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An orthotropic yield criterion in a 3-D general stress state

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Abstract

An anisotropic yield criterion with a general representation was suggested. The yield criterion was derived from the use of the invariants of the stress tensor, similar in constructing an isotropic yield criterion, but which contains a "three-yield-system hypothesis" specifying the state of anisotropy. When applied to rolled sheet metals, such as high strength steels and aluminum alloys, the criterion can be treated in an analytical form to facilitate analyses of engineering problems under a general triaxial stress state. For this specified form, anisotropic properties of the predicted yield surface were characterized by seven experimental results obtained from three standard uniaxial-tension tests and one equibiaxial-tension test. When the applied material becomes isotropic it is transformed back to the form of the von-Mises's criterion. Since the convexity of the yield criterion was proven in its general type, the characterized criterion is valid as a plastic potential in the implementation of finite element programs. It was shown, in full agreement with experimental data, that the accuracy of predicted yield surface was similar to that of predicted by the polycrystal model. Considering the equibiaxial-tension data, in general, may be not available from material supplies, a formulated relation covered variables of the equibiaxial tension and uniaxial tension was proposed. The relation can be used to calculate the equibiaxial-tension yield stress from the experimental data in uniaxial tensions. Several calculated results showed very close to the experimental results.

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1. Introduction

In the structure engineering, a proposed yield criterion is principally used for representing the strength of solids as engineering materials. In the metal forming engineering, such as the panel forming in automotive industry, a valid yield criterion is more often addressed for establishing the constitutive relations of material to simulate a forming process. Since many different problems in engineering applications all deal with the use of a yield criterion, an appropriate yield criterion, in general, should be applicable not only as a measurement to predict the yielding behavior of a material on a macroscopic scale, but also as a plastic potential function further to constitute plastic stress–strain relations of the material. When constructing a yield criterion, we need primarily to find out an acceptable relation between a yielding state and an externally loaded force. From the commonly accepted point of view, if a material is isotropic, its yielding behavior, in general, depends only on the magnitude of a loading force regardless of the loading direction. The yielding state of this kind of material can be described in principle with the invariants of the stress tensor as (Hill, 1950)

$$f(J_1, J_2, J_3) = 0, \tag{1}$$

where J_1, J_2 and J_3 are the first, second and third invariants of the stress tensor.

In plasticity, it is often more common to use the invariants of the deviatoric stress tensor for describing the yield state and plastic deformation. The relations between the invariants of the stress tensor and the deviatoric stress tensor can be given as (Slater, 1977)

$$\begin{aligned} J_1 &= J_1' + 3\sigma_m = 3\sigma_m, \\ J_2 &= J_2' - \frac{1}{3}J_1^2, \\ J_3 &= J_3' - \frac{1}{3}J_1J_2' + \frac{1}{27}J_1^3, \end{aligned} \tag{2}$$

where J_1', J_2' and J_3' are the first, second and third invariants of the deviatoric stress tensor, and σ_m is the average of the normal stress components, or the hydrostatic stress component.

The relations of Eq. (2) show that only three invariants of the stress tensor and the deviatoric stress tensor are independent. That is, Eq. (1) can be represented by the three independent invariants as

$$f(J_1, J_2', J_3') = 0. \tag{3}$$

Many existing proposed isotropic yield criteria have been developed based on this concept (Drucker, 1949, 1953; Maitra and Majumdar, 1973; Lade and Duncan, 1975; Desai et al., 1981; Wang and Hu, 1989; Iyer and Lissenden, 2003). The primary difference among these yield criteria is in the number of the independent invariant used. Some of the criteria involve one independent invariant, while others involve two, or all three independent invariants. In the analysis of isotropic ductile metals, only the second invariant of the deviatoric stress tensor is considered whenever the effects of the hydrostatic stress and strength-differential in compression and tension can be disregarded (Hu and Wang, 2005), e.g. the well-known von-Mises's criterion (1913).

