1.) gostota $f(x;a,b) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} \cdot \mathbb{1}_{(0,\infty)}(x) =$

$(a,b) \in (0,\infty)\times(0,\infty)$

$$= \begin{cases} \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx} & ;\ x>0 \\ 0 & ;\ \text{sicer} \end{cases}$$ = Gama porazdelitev, ki je definirana na $(0,\infty)$

a) Kompletna zadostna statistika

Vemo: Če $f(\vec{x},\vartheta)$ pripada eksponentni družini, torej če lahko f zapišemo kot $f(\vec{x};\vartheta) = C(\vartheta)\, e^{\langle T(\vec{x}), Q(\vartheta)\rangle} h(\vec{x})$ in ima eksponentna družina poln rang (recimo, če je $Q(\vartheta)$ bijekcija) sledi, da je $T(\vec{x})$ kompletna zadostna statistika.

$$f(\vec{x};a,b) = \prod_{i=1}^{n} \frac{b^a}{\Gamma(a)} x_i^{a-1} e^{-bx_i} = \prod_{i=1}^{n} \frac{b^a}{\Gamma(a)} e^{\ln x_i^{a-1}} e^{-bx_i} =$$

$$= \left(\frac{b^a}{\Gamma(a)}\right)^n \prod_{i=1}^{n} e^{(a-1)\ln x_i - bx_i} = \left(\frac{b^a}{\Gamma(a)}\right)^n e^{(a-1)\sum_{i=1}^{n}\ln x_i - b\sum_{i=1}^{n} x_i} =$$

$$= \underbrace{\left(\frac{b^a}{\Gamma(a)}\right)^n}_{C(\vartheta)} e^{\langle \underbrace{((a-1), -b)}_{Q(\vartheta)}, \underbrace{(\sum_{i=1}^{n}\ln x_i, \sum_{i=1}^{n} x_i)}_{T(\vec{x})}\rangle}$$

Ker je $Q(\vartheta)$ bijekcija ima eksponentna družina poln rang.

Sledi, da je $T(\vec{x}) = (\sum_{i=1}^{n}\ln(x_i), \sum_{i=1}^{n} x_i)$ kompletna zadostna statistika.

Izračunamo v R in dobimo: $T(\vec{x}) = (-250.502, 16.6645)$

b) Problem cenilke največjega verjetja

$$L = \prod_{i=1}^{n} f(x_i; a,b) = \prod_{i=1}^{n} \frac{b^a}{\Gamma(a)} x_i^{a-1} e^{-bx_i}$$

$$l = \ln L = \sum_{i=1}^{n} \ln\left(\frac{b^a}{\Gamma(a)} x_i^{a-1} e^{-bx_i}\right) = \sum_{i=1}^{n}\left[\ln\left(\frac{b^a}{\Gamma(a)} x_i^{a-1}\right) - bx_i\right] =$$

$$= \sum_{i=1}^{n}\left[\ln b^a - \ln\Gamma(a) + (a-1)\ln x_i - bx_i\right] =$$

$$= n(a\ln b - \ln\Gamma(a)) + (a-1)\sum_{i=1}^{n}\ln(x_i) - b\sum_{i=1}^{n} x_i$$

$$\frac{\partial l}{\partial a} = n\cdot\ln b - \frac{n}{\Gamma(a)}\cdot\Gamma'(a) + \sum_{i=1}^{n}\ln(x_i) = 0$$ → Označimo $\Psi(a) = \frac{\Gamma'(a)}{\Gamma(a)}$ = digamma

$$\frac{\partial l}{\partial b} = \frac{na}{b} - \sum_{i=1}^{n} x_i = 0 \Rightarrow \sum_{i=1}^{n} x_i = \frac{na}{b} \Rightarrow \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{a}{b} \Rightarrow \bar{x} = \frac{a}{b}$$

