The Three Body Problem

(restricted-approach)

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Topics

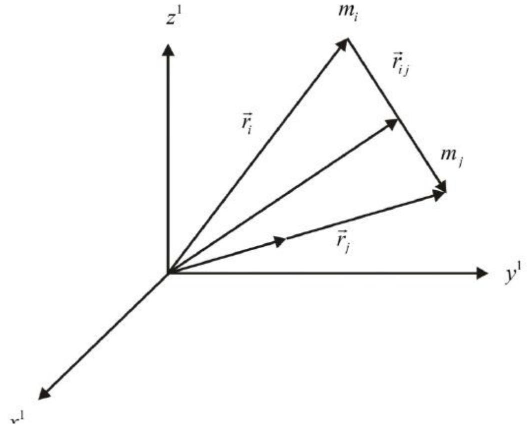
The Three-Body Problem

The classical gravitational three body problem appear exclusively in Nature, like the earth-moon and sun system. This problem was the main investigation topic by the scientistsof the 18th and 19th centuries. Today, The three body problem still unsolved, however, using the modern computational programs and techniques has enabled us to get an insight about the motion itself.

General Analysis of n body problem:

The analysis of n body requires the following, if at a given instant "n" bodies of known masses m_i, i=1,…, n, their positions and velocities are to be identified as a function of time. Also, these masses only move under the effect of the mutual attraction created by gravity.

Assume: $(x'\ y'\ z')$ Inertial frame of reference as in the given figure.



$$\overrightarrow{r_{ij}} = \overrightarrow{r_j} - \overrightarrow{r_i} \qquad (1)$$

Assume f_{ij} be the effective force on the ith particle due to jth particle as i ~= j and (i=1, 2…n). So we can get the total force on the ith particle as the following

$$m_i\ddot{\vec{r}}_i = \sum_{j=1}^{n} \vec{f}_{ij} \qquad j \neq i \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_(2) \qquad \text{Here,} \quad \overline{f}_{ij} = -\overline{f}_{ji} \quad \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_(3)$$

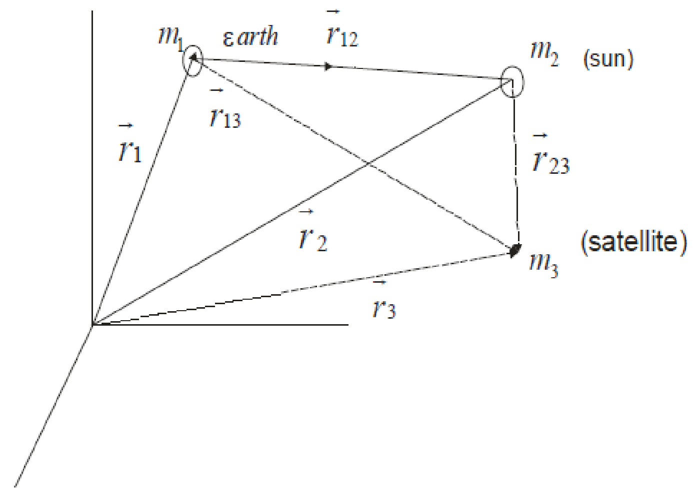
$$\vec{f}_{ij} = \frac{G\ m_i m_j}{r_{ij}^3}\vec{r}_{ij} \quad \_\_\_\_\_\_\_\_\_\_\_\_\_(4) \Rightarrow m_i\ddot{\vec{r}}_i = \sum_{j \neq i}^{n}\left[G\frac{m_i m_j}{r_{ij}^3}\vec{r}_{ij}\right] \quad \_\_\_\_\_\_\_\_\_\_\_\_(5)$$

So, if we have "n" bodies , we would have 3n 2nd coupled odes, that have no analytical solution.

Three body geometry.

$$\vec{r}_{13} = \vec{r}_3 - \vec{r}_1 \qquad \text{\{In the IRF\}}$$

$$\ddot{\vec{r}}_{13} = \ddot{\vec{r}}_3 - \ddot{\vec{r}}_1$$



The EOMs

$$m_3\ddot{\vec{r}}_3 = -\frac{m_3 m_1 G}{r_{13}^3}\vec{r}_{13} - \frac{m_2 m_3 G}{r_{23}^3}\vec{r}_{23}$$

$$m_1\ddot{\vec{r}}_1 = \frac{G m_1 m_3 \vec{r}_{13}}{r_{13}^3} + \frac{G m_1 m_2 \vec{r}_{12}}{r_{12}^3}$$

$$\ddot{\vec{r}}_{13} = -\frac{m_1 G}{r_{13}^3}\vec{r}_{13} - \frac{m_2 G}{r_{23}^3}\vec{r}_{23} - \frac{G m_3}{r_{13}^3}\vec{r}_{13} - \frac{G m_2}{r_{12}^3}\vec{r}_{12}$$

as $\quad \vec{r}_{23} = -\vec{r}_{32}$

$$\ddot{\vec{r}}_{13} = -\frac{(m_1+m_3)G}{r_{13}^3}\vec{r}_{13} - \left[\frac{G m_2}{r_{23}^3}\vec{r}_{23} + \frac{G m_2}{r_{12}^3}\vec{r}_{12}\right]$$

$$\ddot{\vec{r}}_{13} = -G\frac{(m_1+m_3)}{r_{13}^3}\vec{r}_{13} + G m_2\left[\frac{\vec{r}_{32}}{r_{32}^3} - \frac{\vec{r}_{12}}{r_{12}^3}\right]$$

In this form, we can see that the 1st term is the earth's direct influence on the third body, while the 2nd term is the earth's and sun's combined effect that acts as a disturbing term. The 1st term on the right term is the sun's direct influence on the third body, while the 2nd is the earth's indirect influence on the third body. Thus, any slight change in earth's position causes earth's acceleration to change and contribute to the acceleration of the third body.

As we have seen the acceleration of the i^{th} body due to the forces of (n-1) bodies is given by

$$\ddot{\vec{r}}_i = -G\sum_{\substack{j=1\\ i\neq j}}^{n}\frac{m_j}{r_{ji}^3}\vec{r}_{ji} \qquad \vec{r}_j - \vec{r}_i = \vec{r}_{ij}\,; \quad i = 1,\dots n$$

It is useful to look at the system from the center of mass frame so that we can get some insight about the motion. This frame called **Barycentric Formula**.

Position vector of the i^{th} particle in that new frame.

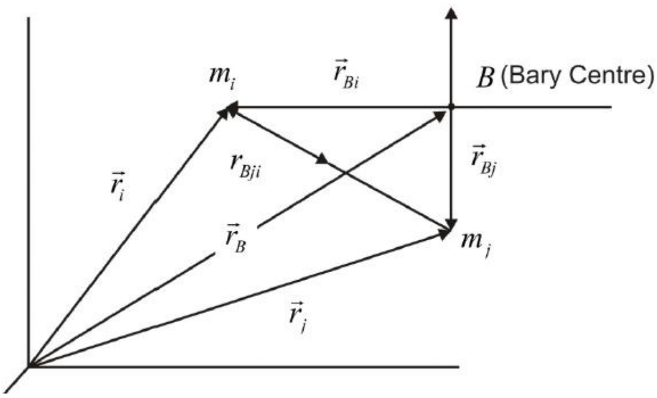
$$\vec{r}_{Bi} = \vec{r}_i - \vec{r}_B$$

$$\vec{r}_{Bj} = \vec{r}_j - \vec{r}_B$$

$$\vec{r}_{Bi} - \vec{r}_{Bj} = \vec{r}_{Bji}$$

$$\vec{r}_{Bji} = (\vec{r}_i - \vec{r}_B) - (\vec{r}_j - \vec{r}_B)$$

$$\vec{r}_{Bji} = (\vec{r}_i - \vec{r}_j) = \vec{r}_{ji}$$



Acceleration of i^{th} particle $$\ddot{\vec{r}}_{Bi} = \ddot{\vec{r}}_i - \ddot{\vec{r}}_B = \ddot{\vec{r}}_i$$

Thus, it is clear that the equation of motion are totally independent of the inertial reference frame.

The generalized equation of motion in Barycentric frame. $$\ddot{\vec{r}}_{Bs} = \ddot{\vec{r}}_s = -G \sum_{\substack{j=1 \\ j\neq s}}^{n} \frac{m_j \vec{r}_{js}}{r_{js}^3}$$

Studing the general properties of the motion.

1-from equation (2), Summing over $i = 1, 2 \dots n$

$$\sum_{i=1}^{n} m_i \ddot{\vec{r}}_i = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \vec{f}_{ij} = 0$$ Since all the forces are in opposite pairs.

$$\Rightarrow \sum_{i=1}^{n} m_i \dot{\vec{r}}_i = \vec{c}_1 \Rightarrow \sum_{i=1}^{n} m_i \vec{r}_i = \vec{c}_1 t + \vec{c}_2$$

Let the center of mass to be at R.

$$\sum_{i=1}^{n} m_i \dot{\vec{r}}_i = \left(\sum_{i=1}^{n} m_i \right) \dot{\vec{R}} = \vec{C}_1$$

$$\sum_{i=1}^{n} m_i \vec{r}_i = \left(\sum_{i=1}^{n} m_i \right) \vec{R} = \vec{c}_1 t + \vec{c}_2$$

$$\therefore \dot{\vec{R}} = \text{Constant}$$

Therefore, the velocity of center of mass of n particles is constant

2-Take the Cross Product of (5) by r_i and summing over $i = 1, 2, \dots n$;

$$\Rightarrow m_i \ddot{\vec{r}}_i = \sum^{n} \left[G \frac{m_i m_j}{r^3} \vec{r}_{ij} \right] \qquad \text{———————— (5)}$$

$$\sum_{i=1}^{n} \vec{r}_i \times m_i \ddot{\vec{r}}_i = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} G \frac{m_i m_j}{r_{ij}^3} \vec{r}_i \times \vec{r}_{ij}$$

$$\vec{r}_{ij} = \vec{r}_j - \vec{r}_i \qquad \Rightarrow \vec{r}_i \times \vec{r}_{ij} = \vec{r}_i \times (\vec{r}_j - r_i) = \vec{r}_i \times \vec{r}_j = \vec{g}_{ij}$$

$$\Rightarrow \sum_{i=1}^{n} (\vec{r}_i \times m_i \ddot{\vec{r}}_i) = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} G \frac{m_i m_j}{r_{ij}^3} \vec{g}_{ij} = \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{d}_{ij}$$

$\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \vec{d}_{ij} = 0$ because these terms appear in pairs.

$$\Rightarrow \sum_{i=1}^{n} \left[\vec{r}_i \times m_i \ddot{\vec{r}}_i \right] = 0 \qquad \Rightarrow \frac{d}{dt} \left[\sum_{i=1}^{n} \vec{r}_i \times m_i \dot{\vec{r}}_i \right] = 0$$

$$\Rightarrow \frac{d\vec{H}}{dt} = 0$$ where H is the angular momentum.

