OPTIMISATION
(a crash course in differential calculus)

Filippo Cavallari

filippo.cavallari@southwales.ac.uk

WHAT IS A FUNCTION?

Given two sets A and B, a **function** from A to B is a relationship that associates to <u>every</u> element of A <u>one and only one</u> element of B.

We will focus on the case when A and B are either the set of real numbers $\mathbb{R}$ or a subset of it.

EXAMPLES OF FUNCTIONS

i. $f(x) = \mathrm{x}$

ii. $f(x) = 3\mathrm{x} + 2$

iii. $f(x) = x^2 + \mathrm{x} - 20$

iv. $f(x) = x^3 - 6x^2 + 11\mathrm{x} - 6$

v. $f(x) = \frac{1}{x}$

vi. $f(x) = \frac{1}{x-5} + 4$

vii. $f(x) = \sqrt{x}$

viii. $f(x) = \sqrt[3]{x}$

ix. $f(x) = \sqrt[4]{x^3}$

x. $f(x) = e^x$

xi. $f(x) = e^{2x}$

xii. $f(x) = e^{3x-1} + 2$

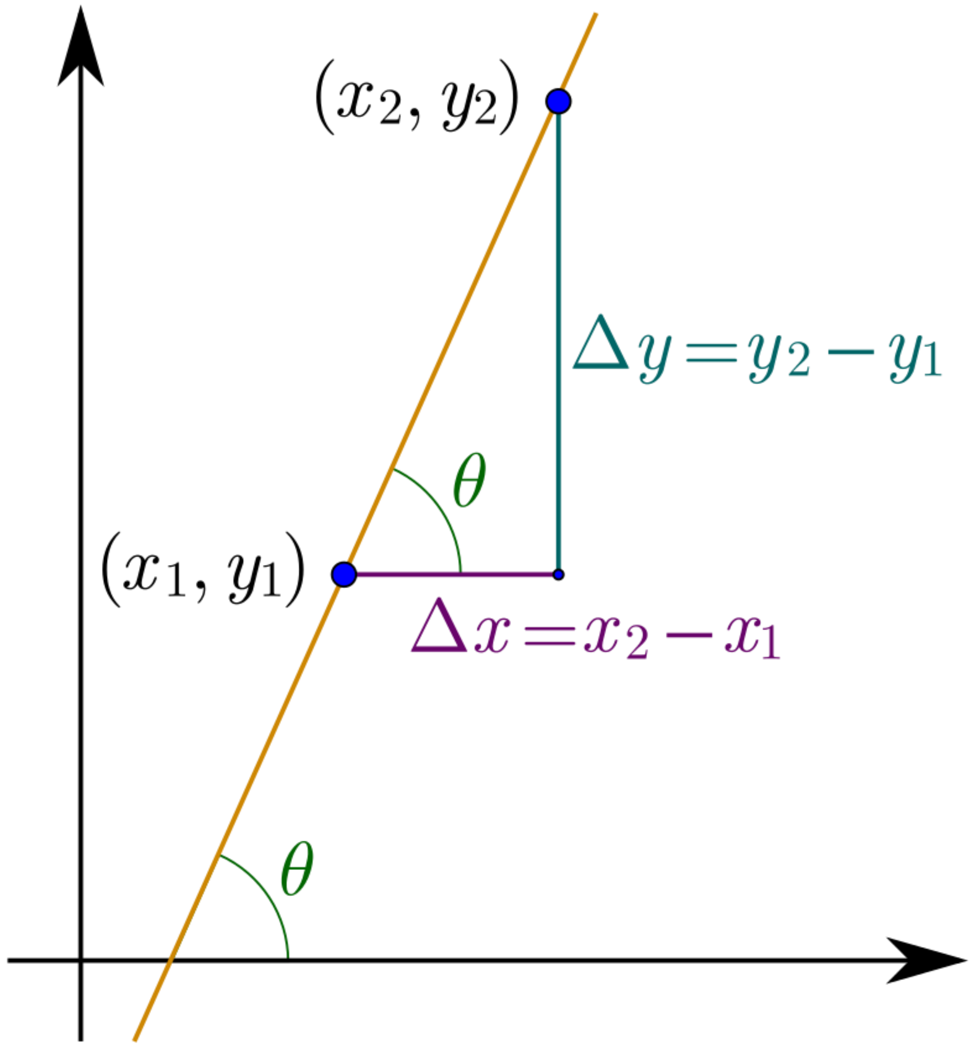
xiii. $f(x) = \log x$

xiv. $f(x) = -\log x$

xv. $f(x) = \log(1 - x)$

xvi. $f(x) = \sinh \mathrm{x}$

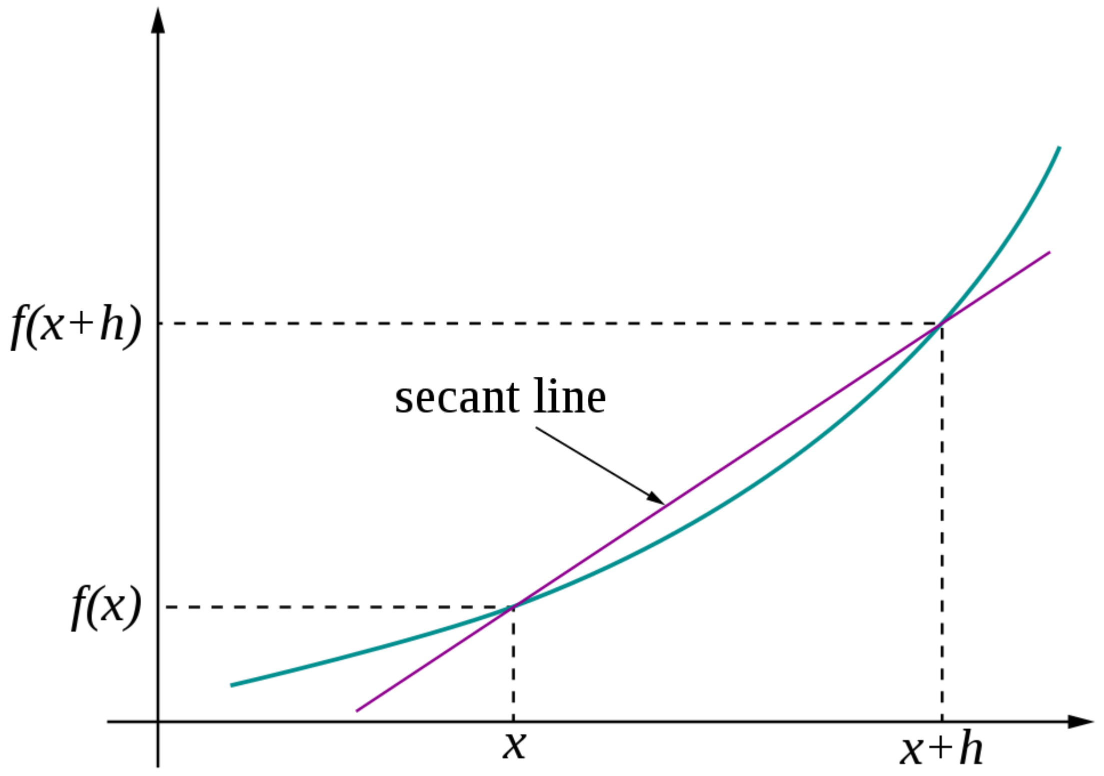