Nevertheless, the isotropic property of a material can only be considered as an ideal and approximated description of the material properties. Yielding behaviors of some materials exhibit strong directional dependencies, such as with rolled sheet metals. The well-developed isotropic yield criteria are invalid in the analysis of anisotropic materials. There is a need for the development of anisotropic plasticity and its associated yield criterion. In the field of anisotropic yield criteria, the most well-known work is of the Hill's quadratic formulation (1948). Hill's criterion contains six parameters specifying the state of anisotropy but is similar in form to the Mises's criterion for isotropic metals. Because many experimental results have shown that some materials exhibit the so-called "anomalous behavior" that causes problems with the use of the Hill's (1948) criterion (Woodthorpe and Pearce, 1970; Dodd and Caddell, 1984), later on, more general types of anisotropic yield criteria were introduced to cover a wider range of experimental observations (Hill, 1979; Hosford, 1979, 1985; Gotoh, 1977; Barlat and Lian, 1989; Barlat et al., 1991; Karafillis and Boyce, 1993; Barlat et al., 1997a,b; etc.; Bron and Besson, 2004; Darrieulat and Montheillet, 2003; Stoughton and Yoon, 2004). Generally, each of the criteria possesses certain physical justifications and provides, accordingly, a mathematical representation. Most of the proposed yield criteria have attempted to take into account more experimental results, or to change the polynomial type of yield criteria by using higher polynomial orders instead of the 2^{nd} order to refine the accuracy of the Hill's (1948) criterion. Although some of the criteria, introduced to replace the Hill's (1948) yield criterion, can better represent specified material properties, such as the description of yield surface for aluminum alloys (Barlat et al., 1997a,b; Wu et al., 2003), most of them cannot be expressed in an analytical form. A review of engineering applications about the proposed criteria shows that the Hill's (1948) yield criterion is still the most popular criterion for describing the behavior of orthotropic materials in the majority of commercial finite element programs currently in use by various industries; even though the Hill's criterion actually has limitations for analyzing some engineering materials. From a practical point of view, the analytical simplicity and proven convexity of the Hill's criterion is the principal reason that it is the preferred criterion for implementation in engineering applications. While other yield criteria may be considered for implementation, a basic concern is that the associated functions are too complicated to be used easily (Lademo et al., 1999; Barlat et al., 2003). Efforts to simplify the representation of yield criterion without a loss of accuracy should be of benefit allowing wider consideration for use in engineering applications.

Based on a physical hypothesis, and focused on rolled sheet metal applications, the author previously proposed an anisotropic yield criterion with a relatively simple and analytical form under the plane stress state (Hu, 2003). This criterion has been found fully to account for the experimental properties of rolled sheets in the equibiaxial tension and in the uniaxial tension at the rolling, transverse and diagonal directions. It can also give fair descriptions of the yield behavior when the principal-stress axes are out of these experimental principal axes. While focusing on a specified principal stress coordinate, the rolling vs. transverse directions, the criterion transforms to a form similar to the Gotoh fourth order criterion (Gotoh, 1977). This yield criterion can also be recovered to the Mises's yield criterion when the material is isotropic. However, even though the plane stress assumption is acceptable in stamping process simulations for most cases, the triaxial stress state often must be considered for analyzing some specified cases, such as the yielding with the effects of contact pressure (Smith et al., 2003) and the friction force. To serve extensive analyses required in engineering, a more general anisotropic yield criterion with a relative simple and analytical form is needed. Simultaneously, its predicting accuracy must meet engineering requirements.

We shall now introduce an approach for constructing a general yield criterion applied to an orthotropic and anisotropic material. The physical concept of the invariants of the stress tensor in the use of isotropic material was adapted so that the concept can be further applied to an anisotropic material in describing its yielding behavior. The constructed yield criterion will describe the yielding phenomenon in terms of the preferred orientation of textured aggregates and considers the preferred yielding possibilities arising from load acting on the polycrystalline aggregate in different directions. Focusing on rolled sheet metal applications, a characterized yield function was constituted. The yield criterion can be expressed as an analytical form in 3-D stress space. To assure the accuracy of the criterion in engineering applications, extensive experimental results have been reviewed, including applications in high strength steels and aluminum alloys. The yield criterion reverts back to the Mises's yield criterion when the material becomes isotropic.