Thus, the angular momentum of n bodies doesn't change with time.

3- Take the dot product (5) by r_i **and summing from** $i = 1, ..., n$.

$$\sum_{i=1}^{n} m_i \ddot{\vec{r}}_i . \dot{\vec{r}}_i = \sum_{i=1}^{n} \sum_{j=1}^{n} \kappa \frac{m_i m_j}{r_{ij}^3} \vec{r}_{ij} . \dot{\vec{r}}_i$$

$$\dot{\vec{r}}_i . \vec{r}_{ij} = \dot{\vec{r}}_i . (\vec{r}_J - \vec{r}_i)$$

$$\dot{\vec{r}}_j . \vec{r}_{ji} = \dot{\vec{r}}_j . (\vec{r}_i - \vec{r}_j)$$

$$\dot{\vec{r}}_i . \vec{r}_{rj} + \dot{\vec{r}}_j . \vec{r}_{ji} = \dot{\vec{r}}_i . (\vec{r}_j - \vec{r}_i) + \dot{\vec{r}}_j . (\vec{r}_i - \vec{r}_j)$$

$$= -(\dot{\vec{r}}_j - \dot{\vec{r}}_i) . (\vec{r}_j - \vec{r}_i) \vec{r}_j - \vec{r}_i) = (\dot{\vec{r}}_i - \dot{\vec{r}}_j) . (\vec{r}_j - \vec{r}_i)$$

$$\sum_{i=1}^{n} \left(m_i \ddot{\vec{r}}_i . \dot{\vec{r}}_i \right) = -\tfrac{1}{2} \sum_{i=1}^{n} \sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{G m_i m_j}{r_{ij}^3} \left(\underline{\dot{\vec{r}}_j - \dot{\vec{r}}_i} \right) . \left(\underline{\vec{r}_j - \vec{r}_i} \right)$$

Integrating once

$$\Rightarrow \sum_{i=1}^{n} \left(m_i \dot{\vec{r}}_i . \dot{\vec{r}}_i \right) = -\tfrac{1}{2} \int \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{G m_i m_j}{r_{ij}^3} (\dot{\vec{r}}_j - \dot{\vec{r}}_i) . (\vec{r}_j - \vec{r}_i) dt + E$$

$$= -\tfrac{1}{2}\int \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{km_i m_j}{r_{ij}^3}\dot{\vec{r}}_{ij}.\vec{r}_{ij}dt\right)+E$$

$$= -\tfrac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\int\left(\frac{km_i m_j}{r_{ij}^3}r_{ij}\frac{dr_{ij}}{dt}\right)dt+E$$

$$= -\tfrac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\int\left(\frac{Gm_i m_j}{r_{ij}^2}\right)dr_{ij}+E$$

$$= \tfrac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{Gm_i m_j}{r_{ij}}+E \qquad \frac{d}{dt}\left(\vec{r}_i.\vec{r}_i\right)=\dot{\vec{r}}_i.\vec{r}_i+\dot{\vec{r}}_i.\vec{r}_i=2\dot{\vec{r}}_i.\vec{r}_i=\frac{d}{dt}\left(r^2\right)=2r\frac{dr}{dt}$$

Where E is a scalar constant

$$\frac{1}{2}\sum_{i=1}^{n}m_i v_i^2=\tfrac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\frac{Gm_i m_j}{r_{ij}}\right)+E$$

$$\tfrac{1}{2}\sum_{i=1}^{n}m_i v^2{}_i=-U+E \;\Rightarrow T=U+E$$

Where $T=\tfrac{1}{2}\sum_{i=1}^{n}m_i v^2{}_i$ and $U=-\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\frac{Gm_i m_j}{r_{ij}}$

Which indicates the total energy is constant.

Energy for 3 bodies given as the following. $U = -k\left[\frac{m_1m_2}{r_{12}}+\frac{m_2m_3}{r_{23}}+\frac{m_1m_3}{r_{13}}\right]$

Number of integration constants calcualted .

1) C.M. is not accelerating, C_1, C_2 gives six scalars constants.
2) angular momentum is constant, gives three scalars constants.
3) Total energy is constant, gives 1 scalar constant.

So, 10 constants of integration are identified. A system of "3n" 2nd ODE can be written for n body problem which gives 6n motion integrals. For the three body we have 18 constants, that 8 are impossible to be identified.

If we consider the two body problem, this is expressed by a structure of 12 order 2nd odes. Linear momentum conservation gives 6 integral constants that are the initial position and velocity of the COM. So, the solution available for this system will not be completed without adding constrains. For this system, by assuming one of the masses is small and moving the coordinate system to barycentre frame we will be able to solve for the relative motion between of the two bodies.

Restricted three-body problem

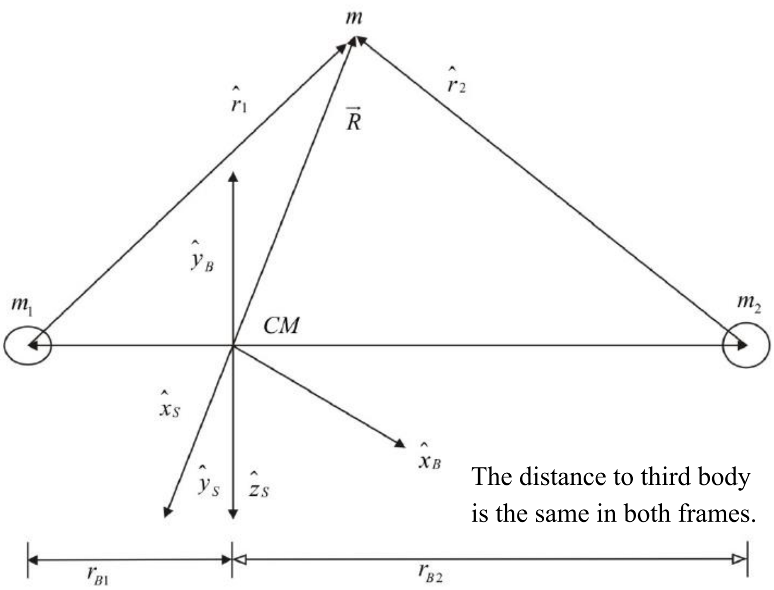
Constrains of motion.

(1) Two bodies (primary and secondary) of the three move around each other in a circular orbit around their COM.That is why it is called circular restricted.

(2) The 3rd body mass is so small that can be neglected compared to the other two masses.

For this system, we would look for the relative position of the third body to the major bodies. So, using a rotating frame called Synodic or co-rotating frame that rotates with the major bodies. Its origin lies on the COM and rotates by w_s . The y axis of that frame lies on the plane of the three masses.

The absolute acceleration of m will be given relative to barycenter R.F. But it is impossible to be calculated. What we can get as in the 2-body problem is the relative motion of m relative to the two mases that is relative to synodic reference frame.



The distance to third body is the same in both frames.

Acceleration relative to Bray R.F

$$\frac{d^2\vec{R}}{dt^2} = -\frac{Gm_1\vec{r}_{1s}}{r_{1s}^3} - \frac{Gm_2\vec{r}_{2s}}{r_{2s}^3}$$

The relation between acceleration in thetwo frames can be given by the relation

$$\ddot{\vec{R}}_{\text{int}} = \ddot{\vec{r}}_s + \dot{\vec{\omega}}_s \times \vec{r}_s + \vec{\omega}_s \times (\vec{\omega}_s \times \vec{r}_s) + 2\vec{\omega}_s \times \vec{v}_s + \ddot{\vec{R}}_{os}$$

$\ddot{\vec{R}}_{os}$ Acceleration of the origin of synodic R.F , however it is anchor at the barycenter, so this term = 0

$\dot{\vec{\omega}}_s$ is the rate of change of angular velocity, but here as the 2 masses move in circular orbit, this term =0.

$2\vec{\omega}_s \times \vec{v}_s$ this is the coriolis term

$\ddot{\vec{r}}_s$ Is the acceleration in the synodic R.F.

