookrightarrow$ DNE

$\therefore$ form one application of L'HR, the original limit DNE

Type ∞ / ∞:

E.x.1

$$\lim_{x\to\infty} \frac{\ln x}{x} \overset{LHR}{=} \frac{\frac{1}{x}}{1} = 0$$

E.x.2

$$\lim_{x\to\infty} \frac{x^2+4x+1}{3x^2-1} \overset{LHR}{=} \lim_{x\to\infty} \frac{2x+4}{6x}$$

$$\overset{LHR}{=} \lim_{x\to\infty} \frac{2}{6} = \frac{1}{3}$$

Type $0 \cdot \infty$

Make it look like the form $\pm\frac{\infty}{\infty}$. Use reciprocal

E.x.1

$$\lim_{x\to 0^+} x\ln(x) = \lim_{x\to 0^+} \frac{\ln(x)}{\frac{1}{x}} \quad "\left(\frac{-\infty}{\infty}\right)$$

$$\overset{LHR}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{\frac{1}{-x^2}} = \frac{1}{x} \cdot -x^2$$

$$= -x = 0$$

E.x.2

$$\lim_{x\to-\infty} e^x x^{\frac{5}{3}} = \lim_{x\to-\infty} \frac{x^{\frac{5}{3}}}{e^{-x}}$$

$$\overset{LHR}{=} \lim_{x\to-\infty} \frac{\frac{5}{3}x^{\frac{2}{3}}}{-e^{-x}} \quad \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{LHR}{=} \lim_{x\to-\infty} \frac{\frac{10}{9}x^{-\frac{1}{3}}}{e^{-x}} \quad \text{"}\frac{0}{\infty}\text{"}$$

$$= 0$$

Type $\infty - \infty$:

Combine them into one form, so it looks like previous form.

E.x 1 "$\infty + (-\infty)$"

$$\lim_{x\to\infty} \ln(3x) + \ln\left(\frac{17}{x+7}\right) = \lim_{x\to\infty} \ln\left(3x\cdot\frac{17}{x+7}\right)$$

$$= \lim_{x\to\infty} \ln\left(\frac{51x}{x+7}\right)$$

$$= \ln \lim_{x\to\infty} \frac{51x}{x+7} \quad \text{"}\frac{\infty}{\infty}\text{"}$$

$$\overset{LHR}{=} \ln\left(\frac{51}{1}\right)$$

$$= \ln(51)$$

E.x.2

$$\lim_{x\to0} \left[\cot(x) - \frac{1}{x}\right] = \lim_{x\to0} \left[\underbrace{\frac{1}{\tan x}}_{\infty} - \underbrace{\frac{1}{x}}_{\infty}\right]$$

$$= \lim_{x\to0} \left[\frac{x-\tan x}{x\tan x}\right]$$

$$\overset{LHR}{=} \lim_{x\to 0} \left[\frac{1-\sec^2 x}{\tan x + x\sec^2 x} \right]$$

$$\overset{LHR}{=} \lim_{x\to 0}$$

$$\overset{LHR}{=} \lim_{x\to 0} \frac{-2\sec^2 x \tan x}{\sec^2 x + \sec^2 x + 2x\sec^2 x \cdot \tan x}$$

$$= 0$$

Type 1^∞, ∞^∞, and 0^0

Rewrite to $e^{\ln(\text{indt})}$

Thank to log laws, expont $\ln(\text{indt})$ will become of type "$0 \cdot \infty$", then use continuity to distribute in the limit.

Ex 1

$$\lim_{x\to 0^+} x^x \quad \text{"}0^0\text{"}$$

$$= \lim_{x\to 0^+} e^{\ln(x^x)} = \lim_{x\to 0^+} e^{x\ln x}$$

$$\overset{cty}{=} e^{\lim_{x\to 0^+}(x\ln x) = 0} = 1$$

Ex.2

$$\lim_{x\to\infty} \left(1+\frac{P}{x}\right)^x \quad \text{"}1^\infty\text{"}$$

$$= \lim_{x\to\infty} e^{\ln\left(1+\frac{P}{x}\right)^x}$$

$$= \lim_{x\to\infty} e^{x\ln\left(1+\frac{P}{x}\right)}$$

$\overset{\text{cty}}{=} e^{\lim_{x\to\infty} x\ln(1+\frac{P}{x})}$ " $\infty \cdot 0$ " $\ln 1 = 0$

Eval $\lim_{x\to\infty} x\ln(1+\frac{P}{x})$

$$= \lim_{x\to\infty} \frac{\ln(1+\frac{P}{x})}{\frac{1}{x}} \qquad \text{" } \frac{0}{0} \text{ "}$$

$$\overset{\text{LHR}}{=} \lim_{x\to\infty} \frac{\frac{1}{1+P/x} \cdot \frac{-P}{x^2}}{-\frac{1}{x^2}}$$

$$\overset{\text{LHR}}{=} \lim_{x\to\infty} \frac{P}{1+\frac{P}{x}}$$

$$= \frac{P}{1} = P$$

$\therefore \lim_{x\to\infty} (1+\frac{P}{x})^x = e^P$

Note: this gives $(1+\frac{1}{x})^x = e^1$

Practice Problems:

$$\lim_{x\to -4} \frac{\sin(\pi x)}{x^2-16}$$

$$\lim_{x\to\infty} (xe^{\frac{1}{x}} - x)$$

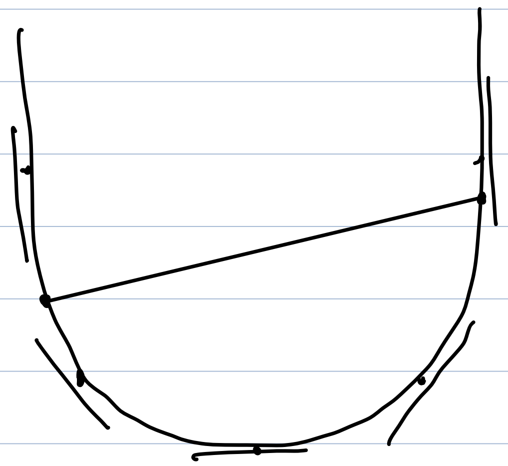
$$\lim_{x\to 0} (\cos(x))^{\csc(x)}$$

Mean Girls Limit (use L'HR)