$$\hookrightarrow b = \frac{na}{\sum x_i} = \frac{a}{\bar{x}}$$

$$\rightarrow n\cdot\ln\left(\frac{a}{\bar{x}}\right) - n\cdot\Psi(a) + \sum_{i=1}^{n}\ln(x_i) = 0$$

$$n\cdot(\Psi(a) - \ln a) = \sum_{i=1}^{n}\ln(x_i) - n\ln\bar{x} \quad /:n$$

$$\Psi(a) - \ln a = \frac{1}{n}\sum_{i=1}^{n}\ln(x_i) - \ln\bar{x}$$

Definiramo: $g(a) = \Psi(a) - \ln a - \frac{1}{n}\sum_{i=1}^{n}\ln(x_i) + \ln\bar{x}$

Iščemo ničle funkcije $g(a)$, kar rešujemo numerično z Newton-Raphsonovo metodo:

Potrebujemo: $g'(a) = \psi'(a) - \frac{1}{a} \longrightarrow$ trigamma

Imamo rekurzivno formulo: $a_{n+1} = a_n - \frac{g(a_n)}{g'(a_n)}$ za a_0 vzamemo 0.1

Poračunamo v R in dobimo: $a = 0.8284439$ $b = 4.971306$

c) $\hat{\vartheta}_{mm}$ po metodi momentov

$$E(X) = \frac{a}{b} \quad Var(X) = \frac{a}{b^2} \Rightarrow E(X^2) = Var(X) + E(X)^2 = \frac{a}{b^2} + \frac{a^2}{b^2} = \frac{a+a^2}{b^2}$$

$$\mu_1 = \frac{a}{b} \quad \mu_2 = \frac{a(1+a)}{b^2}$$

$$\hookrightarrow a = \mu_1 \cdot b \hookrightarrow \mu_2 = \frac{\mu_1 b(1+\mu_1 b)}{b^2} \Rightarrow b^2 \mu_2 = \mu_1 b + \mu_1^2 b^2$$

$$\Rightarrow b^2(\mu_2 - \mu_1^2) - \mu_1 b = 0$$ → kvadratna funkcija v b

$$\Rightarrow b_{1,2} = \frac{\mu_1 \pm \sqrt{\mu_1^2 - 4(\mu_2 - \mu_1^2)\cdot 0}}{2(\mu_2 - \mu_1^2)} = \frac{\mu_1 \pm \mu_1}{2(\mu_2 - \mu_1^2)} \stackrel{+}{=} \frac{2\mu_1}{2(\mu_2 - \mu_1^2)} = \frac{\mu_1}{\mu_2 - \mu_1^2}$$

$$\Rightarrow \hat{a} = \frac{\hat{\mu}_1^2}{\hat{\mu}_2 - \hat{\mu}_1^2} \quad \hat{b} = \frac{\hat{\mu}_1}{\hat{\mu}_2 - \hat{\mu}_1^2} \Rightarrow \hat{\vartheta}_{mm} = (\hat{a}, \hat{b})$$

Komentar: Cenilke, dobljene po metodi momentov so dosledne, ker imamo zvezno funkcijo momentov. ~~Cenilke niso nujno nepristranske~~

Dobljene cenilke niso nepristranske, saj funkciji momentov nista linearni funkciji.

Poračunamo cenilko v R in dobimo:

$\hat{a} = 0.8086602$ $\hat{b} = 4.852589$

~~Dobljeni cenilki sta precej podobni cenilkam dobljenim po~~

Dobljena cenilka (oz. elementi cenilk) poračunana po metodi momentov je precej podobna oceni cenilke po metodi največjega verjetja, zato lahko ocenimo, da je $\hat{\vartheta}_{mm}$ precej dobra ocena resničnega ϑ.

d) Metoda delta za dvorazsežno normalno aproksimacijo $\hat{\theta}_{MM}$

2-razsežni CLI: $\sqrt{n}\left(\begin{bmatrix}\bar{X}\\ \bar{X^2}\end{bmatrix}-\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}\right)\xrightarrow{D} N(0,\Sigma)$

