Chapter 9

(Greedy Technique)

9

Greedy Technique

Greed, for lack of a better word, is good! Greed is right! Greed works!

—Michael Douglas, US actor in the role of Gordon Gecko,
in the film *Wall Street*, 1987

Let us revisit the ***change-making problem*** faced, at least subconsciously, by millions of cashiers all over the world: give change for a specific amount n with the least number of coins of the denominations $d_1 > d_2 > \cdots > d_m$ used in that locale. (Here, unlike Section 8.1, we assume that the denominations are ordered in decreasing order.) For example, the widely used coin denominations in the United States are $d_1 = 25$ (quarter), $d_2 = 10$ (dime), $d_3 = 5$ (nickel), and $d_4 = 1$ (penny). How would you give change with coins of these denominations of, say, 48 cents? If you came up with the answer 1 quarter, 2 dimes, and 3 pennies, you followed—consciously or not—a logical strategy of making a sequence of best choices among the currently available alternatives. Indeed, in the first step, you could have given one coin of any of the four denominations. "Greedy" thinking leads to giving one quarter because it reduces the remaining amount the most, namely, to 23 cents. In the second step, you had the same coins at your disposal, but you could not give a quarter, because it would have violated the problem's constraints. So your best selection in this step was one dime, reducing the remaining amount to 13 cents. Giving one more dime left you with 3 cents to be given with three pennies.

Is this solution to the instance of the change-making problem optimal? Yes, it is. In fact, one can prove that the greedy algorithm yields an optimal solution for every positive integer amount with these coin denominations. At the same time, it is easy to give an example of coin denominations that do not yield an optimal solution for some amounts—e.g., $d_1 = 25$, $d_2 = 10$, $d_3 = 1$ and $n = 30$.

The approach applied in the opening paragraph to the change-making problem is called ***greedy***. Computer scientists consider it a general design technique despite the fact that it is applicable to optimization problems only. The greedy approach suggests constructing a solution through a sequence of steps, each expanding a partially constructed solution obtained so far, until a complete solution

to the problem is reached. On each step—and this is the central point of this technique—the choice made must be:

- *feasible*, i.e., it has to satisfy the problem's constraints
- *locally optimal*, i.e., it has to be the best local choice among all feasible choices available on that step
- *irrevocable*, i.e., once made, it cannot be changed on subsequent steps of the algorithm

These requirements explain the technique's name: on each step, it suggests a "greedy" grab of the best alternative available in the hope that a sequence of locally optimal choices will yield a (globally) optimal solution to the entire problem. We refrain from a philosophical discussion of whether greed is good or bad. (If you have not seen the movie from which the chapter's epigraph is taken, its hero did not end up well.) From our algorithmic perspective, the question is whether such a greedy strategy works or not. As we shall see, there are problems for which a sequence of locally optimal choices does yield an optimal solution for every instance of the problem in question. However, there are others for which this is not the case; for such problems, a greedy algorithm can still be of value if we are interested in or have to be satisfied with an approximate solution.

In the first two sections of the chapter, we discuss two classic algorithms for the minimum spanning tree problem: Prim's algorithm and Kruskal's algorithm. What is remarkable about these algorithms is the fact that they solve the same problem by applying the greedy approach in two different ways, and both of them always yield an optimal solution. In Section 9.3, we introduce another classic algorithm—Dijkstra's algorithm for the shortest-path problem in a weighted graph. Section 9.4 is devoted to Huffman trees and their principal application, Huffman codes—an important data compression method that can be interpreted as an application of the greedy technique. Finally, a few examples of approximation algorithms based on the greedy approach are discussed in Section 12.3.

As a rule, greedy algorithms are both intuitively appealing and simple. Given an optimization problem, it is usually easy to figure out how to proceed in a greedy manner, possibly after considering a few small instances of the problem. What is usually more difficult is to prove that a greedy algorithm yields an optimal solution (when it does). One of the common ways to do this is illustrated by the proof given in Section 9.1: using mathematical induction, we show that a partially constructed solution obtained by the greedy algorithm on each iteration can be extended to an optimal solution to the problem.

The second way to prove optimality of a greedy algorithm is to show that on each step it does at least as well as any other algorithm could in advancing toward the problem's goal. Consider, as an example, the following problem: find the minimum number of moves needed for a chess knight to go from one corner of a 100×100 board to the diagonally opposite corner. (The knight's moves are L-shaped jumps: two squares horizontally or vertically followed by one square in

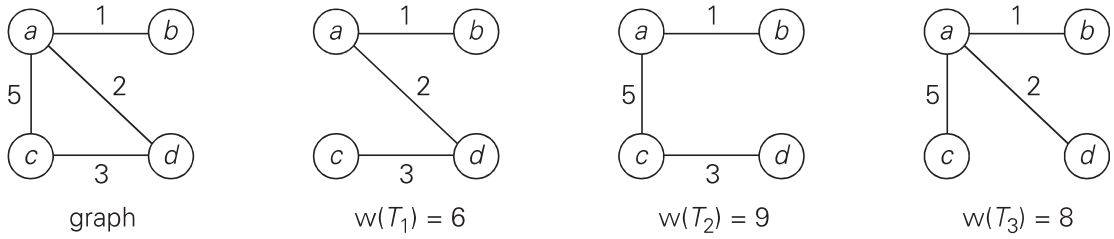


FIGURE 9.2 Graph and its spanning trees, with T_1 being the minimum spanning tree.

