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Well-posedness of the Navier-Stokes Phase-Field Crystal model

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Abstract

The Navier-Stokes Phase-Field Crystal model is a sixth-order non-linear parabolic system of partial differential equations describing the evolution of a colloidal suspension, which can be thought of as a mixture consisting of solid crystal particles continuously immersed within a fluid. We define the notion of a weak solution for our system and carry out an initial energy estimate to demonstrate that over time, the energy of our system is a non-increasing function. The majority of our effort is then put into proving the existence of a (global) weak solution to our system. Finally, we prove that such a solution is unique in the two-dimensional case.

Declaration: The work contained in this thesis is my own work unless otherwise stated.

Contents

1 Introduction

The Navier-Stokes equations have been a central object of interest to both pure mathematicians and physicists alike ever since their introduction by C.L. Navier (1822) and G.G Stokes (1850). Physically, the system describes the motion of a viscous and incompressible [1] fluid on a three-dimensional domain Ω. A great deal of mathematical theory has been developed on the existence, uniqueness and regularity associated to different classes of solutions to this system over the past 150 years. Perhaps the most notable problem in this area of study is one which still remains; namely the Navier-Stokes existence and smoothness problem which holds the status of being amongst the famous Millennium Prize Problems in Mathematics.

However, the aim of this project is not to attain the $1,000,000 prize associated to the Millenium problem. Rather, we turn our attention to the Navier-Stokes Phase-Field Crystal model (NS-PFC) derived by S. Praetorius and A. Voigt [12], which is a non-linear sixth-order parabolic system of partial differential equations [11]. The NS-PFC system is a diffuse-interface model which describes the evolution of a colloidal suspension (a mixture where colloidal particles are immersed in a fluid) contained in a three-dimensional domain Ω. The NS-PFC model also accounts for hydrodynamic interactions which take place between particles. These colloidal particles are modelled as a highly viscous fluid and so we may describe the system as a mixture of two fluids, which are both assumed to be incompressible, viscous and equal in density.

A model such as the NS-PFC system which involves a mixture of two fluids is an example of two phase flow. The density (or phase) function $\psi \equiv \psi(x,t) \in [-1,1]$ is used to denote the difference in concentrations between the fluid and the particles. ψ is defined as $\psi(x,t) := \varphi_1(x,t) - \varphi_2(x,t)$, where $\varphi_i : \Omega \times [0,T] \to [0,1]$ denotes the concentration of fluid i at a given point in space and time. Thus, $\psi(x_0,t_0) = -1$ indicates a pure presence of fluid 1 at position x_0 and time t_0, whereas a value of $+1$ would indicate a pure presence of fluid 2. The density function ψ is also modelled as varying continuously along the interface of the two fluids. This means that rather than jumping from a value of -1 to $+1$ (i.e. a jump discontinuity) when travelling across the interface of the two fluids, the function ψ instead does so in a continuous manner. It will be seen that the difficulty in the analysis of such a system typically arises due to the presence of non-linear terms. These terms present new challenges in the way of proving the existence and uniqueness of weak solutions, compared to the classical Navier-Stokes equations.

Much mathematical analysis has been performed on similar diffuse-interface models. For example, [15] details the existence and well-posedness of a Functionalized Cahn-Hilliard-Navier-Stokes system. The system in [15] is similar to that of the NS-PFC, with the key differences being the choice of forcing term (which describes the hydrodynamic interactions) and free-energy. One may also refer to ([1], [4], [9], [10]) for the analysis of similar diffuse-interface models. However, to the best of our knowledge, **there are currently no known results on the existence and uniqueness of weak solutions to the NS-PFC model**.

The aim of this project is to prove the existence of a weak solution (to be defined) for the NS-PFC system under certain regularity conditions. Moreover, we will prove the uniqueness of weak solutions in two-dimensions.

[1]Incompressibility means that the density of the fluid is constant. In theory one might also study the compressible form of the Navier-Stokes equations but this will not be our focus.

2 The Navier-Stokes PFC model for colloidal suspensions

We study the PFC model [12] which is described by the following system of equations:

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla\cdot\tilde{\sigma} - M_1\psi\nabla\mu, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\partial_t\psi + \mathbf{u}\cdot\nabla\psi = \nabla\cdot(M(\psi)\nabla\mu), \tag{2.3}$$

$$\mu = \frac{\delta\mathcal{F}_{sh}[\psi]}{\delta\psi}, \tag{2.4}$$

in $\Omega\times[0,T]$, with

$$\tilde{\sigma} = -\tilde{p}I + \frac{1}{Re_f}(1+\tilde{\eta}(\psi))(\nabla\mathbf{u}+\nabla^{\mathbf{T}}\mathbf{u}), \tag{2.5}$$

$$\frac{\delta\mathcal{F}_{sh}[\psi]}{\delta\psi} = \psi^3 + (r + (1+\Delta)^2)\psi, \tag{2.6}$$

subject to the boundary conditions

$$\mathbf{u} = 0, \quad \nabla\mu\cdot\mathbf{n} = \nabla\psi\cdot\mathbf{n} = \nabla\Delta\psi\cdot\mathbf{n} = 0, \quad \text{on } \partial\Omega\times[0,T], \tag{2.7}$$

where $\mathbf{n}$ denotes the outward unit normal vector on $\partial\Omega$. The initial conditions are given by

$$\mathbf{u}|_{t=0} = \mathbf{u_0}, \quad \psi|_{t=0} = \psi_0. \tag{2.8}$$

Here, Ω is an open and bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. In solving the above system we look for:

- a velocity field $\mathbf{u} = (u_1, u_2, u_3) : \Omega\times[0,T] \to \mathbb{R}^3$ which tells us the velocity of the fluid at any given point in our domain at any given time,
- the (rescaled) pressure $\tilde{p} : \Omega\times[0,T] \to \mathbb{R}$ which tells us the mechanical pressure exerted by the fluid at any given point in our domain at any given time,
- the phase function $\psi : \Omega\times[0,T] \to \mathbb{R}$ which tells us the difference in concentrations between the background fluid and the particles at any given point in our domain at any given time. Recall that we are considering a colloidal suspension, i.e. a mixture where solid (colloidal) particles are continuously immersed within a background fluid. The particles are modelled as a highly viscous fluid and thus from on analytical viewpoint we are investigating a mixture of two fluids.

2.1 Interpretation of the PFC model [12]

It is important to understand each of the equations (2.1) $-$ (2.8) before we can proceed with the mathematical analysis. To this end, we give a short description of each equation.

Equations (2.1)-(2.2)
(2.1) (commonly referred to as the Navier-Stokes momentum equation) arises as a consequence of the principle of conservation of momentum. The expression $\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u}$ on the LHS of (2.1) describes the inertial forces, whereas $\nabla\cdot\tilde{\sigma}$ denotes the combination of pressure and viscous forces. The final term on the RHS of (2.1) represents the forces exerted on the fluid due to hydrodynamic interactions between particles (see [12] for a detailed derivation of this term). It will be seen that

the value of the constant $M_1 := \frac{\omega}{\Gamma Pe}\frac{q_0}{2\pi A}$ [12] is not important for our analysis and thus we will assume $M_1 = 1$ without loss of generality. (2.2) is known as the 'divergence-free' condition and simply states that the divergence of the velocity field must be 0 everywhere. This is also sometimes known as the incompressible equation, and implies that in any volume the same amount of fluid enters as that which leaves.

Equation (2.3)
This equation is driven by the Swift-Hohenberg (SH) free energy functional ([5] [14])

$$\mathcal{F}_{sh}[\psi] = \int_\Omega \frac{1}{4}\psi^4 + \frac{1}{2}\psi(r + (1+\Delta)^2)\psi \,\mathrm{d}x, \tag{2.9}$$

and for this reason may be referred to as the Swift-Hohenberg energy equation. In [12] the authors mention that one may choose one of two free-energies when considering the NS-PFC model; either $\mathcal{F}_{sh}$ or $\mathcal{F}_{vpfc}$. For our purposes we will be using the former. We note that

$$\mathcal{F}_{sh}[\psi] = \int_\Omega \frac{1}{4}\psi^4 + \frac{r+1}{2}\psi^2 + 2\Delta\psi + \Delta^2\psi \,\mathrm{d}x = \int_\Omega F(\psi) + 2\Delta\psi + \Delta^2\psi \,\mathrm{d}x.$$

Due to the choice of the polynomial energy density $F(\psi)$ in $\mathcal{F}_{sh}[\psi]$ and the lack of maximum principle for sixth-order parabolic equations, the phase field ψ will take values in $\mathbb{R}$ instead of $[-1, 1]$. In fact, polynomial energy densities are Taylor approximations of logarithmic energy densities. The latter guarantees that ψ remains in the physical interval $[-1, 1]$ (see [8], [9]). However, this goes beyond the scope of this project. The total energy of the system is given by

$$\mathfrak{E}(\mathbf{u}(t), \psi(t)) := \frac{1}{2}\|\mathbf{u}(t)\|^2 + \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{sh}[\psi]. \tag{2.10}$$

Finally, we note that the term $M(\psi)$ is known as the mobility function [12] and $r < 0$ is a constant.

Equation (2.4)
The function μ, which represents the chemical potential, is defined as the variational derivative of $\mathcal{F}_{sh}[\psi]$ (the Swift-Hohenberg free-energy functional ([5], [14]). In thermodynamics, the chemical potential is a function which measures the rate of change of free energy with respect to the number of particles in the system. The NS-PFC model being based on density functional theory however, instead expresses the chemical potential as a variational derivative. The definition of the variational derivative will not be important for our purposes but for a rigorous treatment we refer the reader to [3], for example.

Equations (2.5)-(2.6)
The term $\tilde{\sigma}$ is known as the stress-tensor and appears in the Cauchy-momentum equation, from which the Navier-Stokes equations are derived. Re_f is the Reynolds number for the fluid and $\tilde{\eta}$ represents the viscosity perturbation [12]. For simplicity we define

$$\nu(\psi) := \frac{1}{Re_f}(1 + \tilde{\eta}(\psi)), \tag{2.11}$$

$$D(\mathbf{u}) := \frac{1}{2}(\nabla\mathbf{u} + \nabla^T\mathbf{u}). \tag{2.12}$$

The quantity $D(\mathbf{u})$ is known as the symmetric component of the velocity gradient. (2.6) is the variational derivative of $\mathcal{F}_{sh}[\psi]$ which was mentioned in (2.4).

Equations (2.7) - (2.8)
The condition $\mathbf{u} = 0$ on $\partial\Omega \times [0, T]$ in particular is known as the 'no-slip' condition and means that

the particles on the boundary are stuck there forever [7]. The condition $\nabla\mu \cdot \mathbf{n} = 0$ on $\partial\Omega \times [0,T]$ implies together with (2.3) the conservation of mass, namely $\frac{1}{|\Omega|}\int_\Omega \psi(t)\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega \psi_0\,\mathrm{d}x$, for all $t \in [0,T]$. The condition $\nabla\psi \cdot \mathbf{n} = 0$ on $\partial\Omega \times [0,T]$ entails that the interface is orthogonal to the boundary. The condition $\nabla\Delta\psi \cdot \mathbf{n} = 0$ is a variational boundary condition.

3 Main results

3.1 Preliminaries

Let us establish the mathematical setting for the analysis which follows. Let $\Omega \subset \mathbb{R}^d$ be a open bounded domain with Lipschitz boundary $\partial\Omega$ (i.e. $\partial\Omega$ is locally the graph of a Lipschitz function). We will be considering the case $d = 3$ when proving the existence of weak solutions, though our analysis will also work for the case $d = 2$. Later, we will prove a uniqueness result for $d = 2$.

3.1.1 Lebesgue and Sobolev function spaces

We denote by X a real Banach or Hilbert space, and by X^* the dual space of X. The boldface character $\mathbf{X}$ is designated to the vectorial space X^d, where d corresponds to the spatial dimension. X^d is the space of $d-$vectors, with each component belonging to X and norm given by $\|\cdot\|_{\mathbf{X}}$. For $n \times n$ matrices with real entries we will utilise the Frobenius inner product $A : B := trace(A^T B)$ and the Frobenius norm $|A|^2 := A : A$. The notation $L^p(\Omega)$ is designated to the space of equivalence classes of measurable functions defined on $\Omega \subset \mathbb{R}^d$, where $f, g \in L^p(\Omega)$ are equivalent if $f = g$ almost everywhere. This (Banach) space is endowed with the norm $\|f\|_{L^p(\Omega)} := (\int_\Omega |f|^p\,\mathrm{d}s)^{1/p}$. Recall that the case $p = 2$ is of particular significance; indeed, $L^2(\Omega)$ is Hilbert when furnished with the inner product $(f,g)_{L^2(\Omega)} = \int_\Omega fg\,\mathrm{d}x$. We denote by $\overline{f}$ the mean value of f over our domain Ω, i.e. $\overline{f} = |\Omega|^{-1}\int_\Omega f\,\mathrm{d}x$. We also introduce the space $D(\Omega) := C_c^\infty(\Omega)$, which is the space of infinitely differentiable functions on Ω with compact support. For $k \in \mathbb{N}$ and $p \in [0, +\infty]$ we introduce the Sobolev spaces

$$W^{k,p}(\Omega) \equiv \{f : D^\alpha f \in L^p(\Omega) \text{ for } 0 \le \alpha \le k\},$$

where the derivatives are understood in the distributional sense. For $p < \infty$ the associated norm is given by

$$\|\mathbf{u}\|_{W^{k,p}(\Omega)} = (\sum_{0\le|\alpha|\le k} \|D^\alpha \mathbf{u}\|^p_{L^p(\Omega)})^{1/p},$$

whereas for $p = \infty$ we have

$$\|\mathbf{u}\|_{W^{k,\infty}(\Omega)} = \max_{0\le\alpha\le k}\left[\operatorname*{ess\,sup}_{x\in\Omega} |(D^\alpha \mathbf{u})(x)|\right].$$

In particular, we will make use of the Hilbert spaces $H^k(\Omega) \equiv W^{k,2}(\Omega)$ endowed with the inner product $(\mathbf{u},\mathbf{v})_{H^k(\Omega)} = \sum_{0\le|\alpha|\le k}(D^\alpha\mathbf{u}, D^\alpha\mathbf{v})_{L^2(\Omega)}$. The induced norm is denoted by $\|\cdot\|_{H^k}$. Next, suppose $I \subset \mathbb{R}^+$ is an interval and X is a Banach space. We define $L^p(I;X)$ to be the space of functions $u : I \to X$ such that

$$\|u\|_{L^p(I;X)} := \left(\int_I \|u(s)\|_X^p\,ds\right)^{\frac{1}{p}} < \infty, \quad \text{for } p < \infty \text{ and}$$

$$\|u\|_{L^\infty(I;X)} := \operatorname*{ess\,sup}_{x\in I} \|u(s)\|_X < \infty, \quad \text{for } p = \infty.$$

Note that $L^p(I;X)$ is Banach and in particular $L^2(I;X)$ is Hilbert when attributed with the inner product $(u,v) = \int_I (u(s),v(s))_X\,ds$.

3.1.2 Navier-Stokes and PFC specific function spaces

We introduce the spaces

$$\begin{aligned}
&\mathbf{H}_\sigma = \{\mathbf{v} \in [L^2(\Omega)]^3 : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},\\
&\mathbf{V}_\sigma = \{\mathbf{v} \in [H^1(\Omega)]^3 : \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega,\ \mathbf{v} = 0 \text{ on } \partial\Omega\},\\
&\mathbf{W}_\sigma = \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma,\\
&L^2_0(\Omega) = \{u \in L^2(\Omega) :\ |\Omega|^{-1}\overline{u} = 0\},\\
&H^1_0(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega\},\\
&H^2_N(\Omega) = \{u \in H^2(\Omega) : \nabla u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},\\
&H^4_N(\Omega) = \{u \in H^4(\Omega) : \nabla u \cdot \mathbf{n} = \nabla\Delta u \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\},
\end{aligned}$$

where $\mathbf{H}_\sigma$ is considered with the same inner product structure as $L^2(\Omega)$ and $\mathbf{V}_\sigma$ is endowed with $(\mathbf{u},\mathbf{v})_{\mathbf{V}_\sigma} := \int_\Omega \nabla\mathbf{u} : \nabla\mathbf{v}\, \mathrm{d}x$, (along with the corresponding induced norm). We also make mention of the tri-linear continuous form b, defined as

$$b : \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega) \to \mathbb{R},$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{v}\cdot\mathbf{w}\, \mathrm{d}x.$$

This form satisfies

$$b(\mathbf{u},\mathbf{u},\mathbf{u}) = 0,\ \text{for each } u \in \mathbf{V}_\sigma \tag{T1}$$

$$b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0,\ \text{for all } \mathbf{u} \in \mathbf{V}_\sigma, \mathbf{v} \in \mathbf{H}^1(\Omega). \tag{T2}$$

3.1.3 Useful inequalities and results

The following well-known results will be used without proof in what follows:

- **Holder's inequality for triple products:** For $p, q, r \in [1,\infty]$ with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$ and $f, g, h \in L^p(\Omega), L^q(\Omega), L^r(\Omega)$ respectively, we have

$$\|fgh\|_{L^1(\Omega)} \le \|f\|_{L^p(\Omega)}\|g\|_{L^q(\Omega)}\|h\|_{L^r(\Omega)}. \tag{HT}$$

- **Young's inequality (version 1):** If $a, b \ge 0$ and $p, q \in [1,\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$ then

$$ab \le \frac{a^p}{p} + \frac{b^q}{q}. \tag{Y1}$$

- **Young's inequality (version 2):** If $a, b \ge 0$ then for any $\epsilon > 0$ we have

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon} b^2. \tag{Y2}$$

- **Korn's inequality [9]:**

$$\|\nabla\mathbf{u}\| \le \sqrt{2}\|D\mathbf{u}\|,\ \forall\, \mathbf{u} \in \mathbf{V}_\sigma. \tag{KI}$$

- **The Poincare-Wirtinger inequality [15]:** There exists a constant $C > 0$ depending only on Ω such that

$$\|u - \overline{u}\| \le C\|\nabla u\|,\ \ \forall\, u \in H^1(\Omega), \tag{PW}$$

  where $\overline{u} := \frac{1}{|\Omega|}\int_\Omega u\, \mathrm{d}x$.

- **Ladyzhenskaya's inequality [8]:**

$$\|u\|_{L^4(\Omega)} \leq C\|u\|^{\frac{1}{2}}\|u\|^{\frac{1}{2}}_{H^1(\Omega)}, \ \forall\, u \in H^1(\Omega), \text{ when } d = 2. \tag{LZ}$$

- **Brezis-Gallouet inequality [8]:**

$$\|u\|_{L^\infty(\Omega)} \leq C\|u\|_{H^1(\Omega)} \ln^{\frac{1}{2}}\left(e\frac{\|u\|_{H^2(\Omega)}}{\|u\|_{H^1(\Omega)}}\right), \ \forall\, u \in H^2(\Omega), \text{ when } d = 2. \tag{BG}$$

- **The following Gagliardo-Nirenburg interpolation inequality:**

$$\|u\|_{L^3(\Omega)} \leq C\|u\|^{\frac{1}{2}}_{L^2(\Omega)}\|u\|^{\frac{1}{2}}_{H^1(\Omega)}, \ \forall\, u \in H^1(\Omega). \tag{GN}$$

- **Sobolev embedding** [7]: Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain and given $p \in [1, \infty)$ denote by p^* the Sobolev conjugate of p given by $\frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}$. The following compact embeddings hold:

$$\begin{aligned} &W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) \quad \forall q \in [1, p^*), \text{ if } p < n,\\ &W^{1,p}(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega) \quad \forall q \in [1, \infty), \text{ if } p = n,\\ &W^{1,p}(\Omega) \overset{c}{\hookrightarrow} C(\bar{\Omega}), \text{ if } p > n. \end{aligned} \tag{S1}$$

When $d = 3$, the following continuous embedding also holds:

$$H^1(\Omega) \hookrightarrow L^6(\Omega). \tag{S2}$$

- **Green's first identity (integration by parts):** If $\mathbf{u} \in H^1(\Omega; \mathbb{R}^n), v \in H^1(\Omega)$ then

$$\int_\Omega \operatorname{div}(\mathbf{u})\, v \,\mathrm{d}x = -\int_\Omega \mathbf{u} \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} (\mathbf{u} \cdot \mathbf{n}) v \,\mathrm{d}\sigma,$$

where $\mathbf{n}$ denotes the outward unit normal vector at $\partial\Omega$. A particular case of this identity which will be useful is the following: if $u \in H^2(\Omega), v \in H^1(\Omega)$ then

$$\int_\Omega v\Delta u \,\mathrm{d}x = -\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x + \int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) v \,\mathrm{d}\sigma.$$

- **Regularity for the Stokes operator (see [15], [9]; Appendix B):**
Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain with class C^2 boundary and that $f \in \mathbf{H}_\sigma$. Then there exists a unique $\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma$ and $p \in H^1(\Omega) \cap L^2_0(\Omega)$ such that $-\Delta\mathbf{u} + \nabla p = \mathbf{f}$ a.e. in Ω. Furthermore, there exists a constant $C \in \mathbb{R}$ such that

$$\|\mathbf{u}\|_{\mathbf{H}^2(\Omega)} + \|p\|_V \leq C\|\mathbf{f}\|. \tag{SR}$$

- **Regularity for the Laplace operator (see [2], Theorem III.4.3):**
Assume $\Omega \subset \mathbb{R}^3$ is a bounded domain of class C^2 and that $f \in L^2_0(\Omega)$. Then there exists a unique $u \in H^2_N(\Omega) \cap L^2_0(\Omega)$ satisfying $-\Delta u = f$ a.e. in Ω. Furthermore, there exists a constant $C \in \mathbb{R}$ such that

$$\|u\|_{H^2(\Omega)} \leq C\|\Delta u\|. \tag{LR-1}$$

Moreover, if Ω is of class C^4 and $f \in L^2_0(\Omega)$ then there exists a unique $u \in H^4_N(\Omega) \cap L^2_0$ satisfying $-\Delta^2 u = f$ a.e. in Ω. Furthermore, there exists a constant $C \in \mathbb{R}$ such that

$$\|u\|_{H^4(\Omega)} \leq C\|\Delta^2 u\|. \tag{LR-2}$$

3.1.4 Assumptions

We will assume $\nu, M \in C^2(\mathbb{R})$ and that there exist real constants $\nu^*, \nu_*.M^*, M_*$ such that

$$0 < \nu_* < \nu(s) < \nu^*, \quad \forall\, s \in \mathbb{R}, \tag{B1}$$

$$0 < M_* < M(s) < M^*, \quad \forall\, s \in \mathbb{R}. \tag{B2}$$

Furthermore, we will assume $M_1 = 1$ without loss of generality and that $r < 0$ is a constant.

3.2 Statement of main results

We now mention the main results of this thesis. Firstly, we define weak solutions for our system:

Definition 3.1. (Weak Solution)
Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain of class C^4 and $T > 0$. Assume our initial data is given by $(\mathbf{u}_0, \psi_0) \in \mathbf{H}_\sigma \times H^2_N(\Omega)$. The triple $(\mathbf{u}, \psi, \mu)$ is known as a weak solution to the system $(2.1) - (2.8)$ on $[0, T]$ if it satisfies the following properties:

$$\mathbf{u} \in L^\infty(0, T; \mathbf{H}_\sigma) \cap L^2(0, T; \mathbf{V}_\sigma) \cap W^{1,\frac{4}{3}}(0, T; \mathbf{V}^*_\sigma), \tag{3.1}$$
$$\psi \in L^2(0, T; H^4_N(\Omega)) \cap L^\infty(0, T; H^2_N(\Omega)), \tag{3.2}$$
$$\partial_t \psi \in L^2(0, T; (H^1(\Omega))^*), \tag{3.3}$$
$$\mu \in L^2(0, T, H^1(\Omega)). \tag{3.4}$$

Moreover, these functions must satisfy the weak formulation:

$$\langle \partial_t \mathbf{u}, \mathbf{v} \rangle_{(\mathbf{V}_\sigma)^* \times \mathbf{V}_\sigma} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + (2\nu(\psi) D(\mathbf{u}), \nabla \mathbf{v}) = -(\psi \nabla \mu, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}_\sigma, \tag{3.5}$$
$$\langle \partial_t \psi, v \rangle_{(H^1)^* \times H^1} + ((\mathbf{u} \cdot \nabla \psi), v) + (M(\psi) \nabla \mu, \nabla v) = 0 \quad \forall\, v \in H^1(\Omega), \tag{3.6}$$

for almost every $t \in [0, T]$, where

$$\mu = \psi^3 + (r + (1 + \Delta)^2)\psi, \quad \text{almost everywhere in } \Omega \times [0, T]. \tag{3.7}$$

The following initial conditions are also satisfied:

$$\mathbf{u}|_{t=0} = \mathbf{u_0}, \quad \psi|_{t=0} = \psi_0. \tag{3.8}$$

The triple $(\mathbf{u}, \psi, \mu)$ is known as a **global weak solution** if it is a weak solution on $[0, T]$ for any $T > 0$.

Remark on the construction of the above definition:
The function spaces above are derived from the estimates carried out in section 5, and partly from the next lemma. The weak formulation is obtained by multiplying (2.1) and (2.3) by suitable test functions, integrating over the domain Ω and simplifying the resulting equations. In particular, we note that by choosing a test function in $\mathbf{V}_\sigma$ for (2.1), the pressure vanishes from the weak formulation. This is the advantage of considering divergence-free Sobolev spaces. In fact, it is true that the pressure can be recovered once we have a weak solution, thanks to a result due to De Rham (see [2], Theorem IV.2.4).

The function spaces in the above definition are partly derived from the following energy estimate.

Lemma 3.2. (Initial Energy Estimate)
Suppose $(\mathbf{u}, \psi, \mu)$ is a smooth solution to problem (2.1) – (2.8). Then these functions satisfy the following energy equation for all time $t > 0$:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{2}\|\mathbf{u}(\mathbf{t})\|^{\mathbf{2}} + \mathcal{F}_{\mathbf{sh}}[\psi(\mathbf{t})]\right] + \int_\Omega (2\nu(\psi)|D(\mathbf{u})|^2 + M(\psi)|\nabla\mu|^2)\mathrm{d}x = 0,$$

We then work towards the existence of a global weak solution.

Theorem 3.3. (Existence of a Global Weak Solution)
Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain of class C^4 and assume our initial data satisfies $(\mathbf{u}_0, \psi_0) \in \mathbf{H}_\sigma \times H^2_N(\Omega)$. There exists a triple $(\mathbf{u}, \psi, \mu)$ which is a global weak solution to the system (2.1) – (2.8) according to Definition 3.1.

Finally, we will prove the uniqueness of our weak solution in two dimensions.

Theorem 3.4. (Uniqueness in two dimensions)
Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^4 and assume our initial data satisfies $(\mathbf{u}_0, \psi_0) \in \mathbf{H}_\sigma \times H^2_N(\Omega)$. Furthermore, suppose that $(\mathbf{u}_1, \psi_1, \mu_1)$ and $(\mathbf{u}_2, \psi_2, \mu_2)$ are two weak solutions to (2.1) – (2.8) on $[0, T]$ where $T > 0$ is fixed and originating from the same initial datum $(\mathbf{u}_0, \psi_0)$. Then it must hold that $(\mathbf{u}_1, \psi_1, \mu_1) = (\mathbf{u}_2, \psi_2, \mu_2)$ a.e. in $\Omega \times [0, T]$.

4 Towards the existence of a global weak solution

4.1 An initial energy identity

Before we dive into the proof of existence, we first establish an energy law which is satisfied by any smooth solution to our system. This result demonstrates that the energy of the system is dissipated over time.