Taking
$$\vec{r}_s = x\hat{e}_1 + y\,\hat{e}_2 + z\,\hat{e}_3$$
$$\vec{\omega}_s = \omega_s\,\hat{e}_3 \qquad\qquad \Rightarrow \ddot{\vec{r}}_s = \ddot{x}\,\hat{e}_1 + \ddot{y}\,\hat{e}_2 + \ddot{z}\,\hat{e}_3$$

where $\hat{e}_1, \hat{e}_2, \hat{e}_3$ are unit vectors along the axes of the synodic reference axes.

$$\vec{\omega}_s \times (\vec{\omega}_s \times \vec{r}_s) = \omega_s\,\hat{e}_3 \times \left[\,\omega_s\,\hat{e}_3 \times \{x\,\hat{e}_1 + y\,\hat{e}_2 + z\,\hat{e}_3\}\,\right]$$
$$= \omega_s^2\;\hat{e}_3 \times \left[\,x\,\hat{e}_2 - y\,\hat{e}_1 + 0\,\right] = \omega_s^2\left[\,-x\,\hat{e}_1 - y\,\hat{e}_2\,\right]$$
$$= -\omega_s^2[x\,\hat{e}_1 + y\,\hat{e}_2]$$
$$\vec{\omega}_s \times \vec{v}_s = \omega_s\,\hat{e}_3 \times [\dot{x}\,\hat{e}_1 + \dot{y}\,\hat{e}_2 + \dot{z}\,\hat{e}_3]$$
$$= \omega_s[\dot{x}\,\hat{e}_2 - \dot{y}\,\hat{e}_1]$$

Finally,
$$\ddot{\vec{R}}_{\text{int}} = \left(\ddot{x}\,\hat{e}_1 + \ddot{y}\,\hat{e}_2 + \ddot{z}\,\hat{e}_3\right) - \omega_s^2\left(x\,\hat{e}_1 + y\,\hat{e}_2\right) + 2\omega_s\left(\dot{x}\,\hat{e}_2 - \dot{y}\,\hat{e}_1\right)$$
$$= \left(\ddot{x} - \omega_s^2 x - 2\omega_s\dot{y}\right)\hat{e}_1 + \left(\ddot{y} - \omega_s^2 y + 2\omega_s\dot{x}\right)\hat{e}_2 + \ddot{z}\,\hat{e}_3$$

<u>*Now*</u> $$\ddot{\vec{R}} = -\frac{Gm_1\,\vec{r}_{1s}}{r_{1s}^3} - \frac{Gm_2\,\vec{r}_{2s}}{r_{2s}^3} = \nabla\left[\frac{\mu_1}{r_{1s}} + \frac{\mu_2}{r_{2s}}\right]$$

$$\nabla\left(\frac{\mu}{r_{1s}} + \frac{\mu}{r_{2s}}\right) = ? \qquad\qquad \nabla = \left[\frac{\partial}{\partial x}\hat{e}_1 + \frac{\partial}{\partial y}\hat{e}_2 + \frac{\partial}{\partial z}\hat{e}_3\right]$$

$$\Rightarrow \nabla\frac{\mu}{r_{1s}} = \frac{\partial}{\partial x}\left(\frac{\mu}{r_{1s}}\right)\hat{e}_1 + \frac{\partial}{\partial y}\left(\frac{\mu}{r_{1s}}\right)\hat{e}_2 + \frac{\partial}{\partial z}\left(\frac{\mu}{r_{1s}}\right)\hat{e}_3$$
$$= -\frac{\mu}{r_{1s}^2}\cdot\frac{\partial r_{1s}}{\partial x}\hat{e}_1 - \frac{\mu}{r_{1s}^2}\frac{\partial r_{1s}}{\partial y}\hat{e}_2 - \frac{\mu}{r_{1s}^2}\frac{\partial r_{1s}}{\partial z}\hat{e}_3$$
$$= -\frac{\mu}{r_{1s}^2}\left[\frac{\partial r_{1s}}{\partial x}\hat{e}_1 + \frac{\partial r_{1s}}{\partial y}\hat{e}_2 + \frac{\partial r_{1s}}{\partial z}\hat{e}_3\right]$$

$$\vec{r}_{Bsat/B}-\vec{r}_{B1/B}=\vec{r}_{Bsat/s}-\vec{r}_{B1/s}=\vec{r}-\vec{r}_{B1}$$

$$\vec{r}_{1s}\Rightarrow x\hat{e}_1+y\hat{e}_2+z\hat{e}_3-r_{B1}\hat{e}_1$$

$$=\left(x-r_{B1}\right)\hat{e}_1+y\hat{e}_2+z\hat{e}_3$$

$$\Rightarrow r_{1s}^2=\left(x-r_{B1}\right)^2+y^2+z^2$$

$$\left.\Rightarrow 2\frac{\partial r_{1s}}{\partial x}r_{1s}=2\left(x-r_{B1}\right)\Rightarrow\frac{\partial r_{1s}}{\partial x}=\frac{x-r_{B1}}{r_{1s}}\right\}$$

Similarly $\dfrac{\partial r_{1s}}{\partial y}=\dfrac{y}{r_{1s}}$ $\qquad\dfrac{\partial r_{1s}}{\partial z}=\dfrac{z}{r_{1s}}$

Thus,

$$\ddot{\vec{R}}_{int}=-\frac{\mu_1}{r_{1s}^3}\left\{\left(x-r_{B1}\right)\hat{e}_1+y\hat{e}_2+z\hat{e}_3\right\}-\frac{\mu_2}{r_{2s}^3}\left[\left(x+r_{B2}\right)\hat{e}_1+y\hat{e}_2+z\hat{e}_3\right]$$

$$=--\left[\frac{\mu_1}{r_{1s}^3}\left(x-r_{B1}\right)+\frac{\mu_2}{r_{2s}^3}\left(x+r_{B2}\right)\right]\hat{e}_1-\left[\frac{\mu_{1y}}{r_{1s}^3}+\frac{\mu_{2y}}{r_{2s}^3}\right]\hat{e}_2-\left[\frac{\mu_1 z}{r_{1s}^3}+\frac{\mu_2 z}{r_{2s}^3}\right]\hat{e}_3$$

Thus, $$\ddot{x}-\omega_S^2x-2\omega_S\dot{y}=-\frac{\mu_1}{r_{1S}^3}\left(x-r_{B1}\right)-\frac{\mu_2}{r_{2S}^3}\left(x+r_{B2}\right)$$

$$\ddot{y}-\omega_S^2+2\omega_S\dot{x}=-\frac{\mu_1 y}{r_{1S}^3}-\frac{\mu_2 y}{r_{2S}^3}$$

$$\ddot{z}=-\frac{\mu_1 z}{r_{1S}^3}-\frac{\mu_2 z}{r_{2S}^3}$$

Theses 3 equation describe the relative motion of the 3rd body with respect to the other two bodies.

Multiplying these equations by $2x^{\cdot}$, $2y^{\cdot}$, *and* $2z^{\cdot}$ and summing, gives

$$2\dot{x}\,\ddot{x}+2\dot{y}\,\ddot{y}+2\dot{z}\ddot{z}-2\omega_S^2\left(x\dot{x}+y\dot{y}\right)=2\dot{x}\frac{\partial}{\partial_x}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)+2\dot{y}\frac{\partial}{\partial y}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)+2\dot{z}\frac{\partial}{\partial z}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)$$

The left-hand side can be written as $\frac{d}{dt}[\dot{x}^2+\dot{y}^2+\dot{z}^2-w_s^2(x^2+y^2)]$

As

$$\nabla\left(\frac{\mu_1}{r_{1s}}\right)=\left[\frac{\partial}{\partial x}\hat{e}_1+\frac{\partial}{\partial y}\hat{e}_2+\frac{\partial}{\partial z}\hat{e}_3\right]\left(\frac{\mu_1}{r_{1s}}\right)$$

$$= \;=\mu_1\left[\frac{\partial}{\partial x}\left(\frac{1}{r_{1s}}\right)\hat{e}_1+\frac{\partial}{\partial y}\left(\frac{1}{r_{1s}}\right)\hat{e}_2+\frac{\partial}{\partial z}\left(\frac{1}{\partial_{1s}}\right)\hat{e}_3\right]$$

$$=-\mu_1\left[\frac{1}{r_{1s}^2}\frac{\partial r_{1s}}{\partial x}\hat{e}_1+\frac{1}{r_{1s}^2}.\frac{\partial r_{1s}}{dy}\hat{e}_2+\frac{1}{r_{1s}^2}.\frac{\partial r_{1s}}{\partial z}\hat{e}_3\right]$$

$$=-\frac{\mu_1}{r_{1s}^2}\left[\frac{x-x_{B1}}{r_{1s}}\hat{e}_1+\frac{y}{r_{1s}}\hat{e}_2+\frac{z}{r_{1s}}\hat{e}_3\right]$$

$$=-\frac{\mu_1}{r_{1s}^3}\left[\left(x-x_{B1}\right)\hat{e}_1+y\hat{e}_2+z\hat{e}_3\right]$$

Similarly

$$\nabla\left(\frac{\mu_2}{r_{2s}}\right)=-\frac{\mu_2}{r_{2s}^3}\left\{\left(x+x_{B2}\right)\hat{e}_1+y\hat{e}_2+z\hat{e}_3\right\}$$

$$\nabla\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)=\left[-\frac{\mu_1}{r_{1s}^3}(x-x_{B1})-\frac{\mu_2}{r_{2s}^3}\left(x+x_{B2}\right)\right]\hat{e}_1+\left[-\frac{\mu_1}{r_{1s}^3}y-\frac{\mu_2}{r_{2s}^3}y\right]\hat{e}_2+\left[-\frac{\mu_1}{r_{1s}^3}z-\frac{\mu_2}{r_{2s}^3}\right]\hat{e}_3$$

$$\Rightarrow\frac{\partial}{\partial x}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)=-\frac{\mu_1}{r_{2s}^3}\left(x-x_{B1}\right)-\frac{\mu_2}{r_{2s}^3}\left(x+x_{B2}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)=-\frac{\mu_1}{r_{1s}^3}y-\frac{\mu_2}{r_{2s}^3}y$$

$$\frac{\partial}{\partial z}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)=-\frac{\mu_1}{r_{1s}^3}z-\frac{\mu_2}{r_{2s}^3}z$$

Therefore, the right hand side can be simplified to $2\frac{d}{dt}f(x,y,z)$, where $f(x,y,z)=\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}$

