Chapter 2

(Fundamentals of the Analysis of Algorithm Efficiency)

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Fundamentals of the Analysis of Algorithm Efficiency

I often say that when you can measure what you are speaking about and express it in numbers you know something about it; but when you cannot express it in numbers your knowledge is a meagre and unsatisfactory kind: it may be the beginning of knowledge but you have scarcely, in your thoughts, advanced to the stage of science, whatever the matter may be.

—Lord Kelvin (1824–1907)

Not everything that can be counted counts, and not everything that counts can be counted.

—Albert Einstein (1879–1955)

This chapter is devoted to analysis of algorithms. The *American Heritage Dictionary* defines "analysis" as "the separation of an intellectual or substantial whole into its constituent parts for individual study." Accordingly, each of the principal dimensions of an algorithm pointed out in Section 1.2 is both a legitimate and desirable subject of study. But the term "analysis of algorithms" is usually used in a narrower, technical sense to mean an investigation of an algorithm's efficiency with respect to two resources: running time and memory space. This emphasis on efficiency is easy to explain. First, unlike such dimensions as simplicity and generality, efficiency can be studied in precise quantitative terms. Second, one can argue—although this is hardly always the case, given the speed and memory of today's computers—that the efficiency considerations are of primary importance from a practical point of view. In this chapter, we too will limit the discussion to an algorithm's efficiency.

We start with a general framework for analyzing algorithm efficiency in Section 2.1. This section is arguably the most important in the chapter; the fundamental nature of the topic makes it also one of the most important sections in the entire book.

In Section 2.2, we introduce three notations: O ("big oh"), Ω ("big omega"), and Θ ("big theta"). Borrowed from mathematics, these notations have become *the* language for discussing the efficiency of algorithms.

In Section 2.3, we show how the general framework outlined in Section 2.1 can be systematically applied to analyzing the efficiency of nonrecursive algorithms. The main tool of such an analysis is setting up a sum representing the algorithm's running time and then simplifying the sum by using standard sum manipulation techniques.

In Section 2.4, we show how the general framework outlined in Section 2.1 can be systematically applied to analyzing the efficiency of recursive algorithms. Here, the main tool is not a summation but a special kind of equation called a recurrence relation. We explain how such recurrence relations can be set up and then introduce a method for solving them.

Although we illustrate the analysis framework and the methods of its applications by a variety of examples in the first four sections of this chapter, Section 2.5 is devoted to yet another example—that of the Fibonacci numbers. Discovered 800 years ago, this remarkable sequence appears in a variety of applications both within and outside computer science. A discussion of the Fibonacci sequence serves as a natural vehicle for introducing an important class of recurrence relations not solvable by the method of Section 2.4. We also discuss several algorithms for computing the Fibonacci numbers, mostly for the sake of a few general observations about the efficiency of algorithms and methods of analyzing them.

The methods of Sections 2.3 and 2.4 provide a powerful technique for analyzing the efficiency of many algorithms with mathematical clarity and precision, but these methods are far from being foolproof. The last two sections of the chapter deal with two approaches—empirical analysis and algorithm visualization—that complement the pure mathematical techniques of Sections 2.3 and 2.4. Much newer and, hence, less developed than their mathematical counterparts, these approaches promise to play an important role among the tools available for analysis of algorithm efficiency.

2.1 The Analysis Framework

In this section, we outline a general framework for analyzing the efficiency of algorithms. We already mentioned in Section 1.2 that there are two kinds of efficiency: time efficiency and space efficiency. ***Time efficiency***, also called ***time complexity***, indicates how fast an algorithm in question runs. ***Space efficiency***, also called ***space complexity***, refers to the amount of memory units required by the algorithm in addition to the space needed for its input and output. In the early days of electronic computing, both resources—time and space—were at a premium. Half a century

size. In such situations, it is preferable to measure size by the number b of bits in the n's binary representation:

$$b = \lfloor \log_2 n \rfloor + 1. \tag{2.1}$$

This metric usually gives a better idea about the efficiency of algorithms in question.

Units for Measuring Running Time

The next issue concerns units for measuring an algorithm's running time. Of course, we can simply use some standard unit of time measurement—a second, or millisecond, and so on—to measure the running time of a program implementing the algorithm. There are obvious drawbacks to such an approach, however: dependence on the speed of a particular computer, dependence on the quality of a program implementing the algorithm and of the compiler used in generating the machine code, and the difficulty of clocking the actual running time of the program. Since we are after a measure of an *algorithm*'s efficiency, we would like to have a metric that does not depend on these extraneous factors.

One possible approach is to count the number of times each of the algorithm's operations is executed. This approach is both excessively difficult and, as we shall see, usually unnecessary. The thing to do is to identify the most important operation of the algorithm, called the ***basic operation***, the operation contributing the most to the total running time, and compute the number of times the basic operation is executed.

As a rule, it is not difficult to identify the basic operation of an algorithm: it is usually the most time-consuming operation in the algorithm's innermost loop. For example, most sorting algorithms work by comparing elements (keys) of a list being sorted with each other; for such algorithms, the basic operation is a key comparison. As another example, algorithms for mathematical problems typically involve some or all of the four arithmetical operations: addition, subtraction, multiplication, and division. Of the four, the most time-consuming operation is division, followed by multiplication and then addition and subtraction, with the last two usually considered together.[2]

Thus, the established framework for the analysis of an algorithm's time efficiency suggests measuring it by counting the number of times the algorithm's basic operation is executed on inputs of size n. We will find out how to compute such a count for nonrecursive and recursive algorithms in Sections 2.3 and 2.4, respectively.

Here is an important application. Let c_{op} be the execution time of an algorithm's basic operation on a particular computer, and let $C(n)$ be the number of times this operation needs to be executed for this algorithm. Then we can estimate

2. On some computers, multiplication does not take longer than addition/subtraction (see, for example, the timing data provided by Kernighan and Pike in [Ker99, pp. 185–186]).

the running time $T(n)$ of a program implementing this algorithm on that computer by the formula

$$T(n) \approx c_{op}C(n).$$

Of course, this formula should be used with caution. The count $C(n)$ does not contain any information about operations that are not basic, and, in fact, the count itself is often computed only approximately. Further, the constant c_{op} is also an approximation whose reliability is not always easy to assess. Still, unless n is extremely large or very small, the formula can give a reasonable estimate of the algorithm's running time. It also makes it possible to answer such questions as "How much faster would this algorithm run on a machine that is 10 times faster than the one we have?" The answer is, obviously, 10 times. Or, assuming that $C(n) = \frac{1}{2}n(n-1)$, how much longer will the algorithm run if we double its input size? The answer is about four times longer. Indeed, for all but very small values of n,

$$C(n) = \frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \approx \frac{1}{2}n^2$$

and therefore

$$\frac{T(2n)}{T(n)} \approx \frac{c_{op}C(2n)}{c_{op}C(n)} \approx \frac{\frac{1}{2}(2n)^2}{\frac{1}{2}n^2} = 4.$$

Note that we were able to answer the last question without actually knowing the value of c_{op}: it was neatly cancelled out in the ratio. Also note that $\frac{1}{2}$, the multiplicative constant in the formula for the count $C(n)$, was also cancelled out. It is for these reasons that the efficiency analysis framework ignores multiplicative constants and concentrates on the count's ***order of growth*** to within a constant multiple for large-size inputs.