Lemma 3.2. (Initial Energy Estimate)
Suppose $(\mathbf{u}, \psi, \mu)$ is a smooth solution to problem (2.1) – (2.8). Then these functions satisfy the following energy equation for all time $t > 0$:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{1}{2}\|\mathbf{u}(\mathbf{t})\|^{\mathbf{2}} + \mathcal{F}_{\mathbf{sh}}[\psi(\mathbf{t})]\right] + \int_\Omega (2\nu(\psi)|D(\mathbf{u})|^2 + M(\psi)|\nabla\mu|^2)\,\mathrm{d}x = 0, \tag{EE}$$

Proof. We begin by taking the inner product of (2.1) with $\mathbf{u}$ and integrating over Ω:

$$\underbrace{\int_\Omega (\partial_t \mathbf{u})\,\mathbf{u}\,\mathrm{d}x}_{I_1} + \underbrace{\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\,\mathbf{u}\,\mathrm{d}x}_{I_2} = \underbrace{\int_\Omega (\nabla\cdot\tilde{\sigma})\mathbf{u}\,\mathrm{d}x}_{I_3} - \underbrace{\int_\Omega \psi\nabla\mu\cdot\mathbf{u}\,\mathrm{d}x}_{I_4}. \tag{E1.1}$$

Next we simplify each term, starting with I_1:

$$I_1 = \int_\Omega \partial_t\left(\frac{|\mathbf{u}|^2}{2}\right)\mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega |\mathbf{u}|^2 \mathrm{d}x = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{u}\|^2_{L^2(\Omega)},$$

$$I_2 = \int_\Omega u_j\partial_j u_i u_i\,\mathrm{d}x = \int_\Omega u_j\partial_j\left(\frac{1}{2}|u_i|^2\right)\,\mathrm{d}x \equiv \frac{1}{2}\int_\Omega \mathbf{u}\cdot\nabla(|\mathbf{u}|^2)\,\mathrm{d}x = ...,$$

where we have adopted the Einstein summation convention in the first integral. Now using Green's second identity,

$$... = -\frac{1}{2}\int_\Omega \mathrm{div}(\mathbf{u})\cdot|\mathbf{u}|^2\,\mathrm{d}x + \frac{1}{2}\int_{\partial\Omega}\mathbf{u}(|\mathbf{u}|^2\cdot\mathbf{n})\,\mathrm{d}\sigma = 0$$

since $\mathbf{u} = 0$ on $\partial\Omega$ (no-slip condition) and div $\mathbf{u} = 0$ everywhere. Noting that $\tilde{\sigma}$ is a matrix, we compute

$$I_3 = \int_\Omega (\nabla \cdot \tilde{\sigma}) \cdot \mathbf{u} \,\mathrm{d}x = -\int_\Omega \tilde{\sigma} : \nabla \mathbf{u} \,\mathrm{d}x + \underbrace{\int_{\partial\Omega} \mathbf{u}(\tilde{\sigma} \cdot n)\, d\sigma}_{=0 \text{ due to (2.7)}} = \underbrace{\int_\Omega \tilde{p}\mathbf{I} : \nabla \mathbf{u} \,\mathrm{d}x}_{I_{3A}} - \underbrace{\int_\Omega 2\nu(\psi) D(\mathbf{u}) : \mathbf{u} \,\mathrm{d}x}_{I_{3B}}.$$

I_{3A} vanishes due to the divergence-free condition imposed on $\mathbf{u}$. Due to the symmetry of $D(\mathbf{u})$, the expression $D(\mathbf{u}) : \nabla \mathbf{u}$ in I_{3B} is equal to $D(\mathbf{u}) : D(\mathbf{u}) = |D(\mathbf{u})|^2$ where $:$ represents the Frobenius inner product. Therefore we can say

$$I_3 = -\int_\Omega 2\nu(\psi)|D(\mathbf{u})|^2 \,\mathrm{d}x.$$

Next, we have

$$I_4 = \int_\Omega \psi \nabla \mu \mathbf{u} \,\mathrm{d}x = -\int_\Omega \mathrm{div}(\mathbf{u}\psi)\mu \,\mathrm{d}x + \underbrace{\int_{\partial\Omega} (\psi \mathbf{u} \cdot \mathbf{n})\mu \,\mathrm{d}\sigma}_{=0 \text{ due to (2.7)}}$$

$$= -\int_\Omega [\psi \mathrm{div}(\mathbf{u}) + (u \cdot \nabla\psi)]\mu \,\mathrm{d}x = -\int_\Omega (\mathbf{u} \cdot \nabla\psi)]\mu \,\mathrm{d}x$$

due to the divergence-free condition. Thus, assembling $I_1 - I_4$ we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2_{L^2(\Omega)} + \int_\Omega 2\nu(\psi)|D(\mathbf{u})|^2 \,\mathrm{d}x - \int_\Omega (\mathbf{u} \cdot \nabla\psi)\mu \,\mathrm{d}x = 0. \tag{A}$$

Multiplying (2.3) by μ and integrating over Ω, we have

$$\int_\Omega \mu \partial_t \psi \,\mathrm{d}x + \int_\Omega (\mathbf{u} \cdot \nabla\psi)\mu \,\mathrm{d}x + \int_\Omega M(\psi)|\nabla\mu|^2 \,\mathrm{d}x = 0, \tag{E1.2}$$

where we have used Green's second identity again to simplify the latter term. Next we work towards showing that $\int_\Omega \mu\partial_t\psi \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{F}_{sh}[\psi]$. We have

$$\int_\Omega \mu\partial_t\psi \,\mathrm{d}x = \int_\Omega \partial_t\psi(\psi^3 + (r + (1+\Delta)^2)\psi) \,\mathrm{d}x = \underbrace{\int_\Omega (\partial_t\psi)\,\psi^3 + r\psi\partial_t\psi \,\mathrm{d}x}_{I_5} + \underbrace{\int_\Omega (\partial_t\psi)(1+\Delta)^2\psi \,\mathrm{d}x}_{I_6}.$$

Using the chain rule we can rewrite the former term as

$$I_5 = \int_\Omega \partial_t \left(\frac{\psi^4}{4} + \frac{r\psi^2}{2}\right) \mathrm{d}x.$$

Now we expand the brackets in I_6, giving us

$$I_6 = \int_\Omega \partial_t\psi(1 + 2\Delta + \Delta^2)\psi \,\mathrm{d}x = \underbrace{\int_\Omega (\partial_t\psi)\psi \,\mathrm{d}x}_{I_{6A}} + 2\underbrace{\int_\Omega (\partial_t\psi)\Delta\psi \,\mathrm{d}x}_{I_{6B}} + \underbrace{\int_\Omega (\partial_t\psi)\Delta^2\psi \,\mathrm{d}x}_{I_{6C}}.$$

Another application of the chain rule yields $I_{6A} = \int_\Omega \partial_t(\frac{\psi^2}{2}) \,\mathrm{d}x$. Using Green's formula on the next term and noting that the identity $\partial_t\nabla\psi = \nabla\partial_t\psi$ holds true here (as we are assuming ψ is smooth), we observe that

$$I_{6B} = -2\int_\Omega \nabla(\partial_t\psi)\nabla\psi \,\mathrm{d}x + 2\underbrace{\int_{\partial\Omega} (\nabla\psi \cdot \mathbf{n})\partial_t\psi \, d\sigma}_{=0 \text{ due to (2.7)}} \overset{\partial_t\nabla\psi = \nabla\partial_t\psi}{=} -\int_\Omega \partial_t|\nabla\psi|^2 \,\mathrm{d}x.$$

Similarly,

$$I_{6C} = -\int_\Omega \partial_t(\nabla\psi)\nabla\Delta\psi \,\mathrm{d}x + \underbrace{\int_{\partial\Omega}(\nabla\Delta\psi\cdot\mathbf{n})\partial_t\psi \, d\sigma}_{=0 \text{ due to (2.7)}} = -\int_\Omega \nabla(\partial_t\psi)\nabla\Delta\psi \,\mathrm{d}x.$$

Applying Green's second identity once more we have

$$I_{6C} = \int_\Omega \partial_t\Delta\psi\Delta\psi \,\mathrm{d}x - \underbrace{\int_{\partial\Omega}\nabla(\partial_t\psi\cdot\mathbf{n})\Delta\psi \, d\sigma}_{=0 \text{ due to (2.7)}} = \int_\Omega \partial_t\left(\frac{|\Delta\psi|^2}{2}\right) \mathrm{d}x.$$

Adding our integrals together we see that

$$\int_\Omega \mu\partial_t\psi \,\mathrm{d}x = \int_\Omega \partial_t\left(\frac{\psi^4}{4} + \frac{r\psi^2}{2} + \frac{\psi^2}{2} - |\nabla\psi|^2 + \frac{|\Delta\psi|^2}{2}\right) \mathrm{d}x.$$

Applying Green's second identity to the final two terms in the above integrand (the boundary terms will vanish again due to (2.7)) yields

$$\begin{aligned}\int_\Omega \mu\partial_t\psi \,\mathrm{d}x &= \int_\Omega \partial_t\left(\frac{\psi^4}{4} + \frac{r\psi^2}{2} + \frac{\psi^2}{2} + \psi\Delta\psi + \frac{1}{2}\Delta^2\psi\right) \mathrm{d}x \\ &= \int_\Omega \partial_t\left(\frac{\psi^4}{4} + \frac{1}{2}\psi(r+(1+\Delta)^2)\psi\right) \mathrm{d}x = \frac{\mathrm{d}}{\mathrm{dt}}\int_\Omega \frac{\psi^4}{4} + \frac{1}{2}\psi(r+(1+\Delta)^2)\psi \,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{dt}}\mathcal{F}_{sh}[\psi].\end{aligned}$$

Thus, $(E1.2)$ is equivalent to

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{F}_{sh}[\psi] + \int_\Omega (\mathbf{u}\cdot\nabla\psi)\mu \,\mathrm{d}x + \int_\Omega M(\psi)|\nabla\mu|^2 \,\mathrm{d}x = 0. \tag{B}$$

Finally, adding equations (A) and (B) gives us the final energy estimate

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|u\|^2 + \int_\Omega 2\nu(\psi)|D(\mathbf{u})|^2 \,\mathrm{d}x - \int_\Omega (u\cdot\nabla\psi)\mu \,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{dt}}\mathcal{F}_{sh}[\psi] \\ + \int_\Omega (u\cdot\nabla\psi)\mu \,\mathrm{d}x + \int_\Omega M(\psi)|\nabla\mu|^2 \,\mathrm{d}x = 0,\end{aligned}$$

i.e.

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\|u\|^2 + \mathcal{F}_{sh}[\psi]\right) + \int_\Omega (2\nu(\psi)|D(\mathbf{u})|^2 + M(\psi)|\nabla\mu|^2)\mathrm{d}x = 0.$$

□

4.2 The route ahead

Let us now review our plan of action which we will follow in order to tackle the proof of Theorem 3.3.

Step 1: Existence of an orthonormal basis in $\mathbf{H}_\sigma$ and $L^2(\Omega)$

Through the study of two separate PDEs (Stokes problem and Laplace problem), we will infer the existence of a sequence of eigenfunctions $\{\mathbf{v}_k\}_{k=1}^\infty \subset \mathbf{H}^2(\Omega)\cap\mathbf{V}_\sigma$ and $\{w_k\}_{k=1}^\infty \subset H^2_N(\Omega)$, where the former defines an orthonormal basis in $\mathbf{V}_\sigma$ and the latter an orthonormal basis in $L^2(\Omega)$.

Step 2: Constructing a Faedo-Galerkin approximation

Defining $\mathbf{V}_n$ as the span of the first n eigenfunctions in the sequence $\{\mathbf{v}_k\}_k$ and W_n as the span of the first n eigenfunctions in the sequence $\{w_k\}_k$, we look for a solution $(\mathbf{u}_n, \psi_n, \mu_n) \in \mathbf{V}_n \times W_n \times W_n$ which satisfy the **approximating problem (AP)**. The approximating problem is identical to the weak formulation with the caveat that the test functions we consider must also lie in $\mathbf{V}_n$ or W_n (depending on the equation one considers). During this step we will use the properties of our approximate solution $(\mathbf{u}_n, \psi_n, \mu_n)$ to simplify the approximating problem into a system of ODEs.

Step 3: Proving the existence of a weak solution to (AP) on $[0, T]$

We will justify the application of the Cauchy-Lipschitz theorem for non-linear systems of ODEs on our approximating problem. This will prove the existence of a solution to **(AP)** on $[0, T_n)$ for some $0 < T_n < T$ which is attained from the Cauchy-Lipschitz result. Then, we will carry out a series of estimates to deduce that our solution can be extended to the entire interval $[0, T]$.

Step 4: Further estimates

We proceed to carry out further estimates and deduce that our weak solution to **(AP)** is bounded in certain spaces. These estimates are crucial for the next and final step, where they will be used to apply certain compactness results which will allow us to pass to the limit.

Step 5: Passing to the limit

We collect our estimates from the previous two steps to conclude that our functions $(\mathbf{u}_n, \psi_n, \mu_n)$ are bounded in suitable spaces and therefore possess convergent subsequences. This allows us to identify the existence of a candidate limit $(\mathbf{u}, \psi, \mu)$ and the function spaces which it inhabits. Then, we show that the candidate limit also satisfies the original weak formulation. We conclude the proof by showing that we do in fact have a global weak solution.

5 Proving the existence of a global weak solution

5.1 Step 1: Existence of an orthonormal basis in H_σ and $L^2(\Omega)$

In order to use a Faedo-Galerkin approximation scheme, we firstly need to show the existence of eigenfunctions corresponding to two operators, namely the Laplacian and Stokes operators. We will make use of the following theorem ([6], Chapter 6.5):

Theorem X (Existence of eigenbases associated with general elliptic operators) ([6])

Let V, H be two Hilbert spaces such that H is separable, V is dense in H and $V \overset{c}{\hookrightarrow} H$. Assume a is a bilinear form in V that is continuous, symmetric and weakly coercive in V, i.e. $\exists\ \lambda_0 \geq 0,\ \alpha > 0$ such that $a(u,u) + \lambda_0 \|u\|_H^2 \geq \alpha \|u\|_V^2,\ \forall v \in V$. Then:

- there exists a sequence $\{\lambda_n\}_{n\geq 1} \subset (-\lambda_0, \infty)$, with $\lambda_n \to \infty$ as $n \to \infty$ and a sequence of functions $\{u_n\}_{n\geq 1} \subset V$ such that $a(u_n, v) = \lambda_n (u_n, v)$ for each $v \in V,\ n \in \mathbb{N}$,
- the sequence $\{u_n\}_{n\geq 1}$ is an orthonormal basis of H and an orthogonal basis of V with respect to the inner product $((u,v))_V := a(u,v) + \lambda_0 (u,v)$.

Note that the second bullet point in the above theorem, although not mentioned in [6], is a simple corollary of the first bullet point. Indeed, for $n, m \in \mathbb{N}$, we have

$$((u_n, u_m))_V = a(u_n, u_m) + \lambda_0 (u_n, u_m) = (\lambda_n + \lambda_0)(u_n, u_m) = (\lambda_n + \lambda_0)\delta_{nm},$$

by the orthonormality of $\{u_n\}$ in H.

5.1.1 Stokes Problem

We wish to prove the following lemma:

Lemma 5.1. (Existence of an eigenbasis associated to the Stokes problem) Suppose Ω is a bounded domain of class C^2. Then there exists a sequence of eigenfunctions $\{\mathbf{v}_k\}_{k=1}^{\infty} \subset \mathbf{V}_\sigma$ and associated eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$ such that $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \to \infty$ solving $\int_\Omega \nabla \mathbf{v}_k : \nabla \mathbf{w}\, \mathrm{d}x = \lambda_k \int_\Omega \mathbf{v}_k\ \mathbf{w}\, \mathrm{d}x$, for any $\mathbf{w} \in \mathbf{V}_\sigma$. We have that $\{\mathbf{v}_k\}_{k=1}^{\infty}$ forms an orthonormal basis of $\mathbf{H}_\sigma$ and an orthogonal basis of $\mathbf{V}_\sigma$. Additionally, each $\mathbf{v}_k$ also belongs to $\mathbf{H}^2(\Omega)$.

Proof. Consider the eigenvalue problem associated with the Stokes problem:

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \lambda \mathbf{u}, & \text{in } \Omega, \\ \operatorname{div}\mathbf{u} = 0, & \text{in } \Omega, \\ \mathbf{u} = 0, & \text{on } \partial\Omega. \end{cases} \tag{SE}$$

Let us find the weak formulation of (SE). Assuming $\mathbf{u} \in \mathbf{H}^2(\Omega)$, multiplying the first equation by $\mathbf{v} \in \mathbf{V}_\sigma$ and integrating over Ω, we find

$$\int_\Omega -\Delta \mathbf{u}\ \mathbf{v}\, \mathrm{d}x + \underbrace{\int_\Omega \nabla p \cdot \mathbf{v}\, \mathrm{d}x}_{=0} = \lambda \int_\Omega \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}x, \tag{5.1}$$

where the term involving the pressure p vanishes as a consequence of the no slip and divergence-free conditions imposed on $\mathbf{v} \in \mathbf{V}_\sigma$. Integrating by parts in the first term gives

$$\int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, \mathrm{d}x = \lambda \int_\Omega \mathbf{u}\ \mathbf{v}\, \mathrm{d}x.$$

Defining the bilinear form $a : \mathbf{V}_\sigma \times \mathbf{V}_\sigma \to \mathbb{R}$ as $a(u,v) := \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v}\, \mathrm{d}x$, we deduce the weak formulation of (SE) to be

$$\boxed{\text{Find } \mathbf{u} \in \mathbf{V}_\sigma \text{ satisfying } a(\mathbf{u}, \mathbf{v}) = \lambda(\mathbf{u}, \mathbf{v}), \quad \forall\ \mathbf{v} \in \mathbf{V}_\sigma. \quad \text{(WF-S)}}$$

Next, we prove that the bilinear form a is symmetric, continuous and weakly coercive in $\mathbf{V}_\sigma$.

- **Symmetry:** Obvious by properties of the integral.
- **Continuity:** Using Holder's inequality, $|a(u, v)| \leq \|\nabla \mathbf{u}\| \|\nabla \mathbf{v}\| = \|\mathbf{u}\|_{\mathbf{V}_\sigma} \|\mathbf{v}\|_{\mathbf{V}_\sigma}$.
- **Coercivity:** $a(u, u) = \|\nabla \mathbf{u}\|^2 = \|\mathbf{u}\|^2_{\mathbf{V}_\sigma}$.

Therefore, by invoking Theorem X, we have that there exists a sequence of eigenfunctions $\{\mathbf{v}_k\}_{k=1}^\infty \subset \mathbf{V}_\sigma$ and associated eigenvalues $\{\lambda_n\}_{n=1}^\infty$ such that $0 \leq \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \to \infty$ solving (WF-S). Moreover, $\{v_k\}_{k=1}^\infty$ is an orthonormal basis of $\mathbf{H}_\sigma$ and an orthogonal basis of $\mathbf{V}_\sigma$. Notice, however, that this does not completely satisfy the statement of Lemma 5.1 as we still need to show that each element in the sequence $\{v_k\}$ belongs to $\mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma$ and not just $\mathbf{V}_\sigma$. Assuming Ω is a bounded domain of class C^2, we can in fact infer stronger regularity on each v_k. Multiplying the first equation in (SE) by $\mathbf{v} \in [D(\Omega)]^3$ and integrating by parts we achieve $-\Delta \mathbf{u} = \lambda \mathbf{u}$ in $\mathbf{D}'(\Omega)$ and since $\mathbf{u} \in [L^2(\Omega)]^3$ we infer $-\Delta \mathbf{u} \in [L^2(\Omega)]^3$ by comparison. The regularity theory given in (LR-1) implies that $\mathbf{u} \in [H^2(\Omega)]^3$. This means that each v_k belongs to $\mathbf{H}^2(\Omega)$, as required. □

5.1.2 Laplace's problem

We wish to prove the following lemma:

Lemma 5.2. (Existence of an eigenbasis associated to the Laplace problem with Neumann boundary conditions)
Suppose Ω is a bounded domain of class C^2. Then there exists a sequence of eigenfunctions $\{w_k\}_{k=1}^\infty \subset H^2_N(\Omega)$ and associated eigenvalues $\{\lambda_n\}_{n=1}^\infty$ such that $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n \to \infty$ solving $\int_\Omega \nabla w_k \cdot \nabla v \, \mathrm{d}x = \lambda_k \int_\Omega w_k \ v \, \mathrm{d}x$, for any $v \in H^1(\Omega)$. We have that $\{\mathbf{w}_k\}_{k=1}^\infty$ forms an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1(\Omega)$. Additionally, we have that each w_k belongs to $H^2_N(\Omega)$.

Proof. Consider the eigenvalue problem corresponding to the Laplacian with Neumann boundary conditions:

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \partial\Omega. \end{cases} \tag{LE}$$

Again, we begin by finding the weak formulation of (LE). Assuming $u \in H^2_N(\Omega)$, multiplying by $v \in H^1(\Omega)$ and integrating over Ω:

$$\int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x - \underbrace{\int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) v \, d\sigma}_{=0} = \int_\Omega \lambda u \, v \, \mathrm{d}x, \tag{5.2}$$

where the second term vanishes due to the boundary condition. We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ as $a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x$. Noticing that the equation (5.2) is well-posed whenever $u \in H^1(\Omega)$ and $v \in H^1(\Omega)$, we deduce the weak formulation of the above Laplace problem to be:

$$\boxed{\text{Find } u \in H^1(\Omega) \text{ satisfying } a(u, v) = \lambda(u, v), \quad \forall\ v \in H^1(\Omega). \quad \text{(WF-L)}}$$

Next, we prove that the bilinear form a is symmetric, continuous and weakly coercive on $H^1(\Omega) \times H^1(\Omega)$.

- **Symmetry:** Obvious by properties of the integral.
- **Continuity:** $|a(u,v)| \leq \|\nabla u\| \|\nabla v\| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$.
- **Weak coercivity:** $a(u,u) = \|\nabla u\|^2$. Thus for any $\lambda_0 > 0$, we have

$$a(u,u) + \lambda_0 \|u\|^2 = \|\nabla u\|^2 + \lambda_0 \|u\|^2 \geq \min\{1, \lambda_0\} \|u\|^2_{H^1(\Omega)}.$$

Therefore, invoking Theorem X once again, we have that there exists a sequence of eigenfunctions $\{w_k\}_{k=1}^{\infty}$ which form an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1(\Omega)$. Notice however that we still need to show that each element of the sequence $\{w_k\}$ belongs to $H^2_N(\Omega)$ and not just $H^1(\Omega)$. This is easily fixed by recalling the regularity theory of the Laplace operator given in (LR-1), which says that any solution to the above Laplace problem must also lie in $H^2(\Omega)$. It remains to show that each w_k belongs to $H^2_N(\Omega)$. As we already know that each $w_k \in H^2(\Omega)$, we only need to show that they also satisfy $\nabla w_k \cdot \mathbf{n} = 0$ on $\partial\Omega$. Firstly, we remark that taking $v \in D(\Omega)$, it can be shown that $-\Delta w_k = \lambda w_k$ holds true in $D'(\Omega)$. But since $w_k \in H^2(\Omega)$ it follows that the equality $-\Delta w_k = \lambda w_k$ holds true almost everywhere in Ω for each k. Next, recall that each w_k solves (WF-L), namely

$$\int_\Omega \nabla w_k \cdot \nabla v \, \mathrm{d}x = \lambda \int_\Omega w_k \, v \, \mathrm{d}x, \ \forall \, v \in H^1(\Omega).$$

Integrating by parts in the first integral gives us

$$-\int_\Omega \Delta w_k \, v \, \mathrm{d}x + \int_{\partial\Omega} (\nabla w_k \cdot \mathbf{n}) \, v \, \mathrm{d}\sigma = \lambda \int_\Omega w_k \, v \, \mathrm{d}x, \ \forall \, v \in H^1(\Omega).$$

Since $-\Delta w_k = \lambda w_k$ holds true almost everywhere in Ω, we are left with

$$\int_{\partial\Omega} (\nabla w_k \cdot \mathbf{n}) \, v \, \mathrm{d}\sigma = 0, \ \forall \, v \in H^1(\Omega).$$

This implies that $\nabla w_k \cdot \mathbf{n} = 0$ on $\partial\Omega$ which in turn entails that each $w_k \in H^2_N(\Omega)$, completing the proof. $\square$

5.2 Step 2: The Galerkin Approximation

The previous subsection proved the existence of:

- a sequence $\{w_k\}_{k=1}^{\infty} \subset H^2_N(\Omega)$ which forms an orthonormal basis of $L^2(\Omega)$ and an orthogonal basis of $H^1(\Omega)$,
- a sequence $\{\mathbf{v}_k\}_{k=1}^{\infty} \subset \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma$ which forms an orthonormal basis of $\mathbf{H}_\sigma$ and an orthogonal basis of $\mathbf{V}_\sigma$.

We now proceed to construct a Faedo-Galerkin approximation using the above sequences. Our method will be similar to that which can be seen in ([4], [9], [15]), for example. Fix $n \in \mathbb{N}$ and define the subspaces $W_n := \text{span}\{w_1, ..., w_n\} \subset H^2_N(\Omega)$, $\mathbf{V}_n := \text{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\} \subset \mathbf{W}_\sigma$, where $\mathbf{W}_\sigma := \mathbf{H}^2(\Omega) \cap \mathbf{V}_\sigma$. We also define the projections $\mathcal{P}_n : \mathbf{H}_\sigma \to \mathbf{V}_n$ and $\Pi_n : L^2(\Omega) \to W_n$, where $\mathbf{V}_n$ and W_n are considered with the usual inner products in $\mathbf{H}_\sigma$ and $L^2(\Omega)$ respectively.
We are looking for functions taking the form

$$\mathbf{u}_n(x,t) = \sum_{k=1}^{n} A^n_k(t) \mathbf{v}_k(x),$$

$$\psi_n(x,t) = \sum_{k=1}^{n} B_k^n(t) w_k(x),$$

$$\mu_n(x,t) = \sum_{k=1}^{n} C_k^n(t) w_k(x),$$

which solve the approximating problem:

$$\begin{cases} \langle \partial_t \mathbf{u}_n(t), \mathbf{v} \rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}) + (2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{v}) = -(\psi_n \nabla \mu_n, \mathbf{v}), & \forall\, \mathbf{v} \in \mathbf{V}_n, \quad \text{(AP-1)} \\ \langle \partial_t \psi_n, w \rangle_{(H^1)^* \times H^1} + ((\mathbf{u}_n \cdot \nabla \psi_n), w) + (M(\psi_n) \nabla \mu_n, \nabla w) = 0, & \forall\, w \in W_n, \quad \text{(AP-2)} \\ (\mu_n, w) = (\psi_n^3, w) + ((r+1)\psi_n, w) + (2\Delta \psi_n, w) + (\Delta \psi_n, \Delta w), & \forall\, w \in W_n, \quad \text{(AP-3)} \\ \mathbf{u}_n(\cdot, 0) = \mathcal{P}_n(\mathbf{u}_0) =: \mathbf{u}_0^n,\ \psi_n(\cdot, 0) = \Pi_n(\psi_0) =: \psi_0^n, & \text{in } \Omega, \quad \text{(AP-4)} \end{cases}$$

a.e. in $[0,T]$. Next, we simplify each of (AP-1)-(AP-3) by substituting in our expressions for $\mathbf{u}_n, \psi_n$ and μ_n.

Simplifying AP-1

> **Goal:** Achieve a simplification of **(AP-1)**.