Integrating the equation results in

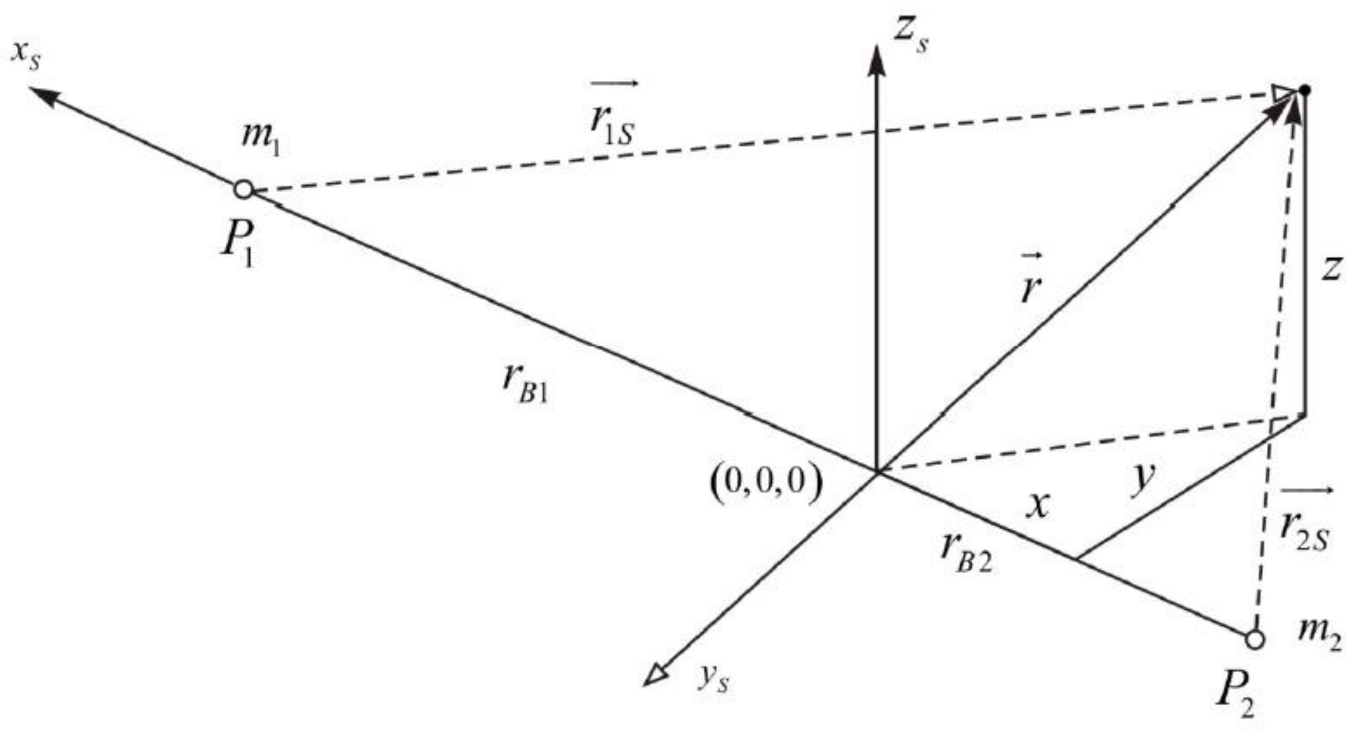
$$\dot{x}^2+\dot{y}^2+\dot{z}^2-\omega_s^2\left(x^2+y^2\right)=\frac{2\mu_1}{r_{1s}}+\frac{2\mu_2}{r_{2s}}-C$$

Where

$$\frac{2\mu_1}{r_{1s}}+\frac{2\mu_2}{r_{2s}}=\int\left[2\frac{\partial}{\partial x}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)\frac{dx}{dt}+2\frac{\partial}{\partial y}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)\frac{dy}{dt}+2\frac{\partial}{\partial z}\left(\frac{\mu_1}{r_{1s}}+\frac{\mu_2}{r_{2s}}\right)\frac{dz}{dt}\right]dt$$

$$=2\int\left(\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\frac{dy}{dt}+\frac{\partial f}{\partial z}\frac{dz}{dt}\right)dt=2f\left(x,y,z\right)$$

This integration constant C is the Jacobi's constant which appears only by moving to the synodic frame.



Circular Restricted 3-body Problem:

It order to manipulate these equations to understand how the motion might looks like, it would be easy to normalize the equations.

For Kepler's third law,

$$T=\frac{2\pi\, a^{3/2}}{\sqrt{G(m_1+m_2)}}=\frac{2\pi}{n} \qquad n^2\, a^3=G\left(m_1+m_2\right)$$

Let $m_1 + m_2 = 1$; $a = 1$, $G = 1$; $n=1$, then $T = (2.pi)$, then $w=1$

$$\dot{\theta} = n = 1 \Rightarrow \theta = t + c_1 \quad \text{at} \quad t = 0 \quad \theta = 0$$

$$\Rightarrow \theta = t$$

Let $m_2 = \mu^* < 1$

$$\Rightarrow m_1 = 1 - m_2 = 1 - \mu^*$$

Also, $m_1 r_{B1} - m_2 r_{B2} = 0$

$$\Rightarrow m_1 r_{B1} = m_2 r_{B2}.$$

$$(1-\mu^*) r_{B2} = \mu^* (1 - r_{B1})$$

$$\Rightarrow \frac{1-\mu^*}{\mu^*} = \frac{1-r_{B1}}{r_{B1}} = \frac{1}{r_{B1}} - 1.$$

$$\Rightarrow \frac{1}{\mu^*} - 1 = \frac{1}{r_{B1}} - 1.$$

$$\Rightarrow \mu^* = r_{B1}.$$

$$\Rightarrow r_{B2} = 1 - r_{B1} = 1 - \mu^*$$

Thus;

$$r_{1.s} = \sqrt{(x-\mu^*)^2 + y^2 + z^2} = r_1$$

$$r_{2.s.} = \sqrt{(x-\mu^*+1)^2 + y^2 + z^2} = r_2$$

Assuming w_s=1 as before in equation

Same as $$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 - \omega_s^2 (x^2 + y^2) = \frac{2\mu}{r_1} + \frac{2\mu_2}{r_2} - C$$

G=1;

$$V^2 = x^2 + y^2 + \frac{2G(1-\mu^*)}{r_1} + \frac{2G(\mu^*)}{r_2} - C.$$ Here V is not the absolute K.E, it is relative

$$V^2 = x^2 + y^2 + \frac{2(1-\mu^*)}{r_1} + \frac{2\mu^*}{r_2} - C$$

$$V^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = (x^2 + y^2) + \frac{2(1-\mu^*)}{r_1} + \frac{2\mu^*}{r_2} - C$$

Putting

$$U = \frac{1}{2}\left(x^2 + y^2\right) + \frac{\left(1-\mu^*\right)}{r_1} + \frac{\mu^*}{r_2}$$

Yields $V^2 = 2U - C$ [Jacobi Integral]

which describes how the 3rd body moves. U is called the Jacobi's function.

$$V^2 \geq 0. \quad \Rightarrow 2U - C \geq 0.$$

i.e.

$$x^2 + y^2 + \frac{2\left(1-\mu^*\right)}{r_1} + \frac{2\mu^*}{r_2} \geq C.$$

$$\phi(x, y, z) = x^2 + y^2 + \frac{2\left(1-\mu^*\right)}{r_1} + \frac{2\mu^*}{r_2} = \mathrm{C}$$

Which is the zero velocity surface ,that the third body can never cross .

Now considering the **Lagrange points**

Lagrange points are the points around the primary body and secondary body at which the third relatively small body can hold its position relative to the larger bodies. At the Lagrangian points all the forces are mangeed to cancel each other that the gravitational forces, the centripetal forces, and Coriolis forces all interact in such a way that allows the third body to hold a stable position relative to the larger bodies.

$$\ddot{x} - \omega_s^2 x - 2\omega_s \dot{y} = -\frac{\mu_1}{r^3}\left(x - r_{B1}\right) - \frac{\mu_2 y}{r_{2s}}\left(x + r_{b2}\right) \qquad (A)$$

$$\ddot{y} - \omega_s^2 y - +2\omega_s \dot{x} = -\frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} \qquad (B)$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3} \qquad (C)$$

[dropping subscript 's']. There's no general solution to those equations. But they can be utilized to solve for the synodic system's Lagrange points.

Therefore, at equilibrium points

$$\dot{x} = \dot{y} = \dot{z} = 0$$
$$\ddot{x} = \ddot{y} = \ddot{z} = 0$$

Yields form the above three equations

$$-\omega^2 x = -\frac{\mu_1}{r_1^3}(x - r_{B1}) - \frac{\mu_2}{r_2^3}(x + r_{B2}) \qquad D$$

$$-\omega^2 y = -\frac{\mu_1}{r_1^3} y - \frac{\mu_2}{r_2^3} y \qquad E$$

$$0 = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3} \qquad F$$

From eq F

$$\left(\frac{\mu_1}{r_1^3} + \frac{\mu_2}{r_2^3}\right) z = 0$$

<u>Since</u>

$$\frac{\mu_1}{r_1^3} > 0 \quad and \quad \frac{\mu_2}{r_2^3} > 0$$
$$\Rightarrow z = 0$$

which indicated that the Lagrange points exist in the plane of the other bodies.

$$\mu_1 = Gm_1 = 1m_1 = 1 - \mu^*$$
$$\mu_2 = Gm_2 = 1m_2 = \mu^*$$

From eq E

$$\Rightarrow -y = -\frac{1-\mu^*}{r_1^3} y - \frac{\mu^*}{r_2^3} y$$

$$\left(\frac{1-\mu^*}{r_1^3} + \frac{\mu^*}{r_2^3} - 1\right) y = 0$$

$$\frac{1-\mu^*}{r_1^3} + \frac{\mu^*}{r_2^3} - 1 = 0 \qquad -----(G)$$

because $y \neq 0$

Similarly

$$-x = -\frac{1-\mu^*}{r_1^3}(x - \mu^*) - \frac{\mu^*}{r_2^3}(x + 1 - \mu^*)$$

$$-x = -\left\{\frac{1-\mu^*}{r_1^3} + \frac{\mu^*}{r_2^3}\right\} x + \frac{\mu^*(1-\mu^*)}{r_1^3} - \frac{\mu^*(1-\mu^*)}{r_2^3}$$