Σ je variančno-kovariančna matrika vektorja $\begin{bmatrix}X\\ X^2\end{bmatrix}$. $\mu_i = E(X^i)$

$$\Sigma = \begin{bmatrix}Var(X) & Cov(X,X^2)\\ Cov(X,X^2) & Var(X^2)\end{bmatrix} \to \text{potrebujemo } E(X^k)$$

$$E(X^k)=\int_0^\infty \frac{b^a x^{a-1}e^{-bx}}{\Gamma(a)}x^k\,dx = \frac{b^a}{\Gamma(a)}\int_0^\infty \frac{x^{a+k-1}e^{-bx}}{\Gamma(a+k)}\cdot\frac{\Gamma(a+k)}{b^{a+k}}\cdot b^{a+k}\,dx$$

$= 1$, ker prepoznamo Gamma$(a+k, b)$

$$= \frac{b^a}{\cancel{\Gamma(a)}}\cdot\frac{\Gamma(a+k)}{b^{a+k}} = \frac{(a+k-1)(a+k-2)\cdot\ldots\cdot(a+1)\cdot(a+k-\cancel{k})}{b^k}$$

$$Var(X^2)=E(X^4)-E(X^2)^2 = \frac{a\cdot(a+1)(a+2)(a+3)}{b^4}-\frac{(a+a^2)^2}{b^4} = \frac{(a^2+a)(a^2+5a+6)-(a^2+a)^2}{b^4} =$$

$$= \frac{(a^2+a)(a^2+5a+6-a^2-a)}{b^4} = \frac{a(a+1)(4a+6)}{b^4}$$

$$Cov(X,X^2)=E(X^3)-E(X)E(X^2) = \frac{a(a+1)(a+2)}{b^3}-\frac{a}{b}\,\frac{(a+a^2)}{b^2} = \frac{a(a+1)(a+2-a)}{b^3} = \frac{2a(a+1)}{b^3}$$

$$\Rightarrow \sqrt{n}\left(\begin{bmatrix}\bar{X}\\ \bar{X^2}\end{bmatrix}-\begin{bmatrix}a/b\\ \frac{a+a^2}{b^2}\end{bmatrix}\right)\xrightarrow{D} N\left(0,\begin{bmatrix}a/b^2 & \frac{2a(a+1)}{b^3}\\ \frac{2a(a+1)}{b^3} & \frac{a(a+1)(4a+6)}{b^4}\end{bmatrix}\right)$$

$$\sqrt{n}\left(g\left(\begin{bmatrix}\bar{X}\\ \bar{X^2}\end{bmatrix}\right)-g\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}\right)\right)\xrightarrow{D} N\left(0, J_g(\mu_1,\mu_2)\cdot\Sigma\cdot J_g(\mu_1,\mu_2)^T\right)$$

$$g\left(\begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix}\right)=\begin{bmatrix}\frac{\mu_1^2}{\mu_2-\mu_1^2}\\ \frac{\mu_1}{\mu_2-\mu_1^2}\end{bmatrix} \Rightarrow \frac{\partial g}{\partial \mu_1}=\begin{bmatrix}\frac{2\mu_1(\mu_2-\mu_1^2)-\mu_1^2(-2\mu_1)}{(\mu_2-\mu_1^2)^2}\\ \frac{(\mu_2-\mu_1^2)-\mu_1(-2\mu_1)}{(\mu_2-\mu_1^2)^2}\end{bmatrix}=\begin{bmatrix}\frac{2\mu_1\mu_2}{(\mu_2-\mu_1^2)^2}\\ \frac{\mu_2+\mu_1^2}{(\mu_2-\mu_1^2)^2}\end{bmatrix}$$