SLOPE OF A LINE

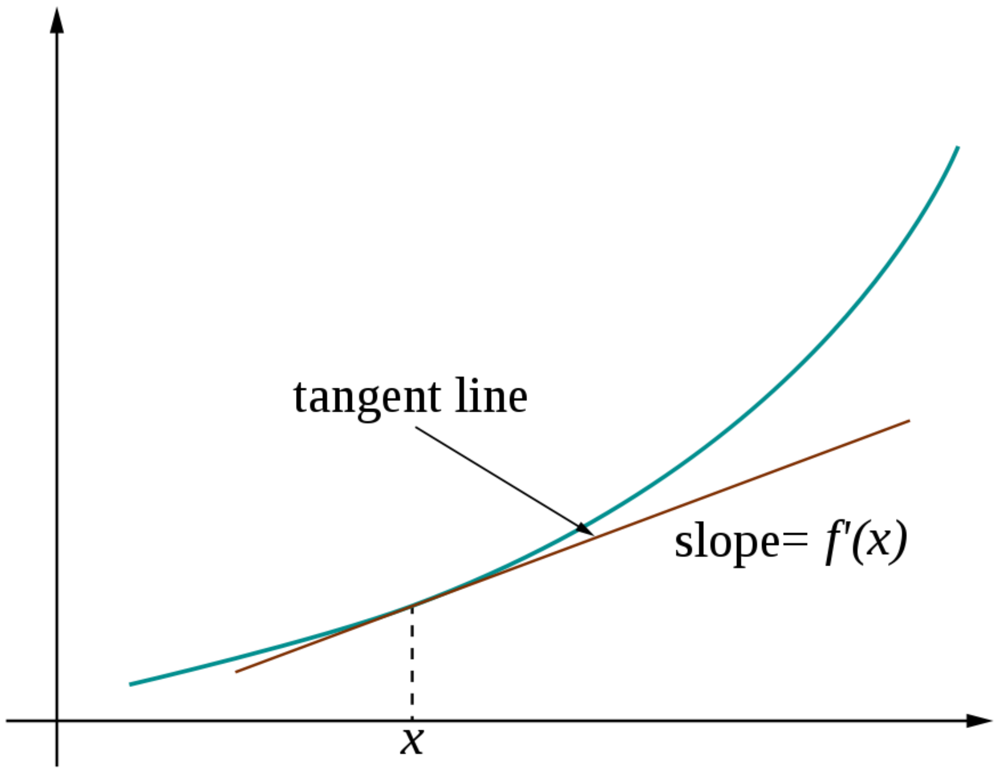


The slope (or gradient) of the line is given by

$$m = \frac{\Delta y}{\Delta x}$$

DERIVATIVE





$$m = \frac{\Delta f(x)}{\Delta x} = \frac{f(x+h) - f(x)}{(x+h) - (x)} = \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \frac{df}{dx}(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

HOW TO COMPUTE A DERIVATIVE

Derivatives of powers

$$\boldsymbol{f(x) = x^r \implies f'(x) = r x^{r-1}}$$

Where r is any real number.

It is useful to remember

$$x^a x^b = x^{a+b} \qquad \frac{x^a}{x^b} = x^{a-b} \qquad (x^a)^b = x^{ab}$$

$$x^0 = 1 \qquad \frac{1}{x^a} = x^{-a} \qquad x^{\frac{1}{a}} = \sqrt[a]{x}$$

EXERCISE

Evaluate the derivative of the following functions

1. $f(x) = x^4$

2. $g(x) = \frac{1}{x^2}$

3. $h(x) = \sqrt[3]{x^2}$

HOW TO COMPUTE A DERIVATIVE

Derivative of exponential

$$\boldsymbol{f(x) = e^x \implies f'(x) = e^x}$$