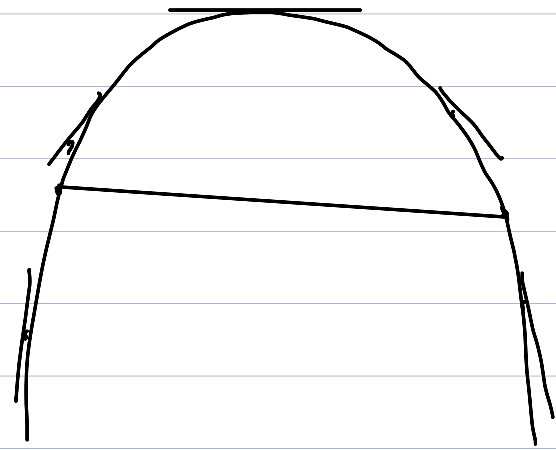
$$\lim_{x \to 0} \frac{\ln(1-x) - \sin x}{1 - \cos^2 x}$$

End of week 9.

Derivative of Derivative : R.O.C of R.O.C.



sec line above curve



sec line below curve

Defn: Concavity

We say a graph is concave up and concave down on an interval I if $\forall a, b \in I$ the secant line connecting $(a, f(a))$ & $(b, f(b))$ sits above / below the curve.

Note: Horizontal Lines are neither CU nor CD.

Thm 10: Second Derivative Test for Concavity

1) If $f''(x) > 0$ $\forall x \in I$, then graph of f is concave up on I.

2) If $f''(x) < 0$ $\forall x \in I$, then graph of f is concave down on I.

Defn: Inflection point (POI)

A point $(c, f(c))$ is called a POI of f is cts at c and the concavity of f changes at point $(c, f(c))$.

These occur when f'' change sign at c.
If f'' is cts at the poi, IVT requires $f''(c) = 0$.

Thm 11: Test for Inflection point.

If f'' is cts at $x=c$, and $(c, f(c))$ is a poi of f, then $f''(c) = 0$.
Note $f''(c) = 0 \not\Rightarrow$ poi.

· $f''(c) = 0$ are candidate poi's.
↳ must change concavity on either side.

E.x.
Find the intervals of concavity & any inflection point of
a) $y = x^3$ and $y = -\frac{1}{x}$

a) $y = x^3 \Rightarrow y' = 3x^2 \Rightarrow y'' = 6x$

y'' is a polynomial → cts everywhere
$y'' = 0$ when $x = 0$.

	-1	$x=0$	1
y''	$-$ve		$+$ve
y	c.d		c.u

∴ y is c.d on $(-\infty, 0)$ and c.u on $(0, \infty)$.
there is a poi at $(0,0)$.

b) $y = -\frac{1}{x} \rightarrow y' = -\frac{1}{x^2} \rightarrow y'' = -\frac{2}{x^3}$

Since $y''(0)$ DNE, we must investigate around it.

	-1	$x=0$	1
y'':	$+$ve		$-$ve
y:	c.u	$\Downarrow$	c.d

not a poi
since f isn't
cts @ $x=0$.

$\therefore$ y is c.u. on $(-\infty, 0)$ and c.d. on $(0, \infty)$, there are no poi's.

We've seen that if $x=c$ is a local extremum, then it's a critical point ($f'=0$ or DNE). But, once we've found cp's we'd like to classify them.

Thm 12: First Derivative Test (FDT)

Assume c is a cp. of f & f is cts at c. If there's an interval (a,b) containing c s.t.

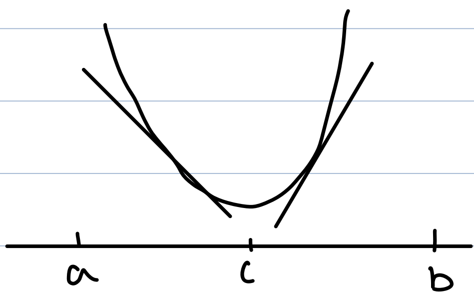
i) $f'(x) < 0$ $\forall x \in (a,c)$ & $f'(x) > 0$ $\forall x \in (c,b)$ then c is a local min.

2) $f'(x) > 0 \ \forall x \in (a, c)$ & $f'(x) < 0 \ \forall x \in (c, b)$ then c is a local max.

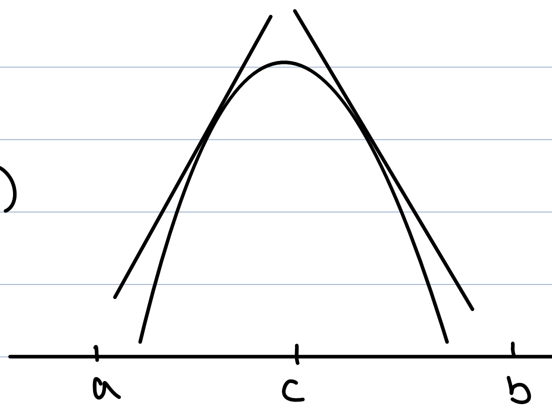
Note: If in neither cases, c is neither local max nor min.

Pictorially:

1)



2)



f'	$-ve$	$+ve$
f''	$+ve$	

f'	$+ve$	$-ve$
f''	$-ve$	

Thm 13: Second Derivative Test (SDT)

Assume $f'(c) = 0$ and that $f''(c)$ is cts at that c. If:

1) $f''(c) > 0$ $\quad$ c is local min
2) $f''(c) < 0$ $\quad$ c is local max
3) $f''(c) = 0$ $\quad$ we have no information.
   ↳ could be any
   ↳ use FDT.

E.x.1.