9.1 Prim's Algorithm

The following problem arises naturally in many practical situations: given n points, connect them in the cheapest possible way so that there will be a path between every pair of points. It has direct applications to the design of all kinds of networks—including communication, computer, transportation, and electrical—by providing the cheapest way to achieve connectivity. It identifies clusters of points in data sets. It has been used for classification purposes in archeology, biology, sociology, and other sciences. It is also helpful for constructing approximate solutions to more difficult problems such the traveling salesman problem (see Section 12.3).

We can represent the points given by vertices of a graph, possible connections by the graph's edges, and the connection costs by the edge weights. Then the question can be posed as the minimum spanning tree problem, defined formally as follows.

DEFINITION A ***spanning tree*** of an undirected connected graph is its connected acyclic subgraph (i.e., a tree) that contains all the vertices of the graph. If such a graph has weights assigned to its edges, a ***minimum spanning tree*** is its spanning tree of the smallest weight, where the ***weight*** of a tree is defined as the sum of the weights on all its edges. The ***minimum spanning tree problem*** is the problem of finding a minimum spanning tree for a given weighted connected graph.

Figure 9.2 presents a simple example illustrating these notions.

If we were to try constructing a minimum spanning tree by exhaustive search, we would face two serious obstacles. First, the number of spanning trees grows exponentially with the graph size (at least for dense graphs). Second, generating all spanning trees for a given graph is not easy; in fact, it is more difficult than finding a *minimum* spanning tree for a weighted graph by using one of several efficient algorithms available for this problem. In this section, we outline ***Prim's algorithm***, which goes back to at least 1957[1] [Pri57].

1. Robert Prim rediscovered the algorithm published 27 years earlier by the Czech mathematician Vojtěch Jarník in a Czech journal.

Prim's algorithm constructs a minimum spanning tree through a sequence of expanding subtrees. The initial subtree in such a sequence consists of a single vertex selected arbitrarily from the set V of the graph's vertices. On each iteration, the algorithm expands the current tree in the greedy manner by simply attaching to it the nearest vertex not in that tree. (By the nearest vertex, we mean a vertex not in the tree connected to a vertex in the tree by an edge of the smallest weight. Ties can be broken arbitrarily.) The algorithm stops after all the graph's vertices have been included in the tree being constructed. Since the algorithm expands a tree by exactly one vertex on each of its iterations, the total number of such iterations is $n - 1$, where n is the number of vertices in the graph. The tree generated by the algorithm is obtained as the set of edges used for the tree expansions.

Here is pseudocode of this algorithm.

ALGORITHM *Prim(G)*

```
//Prim's algorithm for constructing a minimum spanning tree
//Input: A weighted connected graph G = ⟨V, E⟩
//Output: E_T, the set of edges composing a minimum spanning tree of G
V_T ← {v_0}  //the set of tree vertices can be initialized with any vertex
E_T ← ∅
for i ← 1 to |V| − 1 do
    find a minimum-weight edge e* = (v*, u*) among all the edges (v, u)
    such that v is in V_T and u is in V − V_T
    V_T ← V_T ∪ {u*}
    E_T ← E_T ∪ {e*}
return E_T
```

The nature of Prim's algorithm makes it necessary to provide each vertex not in the current tree with the information about the shortest edge connecting the vertex to a tree vertex. We can provide such information by attaching two labels to a vertex: the name of the nearest tree vertex and the length (the weight) of the corresponding edge. Vertices that are not adjacent to any of the tree vertices can be given the ∞ label indicating their "infinite" distance to the tree vertices and a null label for the name of the nearest tree vertex. (Alternatively, we can split the vertices that are not in the tree into two sets, the "fringe" and the "unseen." The fringe contains only the vertices that are not in the tree but are adjacent to at least one tree vertex. These are the candidates from which the next tree vertex is selected. The unseen vertices are all the other vertices of the graph, called "unseen" because they are yet to be affected by the algorithm.) With such labels, finding the next vertex to be added to the current tree $T = \langle V_T, E_T \rangle$ becomes a simple task of finding a vertex with the smallest distance label in the set $V - V_T$. Ties can be broken arbitrarily.

After we have identified a vertex u^* to be added to the tree, we need to perform two operations:

- Move u^* from the set $V - V_T$ to the set of tree vertices V_T.
- For each remaining vertex u in $V - V_T$ that is connected to u^* by a shorter edge than the u's current distance label, update its labels by u^* and the weight of the edge between u^* and u, respectively.[2]

Figure 9.3 demonstrates the application of Prim's algorithm to a specific graph.

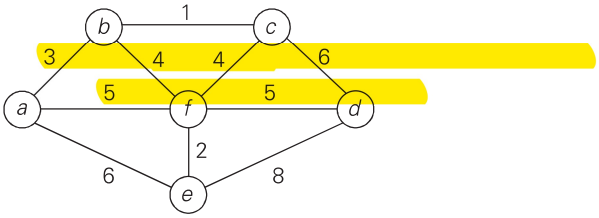
Does Prim's algorithm always yield a minimum spanning tree? The answer to this question is yes. Let us prove by induction that each of the subtrees T_i, $i = 0, \ldots, n-1$, generated by Prim's algorithm is a part (i.e., a subgraph) of some minimum spanning tree. (This immediately implies, of course, that the last tree in the sequence, T_{n-1}, is a minimum spanning tree itself because it contains all n vertices of the graph.) The basis of the induction is trivial, since T_0 consists of a single vertex and hence must be a part of any minimum spanning tree. For the inductive step, let us assume that T_{i-1} is part of some minimum spanning tree T. We need to prove that T_i, generated from T_{i-1} by Prim's algorithm, is also a part of a minimum spanning tree. We prove this by contradiction by assuming that no minimum spanning tree of the graph can contain T_i. Let $e_i = (v, u)$ be the minimum weight edge from a vertex in T_{i-1} to a vertex not in T_{i-1} used by Prim's algorithm to expand T_{i-1} to T_i. By our assumption, e_i cannot belong to any minimum spanning tree, including T. Therefore, if we add e_i to T, a cycle must be formed (Figure 9.4).