By orthogonality of the sequence $\{\mathbf{v}_k\}_{k=1}^n$, (AP-1) is equivalent to:

$$\underbrace{(\partial_t \mathbf{u}_n(t), \mathbf{v}_s)}_{\mathbf{A1}} + \underbrace{b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}_s)}_{\mathbf{A2}} + \underbrace{(2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{v}_s)}_{\mathbf{A3}} = \underbrace{-(\psi_n \nabla \mu_n, \mathbf{v}_s)}_{\mathbf{A4}}, \quad \forall\, s = 1, ..., n. \tag{5.4}$$

Inserting our expressions for $\mathbf{u}_n, \psi_n$ and μ_n, we have:

$$\boxed{\mathbf{A1}} = (\partial_t \left(\sum_{k=1}^{n} \mathrm{A}_k^n(t) \mathbf{v}_k \right), \mathbf{v}_s) = \sum_{k=1}^{n} (\dot{\mathrm{A}}_k^n(t) \mathbf{v}_k, \mathbf{v}_s) = \sum_{k=1}^{n} \dot{\mathrm{A}}_k^n(t) \delta_{ks} = \dot{\mathrm{A}}_s^n(t),$$

$$\boxed{\mathbf{A2}} = b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}_s) = \int_\Omega (\mathbf{u}_n(t) \cdot \nabla) \mathbf{u}_n(t) \cdot \mathbf{v}_s \,\mathrm{d}x = \sum_{k=1}^{n} \sum_{j=1}^{n} \mathrm{A}_k^n(t) \mathrm{A}_j^n(t) \int_\Omega (\mathbf{v}_k \cdot \nabla) \mathbf{v}_j \cdot \mathbf{v}_s \,\mathrm{d}x,$$

$$\boxed{\mathbf{A3}} = (2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{v}_s) = \int_\Omega 2\nu(\psi_n) D(\mathbf{u}_n) : \nabla \mathbf{v}_s \,\mathrm{d}x = \sum_{k=1}^{n} \mathrm{A}_k^n(t) \int_\Omega 2\nu(\psi_n) D(\mathbf{v}_k) : \nabla \mathbf{v}_s \,\mathrm{d}x,$$

$$\boxed{\mathbf{A4}} = -(\psi_n \nabla \mu_n, \mathbf{v}_s) = -\int_\Omega (\psi_n(t)) \left(\sum_{k=1}^{n} \mathrm{C}_k^n(t) \nabla w_k \right) \mathbf{v}_s \,\mathrm{d}x = -\sum_{j=1}^{n} \mathrm{C}_k^n(t) \int_\Omega \psi_n(t) \left(\nabla w_k \cdot \mathbf{v}_s \right) \,\mathrm{d}x$$

$$= -\sum_{k=1}^{n} \mathrm{C}_k^n(t) \int_\Omega \left(\sum_{j=1}^{n} \mathrm{B}_j^n(t) w_j \right) (\nabla w_k \cdot \mathbf{v}_s) \,\mathrm{d}x = -\sum_{k=1}^{n} \sum_{j=1}^{n} \mathrm{C}_k^n(t) \mathrm{B}_j^n(t) \int_\Omega w_j (\nabla w_k \cdot \mathbf{v}_s) \,\mathrm{d}x.$$

Assembling $(A1) - (A4)$ we find that (5.4) simplifies to:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \mathrm{A}_s^n(t) = &-\sum_{k=1}^{n} \sum_{j=1}^{n} \mathrm{A}_k^n(t) \mathrm{A}_j^n(t) \int_\Omega (\mathbf{v}_k \cdot \nabla) \mathbf{v}_j \cdot \mathbf{v}_s \,\mathrm{d}x - \sum_{k=1}^{n} \mathrm{A}_k^n(t) \int_\Omega 2\nu(\psi_n) D(\mathbf{v}_k) : \nabla \mathbf{v}_s \,\mathrm{d}x \\ &- \sum_{k=1}^{n} \sum_{j=1}^{n} \mathrm{C}_k^n(t) \mathrm{B}_j^n(t) \int_\Omega w_j (\nabla w_k \cdot \mathbf{v}_s) \,\mathrm{d}x, \end{aligned} \tag{5.5}$$

Now we will show that each term in the above equation is well-defined. Note that since our summations are all finite, it suffices to verify that the integrals are bounded. Firstly, we have that

$$\left| \int_\Omega (\mathbf{v}_k \cdot \nabla) \mathbf{v}_j \cdot \mathbf{v}_s \,\mathrm{d}x \right| \le C \|\mathbf{v}_k\|_{L^3} \|\nabla \mathbf{v}_j\|_{L^2} \|\mathbf{v}_s\|_{L^6} \le C' \|\mathbf{v}_k\|_{H^1} \|\nabla \mathbf{v}_j\|_{L^2} \|\mathbf{v}_s\|_{H^1} < +\infty,$$

for any $k, j, s \in \{1, ..., n\}$, where we have used the Holder inequality twice to achieve the first inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$ to achieve the second inequality. Next, noting our assumption that $\nu(t) < \nu^*$ for each $t \in \mathbb{R}$, we have

$$\left|\int_\Omega 2\nu(\psi_n) D(\mathbf{v}_k) : \nabla \mathbf{v}_s \,\mathrm{d}x\right| \leq 2\nu^* \int_\Omega |D(\mathbf{v}_k) : \nabla \mathbf{v}_s| \,\mathrm{d}x \leq 2\sqrt{2}\nu^* \int_\Omega |\nabla \mathbf{v}_k : \nabla \mathbf{v}_s| \,\mathrm{d}x$$
$$= 2\sqrt{2}\nu^* (\mathbf{v}_k, \mathbf{v}_s)_{\mathbf{V}_\sigma} < +\infty,$$

for any $k, s \in \{1, ..., n\}$. To achieve the second inequality we used Korn's inequality (see (KI)). Finally, again using the Holder inequality and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we have

$$\left|\int_\Omega w_j (\nabla w_k \cdot \mathbf{v}_s) \,\mathrm{d}x\right| \leq \|w_j\|_{L^3} \|\nabla w_k\|_{L^2} \|\mathbf{v}_s\|_{L^6} \leq \|w_j\|_{H^1} \|w_k\|_{H^1} \|\mathbf{v}_s\|_{H^1} < +\infty,$$

which shows that each term in (5.5) is well-defined.

Simplifying AP-2

Goal: Achieve a simplification of **(AP-2)**.

By orthogonality of the sequence $\{w_k\}_{k=1}^n$, (AP-2) is equivalent to:

$$\underbrace{(\partial_t \psi_n, w_s)}_{\mathbf{A5}} + \underbrace{((\mathbf{u}_n \cdot \nabla \psi_n), w_s)}_{\mathbf{A6}} + \underbrace{(M(\psi_n) \nabla \mu_n, \nabla w_s)}_{\mathbf{A7}} = 0, \quad \forall\ s = 1, ..., n. \tag{5.6}$$

Inserting our expressions for $\mathbf{u}_n, \psi_n$ and μ_n, we have:

$$\boxed{\mathbf{A5}} = (\partial_t \left(\sum_{k=1}^n \mathrm{B}_k^n(t) w_k\right), w_s) = \sum_{k=1}^n (\dot{\mathrm{B}}_k^n(t) w_k, w_s) = \sum_{k=1}^n \dot{\mathrm{B}}_k^n(t) \delta_{ks} = \dot{\mathrm{B}}_s^n(t),$$

$$\boxed{\mathbf{A6}} = \int_\Omega \left[\left(\sum_{k=1}^n \mathrm{A}_k^n(t) \mathbf{v}_k\right) \left(\sum_{j=1}^n \mathrm{B}_j^n(t) \nabla w_j\right)\right] w_s \,\mathrm{d}x = \sum_{k=1}^n \sum_{j=1}^n \mathrm{A}_k^n(t) \mathrm{B}_j^n(t) \int_\Omega (\mathbf{v}_k \cdot \nabla w_j) w_s \,\mathrm{d}x,$$

$$\boxed{\mathbf{A7}} = \int_\Omega M(\psi_n)(\nabla \mu_n \cdot w_s) \,\mathrm{d}x = \int_\Omega M(\psi_n)(\sum_{j=1}^n \mathrm{C}_j^n(t) \nabla w_j) \nabla w_s \,\mathrm{d}x = \sum_{k=1}^n \mathrm{C}_j^n(t) \int_\Omega M(\psi_n) \nabla w_j \nabla w_s \,\mathrm{d}x.$$

Equation (5.6) then simplifies to

$$\frac{\mathrm{d}}{\mathrm{dt}} \mathrm{B}_s^n(t) = -\sum_{k=1}^n \sum_{j=1}^n \mathrm{A}_k^n(t) \mathrm{B}_j^n(t) \int_\Omega (\mathbf{v}_k \cdot \nabla w_j) w_s \,\mathrm{d}x - \sum_{k=1}^n \mathrm{C}_j^n(t) \int_\Omega M(\psi_n) \nabla w_j \nabla w_s \,\mathrm{d}x, \ \forall\ s = 1, ..., n. \tag{5.7}$$

Now we show that each term in the above equation is well-defined. By a very similar computation to that seen in the simplification of **AP-1** we can say that

$$\left|\int_\Omega (\mathbf{v}_k \cdot w_j) w_s \,\mathrm{d}x\right| < +\infty.$$

Moreover by exploiting our assumption on the mobility function (see (B1)-(B2)) and the Holder inequality, we have

$$\left|\int_\Omega M(\psi_n) \nabla w_j \nabla w_s \,\mathrm{d}x\right| \leq M^* \|\nabla w_j\| \|\nabla w_s\| \leq M^* \|w_j\|_{H^1} \|w_s\|_{H^1} < +\infty,$$

which shows that each term in (5.5) is well-defined.

Simplifying AP-3

Goal: Achieve a simplification of **(AP-3)**.

Again by orthogonality of our eigenbases, (AP-3) is equivalent to:

$$\underbrace{(\mu_n, w_s)}_{\mathbf{A8}} + \underbrace{(\psi_n^3, w_s)}_{\mathbf{A9}} + \underbrace{((r + (1+\Delta)^2)\psi_n, w_s)}_{\mathbf{A10}} = 0, \quad \forall\, s = 1, ..., n. \tag{5.8}$$

Inserting our expressions for ψ_n and μ_n, we have:

$$\boxed{\mathbf{A8}} = \int_\Omega \left(\sum_{k=1}^n \mathrm{C}_k^n(t) w_k\right) w_s \,\mathrm{d}x = \mathrm{C}_s^n(t),$$

$$\boxed{\mathbf{A9}} = \int_\Omega \left(\sum_{k=1}^n \mathrm{B}_k^n(t) w_k\right)^3 w_s \,\mathrm{d}x.$$

In order to simplify $\boxed{\mathbf{A10}}$, we first note that

$$(\nabla\psi_n, \nabla w_s) = \sum_{k=1}^n \mathrm{B}_k^n(t)(\nabla w_k, \nabla w_s) = \lambda_s \sum_{k=1}^n \mathrm{B}_k^n(t)(w_k, w_s) = \lambda_s,$$

since the w_k's form an orthonormal basis in $L^2(\Omega)$. Also, since the w_k's solve

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ \nabla u \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \end{cases}$$

we infer that $(\Delta w_k, \Delta w_s) = (-\lambda_k w_k, -\lambda_s w_s) = \lambda_k \lambda_s \delta_{ks}$. Using these facts, we compute:

$$\begin{aligned}\boxed{\mathbf{A10}} &= ((r+1)\psi_n, w_s) + (2\Delta\psi_n, w_s) + (\Delta^2\psi_n, w_s) = (r+1)\mathrm{B}_s^n(t) - 2(\nabla\psi_n, \nabla w_s) + (\Delta\psi_n, \Delta w_s) \\ &= (r+1)\mathrm{B}_s^n(t) - 2\lambda_s \mathrm{B}_s^n(t) + (\sum_{k=1}^n \mathrm{B}_k^n(t)\Delta w_k, \Delta w_s) = (r+1)\mathrm{B}_s^n(t) - 2\lambda_s \mathrm{B}_s^n(t) + \mathrm{B}_s^n(t)\lambda_s^2 \\ &= (r + 1 - 2\lambda_s + \lambda_s^2)\mathrm{B}_s^n(t).\end{aligned}$$

Thus, (5.8) is equivalent to

$$\mathrm{C}_s^n(t) = -\int_\Omega \left(\sum_{k=1}^n \mathrm{B}_k^n(t) w_k\right)^3 w_s \,\mathrm{d}x - (r + 1 - 2\lambda_s + \lambda_s^2)\mathrm{B}_s^n(t), \ \forall\, s = 1, ..., n. \tag{5.9}$$

To conclude the well-posedness of the above equation it suffices to show $\left|\int_\Omega \left(\sum_{k=1}^n w_k\right)^3 w_s \,\mathrm{d}x\right| < +\infty$. First note that by Holder's inequality, we have

$$\left|\int_\Omega \left(\sum_{k=1}^n w_k\right)^3 w_s \,\mathrm{d}x\right| \leq \|w_s\|_{L^2} \|(\sum_{k=1}^n w_k)^3\|_{L^2}. \tag{5.10}$$

Then, we make note of the following estimates for some fixed $k, j \in 1, ..., n$:

$$\|w_k^3\|_{L^2} = \|w_k\|_{L^6}^3 \leq C_s \|w_k\|_{H^1}^3 \leq C_1.$$

Using Young's inequality **(Y)**, the inequality $(a+b)^{1/2} \leq (a^{1/2} + b^{1/2})$ **(C)** and the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ **(S)**, we have

$$\begin{aligned}\|w_k w_j\|_{L^2} = (\int_\Omega w_k^2 w_j^2 \,\mathrm{d}x)^{1/2} &\overset{\mathbf{(Y)}}{\leq} (\frac{1}{2}\|w_k\|_{L^4}^4 + \frac{1}{2}\|w_j\|_{L^4}^4)^{1/2} \\ &\overset{\mathbf{(C)+(S)}}{\leq} \frac{C^2}{\sqrt{2}}(\|w_k\|_{H^1(\Omega)}^2 + \|w_j\|_{H^1(\Omega)}^2) < +\infty,\end{aligned}$$

and

$$\|w_k^2 w_j\|_{L^2} = (\int_\Omega w_k^3 w_k w_j^2 \,\mathrm{d}x)^{1/2} \leq C\|w_k^3\|_{L^3}\|w_j^2\|_{L^2}\|w_k\|_{L^6}$$
$$\overset{\textbf{(S)}}{\leq} C'\|w_k\|_{H^1}^3\|w_j\|_{H^1}^4\|w_k\|_{H^1} < +\infty.$$

By linearity of the integral it follows from the above estimates that $\|(\sum_{k=1}^n w_k)^3\|_{L^2} < +\infty$ and thus from (5.10) we have $\left|\int_\Omega (\sum_{k=1}^n w_k)^3 w_s \,\mathrm{d}x\right| < +\infty$, as required. We have now shown that each term in (5.8) is well-defined.

Transforming the approximating problem (AP)

Goal: Simplify (5.3) into a system of ODEs and solve the resulting system.

Using (5.2), (5.6), (5.8) we deduce that (5.3) is equivalent to:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{dt}}\mathrm{A}_s^n(t) = -\displaystyle\sum_{k=1}^n\sum_{j=1}^n \mathrm{A}_k^n(t)\mathrm{A}_j^n(t)\int_\Omega (\mathbf{v}_k\cdot\nabla)\mathbf{v}_j\cdot\mathbf{v}_s\,\mathrm{d}x - \sum_{k=1}^n \mathrm{A}_k^n(t)\int_\Omega 2\nu(\psi_n)D(\mathbf{v}_k):\nabla\mathbf{v}_s\,\mathrm{d}x & \textbf{(P1)} \\ \qquad -\displaystyle\sum_{k=1}^n\sum_{j=1}^n \mathrm{C}_k^n(t)\mathrm{B}_j^n(t)\int_\Omega w_j(\nabla w_k\cdot\mathbf{v}_s)\,\mathrm{d}x, & \\ \dfrac{\mathrm{d}}{\mathrm{dt}}\mathrm{B}_s^n(t) = -\displaystyle\sum_{k=1}^n\sum_{j=1}^n \mathrm{A}_k^n(t)\mathrm{B}_j^n(t)\int_\Omega (\mathbf{v}_k\cdot\nabla w_j)w_s\,\mathrm{d}x - \sum_{k=1}^n \mathrm{C}_j^n(t)\int_\Omega M(\psi_n)\nabla w_j\nabla w_s\,\mathrm{d}x, & \textbf{(P2)} \\ \mathrm{C}_s^n(t) = -\displaystyle\int_\Omega \left(\sum_{k=1}^n \mathrm{B}_k^n(t)w_k\right)^3 w_s\,\mathrm{d}x - (r+1-2\lambda_s+\lambda_s^2)\mathrm{B}_s^n(t), & \textbf{(P3)} \\ \mathrm{A}_s^n(0) = (\mathbf{u}_0,\mathbf{v}_s),\ \mathrm{B}_s^n(0) = (\psi_0, w_s) \quad \forall\ s=1,..,n. & \textbf{(P4)} \end{cases}$$

Defining the vectors

$$\mathbf{A}^n = (\mathrm{A}_1^n(t),...,\mathrm{A}_n^n(t)),\quad \mathbf{B}^n(t) = (\mathrm{B}_1^n(t),...,\mathrm{B}_n^n(t)),\quad \mathbf{C}^n(t) = (\mathrm{C}_1^n(t),...,\mathrm{C}_n^n(t)),$$
$$\mathcal{A}^n = ((\mathbf{u}_0,\mathbf{v}_1),...,(\mathbf{u}_0,\mathbf{v}_n)),\quad \mathcal{B}^n = ((\psi_0,w_1),...,(\psi_0,w_n)),$$

as well as

$$(\mathbf{F}_1^n(t,\mathbf{A}^n,\mathbf{B}^n,\mathbf{C}^n))_i = -\sum_{k=1}^n\sum_{j=1}^n \mathrm{A}_k^n(t)\mathrm{A}_j^n(t)\int_\Omega (\mathbf{v}_k\cdot\nabla)\mathbf{v}_j\cdot\mathbf{v}_i\,\mathrm{d}x - \sum_{k=1}^n \mathrm{A}_k^n(t)\int_\Omega 2\nu(\psi_n)D(\mathbf{v}_k):\nabla\mathbf{v}_i\,\mathrm{d}x$$
$$-\sum_{k=1}^n\sum_{j=1}^n \mathrm{C}_k^n(t)\mathrm{B}_j^n(t)\int_\Omega w_j(\nabla w_k\cdot\mathbf{v}_i)\,\mathrm{d}x,$$
$$(\mathbf{F}_2^n(t,\mathbf{A}^n,\mathbf{B}^n,\mathbf{C}^n))_i = -\sum_{k=1}^n\sum_{j=1}^n \mathrm{A}_k^n(t)\mathrm{B}_j^n(t)\int_\Omega (\mathbf{v}_k\cdot\nabla w_j)w_i\,\mathrm{d}x - \sum_{k=1}^n \mathrm{C}_j^n(t)\int_\Omega M(\psi_n)\nabla w_j\nabla w_i\,\mathrm{d}x,$$
$$(\mathbf{F}_3^n(t,\mathbf{B}^n))_i = -\int_\Omega \left(\sum_{k=1}^n \mathrm{B}_k^n(t)w_k\right)^3 w_i\,\mathrm{d}x,$$
$$\mathbf{X}_1^n(t) = -\mathrm{diag}\left((r+1-2\lambda_1-\lambda_1^2),...,(r+1-2\lambda_n-\lambda_n^2)\right),$$

we find that the above system is equivalent to:

$$
\textbf{(AP*)}\begin{cases}
\dfrac{\mathrm{d}}{\mathrm{dt}}\mathbf{A}^n = \mathbf{F}_1^n(t, \mathbf{A}^n, \mathbf{B}^n, \mathbf{C}^n), & \textbf{(P1*)}\\
\dfrac{\mathrm{d}}{\mathrm{dt}}\mathbf{B}^n = \mathbf{F}_2^n(t, \mathbf{A}^n, \mathbf{B}^n, \mathbf{C}^n), & \textbf{(P2*)}\\
\mathbf{C}^n = \mathbf{F}_3^n(t, \mathbf{B}^n) + \mathbf{X}_1^n(t)\mathbf{B}^n, & \textbf{(P3*)}\\
\mathbf{A}^n(0) = \mathcal{A}^n, \quad \mathbf{B}^n(0) = \mathcal{B}^n. & \textbf{(P4*)}
\end{cases}
$$

Now, we wish to apply the well-known Cauchy-Lipschitz theorem:

The Cauchy-Lipschitz theorem

Suppose $I = [a, b],\ U \subset \mathbb{R}^k,\ F : I \times U \to \mathbb{R}^k$. Given the initial value problem

$$
\begin{cases}
y'(t) = F(t, y(t)),\\
y(t_0) = y_0,
\end{cases}
$$

where $(t_0, y_0) \in I \times U$, Assume F is continuous at (t_0, y_0) and locally Lipschitz continuous in y and uniformly in t. Then there exists a unique local solution $y(t)$ to the above problem, i.e. a solution which is defined on some $J \subseteq I$.

Note that:

- $\mathbf{X}_1^n(t)$ is continuous on $[0, T]$ (since it is a constant function in t),
- $\mathbf{F}_i^n(t, \mathbf{x}, \mathbf{y}, \mathbf{z})$ are continuous on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n$ for $i = 1, 2, 3$; this follows from the fact that each integral term is finite and thus each $\mathbf{F}_i$ is continuous as a composition of continuous functions.
- $\mathbf{F}_i^n(t, \mathbf{x}, \mathbf{y}, \mathbf{z})$ are locally Lipschitz continuous with respect to $\mathbf{x}, \mathbf{y}, \mathbf{z}$ uniformly in t. This follows from the fact that each $\mathbf{F}_i$ is a composition of locally ($\nu, M \in C^2(\mathbb{R})$) Lipschitz continuous functions.

Thus, taking $I = [0, T],\ k = 2n,\ \mathbf{Y}(t) = (\mathbf{A}^n(t), \mathbf{B}^n(t)),\ F(t, \mathbf{Y}(t)) = (\mathbf{F}_1^n, \mathbf{F}_2^n),\ \mathbf{Y}_0 = (\mathcal{A}^n, \mathcal{B}^n)$, the assumptions of the Cauchy-Lipschitz theorem are met and we can conclude that for each $n \in \mathbb{N}$ there exists T_n such that $0 < T_n < T$ and $\mathbf{A}^n \in C^1([0, T_n); \mathbb{R}^n)$, $\mathbf{B}^n \in C^1([0, T_n); \mathbb{R}^n)$, $\mathbf{C}^n \in C([0, T_n); \mathbb{R}^n)$ solving **(AP*)** uniquely.

5.3 Step 3: Extension to $[0, T]$

In the previous section we successfully proved that given $T > 0$ and $n \in \mathbb{N}$, there exists a constant T_n such that $0 < T_n < T$ and a unique solution $(\mathbf{u}_n, \psi_n, \mu_n)$ solving the approximating problem (5.3) on $[0, T_n)$. We now wish to show that our solution in fact exists on the full interval $[0, T]$. To this end, we carry out two estimates and use this to show that an expression involving the energy of our system can be bounded by a constant independent of n and T. Our estimates follow a similar order to that which can be seen in [15]. Firstly, recall the approximating problem:

$$
\begin{cases}
\langle \partial_t \mathbf{u}_n(t), \mathbf{v} \rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}) + (2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{v}) = -(\psi_n \nabla \mu_n, \mathbf{v}), & \forall\ \mathbf{v} \in \mathbf{V}_n, \quad \text{(AP-1)}\\
\langle \partial_t \psi_n, w \rangle_{(H^1)^* \times H^1} + ((\mathbf{u}_n \cdot \nabla \psi_n), w) + (M(\psi_n) \nabla \mu_n, \nabla w) = 0, & \forall\ w \in W_n, \quad \text{(AP-2)}\\
(\mu_n, w) = (\psi_n^3, w) + ((r+1)\psi_n, w) + (2\Delta \psi_n, w) + (\Delta \psi_n, \Delta w), & \forall\ w \in W_n, \quad \text{(AP-3)}\\
\mathbf{u}_n(\cdot, 0) = \mathcal{P}_n(\mathbf{u}_0) =: \mathbf{u}_0^n,\ \psi_n(\cdot, 0) = \Pi_n(\psi_0) =: \psi_0^n, & \text{in } \Omega, \quad \text{(AP-4)}
\end{cases}
$$

Estimate 1

Goal: Show that for each $n \in \mathbb{N}$,

$$\int_\Omega \psi_n(t)\,\mathrm{d}x = \int_\Omega \psi_0\,\mathrm{d}x,\ \forall\, t \in [0, T_n).$$

We take $w = 1$ in (AP-2).

$$\int_\Omega \partial_t \psi_n(t)\,\mathrm{d}x + \int_\Omega (\mathbf{u}_n(t) \cdot \nabla \psi_n(t))\,\mathrm{d}x + (M(\psi_n(t))\nabla \mu_n(t), 0) = 0.$$

Remark: By definition of the eigenvalue problem (LE) which is solved by the functions $\{w_k\}$, the first eigenfunction is a constant (i.e. $w_1 = c$) and the corresponding eigenvalue is $\lambda_1 = 0$. Without loss of generality we can assume $c = 1$ and therefore it is sensible to take $w = 1$ in (AP-2), since $1 \in W_n$ for any $n \in \mathbb{N}$.

Integrating by parts in the second term and using the divergence-free and no-slip conditions of $\mathbf{u}$ eradicates the second term. The last term vanishes as a consequence of the 0 appearing in the inner product. Using the chain rule in the first term leaves us with

$$\frac{\mathrm{d}}{\mathrm{dt}} \int_\Omega \psi_n(t)\,\mathrm{d}x = 0.$$

Integrating with respect to time we have

$$\int_\Omega \psi_n(t)\,\mathrm{d}x - \int_\Omega \psi_n(0)\,\mathrm{d}x = 0 \implies \int_\Omega \psi_n(t)\,\mathrm{d}x = \int_\Omega \psi_0^n\,\mathrm{d}x,\quad \forall\, t \in [0, T_n).$$

Then note that

$$\int_\Omega \psi_0^n\,\mathrm{d}x = (\psi_0^n, 1) = (\Pi_n(\psi_0), 1) = (\psi_0, \Pi_n(1)) = (\psi_0, 1) = \int_\Omega \psi_0\,\mathrm{d}x,$$

by properties of the projection Π_n. The fact that $\Pi_n(1) = 1$ follows from noticing that $1 \in W_n$ for each $n \in \mathbb{N}$. All in all,

$$\int_\Omega \psi_n(t)\,\mathrm{d}x = \int_\Omega \psi_0\,\mathrm{d}x,\ \forall\, t \in [0, T_n). \tag{E1}$$

Estimate 2

Goal: Show that there exist positive constants $a_1, ..., a_6 \in \mathbb{R}$ as well as a constant C independent of n, T such that

$$\begin{aligned} a_1\|\mathbf{u}_n(t)\|^2 + a_2\|\psi_n(t)\|_{L^4}^4 + a_3\|\psi_n(t)\|_{H^2}^2 + a_4\|\psi_n(t)\|^2 \\ + \int_0^t a_5\|\nabla \mathbf{u}_n(\tau)\|^2 + a_6\|\nabla \mu_n(\tau)\|^2 \mathrm{d}\tau \le C, \end{aligned}$$

for each $t \in [0, T_n)$.

Let $\mathbf{v} = \mathbf{u}_n$ in (AP-1). Note that this choice is sensible since $\mathbf{u}_n \in \mathbf{V}_n$ by construction. We have that

$$\begin{aligned} \int_\Omega \partial_t\left(\mathbf{u}_n(t)\right) \cdot \mathbf{u}_n(t)\,\mathrm{d}x + \underbrace{b(\mathbf{u}_n(t), \mathbf{u}_n(t), \mathbf{u}_n(t))}_{=\,0;\ (\mathrm{T1})} + \int_\Omega 2\nu(\psi_n(t))|D(\mathbf{u}_n(t))|^2\,\mathrm{d}x \\ = -(\psi_n(t)\nabla \mu_n(t), \mathbf{u}_n(t)), \end{aligned}$$

which can be simplified to

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\|\mathbf{u}_n\|^2 + \int_\Omega 2\nu(\psi_n)|D(\mathbf{u}_n)|^2\,\mathrm{d}x = -(\psi_n\nabla\mu_n, \mathbf{u}_n),\ \forall t \in [0, T_n). \tag{5.14}$$

Next, testing (AP-3) with $w = \partial_t\psi_n$ entails

$$\int_\Omega \mu_n(t)\partial_t\psi_n(t)\,\mathrm{d}x = (\psi_n^3(t), \partial_t\psi_n(t)) + ((r + (1+\Delta)^2)\psi_n(t), \partial_t\psi_n(t)) = \frac{\mathrm{d}}{\mathrm{dt}}\mathcal{F}_{sh}[\psi_n(t)]. \tag{5.15}$$

> **Remark:** Taking $w = \partial_t\psi_n$ is permissible since $\psi_n \in W_n$ by definition of the Galerkin expansion and $\partial_t\psi_n(t) = \sum_{k=1}^n \dot{\mathbf{B}}_k^n(t)w_k$ also belongs to W_n (since this is still a linear combination of the eigenfunctions w_k for $k = 1, ..., n$).