Find local extrema of $y = x^3 - 13x + 12$ use FDT / SDT

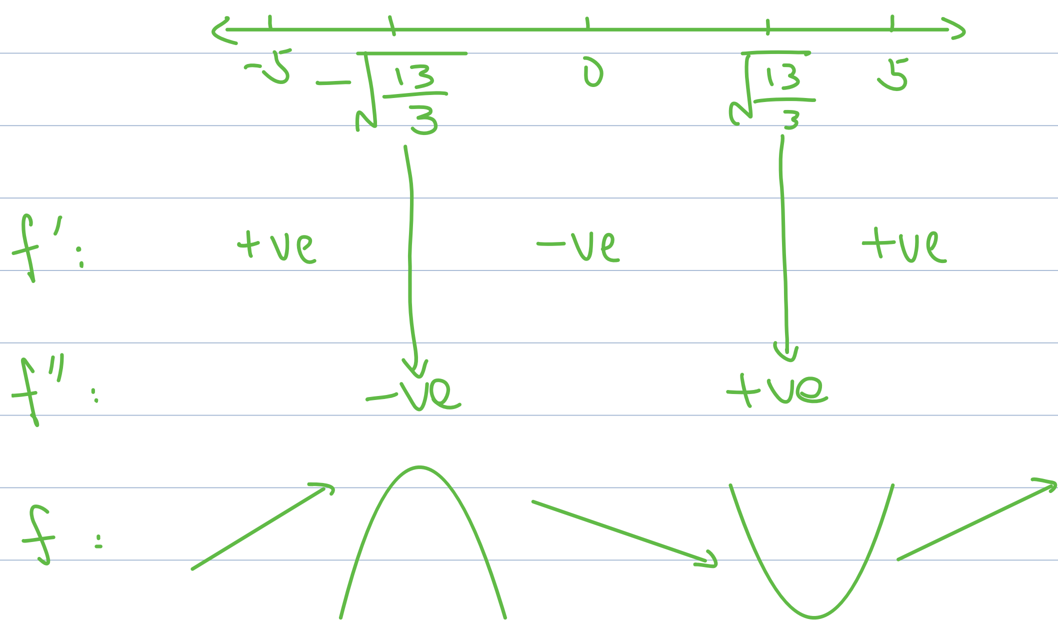
Note: $f' = 3x^2 - 13$ & $f'' = 6x$

We see that we have C.P @ $x = 0$

$\Rightarrow 3x^2 = 13 \Rightarrow x = \pm\sqrt{\frac{13}{3}}$.

Also $f'(x)$ is polynom, $\therefore$ no $x \to f'(x)$ DNE

Do FDT / SDT :



$\therefore$ $x = -\sqrt{\frac{13}{3}}$ is a local max, $\sqrt{\frac{13}{3}}$ is local max.

E.X.2

Find all max/mins of $y = x\sqrt[3]{2-x}$ on $[0,3]$

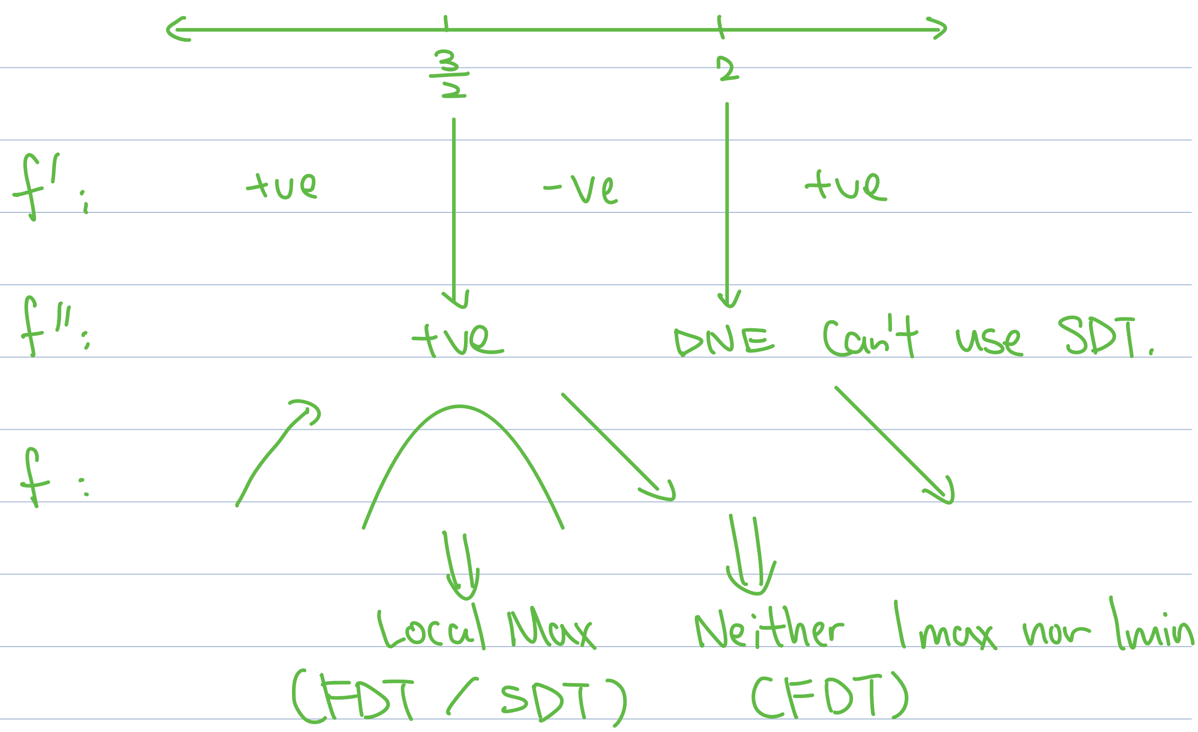
Note: $y' = \dfrac{2-\frac{4}{3}x}{(2-x)^{\frac{2}{3}}}$ & $y'' = \dfrac{\frac{4}{9}x-\frac{4}{3}}{(2-x)^{\frac{5}{3}}}$

We note that $f(0)=0$ & $f(3)=-3$. [Endpoint can't be local extrema].

Now, cp's are where $y'=0 \Rightarrow 2-\frac{4}{3}x=0 \Rightarrow x=\frac{3}{2}$

y' DNE $\Rightarrow x=2$

Conduct FDT / SDT :



Now, $f(\frac{3}{2}) > f(0), > f(3)$

$\therefore$ local & global max @ $x = \frac{3}{2}$.

a global min at $x = 3$. ■

In general, u have freedom to choose FDT / SDT unless told.