In addition to edge $e_i = (v, u)$, this cycle must contain another edge (v', u') connecting a vertex $v' \in T_{i-1}$ to a vertex u' that is not in T_{i-1}. (It is possible that v' coincides with v or u' coincides with u but not both.) If we now delete the edge (v', u') from this cycle, we will obtain another spanning tree of the entire graph whose weight is less than or equal to the weight of T since the weight of e_i is less than or equal to the weight of (v', u'). Hence, this spanning tree is a minimum spanning tree, which contradicts the assumption that no minimum spanning tree contains T_i. This completes the correctness proof of Prim's algorithm.

How efficient is Prim's algorithm? The answer depends on the data structures chosen for the graph itself and for the priority queue of the set $V - V_T$ whose vertex priorities are the distances to the nearest tree vertices. (You may want to take another look at the example in Figure 9.3 to see that the set $V - V_T$ indeed operates as a priority queue.) In particular, if a graph is represented by its weight matrix and the priority queue is implemented as an unordered array, the algorithm's running time will be in $\Theta(|V|^2)$. Indeed, on each of the $|V| - 1$ iterations, the array implementing the priority queue is traversed to find and delete the minimum and then to update, if necessary, the priorities of the remaining vertices.

We can also implement the priority queue as a ***min-heap***. A min-heap is a mirror image of the heap structure discussed in Section 6.4. (In fact, it can be implemented by constructing a heap after negating all the key values given.) Namely, a min-heap is a complete binary tree in which every element is less than or equal

2. If the implementation with the fringe/unseen split is pursued, all the unseen vertices adjacent to u^* must also be moved to the fringe.



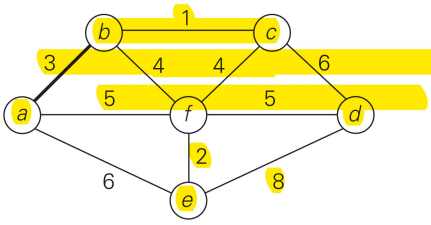
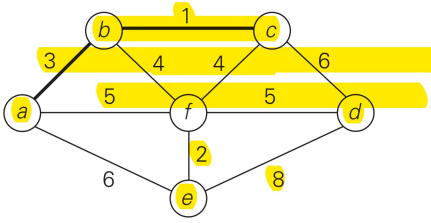
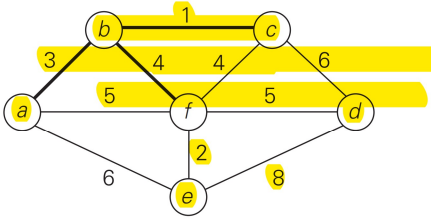
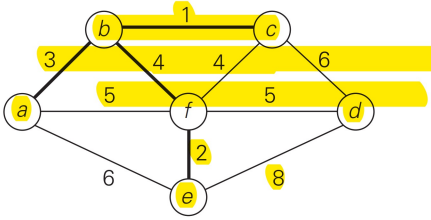
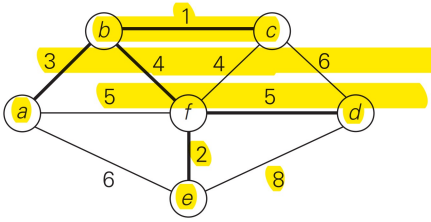
Tree vertices	Remaining vertices	Illustration
a(−, −)	**b(a, 3)** c(−, ∞) d(−, ∞) e(a, 6) f(a, 5)	
b(a, 3)	**c(b, 1)** d(−, ∞) e(a, 6) f(b, 4)	
c(b, 1)	d(c, 6) e(a, 6) **f(b, 4)**	
f(b, 4)	d(f, 5) **e(f, 2)**	
e(f, 2)	**d(f, 5)**	
d(f, 5)		

FIGURE 9.3 Application of Prim's algorithm. The parenthesized labels of a vertex in the middle column indicate the nearest tree vertex and edge weight; selected vertices and edges are shown in bold.

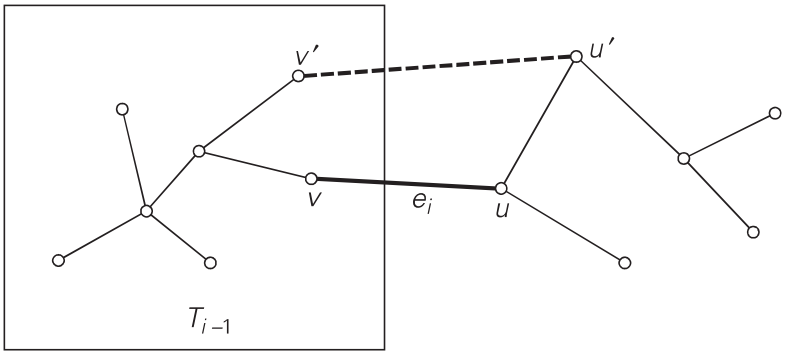


FIGURE 9.4 Correctness proof of Prim's algorithm.

to its children. All the principal properties of heaps remain valid for min-heaps, with some obvious modifications. For example, the root of a min-heap contains the smallest rather than the largest element. Deletion of the smallest element from and insertion of a new element into a min-heap of size n are $O(\log n)$ operations, and so is the operation of changing an element's priority (see Problem 15 in this section's exercises).

