1. Let $f : M \to N$ and $g : N \to L$ both be surjective functions. Prove that $g \circ f$ is surjective. (First, look carefully at what the domain and codomain of $g \circ f$ are. Of course, be sure to follow the definition of surjective EXACTLY, and do not ignore quantifiers. Determine which "for all" is being used and which "for all" is being proved, and which "there exists" is being used and which "there exists" is being proved.)

Proof. To prove that $g \circ f$ is surjective, we must show that for all $\ell \in L$, there exists $m \in M$ such that $(g \circ f)(m) = \ell$.

Let $\ell \in L$ be arbitrary. Since g is surjective, there exists $n \in N$ such that $g(n) = \ell$. Since f is surjective, there exists $m \in M$ such that $f(m) = n$.

Therefore, $(g \circ f)(m) = g(f(m)) = g(n) = \ell$.

Thus, for all $\ell \in L$, there exists $m \in M$ such that $(g \circ f)(m) = \ell$, proving that $g \circ f$ is surjective. $\square$

2. Let $f : M \to N$ and $g : N \to L$ both be injective functions. Prove that $g \circ f$ is injective. (The same types of hints as the previous question, and in fact, at this level of math, the following advice always applies: since the question mentions the word injective, go review the definition first and do NOT ignore quantifiers. Determine which "for all" is being used and which "for all" is being proved.)

Proof. To prove that $g \circ f$ is injective, we must show that for all $m_1, m_2 \in M$, if $(g \circ f)(m_1) = (g \circ f)(m_2)$, then $m_1 = m_2$.

Let $m_1, m_2 \in M$ be arbitrary and assume $(g \circ f)(m_1) = (g \circ f)(m_2)$. Then $g(f(m_1)) = g(f(m_2))$. Since g is injective, this implies $f(m_1) = f(m_2)$. Since f is injective, this implies $m_1 = m_2$.

Thus, for all $m_1, m_2 \in M$, if $(g \circ f)(m_1) = (g \circ f)(m_2)$, then $m_1 = m_2$, proving that $g \circ f$ is injective. $\square$

3. Let $f : M \to N$. Prove: if A and B are subsets of N such that $A \subseteq B$, then $f^{-1}(A) \subseteq f^{-1}(B)$.

Proof. To prove that $f^{-1}(A) \subseteq f^{-1}(B)$, we must show that for all $x \in f^{-1}(A)$, $x \in f^{-1}(B)$.

Let $x \in f^{-1}(A)$ be arbitrary. By definition of preimage, this means $f(x) \in A$. Since $A \subseteq B$, this implies $f(x) \in B$. By definition of preimage, this means $x \in f^{-1}(B)$.

Thus, for all $x \in f^{-1}(A)$, $x \in f^{-1}(B)$, proving that $f^{-1}(A) \subseteq f^{-1}(B)$. $\square$

4. Prove that $[2, 6]$ and $[11, 20]$ are equicardinal. For clarifcation, both sets/intervals mentioned are subsets of $\mathbb{R}$.

Proof. To prove that [2,6] and [11,20] are equicardinal, we need to find a bijective function between them.

Let $f : [2, 6] \to [11, 20]$ be defined by $f(x) = 3x + 5$.

First, let's prove f is injective: Let $x_1, x_2 \in [2, 6]$ and assume $f(x_1) = f(x_2)$. Then $3x_1 + 5 = 3x_2 + 5$. Therefore $x_1 = x_2$, proving f is injective.

Now, let's prove f is surjective: Let $y \in [11, 20]$ be arbitrary. Let $x = (y - 5)/3$. Then $f(x) = y$, and we need to verify $x \in [2, 6]$. When y = 11, x = 2 When y = 20, x = 5 Since f is linear and continuous, $x \in [2, 6]$. Therefore $f(x) = y$ for some $x \in [2, 6]$.

Since f is both injective and surjective, it is bijective. Therefore $[2, 6]$ and $[11, 20]$ are equicardinal. □

5. Prove: if A is countably infinite and B is countably infinite and C is countably infinite and $A \cap B = \emptyset$ and $A \cap C = \emptyset$ and $B \cap C = \emptyset$, prove $A \cup B \cup C$ is countably infinite. (Hint: it will be helpful to look at a past HW key where a formula for a sequence was given.)

Proof. Since A is countably infinite, there exists a bijection $f : \mathbb{N} \to A$. Since B is countably infinite, there exists a bijection $g : \mathbb{N} \to B$. Since C is countably infinite, there exists a bijection $h : \mathbb{N} \to C$.

Define $\emptyset : \mathbb{N} \to A \cup B \cup C$ by: $\phi(n) = \begin{cases} f(k) & \text{if } n = 3k - 2 \text{ for some } k \in \mathbb{N} \\ g(k) & \text{if } n = 3k - 1 \text{ for some } k \in \mathbb{N} \\ h(k) & \text{if } n = 3k \text{ for some } k \in \mathbb{N} \end{cases}$

To prove $\emptyset$ is injective: Let $n_1, n_2 \in \mathbb{N}$ with $\emptyset(n_1) = \emptyset(n_2)$. If n_1, n_2 came from different cases, they would map to different sets (A, B, orC), which are disjoint. If from the same case, f, g, h being injective implies $n_1 = n_2$. Thus $\emptyset$ is injective.

To prove $\emptyset$ is surjective: Let $x \in A \cup B \cup C$. If $x \in A$, then $x = f(k)$ for some k, so $x = \emptyset(3k - 2)$. If $x \in B$, then $x = g(k)$ for some k, so $x = \emptyset(3k - 1)$. If $x \in C$, then $x = h(k)$ for some k, so $x = \emptyset(3k)$. Thus $\emptyset$ is surjective.

Therefore $A \cup B \cup C$ is countably infinite. □