Curve Sketching Process:

1) Identify function domain, and possibly function's values at endpoints.

2) Identify x and y intercepts

3) Identify horizontal asymptotes.

4) Identify any holes and/or vertical asymptotes. $\left(\lim_{x \to a^{\pm}} f(x)\right)$

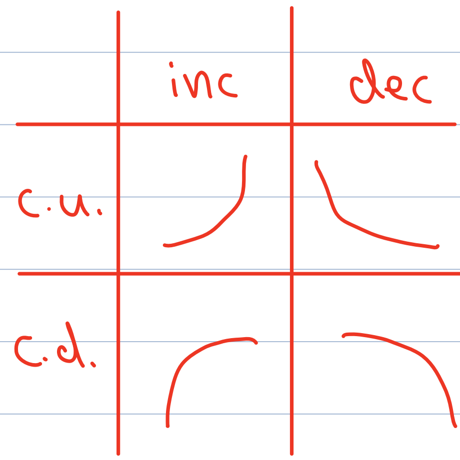
5) Find all cp's ($f' = 0$ or DNE)

6) Find where $f'' = 0$ or DNE.

7) Investigate the intervals divided by the points from 5) and 6) for concavity & inc / dec.

8) Identify any local extrema & poi from 7.

9) Plot



E.x. 1.

Sketch $f(x) = \dfrac{x^2 - 1}{x^2 + 3x}$, $f'(x) = \dfrac{3x^2 + 2x + 3}{x^2(x+3)^2}$

$$f''(x) = \frac{-6(x+1)(x^2+3)}{x^3(x+3)^3}$$

1)

First note $f(x) = \dfrac{(x-1)(x+1)}{x(x+3)}$

So: $D: x \in (-\infty, -3) \cup (-3, 0) \cup (0, \infty)$

2) x-int $(y=0)$

$0 = \dfrac{(x-1)(x+1)}{x(x+3)} \Rightarrow 0 = (x-1)(x+1) \Rightarrow x = \pm 1$

$x(x+3)$

y-int $(x=0)$
↳ $x=0$ not in domain
↳ no y-ints

3) $\lim_{x\to\pm\infty} \frac{x^2+1}{x^2+3x} = 1$ LHR, dominating factor

$\therefore$ H.A @ $y=1$ as $x\to\pm\infty$

4) We analyze near $x=-3$ & $x=0$

$\lim_{x\to0^-} \frac{x^2-1}{x(x+3)} = +\infty$ ⟵ V.A

$\lim_{x\to0^+} \frac{x^2-1}{x(x+3)} = -\infty$ ⟵ V.A

$\lim_{x\to-3^-} \frac{x^2-1}{x(x+3)} = \infty$ ⟵ V.A

$\lim_{x\to-3^+} \frac{x^2-1}{x(x+3)} = -\infty$ ⟵ V.A

5) $f' = \frac{3x^2+2x+3}{x^2(x+3)^2}$

cp's: when $f'=0 \implies 3x^2+2x+3=0$

$x = \frac{-2\pm\sqrt{2^2-4(3\cdot3)}}{2(3)} \to \sqrt{-20}$

$\Rightarrow$ no sdns, $f'(x) \neq 0$.

when f' DNE: $x^2(x+3)^2 = 0 \Rightarrow x = -3, 0$

not in domain, so not c.p.

$\Rightarrow$ no cp's

$\Rightarrow$ no local extrema

Note: still investigate inc/dec on either side of $x=0$ & $x=-3$.

6) $f''(x) = \dfrac{-6(x+1)(x^2+3)}{x^3(x+3)^3}$

$f''(x) = 0 \Rightarrow -6(x+1)(x^2+3) = 0 \Rightarrow x = -1$

candidate poi

$f''(x)$ DNE $\Rightarrow x^3(x+3)^3 = 0 \Rightarrow x = 0, -3$

not in domain, so not candidate poi.

Note: still investigate concavity around $x=0$ & $x=-3$.

7)