Orders of Growth

Why this emphasis on the count's order of growth for large input sizes? A difference in running times on small inputs is not what really distinguishes efficient algorithms from inefficient ones. When we have to compute, for example, the greatest common divisor of two small numbers, it is not immediately clear how much more efficient Euclid's algorithm is compared to the other two algorithms discussed in Section 1.1 or even why we should care which of them is faster and by how much. It is only when we have to find the greatest common divisor of two large numbers that the difference in algorithm efficiencies becomes both clear and important. For large values of n, it is the function's order of growth that counts: just look at Table 2.1, which contains values of a few functions particularly important for analysis of algorithms.

The magnitude of the numbers in Table 2.1 has a profound significance for the analysis of algorithms. The function growing the slowest among these is the logarithmic function. It grows so slowly, in fact, that we should expect a program

TABLE 2.1 Values (some approximate) of several functions important for analysis of algorithms

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

implementing an algorithm with a logarithmic basic-operation count to run practically instantaneously on inputs of all realistic sizes. Also note that although specific values of such a count depend, of course, on the logarithm's base, the formula

$$\log_a n = \log_a b \log_b n$$

makes it possible to switch from one base to another, leaving the count logarithmic but with a new multiplicative constant. This is why we omit a logarithm's base and write simply $\log n$ in situations where we are interested just in a function's order of growth to within a multiplicative constant.

On the other end of the spectrum are the exponential function 2^n and the factorial function $n!$ Both these functions grow so fast that their values become astronomically large even for rather small values of n. (This is the reason why we did not include their values for $n > 10^2$ in Table 2.1.) For example, it would take about $4 \cdot 10^{10}$ years for a computer making a trillion (10^{12}) operations per second to execute 2^{100} operations. Though this is incomparably faster than it would have taken to execute 100! operations, it is still longer than 4.5 billion ($4.5 \cdot 10^9$) years—the estimated age of the planet Earth. There is a tremendous difference between the orders of growth of the functions 2^n and $n!$, yet both are often referred to as "exponential-growth functions" (or simply "exponential") despite the fact that, strictly speaking, only the former should be referred to as such. The bottom line, which is important to remember, is this:

> Algorithms that require an exponential number of operations are practical for solving only problems of very small sizes.

Another way to appreciate the qualitative difference among the orders of growth of the functions in Table 2.1 is to consider how they react to, say, a twofold increase in the value of their argument n. The function $\log_2 n$ increases in value by just 1 (because $\log_2 2n = \log_2 2 + \log_2 n = 1 + \log_2 n$); the linear function increases twofold, the linearithmic function $n \log_2 n$ increases slightly more than twofold; the quadratic function n^2 and cubic function n^3 increase fourfold and

9. For each of the following pairs of functions, indicate whether the first function of each of the following pairs has a lower, same, or higher order of growth (to within a constant multiple) than the second function.

a. $n(n+1)$ and $2000n^2$ **b.** $100n^2$ and $0.01n^3$

c. $\log_2 n$ and $\ln n$ **d.** $\log_2^2 n$ and $\log_2 n^2$

e. 2^{n-1} and 2^n **f.** $(n-1)!$ and $n!$

10. *Invention of chess*

a. According to a well-known legend, the game of chess was invented many centuries ago in northwestern India by a certain sage. When he took his invention to his king, the king liked the game so much that he offered the inventor any reward he wanted. The inventor asked for some grain to be obtained as follows: just a single grain of wheat was to be placed on the first square of the chessboard, two on the second, four on the third, eight on the fourth, and so on, until all 64 squares had been filled. If it took just 1 second to count each grain, how long would it take to count all the grain due to him?

b. How long would it take if instead of doubling the number of grains for each square of the chessboard, the inventor asked for adding two grains?

2.2 Asymptotic Notations and Basic Efficiency Classes

As pointed out in the previous section, the efficiency analysis framework concentrates on the order of growth of an algorithm's basic operation count as the principal indicator of the algorithm's efficiency. To compare and rank such orders of growth, computer scientists use three notations: O (big oh), Ω (big omega), and Θ (big theta). First, we introduce these notations informally, and then, after several examples, formal definitions are given. In the following discussion, $t(n)$ and $g(n)$ can be any nonnegative functions defined on the set of natural numbers. In the context we are interested in, $t(n)$ will be an algorithm's running time (usually indicated by its basic operation count $C(n)$), and $g(n)$ will be some simple function to compare the count with.

Informal Introduction

Informally, $O(g(n))$ is the set of all functions with a lower or same order of growth as $g(n)$ (to within a constant multiple, as n goes to infinity). Thus, to give a few examples, the following assertions are all true:

$$n \in O(n^2), \qquad 100n + 5 \in O(n^2), \qquad \frac{1}{2}n(n-1) \in O(n^2).$$

Indeed, the first two functions are linear and hence have a lower order of growth than $g(n) = n^2$, while the last one is quadratic and hence has the same order of growth as n^2. On the other hand,

$$n^3 \notin O(n^2), \qquad 0.00001n^3 \notin O(n^2), \qquad n^4 + n + 1 \notin O(n^2).$$

Indeed, the functions n^3 and $0.00001n^3$ are both cubic and hence have a higher order of growth than n^2, and so has the fourth-degree polynomial $n^4 + n + 1$.

The second notation, $\Omega(g(n))$, stands for the set of all functions with a higher or same order of growth as $g(n)$ (to within a constant multiple, as n goes to infinity). For example,

$$n^3 \in \Omega(n^2), \qquad \frac{1}{2}n(n-1) \in \Omega(n^2), \qquad \text{but } 100n + 5 \notin \Omega(n^2).$$

Finally, $\Theta(g(n))$ is the set of all functions that have the same order of growth as $g(n)$ (to within a constant multiple, as n goes to infinity). Thus, every quadratic function $an^2 + bn + c$ with $a > 0$ is in $\Theta(n^2)$, but so are, among infinitely many others, $n^2 + \sin n$ and $n^2 + \log n$. (Can you explain why?)