2. Three-yield-system hypothesis of textured aggregates

Suppose that an isotropic metal is loaded by a force and reaches its elastic deformation limit. Its yielding state can then be expressed by a second order stress tensor as

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \tag{4}$$

where σ_{11}, σ_{22} and σ_{33} are the normal stress components, and σ_{12}, σ_{23} and σ_{13} are the shearing stress components. The yielding in question depends only on the magnitude of the loaded force without the effect of loading direction, in which different loading directions vary only the magnitude of each stress component, and not the stress tensor. It implies that only an equivalent yield system is present when the loaded force reaches its elastic limit of deformation.

However, if the loaded material becomes orthotropic anisotropy its yielding state will depend not only on the magnitude of the loaded force but also on the loading direction. Suppose that a stretching force acts on the material in three orthotropic directions, respectively, because the existence of anisotropy, three different yield systems with three different stress tensors at an equivalently yielding state could arise alternatively by changes in the loading direction. If the three orthotropic loading directions are changed three other differing yield systems will be generated simultaneously. Because there is no method to define how many different directions a generally loading process could involve, it is difficult to construct a continuous function of the stress tensor that can represent a prompt yielding state dealing with both the magnitude of the loaded force and the loading direction. However, when allowing for an acceptable level of accuracy in engineering applications, we can simplify the description of anisotropic stress tensor using the following hypothesis:

For a specific principal stress coordinate, we can presume that only three equivalent yield systems and three associated stress tensors will be exhibited. Different loading types, no change of the principal stress coordinate, vary only the magnitude of each stress component of stress tensors. The three stress tensors in describing different yield systems can be given as

$$\sigma_{ij}^{(1)} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \tag{5a}$$

$$\sigma_{ij'}^{(2)} = \begin{bmatrix} \sigma_{11''} & \sigma_{12''} & \sigma_{13''} \\ \sigma_{21''} & \sigma_{22''} & \sigma_{23''} \\ \sigma_{31''} & \sigma_{32''} & \sigma_{33''} \end{bmatrix}, \tag{5b}$$

$$\sigma_{ij''}^{(3)} = \begin{bmatrix} \sigma_{11'''} & \sigma_{12'''} & \sigma_{13'''} \\ \sigma_{21'''} & \sigma_{22'''} & \sigma_{23'''} \\ \sigma_{31'''} & \sigma_{32'''} & \sigma_{33'''} \end{bmatrix}, \tag{5c}$$

where $\sigma_{ij}^{(1)}$, $\sigma_{ij'}^{(2)}$ and $\sigma_{ij''}^{(3)}$ denote three stress tensors with respect to three different yield systems under an equivalently yielding state.

The validity of assuming a three-yield-system on a fixed principal-axis coordinate can be demonstrated in the following example: Suppose that we are dealing with a rolled sheet metal. Directions of a specified principal-axis coordinate are identical to the rolling, transverse and through thickness directions. Then, the three yield systems may arise from three uniaxial tensions loaded in the rolling, transverse and through thickness directions, respectively.

By considering a general stress state and an arbitrary principal stress coordinate, relations of the stress components between two stress tensors are assumed

$$\begin{aligned}&\frac{\sigma_{11''}-\sigma_{22''}}{\sigma_{11}-\sigma_{22}}=a_{11},\quad \frac{\sigma_{22''}-\sigma_{33''}}{\sigma_{22}-\sigma_{33}}=a_{22},\quad \frac{\sigma_{33''}-\sigma_{11''}}{\sigma_{33}-\sigma_{11}}=a_{33},\\ &\frac{\sigma_{12''}}{\sigma_{12}}=a_{12},\quad \frac{\sigma_{23''}}{\sigma_{23}}=a_{23},\quad \frac{\sigma_{31''}}{\sigma_{31}}=a_{13},\end{aligned} \tag{6a}$$