Substituting

$$\frac{1-\mu^*}{r_1^3}+\frac{\mu^*}{r_2^3}=1 \text{ from Eq. (G)}$$

$$-x=-x+\mu^*\left(1-\mu^*\right)\left\{\frac{1}{r_1^3}-\frac{1}{r_2^3}\right\} \Rightarrow \frac{1}{r_1^3}-\frac{1}{r_2^3}=0$$

$$\Rightarrow \mu^*\left(1-\mu^*\right)\left\{\frac{1}{r_1^3}-\frac{1}{r_2^3}\right\}=0 \qquad \Rightarrow \frac{1}{r_1^3}=\frac{1}{r_2^3}$$

$$\Rightarrow r_1=r_2=k=1$$

$$\Rightarrow \frac{1-\mu^*}{k^3}+\frac{\mu^*}{k^3}=1$$

$$\Rightarrow r_1=r_2=r_{AB}=1$$

$$1=k^3$$

Equation D, E and F can be solved without considering the normalization, but will lead to the same conclusion that

$$\frac{1}{r_1}=\frac{1}{r_3}=\frac{1}{r_{12}} \Rightarrow r_1=r_2=r_{12}$$

That the third body m_3 is positioned on the vertices of equilateral triangle.

Form Pythagorean theorem.

$$\left(x-r_{B1}\right)^2+y^2=r_1^2=r_{12}^2$$

$$also\ \left(x-r_{B2}\right)^2+y^2=r_2^2=r_{12}^2$$

$$and\ \ r_{B1}+r_{B2}=r_{12}$$

$$\Rightarrow r_{B1}=r_{12}-r_{B2}$$

$$=r_{12}-\frac{\mu_1}{\mu_2}r_{B1}$$

$$r_{B1}\left(1+\frac{\mu_1}{\mu_2}\right)=r_{12}$$

$$r_{B1}=\frac{r_{12}\mu_2}{\left(\mu_1+\mu_2\right)}$$

$$\Rightarrow x-r_{B1}=\pm\left(x+r_{B2}\right)$$

$$x + x = r_{B1} - r_{B2}$$

$$2x = r_{B1} - r_{B1}\frac{\mu_1}{\mu_2}$$

$$= r_{B1}\left(1 - \frac{\mu_1}{\mu_2}\right)$$

$$= \frac{r_{12}\mu_2}{\mu_2 + \mu_1} \times \frac{\mu_2 - \mu_1}{\mu_2}$$

$$x = \frac{r_{12}}{2}\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}$$

Similarly;

$$y^2 = r_{12}^2 - \left(x + r_{B2}\right)^2$$

$$= r_{12}^2 - \left[\frac{r_{12}}{2}\left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\right) + r_{B1}\frac{\mu_1}{\mu_2}\right]^2$$

$$= r_{12}^2 - \left[\frac{r_{12}}{2}\left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\right) + \frac{r_{12}\mu_2}{\mu_2 + \mu_1}\frac{\mu_1}{\mu_2}\right]^2$$

$$= r_{12}^2\left[1 - \left[\frac{1}{2}\left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\right) + \frac{\mu_1}{\mu_2 + \mu_1}\right]^2\right]$$

$$= r_{12}^2\left[1 - \left(\frac{\mu_2 - \mu_1 + 2\mu_1}{2\left(\mu_2 + \mu_1\right)}\right)^2\right]$$

$$y = \pm\frac{\sqrt{3}}{2}r_{12}$$

Therefore, we get two equilibrium points L4 and L5 whose coordinates are:

$$\left(x, y\right) \equiv \left[\frac{r_{12}}{2}\left(\frac{\mu_2 - \mu_1}{\mu_2 + \mu_1}\right), \quad \pm\frac{\sqrt{3}}{2}r_{12}\right]$$

From these equations we can get 3 more equilibrium points, if we assume z=0 and y=0

$$-\omega^2 x = -\frac{\mu_1}{r_1^3}\left(x - r_{B1}\right) - \frac{\mu_2}{r_2^3}\left(x + r_{B2}\right) \qquad D$$

$$-\omega^2 y = -\frac{\mu_1}{r_1^3}y - \frac{\mu_2}{r_2^3}y \qquad E$$

$$0 = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3} \qquad F$$

The equations reduced to

$$r_{1s}^2 = r_1^2 = \left(x - r_{B1}\right)^2 = \left|x - r_{B1}\right|^2$$

$$r_{2s}^2 = r_2^2 = \left(x + r_{B2}\right)^2 = \left|x + r_{B2}\right|^2$$

$$\Rightarrow -\omega^2 x = -\frac{\mu}{r_{12}^3}x = -\frac{\mu_1}{r_1^3}\left(x - r_{B1}\right) - \frac{\mu_2}{r_2^3}\left(x + r_{B2}\right)$$

Putting,

$$r_{B2} = \frac{\mu_1}{\mu_2} r_{B1} \quad \text{and} \quad r_{B1} + r_{B2} = r_{12}$$

$$\Rightarrow r_{B1} + \frac{\mu_1}{\mu_2} r_{B1} = r_{12}$$

$$\frac{\mu_1}{\mu_1 + \mu_2} = \mu^*$$

$$\boxed{\Rightarrow r_{B1} = \frac{\mu_2}{\mu_1 + \mu_2} r_{12}}$$

$$r_{B1} = \mu^* r_{12} = \mu^* r$$

and $\quad r_{B2} = \left(1 - \mu^*\right) r_{12} = \left(1 - \mu^*\right) r$

Also

$$r_{B2} = \frac{\mu_1}{\mu_2} r_{B1} = \frac{\mu_1}{\mu_1 + \mu_2} r_{12}$$

Thus,

$$\frac{\mu x}{r_{12}^3} = \frac{\mu_1\left(x - \mu^* r\right)}{r_1^3} + \frac{\mu_2\left(x + (1 - \mu^*) r\right)}{r_2^3}$$

$$\Rightarrow \frac{x}{r_{12}^3} = \frac{\frac{\mu_1}{\mu}\left(\frac{x}{r} - \mu^*\right) r}{r_1^3} + \frac{\frac{\mu_2}{\mu}\left(\frac{x}{r} + 1 - \mu^*\right) r}{r_2^3}$$

$$\frac{x}{r_{12}^3} = \frac{\left(1 - \mu^*\right)\left(X - \mu^*\right) r}{\left(x - r_{B1}\right)^3} + \frac{\mu^*\left(X + 1 - \mu^*\right) r}{\left(x + r_{B2}\right)^3} \quad \left[\text{Here } X = \frac{x}{r_{12}}\right]$$

$$\text{Now } x - r_{B1} = x - \mu^* r_{12} = \left(\frac{x}{r_{12}} - \mu^*\right) r = \left(X - \mu^*\right) r_{12}$$

$$x + r_{B2} = x + (1 - \mu^*) r_{12} = \left(X + 1 - \mu^*\right) r_{12}$$

$$\text{Now } x - r_{B1} = x - \mu^* r_{12} = \left(\frac{x}{r_{12}} - \mu^*\right) r = \left(X - \mu^*\right) r_{12}$$

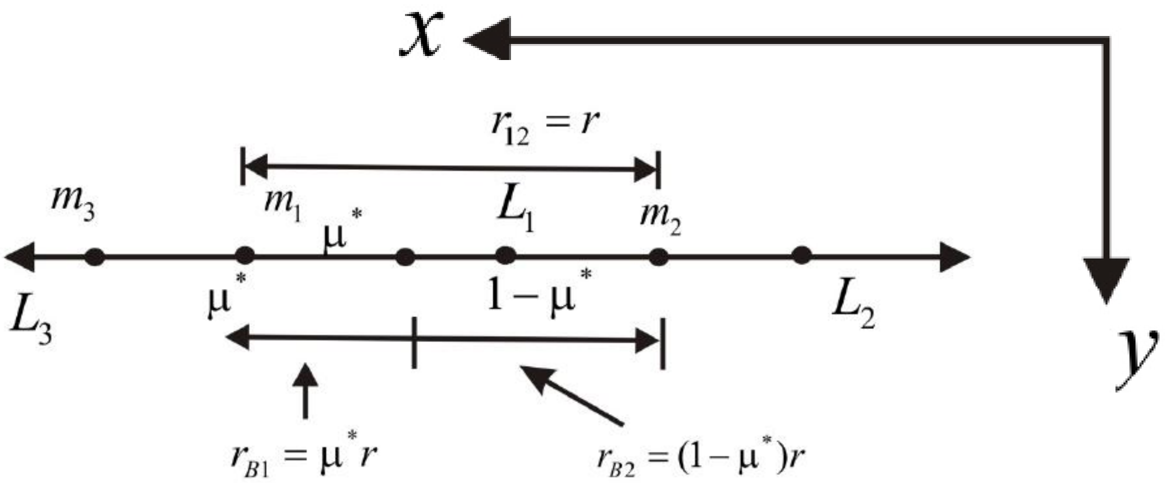
$$x + r_{B2} = x + (1 - \mu^*) r_{12} = \left(X + 1 - \mu^*\right) r_{12}$$

thus,

$$\Rightarrow \frac{x}{r_{12}^3} = \frac{\left(1 - \mu^*\right)\left(X - \mu^*\right) r_{12}}{\left|x - r_{B1}\right|^3} + \frac{\mu^*\left(X + 1 - \mu^*\right) r_{12}}{\left|x + r_{B2}\right|^3}$$

$$\frac{x}{r_{12}^3} = \frac{\left(1 - \mu^*\right)\left(X - \mu^*\right) r_{12}}{r_{12}^3 \left|\frac{x}{r_{12}} - \mu^*\right|^3} + \frac{\mu^*\left(X + 1 - \mu\right) r_{12}}{r_{12}^3 \left|\frac{x}{r_{12}} + 1 - \mu^*\right|^3}$$

$$\Rightarrow X=\frac{\left(1-\mu^{*}\right)\left(X-\mu^{*}\right)}{\left|X-\mu^{*}\right|^{3}}+\frac{\mu^{*}\left(X+1-\mu^{*}\right)}{\left|X+1-\mu^{*}\right|^{3}}$$



Condition (1)

$$\text{if } x-r_{B1}>0 \quad \text{then } \left|x-r_{B1}\right|=x-r_{B1}.$$

$$\Rightarrow x-r_{B2}>0$$

$$\Rightarrow X=\frac{1-\mu^{*}}{\left(X-\mu^{*}\right)^{2}}+\frac{\mu^{*}}{\left(X+1-\mu^{*}\right)^{2}}$$

$$\boxed{\Rightarrow\left(X-\mu^{*}\right)\left(X+1-\mu^{*}\right)^{2} X=\left(1-\mu^{*}\right)\left(X+1-\mu^{*}\right)^{2}+\mu^{*}\left(X-\mu^{*}\right)^{2}}$$

Thus, we can solve for different positions of m_3 individually.