	-4		-1		1
inc/dec:		-3		0	
f'	+ve		+ve		+ve

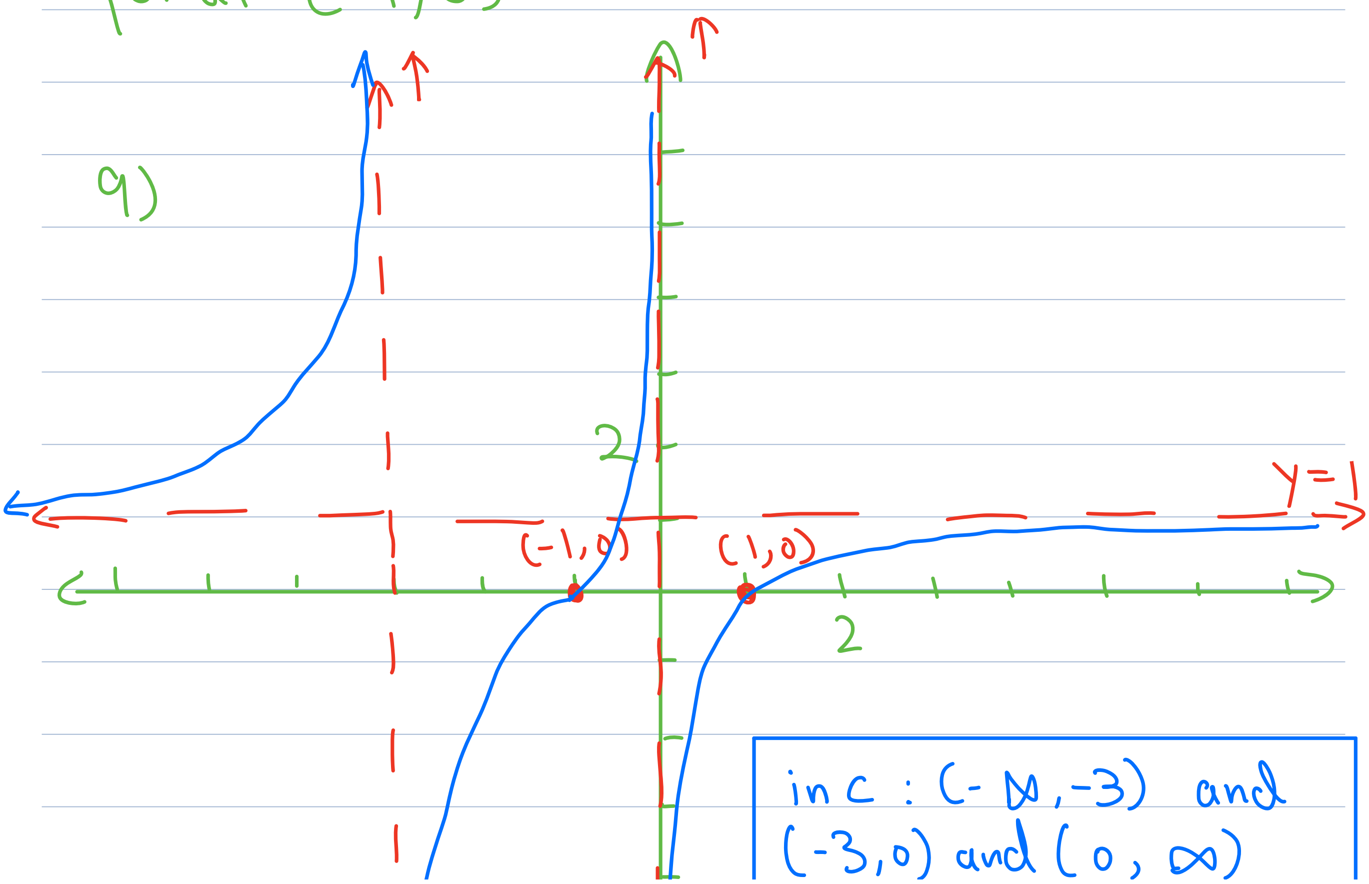
f:	↗	↗	↗	
concavity:	-4, -3	-2, -1	$-\frac{1}{2}$, 0	1
f'':	+ve	−ve	+ve	−ve
f:	∪	∩	∪	∩

For reference: $f(-1) = 0$

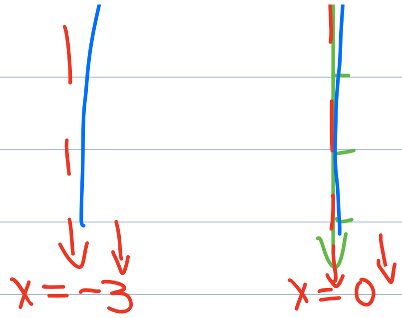
$\therefore$ poi @ $(-1, 0)$

8) We found there're no local extrema & there's a poi at $(-1, 0)$

9)



inc: $(-\infty, -3)$ and $(-3, 0)$ and $(0, \infty)$



dec : no where

c.u: $(-\infty,-3)$ and $[-1,0)$

c.d: $(-3,-1]$ and $(0,\infty)$

Ex. 2

Sketch $f(x)=\dfrac{e^x(x-2)}{x^2-2x}$ given $f'(x)=\dfrac{e^x(x-1)(x-2)}{x^3-2x^2}$

$f''(x)=\dfrac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3}$

1) Note: $f(x)=\dfrac{e^x(x-2)}{x(x-2)}$

Domain: $x\in(-\infty,0)\cup(0,2)\cup(2,\infty)$

2) x-int: $(y=0)$

$0=\dfrac{e^x(x-2)}{x(x-2)} \Rightarrow 0=\underbrace{e^x}_{\neq 0}(x-2)$

$x=2$ isn't in domain.

$\Rightarrow$ no x-int.

y-int: $(x=0)$

N/A, $x=0$ not in domain $\Rightarrow$ no y-int.

3) $\lim_{x\to\infty} \frac{e^x(x-2)}{x(x-2)} = \lim_{x\to\infty} \frac{e^x}{x}$ "$\frac{\infty}{\infty}$"

$\stackrel{LHR}{=} \lim_{x\to\infty} \frac{e^x}{1} = +\infty$

$\lim_{x\to-\infty} \frac{e^x(x-2)}{x(x-2)} = \lim_{x\to-\infty} \frac{e^x}{x}$ "$\frac{0}{\infty}$" not indeterminate

$= 0$

$\therefore$ H.A @ $y=0$ as $x\to-\infty$.

4) at $x=0$:

$\lim_{x\to0^-} \frac{e^x(x-2)}{x(x-2)} = \lim_{x\to0^-} \frac{e^x}{x}$ "$-\infty$"

$\to$ +ve

$\to$ -ve

$\lim_{x\to0^+} \frac{e^x(x-2)}{x(x-2)} = \lim_{x\to0^+} \frac{e^x}{x}$ "$+\infty$"

at $x=2$:

$\lim_{x\to2} \frac{e^x(x-2)}{x(x-2)} = \lim_{x\to2} \frac{e^x}{x} = \frac{e^2}{2}$

$\therefore$ V.A @ $x=0$, hole @ $\left(2, \frac{e^2}{2}\right)$

5) $f'(x) = \frac{e^x(x-1)(x-2)}{x^3-2x^2} = \frac{e^x(x-1)(x-2)}{x^2(x-2)}$

cp's: not in domain.

$f' = 0$: $e^x(x-1)(x-2) = 0 \Rightarrow x = 1, \cancel{2}$

f' DNE: $x^2(x-2) = 0 \Rightarrow x = \cancel{0, 2}$ not in domain.

$\therefore$ there's a cp at $x = 1$.