$$\begin{aligned}&\frac{\sigma_{11'''}-\sigma_{22'''}}{\sigma_{11}-\sigma_{22}}=b_{11},\quad \frac{\sigma_{22'''}-\sigma_{33'''}}{\sigma_{22}-\sigma_{33}}=b_{22},\quad \frac{\sigma_{33'''}-\sigma_{11'''}}{\sigma_{33}-\sigma_{11}}=b_{33},\\ &em\frac{\sigma_{12'''}}{\sigma_{12}}=b_{12},\quad \frac{\sigma_{23'''}}{\sigma_{23}}=b_{23},\quad \frac{\sigma_{31'''}}{\sigma_{31}}=b_{13},\end{aligned} \tag{6b}$$

where a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33}, b_{11}, b_{12}, b_{13}, b_{22}, b_{23} and b_{33} are the anisotropic coefficients that can be determined through experimentation. When the material is isotropic, all coefficients are equal to 1.

From Eqs. (6a) and (6b), a general anisotropic material will likely exhibit twelve relevantly anisotropic variables which describe the yielding behavior under a general stress state. It also implies that at least twelve relations with the variables of experimental results have to be provided in order to determine these pending coefficients.

3. A general anisotropic yield criterion

Based on the foregoing hypothesis, a general anisotropic yield criterion must take into account at least three stress tensors. From Eq. (3), a complete yield criterion must consider the effects of three invariants of the stress and deviatoric stress tensors as

$$f\left(J_1^{(1)},J_1^{(2)},J_1^{(3)},J_2'^{(1)},J_2'^{(2)},J_2'^{(3)},J_3'^{(1)},J_3'^{(2)},J_3'^{(3)}\right)=0. \tag{7}$$

Previous studies have shown that the first invariant of the stress tensor denotes the effect of yielding with the hydrostatic stress component. The third invariant of the deviatoric stress tensor denotes the effect of yielding by the different loading states (Hu and Wang, 2002, 2005). In the analysis of a stamping process, since the effects of these two components are generally small enough to be disregarded, (such as the Mises's yield criterion in the use of isotropic ductile metals), Eq. (7) can be simplified as

$$f\left(J_2'^{(1)},J_2'^{(2)},J_2'^{(3)}\right)=0, \tag{8}$$

where $J_2'^{(1)},J_2'^{(2)},J_2'^{(3)}$ are the second invariants of the deviatoric stress tensor under differing yielding systems.

Based on the stress tensor matrixes of Eqs. (5a), (5b), (5c) and the relations of Eqs. (6a) and (6b), the three second invariants of the deviatoric stress tensors can be expressed

$$\begin{pmatrix} J_2'^{(1)} \\ J_2'^{(2)} \\ J_2'^{(3)} \end{pmatrix} = \frac{1}{6} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ a_{11}^2 & a_{22}^2 & a_{33}^2 & a_{12}^2 & a_{23}^2 & a_{31}^2 \\ b_{11}^2 & b_{22}^2 & b_{33}^2 & b_{12}^2 & b_{23}^2 & b_{31}^2 \end{bmatrix} \begin{pmatrix} (\sigma_{11}-\sigma_{22})^2 \\ (\sigma_{22}-\sigma_{33})^2 \\ (\sigma_{33}-\sigma_{11})^2 \\ 6\sigma_{12}^2 \\ 6\sigma_{23}^2 \\ 6\sigma_{31}^2 \end{pmatrix}. \tag{9}$$

From Eqs. (8) and (9), a probable form of a yield function applied to a general anisotropic material can be constructed as

$$f(\sigma_{ij}) = (J_2'^{(1)})^2 + (J_2'^{(2)})^2 + (J_2'^{(3)})^2 - \frac{1}{C^2} = 0, \tag{10}$$

where C is a constant.

4. Convexity

The assurance of the convexity with respect to the predicted yield surface is an important requirement in the use of finite element analysis. Generally, it has to be proven mathematically, where if the function can be shown its Hessian Matrix H_{ij} being positive semi-definite (i.e., its eigenvalues all are positive or zero), the yield surface should be convex. Fortunately, for the proposed function its convexity can be proven by a simpler method. First, we define some variables as

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11}-\sigma_m \\ \sigma_{22}-\sigma_m \end{pmatrix}, \tag{11}$$

$$\sigma_m = \frac{\sigma_{11}+\sigma_{22}+\sigma_{33}}{3}, \tag{12}$$

$$\begin{pmatrix} \sigma_{11}-\sigma_{33} \\ \sigma_{22}-\sigma_{33} \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix}. \tag{13}$$