Hopefully, this informal introduction has made you comfortable with the idea behind the three asymptotic notations. So now come the formal definitions.

O-notation

DEFINITION A function $t(n)$ is said to be in $O(g(n))$, denoted $t(n) \in O(g(n))$, if $t(n)$ is bounded above by some constant multiple of $g(n)$ for all large n, i.e., if there exist some positive constant c and some nonnegative integer n_0 such that

$$t(n) \le cg(n) \quad \text{for all } n \ge n_0.$$

The definition is illustrated in Figure 2.1 where, for the sake of visual clarity, n is extended to be a real number.

As an example, let us formally prove one of the assertions made in the introduction: $100n + 5 \in O(n^2)$. Indeed,

$$100n + 5 \le 100n + n \text{ (for all } n \ge 5) = 101n \le 101n^2.$$

Thus, as values of the constants c and n_0 required by the definition, we can take 101 and 5, respectively.

Note that the definition gives us a lot of freedom in choosing specific values for constants c and n_0. For example, we could also reason that

$$100n + 5 \le 100n + 5n \text{ (for all } n \ge 1) = 105n$$

to complete the proof with $c = 105$ and $n_0 = 1$.

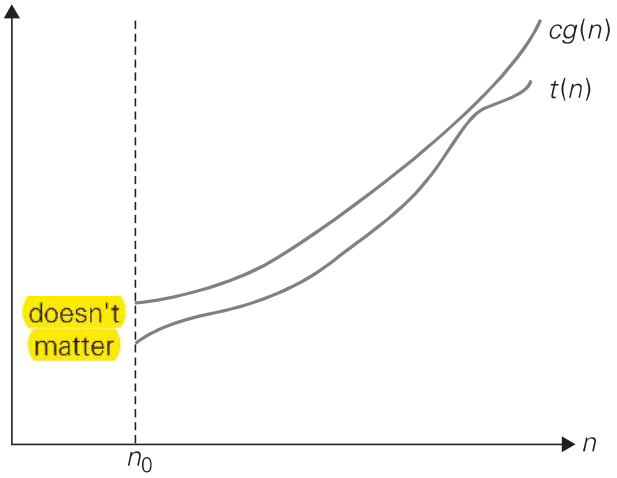


FIGURE 2.1 Big-oh notation: $t(n) \in O(g(n))$.

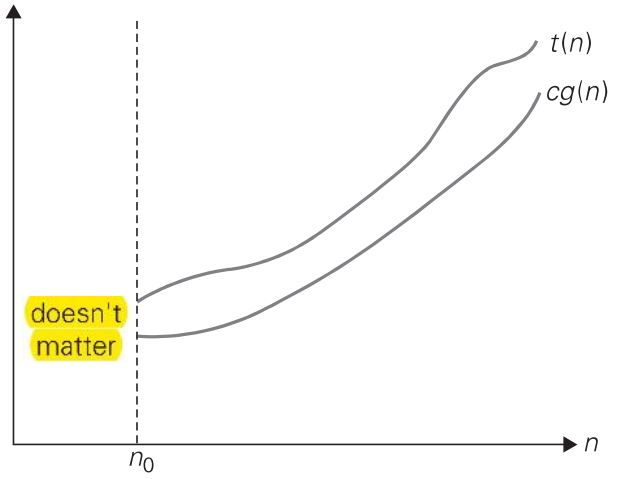


FIGURE 2.2 Big-omega notation: $t(n) \in \Omega(g(n))$.

Ω-notation

DEFINITION A function $t(n)$ is said to be in $\Omega(g(n))$, denoted $t(n) \in \Omega(g(n))$, if $t(n)$ is bounded below by some positive constant multiple of $g(n)$ for all large n, i.e., if there exist some positive constant c and some nonnegative integer n_0 such that

$$t(n) \geq cg(n) \quad \text{for all } n \geq n_0.$$

The definition is illustrated in Figure 2.2.

Here is an example of the formal proof that $n^3 \in \Omega(n^2)$:

$$n^3 \geq n^2 \quad \text{for all } n \geq 0,$$

i.e., we can select $c = 1$ and $n_0 = 0$.

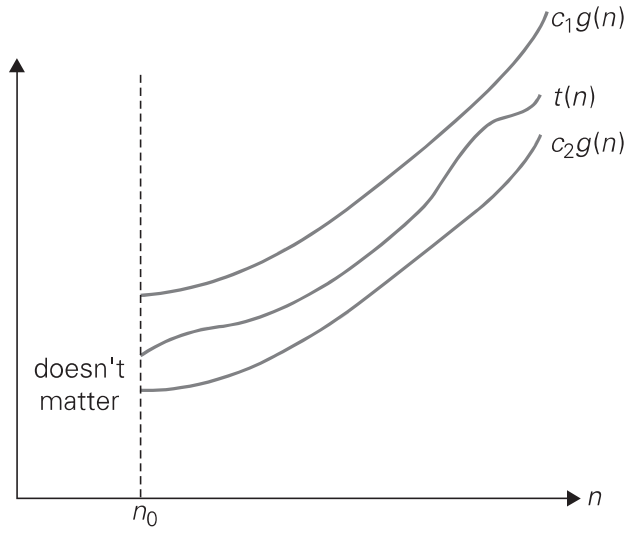


FIGURE 2.3 Big-theta notation: $t(n) \in \Theta(g(n))$.

Θ-notation

DEFINITION A function $t(n)$ is said to be in $\Theta(g(n))$, denoted $t(n) \in \Theta(g(n))$, if $t(n)$ is bounded both above and below by some positive constant multiples of $g(n)$ for all large n, i.e., if there exist some positive constants c_1 and c_2 and some nonnegative integer n_0 such that

$$c_2g(n) \le t(n) \le c_1g(n) \quad \text{for all } n \ge n_0.$$

The definition is illustrated in Figure 2.3.

For example, let us prove that $\frac{1}{2}n(n-1) \in \Theta(n^2)$. First, we prove the right inequality (the upper bound):

$$\frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \le \frac{1}{2}n^2 \quad \text{for all } n \ge 0.$$

Second, we prove the left inequality (the lower bound):

$$\frac{1}{2}n(n-1) = \frac{1}{2}n^2 - \frac{1}{2}n \ge \frac{1}{2}n^2 - \frac{1}{2}n\frac{1}{2}n \ (\text{for all } n \ge 2) = \frac{1}{4}n^2.$$

Hence, we can select $c_2 = \frac{1}{4}$, $c_1 = \frac{1}{2}$, and $n_0 = 2$.

