1. Find a piecewise formula for the sequence a_n whose first terms are

   $1, 1, 1, 1, 2, 2, 2, 2, 3, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, \ldots$

   [1]

   The piecewise formula for this sequence can be expressed as:

$$a_n = \begin{cases} \frac{n+3}{4} \text{ if } n+3 \text{ is a multiple of } 4; \\ \frac{n+2}{4} \text{ if } n+2 \text{ is a multiple of } 4; \\ \frac{n+1}{4} \text{ if } n+1 \text{ is a multiple of } 4; \\ \frac{n}{4} \text{ if } n \text{ is a multiple of } 4. \end{cases}$$

[1] You may not have a recursive formula. Your formula must work for all positive integers n, not just up to $n \leq 20$. You should certainly **check** that your formula works for $n = 5$ and for $n = 18$ and so on, but your formula should also work for $n = 2876387$. Hint: create a piecewise formula with 4 pieces, and look at the key for previous HW where there was a three-piece formula.

2. Given the hypotheses:

   - H1: If Jax does not make jewlery, then Carl does not go camping. $\boxed{\neg j \to \neg c}$
   - H2: If Bo does not bake cakes, then Jax makes jewelry. $\boxed{\neg b \to j}$
   - H3: If Kofi does not fly kites, then Fiona goes fishing. $\boxed{\neg k \to f}$
   - H4: If Al does not like Archery, then Carl goes camping. $\boxed{\neg a \to c}$
   - H5: If Bo bakes cakes, then Al does not like archery. $\boxed{b \to \neg a}$
   - H6: If Carl goes camping, Kofi flies kites. $\boxed{c \to k}$
   - H7: If Jax makes jewelry, then Matthijs records music. $\boxed{j \to m}$
   - H8: If Kofi flies kites, then Matthijs does not record music. $\boxed{k \to \neg m}$

   Prove: If Al likes archery, then Fiona goes fishing.

   Proof. Assume Al likes Archer ("Al does not like Archery" is false).

   Using H5, we know that if Bo bakes cakes, then Al does not like Archery. Since Al likes Archery, Bo must not bake cakes. Then, using H2, we know that if Bo does not make cakes, then Jax must make jewelry. Since Bo does not bake cakes, Jax must make jewelry. Since Jax makes jewelrey, we can apply H7 to conclude that Matthijs records music. By H8, if Kofi flies kites, it would contradict our previous finding that Matthijs records music. Therefore, Kofi must not be flying kites. Finally, using H3, we know that if Kofi does not fly kites, then Fiona goes fishing. Since Kofi does not fly kites, we can conclude that Fiona goes fishing.

   □

   Proof with symbols:

   - Let j be the statement, "Jax makes jewelrey".
   - Let b be the statement, "Bo bakes cakes".
   - Let k be the statement, "Kofi flies kites".
   - Let a be the statement, "Al likes Archery".
   - Let c be the statement, "Carl goes camping".
   - Let m be the statement, "Matthijs records music".

   Proof. Assume a is true ($\neg a$ is false).

   Using H5, we know that if b is true, then $\neg a$ is true. Since $\neg a$ is false, b must be false ($\neg b$ must be true). Then, using H2, we know that if $\neg b$ is true, then j is true. Since $\neg b$ is true, j must be true. Since j is true, we can apply H7 to conclude that m is true. By H8, if k were true, it would contradict our previous finding that m is true. Therefore, k must be false ($\neg k$ must be true). Finally, using H3, we know that if $\neg k$ is true, then f is true. Since $\neg k$ is true, f must be true.

   □

3. Prove: for all $a \in \mathbb{Z}$, for all $b \in \mathbb{Z}$, for all $c \in \mathbb{Z}$, if a divides b, and b divides c, then a divides c.

   Symbolic representation: $\forall a \in \mathbb{Z}, \forall b \in \mathbb{Z}, \forall c \in \mathbb{Z}, (a|b \wedge b|c) \to a|c$.