Taking $w = \mu_n$ in (AP-2) additionally yields

$$\int_\Omega \partial_t\psi_n\mu_n\,\mathrm{d}x + ((\mathbf{u}_n\cdot\nabla\psi_n), \mu_n) + (M(\psi_n)\nabla\mu_n, \nabla\mu_n) = 0. \tag{5.16}$$

Exploiting (5.15), the above equation simplifies to

$$\frac{\mathrm{d}}{\mathrm{dt}}\mathcal{F}_{sh}[\psi_n] + ((\mathbf{u}_n\cdot\nabla\psi_n), \mu_n) + (M(\psi_n)\nabla\mu_n, \nabla w) = 0. \tag{5.17}$$

Adding (5.14) and (5.17) together gives

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\frac{1}{2}\|\mathbf{u}_n\|^2 + \mathcal{F}_{sh}[\psi_n]\right) + \int_\Omega (2\nu(\psi_n)|D(\mathbf{u}_n)|^2 + M(\psi_n)|\nabla\mu_n|^2)\,\mathrm{d}x = 0,\ \ \forall\ t \in [0, T_n).$$

Integrating with respect to time gives us

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_n(t)\|^2 + \mathcal{F}_{sh}[\psi_n(t)] + \int_0^t\int_\Omega (2\nu(\psi_n(\tau))|D(\mathbf{u}_n(\tau))|^2 + M(\psi_n(\tau))|\nabla\mu_n(\tau)|^2)\,\mathrm{d}x\mathrm{d}\tau \\ = \frac{1}{2}\|\mathbf{u}_n(0)\|^2 + \mathcal{F}_{sh}[\psi_n(0)],\ \ \forall\, t \in [0, T_n).\end{aligned} \tag{E2}$$

Now we need to bound $\frac{1}{2}\|\mathbf{u}_n(0)\|^2 + \mathcal{F}_{sh}[\psi_n(0)]$ by a constant independent of n. To this end, firstly note that using Parseval's identity we have

$$\|\mathbf{u}_n(0)\|^2 = \left\|\sum_{k=1}^n (\mathbf{u}_0, \mathbf{v}_k)\mathbf{v}_k\right\|^2 = \sum_{k=1}^n |(\mathbf{u}_0, \mathbf{v}_k)|^2 \le \sum_{k=1}^\infty |(\mathbf{u}_0, \mathbf{v}_k)|^2 = \|\mathbf{u}_0\|^2, \tag{5.18}$$

and similarly

$$\|\psi_n(0)\|_{H^2}^2 \le \|\psi_0\|_{H^2}^2. \tag{5.19}$$

These inequalities allow us to estimate

$$\begin{aligned}&\frac{1}{2}\|\mathbf{u}_n(0)\|^2 + \int_\Omega \frac{1}{4}\psi_n^4(0) + \frac{1}{2}\psi_n(0)(r + (1+\Delta)^2)\psi_n(0)\,\mathrm{d}x \\ &\le \frac{1}{2}\|\mathbf{u}_0\|^2 + \frac{1}{4}\int_\Omega \psi_n^4(0)\,\mathrm{d}x + \frac{(r+1)}{2}\int_\Omega \psi_n^2(0)\,\mathrm{d}x + \int_\Omega \psi_n(0)\Delta\psi_n(0)\,\mathrm{d}x + \frac{1}{2}\int_\Omega \psi_n(0)\Delta^2\psi_n(0)\,\mathrm{d}x\end{aligned} \tag{1}$$

Integrating by parts in the latter two integrals and using the fact that $r < 0$ gives us

$$\begin{aligned}(1) &\le \frac{1}{2}\|\mathbf{u}_0\|^2 + \frac{1}{4}\|\psi_n(0)\|_{L^4(\Omega)}^4 + \frac{1}{2}\|\psi_n(0)\|^2 - \|\nabla\psi_n(0)\|^2 + \frac{1}{2}\|\Delta\psi_n(0)\|^2 \\ &\le \frac{1}{2}\|\mathbf{u}_0\|^2 + \frac{1}{4}\|\psi_n(0)\|_{L^4(\Omega)}^4 + \frac{1}{2}\|\psi_n(0)\|^2 + \frac{1}{2}\|\Delta\psi_n(0)\|^2.\end{aligned}$$

Recall that in three dimensions we have the Sobolev Embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ for $p \in (1, 6]$, so using (5.18) and (5.19) the following inequalities hold:

- $\|\psi_n(0)\|_{L^4}^4 \leq C_s^4 \|\psi_n(0)\|_{H^1}^4 \leq C_s^4 \|\psi_n(0)\|_{H^2}^4 \leq C_s^4 \|\psi_0\|_{H^2}^4$,
- $\|\Delta\psi_n(0)\|_{L^2}^2 \leq \|\psi_n(0)\|_{H^2}^2 \leq \|\psi_0\|_{H^2}^2$,
- $\|\psi_n(0)\|^2 \leq \|\psi_n(0)\|_{H^2}^2 \leq \|\psi_0\|_{H^2}^2$.

Applying the above three estimates gives us

$$\frac{1}{2}\|\mathbf{u}_n(0)\|^2 + \mathcal{F}_{sh}[\psi_n(0)] \leq \frac{1}{2}\|\mathbf{u}_0\|^2 + \frac{1}{4}C_s^4\|\psi_0\|_{H^2}^4 + \frac{1}{2}\|\psi_0\|_{H^2}^2 + \frac{1}{2}\|\psi_0\|_{H^2}^2 =: C(\Omega, \psi_0, \mathbf{u}_0). \quad (5.20)$$

Finally we need to bound the expression $\frac{1}{2}\|\mathbf{u}_n(t)\|^2 + \mathcal{F}_{sh}[\psi_n(t)]$ from below by an expression involving the norms of $\mathbf{u}_n$ and ψ_n in appropriate spaces (to be determined). In a similar way to above, we can rewrite the free-energy term $\mathcal{F}_{sh}[\psi_n(t)]$ in the following way:

$$\begin{aligned}\mathcal{F}_{sh}[\psi_n(t)] &= \int_\Omega \frac{1}{4}\psi_n^4 + \frac{1}{2}\psi_n(r + (1+\Delta)^2)\psi_n \,\mathrm{d}x \\ &= \frac{1}{4}\|\psi_n\|_{L^4}^4 \;+\; \frac{(r+1)}{2}\|\psi_n\|^2 \;+\; \underbrace{\int_\Omega \psi_n\Delta\psi_n \,\mathrm{d}x}_{\textbf{(D)}} \;+\; \frac{1}{2}\|\Delta\psi_n\|^2 \end{aligned} \quad (5.21)$$

We can estimate **(D)** from below using Holder and Young's inequalities:

$$\left|\int_\Omega \psi_n\Delta\psi_n \,\mathrm{d}x\right| \leq \|\psi_n\|\|\Delta\psi_n\| \leq \frac{1}{4}\|\Delta\psi_n\|^2 + \|\psi_n\|^2$$

and therefore

$$\int_\Omega \psi_n\Delta\psi_n \,\mathrm{d}x \geq -\frac{1}{4}\|\Delta\psi_n\|^2 - \|\psi_n\|^2.$$

This gives us the following bound:

$$\begin{aligned}\mathcal{F}_{sh}[\psi_n(t)] &\geq \frac{1}{4}\|\psi_n\|_{L^4}^4 + \frac{(r+1)}{2}\|\psi_n\|^2 - \frac{1}{4}\|\Delta\psi_n\|^2 - \|\psi_n\|^2 + \frac{1}{2}\|\Delta\psi_n\|^2 \\ &= \frac{1}{4}\|\psi_n\|_{L^4}^4 + \frac{(r+1)}{2}\|\psi_n\|^2 + \frac{1}{4}\|\Delta\psi_n\|^2 - \|\psi_n\|^2 \end{aligned} \quad (5.22)$$

In order to involve the H^2 norm in our bound we can recall that as a consequence of the regularity theory of the Laplacian (see (LR-1)), the function

$$u \mapsto (\|u\|^2 + \|\Delta u\|^2)^{\frac{1}{2}}$$

defines a norm on $H^2_N(\Omega)$ equivalent to the standard norm. This implies that there exists a constant $c > 0$ such that

$$c\|u\|_{H^2(\Omega)}^2 \leq \|u\|^2 + \|\Delta u\|^2. \quad \text{(E)}$$

Moreover, (E) gives us that

$$\frac{1}{2}\|\psi_n\|^2 + \frac{1}{4}\|\Delta\psi_n\|^2 = \frac{1}{4}\|\psi_n\|^2 + \frac{1}{4}\|\psi_n\|^2 + \frac{1}{4}\|\Delta\psi_n\|^2 \overset{(E)}{\geq} \frac{1}{4}\|\psi_n\|^2 + \frac{c}{4}\|\psi_n\|_{H^2(\Omega)}^2.$$

Utilising the above inequality in (5.22) gives us

$$\mathcal{F}_{sh}[\psi_n(t)] \geq \frac{1}{4}\|\psi_n\|_{L^4}^4 + \frac{r}{2}\|\psi_n\|^2 + \frac{c}{4}\|\psi_n\|_{H^2}^2 - \|\psi_n\|^2 + \frac{1}{4}\|\psi_n\|^2. \quad (5.23)$$

Next, we bound the norms with negative coefficients ($\frac{r}{2}\|\psi_n\|^2$ and $-\|\psi_n\|^2$) from above. This is crucial as otherwise we will not be able to establish the boundedness of our functions in any spaces from this estimate. To this end, note that by Young's inequality we have

$$\|\psi_n\|^2 = \int_\Omega |\psi_n|^2 \,\mathrm{d}x \leq \frac{1}{2\epsilon}\int_\Omega |\psi_n|^4 \,\mathrm{d}x + \frac{\epsilon}{2}|\Omega| \quad (5.24)$$

Using $\epsilon = 8$ rewards us with

$$-\|\psi_n\|^2 \geq -\frac{1}{16}\|\psi_n\|_{L^4}^4 - 4|\Omega|.$$

Another application of (5.25) with $\epsilon = -4r$ (> 0 since $r < 0$) yields

$$r\|\psi_n\|^2 \geq -\frac{1}{8}\|\psi_n\|_{L^4}^4 - 2r^2|\Omega|.$$

The above two identities together imply

$$\begin{aligned}\frac{1}{4}\|\psi_n\|_{L^4}^4 - \|\psi_n\|^2 + \frac{r}{2}\|\psi_n\|^2 &\geq \left(\frac{1}{4} - \frac{1}{16} - \frac{1}{16}\right)\|\psi_n\|_{L^4}^4 - (4+r^2)|\Omega| \\ &= \frac{1}{8}\|\psi_n\|_{L^4}^4 - (4+r^2)|\Omega|. \end{aligned} \tag{5.25}$$

Implementing (5.25) into (5.22) gives way to the following bound:

$$\mathcal{F}_{sh}[\psi_n(t)] \geq \frac{1}{8}\|\psi_n\|_{L^4}^4 + \frac{c}{4}\|\psi_n\|_{H^2}^2 + \frac{1}{4}\|\psi_n\|^2 - (4+r^2)|\Omega|. \tag{FE-UB}$$

Returning to (E2) and applying (FE-UB) and (5.20) yields the following estimate:

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_n(t)\|^2 + \frac{1}{8}\|\psi_n(t)\|_{L^4}^4 - (4+r^2)|\Omega| + \frac{c}{4}\|\psi_n(t)\|_{H^2}^2 + \frac{1}{4}\|\psi_n(t)\|^2 \\ + \int_0^t\int_\Omega (2\nu(\psi_n(\tau))|D(\mathbf{u}_n(\tau))|^2 + M(\psi_n(\tau))|\nabla\mu_n(\tau)|^2)\,\mathrm{d}x\mathrm{d}\tau \leq C(\Omega, \mathbf{u}_0, \psi_0),\end{aligned}$$

for each $t \in [0, T_n)$. Recalling Korn's inequality and our assumptions on the viscosity and mobility functions:

- Korn's inequality: $\|\nabla\mathbf{u}\| \leq \sqrt{2}\|D\mathbf{u}\|$, $\forall\ u \in \mathbf{V}_\sigma$, and
- that there exist constants ν_*, ν^*, M^*, M_* such that $0 < \nu_* < \nu(s) < \nu^*,\ 0 < M_* < M(s) < M^*$, for each $s \in \mathbb{R}$,

we deduce that for each $t \in [0, T_n)$,

$$\begin{aligned}&\int_0^t\int_\Omega (2\nu(\psi_n(\tau))|D(\mathbf{u}_n(\tau))|^2 + M(\psi_n(\tau))|\nabla\mu_n(\tau)|^2)\,\mathrm{d}x\mathrm{d}\tau \geq \\ &\int_0^t\int_\Omega (2\nu_*|\nabla\mathbf{u}_n(\tau))|^2 + M_*|\nabla\mu_n(\tau)|^2)\,\mathrm{d}x\mathrm{d}\tau = \int_0^t 2\nu_*\|\nabla\mathbf{u}_n(\tau)\|^2 + M_*\|\nabla\mu_n(\tau)\|^2\mathrm{d}\tau.\end{aligned}$$

This gives us

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_n(t)\|^2 + \frac{1}{8}\|\psi_n(t)\|_{L^4}^4 - (4+r^2)|\Omega| + \frac{c}{4}\|\psi_n(t)\|_{H^2}^2 + \frac{1}{4}\|\psi_n(t)\|^2 \\ + \int_0^t 2\nu_*\|\nabla\mathbf{u}_n(\tau)\|^2 + M_*\|\nabla\mu_n(\tau)\|^2\mathrm{d}\tau \leq C(\Omega, \mathbf{u}_0, \psi_0),\ \forall\ t \in [0, T_n).\end{aligned}$$

Collating the constant term $(4+r^2)|\Omega|$ with the constant on the RHS rewards us with

$$\begin{aligned}\frac{1}{2}\|\mathbf{u}_n(t)\|^2 + \frac{1}{8}\|\psi_n(t)\|_{L^4}^4 + \frac{c}{4}\|\psi_n(t)\|_{H^2}^2 + \frac{1}{4}\|\psi_n(t)\|^2 \\ + \int_0^t 2\nu_*\|\nabla\mathbf{u}_n(\tau)\|^2 + M_*\|\nabla\mu_n(\tau)\|^2\mathrm{d}\tau \leq C'(\Omega, \mathbf{u}_0, \psi_0),\end{aligned} \tag{E2}$$

for each $t \in [0, T_n)$.

Remark:
The above constant C' is independent of n and T. This means that our solution $(\mathbf{u}_n, \psi_n, \mu_n)$ can be extended to the full interval $[0, T]$ and thus the goal of this subsection has been attained.

5.4 Step 4: Further estimates

We need to carry out further estimates to assert the boundedness of $(\mathbf{u}_n, \psi_n)$ in appropriate spaces. The estimates we establish here will be crucial for the next step, where they will allow us to use compactness results when passing to the limit in the approximating problem and weak formulation.

Estimate 3

Goal: Show that $\int_0^T \|\mu_n(\tau)\|_{H^1}^2 \mathrm{d}\tau$ can be bounded by a constant independent of n.

Taking $w = 1$ in

$$\boxed{\text{(AP-3)}: \ (\mu_n, w) = (\psi_n^3, w) + ((r+1)\psi_n, w) + (2\Delta\psi_n, w) + (\Delta\psi_n, \Delta w), \ \forall\, w \in W_n.}$$

yields:

$$\begin{aligned}\int_\Omega \mu_n(t)\,\mathrm{d}x &= \int_\Omega \psi_n^3\,\mathrm{d}x + (r+1)\int_\Omega \psi_n\,\mathrm{d}x \\ &= \int_\Omega \psi_n^3\,\mathrm{d}x + (r+1)\int_\Omega \psi_0\,\mathrm{d}x,\end{aligned}$$

where we have used Estimate 1 to achieve the second equality. Utilising the Sobolev embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ and (E2) grants us

$$\begin{aligned}\left|\int_\Omega \mu_n\,\mathrm{d}x\right| &\le \|\psi_n\|_{L^3}^3 + |r+1|\|\psi_0\|_{L^1} \le C^3\|\psi_n\|_{H^1}^3 + C'|r+1|\|\psi_0\| \\ &\le C^3\|\psi_n\|_{H^2}^3 + |r+1|\|\psi_0\| \le C''(\Omega, \psi_0), \ \forall\, t \in [0, T_n).\end{aligned} \tag{5.26}$$

We will use (5.26) to control $\int_0^t \|\mu_n(\tau)\|_{H^1(\Omega)}^2 \mathrm{d}\tau$. Firstly,

$$\int_0^t \|\mu_n(\tau)\|_{H^1}^2 \mathrm{d}\tau = \int_0^\tau \|\mu_n(\tau)\|^2 \mathrm{d}\tau + \underbrace{\int_0^t \|\nabla\mu_n(\tau)\|^2 \mathrm{d}\tau}_{\le C'(\Omega, \mathbf{u}_0, \psi_0) \text{ by (E2)}}.$$

Defining $\overline{\mu_n(\tau)} := \frac{1}{|\Omega|}\int_\Omega \mu_n(\tau)\,\mathrm{d}x$, we have

$$\begin{aligned}\int_0^\tau \|\mu_n(\tau)\|^2 \mathrm{d}\tau &= \int_0^t \|\mu_n(\tau) - \overline{\mu_n(\tau)} + \overline{\mu_n(\tau)}\|^2 \mathrm{d}\tau \le \int_0^t (\|\mu_n(\tau) - \overline{\mu_n(\tau)}\| + \|\overline{\mu_n(\tau)}\|)^2 \mathrm{d}\tau \\ &= \int_0^t \|\mu_n(\tau) - \overline{\mu_n(\tau)}\|^2 \mathrm{d}\tau + \int_0^t 2\|\mu_n(\tau) - \overline{\mu_n(\tau)}\|\|\overline{\mu_n(\tau)}\| \mathrm{d}\tau + \int_0^t \|\overline{\mu_n(\tau)}\|^2 \mathrm{d}\tau\end{aligned}$$

Using the Poincare-Wirtinger ((PW)) and Young's inequalities gives us

$$\begin{aligned}\int_0^t \|\mu_n(\tau)\|^2 \mathrm{d}\tau &\le C^2 \int_0^t \|\nabla\mu_n(\tau)\|^2 \mathrm{d}\tau + \int_0^t \|\mu_n(\tau) - \overline{\mu_n(\tau)}\|^2 \mathrm{d}\tau + \int_0^t \|\overline{\mu_n(\tau)}\|^2 \mathrm{d}\tau + \int_0^t \|\overline{\mu_n(\tau)}\|^2 \mathrm{d}\tau \\ &\le C(\Omega, \mathbf{u}_0, \psi_0) + C^2 \underbrace{\int_0^t \|\nabla\mu_n(\tau)\|^2 \mathrm{d}\tau}_{\le C(\Omega, \mathbf{u}_0, \psi_0);\ \text{due to (E2)}} + 2\int_0^t \|\overline{\mu_n(\tau)}\|^2 \mathrm{d}\tau, \ \forall\, t \in [0, T].\end{aligned}$$

Now,

$$\|\overline{\mu_n(\tau)}\|^2 \equiv \|\overline{\mu_n(\tau)}\|^2_{L^2(\Omega)} = \|\frac{1}{\Omega}\int_\Omega \mu_n(\tau)\,\mathrm{d}x\|^2 = \frac{1}{\Omega}\left|\int_\Omega \mu_n(\tau)\right|^2 \le C(\Omega,\psi_0),\ \forall\, t\in[0,T],$$

which implies

$$\int_0^T \|\mu_n(\tau)\|^2_{H^1}\mathrm{d}\tau \le K(\Omega,\mathbf{u}_0,\psi_0,T). \tag{E3}$$

Remark: Our estimates so far allow us to deduce that

- $\mathbf{u}_n$ is bounded in $L^\infty(0,T;\mathbf{H}_\sigma)\cap L^2(0,T;\mathbf{V}_\sigma)$,
- ψ_n is bounded in $L^\infty(0,T;L^4(\Omega))\cap L^\infty(0,T;H^2(\Omega))$,
- μ_n is bounded in $L^2(0,T;H^1(\Omega))$.

It remains to show that $\partial_t\mathbf{u}_n$ and $\partial_t\psi_n$ are bounded in appropriate spaces!

Estimate 4

Goal: Show that $\int_0^T \|\partial_t\psi_n(\tau)\|^2_{(H^1)^*}\mathrm{d}\tau$ can be bounded by a constant independent of n.

Consider the second equation in the approximating problem (AP), namely

$$\boxed{\text{(AP-2): } \langle\partial_t\psi_n, w\rangle_{(H^1)^*\times H^1} + ((\mathbf{u}_n\cdot\nabla\psi_n), w) + (M(\psi_n)\nabla\mu_n,\nabla w) = 0,\ \forall\, w\in W_n.}$$

We begin with the following estimates. Let $w\in W_n$. Then

$$|(\mathbf{u}_n\cdot\nabla\psi_n), w)| = \left|\int_\Omega(\mathbf{u}_n\cdot\nabla\psi_n)\cdot w\,\mathrm{d}x\right| \le \|\mathbf{u}_n\|_{L^3}\|\nabla\psi_n\|\|w\|_{L^6} \le C_s\|\mathbf{u}_n\|_{L^3}\|\nabla\psi_n\|\|w\|_{H^1},$$

where C_s is the constant originating from the Sobolev embedding $H^1(\Omega)\hookrightarrow L^6(\Omega)$. Applying the interpolation inequality (GN) gives us

$$|(\mathbf{u}_n\cdot\nabla\psi_n), w)| \le C\|\mathbf{u}_n\|^{\frac{1}{2}}\|\mathbf{u}_n\|^{\frac{1}{2}}_{H^1}\|\psi_n\|_{H^2}\|w\|_{H^1}. \tag{5.27}$$

Also,

$$|(M(\psi_n)\nabla\mu_n,\nabla w)| = |\int_\Omega M(\psi_n)\nabla\mu_n\cdot\nabla w\,\mathrm{d}x| \le M^*\|\nabla\mu_n\|\|w\|_{H^1}. \tag{5.28}$$

Now let $w\in H^1(\Omega)$. Then w can be expressed as $w = y+z$ for some $y\in W_n,\ z\in W_n^\perp$, where $W_n^\perp$ is the orthogonal complement of the subspace W_n. Note that this representation holds true because $H^1(\Omega)$ (with the usual inner product structure) is a Hilbert space and W_n is a closed linear subspace of $H^1(\Omega)$. It follows from elementary functional analysis that the space $H^1(\Omega)$ can be represented as $W_n\oplus W_n^\perp$, from which the above representation follows. Then,

$$\begin{aligned}|\langle\partial_t\psi_n, w\rangle_{(H^1)^*\times(H^1)}| &= |(\partial_t\psi_n, w)| = |(\partial_t\psi_n, y)| = |-((\mathbf{u}_n\cdot\nabla\psi_n), y) - (M(\psi_n)\nabla\mu_n,\nabla y)|\\ &\le C\|\mathbf{u}_n\|^{\frac{1}{2}}\|\mathbf{u}_n\|^{\frac{1}{2}}_{H^1}\|\psi_n\|_{H^2}\|y\|_{H^1} + M^*\|\nabla\mu_n\|\|y\|_{H^1}\\ &\le \left(C\|\mathbf{u}_n\|^{\frac{1}{2}}\|\mathbf{u}_n\|^{\frac{1}{2}}_{H^1}\|\psi_n\|_{H^2} + M^*\|\nabla\mu_n\|\right)\|w\|_{H^1},\end{aligned}$$

where we have made use of (5.27), (5.28) and the fact $\|y\|_{H^1}\le\|w\|_{H^1}$. Dividing through by $\|w\|_{H^1}$ and squaring both sides yields:

$$\begin{aligned}\left(\frac{|\langle\partial_t\psi_n, w\rangle_*|}{\|w\|_{H^1}}\right)^2 &\le (C\|\mathbf{u}_n\|^{\frac{1}{2}}\|\mathbf{u}_n\|^{\frac{1}{2}}_{H^1}\|\psi_n\|_{H^2} + M^*\|\nabla\mu_n\|)^2\\ &= 2C^2\|\mathbf{u}_n\|\|\mathbf{u}_n\|_{H^1}\|\psi_n\|^2_{H^2} + 2(M^*)^2\|\nabla\mu_n\|^2|,\end{aligned}$$

where we have used Young's inequality $(a+b)^2 \le 2a^2 + 2b^2$ to achieve the above equality. By definition of the operator norm we have that

$$\|\partial_t \psi_n\|^2_{(H^1)^*} = \sup_{0\neq w \in H^1(\Omega)} \left(\frac{|\langle \partial_t \psi_n, w\rangle_*|}{\|w\|_{H^1}} \right)^2 ,$$

and so taking the appropriate supremum over the above inequality gives us

$$\|\partial_t \psi_n\|^2_{(H^1)^*} \le C^2 \|\mathbf{u}_n\| \|\mathbf{u}_n\|_{H^1} \|\psi_n\|^2_{H^2} + (M^*)^2 \|\nabla \mu_n\|^2. \tag{5.29}$$

Integrating with respect to time from 0 to $t \in [0,T]$ results in

$$\int_0^t \|\partial_t \psi_n(\tau)\|^2_{(H^1)^*} \mathrm{d}\tau \le C^2 \underbrace{\int_0^t \|\mathbf{u}_n\| \|\mathbf{u}_n\|_{H^1} \|\psi_n\|^2_{H^2} \mathrm{d}\tau}_{\textbf{(E4A)}} + M^* \underbrace{\int_0^t \|\nabla \mu_n(\tau)\|^2 \mathrm{d}\tau}_{\textbf{(E4B)}} .$$

Using the inequalities due to Young and Holder as well as the trivial estimate $\|\mathbf{u}_n\| \le \|\mathbf{u}_n\|_{H^1}$, we deduce the following estimates:

$$\textbf{(E4A)} \le \sup_{\tau\in[0,T]} \|\psi_n(\tau)\|^2_{H^2} \int_0^t \|\mathbf{u}_n\|^2_{H^1} \mathrm{d}\tau = \|\psi_n\|^2_{L^\infty(0,T;H^2(\Omega))} \|\mathbf{u}_n\|^2_{L^2(0,T;H^1(\Omega))} < K_1(\Omega, \mathbf{u}_0, \psi_0, T),$$

since $\mathbf{u}_n, \psi_n$ are bounded in $L^2(0,T;\mathbf{V}_\sigma)$ and in $L^\infty(0,T;H^2(\Omega))$ respectively.

$$\textbf{(E4B)} \le M^* \|\mu_n\|^2_{L^2(0,T;H^1(\Omega))} < K_2(\Omega, \mathbf{u}_0, \psi_0, T),$$ since μ_n is bounded in $L^2(0,T;H^1(\Omega))$.

All in all, we find

$$\int_0^t \|\partial_t \psi_n(\tau)\|^2_{(H^1)^*} \mathrm{d}\tau \le K(\Omega, \mathbf{u}_0, \psi_0, T), \tag{E4}$$

implying that $\partial_t \psi_n$ is bounded in $L^2(0,T;(H^1)^*)$.

Estimate 5

Goal: Show that $\int_0^T \|\partial_t \mathbf{u}_n(\tau)\|^{\frac{4}{3}} \, \mathrm{d}\tau$ can be bounded by a constant independent of n.

Let $\mathbf{w} \in \mathbf{V}_\sigma$. Then $\mathbf{w} = \mathbf{y} + \mathbf{z}$ for some $\mathbf{y} \in \mathbf{V}_n$, $\mathbf{z} \in \mathbf{V}_n^\perp$. Consider the first equation in the approximating problem, namely

$$\boxed{\textbf{(AP-1):}\ \langle \partial_t \mathbf{u}_n(t), \mathbf{v}\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}) + (2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{v}) = -(\psi_n \nabla \mu_n, \mathbf{v}),\ \forall\ \mathbf{v} \in \mathbf{V}_n.}$$

We can estimate the action of the functional $\partial_t \mathbf{u}_n$ as follows:

$$\begin{aligned} |\langle \partial_t \mathbf{u}_n, \mathbf{w}\rangle_*| = |(\partial_t \mathbf{u}_n, \mathbf{w})| = |(\partial_t \mathbf{u}_n, \mathbf{y})| &= |(\psi_n \nabla \mu_n, \mathbf{y}) - (2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{y}) - b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{y})| \\ &\le |(\psi_n \nabla \mu_n, \mathbf{y})| + |(2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{y})| + |b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{y})| \\ &\le \|\psi_n \nabla \mu_n\| \|\mathbf{y}\| + 2\nu^* \|D(\mathbf{u}_n)\| \|\nabla \mathbf{y}\| + \|\mathbf{u}_n\|_{L^3} \|\nabla \mathbf{u}_n\| \|\mathbf{y}\|_{L^6} && \text{(Holder)} \\ &\le \left(\|\psi_n \nabla \mu_n\| + 2\nu^* \|D(\mathbf{u}_n)\| + \|\mathbf{u}_n\|_{L^3} \|\nabla \mathbf{u}_n\| \right) \|\mathbf{y}\|_{H^1} && (H^1(\Omega) \hookrightarrow L^6(\Omega)) \end{aligned}$$

Following a similar path to the previous estimate, we have

$$\sup_{0\neq \mathbf{w}\in \mathbf{V}_\sigma} \left(\frac{|\langle \partial_t \mathbf{u}_n(t), \mathbf{w}\rangle_*|}{\|\mathbf{w}\|_{\mathbf{V}_\sigma}} \right) \equiv \|\partial_t \mathbf{u}_n(t)\|_{(\mathbf{V}_\sigma)^*} \le \|\mathbf{u}_n\|_{L^3} \|\nabla \mathbf{u}_n\| + 2\nu^* \|D(\mathbf{u}_n)\| + \|\psi_n \nabla \mu_n\|).$$

Raising both sides to the power of 4/3 and using the well-known inequality $|a+b|^p \leq 2^{p-1}\{|a|^p + |b|^p\}$ (for $a, b \in \mathbb{R},\ 1 \leq p < +\infty$) gives us

$$\|\partial_t \mathbf{u}_n(t)\|^{\frac{4}{3}}_{(\mathbf{V}_\sigma)^*} \leq 2^{\frac{2}{3}} \left(\|\mathbf{u}_n\|^{\frac{4}{3}}_{L^3} \|\nabla \mathbf{u}_n\|^{\frac{4}{3}} + (2\nu^*)^{\frac{4}{3}} \|D(\mathbf{u}_n)\|^{\frac{4}{3}} + \|\psi_n \nabla \mu_n\|^{\frac{4}{3}} \right).$$

Integrating with respect to time from 0 to $t \in [0, T]$,

$$\begin{aligned}\int_0^t \|\partial_t \mathbf{u}_n(\tau)\|^{\frac{4}{3}}\, \mathrm{d}\tau &\leq 2^{\frac{2}{3}} \left\{ \int_0^t \|\mathbf{u}_n(\tau)\|^{\frac{4}{3}}_{L^3} \|\nabla \mathbf{u}_n(\tau)\|^{\frac{4}{3}} \mathrm{d}\tau + (2\nu^*)^{\frac{4}{3}} \int_0^t \|D(\mathbf{u}_n(\tau))\|^{\frac{4}{3}} \mathrm{d}\tau + \int_0^t \|\psi_n \nabla \mu_n\|^{\frac{4}{3}}\, \mathrm{d}\tau \right\} \\ &\leq 2^{\frac{2}{3}} \left\{ \int_0^t C_g \|\mathbf{u}_n(\tau)\|^{\frac{2}{3}} \|\mathbf{u}_n(\tau)\|^{\frac{2}{3}}_{H^1} \|\nabla \mathbf{u}_n(\tau)\|^{\frac{4}{3}}\, \mathrm{d}\tau + C_2(\Omega, \mathbf{u}_0, \psi_0) + \int_0^t \|\psi_n \nabla \mu_n\|^{\frac{4}{3}}\, \mathrm{d}\tau \right\},\end{aligned}$$

where the constant C_g appears as a consequence of the (GN) inequality and the second integral is bounded by C_2 due to the estimate (E2). Continuing our estimates, we find

$$\int_0^t \|\mathbf{u}_n(\tau)\|^{\frac{2}{3}} \|\mathbf{u}_n(\tau)\|^{\frac{2}{3}}_{H^1} \|\nabla \mathbf{u}_n(\tau)\|^{\frac{4}{3}}\, \mathrm{d}\tau \leq \sup_{\tau \in [0,T]} \|\mathbf{u}_n(\tau)\|^{\frac{2}{3}} \int_0^t \|\mathbf{u}_n(\tau)\|^2_{H^1}\, \mathrm{d}\tau. \tag{5.30}$$

Using the Holder inequality and the embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$,

$$\int_0^t \|\psi_n \nabla \mu_n\|^{\frac{4}{3}}\, \mathrm{d}\tau \leq \int_0^t \|\psi_n\|^{\frac{4}{3}}_{L^\infty} \|\nabla \mu_n\|^{\frac{4}{3}}\, \mathrm{d}\tau \leq \sup_{\tau \in [0,T]} \|\psi_n(\tau)\|^{\frac{4}{3}}_{H^2} \int_0^t \|\mu_n(\tau)\|^{\frac{4}{3}}_{H^1}\, \mathrm{d}\tau.$$

Now since

- $\mathbf{u}_n$ is bounded in $L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$,
- μ_n is bounded in $L^2(0, T; H^1(\Omega)) \hookrightarrow L^{\frac{4}{3}}(0, T; H^1(\Omega))$,
- ψ_n is bounded in $L^\infty(0, T; H^2(\Omega))$,

we obtain that

$$\int_0^T \|\partial_t \mathbf{u}_n(\tau)\|^{\frac{4}{3}}\, \mathrm{d}\tau \leq C, \tag{E5}$$

where C is a constant depending on $\Omega, \mathbf{u}_0, \psi_0, T$ but not n. We conclude that $\partial_t \mathbf{u}_n$ is bounded in $L^{\frac{4}{3}}(0, T; (\mathbf{V}_\sigma)^*)$.