Useful Property Involving the Asymptotic Notations

Using the formal definitions of the asymptotic notations, we can prove their general properties (see Problem 7 in this section's exercises for a few simple examples). The following property, in particular, is useful in analyzing algorithms that comprise two consecutively executed parts.

THEOREM If $t_1(n) \in O(g_1(n))$ and $t_2(n) \in O(g_2(n))$, then
$t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\})$.

(The analogous assertions are true for the Ω and Θ notations as well.)

PROOF The proof extends to orders of growth the following simple fact about four arbitrary real numbers a_1, b_1, a_2, b_2: if $a_1 \le b_1$ and $a_2 \le b_2$, then $a_1 + a_2 \le 2\max\{b_1, b_2\}$.

Since $t_1(n) \in O(g_1(n))$, there exist some positive constant c_1 and some nonnegative integer n_1 such that

$$t_1(n) \le c_1 g_1(n) \quad \text{for all } n \ge n_1.$$

Similarly, since $t_2(n) \in O(g_2(n))$,

$$t_2(n) \le c_2 g_2(n) \quad \text{for all } n \ge n_2.$$

Let us denote $c_3 = \max\{c_1, c_2\}$ and consider $n \ge \max\{n_1, n_2\}$ so that we can use both inequalities. Adding them yields the following:

$$\begin{aligned} t_1(n) + t_2(n) &\le c_1 g_1(n) + c_2 g_2(n) \\ &\le c_3 g_1(n) + c_3 g_2(n) = c_3[g_1(n) + g_2(n)] \\ &\le c_3 2\max\{g_1(n), g_2(n)\}. \end{aligned}$$

Hence, $t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\})$, with the constants c and n_0 required by the O definition being $2c_3 = 2\max\{c_1, c_2\}$ and $\max\{n_1, n_2\}$, respectively. ■

So what does this property imply for an algorithm that comprises two consecutively executed parts? It implies that the algorithm's overall efficiency is determined by the part with a higher order of growth, i.e., its least efficient part:

$$\left.\begin{array}{|c|}\hline t_1(n) \in O(g_1(n)) \\ \hline t_2(n) \in O(g_2(n)) \\ \hline \end{array}\right\} \quad t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\}).$$

For example, we can check whether an array has equal elements by the following two-part algorithm: first, sort the array by applying some known sorting algorithm; second, scan the sorted array to check its consecutive elements for equality. If, for example, a sorting algorithm used in the first part makes no more than $\frac{1}{2}n(n-1)$ comparisons (and hence is in $O(n^2)$) while the second part makes no more than $n - 1$ comparisons (and hence is in $O(n)$), the efficiency of the entire algorithm will be in $O(\max\{n^2, n\}) = O(n^2)$.

Using Limits for Comparing Orders of Growth

Though the formal definitions of O, Ω, and Θ are indispensable for proving their abstract properties, they are rarely used for comparing the orders of growth of two specific functions. A much more convenient method for doing so is based on

computing the limit of the ratio of two functions in question. Three principal cases may arise:

$$\lim_{n\to\infty}\frac{t(n)}{g(n)}=\begin{cases}0 & \text{implies that } t(n) \text{ has a smaller order of growth than } g(n),\\ c & \text{implies that } t(n) \text{ has the same order of growth as } g(n),\\ \infty & \text{implies that } t(n) \text{ has a larger order of growth than } g(n).^{3}\end{cases}$$

Note that the first two cases mean that $t(n)\in O(g(n))$, the last two mean that $t(n)\in\Omega(g(n))$, and the second case means that $t(n)\in\Theta(g(n))$.

The limit-based approach is often more convenient than the one based on the definitions because it can take advantage of the powerful calculus techniques developed for computing limits, such as L'Hôpital's rule

$$\lim_{n\to\infty}\frac{t(n)}{g(n)}=\lim_{n\to\infty}\frac{t'(n)}{g'(n)}$$

and Stirling's formula

$$n!\approx\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{for large values of } n.$$

Here are three examples of using the limit-based approach to comparing orders of growth of two functions.

EXAMPLE 1 Compare the orders of growth of $\frac{1}{2}n(n-1)$ and n^2. (This is one of the examples we used at the beginning of this section to illustrate the definitions.)

$$\lim_{n\to\infty}\frac{\frac{1}{2}n(n-1)}{n^2}=\frac{1}{2}\lim_{n\to\infty}\frac{n^2-n}{n^2}=\frac{1}{2}\lim_{n\to\infty}(1-\frac{1}{n})=\frac{1}{2}.$$

Since the limit is equal to a positive constant, the functions have the same order of growth or, symbolically, $\frac{1}{2}n(n-1)\in\Theta(n^2)$. ■

EXAMPLE 2 Compare the orders of growth of $\log_2 n$ and $\sqrt{n}$. (Unlike Example 1, the answer here is not immediately obvious.)

$$\lim_{n\to\infty}\frac{\log_2 n}{\sqrt{n}}=\lim_{n\to\infty}\frac{(\log_2 n)'}{(\sqrt{n})'}=\lim_{n\to\infty}\frac{(\log_2 e)\frac{1}{n}}{\frac{1}{2\sqrt{n}}}=2\log_2 e\lim_{n\to\infty}\frac{1}{\sqrt{n}}=0.$$

Since the limit is equal to zero, $\log_2 n$ has a smaller order of growth than $\sqrt{n}$. (Since $\lim_{n\to\infty}\frac{\log_2 n}{\sqrt{n}}=0$, we can use the so-called ***little-oh notation***: $\log_2 n\in o(\sqrt{n})$. Unlike the big-Oh, the little-oh notation is rarely used in analysis of algorithms.) ■

3. The fourth case, in which such a limit does not exist, rarely happens in the actual practice of analyzing algorithms. Still, this possibility makes the limit-based approach to comparing orders of growth less general than the one based on the definitions of O, Ω, and Θ.

EXAMPLE 3 Compare the orders of growth of $n!$ and 2^n. (We discussed this informally in Section 2.1.) Taking advantage of Stirling's formula, we get

$$\lim_{n\to\infty} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{2^n} = \lim_{n\to\infty} \sqrt{2\pi n}\frac{n^n}{2^n e^n} = \lim_{n\to\infty} \sqrt{2\pi n}\left(\frac{n}{2e}\right)^n = \infty.$$

Thus, though 2^n grows very fast, $n!$ grows still faster. We can write symbolically that $n! \in \Omega(2^n)$; note, however, that while the big-Omega notation does not preclude the possibility that $n!$ and 2^n have the same order of growth, the limit computed here certainly does. ■

Basic Efficiency Classes

Even though the efficiency analysis framework puts together all the functions whose orders of growth differ by a constant multiple, there are still infinitely many such classes. (For example, the exponential functions a^n have different orders of growth for different values of base a.) Therefore, it may come as a surprise that the time efficiencies of a large number of algorithms fall into only a few classes. These classes are listed in Table 2.2 in increasing order of their orders of growth, along with their names and a few comments.