If a graph is represented by its adjacency lists and the priority queue is implemented as a min-heap, the running time of the algorithm is in $O(|E| \log |V|)$. This is because the algorithm performs $|V| - 1$ deletions of the smallest element and makes $|E|$ verifications and, possibly, changes of an element's priority in a min-heap of size not exceeding $|V|$. Each of these operations, as noted earlier, is a $O(\log |V|)$ operation. Hence, the running time of this implementation of Prim's algorithm is in

$$(|V| - 1 + |E|)O(\log |V|) = O(|E| \log |V|)$$

because, in a connected graph, $|V| - 1 \leq |E|$.

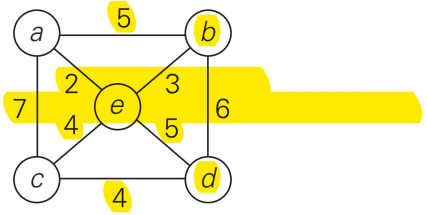
In the next section, you will find another greedy algorithm for the minimum spanning tree problem, which is "greedy" in a manner different from that of Prim's algorithm.

Exercises 9.1

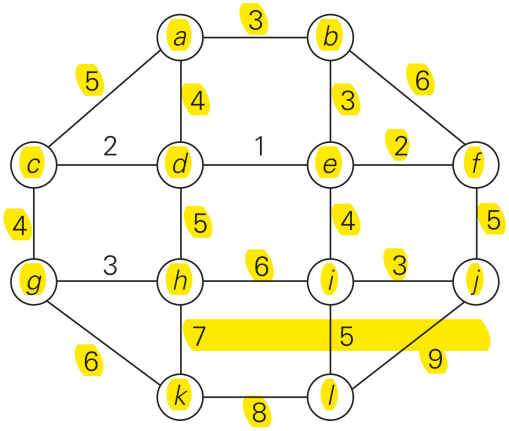
1. Write pseudocode of the greedy algorithm for the change-making problem, with an amount n and coin denominations $d_1 > d_2 > \cdots > d_m$ as its input. What is the time efficiency class of your algorithm?
2. Design a greedy algorithm for the assignment problem (see Section 3.4). Does your greedy algorithm always yield an optimal solution?
3. *Job scheduling* Consider the problem of scheduling n jobs of known durations $t_1, t_2, \ldots, t_n$ for execution by a single processor. The jobs can be executed in any order, one job at a time. You want to find a schedule that minimizes

b. weights can be put on both cups of the scale.

9. a. Apply Prim's algorithm to the following graph. Include in the priority queue all the vertices not already in the tree.



b. Apply Prim's algorithm to the following graph. Include in the priority queue only the fringe vertices (the vertices not in the current tree which are adjacent to at least one tree vertex).



10. The notion of a minimum spanning tree is applicable to a connected weighted graph. Do we have to check a graph's connectivity before applying Prim's algorithm, or can the algorithm do it by itself?

11. Does Prim's algorithm always work correctly on graphs with negative edge weights?

12. Let T be a minimum spanning tree of graph G obtained by Prim's algorithm. Let G_{new} be a graph obtained by adding to G a new vertex and some edges, with weights, connecting the new vertex to some vertices in G. Can we construct a minimum spanning tree of G_{new} by adding one of the new edges to T? If you answer yes, explain how; if you answer no, explain why not.

13. How can one use Prim's algorithm to find a spanning tree of a connected graph with no weights on its edges? Is it a good algorithm for this problem?

14. Prove that any weighted connected graph with distinct weights has exactly one minimum spanning tree.

15. Outline an efficient algorithm for changing an element's value in a min-heap. What is the time efficiency of your algorithm?

9.2 Kruskal's Algorithm

In the previous section, we considered the greedy algorithm that "grows" a minimum spanning tree through a greedy inclusion of the nearest vertex to the vertices already in the tree. Remarkably, there is another greedy algorithm for the minimum spanning tree problem that also always yields an optimal solution. It is named ***Kruskal's algorithm*** after Joseph Kruskal, who discovered this algorithm when he was a second-year graduate student [Kru56]. Kruskal's algorithm looks at a minimum spanning tree of a weighted connected graph $G = \langle V, E \rangle$ as an acyclic subgraph with $|V| - 1$ edges for which the sum of the edge weights is the smallest. (It is not difficult to prove that such a subgraph must be a tree.) Consequently, the algorithm constructs a minimum spanning tree as an expanding sequence of subgraphs that are always acyclic but are not necessarily connected on the intermediate stages of the algorithm.

The algorithm begins by sorting the graph's edges in nondecreasing order of their weights. Then, starting with the empty subgraph, it scans this sorted list, adding the next edge on the list to the current subgraph if such an inclusion does not create a cycle and simply skipping the edge otherwise.

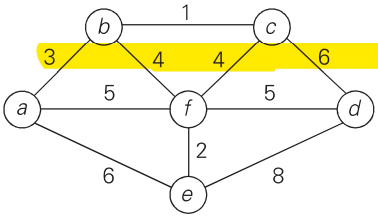
ALGORITHM *Kruskal(G)*

```
//Kruskal's algorithm for constructing a minimum spanning tree
//Input: A weighted connected graph G = ⟨V, E⟩
//Output: E_T, the set of edges composing a minimum spanning tree of G
sort E in nondecreasing order of the edge weights w(e_{i_1}) ≤ ··· ≤ w(e_{i_|E|})
E_T ← ∅;  ecounter ← 0        //initialize the set of tree edges and its size
k ← 0                         //initialize the number of processed edges
while ecounter < |V| − 1 do
    k ← k + 1
    if E_T ∪ {e_{i_k}} is acyclic
        E_T ← E_T ∪ {e_{i_k}};  ecounter ← ecounter + 1
return E_T
```

The correctness of Kruskal's algorithm can be proved by repeating the essential steps of the proof of Prim's algorithm given in the previous section. The fact that E_T is actually a tree in Prim's algorithm but generally just an acyclic subgraph in Kruskal's algorithm turns out to be an obstacle that can be overcome.

