6.) $p_i = P(\text{foreign}_i = 1) = P(X_i = 1) = 1 - e^{-e^{\beta_0 + \beta_1 \text{weight}_i + \beta_2 \text{mpg}_i}} = 1 - e^{-e^{z_i\beta}}$

Definiramo: $Z = \begin{bmatrix} 1 & w_1 & mpg_1 \\ \vdots & \vdots & \vdots \\ 1 & w_n & mpg_n \end{bmatrix}$ in $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}$

Vemo: $X_i \overset{NEP}{\sim} Bin(1, p_i)$ in zanj velja:

$g(p_i) = \beta_0 + z_{i1}\beta_1 + z_{i2}\beta_2 = z_i\beta$ $\quad p_i = g^{-1}(z_i\beta) \to$ naša funkcija je cloglog

d) Funkcija verjetja

$f(x_i) = p_i^{x_i}(1-p_i)^{1-x_i}$

$$L = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} p_i^{x_i}(1-p_i)^{1-x_i} = \prod_{i=1}^{n} e^{x_i \ln p_i + (1-x_i)\ln(1-p_i)} =$$

$$= e^{\sum_{i=1}^{n}[x_i \ln p_i + (1-x_i)\ln(1-p_i)]}$$

e) $\beta_0, \beta_1, \beta_2$ so ΠNV

$$l(\beta; X, Z) = \ln L = \sum_{i=1}^{n}[x_i \ln p_i + (1-x_i)\ln(1-p_i)]$$

$$\frac{\partial l}{\partial \beta} = \sum_{i=1}^{n} \frac{\partial[x_i \ln p_i + (1-x_i)\ln(1-p_i)]}{\partial p_i} \cdot \frac{\partial p_i}{\partial \beta} = \sum_{i=1}^{n}\left(\frac{x_i}{p_i} - \frac{1-x_i}{1-p_i}\right) \underbrace{e^{-e^{z_i\beta}}}_{(1-p_i)} e^{z_i\beta} \cdot z_i = (*)$$

$$\frac{\partial p_i}{\partial \beta} = -e^{-e^{z_i\beta}}(-1)e^{z_i\beta} \cdot z_i = e^{-e^{z_i\beta}} e^{z_i\beta} \cdot z_i$$

$$(*) = \sum_{i=1}^{n}\left(\frac{x_i - \cancel{x_i p_i} - p_i + \cancel{x_i p_i}}{p_i \cancel{(1-p_i)}}\right)\cancel{(1-p_i)} e^{z_i\beta} \cdot z_i =$$

$$= \sum_{i=1}^{n}\left(\frac{x_i}{p_i} - 1\right) e^{z_i\beta} \cdot z_i = \text{grad}_\beta(l) = Z^T \begin{bmatrix} e^{z_1\beta}\frac{x_1}{p_1} \\ \vdots \\ e^{z_n\beta}\frac{x_n}{p_n} \end{bmatrix} - Z^T \begin{bmatrix} e^{z_1\beta} \\ \vdots \\ e^{z_n\beta} \end{bmatrix}$$

$$H_\beta(l) = \frac{\partial^2 l}{\partial^2 \beta} = \sum_{i=1}^{n} z_i e^{z_i\beta} \cdot \frac{x_i}{p_i} \cdot z_i + \sum_{i=1}^{n} z_i e^{z_i\beta} x_i \cdot \frac{\partial}{\partial \beta}(p_i^{-1}) - \sum_{i=1}^{n} z_i e^{z_i\beta} \cdot z_i \overset{(**)}{=}$$

$$\frac{\partial(p_i^{-1})}{\partial \beta} = -p_i^{-2} \cdot \frac{\partial p_i}{\partial \beta} = -p_i^{-2} \cdot e^{-e^{z_i\beta}} \cdot e^{z_i\beta} \cdot z_i$$

$$\overset{(**)}{=} \sum_{i=1}^{n} z_i e^{z_i\beta} \cdot \frac{x_i}{p_i} \cdot z_i - \sum_{i=1}^{n} z_i e^{z_i\beta} \frac{x_i}{p_i^2} e^{-e^{z_i\beta}} e^{z_i\beta} \cdot z_i - \sum_{i=1}^{n} z_i e^{z_i\beta} \cdot z_i =$$

$$= Z^T \begin{bmatrix} e^{z_1\beta}\left(\frac{x_1}{p_1} - \frac{x_1}{p_1^2}(1-p_1)e^{z_1\beta} - 1\right) & & \\ & \ddots & \\ & & e^{z_n\beta}\left(\frac{x_n}{p_n} - \frac{x_n}{p_n^2}(1-p_n)e^{z_n\beta} - 1\right) \end{bmatrix} \cdot Z$$

Rešujemo numerično z Newton-Raphsonovo metodo:
nastavimo začetni približek $\hat{\beta}_0 = 0$ in potem rekurzivno računamo:

$$\hat{\beta}_k = \hat{\beta}_{k-1} - [H_\beta(l(\hat{\beta}_{k-1}))]^{-1} \cdot grad_\beta(l(\hat{\beta}_{k-1}))$$

Implementiramo v R in dobimo:

$\beta_0 = 12.595884184$ $\beta_1 = -0.003895763$ $\beta_2 = -0.147706589$

Če primerjamo z ocenami, ki jih vrne vgrajena funkcija glm za link = cloglog:

$\beta_0 = 12.595809982$ $\beta_1 = -0.003895713$ $\beta_2 = -0.147708206$

vidimo, da so naše ocene zelo dobre.