You could raise a concern that classifying algorithms by their asymptotic efficiency would be of little practical use since the values of multiplicative constants are usually left unspecified. This leaves open the possibility of an algorithm in a worse efficiency class running faster than an algorithm in a better efficiency class for inputs of realistic sizes. For example, if the running time of one algorithm is n^3 while the running time of the other is $10^6 n^2$, the cubic algorithm will outperform the quadratic algorithm unless n exceeds 10^6. A few such anomalies are indeed known. Fortunately, multiplicative constants usually do not differ that drastically. As a rule, you should expect an algorithm from a better asymptotic efficiency class to outperform an algorithm from a worse class even for moderately sized inputs. This observation is especially true for an algorithm with a better than exponential running time versus an exponential (or worse) algorithm.

Exercises 2.2

1. Use the most appropriate notation among O, Θ, and Ω to indicate the time efficiency class of sequential search (see Section 2.1)
   - **a.** in the worst case.
   - **b.** in the best case.
   - **c.** in the average case.
2. Use the informal definitions of O, Θ, and Ω to determine whether the following assertions are true or false.

11. *Lighter or heavier?* You have $n > 2$ identical-looking coins and a two-pan balance scale with no weights. One of the coins is a fake, but you do not know whether it is lighter or heavier than the genuine coins, which all weigh the same. Design a $\Theta(1)$ algorithm to determine whether the fake coin is lighter or heavier than the others.

12. *Door in a wall* You are facing a wall that stretches infinitely in both directions. There is a door in the wall, but you know neither how far away nor in which direction. You can see the door only when you are right next to it. Design an algorithm that enables you to reach the door by walking at most $O(n)$ steps where n is the (unknown to you) number of steps between your initial position and the door. [Par95]

2.3 Mathematical Analysis of Nonrecursive Algorithms

In this section, we systematically apply the general framework outlined in Section 2.1 to analyzing the time efficiency of nonrecursive algorithms. Let us start with a very simple example that demonstrates all the principal steps typically taken in analyzing such algorithms.

EXAMPLE 1 Consider the problem of finding the value of the largest element in a list of n numbers. For simplicity, we assume that the list is implemented as an array. The following is pseudocode of a standard algorithm for solving the problem.

ALGORITHM *MaxElement*$(A[0..n-1])$

```
//Determines the value of the largest element in a given array
//Input: An array A[0..n − 1] of real numbers
//Output: The value of the largest element in A
maxval ← A[0]
for i ← 1 to n − 1 do
    if A[i] > maxval
        maxval ← A[i]
return maxval
```

The obvious measure of an input's size here is the number of elements in the array, i.e., n. The operations that are going to be executed most often are in the algorithm's **for** loop. There are two operations in the loop's body: the comparison $A[i] > maxval$ and the assignment $maxval \leftarrow A[i]$. Which of these two operations should we consider basic? Since the comparison is executed on each repetition of the loop and the assignment is not, we should consider the comparison to be the algorithm's basic operation. Note that the number of comparisons will be the same for all arrays of size n; therefore, in terms of this metric, there is no need to distinguish among the worst, average, and best cases here.

Let us denote $C(n)$ the number of times this comparison is executed and try to find a formula expressing it as a function of size n. The algorithm makes one comparison on each execution of the loop, which is repeated for each value of the loop's variable i within the bounds 1 and $n-1$, inclusive. Therefore, we get the following sum for $C(n)$:

$$C(n) = \sum_{i=1}^{n-1} 1.$$

This is an easy sum to compute because it is nothing other than 1 repeated $n-1$ times. Thus,

$$C(n) = \sum_{i=1}^{n-1} 1 = n - 1 \in \Theta(n).$$ ■

Here is a general plan to follow in analyzing nonrecursive algorithms.

General Plan for Analyzing the Time Efficiency of Nonrecursive Algorithms

1. Decide on a parameter (or parameters) indicating an input's size.
2. Identify the algorithm's basic operation. (As a rule, it is located in the innermost loop.)
3. Check whether the number of times the basic operation is executed depends only on the size of an input. If it also depends on some additional property, the worst-case, average-case, and, if necessary, best-case efficiencies have to be investigated separately.
4. Set up a sum expressing the number of times the algorithm's basic operation is executed.[4]
5. Using standard formulas and rules of sum manipulation, either find a closed-form formula for the count or, at the very least, establish its order of growth.

Before proceeding with further examples, you may want to review Appendix A, which contains a list of summation formulas and rules that are often useful in analysis of algorithms. In particular, we use especially frequently two basic rules of sum manipulation

$$\sum_{i=l}^{u} ca_i = c \sum_{i=l}^{u} a_i, \qquad \textbf{(R1)}$$

$$\sum_{i=l}^{u} (a_i \pm b_i) = \sum_{i=l}^{u} a_i \pm \sum_{i=l}^{u} b_i, \qquad \textbf{(R2)}$$

4. Sometimes, an analysis of a nonrecursive algorithm requires setting up not a sum but a recurrence relation for the number of times its basic operation is executed. Using recurrence relations is much more typical for analyzing recursive algorithms (see Section 2.4).

and two summation formulas

$$\sum_{i=l}^{u} 1 = u - l + 1 \quad \text{where } l \le u \text{ are some lower and upper integer limits,} \tag{S1}$$

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2 \in \Theta(n^2). \tag{S2}$$

Note that the formula $\sum_{i=1}^{n-1} 1 = n - 1$, which we used in Example 1, is a special case of formula (S1) for $l = 1$ and $u = n - 1$.

EXAMPLE 2 Consider the ***element uniqueness problem***: check whether all the elements in a given array of n elements are distinct. This problem can be solved by the following straightforward algorithm.

ALGORITHM *UniqueElements*$(A[0..n-1])$

```
//Determines whether all the elements in a given array are distinct
//Input: An array A[0..n − 1]
//Output: Returns "true" if all the elements in A are distinct
//        and "false" otherwise
for i ← 0 to n − 2 do
    for j ← i + 1 to n − 1 do
        if A[i] = A[j] return false
return true
```

The natural measure of the input's size here is again n, the number of elements in the array. Since the innermost loop contains a single operation (the comparison of two elements), we should consider it as the algorithm's basic operation. Note, however, that the number of element comparisons depends not only on n but also on whether there are equal elements in the array and, if there are, which array positions they occupy. We will limit our investigation to the worst case only.