6) $f''(x) = \dfrac{e^x(x^2-2x+2)(x-2)}{x^4-2x^3} = \dfrac{e^x(x^2-2x+2)(x-2)}{x^3(x-2)}$

$f'' = 0$: $e^x\underbrace{(x^2-2x+2)}_{b^2-4ac<0}\underbrace{(x-2)}_{x\neq 2} = 0$

f'' DNE: $x^3(x-2)$ $\quad x \neq 0,\ x \neq 2$

$\Rightarrow$ no candidate poi

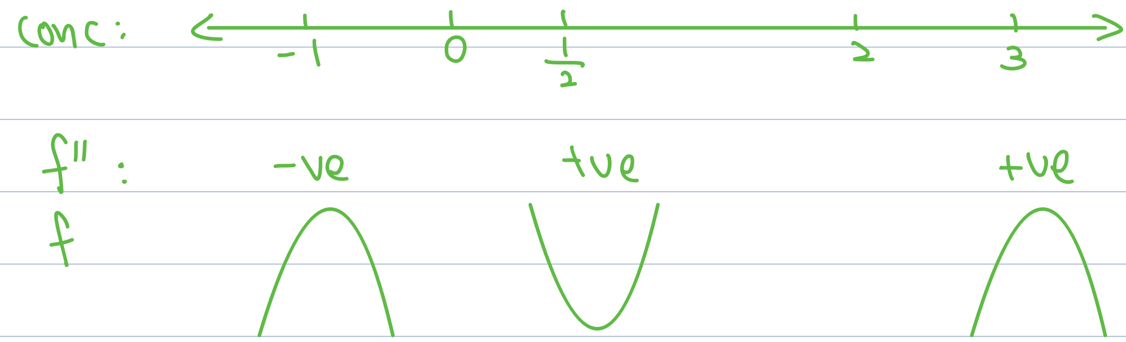
$\Rightarrow$ no poi

7) inc / dec:

x	-1	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	3
f':	$-$ve		$-$ve		$+$ve		$+$ve
f:	$\searrow$		$\searrow$	$\downarrow$	$\nearrow$		$\nearrow$

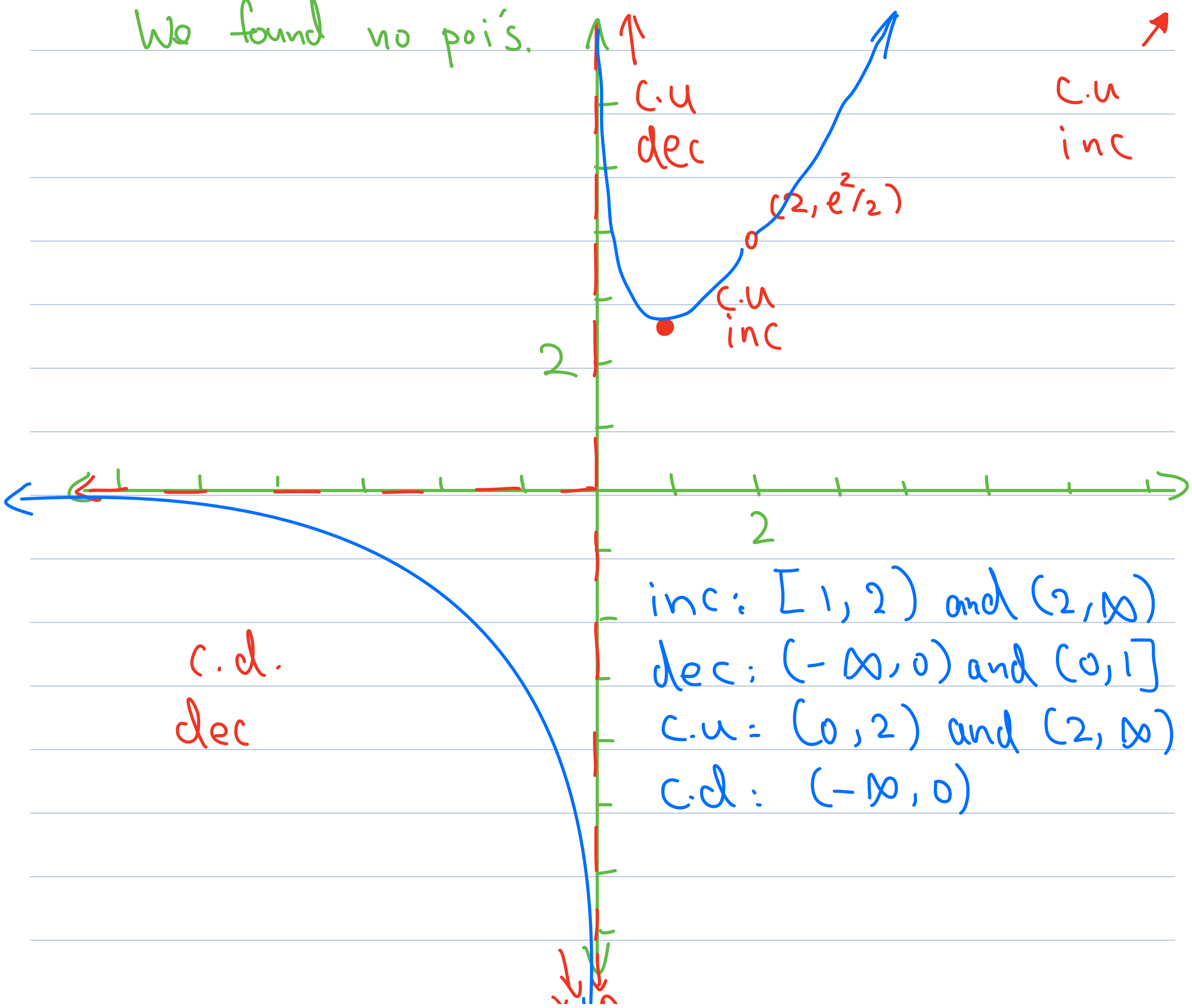
FDT classify $x=1$ as a local min.

(Note: SDT, $f''(1) > 0 \Rightarrow$ c.u. local min.



We already concluded no pois.