Figure 9.5 demonstrates the application of Kruskal's algorithm to the same graph we used for illustrating Prim's algorithm in Section 9.1. As you trace the algorithm's operations, note the disconnectedness of some of the intermediate subgraphs.

Applying Prim's and Kruskal's algorithms to the same small graph by hand may create the impression that the latter is simpler than the former. This impression is wrong because, on each of its iterations, Kruskal's algorithm has to check whether the addition of the next edge to the edges already selected would create a



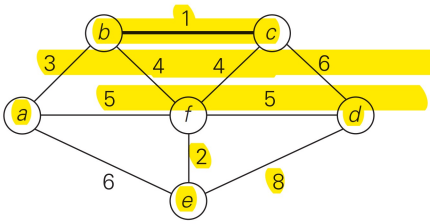
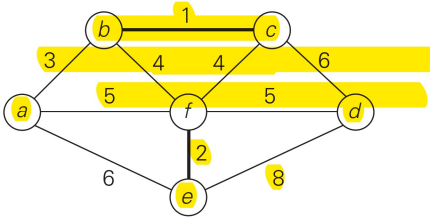
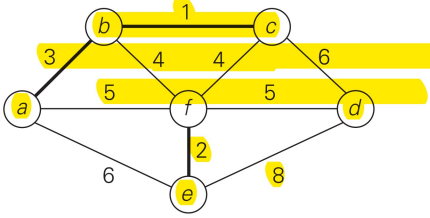
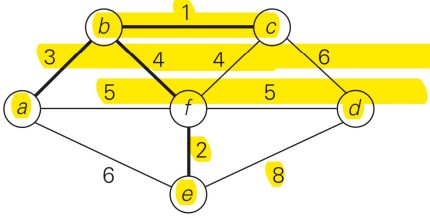
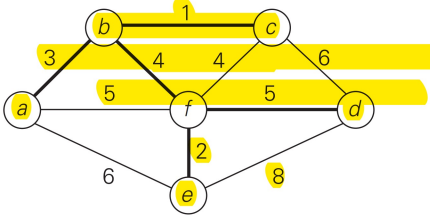
Tree edges	Sorted list of edges	Illustration
	bc ef ab bf cf af df ae cd de **1** 2 3 4 4 5 5 6 6 8	
bc 1	bc **ef** ab bf cf af df ae cd de 1 **2** 3 4 4 5 5 6 6 8	
ef 2	bc ef **ab** bf cf af df ae cd de 1 2 **3** 4 4 5 5 6 6 8	
ab 3	bc ef ab **bf** cf af df ae cd de 1 2 3 **4** 4 5 5 6 6 8	
bf 4	bc ef ab bf cf af **df** ae cd de 1 2 3 4 4 5 **5** 6 6 8	
df 5		

FIGURE 9.5 Application of Kruskal's algorithm. Selected edges are shown in bold.

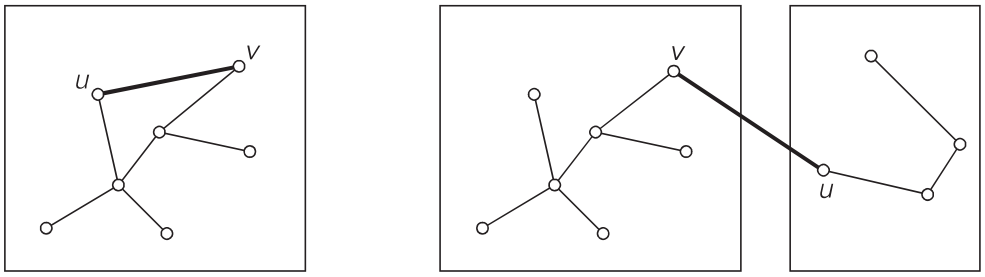


FIGURE 9.6 New edge connecting two vertices may (a) or may not (b) create a cycle.

cycle. It is not difficult to see that a new cycle is created if and only if the new edge connects two vertices already connected by a path, i.e., if and only if the two vertices belong to the same connected component (Figure 9.6). Note also that each connected component of a subgraph generated by Kruskal's algorithm is a tree because it has no cycles.

In view of these observations, it is convenient to use a slightly different interpretation of Kruskal's algorithm. We can consider the algorithm's operations as a progression through a series of forests containing *all* the vertices of a given graph and *some* of its edges. The initial forest consists of $|V|$ trivial trees, each comprising a single vertex of the graph. The final forest consists of a single tree, which is a minimum spanning tree of the graph. On each iteration, the algorithm takes the next edge (u, v) from the sorted list of the graph's edges, finds the trees containing the vertices u and v, and, if these trees are not the same, unites them in a larger tree by adding the edge (u, v).

Fortunately, there are efficient algorithms for doing so, including the crucial check for whether two vertices belong to the same tree. They are called ***union-find*** algorithms. We discuss them in the following subsection. With an efficient union-find algorithm, the running time of Kruskal's algorithm will be dominated by the time needed for sorting the edge weights of a given graph. Hence, with an efficient sorting algorithm, the time efficiency of Kruskal's algorithm will be in $O(|E| \log |E|)$.