By definition, the worst case input is an array for which the number of element comparisons $C_{worst}(n)$ is the largest among all arrays of size n. An inspection of the innermost loop reveals that there are two kinds of worst-case inputs—inputs for which the algorithm does not exit the loop prematurely: arrays with no equal elements and arrays in which the last two elements are the only pair of equal elements. For such inputs, one comparison is made for each repetition of the innermost loop, i.e., for each value of the loop variable j between its limits $i + 1$ and $n - 1$; this is repeated for each value of the outer loop, i.e., for each value of the loop variable i between its limits 0 and $n - 2$. Accordingly, we get

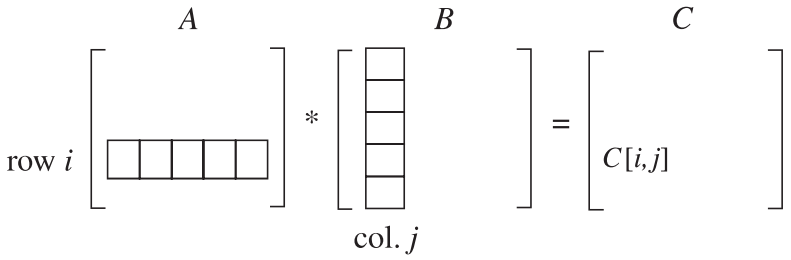
$$C_{worst}(n) = \sum_{i=0}^{n-2} \sum_{j=i+1}^{n-1} 1 = \sum_{i=0}^{n-2} [(n-1) - (i+1) + 1] = \sum_{i=0}^{n-2} (n-1-i)$$
$$= \sum_{i=0}^{n-2} (n-1) - \sum_{i=0}^{n-2} i = (n-1) \sum_{i=0}^{n-2} 1 - \frac{(n-2)(n-1)}{2}$$
$$= (n-1)^2 - \frac{(n-2)(n-1)}{2} = \frac{(n-1)n}{2} \approx \frac{1}{2} n^2 \in \Theta(n^2).$$

We also could have computed the sum $\sum_{i=0}^{n-2} (n-1-i)$ faster as follows:

$$\sum_{i=0}^{n-2} (n-1-i) = (n-1) + (n-2) + \cdots + 1 = \frac{(n-1)n}{2},$$

where the last equality is obtained by applying summation formula (S2). Note that this result was perfectly predictable: in the worst case, the algorithm needs to compare all $n(n-1)/2$ distinct pairs of its n elements. ■

EXAMPLE 3 Given two $n \times n$ matrices A and B, find the time efficiency of the definition-based algorithm for computing their product $C = AB$. By definition, C is an $n \times n$ matrix whose elements are computed as the scalar (dot) products of the rows of matrix A and the columns of matrix B:



where $C[i, j] = A[i, 0]B[0, j] + \cdots + A[i, k]B[k, j] + \cdots + A[i, n-1]B[n-1, j]$ for every pair of indices $0 \le i, j \le n-1$.

ALGORITHM *MatrixMultiplication*$(A[0..n-1, 0..n-1], B[0..n-1, 0..n-1])$

```
//Multiplies two square matrices of order n by the definition-based algorithm
//Input: Two n × n matrices A and B
//Output: Matrix C = AB
for i ← 0 to n − 1 do
    for j ← 0 to n − 1 do
        C[i, j] ← 0.0
        for k ← 0 to n − 1 do
            C[i, j] ← C[i, j] + A[i, k] ∗ B[k, j]
return C
```

Exercises 2.3

1. Compute the following sums.
   a. $1 + 3 + 5 + 7 + \cdots + 999$
   b. $2 + 4 + 8 + 16 + \cdots + 1024$
   c. $\sum_{i=3}^{n+1} 1$
   d. $\sum_{i=3}^{n+1} i$
   e. $\sum_{i=0}^{n-1} i(i+1)$
   f. $\sum_{j=1}^{n} 3^{j+1}$
   g. $\sum_{i=1}^{n} \sum_{j=1}^{n} ij$
   h. $\sum_{i=1}^{n} 1/i(i+1)$

2. Find the order of growth of the following sums. Use the $\Theta(g(n))$ notation with the simplest function $g(n)$ possible.
   a. $\sum_{i=0}^{n-1} (i^2+1)^2$
   b. $\sum_{i=2}^{n-1} \lg i^2$
   c. $\sum_{i=1}^{n} (i+1)2^{i-1}$
   d. $\sum_{i=0}^{n-1} \sum_{j=0}^{i-1} (i+j)$

3. The sample variance of n measurements $x_1, \ldots, x_n$ can be computed as either

$$\frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \quad \text{where } \bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}$$

   or

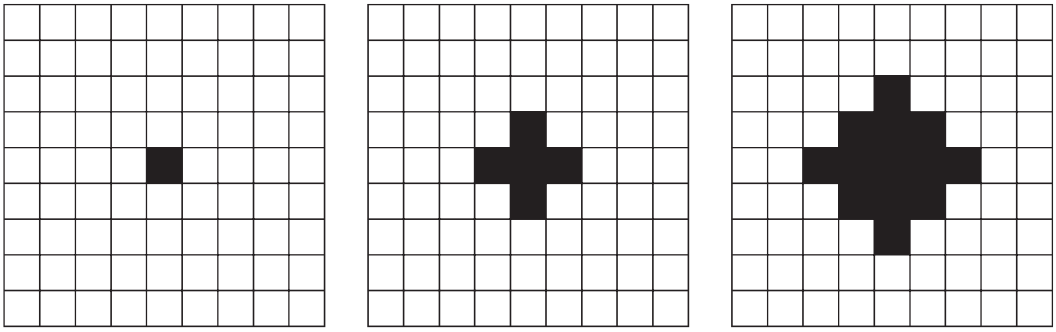
$$\frac{\sum_{i=1}^{n} x_i^2 - (\sum_{i=1}^{n} x_i)^2/n}{n-1}.$$

   Find and compare the number of divisions, multiplications, and additions/subtractions (additions and subtractions are usually bunched together) that are required for computing the variance according to each of these formulas.

4. Consider the following algorithm.

```
ALGORITHM Mystery(n)
    //Input: A nonnegative integer n
    S ← 0
    for i ← 1 to n do
        S ← S + i * i
    return S
```

   a. What does this algorithm compute?
   b. What is its basic operation?
   c. How many times is the basic operation executed?
   d. What is the efficiency class of this algorithm?
   e. Suggest an improvement, or a better algorithm altogether, and indicate its efficiency class. If you cannot do it, try to prove that, in fact, it cannot be done.