8) We found a local min @ $x = 1$.
We'd find $f(1) = e$, so local min @ $(1, e)$.
We found no poi's.



inc: $[1, 2)$ and $(2, \infty)$
dec: $(-\infty, 0)$ and $(0, 1]$
c.u: $(0, 2)$ and $(2, \infty)$
c.d: $(-\infty, 0)$

$$L_a^f(x) = f(a) + f'(a)(x-a)$$

This was just a tangent line to f @ $(a, f(a))$

Key feature:

- $L_a^f(a) = f(a)$
- $L_a^{f'}(a) = f'(a)$
- $\underbrace{\left| f(x) - L_a^f(x) \right|}_{\text{Error.}} \leq \frac{M}{2}(x-a)^2$ $\quad |f''(x)| \leq M$

We did this to approximate complicated functions with a linear one.

Now, what if we want an n^{th} degree polynomial which is a better approx which makes up to the n^{th} derivative & the fcns value at $x = a$.

Construct:

Center

$$T_{n,a}(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + C_3(x-a)^3 + \ldots + C_n(x-a)^n$$

Taylor Degree

First, we want $T_{n,a}(a) = f(a)$

$$T_{n,a}(a) = C_0 + C_1(a-a) + \ldots C_n(a-a)^n$$
$$= C_0$$
$$\Rightarrow C_0 = f(a)$$

Next, we want $T_{n,a}'(a) = f'(a)$

$$T_{n,a}'(x) = 0 + C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + \ldots$$
$$T_{n,a}'(a) = 0 + C_1 + 2C_2(x-a) + 3C_3(x-a)^2 +$$
$$= 0 + C_1 + 0 + \ldots + 0$$
$$= C_1$$
$$\Rightarrow C_1 = f'(a)$$

Now, $T_{n,a}''(a) = f''(a)$

$$T_{n,a}''(x) = 0 + 2C_2 + 6C_3(x-a)$$
$$T_{n,a}''(a) = 2C_2 + 6C_3(a-a) + \ldots.$$
$$\Rightarrow 2C_2 = f''(a)$$
$$\Rightarrow C_2 = \frac{f''(a)}{2}$$

Under the demand that $T_{n,a}'''(a) = f'''(a)$

We find $\quad C_3 = \dfrac{f'''(a)}{6} = \dfrac{f'''(a)}{3!}$

The k^{th} demand leads to:

$$C_k = \frac{f^{(k)}(a)}{k!}$$

Defn: Taylor Polynomials

If f is n-times diff'ble at $x=a$, we say the 'n^{th} degree Taylor theorem for f centered at $x=a$' is the polynomial:

$$T_{n,a}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

Note: If $a=0$, we call it a 'Maclaurin' polynomial.

Ex. Find $T_{5,0}(x)$ for $f(x) = \sin(x)$

First, note $f(0) = 0. \Rightarrow T_{0,0}(x) = 0$

Now, $f'(x) = \cos(x) \Rightarrow f'(0) = 1$

$f''(x) = -\sin(x) \Rightarrow f''(0) = 0$

$f'''(x) = -\cos(x) \Rightarrow f'''(0) = -1$

$f''''(x) = \sin(x) \Rightarrow f''''(0) = 0$

repeats

$\hookrightarrow f^{(5)}(0) = 1,\ f^{(6)} = 0, \dots$

$$T_{1,0}(x) = 0 + \frac{1}{1!}(x-0)^1 \Rightarrow T_{1,0}(x) = x$$

$$T_{2,0}(x) = 0 + \frac{1}{1!}(x-0)^1 + \frac{0}{2!}(x-0)^2 \Rightarrow T_{2,0}(x) = x$$

$$T_{3,0}(x) = 0 + \frac{1}{1!}(x-0)^1 + \frac{0}{2!}(x-0)^2 + \frac{-1}{3!}(x-0)^3$$
$$= x - \frac{1}{6}x^3$$

$$T_{4,0}(x) = 0 + \frac{1}{1!}(x-0)^1 + \frac{0}{2!}(x-0)^2 + \frac{-1}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4$$
$$= x - \frac{1}{6}x^3$$

$$T_{5,0}(x) = 0 + \frac{1}{1!}(x-0)^1 + \frac{0}{2!}(x-0)^2 + \frac{-1}{3!}(x-0)^3 + \frac{0}{4!}(x-0)^4$$
$$+ \frac{1}{5!}(x-0)^5$$
$$= x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

E.x.2.

Find $T_{5,0}(x)$ for $f(x) = \cos(x)$

We have: $f(0) = 1$.

- $f'(x) = -\sin(x) \Rightarrow f'(0) = 0$
- $f''(x) = -\cos(x) \Rightarrow f''(0) = -1$
- $f^{(3)}(x) = \sin(x) \Rightarrow f^{(3)}(0) = 0$
- $f^{(4)}(x) = \cos(x) \Rightarrow f^{(4)}(0) = 1$

$$T_{5,0}(x) = 1 + \frac{0}{1!}(x-0)^1 + \frac{-1}{2!}(x-0)^2 + \frac{0}{3!}(x-0)^3 + \frac{1}{4!}(x-0)^4$$
$$+ \frac{0}{5!}(x-0)^5$$

$$\therefore T_{5,0}(x) = 1 - \frac{1}{2}(x-0)^2 + \frac{1}{24}(x)^4$$

$T_{3,0}(1)$ for $f(x) = e^x$

$$T_{3,0}(x) = 1 + \frac{1}{1!}(x-0)^1 + \frac{1}{2!}(x-0)^2 + \frac{1}{3!}(x-0)^3$$
$$= 1 + 1\cdot x + \frac{1}{2}\cdot x^2 + \frac{1}{6}\cdot x^3$$
$$T_{3,0}(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = \frac{8}{3}$$

Concern about error:

Defn: Taylor Remainder

Assume f is n times diff'ble at $x=a$. Then,
$$R_{n,a}(x) = f(x) - T_{n,a}(x)$$
is called the n^{th} degree Taylor Remainder fcn centered at $x = a$.