$$\frac{\partial g}{\partial \mu_2}=\begin{bmatrix}\frac{0-\mu_1^2}{(\mu_2-\mu_1^2)^2}\\ \frac{0-\mu_1}{(\mu_2-\mu_1^2)^2}\end{bmatrix}=\begin{bmatrix}-\frac{\mu_1^2}{(\mu_2-\mu_1^2)^2}\\ -\frac{\mu_1}{(\mu_2-\mu_1^2)^2}\end{bmatrix} \Rightarrow J_g(\mu_1,\mu_2)=\begin{bmatrix}\frac{2\mu_1\mu_2}{(\mu_2-\mu_1^2)^2} & -\frac{\mu_1^2}{(\mu_2-\mu_1^2)^2}\\ \frac{\mu_2+\mu_1^2}{(\mu_2-\mu_1^2)^2} & -\frac{\mu_1}{(\mu_2-\mu_1^2)^2}\end{bmatrix}$$

~~Označimo~~

$$\Sigma=\begin{bmatrix}\mu_2-\mu_1^2 & \frac{2\mu_2(\mu_2-\mu_1^2)}{\mu_1}\\ \frac{2\mu_2(\mu_2-\mu_1^2)}{\mu_1} & \frac{\mu_2(6\mu_2-2\mu_1^2)(\mu_2-\mu_1^2)}{\mu_1^2}\end{bmatrix} \qquad J_g^T(\mu_1,\mu_2)=\begin{bmatrix}\frac{2\mu_1\mu_2}{(\mu_2-\mu_1^2)^2} & \frac{\mu_2+\mu_1^2}{(\mu_2-\mu_1^2)^2}\\ -\frac{\mu_1^2}{(\mu_2-\mu_1^2)^2} & -\frac{\mu_1}{(\mu_2-\mu_1^2)^2}\end{bmatrix}$$

Označimo $\Sigma_D = J_g(\mu_1,\mu_2)\cdot\Sigma\cdot J_g^T(\mu_1,\mu_2)$ = izračunano v Mathematici = $\nearrow$

$$\to \Sigma_D = \cancel{\begin{bmatrix} \dfrac{-4\mu_1^4\mu_2^2+8\mu_1^2\mu_2^2-6\mu_2^2+2\mu_1\mu_2}{\mu_1^2(\mu_1^2-\mu_2)^3} & \dfrac{-2\mu_1^3\mu_2+4\mu_1^2\mu_2^2+2\mu_1\mu_2^2-6\mu_2^2-2\mu_1^5\mu_2+2\mu_1^3\mu_2+2\mu_1^2\mu_2}{\mu_1^2(\mu_1^2-\mu_2)^3} \\ \dfrac{-2\mu_1^3\mu_2^2+4\mu_1^2\mu_2^2+2\mu_1\mu_2^2-6\mu_2^2-2\mu_1^5\mu_2+2\mu_1^3\mu_2+2\mu_1^2\mu_2}{\mu_1^2(\mu_1^2-\mu_2)^3} & \dfrac{-\mu_1^6-\mu_1^2\mu_2^2+4\mu_1\mu_2^2-6\mu_2^2-2\mu_1^4\mu_2+4\mu_1^3\mu_2+2\mu_1^2\mu_2}{\mu_1^2(\mu_1^2-\mu_2)^3} \end{bmatrix}}$$

$$= \begin{bmatrix} \dfrac{2\mu_1^2\mu_2}{\mu_1^4+\mu_2^2-2\mu_1^2\mu_2} & \dfrac{2\mu_1\mu_2}{\mu_1^4+\mu_2^2-2\mu_1^2\mu_2} \\ 2\mu_1\mu_2/\mu_1^4+\mu_2^2-2\mu_1^2\mu_2 & -\mu_1^2+3\mu_2/\mu_1^4+\mu_2^2-2\mu_1^2\mu_2 \end{bmatrix}$$

Po metodi delta torej dobimo: $\sqrt{n}\left(\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}-\begin{bmatrix}a\\ b\end{bmatrix}\right)$

$$\cancel{\sqrt{n}\left(\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}-\right)} \; \cancel{\sqrt{n}\left(\begin{bmatrix}\frac{\bar{x}^2}{\bar{x^2}-\bar{x}^2}\\ \frac{\bar{x}}{\bar{x^2}-\bar{x}^2}\end{bmatrix}-\begin{bmatrix}\frac{\mu_1^2}{\mu_2-\mu_1^2}\\ \frac{\mu_1}{\mu_2-\mu_1^2}\end{bmatrix}\right)} \xrightarrow{D} N(0, \Sigma_D)$$

Če želimo, lahko dobimo tudi standardno normalno porazdelitev – s standardizacijo.