Derivative of natural logarithm

$$\boldsymbol{f(x) = log\, x \implies f'(x) = \frac{1}{x}}$$

It is useful to remember

$$log(xy) = log\, x + log\, y \qquad\qquad log\, x^a = a \log x$$

ALGEBRA OF DERIVATIVES

- Constant Rule

$$f(x) = k \implies f'(x) = 0$$

- Sum Rule

$$[\alpha f(x) + \beta g(x)]' = \alpha f'(x) + \beta g'(x)$$

- Product Rule

$$[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$$

- Quotient Rule

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x) - f(x)g'(x)}{g^2(x)}$$

EXERCISE

Evaluate the derivative of the following functions

1. $f_1(x) = 5x^6$
2. $f_2(x) = 4x - x^2$
3. $f_3(x) = xe^x$
4. $f_4(x) = x \log x$
5. $f_5(x) = \frac{e^x}{x}$
6. $f_6(x) = x^7 - \frac{\log x}{x^2} + 4 - \sqrt{x}e^x$

ALGEBRA OF DERIVATIVES

- Chain Rule

$$[\boldsymbol{f}(\boldsymbol{g}(\boldsymbol{x}))]' = \boldsymbol{f}'(\boldsymbol{g}(\boldsymbol{x}))\boldsymbol{g}'(\boldsymbol{x})$$