Remark: The inequality (5.30) explains why we raised the operator norm to the power of $\frac{4}{3}$ and nothing greater. Indeed, the integral term in (5.30) is equivalent to $\|\mathbf{u}_n\|_{L^2(0,T;H^1(\Omega))}$ which we know can be bounded by a constant independent of n. If we had used a power greater than $\frac{4}{3}$ we would instead be left with $\|\mathbf{u}_n\|_{L^r(0,T;H^1(\Omega))}$ for $r > 2$, which is problematic as we do not know if $\mathbf{u}_n$ is bounded in such a space!

Estimate 6

Goal: Show that $\int_0^T \|\psi_n(\tau)\|^2_{H^4}\, \mathrm{d}\tau$ can be bounded by a constant independent of n.

We finally derive a higher order spatial estimate for ψ_n. Considering the following problem

$$\begin{cases} -\Delta^2 \psi_n = \mu_n - \Pi_n(\psi_n^3) - (r+1)\psi_n - 2\Delta\psi_n, & \text{in } \Omega, \\ \nabla \psi_n \cdot \mathbf{n} = \nabla \Delta \psi_n \cdot \mathbf{n} = 0, & \text{on } \partial\Omega, \end{cases}$$

and recalling the regularity theory for the bi-Laplacian problem (see (LR-2)), we have that (since Ω is a C^4 domain) there exists a constant $C > 0$ such that

$$\|\psi_n\|_{H^4(\Omega)} \leq C(\|\Delta^2 \psi_n\| + \|\psi_n\|). \tag{5.31}$$

Proceeding with this inequality, we deduce

$$\begin{aligned}\|\psi_n\|_{H^4(\Omega)} &\leq C(\|\Delta^2\psi_n\| + \|\psi_n\|)\\ &\leq C(\|\mu_n - \Pi_n\left(\psi_n^3\right) - (r+1)\psi_n - 2\Delta\psi_n\| + \|\psi_n\|)\\ &\leq C(\|\mu_n - \overline{\mu_n}\| + |\overline{\mu_n}| + \|\Pi_n(\psi_n)\|_{L^6}^3 + 2\|\psi_n\| + 2\|\Delta\psi_n\|),\end{aligned}$$

where we have added and subtracted $\overline{\mu_n}$ to achieve the last inequality. Note that $\|\Pi_n(\psi_n^3)\| \leq \|\psi_n^3\| = \|\psi_n\|_{L^6}^3$. Using this fact along with the Poincare-Wirtinger inequality ((PW)) gives us

$$\begin{aligned}\|\psi_n\|_{H^4(\Omega)} &\leq C\left(\|\nabla\mu_n\| + |\overline{\mu_n}| + C_s\|\psi_n\|_{H^2}^3 + 2\|\psi_n\|_{H^2} + 2\|\psi_n\|_{H^2}\right)\\ &\leq C_1\|\nabla\mu_n\| + C_2(\Omega,\psi_0,\mathbf{u}_0),\end{aligned}$$

where the constant C_2 arises as a consequence of the bounds previously obtained on ψ_n and μ_n. Squaring and integrating from 0 to T,

$$\int_0^T \|\psi_n(\tau)\|_{H^4}^2\,\mathrm{d}\tau \leq C\int_0^T \|\nabla\mu_n(\tau)\|^2\,\mathrm{d}\tau + C_2^2\int_0^T 1\,\mathrm{d}\tau + 2C_1C_2\int_0^T \|\nabla\mu_n(\tau)\|\,\mathrm{d}\tau.$$

Since

- $L^2(0,T;H^1(\Omega)) \hookrightarrow L^1(0,T;H^1(\Omega))$,
- μ_n is bounded in $L^2(0,T;H^1(\Omega))$,

we conclude that

$$\int_0^T \|\psi_n(\tau)\|_{H^4}^2\,\mathrm{d}\tau \leq K(\Omega,\mathbf{u}_0,\psi_0,T), \tag{E6}$$

which implies that ψ_n is bounded in $L^2(0,T;H^4(\Omega))$.

To summarise the previous six estimates, we have found that for any $T\in(0,\infty)$,

- $\mathbf{u}_n$ is bounded in $L^\infty(0,T;\mathbf{H}_\sigma)\cap L^2(0,T;\mathbf{V}_\sigma)\cap W^{1,\frac{4}{3}}(0,T;(\mathbf{V}_\sigma)^*)$,
- ψ_n is bounded in $L^\infty(0,T;H^2(\Omega))\cap L^2(0,T;H^4(\Omega))$,
- $\partial_t\psi_n$ is bounded in $L^2(0,T;(H^1)^*)$,
- μ_n is bounded in $L^2(0,T;H^1(\Omega))$.

5.5 Step 5: Passing to the limit

The previous section successfully proved that given $T>0$ and $n\in\mathbb{N}$, there exists a unique solution $(\mathbf{u}_n,\psi_n,\mu_n)$ solving the following approximating problem (5.3) on $[0,T]$

$$\begin{cases}\langle\partial_t\mathbf{u}_n(t),\mathbf{v}\rangle_{\mathbf{V}_\sigma^*\times\mathbf{V}_\sigma} + b(\mathbf{u}_n,\mathbf{u}_n,\mathbf{v}) + (2\nu(\psi_n)D(\mathbf{u}_n),\nabla\mathbf{v}) = -(\psi_n\nabla\mu_n,\mathbf{v}), & \forall\ \mathbf{v}\in\mathbf{V}_n, \quad \text{(AP-1)}\\ \langle\partial_t\psi_n,w\rangle_{(H^1)^*\times H^1} + ((\mathbf{u}_n\cdot\nabla\psi_n),w) + (M(\psi_n)\nabla\mu_n,\nabla w) = 0, & \forall\ w\in W_n, \quad \text{(AP-2)}\\ (\mu_n,w) = (\psi_n^3,w) + ((r+1)\psi_n,w) + (2\Delta\psi_n,w) + (\Delta\psi_n,\Delta w), & \forall\ w\in W_n, \quad \text{(AP-3)}\\ \mathbf{u}_n(\cdot,0) = \mathcal{P}_n(\mathbf{u}_0) =: \mathbf{u}_0^n,\ \psi_n(\cdot,0) = \Pi_n(\psi_0) =: \psi_0^n, & \text{in}\ \ \Omega. \quad \text{(AP-4)}\end{cases}$$

In order to complete the proof of existence of a global weak solution, we need to prove:

1. **existence of the limit;** i.e., that there exists a subsequence $(\mathbf{u}_{n_j}, \psi_{n_j}, \mu_{n_j})$ converging to a limit $(\mathbf{u}, \psi, \mu)$,

2. **suitability of the limit;** i.e., that $(\mathbf{u}, \psi, \mu)$ globally satisfies the weak formulation:

$$\begin{cases} \langle \partial_t \mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + (2\nu(\psi) D(\mathbf{u}), \nabla \mathbf{v}) = -(\psi \nabla \mu, \mathbf{v}), & \forall\ \mathbf{v} \in \mathbf{V}_\sigma, \quad \text{(WF-1)} \\ \langle \partial_t \psi, w\rangle_{(H^1)^* \times H^1} + ((\mathbf{u} \cdot \nabla \psi), w) + (M(\psi)\nabla\mu, \nabla w) = 0, & \forall\ w \in H^1(\Omega), \quad \text{(WF-2)} \\ \mu = \psi^3 + (r + (1+\Delta)^2)\psi, & \text{a.e. in } \ \Omega \times [0,T], \quad \text{(WF-3)} \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0, \ \psi(\cdot, 0) = \psi_0, & \text{in } \ \Omega. \quad \text{(WF-4)} \end{cases}$$

We will approach each of these two tasks separately.

5.5.1 Existence of the limit

Firstly, the spaces $L^2(0,T;\mathbf{V}_\sigma)$, $L^2(0,T;H^4(\Omega))$, $L^2(0,T;(H^1)^*)$, $L^2(0,T;H^1(\Omega))$ are all Hilbert which implies that we have subsequences $(\mathbf{u}_{n_j}, \psi_{n_j}, \mu_{n_j})$ of $(\mathbf{u}_n, \psi_n, \mu_n)$ (our solution to (AP)) that converge weakly to some $(\mathbf{u}, \psi, \mu)$, where

- $\mathbf{u} \in L^2(0,T;\mathbf{V}_\sigma)$,
- $\psi \in L^2(0,T;H^4_N(\Omega))$,
- $\partial_t \psi_n \in L^2(0,T;(H^1)^*)$,
- $\mu \in L^2(0,T;H^1(\Omega))$.

Additionally,

$$\mathbf{u}_{n_j} \rightharpoonup \mathbf{u} \text{ in } L^2(0,T;\mathbf{V}_\sigma), \tag{L1}$$

$$\psi_{n_j} \rightharpoonup \psi \text{ in } L^2(0,T;H^4(\Omega)), \tag{L2}$$

$$\partial_t \psi_{n_j} \rightharpoonup \partial_t \psi \text{ in } L^2(0,T;(H^1)^*), \tag{L3}$$

$$\mu_{n_j} \rightharpoonup \mu \text{ in } L^2(0,T;H^1(\Omega)). \tag{L4}$$

Next, noting that

$$L^{\frac{4}{3}}(0,T;(\mathbf{V}_\sigma)^*) = (L^4(0,T;\mathbf{V}_\sigma))^*,$$

$$L^\infty(0,T;H^2(\Omega)) = (L^1(0,T;(H^2(\Omega))^*)^*,$$

$$L^\infty(0,T;\mathbf{H}_\sigma) = (L^1(0,T;(\mathbf{H}_\sigma)^*)^*,$$

and recalling a well known compactness result:

Theorem 5.3: Banach-Alaoglu weak-* compactness theorem ([13], Theorem 4.18)

Let X be a separable Banach space and let $\{f_n\}_{n\geq 1}$ be a bounded sequence in X^*. Then f_n has a weakly-* convergent subsequence.

we additionally infer that

$$\mathbf{u}_{n_j} \overset{*}{\rightharpoonup} \mathbf{u} \text{ in } L^\infty(0,T;\mathbf{H}_\sigma), \tag{L5}$$

$$\partial_t \mathbf{u}_{n_j} \overset{*}{\rightharpoonup} \partial_t \mathbf{u} \text{ in } L^{\frac{4}{3}}(0,T;(\mathbf{V}_\sigma)^*), \tag{L6}$$

$$\psi_{n_j} \overset{*}{\rightharpoonup} \psi \text{ in } L^\infty(0,T;H^2(\Omega)). \tag{L7}$$

Also, we recall the famous Aubin-Lions theorem

Theorem 5.4: Aubin-Lions theorem ([2], Theorem II.5.16)

Let $X_0,\ X,\ X_1$ be three Banach spaces such that $X_0 \overset{c}{\hookrightarrow} X \hookrightarrow X_1$. Then, the set

$$E = \{\mathbf{u} \in L^p(0,T;X_0),\ \partial_t \mathbf{u} \in L^q(0,T;X_1)\}$$

with $1 \leq p,\ q < \infty$ is compactly embedded in $L^p(0,T;X)$. In the case of $p = \infty,\ q > 1$ we have $E \overset{c}{\hookrightarrow} C([0,T];X)$.

and choose X_0, X, X_1, p, q carefully in accordance with this result.

- Taking $X_0 = H^4(\Omega),\ X = H^3(\Omega),\ X_1 = (H^1)^*,\ p = q = 2$ yields that there exists a subsequence $\psi_{n_j} \to \psi$ strongly in $L^2(0,T;H^3(\Omega))$.
- Taking $X_0 = H^2(\Omega),\ X = H^1(\Omega),\ X_1 = (H^1)^*,\ p = \infty,\ q = 2$ yields that there exists a subsequence $\psi_{n_j} \to \psi$ strongly in $C([0,T];H^1(\Omega))$.
- Taking $X_0 = H^1(\Omega),\ X = L^r(\Omega)\ (1 \leq r < 6),\ X_1 = L^2(\Omega),\ p = 2,\ q = 4/3$ yields that there exists a subsequence $\mathbf{u}_{nj} \to \mathbf{u}$ strongly in $L^2(0,T;L^r(\Omega))$ for $1 \leq r < 6$.

All in all, we have the existence of subsequences (denoted by $\mathbf{u}_{n_j}, \psi_{n_j}$ again for convenience) such that:

$$\psi_{n_j} \to \psi \text{ strongly in } C([0,T];H^1(\Omega)) \cap L^2(0,T;H^3(\Omega)), \tag{L8}$$

$$\mathbf{u}_{n_j} \to \mathbf{u} \text{ strongly in } L^2(0,T;L^r(\Omega)),\ \text{for } 1 \leq r < 6. \tag{L9}$$

5.5.2 Suitability of the limit

We now show that our limit $(\mathbf{u}, \psi, \mu)$ also satisfies the weak formulation. It is sufficient to show that for any $\varphi \in D(0,T)$, we have

$$\begin{cases} \displaystyle\int_0^T \Big[\langle \partial_t \mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u},\mathbf{u},\mathbf{v}) + (2\nu(\psi)D(\mathbf{u}), \nabla \mathbf{v}) + (\psi \nabla\mu, \mathbf{v})\Big] \varphi(t)\, \mathrm{d}t = 0, & \forall\ \mathbf{v} \in \mathbf{V}_\sigma, \\ \displaystyle\int_0^T \Big[\langle \partial_t \psi, w\rangle_{(H^1)^* \times H^1} + ((\mathbf{u}\cdot\nabla\psi), w) + (M(\psi)\nabla\mu, \nabla w)\Big] \varphi(t) \mathrm{d}t = 0, & \forall\ w \in H^1(\Omega), \\ \mu = \psi^3 + (r + (1+\Delta)^2)\psi, & \text{a.e. in } \ \Omega \times [0,T], \\ \mathbf{u}(\cdot,0) = \mathbf{u}_0,\ \psi(\cdot,0) = \psi_0, & \text{in } \ \Omega. \end{cases}$$

We know that for sufficiently large n the triple $(\mathbf{u}_n, \psi_n, \mu_n)$ (our solution to (AP)) solves:

$$\begin{cases} \displaystyle\int_0^T \Big[\langle \partial_t \mathbf{u}_n(t), \mathbf{v}\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}) + (2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{v}) + (\psi_n \nabla \mu_n, \mathbf{v})\Big] \varphi(t)\, \mathrm{d}t = 0, & \forall\, \mathbf{v} \in \mathbf{V}_n, \\ \displaystyle\int_0^T \Big[\langle \partial_t \psi_n, w\rangle_{(H^1)^* \times H^1} + ((\mathbf{u}_n \cdot \nabla \psi_n), w) + (M(\psi_n)\nabla \mu_n, \nabla w)\Big] \varphi(t)\, \mathrm{d}t = 0, & \forall\, w \in W_n, \\ (\mu_n, w) = (\psi_n^3, w) + ((r+1)\psi_n, w) + (2\Delta \psi_n, w) + (\Delta \psi_n, \Delta w), & \forall\, w \in W_n, \\ \mathbf{u}_n(\cdot, 0) = \mathcal{P}_n(\mathbf{u}_0) =: \mathbf{u}_0^n,\ \psi_n(\cdot, 0) = \Pi_n(\psi_0) =: \psi_0^n, & \text{in } \Omega. \end{cases}$$

Our next step is to prove nine items, which we label as **Z1**-**Z9**. We will then use these nine items to complete the proof. Firstly, we will show the existence of a subsequence $\{(\mathbf{u}_{n_j}, \psi_{n_j}, \mu_{n_j})\}_{j\geq 1}$ such that for any $s \in \{1, ..., n_j\}$ and $\varphi \in D(0,T)$, we have that as $j \to \infty$:

$$\int_0^T \langle \partial_t \mathbf{u}_{n_j}(t), \mathbf{v}_s\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma}\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T \langle \partial_t \mathbf{u}(t), \mathbf{v}_s\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma}\, \varphi(t)\, \mathrm{d}t, \qquad \boxed{\textbf{Z1}}$$

$$\int_0^T ((\mathbf{u}_{n_j} \cdot \nabla)\mathbf{u}_{n_j}, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t, \qquad \boxed{\textbf{Z2}}$$

$$\int_0^T (2\nu(\psi_{n_j}) D(\mathbf{u}_{n_j}), \nabla \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T (2\nu(\psi) D(\mathbf{u}), \nabla \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t, \qquad \boxed{\textbf{Z3}}$$

$$\int_0^T (\psi_{n_j} \nabla \mu_{n_j}, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T (\psi \nabla \mu, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t, \qquad \boxed{\textbf{Z4}}$$

$$\int_0^T \langle \partial_t \psi_{n_j}, w_s\rangle_{(H^1)^* \times H^1}\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T \langle \partial_t \psi, w_s\rangle_{(H^1)^* \times H^1}\, \varphi(t)\, \mathrm{d}t, \qquad \boxed{\textbf{Z5}}$$

$$\int_0^T ((\mathbf{u}_{n_j} \cdot \nabla \psi_{n_j}), w_s)\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T ((\mathbf{u} \cdot \nabla \psi), w_s)\, \varphi(t)\, \mathrm{d}t, \qquad \boxed{\textbf{Z6}}$$

$$\int_0^T (M(\psi_{n_j}) \nabla \mu_{n_j}, \nabla w_s)\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T (M(\psi) \nabla \mu, \nabla w_s)\, \varphi(t)\, \mathrm{d}t, \qquad \boxed{\textbf{Z7}}$$

$$\mu = \psi^3 + (r + (1+\Delta)^2)\psi, \quad \text{a.e. in } \Omega \times [0,T], \qquad \boxed{\textbf{Z8}}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0,\ \psi(\cdot, 0) = \psi_0, \ \text{ in } \Omega. \qquad \boxed{\textbf{Z9}}$$

In what follows we fix the arbitrary elements $\varphi \in D(0,T)$, $\mathbf{v}_s \in V_{n_j}$, $w_s \in W_{n_j}$.

$\boxed{\textbf{Z1}}$ — **Goal:** Show that as $j \to \infty$,

$$\int_0^T \langle \partial_t \mathbf{u}_{n_j}(t), \mathbf{v}_s\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma}\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T \langle \partial_t \mathbf{u}(t), \mathbf{v}_s\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma}\, \varphi(t)\, \mathrm{d}t.$$

(L6) implies that

$$\int_0^T \langle \partial_t \mathbf{u}_{n_j}, \mathbf{v}\rangle_{(\mathbf{V}_\sigma)^* \times \mathbf{V}_\sigma}\, \mathrm{d}t \longrightarrow \int_0^T \langle \partial_t \mathbf{u}, \mathbf{v}\rangle_{(\mathbf{V}_\sigma)^* \times \mathbf{V}_\sigma}\, \mathrm{d}t, \quad \text{as } j \to \infty,\ \forall\, v \in L^4(0,T; \mathbf{V}_\sigma).$$

Since $\mathbf{v}_s \varphi \in L^\infty(0,T; \mathbf{V}_\sigma) \hookrightarrow L^4(0,T; \mathbf{V}_\sigma)$, the result follows.

$\boxed{\textbf{Z2}}$ — **Goal:** Show that as $j \to \infty$,

$$\int_0^T ((\mathbf{u}_{n_j} \cdot \nabla)\mathbf{u}_{n_j}, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T ((\mathbf{u} \cdot \nabla)\mathbf{u}, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t.$$

We follow the technique seen in ([2], Theorem V.1.4). Firstly, we can show that $\{(\mathbf{u}_{n_j} \cdot \nabla)\mathbf{u}_{n_j}\}_j$ is bounded in $L^{\frac{4}{3}}(0,T;[L^{\frac{6}{5}}(\Omega)]^3)$. Observe that

$$\|(\mathbf{u}_{n_j} \cdot \nabla)\mathbf{u}_{n_j}\|^{\frac{6}{5}}_{L^{\frac{6}{5}}(\Omega)} = \int_\Omega \sum_{k=1}^{3} |\mathbf{u}^l_{n_j}\partial_l \mathbf{u}^k_{n_j}|^{\frac{6}{5}}\,\mathrm{d}x,$$

where $\mathbf{u}^l_{n_j}$ is used to denote the l'th component of $\mathbf{u}_{n_j}$ and the Einstein summation convention has been adopted. For each $k, j \in \{1,2,3\}$, we have

$$\int_\Omega |\mathbf{u}^k_{n_j}\partial_k \mathbf{u}^j_{n_j}|^{\frac{6}{5}}\,\mathrm{d}x \le \sup_{x\in\Omega} |\mathbf{u}^k_{n_j}|^{\frac{6}{5}} \int_\Omega |\partial_k \mathbf{u}^j_{n_j}|^{\frac{6}{5}}\,\mathrm{d}x \le C,$$

where C is independent of j, since $\mathbf{u}_n$ is bounded in $L^\infty(0,T;\mathbf{H}_\sigma)$ and $L^2(0,T;\mathbf{V}_\sigma) \hookrightarrow L^{\frac{6}{5}}(0,T;\mathbf{V}_\sigma)$. This is sufficient to show that $\|(\mathbf{u}_{n_j}\cdot\nabla)\mathbf{u}_{n_j}\|^{\frac{6}{5}}_{L^{\frac{6}{5}}(\Omega)} \le C$ where C is independent of j and thus we deduce in particular that

$$\|(\mathbf{u}_{n_j} \cdot \nabla)\mathbf{u}_{n_j}\|_{L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}\Omega)} \le C,$$

where by abuse of notation C is again used to denote a constant independent of n. Thus, we have that $(\mathbf{u}_{n_j} \cdot \nabla)\mathbf{u}_{n_j}$ is bounded in $L^{\frac{4}{3}}(0,T;L^{\frac{6}{5}}\Omega) = (L^4(0,T;L^6(\Omega))^*$. This implies that $(\mathbf{u}_{n_j} \cdot \nabla)\mathbf{u}_{n_j} \longrightarrow f$ weakly in $L^{\frac{4}{3}}(0,T;[L^{\frac{6}{5}}(\Omega)]^3)$. It remains to show that $f = (\mathbf{u}\cdot\nabla)\mathbf{u}$. To this end, we will make use of (L9), which tells us in particular that there exists a subsequence $\mathbf{u}_{n_j}$ such that $\mathbf{u}_{n_j} \to \mathbf{u}$ strongly in $L^2(0,T;[L^2(\Omega)]^3)$. We also know from (L1) that $\nabla\mathbf{u}_{n_j} \rightharpoonup \nabla\mathbf{u}$ weakly in $L^2(0,T;[L^2(\Omega)]^9)$. Recall the following elementary result from analysis (see [2], Proposition II.2.12.):

Lemma 5.5: Weak convergence of a bilinear function

Let E, F, G be three Banach spaces and suppose that $B : E \times F \to G$ is a continuous bilinear function. If $(f_n)_{n\ge1} \subseteq E$ strongly converges to f and $(g_n)_{n\ge1} \subseteq F$ weakly converges to g then the sequence $\{B(f_n, g_n)\}_{n\ge1}$ weakly converges to $B(f,g)$ in the space G.

Applying the above result with the bilinear function $B(f,g) := (f\cdot\nabla)g$ and the spaces $E = L^2(0,T;[L^2(\Omega)]^3),\ F = L^2(0,T;[L^2(\Omega)]^9),\ G = L^1(0,T;[L^1(\Omega)]^3)$ allows us to conclude that the sequence $(\mathbf{u}_{n_j}\cdot\nabla)\mathbf{u}_{n_j} \rightharpoonup (\mathbf{u}\cdot\nabla)\mathbf{u}$ weakly in $L^1(0,T;[L^1(\Omega)]^3)$. However, we also know that $(\mathbf{u}_{n_j}\cdot\nabla)\mathbf{u}_{n_j} \longrightarrow f$ weakly in $L^{\frac{4}{3}}(0,T;[L^{\frac{6}{5}}(\Omega)]^3)$. Since both of these spaces are embedded in the space $D'([0,T]\times\Omega)$ in which limits of sequences are unique, we conclude that $f = (\mathbf{u}\cdot\nabla)\mathbf{u}$, as required.

$\boxed{\textbf{Z3}}$ — **Goal:** Show that as $j \to \infty$,

$$\int_0^T (2\nu(\psi_{n_j})D(\mathbf{u}_{n_j}), \nabla\mathbf{v}_s)\,\varphi(t)\,\mathrm{d}t \longrightarrow \int_0^T (2\nu(\psi)D(\mathbf{u}), \nabla\mathbf{v}_s)\,\varphi(t)\,\mathrm{d}t.$$

It suffices to show that as $j \to \infty$,

$$I_j := \int_0^T (\nu(\psi_{n_j})D(\mathbf{u}_{n_j}) - \nu(\psi)D(\mathbf{u}), \nabla\mathbf{v}_s)\varphi(t)\,\mathrm{d}t \longrightarrow 0.$$

Adding and subtracting $\nu(\psi)D(\mathbf{u}_{n_j})$ gives

$$\begin{aligned} I_3 &= \int_0^T (\nu(\psi_{n_j})D(\mathbf{u}_{n_j}) - \nu(\psi)D(\mathbf{u}_{n_j}), \nabla \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t + \int_0^T (\nu(\psi)(D(\mathbf{u}_{n_j}) - D(\mathbf{u})), \nabla \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t \\ &= \underbrace{\int_0^T (\nu(\psi_{n_j}) - \nu(\psi), D(\mathbf{u}_{n_j})\nabla \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t}_{\textbf{(1)}} + \underbrace{\int_0^T (D(\mathbf{u}_{n_j}) - D(\mathbf{u}), \nu(\psi)\nabla \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t}_{\textbf{(2)}} \end{aligned}$$

Now we estimate **(1)** and **(2)** separately. Using Holder's inequality firstly in space and then in time gives us

$$\begin{aligned} \textbf{(1)} &\leq \int_0^T \|D(\mathbf{u}_{n_j})\nabla \mathbf{v}_s \varphi(t)\| \|\nu(\psi_{n_j}) - \nu(\psi)\|\, \mathrm{d}t \\ &\leq \left(\int_0^T \|D(\mathbf{u}_{n_j})\nabla \mathbf{v}_s \varphi(t)\|^2\, \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \|\nu(\psi_{n_j}) - \nu(\psi)\|^2\, \mathrm{d}t\right)^{\frac{1}{2}}. \end{aligned}$$

Thanks to the following observations:

- $D(\mathbf{u}_{n_j})\nabla \mathbf{v}_s \varphi$ is bounded in $L^2(0,T;[L^2(\Omega)]^9)$, and
- due to (L8) and our assumption $\nu \in C^2(\mathbb{R})$, it follows that $\nu(\psi_{n_j}) \to \nu(\psi)$ strongly in $C([0,T];L^2(\Omega))$,

we are able to conclude that

$$\left(\int_0^T \|D(\mathbf{u}_{n_j})\nabla \mathbf{v}_s \varphi(t)\|^2\, \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \|\nu(\psi_{n_j}) - \nu(\psi)\|^2\, \mathrm{d}t\right)^{\frac{1}{2}} \longrightarrow 0,$$

as $j \to \infty$. Additionally, since $D(\mathbf{u}_{n_j}) \rightharpoonup D(\mathbf{u})$ weakly in $L^2(0,T;[L^2(\Omega)]^9)$ (this is a consequence of (L1)) and the fact that $\nu(\psi)\nabla \mathbf{v}_s \varphi \in L^2(0,T;[L^2(\Omega)]^3)$, we have that **(2)** $\to 0$ as $j \to \infty$. We have shown that $I_3 \to 0$ as $j \to \infty$, as required.