$n = 0$ $n = 1$ $n = 2$

13. *Page numbering* Find the total number of decimal digits needed for numbering pages in a book of 1000 pages. Assume that the pages are numbered consecutively starting with 1.

2.4 Mathematical Analysis of Recursive Algorithms

In this section, we will see how to apply the general framework for analysis of algorithms to recursive algorithms. We start with an example often used to introduce novices to the idea of a recursive algorithm.

EXAMPLE 1 Compute the factorial function $F(n) = n!$ for an arbitrary nonnegative integer n. Since

$$n! = 1 \cdot \ldots \cdot (n-1) \cdot n = (n-1)! \cdot n \quad \text{for } n \geq 1$$

and $0! = 1$ by definition, we can compute $F(n) = F(n-1) \cdot n$ with the following recursive algorithm.

ALGORITHM $F(n)$

```
//Computes n! recursively
//Input: A nonnegative integer n
//Output: The value of n!
if n = 0 return 1
else return F(n − 1) ∗ n
```

For simplicity, we consider n itself as an indicator of this algorithm's input size (rather than the number of bits in its binary expansion). The basic operation of the algorithm is multiplication,[5] whose number of executions we denote $M(n)$. Since the function $F(n)$ is computed according to the formula

$$F(n) = F(n-1) \cdot n \quad \text{for } n > 0,$$

5. Alternatively, we could count the number of times the comparison $n = 0$ is executed, which is the same as counting the total number of calls made by the algorithm (see Problem 2 in this section's exercises).

the number of multiplications $M(n)$ needed to compute it must satisfy the equality

$$M(n) = \underbrace{M(n-1)}_{\substack{\text{to compute} \\ F(n-1)}} + \underbrace{1}_{\substack{\text{to multiply} \\ F(n-1) \text{ by } n}} \quad \text{for } n > 0.$$

Indeed, $M(n-1)$ multiplications are spent to compute $F(n-1)$, and one more multiplication is needed to multiply the result by n.

The last equation defines the sequence $M(n)$ that we need to find. This equation defines $M(n)$ not explicitly, i.e., as a function of n, but implicitly as a function of its value at another point, namely $n-1$. Such equations are called ***recurrence relations*** or, for brevity, ***recurrences***. Recurrence relations play an important role not only in analysis of algorithms but also in some areas of applied mathematics. They are usually studied in detail in courses on discrete mathematics or discrete structures; a very brief tutorial on them is provided in Appendix B. Our goal now is to solve the recurrence relation $M(n) = M(n-1) + 1$, i.e., to find an explicit formula for $M(n)$ in terms of n only.

Note, however, that there is not one but infinitely many sequences that satisfy this recurrence. (Can you give examples of, say, two of them?) To determine a solution uniquely, we need an ***initial condition*** that tells us the value with which the sequence starts. We can obtain this value by inspecting the condition that makes the algorithm stop its recursive calls:

if $n = 0$ **return** 1.

This tells us two things. First, since the calls stop when $n = 0$, the smallest value of n for which this algorithm is executed and hence $M(n)$ defined is 0. Second, by inspecting the pseudocode's exiting line, we can see that when $n = 0$, the algorithm performs no multiplications. Therefore, the initial condition we are after is

$$M(0) = 0.$$

the calls stop when $n = 0$ — no multiplications when $n = 0$

Thus, we succeeded in setting up the recurrence relation and initial condition for the algorithm's number of multiplications $M(n)$:

$$\begin{aligned} M(n) &= M(n-1) + 1 \quad \text{for } n > 0, \\ M(0) &= 0. \end{aligned} \tag{2.2}$$

Before we embark on a discussion of how to solve this recurrence, let us pause to reiterate an important point. We are dealing here with two recursively defined functions. The first is the factorial function $F(n)$ itself; it is defined by the recurrence

$$\begin{aligned} F(n) &= F(n-1) \cdot n \quad \text{for every } n > 0, \\ F(0) &= 1. \end{aligned}$$

The second is the number of multiplications $M(n)$ needed to compute $F(n)$ by the recursive algorithm whose pseudocode was given at the beginning of the section.

As we just showed, $M(n)$ is defined by recurrence (2.2). And it is recurrence (2.2) that we need to solve now.

Though it is not difficult to "guess" the solution here (what sequence starts with 0 when $n = 0$ and increases by 1 on each step?), it will be more useful to arrive at it in a systematic fashion. From the several techniques available for solving recurrence relations, we use what can be called the ***method of backward substitutions***. The method's idea (and the reason for the name) is immediately clear from the way it applies to solving our particular recurrence:

$$\begin{aligned} M(n) &= M(n-1) + 1 && \text{substitute } M(n-1) = M(n-2) + 1 \\ &= [M(n-2) + 1] + 1 = M(n-2) + 2 && \text{substitute } M(n-2) = M(n-3) + 1 \\ &= [M(n-3) + 1] + 2 = M(n-3) + 3. \end{aligned}$$

After inspecting the first three lines, we see an emerging pattern, which makes it possible to predict not only the next line (what would it be?) but also a general formula for the pattern: $M(n) = M(n-i) + i$. Strictly speaking, the correctness of this formula should be proved by mathematical induction, but it is easier to get to the solution as follows and then verify its correctness.

What remains to be done is to take advantage of the initial condition given. Since it is specified for $n = 0$, we have to substitute $i = n$ in the pattern's formula to get the ultimate result of our backward substitutions:

$$M(n) = M(n-1) + 1 = \cdots = M(n-i) + i = \cdots = M(n-n) + n = n.$$

You should not be disappointed after exerting so much effort to get this "obvious" answer. The benefits of the method illustrated in this simple example will become clear very soon, when we have to solve more difficult recurrences. Also, note that the simple iterative algorithm that accumulates the product of n consecutive integers requires the same number of multiplications, and it does so without the overhead of time and space used for maintaining the recursion's stack.

The issue of time efficiency is actually not that important for the problem of computing $n!$, however. As we saw in Section 2.1, the function's values get so large so fast that we can realistically compute exact values of $n!$ only for very small n's. Again, we use this example just as a simple and convenient vehicle to introduce the standard approach to analyzing recursive algorithms. ■

Generalizing our experience with investigating the recursive algorithm for computing $n!$, we can now outline a general plan for investigating recursive algorithms.

General Plan for Analyzing the Time Efficiency of Recursive Algorithms

1. Decide on a parameter (or parameters) indicating an input's size.
2. Identify the algorithm's basic operation.

3. Check whether the number of times the basic operation is executed can vary on different inputs of the same size; if it can, the worst-case, average-case, and best-case efficiencies must be investigated separately.
4. Set up a recurrence relation, with an appropriate initial condition, for the number of times the basic operation is executed.
5. Solve the recurrence or, at least, ascertain the order of growth of its solution.