c) Fisherjeva informacijska matrika : $FI(\vartheta) = -E(H_\vartheta(\ln f)(X;\vartheta))$

$$nFI(\vartheta) = -E[H_\vartheta(l)(X;\vartheta)]$$

Zanima nas torej $E[H_\beta(l)]$. Vemo: $E[X_i] = n \cdot p_i = p_i$, ker je $X_i \sim Bin(1, p_i)$

Za vsi velja torej $E[H_\beta(l)] = z^T E[e^{z_i\beta}(\frac{x_i}{p_i} - \frac{x_i}{p_i^2}(1-p_i)e^{z_i\beta} - 1)] \cdot z =$

$$= z^T e^{z_i\beta}(\frac{p_i}{p_i} - \frac{p_i}{p_i^2}(1-p_i)e^{z_i\beta} - 1) \cdot z = -z^T e^{z_i\beta}(\frac{1-p_i}{p_i}e^{z_i\beta}) \cdot z$$

Sledi:

$$FI = -\frac{1}{n}E[H_\beta(l)(X;\beta)] = +\frac{1}{n} z^T \begin{bmatrix} e^{2z_1\beta}\frac{1-p_1}{p_1} & & \\ & \ddots & \\ & & e^{2z_n\beta}\frac{(1-p_n)}{p_n} \end{bmatrix} \cdot z$$

Izračunamo v R in dobimo:

$$FI = \begin{bmatrix} 0.1428459 & 317.2122 & 3.756256 \\ 317.2122 & 729062.8464 & 7867.1676 \\ 3.756256 & 7867.1679 & 114.784113 \end{bmatrix}$$

d) Standardne napake za parametre β_0, β_1 in β_2

Najprej moramo izračunati $\frac{FI^{-1}}{n}$. Standardne napake za parametre β_0, β_1 in β_2 so potem koreni diagonalnih elementov izračunane matrike torej $\sqrt{\frac{FI^{-1}_{ii}}{n}}$.

Implementiramo v R in dobimo:

$\sigma_0 = 2.998616$ $\sigma_1 = 0.0009714016$ $\sigma_2 = 0.03811046$

Če primerjamo to z vrednostmi dobljenimi z vgrajeno metodo:

$\sigma_0 = 2.998545$ $\sigma_1 = 0.0009713588$ $\sigma_2 = 0.03811120$

vidimo, da so naši približki zelo dobri.

e) Na standarden način preizkusite $H_0: \beta_1 = \beta_2 = 0$ - z metodo razmerja verjetij:

$$\ell(\beta; z, X) = \sum_{i=1}^{n} [x_i \ln p_i + (1-x_i)\ln(1-p_i)]$$

$$\Lambda = \frac{\sup_{\beta \in H_0} \mathcal{L}(\beta; X, z)}{\sup_{\beta \in \Theta} \mathcal{L}(\beta; X, z)} = \frac{\mathcal{L}(\tilde{\beta}; X, z)}{\mathcal{L}(\hat{\beta}; X, z)}$$

$$\log \Lambda = \underbrace{\ell(\tilde{\beta}; X, z)}_{\text{model pod hipotezo } H_0} - \underbrace{\ell(\hat{\beta}; X, z)}_{\text{polni model}}$$

pod H_0: $$\ell(\beta; X, z) = \sum_{i=1}^{n} [x_i \ln(1-e^{-e^{\beta_0}}) + (1-x_i)\cancel{\ln}(\cancel{1-1+}e^{-e^{\beta_0}})] =$$

$$= \sum_{i=1}^{n} [x_i \ln(1-e^{-e^{\beta_0}}) - (1-x_i)e^{\beta_0}] =$$

$$\frac{\partial \ell}{\partial \beta_0} = \sum_{i=1}^{n} \left[\frac{x_i}{1-e^{-e^{\beta_0}}} (-1) e^{-e^{\beta_0}} (-1) e^{\beta_0} \beta_0 - (1-x_i) e^{\beta_0} \beta_0\right] = 0$$

$$\Rightarrow \sum_{i=1}^{n} \left[x_i e^{\beta_0} \beta_0 \left(\frac{e^{-e^{\beta_0}}}{1-e^{-e^{\beta_0}}} + 1\right) - e^{\beta_0}\beta_0\right] = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i e^{\beta_0} \beta_0 \left(\frac{\cancel{e^{-e^{\beta_0}}} + 1 - \cancel{e^{-e^{\beta_0}}}}{1-e^{-e^{\beta_0}}}\right) = n \cdot e^{\beta_0}\beta_0$$

$$\Rightarrow \frac{\sum_{i=1}^{n} x_i}{n} = \frac{\cancel{e^{\beta_0} \beta_0}}{\cancel{e^{\beta_0} \beta_0}} \cdot \frac{1-e^{-e^{\beta_0}}}{1} \Rightarrow \bar{X} = 1-e^{-e^{\beta_0}}$$

$$g^{-1}(\tilde{\beta}_0) = \bar{X} \qquad \tilde{\beta}_0 = g(\bar{X})$$

Vemo tudi: $-2\ln\Lambda \sim \chi^2_{3-1} = \chi^2_2 \Rightarrow$ naš test $= \begin{cases} H_0 \text{ zavrnemo}; & -2\ln\Lambda > \chi^2_2 \\ H_0 \text{ ne zavrnemo}; & \text{sicer} \end{cases}$

V R poračunamo vrednosti za poln in omejen model ter testno statistiko in dobimo:

$-2\ln\Lambda = 85.4191 > 5.9915 = \chi^2_2$,

torej ničelno hipotezo zavrnemo.