   Proof. Let a, b, and c be integers. Assume a divides b and b divides c. This means there exists integers k and m such that $b = ak$ and $c = bm$. Substituting $b = ak$ into $c = bm$, we get:

$$c = bm = (ak)m = a(km).$$

   We know km is an integer because the set of integers is closed under multiplication, so we can conclude that a divides c. Therefore, if a divides b and b divides c, then a divides c. $\square$

4. Prove: for all $c \in \mathbb{Z}$, for all $d \in \mathbb{Z}$, if c is even and d is odd, then $c - d$ is odd.[2]

Proof. Let c and d be integers. Assume c is even and d is odd. This means there exists integers k and m such that

$$c = 2k \text{ and } d = 2m + 1.$$

Substituting $c = 2k$ and $d = 2m + 1$ into $c - d$, we get:

$$c - d = 2k - (2m + 1) = 2k - 2m - 1 = 2(k - m) - 1.$$

Let $w = (k - m)$. We know that $w \in \mathbb{Z}$ because k and m are integers, and the set of integers is closed under subtraction. Thus, we can conclude that $c - d$ is odd. Therefore, if c is even and d is odd, then $c - d$ is odd. $\square$

[2]Note for this question: apply **real** algebra, so note that order of operations matters. As another note apply the definition of odd that is given. Do NOT make up your own "alternate" definition of odd.

5. Using the hypotheses:

   I have given each of these hypotheses a name, H1 through H7, to more clearly reference them in the proof and preserve conciseness.

   - For all $y \in Y$, if y is a poet, then y does not skateboard.
   - For all $s \in V$, for all $t \in W$, if s gives t chocolate, then t tap dances.
   - Every element in the set C is also in the set V.
   - Every element in the set C is also in the set Y.
   - Every element in the set A is also in the set V.
   - Every element in the set B is also in the set W.
   - For all $h \in B$, if h tap dances, then h is happy.

   Prove: For all $a \in A$, for all $b \in B$, for all $c \in C$, if c skateboards and a gives b chocolate, then b is happy and c is not a poet.[3]

   Proof. Let $a \in A$, $b \in B$, and $c \in C$. Assume c skateboards and a gives b chocolate. Since every element in the set C is also in the set Y, we conclude that $c \in y$. From H1, we note that if c skateboards, then c cannot be a poet. Thus, c is not a poet. Since every element in the set A is also in the set V, we have $a \in V$. According to H2, we know that because a gives b chocolate, it follows that b tap dances. Since every element in the set B is also in the set W, we have $b \in W$. From H7, we know that since b tap dances, b is happy. Therefore, if c skateboards and a gives b chocolate, then b is happy and c is not a poet. $\square$

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[3] Take it one step at a time. For example, if we knew $\boxed{\text{For all } g \in U \text{, if } g \text{ works hard, then the boss gives } g \text{ a raise}}$ and we knew $\boxed{k \in U}$ was already established, then we'd get the implication $\boxed{\text{if } k \text{ works hard, then the boss gives } k \text{ a raise}}$. My expectation is that you write about this implication that is newly-obtained (newly-proved). Then, now that we know this implication is true, we'd be looking for an opportunity to **use** it. That is, we could do something if we knew $\boxed{k \text{ works hard}}$, and we could also use the implication if we knew $\boxed{\text{the boss does not give } k \text{ a raise}}$. (My point is: slow down, take things one step at a time, and show your work. Using a for all ends up giving you a statement that is *slightly* smaller.) Of course, the question that you have to work on doesn't talk about raises, working hard, etc. I just wanted to discuss based on an example that has different activities than the poetry, skateboard, tap dances, etc. in the problem.

6. Optional (0pts). You walk home from a friend's house and run into another genie. "Oh no, not you again!" you exclaim. The genie lays out 12 coins and a balance while saying, "I'll give you one trillion dollars if you can identify which one of these twelve coins is fake, but you can only use the balance three times." You think for a minute, without even using scratch paper to say, "Oh hey, that's *easy* now! Give me that balance and I'll identify which of these coins is heaviest in no time!" The genie grabs the balance from you and says, "Not so fast! I didn't say whether the fake coin is lighter than the rest or heavier than the rest. Here, have some scratch paper." The genie gives an evil laugh, only to say, "Oh, for the trillion dollars, you have to identify *which* coin is fake, and whether the fake coin is heavier than a genuine coin or lighter than a genuine coin." Explain how using the balance at most three times, you can identify the fake, and whether it is heavier or lighter than the typical coin. Good luck!