Uporabimo razcep Choleskega: $\Sigma_D = V \cdot V^T$ in dobimo:

$$\cancel{V^{-1}\sqrt{n}\left(\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}\begin{bmatrix}\frac{\bar{x}^2}{\bar{x^2}-\bar{x}^2}\\ \frac{\bar{x}}{\bar{x^2}-\bar{x}^2}\end{bmatrix}-\begin{bmatrix}a\\ b\end{bmatrix}\begin{bmatrix}\frac{\mu_1^2}{\mu_2-\mu_1^2}\\ \frac{\mu_1}{\mu_2-\mu_1^2}\end{bmatrix}\right)\xrightarrow{D} N(0, I)} \; V^{-1}\sqrt{n}\left(\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}-\begin{bmatrix}a\\ b\end{bmatrix}\right)\xrightarrow{D} N(0, I)$$

(vendar mora biti za to Σ_d hermitska)

e) Aproksimativno območje zaupanja stopnje zaupanja 0.95

Poskusimo s konstrukcijo Hotellingovega območja zaupanja. $= \frac{1}{n-d}\sum_{i=1}^{n}(x_i-\bar{x})(x_i-\bar{x})^T$ (standardna cenilka za Σ)

Ker so d) vemo, da lahko $\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}$ aproksimiramo s $N\left(\begin{bmatrix}a\\ b\end{bmatrix}, \Sigma_D\right)$, lahko za primerno cenilko $\hat{\Sigma}_D$ konstruiramo območje zaupanja iz pivotne funkcije

$$\left\langle \left(\frac{\hat{\Sigma}_D}{n}\right)^{-1}\left(\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}-\begin{bmatrix}a\\ b\end{bmatrix}\right), \left(\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}-\begin{bmatrix}a\\ b\end{bmatrix}\right)\right\rangle \sim \frac{d(n-1)}{n-d}F_{d,n-d}$$

Če rot v ~~…~~ normalnem primeru naredimo diagonalizacijo $\left(\frac{\hat{\Sigma}_D}{n}\right)^{-1} = Q \cdot \Lambda \cdot Q^T$
↳ (5. naloga)

dobimo: $\left\langle \Lambda Q^T\left(\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}-\begin{bmatrix}a\\ b\end{bmatrix}\right), Q^T\left(\begin{bmatrix}\hat{a}\\ \hat{b}\end{bmatrix}-\begin{bmatrix}a\\ b\end{bmatrix}\right)\right\rangle \leq \frac{d(n-1)}{n-d}F_{d;n-d;\alpha} =: \eta^2 = \eta^2(X)$

Pri tem je seveda Q ~~za~~ ortogonalna matrika lastnih vektorjev $\left(\frac{\hat{\Sigma}_D}{n}\right)^{-1}$,

Λ pa diagonalna matrika z lastnimi vrednostmi na diagonali ; $\lambda_i > 0$

Če rešimo zgornjo neenakost dobimo za rešitev elipsoid s polosmi dolžine $\frac{\eta}{\sqrt{\lambda_1}}$, $\frac{\eta}{\sqrt{\lambda_2}}$ na stolpcih matrike Q s središčem v $(\hat{a}, \hat{b}) = (0.8086602, 4.852589)$

$\frac{\eta}{\sqrt{\lambda_1}} = 0.1971685$, $\frac{\eta}{\sqrt{\lambda_2}} = 2.921562$