1. Let $P = \{a \in \mathbb{Z} : a > 0\}$. Let $D = P \times P$. Consider the binary relation $\sim$ on D defined by the following property: $(a, b) \sim (c, d)$ is true means $a + d = b + c$. Prove that $\sim$ is an equivalence relation on D.

   Proof. To prove that $\sim$ is an equivalence relation on D, we need to show that $\sim$ is reflexive, symmetric, and transitive.

   Reflexive: We need to show that for all $(a, b) \in D$, $(a, b) \sim (a, b)$. By definition, $(a, b) \sim (a, b)$ means $a + b = b + a$, which is true since addition is commutative. Therefore, $\sim$ is reflexive.

   Symmetric: We need to show that for all $(a, b), (c, d) \in D$, if $(a, b) \sim (c, d)$, then $(c, d) \sim (a, b)$. By definition, $(a, b) \sim (c, d)$ means $a + d = b + c$. Since addition is commutative, $a + d = b + c$ implies $c + b = d + a$, which means $(c, d) \sim (a, b)$. Therefore, $\sim$ is symmetric.

   Transitive: We need to show that for all $(a, b), (c, d), (e, f) \in D$, if $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$, then $(a, b) \sim (e, f)$. By definition, $(a, b) \sim (c, d)$ means $a + d = b + c$ and $(c, d) \sim (e, f)$ means $c + f = d + e$. Adding these two equations, we get $a + d + c + f = b + c + d + e$. Simplifying, we get $a + f = b + e$, which means $(a, b) \sim (e, f)$. Therefore, $\sim$ is transitive.

   Since $\sim$ is reflexive, symmetric, and transitive, it is an equivalence relation on D. $\square$

2. Based on these definitions:

   - $B = \{x \in G \mid \text{for all } y \in M, \text{ the post office delivers a letter from } x \text{ to } y\}$
   - $G = \{x \in L : \text{the ice skating performance featured } x\}$

   and these hypotheses:

   - H1: For all $c \in S$, the chancellor calls c.
   - H2: $M \subseteq U$.
   - H3: $T \subseteq G$.
   - H4: $B \subseteq S$.
   - H5: For all $g \in G$, for all $u \in U$, if g plays tennis and the ice skating performance featured u, then the post office delivers a letter from g to u.
   - H6: For all $u \in U$, the ice skating performance featured u.

   prove: For all $h \in T$, if h plays tennis, then the chancellor calls h.

   Proof. Let $h \in T$ be given and assume h plays tennis. We need to show that the chancellor calls h. Since $h \in T$ and $T \subseteq G$ (by H3), we know that $h \in G$. Since $h \in G$ and h plays tennis, by H5 we know: For all $u \in U$, if the ice skating performance featured u, then the post office delivers a letter from h to u. By H6, we know that for all $u \in U$, the ice skating performance featured u. Therefore, for all $u \in U$, the post office delivers a letter from h to u. Since $M \subseteq U$ (by H2), we know that for all $y \in M$, the post office delivers a letter from h to y. This means $h \in B$ (by the definition of B). Since $h \in B$ and $B \subseteq S$ (by H4), we know that $h \in S$. By H1, since $h \in S$, the chancellor calls h. Therefore, for all $h \in T$, if h plays tennis, then the chancellor calls h. $\square$

3. Let $A = \{6, 7\}$ and let $B = \{8, 9\}$. Let $f = \{(6, 8), (7, 8), (7, 9)\}$. Explain why f is not a function from A to B.

Proof. To show that f is not a function from A to B, we need to show that there exists an element in A that is associated with more than one element in B. According to the definition of a function, each element in the domain (set A) must be associated with exactly one element in the codomain (set B). In this case, $f = \{(6, 8), (7, 8), (7, 9)\}$. We see that the element $7 \in A$ is associated with both 8 and 9 in B. Specifically, $(7, 8) \in f$ and $(7, 9) \in f$. This means that 7 is associated with more than one element in B, which violates the definition of a function. Therefore, f is not a function from A to B. $\square$

4. Let $A = \{6, 7\}$ and let $B = \{8, 9\}$. Let $f = \{(6, 8), (7, 8)\}$. Explain why f is a function from A to B.

Proof. To show that f is a function from A to B, we need to verify that each element in A is associated with exactly one element in B. According to the definition of a function, for every input (element in the domain A), there must be a unique output (element in the codomain B).

In this case, $f = \{(6, 8), (7, 8)\}$. We see that:

- The element $6 \in A$ is associated with the element $8 \in B$.
- The element $7 \in A$ is associated with the element $8 \in B$.

Each element in A is associated with exactly one element in B, and there are no elements in A that are associated with more than one element in B. Therefore, f satisfies the definition of a function.

Additionally, the non-numbered part of the definition of a function states that every element in the domain must be mapped to an element in the codomain. In this case, both elements 6 and 7 in A are mapped to elements in B, satisfying this part of the definition as well.

Therefore, f is a function from A to B. $\square$

5. Let $A = \{1, 2, 3\}$. Let $B = \{4, 5\}$. State every function from A to B. (How many functions total do you end up defining?)

Proof. There are $2^3 = 8$ functions from A to B. They are:

- $f_1 = \{(1, 4), (2, 4), (3, 4)\}$
- $f_2 = \{(1, 4), (2, 4), (3, 5)\}$
- $f_3 = \{(1, 4), (2, 5), (3, 4)\}$
- $f_4 = \{(1, 4), (2, 5), (3, 5)\}$
- $f_5 = \{(1, 5), (2, 4), (3, 4)\}$
- $f_6 = \{(1, 5), (2, 4), (3, 5)\}$
- $f_7 = \{(1, 5), (2, 5), (3, 4)\}$
- $f_8 = \{(1, 5), (2, 5), (3, 5)\}$

$\square$

6. OPTIONAL: You are the front desk manager at The Count's Hotel at Transylvania Beach. The hotel has an infinite number of rooms in the following sense: each hotel room has a plaque with a positive integer on it, with no duplication, and for each positive integer, there is a hotel room with that number. Using the PA system, you can use the microphone at the front desk to speak to the occupant in each room. Oh! Each room is occupied, so you have no vacancy.

   Suddenly, a bus from Van Helsing's Charter Vans, Inc. with an infinite number of people pulls up. The number of people in the bus is infinite in the following sense: each person on the bus has an index card with a positive integer written on it (with no duplication), and for each positive integer, there is a person who is assigned that number.

   How can you accommodate all infinite people already in the hotel and all infinite people on the bus? Note, you can't just tell all the people in the hotel to move "an infinite number of spots". Your instructions should give the occupant in hotel room 54601 a specific hotel room to use, and should also give the person number 608 on the bus a specific hotel room to use!

   Proof. To accommodate all the infinite people already in the hotel and all the infinite people on the bus, we can use the following strategy:

   1. Announce to all current hotel occupants to move to the room with double their current room number. Specifically, the occupant in room n should move to room $2n$.
   2. Announce to all people on the bus to occupy the rooms with odd numbers. Specifically, the person with index card m should move to room $2m - 1$.

   This way, every current hotel occupant moves to an even-numbered room, and every person from the bus moves to an odd-numbered room. Since every positive integer is either even or odd, this ensures that each person has a unique room.

   As an example situation, we have the following:

   - The occupant in room 1 moves to room 2.
   - The occupant in room 2 moves to room 4.
   - The occupant in room 3 moves to room 6.
   - The person with index card 1 moves to room 1.
   - The person with index card 2 moves to room 3.
   - The person with index card 3 moves to room 5.

   Therefore, all infinite people already in the hotel and all infinite people on the bus are accommodated. $\square$