Disjoint Subsets and Union-Find Algorithms

Kruskal's algorithm is one of a number of applications that require a dynamic partition of some n element set S into a collection of disjoint subsets $S_1, S_2, \ldots, S_k$. After being initialized as a collection of n one-element subsets, each containing a different element of S, the collection is subjected to a sequence of intermixed union and find operations. (Note that the number of union operations in any such sequence must be bounded above by $n - 1$ because each union increases a subset's size at least by 1 and there are only n elements in the entire set S.) Thus, we are

3. Give a counterexample that shows that Dijkstra's algorithm may not work for a weighted connected graph with negative weights.
4. Let T be a tree constructed by Dijkstra's algorithm in the process of solving the single-source shortest-paths problem for a weighted connected graph G.
   a. True or false: T is a spanning tree of G?
   b. True or false: T is a minimum spanning tree of G?
5. Write pseudocode for a simpler version of Dijkstra's algorithm that finds only the distances (i.e., the lengths of shortest paths but not shortest paths themselves) from a given vertex to all other vertices of a graph represented by its weight matrix.
6. Prove the correctness of Dijkstra's algorithm for graphs with positive weights.
7. Design a linear-time algorithm for solving the single-source shortest-paths problem for dags (directed acyclic graphs) represented by their adjacency lists.
8. Explain how the minimum-sum descent problem (Problem 8 in Exercises 8.1) can be solved by Dijkstra's algorithm.
9. *Shortest-path modeling* Assume you have a model of a weighted connected graph made of balls (representing the vertices) connected by strings of appropriate lengths (representing the edges).
   a. Describe how you can solve the single-pair shortest-path problem with this model.
   b. Describe how you can solve the single-source shortest-paths problem with this model.
10. Revisit the exercise from Section 1.3 about determining the best route for a subway passenger to take from one designated station to another in a well-developed subway system like those in Washington, DC, or London, UK. Write a program for this task.

9.4 Huffman Trees and Codes

Suppose we have to encode a text that comprises symbols from some n-symbol alphabet by assigning to each of the text's symbols some sequence of bits called the ***codeword***. For example, we can use a ***fixed-length encoding*** that assigns to each symbol a bit string of the same length m ($m \geq \log_2 n$). This is exactly what the standard ASCII code does. One way of getting a coding scheme that yields a shorter bit string on the average is based on the old idea of assigning shorter codewords to more frequent symbols and longer codewords to less frequent symbols. This idea was used, in particular, in the telegraph code invented in the mid-19th century by Samuel Morse. In that code, frequent letters such as e ($\cdot$) and a ($\cdot -$) are assigned short sequences of dots and dashes while infrequent letters such as q ($- - \cdot -$) and z ($- - \cdot \cdot$) have longer ones.

Variable-length encoding, which assigns codewords of different lengths to different symbols, introduces a problem that fixed-length encoding does not have. Namely, how can we tell how many bits of an encoded text represent the first (or, more generally, the ith) symbol? To avoid this complication, we can limit ourselves to the so-called ***prefix-free*** (or simply ***prefix***) ***codes***. In a prefix code, no codeword is a prefix of a codeword of another symbol. Hence, with such an encoding, we can simply scan a bit string until we get the first group of bits that is a codeword for some symbol, replace these bits by this symbol, and repeat this operation until the bit string's end is reached.

If we want to create a binary prefix code for some alphabet, it is natural to associate the alphabet's symbols with leaves of a binary tree in which all the left edges are labeled by 0 and all the right edges are labeled by 1. The codeword of a symbol can then be obtained by recording the labels on the simple path from the root to the symbol's leaf. Since there is no simple path to a leaf that continues to another leaf, no codeword can be a prefix of another codeword; hence, any such tree yields a prefix code.

Among the many trees that can be constructed in this manner for a given alphabet with known frequencies of the symbol occurrences, how can we construct a tree that would assign shorter bit strings to high-frequency symbols and longer ones to low-frequency symbols? It can be done by the following greedy algorithm, invented by David Huffman while he was a graduate student at MIT [Huf52].

Huffman's algorithm

Step 1 Initialize n one-node trees and label them with the symbols of the alphabet given. Record the frequency of each symbol in its tree's root to indicate the tree's ***weight***. (More generally, the weight of a tree will be equal to the sum of the frequencies in the tree's leaves.)

Step 2 Repeat the following operation until a single tree is obtained. Find two trees with the smallest weight (ties can be broken arbitrarily, but see Problem 2 in this section's exercises). Make them the left and right subtree of a new tree and record the sum of their weights in the root of the new tree as its weight.

A tree constructed by the above algorithm is called a ***Huffman tree***. It defines—in the manner described above—a ***Huffman code***.

EXAMPLE Consider the five-symbol alphabet {A, B, C, D, _} with the following occurrence frequencies in a text made up of these symbols:

symbol	A	B	C	D	_
frequency	0.35	0.1	0.2	0.2	0.15

The Huffman tree construction for this input is shown in Figure 9.12.