As the point located to the left of m_1.

$$X=\frac{\left(1-\mu^{*}\right)\left(X-\mu^{*}\right)}{\left[X-\mu^{*}\right]^{3}}+\frac{\mu^{*}\left(X-\left(-\left(1-\mu^{*}\right)\right)\right)}{\left[X-\left(-\left(1-\mu^{*}\right)\right)\right]^{3}}$$

$$X=\frac{\left(1-\mu^{*}\right)}{\left[X-\mu^{*}\right]^{2}}+\frac{\mu^{*}}{\left[X+(1-\mu)\right]^{2}} \quad \text{gives point } L_3$$

Right of m_2

$$-X=\frac{\left(1-\mu^{*}\right)\left(-X-\mu^{*}\right)}{\left|-X-\mu^{*}\right|^{3}}+\frac{\mu^{*}\left(-X-\left(-\left(1-\mu^{*}\right)\right)\right)}{\left|-X+(1-\mu)\right|^{3}}$$

$$-X=-\frac{\left(1-\mu^{*}\right)(X+\mu)}{(X+\mu)^{3}}-\frac{\mu^{*}\left(X-(1-\mu)\right)}{\left[X-(1-\mu)\right]^{3}}$$

$$\Rightarrow X=\frac{\left(1-\mu^{*}\right)}{\left(X+\mu^{*}\right)^{2}}+\frac{\mu^{*}}{\left(X-\left(1-\mu^{*}\right)\right)^{2}} \quad \text{gives point } L_2$$

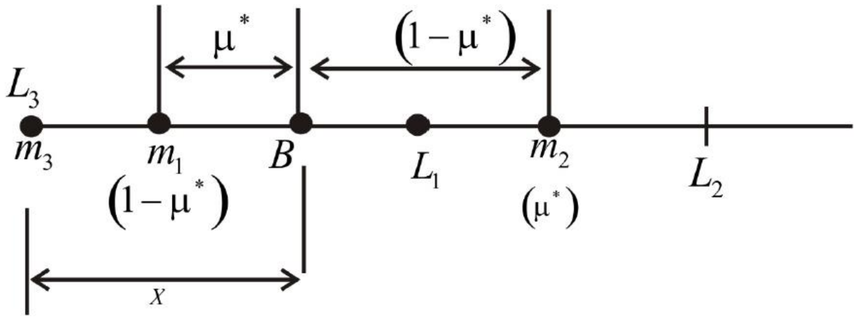
Intermediate position

$$-X=\frac{(1-\mu)\left(-X-\mu^{*}\right)}{\left|-X-\mu^{*}\right|^{3}}+\frac{\mu^{*}\left[-(1-\mu^{*})-(-X)\right]}{\left|-(1-\mu)-(-X)\right|^{3}}$$

$$-X=-\frac{(1-\mu)\left(X+\mu^{*}\right)}{\left|X+\mu^{*}\right|^{3}}-\frac{\mu^{*}(1-\mu-X)}{\left|(1-X)-(-X)\right|^{3}}$$

$$X=\frac{(1-\mu)}{\left(X+\mu^{*}\right)^{2}}-\frac{\mu^{*}}{(X-1+\mu)^{2}}$$

Instead, we can get the same equations easly using the following :



We can write the above equations by considering that the two massive bodies provided the centripital acceleration for the third body. while assuming ω=1.

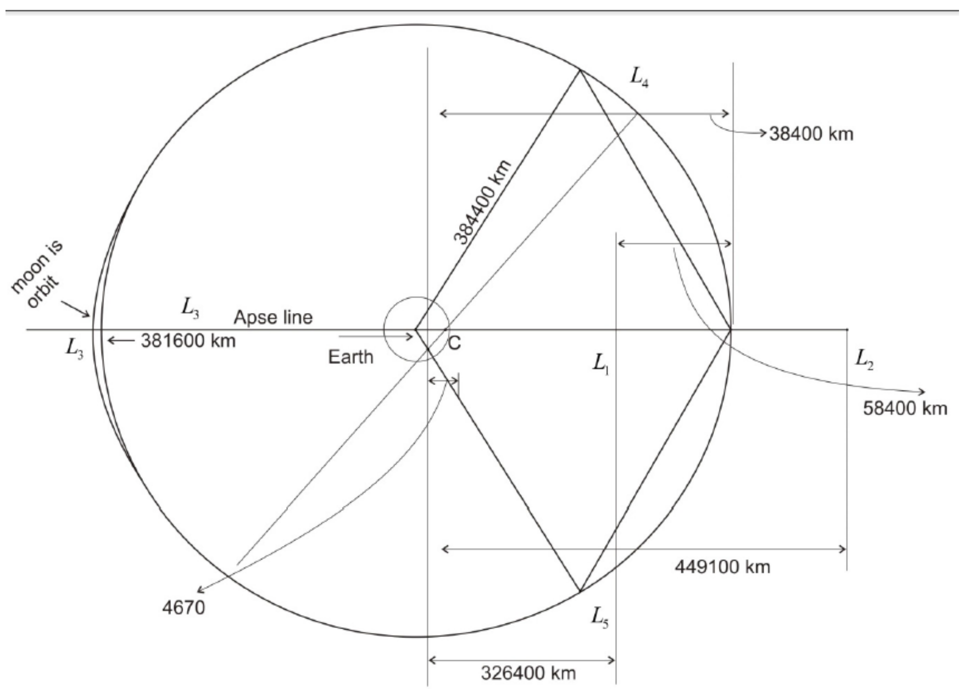
$$(L_3) \quad \omega^2 X = X = \frac{(1-\mu^*)}{(X-\mu^*)^2} + \frac{\mu^*}{(X+1-\mu^*)^2}$$

$$(L_2) \quad \omega^2 X = X = \frac{1-\mu^*}{(X+\mu^*)^2} + \frac{\mu^*}{(X-(1-\mu^*))^2}$$

$$(L_1) \quad \omega^2 X = X = \frac{1-\mu^*}{(X+\mu^*)^2} - \frac{\mu^*}{((1-\mu^*)-X)^2}$$

Thus, we get that five Lagrangian points. That points L_1, L_2, *and* L_3 aren't stable while L_4, L_5 are stable if $m_1/m_2 = 24.96$. For the earth moon system $m_1/m_2 = 81.3$. Hence these points are stable.

The figure below, illustrates with numbers the equilibrium points of earth-moon system.



Zero-Velocity Surfaces

From Jacobi integral we found that $V^2 = 2U - C = \phi - C \geq 0$

Where
$$\phi = x^2 + y^2 + \frac{2(1-\mu^*)}{r_1} + \frac{2\mu^*}{r_2}$$

Moreover, the velocity at the Lagrangian points is zero, then

$$x^2 + y^2 + \frac{2(1-\mu^*)}{r_1} + \frac{2\mu^*}{r_2} = C$$

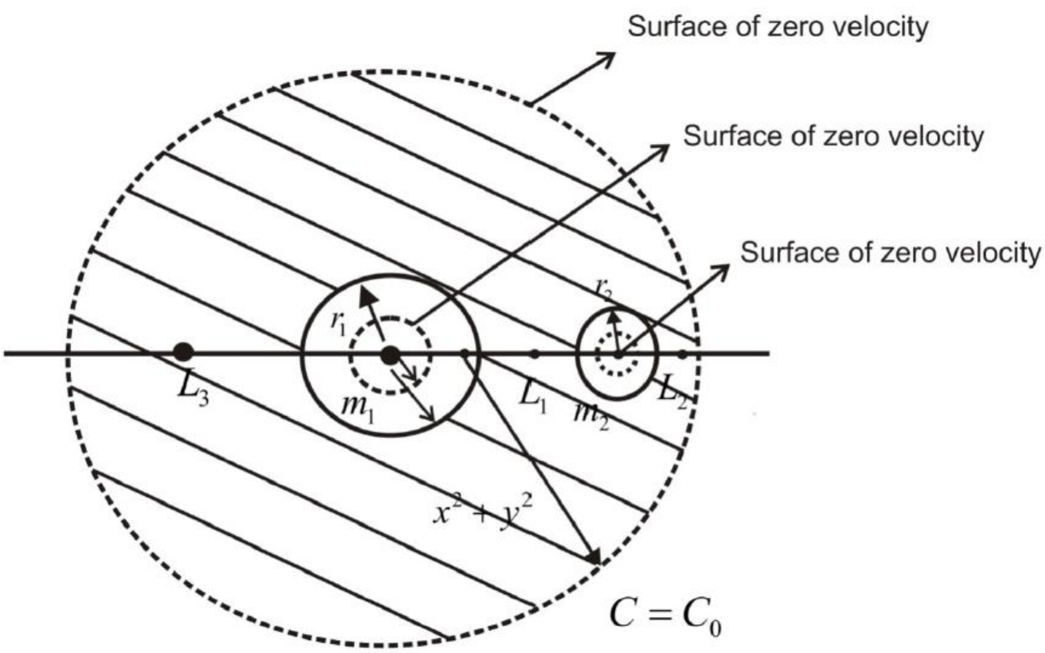
The mass m_3 has the Jacobi integral C, and lies on the surface specified by the Equation above, so the body would have zero velocity relative to the other two bodies. Zero-velocity surfaces are significant because they shape the boundaries of areas where the body m_3 is dynamically excluded.