$\boxed{\textbf{Z4}}$ — **Goal:** Show that as $j \to \infty$,

$$\int_0^T (\psi_{n_j} \nabla \mu_{n_j}, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t \longrightarrow \int_0^T (\psi \nabla \mu, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t.$$

It suffices to show that as $j \to \infty$,

$$I_4 := \int_0^T (\psi_{n_j} \nabla \mu_{n_j} - \psi \nabla \mu, \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t \longrightarrow 0.$$

We will follow a similar technique to that which was demonstrated in $\boxed{\text{Z3}}$. Adding and subtracting $\psi \nabla \mu_n$ in the first component of the inner product, we have

$$\begin{aligned} I_4 &= \int_0^T (\nabla \mu_{n_j}(\psi_{n_j} - \psi), \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t + \int_0^T (\psi(\nabla \mu_{n_j} - \nabla \mu), \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t \\ &= \underbrace{\int_0^T (\psi_{n_j} - \psi, \nabla \mu_{n_j} \cdot \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t}_{I_{4A}} + \underbrace{\int_0^T (\nabla \mu_{n_j} - \nabla \mu, \psi \mathbf{v}_s)\, \varphi(t)\, \mathrm{d}t}_{I_{4B}}. \end{aligned}$$

Using Holder's inequality in space and time gives us

$$I_{4A} \leq \int_0^T \|\psi_{n_j} - \psi\| \|\varphi(\nabla \mu_{n_j} \cdot \mathbf{v}_s)\|\, \mathrm{d}t \leq \left(\int_0^T \|\psi_{n_j} - \psi\|^2\, \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^T \|\varphi(\nabla \mu_{n_j} \cdot \mathbf{v}_s)\|^2\, \mathrm{d}t\right)^{\frac{1}{2}}.$$

Since $\psi_{n_j} \to \psi$ strongly in $C([0,T];H^1(\Omega))$ and $\varphi(\nabla\mu_{n_j}\cdot\mathbf{v}_s)$ is bounded in $L^2(0,T;L^2(\Omega))$, we conclude that $I_{4A} \to 0$ as $j \to \infty$. To see that $I_{4B} \to 0$ as $j \to \infty$, simply recall the item (L4) which says that $\mu_{n_j} \rightharpoonup \mu$ in $L^2(0,T;H^1(\Omega))$ and notice that $\psi\mathbf{v}_s\varphi \in (L^2(0,T;H^1(\Omega)))^* = L^2(0,T;H^1(\Omega))$. We have now shown that $I_4 \to 0$ as $j \to \infty$.

$\boxed{\textsf{Z5}}$ — **Goal:** Show that as $j \to \infty$,

$$\int_0^T \langle \partial_t\psi_{n_j}, w_s\rangle_{(H^1)^*\times H^1}\,\varphi(t)\,\mathrm{d}t \longrightarrow \int_0^T \langle \partial_t\psi, w_s\rangle_{(H^1)^*\times H^1}\,\varphi(t)\,\mathrm{d}t.$$

Item (L3) gives us that as $j \to \infty$,

$$\int_0^T \langle \partial_t\psi_{n_j}, w\rangle_{(H^1)^*\times H^1}\,\mathrm{d}t \longrightarrow \int_0^T \langle \partial_t\psi, w\rangle_{(H^1)^*\times H^1}\,\mathrm{d}t, \quad \forall\, w \in L^2(0,T;H^1(\Omega)).$$

The result follows from the observation that $w_s\varphi \in L^2(0,T;H^1(\Omega))$.

$\boxed{\textsf{Z6}}$ — **Goal:** Show that as $j \to \infty$,

$$\int_0^T ((\mathbf{u}_{n_j}\cdot\nabla\psi_{n_j}), w_s)\,\varphi(t)\,\mathrm{d}t \longrightarrow \int_0^T ((\mathbf{u}\cdot\nabla\psi), w_s)\,\varphi(t)\,\mathrm{d}t.$$

Due to (L8) and (L9) we certainly have that $\mathbf{u}_{n_j}\cdot\nabla\psi_{n_j} \to \mathbf{u}\cdot\nabla\psi$ strongly in $L^2(0,T;L^2(\Omega))$, and therefore also in the weak sense. Since $w_s\varphi \in (L^2(0,T;L^2(\Omega)))^* = L^2(0,T;L^2(\Omega))$, the required result follows.

$\boxed{\textsf{Z7}}$ — **Goal:** Show that as $j \to \infty$,

$$\int_0^T (M(\psi_{n_j})\nabla\mu_{n_j}, \nabla w_s)\,\varphi(t)\,\mathrm{d}t \longrightarrow \int_0^T (M(\psi)\nabla\mu, \nabla w_s)\,\varphi(t)\,\mathrm{d}t.$$

We follow a similar technique to that which was demonstrated in $\boxed{\textbf{Z3}}$ and $\boxed{\textbf{Z4}}$. It suffices to show that as $j \to \infty$,

$$I_7 := \int_0^T (M(\psi_{n_j})\nabla\mu_{n_j} - M(\psi)\nabla\mu, \nabla w_s)\,\varphi(t)\,\mathrm{d}t \longrightarrow 0.$$

Adding and subtracting $M(\psi_{n_j})\nabla\mu$ gives us

$$\begin{aligned} I_7 &= \int_0^T (M(\psi_{n_j})(\nabla\mu_{n_j} - \nabla\mu), \nabla w_s)\,\varphi(t)\,\mathrm{d}t + \int_0^T ((M(\psi_{n_j}) - M(\psi))\nabla\mu, \nabla w_s)\,\varphi(t)\,\mathrm{d}t \\ &= \underbrace{\int_0^T (\nabla\mu_{n_j} - \nabla\mu, M(\psi_{n_j})\nabla w_s)\,\varphi(t)\,\mathrm{d}t}_{I_{7A}} + \underbrace{\int_0^T (M(\psi_{n_j}) - M(\psi), \nabla\mu\cdot\nabla w_s)\,\varphi(t)\,\mathrm{d}t}_{I_{7B}}. \end{aligned}$$

Now we note that $I_{7A} \to 0$ as $j \to \infty$ due to (L4) and the observation that $M(\psi_{n_j})\nabla w_s\varphi \in L^2(0,T;L^2(\Omega))$. The latter fact follows from the assumption that the mobility function $M \in C^2(\mathbb{R})$ and that $\varphi \in D(0,T)$, $w_s \in H^2(\Omega), \psi_{n_j} \in L^\infty(0,T;H^2(\Omega))$ for each j.

$\boxed{\textbf{Z8}}$ — **Goal:** Show that almost everywhere in $\Omega \times [0,T]$, we have

$$\mu = \psi^3 + (r + (1+\Delta)^2)\psi.$$

We know that for each $s \in \{1, ..., n_j\}$ and almost everywhere in $t \in [0,T]$,

$$(\mu_{n_j}, w) = (\psi_{n_j}^3, w) + ((r+1)\psi_{n_j}, w) + (2\Delta\psi_{n_j}, w) + (\Delta\psi_{n_j}, \Delta w), \ \forall\, w \in W_{n_j}.$$

Using (L4) and (L8) we can pass to the limit as $j \to \infty$ to deduce that for a.e. $t \in [0,T]$,

$$(\mu, w) = (\psi^3, w) + ((r+1)\psi, w) + (2\Delta\psi, w) + (\Delta\psi, \Delta w), \ \forall\, w \in H^2(\Omega),$$

where we have used the fact that $\{w_k\}_k$ forms a complete orthonormal system of $L^2(\Omega)$. Recalling that $\psi \in L^2(0,T; H^4(\Omega))$ and integrating by parts in the final two inner products gives us that

$$(\mu, w) = (\psi^3 + (r + (1+\Delta)^2)\psi, w), \ \forall\, w \in H^2(\Omega),$$

almost everywhere in $[0,T]$. Since $H^2(\Omega)$ is dense in $L^2(\Omega)$, the above holds for any $w \in L^2(\Omega)$. This is enough to deduce that

$$\mu = \psi^3 + (r + (1+\Delta)^2)\psi$$

holds almost everywhere in $\Omega \times [0,T]$.

$\boxed{\textbf{Z9}}$ — **Goal:** Show that the following initial conditions are met:

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \ \psi(\cdot, 0) = \psi_0, \ \text{ in } \Omega.$$

We will make use of the following result (see [2], Proposition II.5.11):

Lemma 5.6: A useful embedding ([2], Proposition II.5.11)

Let X, Y be two Banach spaces such that X is continuously, densely embedded in Y. Any element of the set

$$E_{p,q} := \left\{ u \in L^p(0,T;X), \ \frac{du}{dt} \in L^q(0,T;Y) \right\}$$

(defined a.e.) possesses a continuous representation on $[0,T]$ with values in Y, and the embedding $E_{p,q} \hookrightarrow C([0,T];Y)$ is continuous.

We follow a very similar approach to that which can be seen in ([2], Theorem V.1.4). Taking $X = \mathbf{V}_\sigma, \ Y = (\mathbf{V}_\sigma)^*, \ p = 2, \ q = \frac{4}{3}$, we infer that the embedding of $E_{2,\frac{4}{3}}$ (with $X, \ Y$ as chosen) into $C([0,T]; (\mathbf{V}_\sigma)^*)$ is continuous. Since the sequence of elements $\mathbf{u}_{n_j}$ belongs to $E_{2,\frac{4}{3}}$ we are able to deduce that the limit $\mathbf{u} \in C([0,T]; (\mathbf{V}_\sigma)^*)$. Next, applying the Aubin-Lions theorem with $X_0 = \mathbf{H}_\sigma, \ X = \mathbf{V}_\sigma, \ X_1 = (\mathbf{V}_\sigma)^*, \ p = \infty, \ q = \frac{4}{3}$ gives us that the set $F := \{\mathbf{u} \in L^\infty(0,T; \mathbf{H}_\sigma, \ \partial_t \mathbf{u} \in L^{\frac{4}{3}}(0,T; (\mathbf{V}_\sigma)^*)\}$ is compactly embedded in $C([0,T]; (\mathbf{V}_\sigma)^*)$. Since the sequence $\{\mathbf{u}_{n_j}\}_{j \geq 1}$ belongs to F, we can infer that $\mathbf{u}_{n_j} \to \mathbf{u}$ strongly in $C([0,T]; (\mathbf{V}_\sigma)^*)$. This in particular means that $\mathbf{u}_{n_j}(0) \to \mathbf{u}(0)$ in $(\mathbf{V}_\sigma)^*$ as $j \to \infty$. On the other hand, $\mathbf{u}_{n_j}(0) = \mathcal{P}_{n_j}(\mathbf{u}_0) \longrightarrow \mathbf{u}_0$ as $j \to \infty$ since $\mathcal{P}_{n_j}$ is the projection onto a complete orthonormal basis of $\mathbf{H}_\sigma$. Thus, $\mathbf{u}_{n_j}(0) \to \mathbf{u}_0$ in $\mathbf{H}_\sigma$ and therefore also in

$(\mathbf{V}_\sigma)^*$ (by the embedding $\mathbf{H}_\sigma \hookrightarrow (\mathbf{V}_\sigma)^*$). Since limits are unique in $(\mathbf{V}_\sigma)^*$, we conclude that $\mathbf{u}(0) = \mathbf{u}_0$. Finally, recalling that $\psi_{n_j} \to \psi$ in $C([0,T]; H^1(\Omega))$, we infer that $\psi_{n_j}(0) \to \psi(0)$ in $H^1(\Omega)$. Additionally, since $\psi_{n_j}(0) = \Pi_{n_j}(\psi_0) \longrightarrow \psi_0$ in $H^1(\Omega)$ by properties of the projection Π, we infer that $\psi(0) = \psi_0$ by uniqueness of the limit in $H^1(\Omega)$.

Now that we have collected items $\boxed{\textbf{Z1}}$ - $\boxed{\textbf{Z9}}$, we can complete the proof of existence.

Theorem 3.3. (Existence of a Global Weak Solution)
Let $\Omega \subset \mathbb{R}^3$ be an open bounded domain of class C^4 and assume our initial data satisfies $(u_0, \psi_0) \in \mathbf{H}_\sigma \times H^2_N(\Omega)$. There exists a triple (u, ψ, μ) which satisfies the criteria for a global weak solution to the system (2.1) – (2.8).

Proof. We have shown that there exists a sequence of triples $\{(\mathbf{u}_n, \psi_n, \mu_n)\}_{n\geq 1}$ solving the approximating problem:

$$\begin{cases} \displaystyle\int_0^T \Big[\langle \partial_t \mathbf{u}_n(t), \mathbf{v}\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}_n, \mathbf{u}_n, \mathbf{v}) + (2\nu(\psi_n) D(\mathbf{u}_n), \nabla \mathbf{v}) + (\psi_n \nabla \mu_n, \mathbf{v})\Big] \varphi(t)\, \mathrm{d}t = 0, \\ \hfill \forall\, \mathbf{v} \in \mathbf{V}_n, \\ \displaystyle\int_0^T \Big[\langle \partial_t \psi_n, w\rangle_{(H^1)^* \times H^1} + ((\mathbf{u}_n \cdot \nabla \psi_n), w) + (M(\psi_n) \nabla \mu_n, \nabla w)\Big] \varphi(t)\, \mathrm{d}t = 0, \quad \forall\, w \in W_n, \\ (\mu_n, w) = (\psi_n^3, w) + ((r+1)\psi_n, w) + (2\Delta \psi_n, w) + (\Delta \psi_n, \Delta w), \quad \forall\, w \in W_n, \\ \mathbf{u}_n(\cdot, 0) = \mathcal{P}_n(\mathbf{u}_0) =: \mathbf{u}_0^n, \;\; \psi_n(\cdot, 0) = \Pi_n(\psi_0) =: \psi_0^n, \quad \text{in } \Omega. \end{cases}$$

We have also shown that there exists a limit $(\mathbf{u}, \psi, \mu)$ and a subsequence $(\mathbf{u}_{n_j}, \psi_{n_j}, \mu_{n_j})$ converging to this limit. We will show that the limit $(\mathbf{u}, \psi, \mu)$ satisfies an equivalent version of the weak formulation:

$$\begin{cases} \displaystyle\int_0^T \Big[\langle \partial_t \mathbf{u}, \mathbf{v}\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + (2\nu(\psi) D(\mathbf{u}), \nabla \mathbf{v}) + (\psi \nabla \mu, \mathbf{v})\Big] \varphi(t)\, \mathrm{d}t = 0, \;\; \forall\, \mathbf{v} \in \mathbf{V}_\sigma, & \text{(WF-1)} \\ \displaystyle\int_0^T \Big[\langle \partial_t \psi, w\rangle_{(H^1)^* \times H^1} + ((\mathbf{u} \cdot \nabla \psi), w) + (M(\psi) \nabla \mu, \nabla w)\Big] \varphi(t) \mathrm{d}t = 0, \;\; \forall\, w \in H^1(\Omega), & \text{(WF-2)} \\ \mu = \psi^3 + (r + (1+\Delta)^2)\psi, \;\text{ a.e. in } \Omega \times [0,T], & \text{(WF-3)} \\ \mathbf{u}(\cdot, 0) = \mathbf{u}_0, \;\; \psi(\cdot, 0) = \psi_0, \;\text{ in } \Omega, & \text{(WF-4)} \end{cases}$$

Applying $\boxed{\textbf{Z8}}$ and $\boxed{\textbf{Z9}}$ allows us to immediately conclude that (WF-3) and (WF-4) hold. We now turn our attention to (WF-1). Fixing $\mathbf{v}_s \in \mathbf{V}_{n_j}$ and $\varphi \in D(0,T)$, we have from (AP) that for sufficiently large j,

$$\int_0^T \Big[\langle \partial_t \mathbf{u}_{n_j}(t), \mathbf{v}_s\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}_{n_j}, \mathbf{u}_{n_j}, \mathbf{v}_s) + (2\nu(\psi_{n_j}) D(\mathbf{u}_{n_j}), \nabla \mathbf{v}_s) + (\psi_{n_j} \nabla \mu_{n_j}, \mathbf{v}_s)\Big] \varphi(t)\, \mathrm{d}t = 0.$$

Since s is fixed, we can take $j \to \infty$ and exploit $\boxed{\textbf{Z1}}$ - $\boxed{\textbf{Z4}}$, giving us

$$\int_0^T \Big[\langle \partial_t \mathbf{u}(t), \mathbf{v}_s\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}_s) + (2\nu(\psi) D(\mathbf{u}), \nabla \mathbf{v}_s) + (\psi \nabla \mu, \mathbf{v}_s)\Big] \varphi(t)\, \mathrm{d}t = 0.$$

We now need to show that the above also holds if we replace $\mathbf{v}_s$ with any $v \in \mathbf{V}_\sigma$. To this end, let $v \in \mathbf{V}_\sigma$ and $v_s = \mathcal{P}_s v$. We know by properties of the orthogonal projection $\mathcal{P}$ that $\mathbf{v}_s \to \mathbf{v}$ in $\mathbf{V}_\sigma$. This implies that $\mathbf{v}_s \varphi \to \mathbf{v} \varphi$ in $C([0,T]; \mathbf{V}_\sigma)$, which is embedded in $C([0,T]; (L^6(\Omega))^3)$ (by Sobolev embedding). Since we also have that

- $\mathbf{u} \in W^{1,\frac{4}{3}}(0,T; (\mathbf{V}_\sigma)^*)$,

- $(\mathbf{u} \cdot \nabla)\mathbf{u} \in L^{\frac{4}{3}}(0, T; [L^{\frac{6}{5}}(\Omega)]^3$,
- $\nu(\psi)D(\mathbf{u}) \in L^2(0, T; [L^2(\Omega)]^9)$,
- $\psi \nabla \mu \in L^2(0, T; [L^2(\Omega)]^3)$,

we are able to pass to the limit as $s \to \infty$ to obtain

$$\int_0^T \langle \partial_t \mathbf{u}(t), \mathbf{v} \rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} \varphi(t) \, \mathrm{d}t + \int_0^T b(\mathbf{u}, \mathbf{u}, \mathbf{v}) \, \varphi(t) \, \mathrm{d}t \\ + \int_0^T (2\nu(\psi) D(\mathbf{u}), \nabla \mathbf{v}) \, \varphi(t) \, \mathrm{d}t + \int_0^T (\psi \nabla \mu, \mathbf{v}) \, \varphi(t) \, \mathrm{d}t = 0, \quad \forall \, \mathbf{v} \in \mathbf{V}_\sigma.$$

This is enough to show that (WF-1) is satisfied. We can repeat the same process for the second equation in our approximating problem (AP), making use of $\boxed{\textbf{Z5}}$ - $\boxed{\textbf{Z7}}$ and the properties of the limit $(\mathbf{u}, \psi, \mu)$, to deduce that taking $s, j \to \infty$, we have

$$\int_0^T \langle \partial_t \psi, w \rangle_{(H^1)^* \times H^1} \varphi(t) \, \mathrm{d}t + \int_0^T ((\mathbf{u} \cdot \nabla \psi), w) \, \varphi(t) \, \mathrm{d}t + \int_0^T (M(\psi) \nabla \mu, \nabla w) \, \varphi(t) \, \mathrm{d}t = 0,$$

for each $w \in H^1(\Omega)$. Thus, (WF-2) also holds. We have now shown that the limit $(\mathbf{u}, \psi, \mu)$ satisfies the weak formulation on $[0, T]$. To summarise our work so far, we have shown that for any $T > 0$ there exists a triple $(\mathbf{u}, \psi, \mu)$ which solves the weak problem on $[0, T]$. In order to prove the existence of a global weak solution however, we need to show that there exists a triple $(\mathbf{u}, \psi, \mu)$ which is a weak solution on $[0, T]$ for any $T > 0$. To this end, we will employ a similar diagonal argument to that which can be seen in ([2], Chapter V, Section 1.3.6). Recall that given $T > 0$ and $n \in \mathbb{N}$, we proved the existence of a unique solution $(\mathbf{u}_n, \psi_n, \mu_n)$ solving the approximating problem on $[0, T]$. Note that due to previous estimates we have that $(\mathbf{u}_n, \psi_n, \mu_n)$ is defined on $\mathbb{R}^+$. Let $T = 1$ to begin with. Then, by the preceding analysis in this subsection (5.5), there exists a subsequence $(\mathbf{u}_{n_{j1}}, \psi_{n_{j1}}, \mu_{n_{j1}})$ converging to some $(\mathbf{u}_1, \psi_1, \mu_1)$ on $[0, 1]$. Then consider the case $T = 2$. Thanks to our previous estimates and the fact that $(\mathbf{u}_{n_{j1}}, \psi_{n_{j1}}, \mu_{n_{j1}})$ is defined globally, we can extract a subsequence $(\mathbf{u}_{n_{j2}}, \psi_{n_{j2}}, \mu_{n_{j2}})$ from $(\mathbf{u}_{n_{j1}}, \psi_{n_{j1}}, \mu_{n_{j1}})$ which converges to some $(\mathbf{u}_2, \psi_2, \mu_2)$ on $[0, 2]$. Repeating this process will give us a subsequence $(\mathbf{u}_{n_{jK}}, \psi_{n_{jK}}, \mu_{n_{jK}})$ converging to a limit $(\mathbf{u}_K, \psi_K, \mu_K)$ which solves the weak formulation on $[0, K]$. Thus, the subsequence $\{(\mathbf{u}_{n_{jK}}, \psi_{n_{jK}}, \mu_{n_{jK}})\}_{K \geq 1}$ will converge to a solution to the weak formulation on $[0, T]$ for any $T > 0$, which is what we needed to show. $\square$

Remark on the globality of weak solutions in two dimensions:
The analysis we performed in order to assert the existence of weak solutions also works for the case $d = 2$. This realisation allows us to immediately infer the existence of weak solutions in two dimensions. Proving the globality of weak solutions in this case is simpler than the above situation where $d = 3$ since, as we will show in the next section, weak solutions are unique when $d = 2$. We know that for any $T > 0$ there exists a weak solution $(\mathbf{u}_T, \psi_T, \mu_T)$ on $[0, T]$. The uniqueness of such a solution in two dimensions (which is proved in the next section) entails that whenever $T < K$ we have that $(\mathbf{u}_T(t), \psi_T(t), \mu_T(t)) = (\mathbf{u}_K(t), \psi_K(t), \mu_K(t))$ for $t \leq T$. Thus we can define a global weak solution $(\mathbf{u}, \psi, \mu)$ as $(\mathbf{u}(t), \psi(t), \mu(t)) := (\mathbf{u}_T(t), \psi_T(t), \mu_T(t))$ for each $t \leq T$. Indeed, this solution is defined on $\mathbb{R}^+$ since for any $t \in [0, \infty)$ there exists a unique weak solution $(\mathbf{u}_T, \psi_T, \mu_T)$ on $[0, T]$ where $T > t$, so $(\mathbf{u}, \psi, \mu)$ is well-defined on $\mathbb{R}^+$.

6 Uniqueness in two dimensions

The aim of this section is to prove Theorem 3.4, which we recall below for convenience.

Theorem 3.4. (Uniqueness in two dimensions)
Let $\Omega \subset \mathbb{R}^2$ be an open bounded domain of class C^4 and assume our initial data satisfies $(u_0, \psi_0) \in \mathbf{H}_\sigma \times H^2_N(\Omega)$. Furthermore, suppose that (u_1, ψ_1, μ_1) and (u_2, ψ_2, μ_2) are two weak solutions to (2.1) – (2.8) on $[0, T]$ where $T > 0$ is fixed and originating from the same initial datum $(\mathbf{u}_0, \psi_0)$. Then it must hold that $(u_1, \psi_1, \mu_1) = (u_2, \psi_2, \mu_2)$ a.e. in $\Omega \times [0, T]$.

Let us first discuss our approach.

Our plan of action

To begin, we will take an appropriate test function in the first equation of the weak formulation satisfied by the difference of two arbitrary weak solutions. Simplifying the resulting equation will lead to an inequality of the form

$$\frac{1}{2}\|\mathbf{u}(t)\|^2 + 2\nu_* \int_0^T \|\nabla \mathbf{u}(t)\|^2 \,\mathrm{d}t \leq \frac{1}{2}\|\mathbf{u}(0)\|^2 + \sum_{j=1}^{8} I_j.$$

Next, we bound each of the terms $I_1, \ldots I_8$ from above in an appropriate way, leading to an inequality which we will label as **U1**. Then, we consider the second equation of the weak formulation satisfied by the difference of the same two arbitrary weak solutions. Taking an appropriate test function and simplifying once more will lead to an inequality of the form

$$\frac{1}{2}\|\nabla \psi(t)\|^2 + \int_0^T \|\Delta^2 \psi\|^2 \,\mathrm{d}t \leq \frac{1}{2}\|\nabla \psi(0)\|^2 + \sum_{j=9}^{13} I_j.$$

Estimating each of $I_9, \ldots, I_{13}$ from above will reward us with another inequality which we will call **U2**. Finally, we sum up **U1** and **U2** before applying Osgood's Lemma ([8], Appendix B) which will complete the proof.