EXERCISE

Evaluate the derivative of the following functions

1. $g_1(x) = (9 - 7x^2)^5$

2. $g_2(x) = \frac{1}{5-2x}$

3. $g_3(x) = \sqrt{2 - x^3}$

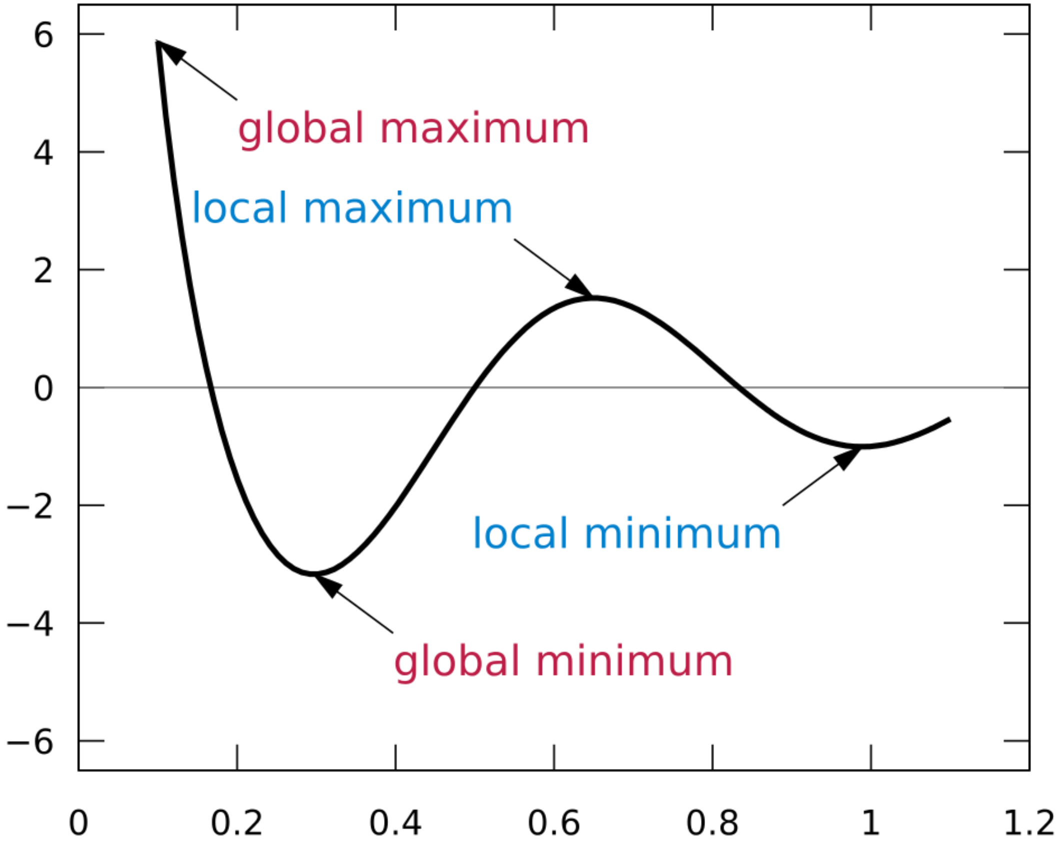
4. $g_4(x) = \log(1 - x^2)$

5. $g_5(x) = e^{-x^2}$

6. $g_6(x) = \sqrt[3]{e^x - \log x}$

WHY DERIVATIVES ARE IMPORTANT?

Differential calculus allow us to find max and min of functions. These are where the derivative is equal to zero.



EXAMPLE

Find the max and min of the function $f(x) = 5x^3 + 2x^2 - 3\mathrm{x}$

FUNCTION OF MORE VARIABLES

Let f be a function that depends on more than one variable. For example it can depend on two variables

$$f(x, y) = x^3 - xy^2 + y$$

f can be interpreted as a family of function depending on one of these variables and indexed by the other. In this case we have two options

$$f(x, y) = f_y(x) \quad \text{or} \quad f(x, y) = f_x(y).$$

The index is constant respect to the other variable.

PARTIAL DERIVATIVES

$$f(x,y) = x^3 - xy^2 + y$$

Hence we can evaluate the derivative of the function considering the index function as a constant. We have two options

$$\frac{\partial f}{\partial x} = \frac{df_y}{dx} = 3x^2 - y^2$$

$$\frac{\partial f}{\partial y} = \frac{df_x}{dy} = 2xy + 1$$

PARTIAL DERIVATIVES

The algebra of derivative is still valid in the case of function of more variables.

EXERCISE

Evaluate the derivative of the following functions

1. $f(x, y) = x^4y^3 - 5xy + 7\sqrt{x}$
2. $g(x, y) = e^{xy+x+y}$
3. $h(x, y) = \log(1 - xy)$