Case I

C is large [$C = C_0$ is the zero velocity surface], this is possible in the following ways.

$$(a) r_1 \to 0 \quad (b) r_2 \to 0 \quad (c) x^2 + y^2 \to \infty$$

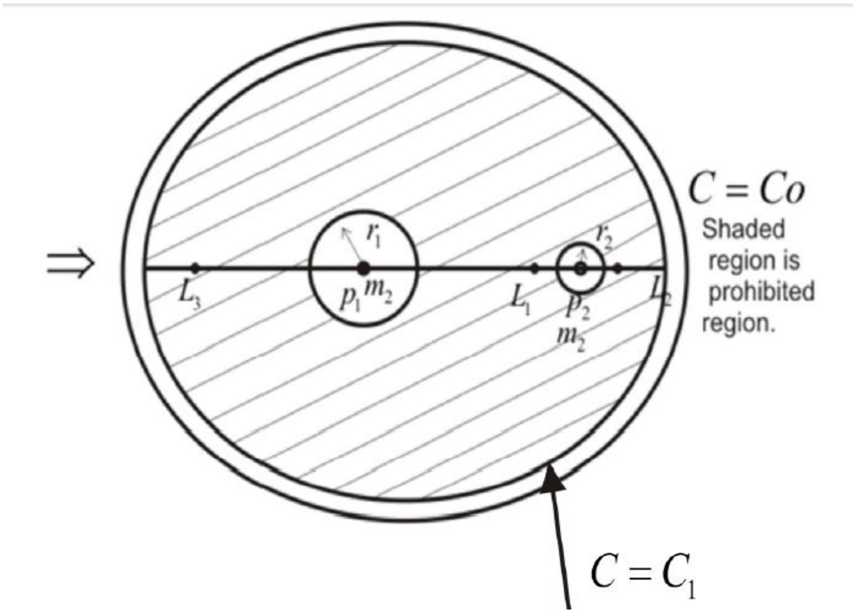
Then the allowed regions for m_3 is a small circle around m_1 and m_2 and at large $x^2 + y^2$, which gives the shaded region is excluded.



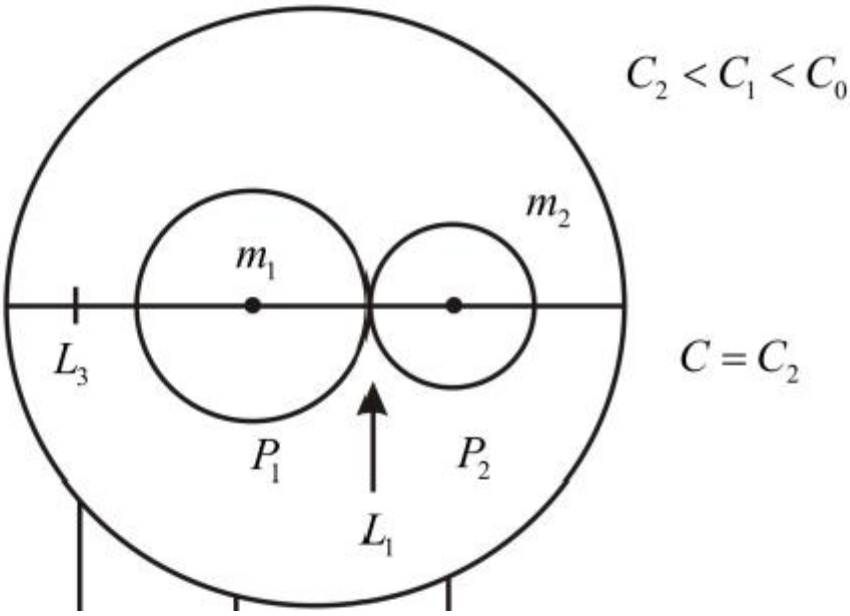
Case II

As C is drops down, $C = C_1 < C_0$ which indicates the following

$$(a) \quad r_1 \ \textit{increases} \quad \text{or} (b) \quad r_2 \ \textit{increases} \quad \text{or} (c) x^2 + y^2 \ \text{shrinks}$$

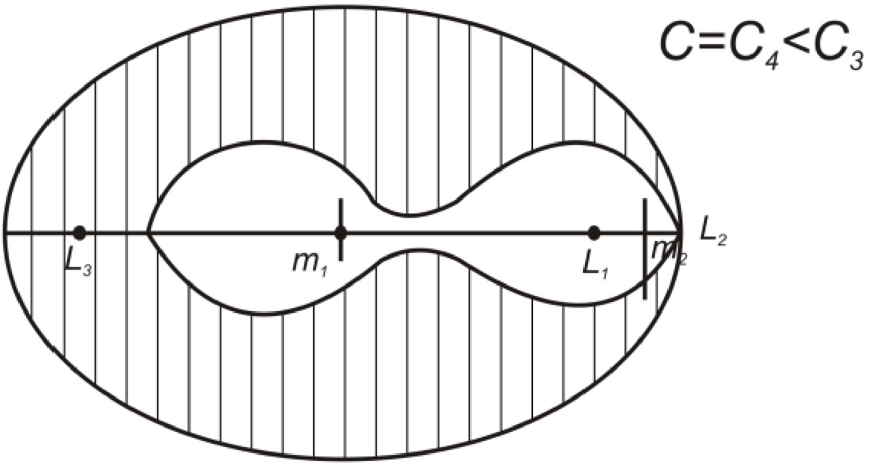


Case (3) : As the value of C is reduced more and more the two inner circles increase in radius and meet each other at L_1 that would allow the third body to transfer from P_1 region to P_2.

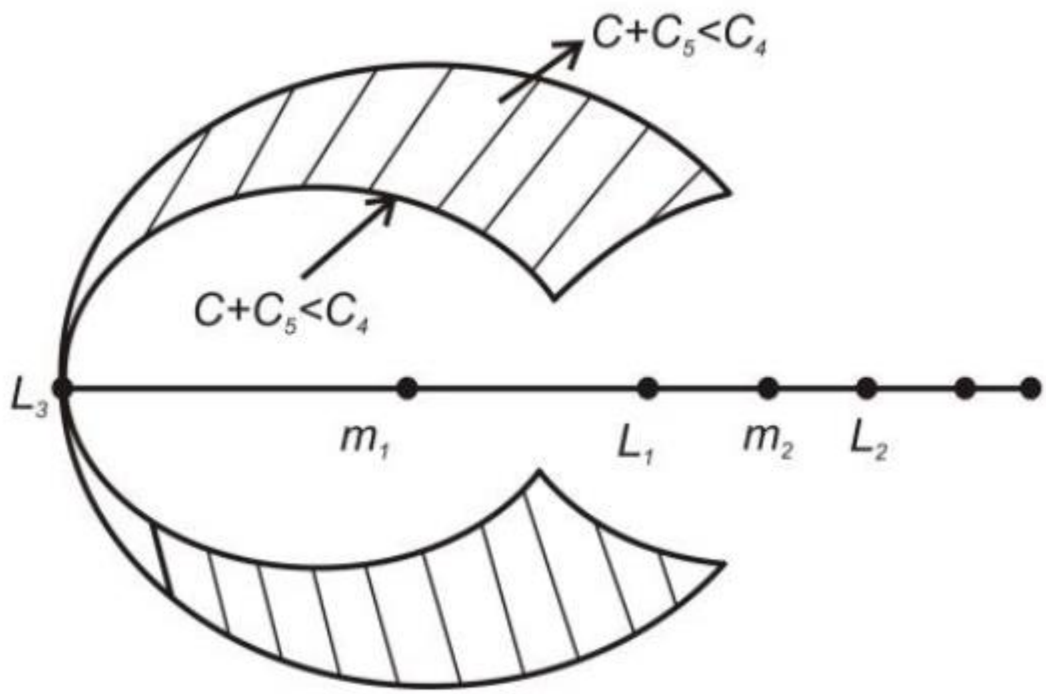


Case (4) As the value of C decrease more and more the two circular areas will mingle up.

Case (5) as C decrease further the L_2 point will be developed.



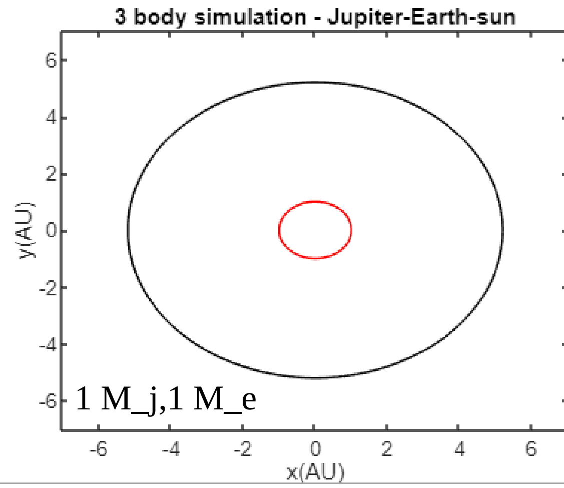
Case (6) as C decreased further the L_3 point will be developed.