EXAMPLE 2 As our next example, we consider another educational workhorse of recursive algorithms: the ***Tower of Hanoi*** puzzle. In this puzzle, we (or mythical monks, if you do not like to move disks) have n disks of different sizes that can slide onto any of three pegs. Initially, all the disks are on the first peg in order of size, the largest on the bottom and the smallest on top. The goal is to move all the disks to the third peg, using the second one as an auxiliary, if necessary. We can move only one disk at a time, and it is forbidden to place a larger disk on top of a smaller one.

The problem has an elegant recursive solution, which is illustrated in Figure 2.4. To move $n > 1$ disks from peg 1 to peg 3 (with peg 2 as auxiliary), we first move recursively $n - 1$ disks from peg 1 to peg 2 (with peg 3 as auxiliary), then move the largest disk directly from peg 1 to peg 3, and, finally, move recursively $n - 1$ disks from peg 2 to peg 3 (using peg 1 as auxiliary). Of course, if $n = 1$, we simply move the single disk directly from the source peg to the destination peg.

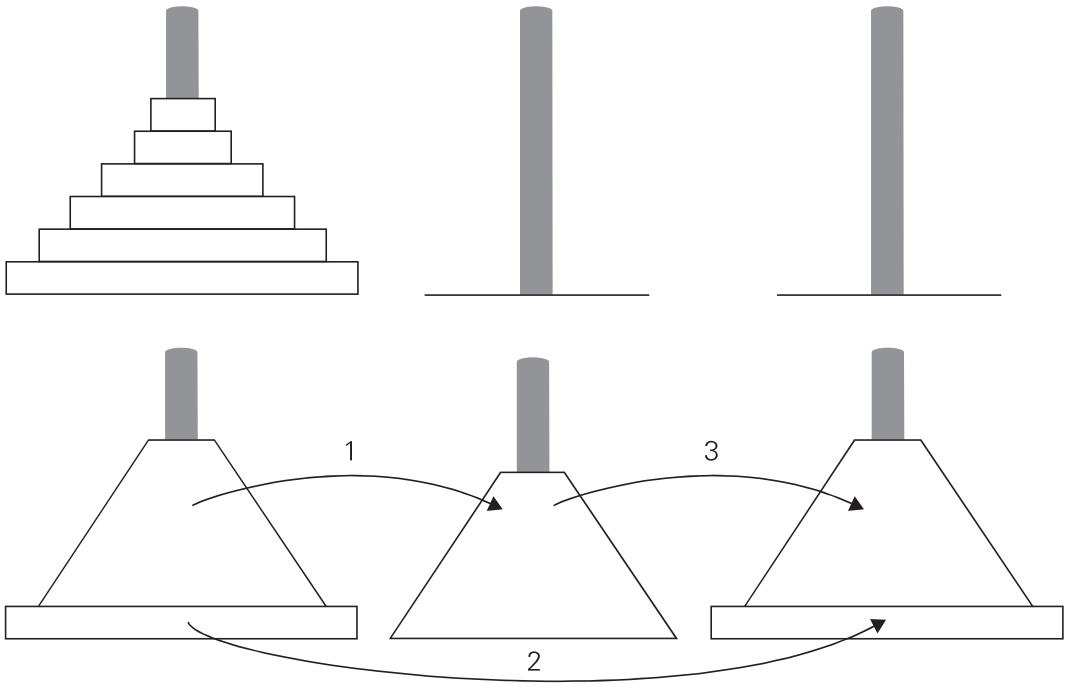


FIGURE 2.4 Recursive solution to the Tower of Hanoi puzzle.

Let us apply the general plan outlined above to the Tower of Hanoi problem. The number of disks n is the obvious choice for the input's size indicator, and so is moving one disk as the algorithm's basic operation. Clearly, the number of moves $M(n)$ depends on n only, and we get the following recurrence equation for it:

$$M(n) = M(n-1) + 1 + M(n-1) \quad \text{for } n > 1.$$

With the obvious initial condition $M(1) = 1$, we have the following recurrence relation for the number of moves $M(n)$:

$$M(n) = 2M(n-1) + 1 \quad \text{for } n > 1, \qquad \textbf{(2.3)}$$
$$M(1) = 1.$$

We solve this recurrence by the same method of backward substitutions:

$$M(n) = 2M(n-1) + 1 \qquad \text{sub. } M(n-1) = 2M(n-2) + 1$$
$$= 2[2M(n-2) + 1] + 1 = 2^2 M(n-2) + 2 + 1 \quad \text{sub. } M(n-2) = 2M(n-3) + 1$$
$$= 2^2[2M(n-3) + 1] + 2 + 1 = 2^3 M(n-3) + 2^2 + 2 + 1.$$

The pattern of the first three sums on the left suggests that the next one will be $2^4 M(n-4) + 2^3 + 2^2 + 2 + 1$, and generally, after i substitutions, we get

$$M(n) = 2^i M(n-i) + 2^{i-1} + 2^{i-2} + \cdots + 2 + 1 = 2^i M(n-i) + 2^i - 1.$$

Since the initial condition is specified for $n = 1$, which is achieved for $i = n - 1$, we get the following formula for the solution to recurrence (2.3):

$$M(n) = 2^{n-1} M(n-(n-1)) + 2^{n-1} - 1$$
$$= 2^{n-1} M(1) + 2^{n-1} - 1 = 2^{n-1} + 2^{n-1} - 1 = 2^n - 1.$$

Thus, we have an exponential algorithm, which will run for an unimaginably long time even for moderate values of n (see Problem 5 in this section's exercises). This is not due to the fact that this particular algorithm is poor; in fact, it is not difficult to prove that this is the most efficient algorithm possible for this problem. It is the problem's intrinsic difficulty that makes it so computationally hard. Still, this example makes an important general point:

> One should be careful with recursive algorithms because their succinctness may mask their inefficiency.

When a recursive algorithm makes more than a single call to itself, it can be useful for analysis purposes to construct a tree of its recursive calls. In this tree, nodes correspond to recursive calls, and we can label them with the value of the parameter (or, more generally, parameters) of the calls. For the Tower of Hanoi example, the tree is given in Figure 2.5. By counting the number of nodes in the tree, we can get the total number of calls made by the Tower of Hanoi algorithm:

$$C(n) = \sum_{l=0}^{n-1} 2^l \text{ (where } l \text{ is the level in the tree in Figure 2.5)} = 2^n - 1.$$