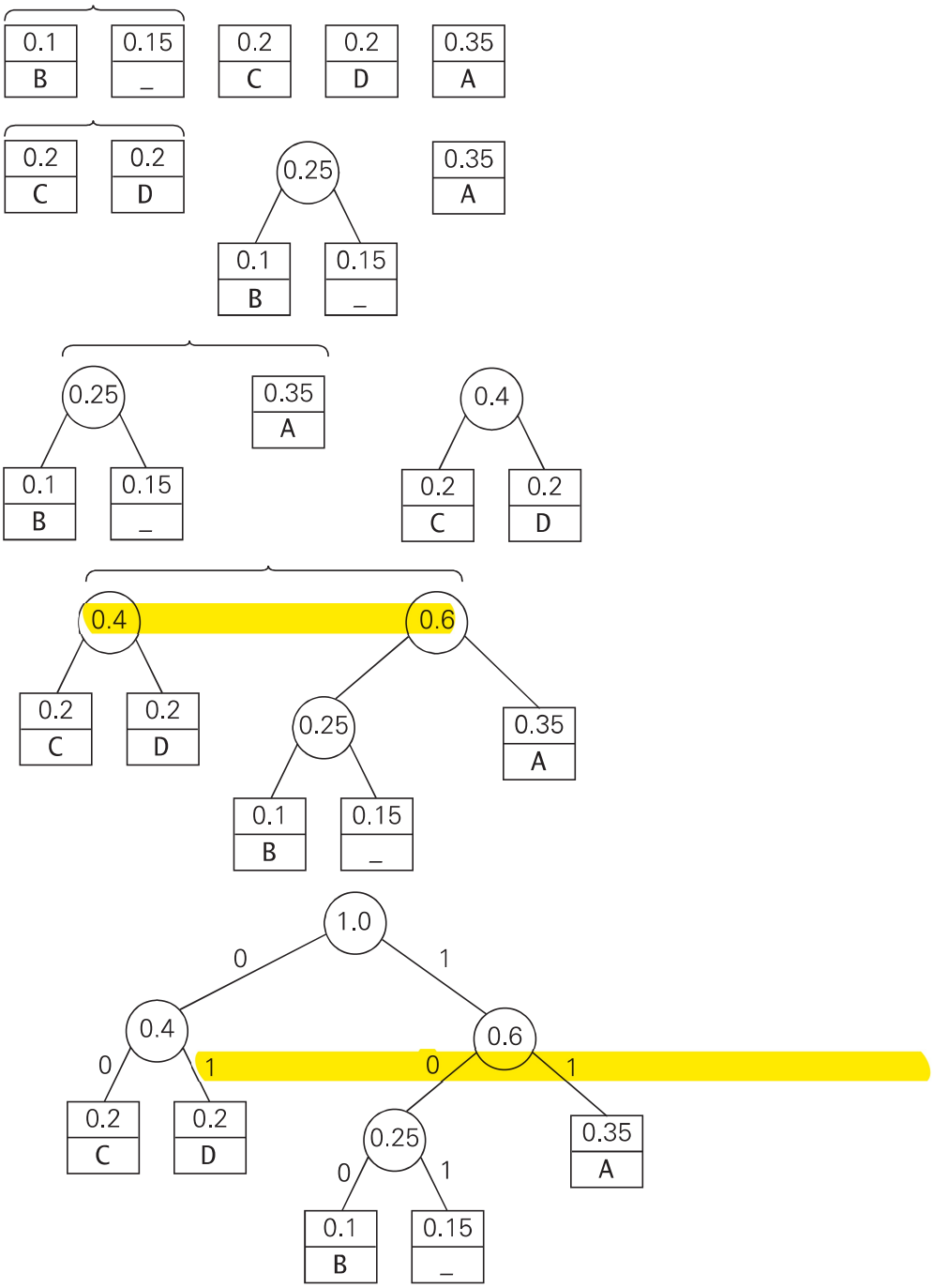


FIGURE 9.12 Example of constructing a Huffman coding tree.

The resulting codewords are as follows:

symbol	A	B	C	D	_
frequency	0.35	0.1	0.2	0.2	0.15
codeword	11	100	00	01	101

Hence, DAD is encoded as 011101, and 10011011011101 is decoded as BAD_AD.

With the occurrence frequencies given and the codeword lengths obtained, the average number of bits per symbol in this code is

$$2 \cdot 0.35 + 3 \cdot 0.1 + 2 \cdot 0.2 + 2 \cdot 0.2 + 3 \cdot 0.15 = 2.25.$$

Had we used a fixed-length encoding for the same alphabet, we would have to use at least 3 bits per each symbol. Thus, for this toy example, Huffman's code achieves the ***compression ratio***—a standard measure of a compression algorithm's effectiveness—of $(3 - 2.25)/3 \cdot 100\% = 25\%$. In other words, Huffman's encoding of the text will use 25% less memory than its fixed-length encoding. (Extensive experiments with Huffman codes have shown that the compression ratio for this scheme typically falls between 20% and 80%, depending on the characteristics of the text being compressed.) ■

Huffman's encoding is one of the most important file-compression methods. In addition to its simplicity and versatility, it yields an optimal, i.e., minimal-length, encoding (provided the frequencies of symbol occurrences are independent and known in advance). The simplest version of Huffman compression calls, in fact, for a preliminary scanning of a given text to count the frequencies of symbol occurrences in it. Then these frequencies are used to construct a Huffman coding tree and encode the text as described above. This scheme makes it necessary, however, to include the coding table into the encoded text to make its decoding possible. This drawback can be overcome by using ***dynamic Huffman encoding***, in which the coding tree is updated each time a new symbol is read from the source text. Further, modern alternatives such as ***Lempel-Ziv*** algorithms (e.g., [Say05]) assign codewords not to individual symbols but to strings of symbols, allowing them to achieve better and more robust compressions in many applications.

It is important to note that applications of Huffman's algorithm are not limited to data compression. Suppose we have n positive numbers $w_1, w_2, \ldots, w_n$ that have to be assigned to n leaves of a binary tree, one per node. If we define the ***weighted path length*** as the sum $\sum_{i=1}^{n} l_i w_i$, where l_i is the length of the simple path from the root to the ith leaf, how can we construct a binary tree with minimum weighted path length? It is this more general problem that Huffman's algorithm actually solves. (For the coding application, l_i and w_i are the length of the codeword and the frequency of the ith symbol, respectively.)