So as the values of C decrease the surface that is allowed for the third body increases, and the Lagrange points starts to appear. Which completed the solution.

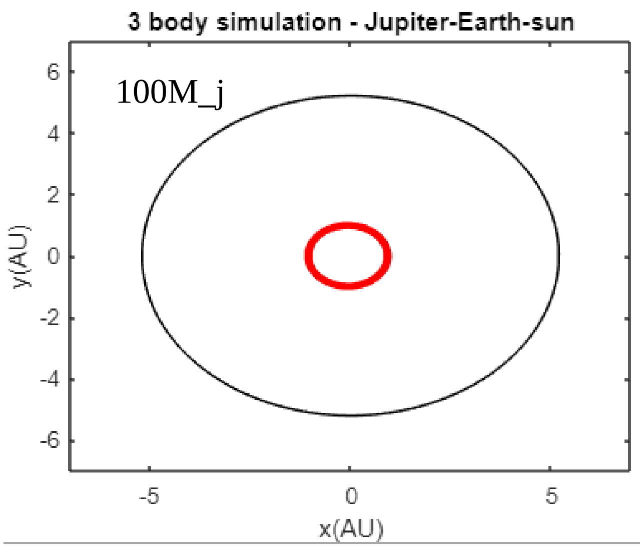
Numerical Solution

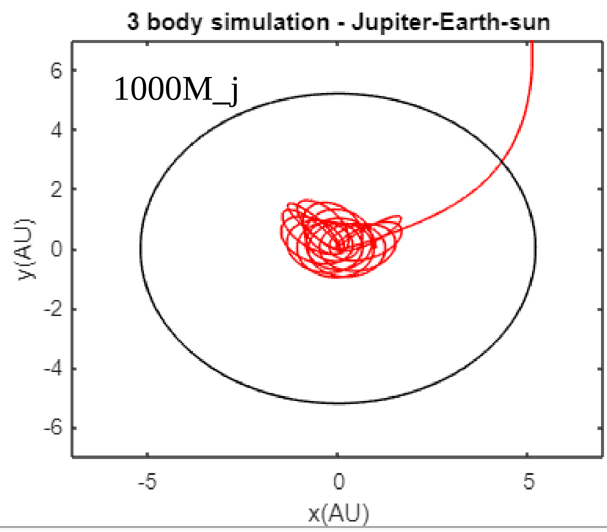
If we consider Earth- Jupiter and sun system, it is clear that this system matches our constrains, as the sun is the primary body while Jupiter is the secondary body and earth is neglectable in mass compared to them. This system is stable as (**Fig1**) illustrates that both earth and Jupiter orbit the sun in a circular orbit. However, if we changed the initial conditions, this system would be chaotic. Here I would be changing both the initial mass of Jupiter and earth to see the effect on their orbits. I will indicate the orbit of **Jupiter** in **black** and earth in red. Mass of Jupiter(**M_j**), mass of earth (**M_e**).

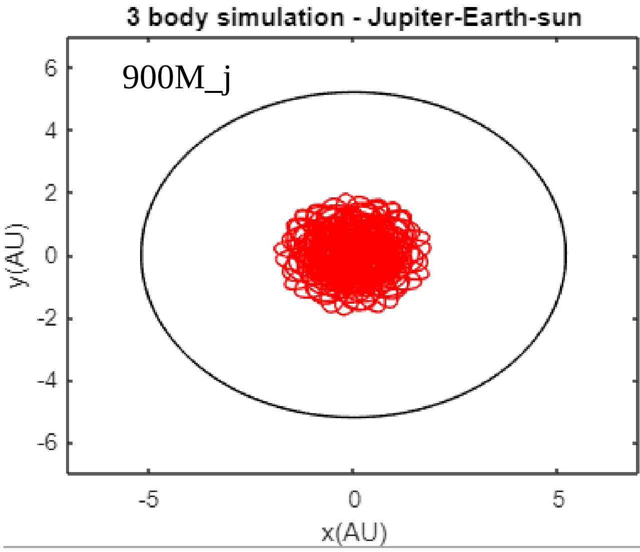


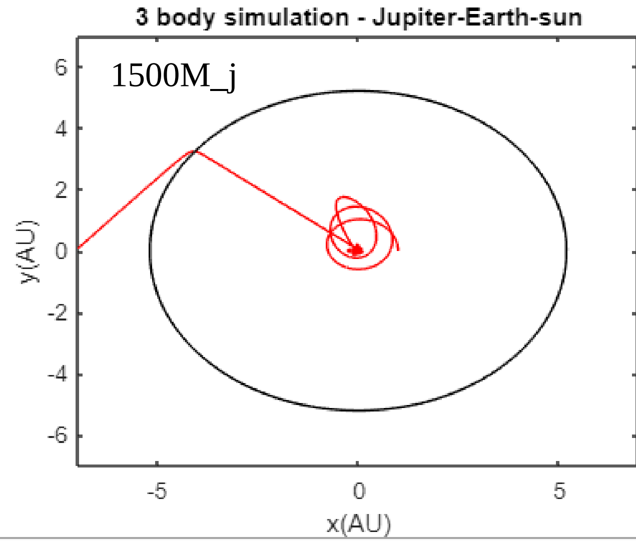
(**Fig1**): The actual mass of earth and Jupiter were used

<u>**The following four figures the initial mass of *Jupiter* was changed.**</u>

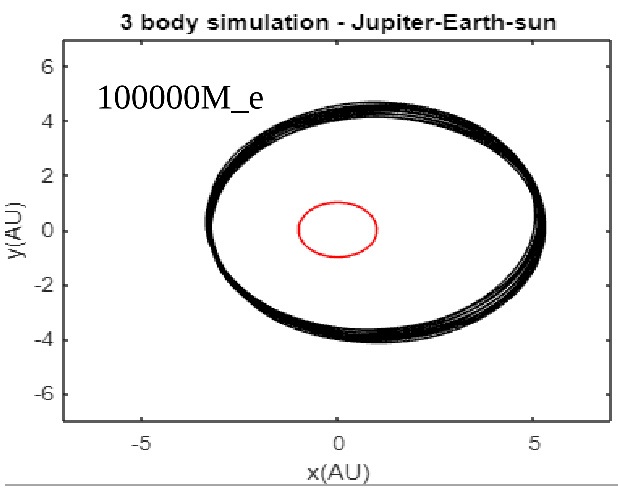


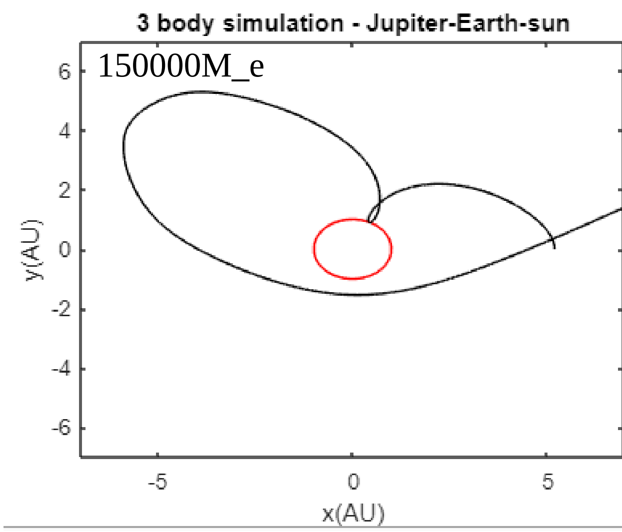


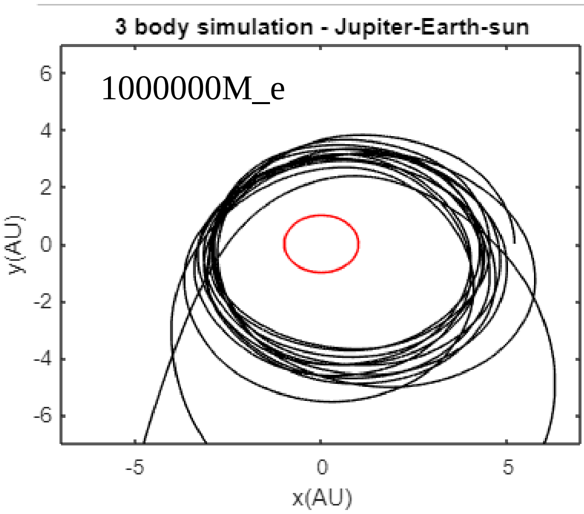


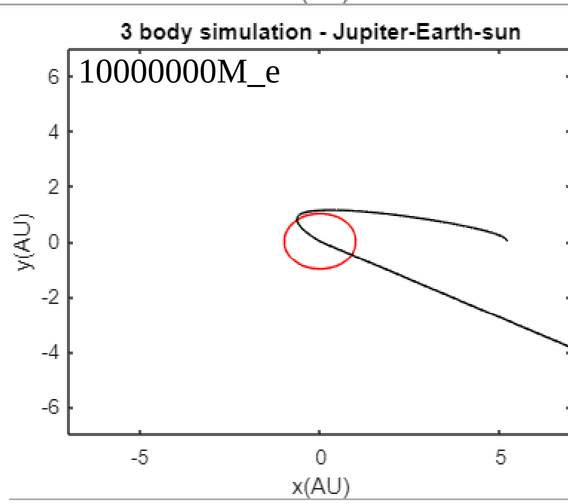


<u>The following four figures the initial mass of *earth* was changed.</u>









Conclusion

Finding the absolute motion of the 3 body is impossible, however with the constrains we impose, a solution is possible. The first constrain that both the primary and secondary bodies move in a circle was successful in reducing the equations when transformed to Synodic reference frame as $\dot{\boldsymbol{w}}$ equals zero. The second constrain was essential in reducing the equations, as the gravitational force produced by the third body could be neglected. Therefore, these two constrains gave us some intuition on how the motion might looks like.

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