6.1 The Navier-Stokes momentum equation

We now begin working in accordance with the above plan. Let $(\mathbf{u}_1, \psi_1, \mu_1)$, $(\mathbf{u}_2, \psi_2, \mu_2)$ be two weak solutions, and define $(\mathbf{u}, \psi, \mu) := (\mathbf{u}_1 - \mathbf{u}_2, \psi_1 - \psi_2, \mu_1 - \mu_2)$ to be the difference of the two solutions. In what follows we will assume $M_1 = M(\psi_1) = M(\psi_2) = 1$, without loss of generality. Recall the first equation in our weak formulation:

$$\langle \partial_t \mathbf{u}, \mathbf{v} \rangle_{(\mathbf{V}_\sigma)^* \times \mathbf{V}_\sigma} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + (2\nu(\psi) D(\mathbf{u}), \nabla \mathbf{v}) = -(\psi \nabla \mu, \mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{V}_\sigma, \text{ a.e. in } [0, T].$$

This is equivalent to:

$$\langle \partial_t \mathbf{u}, \mathbf{v} \rangle_{(\mathbf{V}_\sigma)^* \times \mathbf{V}_\sigma} + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) + (2\nu(\psi) D(\mathbf{u}), \nabla \mathbf{v}) = -(\psi \nabla \mu, \mathbf{v}) \quad \forall\, \mathbf{v} \in L^2(0, T; \mathbf{V}_\sigma), \text{ a.e. in } [0, T].$$

Integrating with respect to time, we have that the following equation is satisfied by $(\mathbf{u}_i, \psi_i, \mu_i)$ for $i = 1, 2$:

$$\int_0^T \langle \partial_t \mathbf{u}_i(t), \mathbf{v}(t) \rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} \,\mathrm{d}t + \int_0^T b(\mathbf{u}_i(t), \mathbf{u}_i(t), \mathbf{v}(t)) \,\mathrm{d}t + \int_0^T (2\nu(\psi_i) D(\mathbf{u}_i(t)), \nabla \mathbf{v}(t)) \,\mathrm{d}t$$
$$+ \int_0^T (\psi_i(t) \nabla \mu_i(t), \mathbf{v}(t)) \,\mathrm{d}t = 0, \quad \forall\, \mathbf{v} \in L^2(0, T; \mathbf{V}_\sigma), \qquad (6.1)$$

Subtracting (6.1) with $i = 2$ from (6.1) with $i = 1$ yields

$$\int_0^T \langle \partial_t \mathbf{u}(t), \mathbf{v}(t)\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} \,\mathrm{d}t + \int_0^T b(\mathbf{u}_1(t), \mathbf{u}_1(t), \mathbf{v}(t)) \,\mathrm{d}t - \int_0^T b(\mathbf{u}_2(t), \mathbf{u}_2(t), \mathbf{v}(t)) \,\mathrm{d}t$$
$$+ \int_0^T (2\nu(\psi_1) D(\mathbf{u}_1(t)), \nabla \mathbf{v}(t)) \,\mathrm{d}t - \int_0^T (2\nu(\psi_2) D(\mathbf{u}_2(t)), \nabla \mathbf{v}(t)) \,\mathrm{d}t + \int_0^T (\psi_1(t) \nabla \mu_1(t), \mathbf{v}(t)) \,\mathrm{d}t$$
$$- \int_0^T (\psi_2(t) \nabla \mu_2(t), \mathbf{v}(t)) \,\mathrm{d}t = 0, \quad \forall\, \mathbf{v} \in L^2(0,T;\mathbf{V}_\sigma). \tag{6.2}$$

Taking $\mathbf{v} = \mathbf{u}$ and noting that $\langle \partial_t \mathbf{u}(t), \mathbf{u}(t)\rangle_{\mathbf{V}_\sigma^* \times \mathbf{V}_\sigma} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}(t)\|^2$, we have

$$\frac{1}{2}\|\mathbf{u}(T)\|^2 + \int_0^T b(\mathbf{u}_1(t), \mathbf{u}_1(t), \mathbf{u}(t)) \,\mathrm{d}t - \int_0^T b(\mathbf{u}_2(t), \mathbf{u}_2(t), \mathbf{u}(t)) \,\mathrm{d}t$$
$$+ \int_0^T (2\nu(\psi_1) D(\mathbf{u}_1(t)), \nabla \mathbf{u}(t)) \,\mathrm{d}t - \int_0^T (2\nu(\psi_2) D(\mathbf{u}_2(t)), \nabla \mathbf{u}(t)) \,\mathrm{d}t$$
$$+ \int_0^T (\psi_1(t) \nabla \mu_1(t), \mathbf{u}(t)) \,\mathrm{d}t - \int_0^T (\psi_2(t) \nabla \mu_2(t), \mathbf{u}(t)) \,\mathrm{d}t = \frac{1}{2}\|\mathbf{u}(0)\|^2, \quad \text{for all } T > 0. \tag{6.3}$$

Using the tri-linearity of the form b, we have that

$$b(\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}) - b(\mathbf{u}_2, \mathbf{u}_2, \mathbf{u}) = b(\mathbf{u}_2, \mathbf{u}, \mathbf{u}) + b(\mathbf{u}, \mathbf{u}_1, \mathbf{u}) = b(\mathbf{u}, \mathbf{u}_1, \mathbf{u}),$$

where we have used the property (T2) pertaining to b to achieve the final equality. Additionally, we note that by adding and subtracting the quantity $2\nu(\psi_1)D(\mathbf{u}_2(t))$, the following equality holds

$$(2\nu(\psi_1) D(\mathbf{u}_1(t)), \nabla \mathbf{u}(t)) - (2\nu(\psi_2) D(\mathbf{u}_2(t)), \nabla \mathbf{u}(t))$$
$$= (2\nu(\psi_1) D(\mathbf{u}(t)), D(\mathbf{u}(t))) + ((2\nu(\psi_1) - 2\nu(\psi_2)) D(\mathbf{u}_2(t)), \nabla \mathbf{u}(t)),$$

which will be useful for the subsequent analysis. Next, we manipulate the Korteweg force term $\psi\nabla\mu$ into a more useful form. We have that

$$\psi\nabla\mu = \nabla(\psi\mu) - \mu\nabla\psi = \nabla(\psi\mu) - (\psi^3 + (r+1)\psi + 2\Delta\psi + \Delta^2\psi)\nabla\psi$$
$$= \nabla(\psi\mu) - \nabla\left(\frac{1}{4}\psi^4\right) - (r+1)\nabla\left(\frac{1}{2}\psi^2\right) - 2\Delta\psi\nabla\psi + \Delta^2\psi\nabla\psi.$$

In component notation,

$$(\Delta^2\psi\nabla\psi)_i = \mathrm{div}(\nabla\Delta\psi)\partial_i\psi = \partial_j(\partial_j\Delta\psi)\partial_i\psi = \partial_j(\partial_j\Delta\psi\partial_i\psi) - \partial_j\Delta\psi\;\partial_i\partial_j\psi.$$

This implies that

$$\Delta^2\psi\nabla\psi = \mathrm{div}(\nabla\psi \otimes \nabla\Delta\psi) - (\nabla\Delta\psi \cdot \nabla)\nabla\psi,$$

where $\mathbb{R}^{n\times n} \ni (v \otimes w)_{ij} := v_i w_j$, for $v, w \in \mathbb{R}^n$. Thus,

$$(\psi_1\nabla\mu_1, \mathbf{u}) = \underbrace{\int_\Omega \{\nabla(\psi_1\mu_1) - \nabla(\frac{1}{4}\psi_1^4) - (r+1)\nabla(\frac{1}{2}\psi_1^2)\}\mathbf{u}\,\mathrm{d}x}_{J_1}$$
$$- 2\int_\Omega \Delta\psi_1\nabla\psi_1 \cdot \mathbf{u}\,\mathrm{d}x + \int_\Omega \Delta^2\psi_1\nabla\psi_1 \cdot \mathbf{u}\,\mathrm{d}x.$$

It follows from integration by parts and the divergence free condition imposed on $\mathbf{u}$ that $J_1 = 0$. We now proceed to reformulate the final two integrals in the above equation using integration by parts.

$$\int_\Omega \Delta^2\psi_1\nabla\psi_1 \cdot \mathbf{u}\,\mathrm{d}x = \int_\Omega \mathrm{div}(\nabla\psi_1 \otimes \nabla\Delta\psi_1) \cdot \mathbf{u}\,\mathrm{d}x - \int_\Omega (\nabla\Delta\psi \cdot \nabla)\nabla\psi \cdot \mathbf{u}\,\mathrm{d}x$$
$$= -\int_\Omega (\nabla\psi_1 \otimes \nabla\Delta\psi_1) : \nabla\mathbf{u}\,\mathrm{d}x - b(\nabla\Delta\psi_1, \nabla\psi, \mathbf{u}),$$

$$\int_\Omega \Delta\psi_1\nabla\psi_1 \cdot \mathbf{u}\,\mathrm{d}x = -\int_\Omega (\nabla\Delta\psi_1 \cdot \mathbf{u})\psi_1\,\mathrm{d}x,$$

where the boundary integrals vanish due to the properties of $\mathbf{u}$ and ψ. Therefore, we have that for $i = 1, 2$

$$(\psi_i \nabla \mu_i, \mathbf{u}) = -\int_\Omega (\nabla \psi_i \otimes \nabla \Delta \psi_i) : \nabla \mathbf{u} \, \mathrm{d}x - b(\nabla \Delta \psi_i, \nabla \psi_i, \mathbf{u}) - 2 \int_\Omega (\nabla \Delta \psi_i \cdot \mathbf{u}) \psi_i \, \mathrm{d}x,$$

which implies

$$\begin{aligned}(\psi_1 \nabla \mu_1, \mathbf{u}) - (\psi_2 \nabla \mu_2, \mathbf{u}) = &\int_\Omega \nabla \psi_2 \otimes \nabla \Delta \psi_2 : \nabla \mathbf{u} \, \mathrm{d}x - \int_\Omega \nabla \psi_1 \otimes \nabla \Delta \psi_1 : \nabla \mathbf{u} \, \mathrm{d}x \\ &+ b(\nabla \Delta \psi_2, \nabla \psi_2, \mathbf{u}) - b(\nabla \Delta \psi_1, \nabla \psi_1, \mathbf{u}) \\ &+ 2 \int_\Omega (\nabla \Delta \psi_2 \cdot \mathbf{u}) \psi_2 \, \mathrm{d}x - 2 \int_\Omega (\nabla \Delta \psi_1 \cdot \mathbf{u}) \psi_1 \, \mathrm{d}x.\end{aligned}$$

Thanks to linearity we can use the following three equivalences:

$$\int_\Omega (\nabla \psi_2 \otimes \nabla \Delta \psi_2 - \nabla \psi_1 \otimes \nabla \Delta \psi_1) : \nabla \mathbf{u} \, \mathrm{d}x = -\int_\Omega \nabla \psi_2 \otimes \nabla \Delta \psi : \nabla \mathbf{u} \, \mathrm{d}x - \int_\Omega \nabla \psi \otimes \nabla \Delta \psi_1 : \nabla \mathbf{u} \, \mathrm{d}x,$$

$$b(\nabla \Delta \psi_2, \nabla \psi_2, \mathbf{u}) - b(\nabla \Delta \psi_1, \nabla \psi_1, \mathbf{u}) = -b(\nabla \Delta \psi, \nabla \psi_2, \mathbf{u}) - b(\nabla \Delta \psi_1, \nabla \psi, \mathbf{u}),$$

$$\int_\Omega (\nabla \Delta \psi_2 \cdot \mathbf{u}) \psi_2 \, \mathrm{d}x - \int_\Omega (\nabla \Delta \psi_1 \cdot \mathbf{u}) \psi_1 \, \mathrm{d}x = -\int_\Omega (\nabla \Delta \psi \cdot \mathbf{u}) \psi_2 \, \mathrm{d}x - \int_\Omega (\nabla \Delta \psi_1 \cdot \mathbf{u}) \psi \, \mathrm{d}x.$$

Therefore, we have

$$\begin{aligned}(\psi_1 \nabla \mu_1, \mathbf{u}) - (\psi_2 \nabla \mu_2, \mathbf{u}) = &-\int_\Omega \nabla \psi_2 \otimes \nabla \Delta \psi : \nabla \mathbf{u} \, \mathrm{d}x - \int_\Omega \nabla \psi \otimes \nabla \Delta \psi_1 : \nabla \mathbf{u} \, \mathrm{d}x \\ &- b(\nabla \Delta \psi, \nabla \psi_2, \mathbf{u}) - b(\nabla \Delta \psi_1, \nabla \psi, \mathbf{u}) - \int_\Omega (\nabla \Delta \psi \cdot \mathbf{u}) \psi_2 \, \mathrm{d}x - \int_\Omega (\nabla \Delta \psi_1 \cdot \mathbf{u}) \psi \, \mathrm{d}x,\end{aligned}$$

and thus that the following inequality is satisfied for all $T > 0$:

$$\begin{aligned}&\frac{1}{2} \|\mathbf{u}(T)\|^2 + \int_0^T 2\nu(\psi_1) \|D\mathbf{u}(t)\|^2 \, \mathrm{d}t \leq \frac{1}{2} \|\mathbf{u}(0)\|^2 + \int_0^T |b(\mathbf{u}, \mathbf{u}_1, \mathbf{u})| \, \mathrm{d}t \\ &+ \int_0^T |(2(\nu(\psi_1) - \nu(\psi_2)) D(\mathbf{u}_2(t)), \nabla \mathbf{u}(t))| \, \mathrm{d}t + \int_0^T (\nabla \psi_2 \otimes \nabla \Delta \psi, \nabla \mathbf{u}) \, \mathrm{d}t + \int_0^T (\nabla \psi \otimes \nabla \Delta \psi_1, \nabla \mathbf{u}) \, \mathrm{d}t \\ &+ \int_0^T b(\nabla \Delta \psi, \nabla \psi_2, \mathbf{u}) \, \mathrm{d}t + \int_0^T b(\nabla \Delta \psi_1, \nabla \psi, \mathbf{u}) \, \mathrm{d}t + \int_0^T (\nabla \Delta \psi \cdot \mathbf{u}, \psi_2) \, \mathrm{d}t + \int_0^T (\nabla \Delta \psi_1 \cdot \mathbf{u}, \psi) \, \mathrm{d}t.\end{aligned}$$

We can more conveniently express the above inequality as

$$\frac{1}{2} \|\mathbf{u}(T)\|^2 + \int_0^T \frac{2\nu_*}{\sqrt{2}} \|\nabla \mathbf{u}(t)\|^2 \, \mathrm{d}t \leq \frac{1}{2} \|\mathbf{u}(0)\|^2 + \sum_{j=1}^{8} I_j, \qquad \text{for all } T > 0, \tag{6.4}$$

where we have used Korn's inequality $\|\nabla \mathbf{u}\| \leq \sqrt{2} \|D\mathbf{u}\|$ for $\mathbf{u} \in \mathbf{V}_\sigma$ and the assumption on the viscosity term $0 < \nu_* < \nu(s)$ for each $s \in \mathbb{R}$, to attain the second term on the left-hand side. We now proceed to estimate each of $I_1, ..., I_8$. We recall the inequalities (HT), (Y1), (Y2), (PW), (LZ), (BG), (GN) , (S1) which will be used repeatedly in what follows. By an abuse of notation we will repeatedly denote by C an arbitrary positive constant. The constants c_j and c_j' will also be used to denote arbitrary positive constants where the index j indicates that the constant arises from the estimate of I_j. Similarly, ϵ_j and ϵ_j' will be used to denote arbitrary positive constants which arise from applying either of Young's inequalities ((Y1) or (Y2)) in the estimation of I_j. We also observe that since $\overline{\psi} = \overline{\psi_1(t)} - \overline{\psi_2(t)} = \overline{\psi_0} - \overline{\psi_0} = 0$, we have that $\|\psi\|_{H^2} \leq C \|\Delta \psi\|$, $\|\psi\|_{H^3} \leq C \|\nabla \Delta \psi\|$, $\|\psi\|_{H^4} \leq C \|\Delta^2 \psi\|$, $\|\psi\|_{H^1} \leq C \|\nabla \psi\|$.

$\boxed{\mathbf{I_1}}$

Using Holder's inequality for triple products (HT),

$$I_1 = \int_0^T |b(\mathbf{u}, \mathbf{u}_1, \mathbf{u})| \, \mathrm{d}t \overset{\text{(HT)}}{\leq} \int_0^T \|\mathbf{u}(t)\|_{L^4} \|\nabla \mathbf{u}_1(t)\| \|\mathbf{u}(t)\|_{L^4} \, \mathrm{d}t = \int_0^T \|\mathbf{u}(t)\|^2_{L^4(\Omega)} \|\nabla \mathbf{u}_1(t)\| \, \mathrm{d}t$$
$$\overset{\text{(LZ)}}{\leq} C \int_0^T \|\mathbf{u}\| \|\mathbf{u}\|_{H^1} \| \|\nabla \mathbf{u}_1(t)\| \, \mathrm{d}t \overset{\text{(Y2)}}{\leq} \frac{C}{4\epsilon} \int_0^T \|\mathbf{u}\|^2 \|\nabla \mathbf{u_1}\|^2 \, \mathrm{d}t + \epsilon \int_0^T \|\mathbf{u}\|^2_{H^1} \, \mathrm{d}t.$$

Since $u = 0$ on $\partial\Omega$, Poincare's inequality yields that $\|\mathbf{u}\| \leq C\|\nabla \mathbf{u}\|$ which in turn leads to $\|\mathbf{u}\|^2_{H^1} \leq C\|\nabla \mathbf{u}\|^2$. Thus,

$$I_1 \leq c_1\epsilon_1 \int_0^T \|\nabla \mathbf{u}\|^2 \, \mathrm{d}t + c_1' \int_0^T \|\mathbf{u}\|^2 \|\nabla \mathbf{u}_1\|^2 \, \mathrm{d}t.$$

$\boxed{\mathbf{I_2}}$

Beginning with an application of (HT),

$$I_2 \leq \int_0^T \|2\nu(\psi_1) - 2\nu(\psi_2)\|_{L^\infty} \|D\mathbf{u_2}\| \|\nabla \mathbf{u}\| \, \mathrm{d}t.$$

Also,

$$\|2\nu(\psi_1) - 2\nu(\psi_2)\|_{L^\infty} = \left\| \int_0^1 \nu'(s\psi_1 + (1-s)\psi_2)\psi(t) \, \mathrm{d}s \right\|_{L^\infty} \leq C\|\psi\|_\infty.$$

Thus,

$$I_2 \leq C \int_0^T \|\psi\|_{L^\infty} \|D\mathbf{u_2}\| \|\nabla \mathbf{u}\| \, \mathrm{d}t \overset{\text{(BG)}}{\leq} C \int_0^T \|\nabla\psi\| \ln^{\frac{1}{2}} \left(e\frac{\|\psi\|_{H^2}}{\|\nabla\psi\|} \right) \|D\mathbf{u_2}\| \|\nabla \mathbf{u}\| \, \mathrm{d}t$$
$$\overset{\text{(Y2)}}{\leq} \epsilon \int_0^T \|\nabla \mathbf{u}\|^2 \, \mathrm{d}t + \frac{C}{4\epsilon} \int_0^T \|\nabla\psi\|^2 \ln \left(e\frac{\|\psi\|_{H^2}}{\|\nabla\psi\|} \right) \|D\mathbf{u_2}\|^2 \, \mathrm{d}t.$$

By exploiting properties of the logarithm and the fact that $\psi \in L^\infty(0,T; H^2(\Omega))$, we finally have

$$I_2 \leq \epsilon_2 \int_0^T \|\nabla \mathbf{u}\|^2 \, \mathrm{d}t + c_2 \int_0^T \|\nabla\psi\|^2 \ln \left(\frac{c_2}{\|\nabla\psi\|^2} \right) \|D\mathbf{u_2}\|^2 \, \mathrm{d}t.$$

$\boxed{\mathbf{I_3}}$

Applying Holder and Young's (Y2) inequalities yields

$$I_3 = \int_0^T (\nabla\psi_2 \otimes \nabla\Delta\psi, \nabla \mathbf{u}) \, \mathrm{d}t \overset{\text{(HT)}}{\leq} \int_0^T \|\nabla\psi_2\|_{L^\infty} \|\nabla\Delta\psi\| \|\nabla \mathbf{u}\| \, \mathrm{d}t$$
$$\overset{\text{(Y2)}}{\leq} \epsilon \int_0^T \|\nabla \mathbf{u}\|^2 \, \mathrm{d}t + \frac{1}{4\epsilon} \int_0^T \|\nabla\psi_2\|^2_{L^\infty} \|\nabla\Delta\psi\|^2 \, \mathrm{d}t.$$

We now turn to estimate $\|\nabla\Delta\psi\|^2$. Firstly, we have

$$\|\nabla\Delta\psi\|^2 = (\nabla\Delta\psi, \nabla\Delta\psi) \overset{IBP}{=} -(\Delta\psi, \Delta^2\psi) \leq \|\Delta\psi\| \|\Delta^2\psi\|. \tag{6.5}$$

Therefore,

$$\|\nabla\psi_2\|^2_{L^\infty} \|\nabla\Delta\psi\|^2 \overset{(6.5)}{\leq} \|\nabla\psi_2\|^2_{L^\infty} \|\Delta\psi\| \|\Delta^2\psi\| \leq \epsilon' \|\Delta^2\psi\|^2 + \frac{1}{4\epsilon'} \|\nabla\psi_2\|^4_{L^\infty} \|\Delta\psi\|^2.$$

Also, we have by the use of integration by parts that

$$\|\Delta\psi\|^2 = (\Delta\psi, \Delta\psi) = -(\nabla\psi, \nabla\Delta\psi) \leq \|\nabla\psi\|\|\nabla\Delta\psi\| \leq \|\nabla\psi\|\|\Delta\psi\|^{\frac{1}{2}}\|\Delta^2\psi\|^{\frac{1}{2}},$$

which entails

$$\|\Delta\psi\|^2 \leq \|\nabla\psi\|^{\frac{4}{3}}\|\Delta^2\psi\|^{\frac{2}{3}}. \tag{6.6}$$

In light of this estimate, we have

$$\begin{aligned}\|\nabla\psi_2\|_{L^\infty}^2\|\nabla\Delta\psi\|^2 &\leq \epsilon'\|\Delta^2\psi\|^2 + \frac{1}{4\epsilon'}\|\nabla\psi_2\|_{L^\infty}^4\|\nabla\psi\|^{\frac{4}{3}}\|\Delta^2\psi\|^{\frac{2}{3}} \\ &= \epsilon'\|\Delta^2\psi\|^2 + \frac{1}{4\epsilon'}\|\nabla\psi_2\|_{L^\infty}^4\|\nabla\psi\|^{\frac{4}{3}}\frac{(3\epsilon')^{\frac{1}{3}}}{(3\epsilon')^{\frac{1}{3}}}\|\Delta^2\psi\|^{\frac{2}{3}} \end{aligned}\tag{6.7}$$

Applying (Y1) with $p = 3/2,\ q = 3$ yields

$$\|\nabla\psi_2\|_{L^\infty}^2\|\nabla\Delta\psi\|^2 \leq 2\epsilon'\|\Delta^2\psi\|^2 + C\|\nabla\psi_2\|_{L^\infty}^6\|\nabla\psi\|^2$$

From the above working, we deduce the following two facts which will be useful for later estimates

$$\|\nabla\psi\|^2 \leq \|\nabla\psi\|^2 + \|\Delta^2\psi\|^2, \tag{Δ}$$

$$\|\nabla\Delta\psi\|^2 \leq 2\epsilon\|\Delta^2\psi\|^2 + C\|\nabla\psi\|^2. \tag{∇}$$

Returning to our estimate, applying (6.7) and taking $\epsilon = \frac{\nu_*}{\sqrt{2}}$ gives us

$$I_3 \leq \frac{\nu_*}{\sqrt{2}}\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + \frac{\epsilon_3\sqrt{2}}{2\nu_*}\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_3\int_0^T \|\nabla\psi_2\|_{L^\infty}^6\|\nabla\psi\|^2\,\mathrm{d}t.$$

$\boxed{\mathbf{I_4}}$

We proceed in a similar fashion to I_3.

$$\begin{aligned} I_4 &= \int_0^T (\nabla\psi\otimes\nabla\Delta\psi_1, \nabla\mathbf{u})\,\mathrm{d}t \overset{\text{(HT)}}{\leq} \int_0^T \|\nabla\Delta\psi_1\|_{L^4}\|\nabla\psi\|_{L^4}\|\nabla\mathbf{u}\|\,\mathrm{d}t \\ &\overset{\text{(LZ)}}{\leq} C\int_0^T \|\nabla\Delta\psi_1\|_{L^4}\|\nabla\psi\|^{\frac{1}{2}}\|\nabla\psi\|_{H^1}^{\frac{1}{2}}\|\nabla\mathbf{u}\|\,\mathrm{d}t \leq C\int_0^T \|\nabla\Delta\psi_1\|_{L^4}\|\Delta\psi\|\|\nabla\mathbf{u}\|\,\mathrm{d}t \\ &\overset{\text{(Y2)}}{\leq} \epsilon\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + \frac{C}{4\epsilon}\int_0^T \|\nabla\Delta\psi_1\|_{L^4}^2\|\Delta\psi\|^2\,\mathrm{d}t. \end{aligned}$$

We now need to estimate $\|\Delta\psi\|^2$. By use of integration by parts,

$$\|\Delta\psi\|^2 = (\Delta\psi, \Delta\psi) = -(\nabla\psi, \nabla\Delta\psi) \leq \|\nabla\psi\|\|\nabla\Delta\psi\|. \tag{6.8}$$

Therefore,

$$\begin{aligned}\|\nabla\Delta\psi_1\|_{L^4}^2\|\Delta\psi\|^2 &\leq \|\nabla\Delta\psi_1\|_{L^4}^2\|\nabla\psi\|\|\nabla\Delta\psi\| \leq \epsilon\|\nabla\Delta\psi\|^2 + \frac{1}{4\epsilon}\|\nabla\Delta\psi_1\|_{L^4}^4\|\nabla\psi\|^2 \\ &\overset{(\nabla)}{\leq} 2\epsilon'\|\Delta^2\psi\|^2 + C\|\nabla\Delta\psi_1\|_{L^4}^4\|\nabla\psi\|^2. \end{aligned}$$

The above estimate allows us to finally conclude that

$$I_4 \leq \epsilon_4\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + c_4\epsilon_4'\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_4'\int_0^T \|\nabla\Delta\psi_1\|_{L^4}^4\|\nabla\psi\|^2\,\mathrm{d}t.$$

$\boxed{\mathbf{I_5}}$

Since $\psi_2 \in L^2(0,T;H^4_N(\Omega))$, we have by (HT) that

$$I_5 = \int_0^T b(\nabla\Delta\psi, \nabla\psi_2, \mathbf{u})\,\mathrm{d}t \overset{\text{(HT)}}{\leq} \int_0^T \|\nabla\Delta\psi\|_{L^4}\|\nabla^2\psi_2\|_{L^2}\|\mathbf{u}\|_{L^4}\,\mathrm{d}t \leq C\int_0^T \|\nabla\Delta\psi\|_{L^4}\|\mathbf{u}\|_{L^4}\,\mathrm{d}t.$$

Furthermore, note that there exists $C>0$ such that $\|\nabla\Delta\psi\|_{H^1} \leq \|\psi\|_{H^4} \leq C\|\Delta^2\psi\|$. This is a consequence of the fact that $\overline{\psi}=0$ and the Poincare-Wirtinger (PW) inequality. Thus,

$$I_5 \leq C\int_0^T \|\nabla\Delta\psi\|^{\frac{1}{2}}\|\Delta^2\psi\|^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}\|\nabla\mathbf{u}\|^{\frac{1}{2}}\,\mathrm{d}t \overset{\text{(Y2)}}{\leq} \epsilon\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + \frac{1}{4\epsilon}\int_0^T \|\nabla\Delta\psi\|\|\Delta^2\psi\|\,\mathrm{d}t,$$

where we have used the regular Poincare inequality to deduce $\|\mathbf{u}\|\|\nabla\mathbf{u}\| \leq \|\nabla\mathbf{u}\|^2$. Using (Y2) and ($\nabla$) yields

$$I_5 \overset{\text{(LZ)}}{\leq} \epsilon_5\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + \left(\epsilon_5 + \frac{2\epsilon_5''}{16\epsilon_5\epsilon_5'}\right)\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_5\int_0^T \|\nabla\psi\|^2\,\mathrm{d}t.$$

Note that since we can choose each of $\epsilon_5, \epsilon_5', \epsilon_5''$ arbitrarily small (but positive), the coefficients of the first two integrals can be made as small as we wish. This is an important remark as we will eventually want to absorb these terms into the left hand side of an inequality, to be seen.