Notes:

- $|R_{n,a}(x)| = \text{Error}$

- $R > 0 \Rightarrow f - T > 0 \Rightarrow f > T \Rightarrow$ underestimate

- $R < 0 \Rightarrow$ overestimate

Thm 1: Taylor's Theorem

Assume f is $n+1$ times diff'ble on interval I containing a. Let $x \in I$. Then $\exists$ c between x and a s.t.
$$f(x) - T_{n,a}(x) = R_{n,a}(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$$

Notes:

- For $n=1$, we have $T_{1,a}(x) = L_a^f(x)$

Taylor's Thm says $R_{1,a}(x) = \frac{f''(c)}{2!}(x-a)^2$ and if $|f''(c)| \le M \ \forall x \in I$, Then

$$|R_{1,a}(x)| \le \frac{M}{2}(x-a)^2$$

- For $n=0$, we have $T_{0,a}(x) = f(a)$.

Taylor's Thm says

$$f(x) - T_{0,a}(x) = R_{0,a}(x) = f'(c)(x-a)$$

$\Rightarrow f(x) - f(a) = f'(c)(x-a)$

$\Rightarrow \quad \frac{f(x)-f(a)}{x-a} = f'(c) \qquad$ MVT!

- Taylor's Thm doesn't say how to find c, just that it exists.

Seek the upper bound on the error of approx.

Corollary: Taylor's Equality

If we have $|f^{n+1}(c)| \le M \ \forall c$ between x & a

then, $\quad |R_{n,a}(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$

$\forall c$ between x & a.

Ex 1.

a) Estimating $\cos(0.1)$ with $T_{5,0}(x)$ of $\cos(x)$.

b) What is an upper bound on the error of the approx in a)?

c) Is $T_{5,0}(x)$ an over or under estimate on $[0,1]$?

d) Based on a)→c), give interval for true value of $\cos(0.1)$.

a) $T_{5,0}(x) = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$

Then $\cos(0.1) = 1 - \frac{1}{2}\cdot 0.01 + \frac{1}{24}\cdot 0.0001 = \frac{238801}{240000}$

b) By Taylor's Thm $\exists\, c \in (0, 0.1) \ni$

$$R_{5,0}(0.1) = \frac{f^6 \; c}{6!}(0.1-0)^6$$

$$\Rightarrow |R_{5,0}(0.1)| = \frac{|f^6(c)|}{720}|0.1-0|^6$$

We examine $|f^6(c)|$. We find $f^6(c) = -\cos(c)$

Then $|-\cos(c)| \le 1$ for $c \in (0, 0.1)$

$\therefore$ $M = 1$ is a valid choice.

$\therefore$ $|R_{5,0}(0.1)| \le \frac{1}{720}|0.1|^6 = \frac{1}{720\,000\,000}$

c) We investigate whether $R > 0$ or $R < 0$.

$$R_{5,0}(x) = \frac{-\cos(c)}{720}x^6$$

Now, for $x \in [0,1]$ we have $c \in [0,1]$

for $c \in [0,1]$, $-\cos(c)$

Also, note that $x^6 \geq 0$ on $[0,1]$

$\therefore R_{5,0}(x) \leq 0$.

$\therefore$ We have an overestimate on $[0,1]$

d) Our approx in a) was $\frac{238801}{240000}$ and in c), this is an overestimate. In b), we said, worst error is $\frac{1}{720\,000\,000}$

So: $\cos(0.1) \in \left[\frac{238801}{240000} - \frac{1}{720\,000\,000}, \frac{238801}{240000}\right]$

Ex2.

a) Estimating $\sqrt[3]{30}$ with $T_{2,27}(x)$ of $\sqrt[3]{x}$

b) What is an upper bound on the error of the approx in a)?

c) What is an upper bound on the error of $T_{2,27}(x)$ for $x \in [20,35]$.

d) What intvl of $x>0$ is $T_{2,27}(x)$ over or under estimate?

$f(a) = \sqrt[3]{27} = 3$.

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} \Rightarrow f'(27) = \frac{1}{27}$$
$$f''(x) = -\frac{2}{9}x^{-\frac{5}{3}} \Rightarrow f''(27) = \frac{-2}{2187}$$

$$T_{2,27}(x) = f(a) + f'(a)(x-27) + \frac{f''(a)}{2!}(x-27)^2$$
$$= 3 + \frac{1}{27}(x-27) - \frac{2}{2187 \cdot 2!}(x-27)^2$$
$$= 3 + \frac{1}{27}(x-27) - \frac{1}{2187}(x-27)^2$$

$$\sqrt[3]{30} = T_{2,27}(30) = 3 + \frac{3}{27} - \frac{9}{2187} = \frac{755}{243}$$

b) By Taylor's Theorem,

$\exists\, c \in (27, 30)$ s.t.

$$R_{2,27}(30) = \frac{f'''(c)}{3!}(30-27)^3$$
$$|R_{2,27}(30)| = \frac{|f'''(c)|}{|3!|}|30-27|^3$$

Now, for references. $f'''(x) = \frac{10}{27}x^{-\frac{8}{3}}$

Now, on $[27, 30]$, we note that

$$|f'''(c)| \leq \frac{10}{27}(27)^{-\frac{8}{3}} = 10 \cdot 27^{-\frac{11}{3}}$$

$$\therefore |R_{2,27}(30)| \leq \frac{10 \cdot 27^{-\frac{11}{3}}}{3!}(30-27)^3 = \frac{5}{19683}$$

c) By Taylor's Inequality:

$$|R_{2,27}(x)| \leq \frac{M|x-27|^3}{3!}$$

where $|f'''(c)| \leq M$ for $c\ [20, 35]$.

We saw $f'''(x) = \frac{10}{27} x^{-\frac{8}{3}}$

Then $|f'''(c)| \leq \left|\frac{10}{27} \cdot 20^{-\frac{8}{3}}\right|$ on $[20, 35]$

So:

$$|R_{2,27}(x)| \leq \frac{10}{27 \cdot 3!}(20)^{-\frac{8}{3}} |x-27|^3$$

Then, we wanna max RHS to find upper bound on Error.

For $x \in [20, 35]$, take $x = 35$ as this maximizes $|x-27|^3$

$$\therefore |R_{2,27}(x)| \leq \frac{10}{162}(20)^{-\frac{8}{3}}(8)^3 = \frac{2560}{81}(20)^{-\frac{8}{3}}$$

$$\approx 0.0107$$

d)

It's Joever