Substituting relations (11)–(13) into Eq. (10), it becomes a form

$$\begin{aligned} f(\sigma_{ij}) = {} & \rho\left(K_1^2 + K_2^2 + K_1K_2 + \sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)^2 \\ & + \left[\alpha_1 K_1^2 + \alpha_2 K_2^2 + \alpha_3 K_1 K_2 + \alpha_4\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right]^2 \\ & + \left[\beta_1 K_1^2 + \beta_2 K_2^2 + \beta_2 K_1 K_2 + \beta_4\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)\right]^2 - 12 = 0, \end{aligned} \tag{14}$$

where ρ, α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4 are positive parameters.

In order to facilitate the analysis, we transform Eq. (14) to its principal stress state. The relations of Eq. (13) becomes

$$\begin{pmatrix} \sigma_{11}^* - \sigma_{33}^* \\ \sigma_{22}^* - \sigma_{33}^* \end{pmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{pmatrix} K_1^* \\ K_2^* \end{pmatrix}, \tag{15}$$

where $\sigma_{11}^*, \sigma_{22}^*$ and σ_{33}^* are the principal stresses.

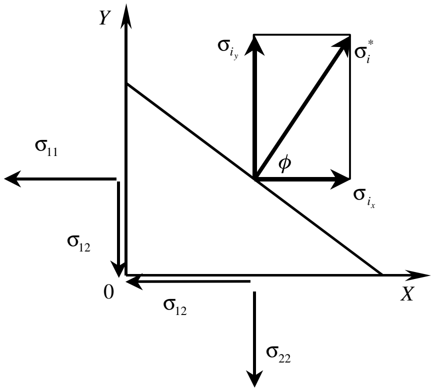


Fig. 1. Coordinate system of equivalent stresses.

If we suppose that the stress σ_{33} is already a principal stress, the representation with a complete principal-axis coordinate can be obtained by rotating the original coordinate axes (K_1, K_2) an angle ϕ (see Fig. 1), which can be obtained from the function

$$\begin{aligned}&\sigma_{11}\cos^2\phi + \sigma_{22}\sin^2\phi + 2\sigma_{12}\sin\phi\cos\phi\\&\quad = \sqrt{(\sigma_{11}\cos\phi + \sigma_{12}\sin\phi)^2 + (\sigma_{12}\cos\phi + \sigma_{22}\sin\phi)^2}.\end{aligned} \tag{16}$$

Thus, Eq. (14) transforms to its principal-axis form as

$$\begin{aligned}f(\sigma_{ij}) = {}&\left[\rho_1^*\left(K_1^*\right)^2 + \rho_2^*\left(K_2^*\right)^2 + \rho_{12}^*K_1^*K_2^*\right]^2\\&+\left[\alpha_1^*\left(K_1^*\right)^2 + \alpha_2^*\left(K_2^*\right)^2 + \alpha_{12}^*K_1^*K_2^*\right]^2\\&+\left[\beta_1^*\left(K_1^*\right)^2 + \beta_2^*\left(K_2^*\right)^2 + \beta_{12}^*K_1^*K_2^*\right]^2 - 12 = 0\end{aligned} \tag{17}$$

where $\rho_1^*, \rho_2^*, \rho_{12}^*, \alpha_1^*, \alpha_2^*, \alpha_{12}^*\beta_1^*, \beta_2^*$ and β_{12}^* are the coefficients, which are the functions of variables ρ, α_1, α_2, α_3, α_4, β_1, β_2, β_3 and β_4.

Because the coefficients $\rho_1^*, \rho_2^*, \alpha_1^*, \alpha_2^*, \beta_1^*$ and β_2^* are all positive, it shows Eq. (17) as the sum of three ellipses formulas with the variables K_1^* and K_2^*.

It has been assumed that the hydrostatic stress component can be superimposed on the yield criterion without the effect of yielding. Changing the hydrostatic stress component σ_m only shifts the location of ellipses but does not vary its shape. This feature can be represented in a defined stress space as shown in Fig. 2. As it is known that the ellipses formulas are convex, and the sum of the three ellipses formulas remains convex, it implies then that the yielding function is convex also.