$\boxed{\mathbf{I_6}}$

Proceeding in a similar way to I_5 with (HT),

$$\begin{aligned} I_6 &= \int_0^T b(\nabla\Delta\psi_1, \nabla\psi_1, \mathbf{u})\,\mathrm{d}t \leq \int_0^T \|\nabla\Delta\psi_1\|_{L^4}\|\nabla^2\psi\|\|\mathbf{u}\|_{L^4}\,\mathrm{d}t \\ &\leq C\int_0^T \|\nabla\Delta\psi_1\|_{L^4}\|\Delta\psi\|\|\mathbf{u}\|^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}_{H^1}\,\mathrm{d}t, \end{aligned}$$

where we have used $\|\nabla^2\psi\| \leq \|\psi\|_{H^2} \leq C\|\Delta\psi\|$. Next, using the estimate $\|\Delta\psi\| \leq \|\nabla\psi\|^{\frac{2}{3}}\|\Delta^2\psi\|^{\frac{1}{3}}$ (see (6.6)) and $\|\mathbf{u}\|_{H^1} \leq C\|\nabla\mathbf{u}\|$ once more gives us that

$$\begin{aligned} I_6 &\leq C\int_0^T \|\nabla\Delta\psi_1\|_{L^4}\|\nabla\psi\|^{\frac{2}{3}}\|\Delta^2\psi\|^{\frac{1}{3}}\|\nabla\mathbf{u}\|\,\mathrm{d}t \\ &\overset{\text{(Y1)}}{\leq} \frac{\epsilon^4}{4}\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + \frac{3C^{\frac{4}{3}}}{4\epsilon^{\frac{4}{3}}}\int_0^T \|\nabla\Delta\psi_1\|^{\frac{4}{3}}_{L^4}\|\nabla\psi\|^{\frac{8}{9}}\|\Delta^2\psi\|^{\frac{4}{9}}\|\mathbf{u}\|^{\frac{2}{3}}\,\mathrm{d}t, \end{aligned}$$

where we have used (Y1) with $p=4,\ q=4/3$ above (i.e., the estimate $ab \leq \frac{a^4}{4} + \frac{b^{\frac{4}{3}}}{\frac{4}{3}}$). Applying (Y1) again with $p=9/2,\ q=9/7$ (i.e., the estimate $ab \leq \frac{a^{\frac{9}{2}}}{\frac{9}{2}} + \frac{b^{\frac{9}{7}}}{\frac{9}{7}}$) to the latter integral yields

$$I_6 \leq \frac{\epsilon^4}{4}\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + \left(\frac{3}{4\epsilon^{\frac{4}{3}}}\right)^{\frac{9}{2}}(\epsilon')^{\frac{9}{2}}\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + \frac{7}{9\epsilon^{\frac{9}{7}}}\int_0^T \|\nabla\Delta\psi_1\|^{\frac{12}{7}}_{L^4}\|\nabla\psi\|^{\frac{8}{7}}\|\mathbf{u}\|^{\frac{6}{7}}\,\mathrm{d}t.$$

Applying (Y1) a final time with $p=7/4,\ q=7/3$ (i.e., the estimate $ab \leq \frac{a^{\frac{7}{4}}}{\frac{7}{4}} + \frac{b^{\frac{7}{3}}}{\frac{7}{3}}$) gives us that

$$I_6 \leq \frac{\epsilon_6^4}{4}\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + c_6(\epsilon_6')^{\frac{9}{2}}\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_6'\int_0^T \|\nabla\Delta\psi_1\|^{\frac{12}{7}}_{L^4}(\|\nabla\psi\|^2 + \|\mathbf{u}\|^2)\,\mathrm{d}t.$$

$\boxed{\mathbf{I_7}}$

Using the fact that $\psi_2 \in L^\infty(0,T;H^2(\Omega)) \hookrightarrow L^\infty(0,T;L^\infty(\Omega))$, we have

$$
\begin{aligned}
I_7 = \int_0^T (\nabla\Delta\psi \cdot \mathbf{u}, \psi_2)\,\mathrm{d}t \le \int_0^T \|\psi_2\|_{L^\infty}\|\nabla\Delta\psi\|\|\mathbf{u}\|\,\mathrm{d}t &\overset{(\mathrm{Y2})}{\le} C\epsilon\int_0^T \|\nabla\Delta\psi\|^2\,\mathrm{d}t + \frac{1}{4\epsilon}\int_0^T \|\mathbf{u}\|^2\,\mathrm{d}t \\
&\le 2c_7\epsilon_7\epsilon_7'\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_7'\int_0^T \|\nabla\psi\|^2\,\mathrm{d}t + \frac{1}{4\epsilon_7}\int_0^T \|\mathbf{u}\|^2\,\mathrm{d}t.
\end{aligned}
$$

$\boxed{\mathbf{I_8}}$

In a similar way to previous estimates, we begin by applying Holder and Young's inequalities.

$$
\begin{aligned}
I_8 = \int_0^T (\nabla\Delta\psi_1 \cdot \mathbf{u}, \psi)\,\mathrm{d}t &\overset{(\mathrm{HT})}{\le} \int_0^T \|\nabla\Delta\psi_1\|_{L^4}\|\mathbf{u}\|\|\psi\|_{L^4}\,\mathrm{d}t \\
&\overset{(\mathrm{Y2})}{\le} \frac{1}{4\epsilon}\int_0^T \|\nabla\Delta\psi_1\|_{L^4}^2\|\mathbf{u}\|^2\,\mathrm{d}t + \epsilon\int_0^T \|\psi\|_{L^4}^2.
\end{aligned}
$$

Using Ladyzhenskaya's inequality (LZ), we have

$$
\|\psi\|_{L^4}^2 \le \|\psi\|\|\psi\|_{H^1} \le \|\psi\|_{H^1}^2 = \|\psi\|^2 + \|\nabla\psi\|^2 \overset{(\mathrm{PW})}{\le} C\|\nabla\psi\|^2. \tag{6.9}
$$

Applying (6.9) gives us

$$
I_8 \le c_8\int_0^T \|\nabla\Delta\psi_1\|_{L^4}^2\|\mathbf{u}\|^2\,\mathrm{d}t + c_8'\int_0^T \|\nabla\psi\|^2\,\mathrm{d}t.
$$

Collecting the above estimates, we have that for all $T > 0$,

$$
\begin{aligned}
\frac{1}{2}\|\mathbf{u}(T)\|^2 + \frac{2\nu_*}{\sqrt{2}}\int_0^T \|\nabla\mathbf{u}(t)\|^2\,\mathrm{d}t \le{}& \frac{1}{2}\|\mathbf{u}(0)\|^2 + c_1\epsilon_1\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + c_1'\int_0^T \|\mathbf{u}\|^2\|\nabla\mathbf{u}_1\|^2\,\mathrm{d}t \\
&+ \epsilon_2\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + c_2\int_0^T \|\nabla\psi\|^2\ln\left(\frac{c_2}{\|\nabla\psi\|^2}\right)\|D\mathbf{u}_2\|^2\,\mathrm{d}t \\
&+ \frac{\nu_*}{\sqrt{2}}\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + \frac{\epsilon_3\sqrt{2}}{2\nu_*}\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_3\int_0^T \|\nabla\psi_2\|_{L^\infty}^6\|\nabla\psi\|^2\,\mathrm{d}t \\
&+ \epsilon_4\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + c_4\epsilon_4'\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_4'\int_0^T \|\nabla\Delta\psi_1\|_{L^4}^4\|\nabla\psi\|^2\,\mathrm{d}t \\
&+ \epsilon_5\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + \left(\epsilon_5 + \frac{2\epsilon_5''}{16\epsilon_5\epsilon_5'}\right)\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_5\int_0^T \|\nabla\psi\|^2\,\mathrm{d}t \\
&+ \frac{\epsilon_6^4}{4}\int_0^T \|\nabla\mathbf{u}\|^2\,\mathrm{d}t + c_6(\epsilon_6')^{\frac{9}{2}}\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_6'\int_0^T \|\nabla\Delta\psi\|_{L^4}^{\frac{12}{7}}(\|\nabla\psi\|^2 + \|\mathbf{u}\|^2)\,\mathrm{d}t \\
&+ c_7\epsilon_7\epsilon_7'\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_7'\int_0^T \|\nabla\psi\|^2\,\mathrm{d}t + \frac{1}{4\epsilon_7}\int_0^T \|\mathbf{u}\|^2\,\mathrm{d}t \\
&+ c_8\int_0^T \|\nabla\Delta\psi_1\|_{L^4}^2\|\mathbf{u}\|^2\,\mathrm{d}t + c_8'\int_0^T \|\nabla\psi\|^2\,\mathrm{d}t.
\end{aligned}
$$

Choosing ϵ_i and ϵ_i' appropriately, the above inequality leads to

$$\frac{1}{2}\|\mathbf{u}(T)\|^2 + a_1\int_0^T \|\nabla\mathbf{u}(t)\|^2\,\mathrm{d}t \leq \frac{1}{2}\|\mathbf{u}(0)\|^2 + f(\epsilon_i,\epsilon_i')\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t$$
$$+\int_0^T \|\nabla\psi\|^2\left\{c_2\ln\left(\frac{c_2}{\|\nabla\psi\|^2}\right)\|D\mathbf{u}_2\|^2 + c_3\|\nabla\psi_2\|_{L^\infty}^6 + c_4'\|\nabla\Delta\psi_1\|_{L^4}^4 + c_5 + c_6'\|\nabla\Delta\psi_1\|_{L^4}^{\frac{12}{7}} + c_7' + c_8'\right\}\mathrm{d}t$$
$$+\int_0^T \|\mathbf{u}(t)\|^2\left\{c_1'\|\nabla\mathbf{u}_1(t)\|^2 + c_6'\|\nabla\Delta\psi_1\|_{L^4}^{\frac{12}{7}} + \frac{1}{4\epsilon_7} + c_8\|\nabla\Delta\psi_1\|_{L^4}^2\right\}\mathrm{d}t, \qquad \boxed{\mathbf{U1}}$$

for all $T>0$, where $a_1>0$ and $f(\epsilon_i,\epsilon_i')>0$ can be made arbitrarily small.

6.2 The Swift-Hohenberg energy equation

Next, we turn our attention to the second equation in the weak formulation, namely

$$\langle\partial_t\psi, v\rangle_{(H^1)^*\times H^1} + ((\mathbf{u}\cdot\nabla\psi),v) + (M(\psi)\nabla\mu,\nabla v) = 0 \quad \forall\, v\in H^1(\Omega).$$

Proceeding in a similar fashion to the prior working and taking $\mathbf{v}=\Delta\psi$ as a test function, we have that the following equations are satisfied for $(\mathbf{u}_i,\psi_i,\mu_i)$ for $i=1,2$:

$$\int_0^T\langle\partial_t\psi_i,\Delta\psi\rangle_{(H^1)^*\times H^1}\,\mathrm{d}t + \int_0^T((\mathbf{u}_i\cdot\nabla\psi_i),\Delta\psi)\,\mathrm{d}t + \int_0^T(\nabla\mu_i,\nabla\Delta\psi)\,\mathrm{d}t = 0, \tag{6.10}$$

Subtracting (6.10) with $i=2$ from (6.10) with $i=1$ and noting that

$$((\mathbf{u}_1\cdot\nabla\psi_1),\Delta\psi) - ((\mathbf{u}_2\cdot\nabla\psi_2),\Delta\psi) = ((\mathbf{u}\cdot\nabla\psi_1),\Delta\psi) + ((\mathbf{u}_2\cdot\nabla\psi),\Delta\psi),$$

yields that for all $T>0$,

$$\frac{1}{2}\|\nabla\psi(T)\|^2 - \int_0^T((\mathbf{u}(t)\cdot\nabla\psi_1(t)),\Delta\psi(t))\,\mathrm{d}t - \int_0^T((\mathbf{u}_2(t)\cdot\nabla\psi(t)),\Delta\psi(t))\,\mathrm{d}t$$
$$-\int_0^T(\nabla\mu(t),\nabla\Delta\psi(t))\,\mathrm{d}t = \frac{1}{2}\|\nabla\psi(0)\|^2. \tag{6.11}$$

We can simplify the term involving the chemical potential μ. Firstly, integrating by parts and using the definition of μ gives us

$$\begin{aligned}(\nabla\mu,\nabla\Delta\psi) &= -(\mu,\Delta^2\psi) = -\int_\Omega\left(\psi_1^3-\psi_2^3+(r+1)\psi+2\Delta\psi+\Delta^2\psi\right)\Delta^2\psi\,\mathrm{d}x\\ &= -\int_\Omega(\psi_1^3-\psi_2^3)\Delta^2\psi\,\mathrm{d}x - (r+1)\int_\Omega\psi\Delta^2\psi\,\mathrm{d}x - 2\int_\Omega\Delta\psi\Delta^2\psi\,\mathrm{d}x - \int_\Omega|\Delta^2\psi|^2\,\mathrm{d}x\\ &= -\int_\Omega(\psi_1^3-\psi_2^3)\Delta^2\psi\,\mathrm{d}x - (r+1)\int_\Omega\psi\Delta^2\psi\,\mathrm{d}x + 2\int_\Omega|\nabla\Delta\psi|^2\,\mathrm{d}x - \int_\Omega|\Delta^2\psi|^2\,\mathrm{d}x.\end{aligned}$$

Thus, (6.11) becomes

$$\frac{1}{2}\|\nabla\psi(T)\|^2 + \int_0^T\|\Delta^2\psi\|^2\,\mathrm{d}t = \frac{1}{2}\|\nabla\psi(0)\|^2 - \int_0^T(\psi_1^3-\psi_2^3,\Delta^2\psi)\,\mathrm{d}t - (r+1)\int_0^T(\psi,\Delta^2\psi)\,\mathrm{d}t$$
$$+\int_0^T(\mathbf{u}\cdot\nabla\psi_1,\Delta\psi)\,\mathrm{d}t + \int_0^T(\mathbf{u}_2\cdot\nabla\psi,\Delta\psi)\,\mathrm{d}t + 2\int_0^T\|\nabla\Delta\psi\|^2\,\mathrm{d}t,$$

for all $T>0$, which can be more conveniently expressed as

$$\frac{1}{2}\|\nabla\psi(T)\|^2 + \int_0^T\|\Delta^2\psi\|^2\,\mathrm{d}t = \frac{1}{2}\|\nabla\psi(0)\|^2 + \sum_{j=9}^{13}I_j, \qquad \text{for all } T>0.$$

We now proceed to estimate $I_9,\ldots,I_{13}$ in a similar spirit to our estimates for $I_1,\ldots,I_8$.

$\boxed{\mathbf{I_9}}$

Firstly, note that $|\psi_1^3 - \psi_2^3| = |(\psi_1 - \psi_2)(\psi_1^2 + \psi_1\psi_2 + \psi_2^3)|$. In light of this equality, the fact that $\psi_i \in L^\infty(0,T;H^2(\Omega)) \hookrightarrow L^\infty(0,T;L^\infty(\Omega))$ and the repeated use of Holder's inequality, we have

$$\begin{aligned}|(\psi_1^3 - \psi_2^3, \Delta^2\psi)| &\leq (|\psi_1 - \psi_2||\psi_1^2 + \psi_1\psi_2 + \psi_2^2|, |\Delta^2\psi|)\\ &\leq \|\psi_1^2 + \psi_1\psi_2 + \psi_2^2\|_{L^\infty}(|\psi_1 - \psi_2|, |\Delta^2\psi|) \leq C(|\psi|, |\Delta^2\psi|)\\ &\leq C\|\psi\|\|\Delta^2\psi\| \overset{\text{(Y2)}}{\leq} \epsilon\|\Delta^2\psi\|^2 + \frac{C}{4\epsilon}\|\nabla\psi\|^2.\end{aligned}$$

Therefore,

$$I_9 = \int_0^T (\psi_1^3 - \psi_2^3, \Delta^2\psi)\,\mathrm{d}t \leq \epsilon_9 \int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t + c_9 \int_0^T \|\nabla\psi\|^2\,\mathrm{d}t.$$

$\boxed{\mathbf{I_{10}}}$

We have

$$(\psi, \Delta^2\psi) \leq \|\psi\|\|\Delta^2\psi\| \leq \frac{1}{4\epsilon}\|\psi\|^2 + \epsilon\|\Delta^2\psi\|^2 \leq \frac{C}{4\epsilon}\|\nabla\psi\|^2 + \epsilon\|\Delta^2\psi\|^2.$$

Thus,

$$I_{10} = -(r+1)\int_0^T (\psi, \Delta^2\psi)\,\mathrm{d}t \leq c_{10}\int_0^T \|\nabla\psi\|^2\,\mathrm{d}t + |r+1|\epsilon_{10}\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t.$$

$\boxed{\mathbf{I_{11}}}$

We have

$$\begin{aligned}I_{11} &= \int_0^T (\mathbf{u}\cdot\nabla\psi_1, \Delta\psi)\,\mathrm{d}t \overset{\text{(HT)}}{\leq} \int_0^T \|\nabla\psi_1\|_{L^\infty}\|\mathbf{u}\|\|\Delta\psi\|\,\mathrm{d}t\\ &\overset{\text{(Y2)}}{\leq} \frac{1}{4\epsilon}\int_0^T \|\mathbf{u}\|^2\|\nabla\psi_1\|_{L^\infty}^2\,\mathrm{d}t + \epsilon\int_0^T \|\Delta\psi\|^2\,\mathrm{d}t\\ &\overset{(\Delta)}{\leq} \frac{1}{4\epsilon}\int_0^T \|\mathbf{u}\|^2\|\nabla\psi_1\|_{L^\infty}^2\,\mathrm{d}t + c_{11}'\int_0^T \|\nabla\psi\|^2\,\mathrm{d}t + \epsilon_{11}\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t.\end{aligned}$$

$\boxed{\mathbf{I_{12}}}$

Integrating by parts, we have

$$\begin{aligned}I_{12} &= \int_0^T (\mathbf{u}_2\cdot\nabla\psi, \Delta\psi)\,\mathrm{d}t = -\int_0^T (\psi\mathbf{u}_2, \nabla\Delta\psi)\,\mathrm{d}t \overset{\text{(HT)}}{\leq} \int_0^T \|\mathbf{u}_2\|_{L^3}\|\psi\|_{L^6}\|\nabla\Delta\psi\|\,\mathrm{d}t\\ &\overset{\text{(S1)+(Y2)}}{\leq} C\epsilon\int_0^T \|\mathbf{u}_2\|_{L^3}^2\|\psi\|_{H^1}^2\,\mathrm{d}t + \frac{1}{4\epsilon}\int_0^T \|\nabla\Delta\psi\|^2\,\mathrm{d}t.\end{aligned}$$

Next, using the inequality (∇) obtained from I_3 and the fact that $\|\psi\|_{H^1}^2 \le C\|\nabla\psi\|^2$ (this is a consequence of Poincare-Wirtinger and that $\overline{\psi} = 0$), we obtain

$$
\begin{aligned}
I_{12} &\le C \int_0^T \|\mathbf{u}_2\|_{L^3}^2 \|\nabla\psi\|^2 \,\mathrm{d}t + \frac{1}{4\epsilon} \int_0^T \left(2\epsilon' \|\Delta^2\psi\|^2 + C\|\nabla\psi\|^2\right) \mathrm{d}t \\
&\le c_{12} \int_0^T \|\mathbf{u}_2\|_{L^3}^2 \|\nabla\psi\|^2 \,\mathrm{d}t + \frac{\epsilon_{12}'}{2\epsilon_{12}} \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t.
\end{aligned}
$$

$\boxed{\mathbf{I_{13}}}$

A simple application of (∇) yields

$$
\begin{aligned}
I_{13} = 2\int_0^T \|\nabla\Delta\psi\|^2 \,\mathrm{d}t &\overset{(\nabla)}{\le} 2\int_0^T 2\epsilon\|\Delta^2\psi\|^2 + C\|\nabla\psi\|^2 \,\mathrm{d}t \\
&\le c_{13} \int_0^T \|\nabla\psi\|^2 \,\mathrm{d}t + 4\epsilon_{13} \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t.
\end{aligned}
$$

6.3 Tying the ends together

We are now ready to complete the proof of uniqueness. Collecting the estimates $I_9, ...I_{13}$, we see that for each $T > 0$,

$$
\begin{aligned}
\frac{1}{2}\|\nabla\psi(T)\|^2 + \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t \le \frac{1}{2}&\|\nabla\psi(0)\|^2 + c_9 \int_0^T \|\nabla\psi\|^2 \,\mathrm{d}t + \epsilon_9 \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t \\
&+ c_{10} \int_0^T \|\nabla\psi\|^2 \,\mathrm{d}t + |r+1|\epsilon_{10} \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t \\
&+ c_{11} \int_0^T \|\mathbf{u}\|^2 \|\nabla\psi_1\|_{L^\infty}^2 \,\mathrm{d}t + c_{11}' \int_0^T \|\nabla\psi\|^2 \,\mathrm{d}t + \epsilon_{11} \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t \\
&+ c_{12} \int_0^T \|\mathbf{u}_2\|_{L^3}^2 \|\nabla\psi\|^2 \,\mathrm{d}t + \frac{\epsilon_{12}'}{2\epsilon_{12}} \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t \\
&+ c_{13} \int_0^T \|\nabla\psi\|^2 \,\mathrm{d}t + 4\epsilon_{13} \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t.
\end{aligned}
$$

Collecting like terms leads to

$$
\begin{aligned}
\frac{1}{2}\|\nabla\psi(T)\|^2 + \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t \le \frac{1}{2}\|\nabla\psi(0)\|^2 + \int_0^T \|\nabla\psi\|^2 \left(c_9 + c_{10} + c_{11}' + c_{12}\|\mathbf{u}_2\|_{L^3}^2 + c_{13}\right) \mathrm{d}t \\
+ (\epsilon_9 + |r+1|\epsilon_{10} + \epsilon_{11} + \frac{\epsilon_{12}'}{2\epsilon_{12}} + 4\epsilon_{13}) \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t + c_{11} \int_0^T \|\mathbf{u}\|^2 \|\nabla\psi_1\|_{L^\infty}^2 \,\mathrm{d}t,
\end{aligned}
$$

for all $T > 0$. Choosing ϵ_i, ϵ_i' appropriately gives us

$$
\begin{aligned}
&\frac{1}{2}\|\nabla\psi(T)\|^2 + a_2 \int_0^T \|\Delta^2\psi\|^2 \,\mathrm{d}t \le \frac{1}{2}\|\nabla\psi(0)\|^2 \\
&\quad + \int_0^T \|\nabla\psi\|^2 \left(c_9 + c_{10} + c_{11}' + c_{12}\|\mathbf{u}_2\|_{L^3}^2 + c_{13}\right) \mathrm{d}t + c_{11} \int_0^T \|\mathbf{u}\|^2 \|\nabla\psi_1\|_{L^\infty}^2 \,\mathrm{d}t, \qquad \boxed{\mathsf{U2}}
\end{aligned}
$$

for all $T > 0$, where $a_2, c_i, c_i' > 0$. Adding $\boxed{\textbf{U1}}$ to $\boxed{\textbf{U2}}$ gives us

$$\frac{1}{2}\|\mathbf{u}(T)\|^2 + \frac{1}{2}\|\nabla\psi(t)\|^2 + a_1\int_0^T \|\nabla\mathbf{u}(t)\|^2\,\mathrm{d}t + a_2\int_0^T \|\Delta^2\psi\|^2\,\mathrm{d}t \le \frac{1}{2}\|\mathbf{u}(0)\|^2 + \frac{1}{2}\|\nabla\psi(0)\|^2$$

$$+\int_0^T \|\nabla\psi\|^2\Bigg\{c_2\ln\left(\frac{c_2}{\|\nabla\psi\|^2}\right)\|D\mathbf{u}_2\|^2 + c_3\|\nabla\psi_2\|_{L^\infty}^6 + c_4'\|\nabla\Delta\psi_1\|_{L^4}^4 + c_5 +$$

$$c_6'\|\nabla\Delta\psi_1\|_{L^4}^{\frac{12}{7}} + c_7' + c_8' + c_9 + c_{10} + c_{11}' + c_{12}\|\mathbf{u}_2\|_{L^3}^2 + c_{13}\Bigg\}\,\mathrm{d}t \tag{A}$$

$$+\int_0^T \|\mathbf{u}(T)\|^2\left\{c_1'\|\nabla\mathbf{u}_1(t)\|^2 + c_6'\|\nabla\Delta\psi_1\|_{L^4}^{\frac{12}{7}} + \frac{1}{4\epsilon_7} + c_8\|\nabla\Delta\psi_1\|_{L^4}^2 + c_{11}\|\nabla\psi_1\|_{L^\infty}^2\right\}\,\mathrm{d}t,$$

for all $T > 0$, where $a_i, c_i, c_i' > 0$. Defining

$$H(t) := c_3\|\nabla\psi_2\|_{L^\infty}^6 + c_4'\|\nabla\Delta\psi_1\|_{L^4}^4 + c_5 + c_6'\|\nabla\Delta\psi_1\|_{L^4}^{\frac{12}{7}} + c_7' + c_8' + c_9 + c_{10} + c_{11}' + c_{12}\|\mathbf{u}_2\|_{L^3}^2 + c_{13},$$

we have

$$(\mathrm{A}) = \int_0^T \|\nabla\psi\|^2\left\{c_3\ln\left(\frac{c_3}{\|\nabla\psi\|^2}\right)\|D\mathbf{u}_2\|^2 + H(t)\right\}\,\mathrm{d}t.$$

Since

$$1 \le \ln\left(e\frac{\|\psi\|_{H^2}}{\|\nabla\psi\|}\right) \le \ln\left(\frac{c}{\|\nabla\psi\|}\right) \le \tilde{C}\ln\left(\frac{c_3}{\|\nabla\psi\|^2}\right)$$

for some $\tilde{C} > 0$ independent of t, we can say that

$$(\mathrm{A}) \le \int_0^T \|\nabla\psi\|^2\left\{c_3\ln\left(\frac{c_3}{\|\nabla\psi\|^2}\right)\|D\mathbf{u}_2\|^2 + \tilde{C}\ln\left(\frac{c_3}{\|\nabla\psi\|^2}\right)H(t)\right\}.$$

Thus, there exists $C > 0$ such that

$$(\mathrm{A}) \le C\int_0^T \|\nabla\psi\|^2\ln\left(\frac{c_3}{\|\nabla\psi\|^2}\right)\{\|D\mathbf{u}_2\|^2 + H(t)\}\,\mathrm{d}t.$$

Defining the following functions which belong to $L^1(0,T)$ (as a consequence of the function spaces inhabited by $(\mathbf{u}_i, \psi_i, \mu_i)$ for $i = 1, 2$):

$$G_1(t) := \|D\mathbf{u}_2\|^2 + H(t),$$
$$G_2(t) := c_1'\|\nabla\mathbf{u}_1(t)\|^2 + c_6'\|\nabla\Delta\psi_1\|_{L^4}^{\frac{12}{7}} + \frac{1}{4\epsilon_7} + c_8\|\nabla\Delta\psi_1\|_{L^4}^2,$$

and noting that $\|\mathbf{u}(0)\| = \|\mathbf{u}_1(0) - \mathbf{u}_2(0)\| = \|\mathbf{u}_0 - \mathbf{u}_0\| = 0 = \|\nabla\psi(0)\|$, we obtain

$$\frac{1}{2}\|\mathbf{u}(T)\|^2 + \frac{1}{2}\|\nabla\psi(T)\|^2 + a_1\int_0^T \|\nabla\mathbf{u}(t)\|^2\,\mathrm{d}t + a_2\int_0^T \|\Delta^2\psi(t)\|^2\,\mathrm{d}t \le$$
$$C\int_0^T \|\nabla\psi\|^2\ln\left(\frac{c_3}{\|\nabla\psi\|^2}\right)G_1(t)\,\mathrm{d}t + \int_0^T \|\mathbf{u}(t)\|^2 G_2(t)\,\mathrm{d}t, \quad \text{for all } T > 0.$$

Additionally defining $F(t) := \frac{1}{2}\|\mathbf{u}(t)\|^2 + \frac{1}{2}\|\nabla\psi(t)\|^2$, the above inequality implies that

$$F(T) \le C\int_0^T \|\nabla\psi\|^2\ln\left(\frac{c_3}{\|\nabla\psi\|^2}\right)G_1(t)\,\mathrm{d}t + \int_0^T \|\mathbf{u}(t)\|^2 G_2(t)\,\mathrm{d}t$$
$$\le C\int_0^T \|\nabla\psi\|^2\ln\left(\frac{c_3}{\|\nabla\psi\|^2}\right)(G_1(t) + G_2(t))\,\mathrm{d}t, \quad \text{for all } T > 0.$$

Notice that $x\ln\!\left(\frac{c}{x}\right)$ is an increasing function for $x < \frac{c}{e}$, so there exists $C > 0$ independent of t such that

$$\|\nabla\psi\|^2 \ln\!\left(\frac{c_3}{\|\nabla\psi\|^2}\right) \leq F(t)\ln\!\left(\frac{C}{F(t)}\right),$$

a.e. in $[0, T]$. Thus, we are rewarded with the inequality

$$F(T) \leq C\int_0^T F(t)\ln\left(\frac{C}{F(t)}\right)(G_1(t) + G_2(t))\,\mathrm{d}t, \quad \text{for all } T > 0.$$

In order to appeal to Osgood's Lemma ([8], Appendix B), we finally define $G(t) := G_1(t) + G_2(t),\ W(s) := s\ln\left(\frac{C}{s}\right)$ for $s \geq 0$. Then the above inequality can be expressed as

$$F(T) \leq C\int_0^T W\left(F(t)\right)G(t)\,\mathrm{d}t, \quad \text{for all } T > 0. \tag{6.12}$$

We are now in a position where we can apply the following result:

Osgood's Lemma ([8], Appendix B)

Suppose $F : [0, T] \to [0, M]$ is a measurable function, $G \in L^1(0, T)$ and $W : [0, M] \to \mathbb{R}^+$ is a continuous and non-decreasing function. Assume further that

$$F(t) \leq \int_0^t W(F(s))G(s)\,\mathrm{d}s, \quad \text{for a.e. } t \in [0, T].$$

Provided that $\int_0^M \frac{1}{W(s)}\,\mathrm{d}s = \infty$, we have that $F(t) = 0$ for a.e. $t \in [0, T]$.

It is straightforward to verify that the following facts hold:

- $F : [0, T] \to [0, M]$ is a measurable function, where $M = \sup_{t\geq 0} F(t) < +\infty$ (since $\nabla\psi,\ \mathbf{u} \in L^\infty(0, T; [L^2(\Omega)]^3)$),
- $G \in L^1(0, T)$ since $G_1, G_2 \in L^1(0, T)$,
- we can choose $C > 0$ sufficiently large such that $W(s) = s\ln\left(\frac{C}{s}\right)$ is a continuous non-decreasing function on $0 \leq s \leq M$ taking values in $\mathbb{R}^+$, where $M := \sup_{t\geq 0} F(t)$,
- $\int_0^a \frac{1}{W(s)}\,\mathrm{d}s = \infty$ for any $a \leq C$,

and thus taking C sufficiently large, we deduce from an application of Osgood's lemma that $F(t) = 0$ for almost every $t \in [0, T]$. This implies that $\|\mathbf{u}(t)\| = \|\nabla\psi(t)\| = 0$ a.e. in $[0, T]$, from which we conclude that $(\mathbf{u}_1, \psi_1, \mu_1) = (\mathbf{u}_2, \psi_2, \mu_2)$ almost everywhere in $\Omega \times [0, T]$, as required. $\square$

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