This problem arises in many situations involving decision making. Consider, for example, the game of guessing a chosen object from n possibilities (say, an integer between 1 and n) by asking questions answerable by yes or no. Different strategies for playing this game can be modeled by ***decision trees***[5] such as those depicted in Figure 9.13 for $n = 4$. The length of the simple path from the root to a leaf in such a tree is equal to the number of questions needed to get to the chosen number represented by the leaf. If number i is chosen with probability p_i, the sum

5. Decision trees are discussed in more detail in Section 11.2.

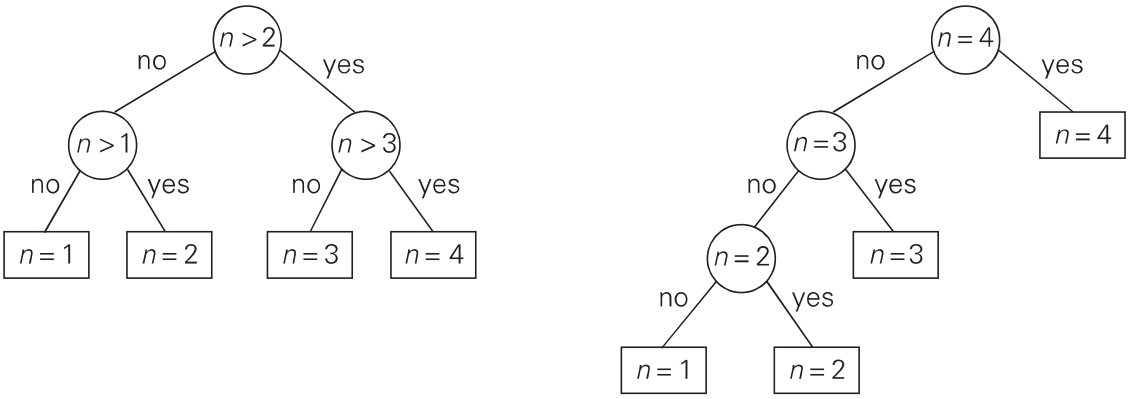


FIGURE 9.13 Two decision trees for guessing an integer between 1 and 4.

$\sum_{i=1}^{n} l_i p_i$, where l_i is the length of the path from the root to the ith leaf, indicates the average number of questions needed to "guess" the chosen number with a game strategy represented by its decision tree. If each of the numbers is chosen with the same probability of $1/n$, the best strategy is to successively eliminate half (or almost half) the candidates as binary search does. This may not be the case for arbitrary p_i's, however. For example, if $n = 4$ and $p_1 = 0.1$, $p_2 = 0.2$, $p_3 = 0.3$, and $p_4 = 0.4$, the minimum weighted path tree is the rightmost one in Figure 9.13. Thus, we need Huffman's algorithm to solve this problem in its general case.

Note that this is the second time we are encountering the problem of constructing an optimal binary tree. In Section 8.3, we discussed the problem of constructing an optimal binary search tree with positive numbers (the search probabilities) assigned to every node of the tree. In this section, given numbers are assigned just to leaves. The latter problem turns out to be easier: it can be solved by the greedy algorithm, whereas the former is solved by the more complicated dynamic programming algorithm.

Exercises 9.4

1. a. Construct a Huffman code for the following data:

symbol	A	B	C	D	_
frequency	0.4	0.1	0.2	0.15	0.15

b. Encode ABACABAD using the code of question (a).

c. Decode 100010111001010 using the code of question (a).

2. For data transmission purposes, it is often desirable to have a code with a minimum variance of the codeword lengths (among codes of the same average length). Compute the average and variance of the codeword length in two

Huffman codes that result from a different tie breaking during a Huffman code construction for the following data:

symbol	A	B	C	D	E
probability	0.1	0.1	0.2	0.2	0.4

3. Indicate whether each of the following properties is true for every Huffman code.
   a. The codewords of the two least frequent symbols have the same length.
   b. The codeword's length of a more frequent symbol is always smaller than or equal to the codeword's length of a less frequent one.
4. What is the maximal length of a codeword possible in a Huffman encoding of an alphabet of n symbols?
5. a. Write pseudocode of the Huffman-tree construction algorithm.
   b. What is the time efficiency class of the algorithm for constructing a Huffman tree as a function of the alphabet size?
6. Show that a Huffman tree can be constructed in linear time if the alphabet symbols are given in a sorted order of their frequencies.
7. Given a Huffman coding tree, which algorithm would you use to get the codewords for all the symbols? What is its time-efficiency class as a function of the alphabet size?
8. Explain how one can generate a Huffman code without an explicit generation of a Huffman coding tree.
9. a. Write a program that constructs a Huffman code for a given English text and encode it.
   b. Write a program for decoding of an English text which has been encoded with a Huffman code.
   c. Experiment with your encoding program to find a range of typical compression ratios for Huffman's encoding of English texts of, say, 1000 words.
   d. Experiment with your encoding program to find out how sensitive the compression ratios are to using standard estimates of frequencies instead of actual frequencies of symbol occurrences in English texts.
10. *Card guessing* Design a strategy that minimizes the expected number of questions asked in the following game [Gar94]. You have a deck of cards that consists of one ace of spades, two deuces of spades, three threes, and on up to nine nines, making 45 cards in all. Someone draws a card from the shuffled deck, which you have to identify by asking questions answerable with yes or no.