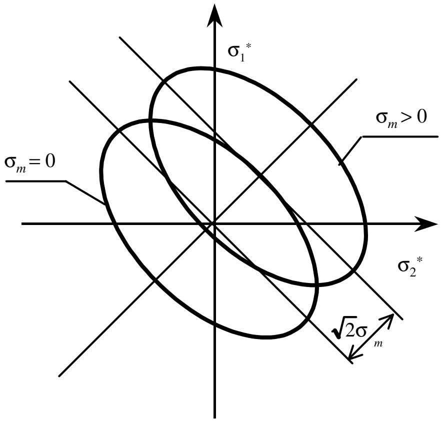


Fig. 2. Ellipses in a defined stress space.

5. Application to rolled sheet metals

To characterize the yield function of Eq. (10) as applied to rolled sheet metals, we need to determine the pending coefficients using experimental results. Suppose the directions of normal stress σ_{11} and σ_{22} are in the sheet plane and the direction of normal stress σ_{33} is through the thickness. Thus, when the directions of stresses σ_{11} and σ_{22} become the principal axes identical to the rolling and transverse directions, in terms of the experiments carried out through the rolling and transverse directions we have

$$4+\left(a_{11}^2+a_{33}^2\right)^2+\left(b_{11}^2+b_{33}^2\right)^2=\frac{36}{\sigma_0^4C^2}, \tag{18}$$

$$4+\left(a_{11}^2+\alpha_{22}^2\right)^2+\left(b_{11}^2+b_{22}^2\right)^2=\frac{36}{\sigma_{90}^4C^2}, \tag{19}$$

$$4+\left(a_{22}^2+a_{33}^2\right)^2+\left(b_{22}^2+b_{33}^2\right)^2=\frac{36}{\sigma_b^4C^2}, \tag{20}$$

$$R_0=\frac{2+\left(a_{11}^2+a_{33}^2\right)a_{11}^2+\left(b_{11}^2+b_{33}^2\right)b_{11}^2}{2+(a_{11}^2+a_{33}^2)a_{33}^2+\left(b_{11}^2+b_{33}^2\right)b_{33}^2}, \tag{21}$$

$$R_{90}=\frac{2+\left(a_{11}^2+a_{22}^2\right)a_{11}^2+\left(b_{11}^2+b_{22}^2\right)b_{11}^2}{2+(a_{11}^2+a_{22}^2)a_{22}^2+\left(b_{11}^2+b_{22}^2\right)b_{22}^2}, \tag{22}$$

where σ_0 and σ_{90} are the yield stresses under the uniaxial tensions in the rolling and transverse directions respectively, R_0 and R_{90} are the ratios of transverse to through thickness increments of the logarithmic strain under σ_0 and σ_{90} uniaxial tension states respectively, and σ_b is the yield stress under the equibiaxial tension state.

We can also obtain some relations dealt with the principal-stress directions of experiments in two diagonal directions against the rolling. Since two yield systems arising from the uniaxial tensions in the two diagonal directions are the same, we have

$$a_{22}^2 + a_{33}^2 = 2, \tag{23}$$

$$a_{12}^2 = 1, \tag{24}$$

$$8^2 + \left(a_{22}^2 + a_{33}^2 + 6a_{12}^2\right)^2 + \left(b_{22}^2 + b_{33}^2 + 6b_{12}^2\right)^2 = \frac{36 \times 16}{\sigma_{45}^4 C^2}, \tag{25}$$

$$R_{45} = \frac{32 - \left(a_{22}^2 + a_{33}^2\right)^2 + \left(6a_{12}^2\right)^2 - \left(b_{22}^2 + b_{33}^2\right)^2 + \left(6b_{12}^2\right)^2}{32 + 2\left[(a_{22}^2 + a_{33}^2 + 6a_{12}^2)(a_{22}^2 + a_{33}^2) + \left(b_{22}^2 + b_{33}^3 + 6b_{12}^2\right)\left(b_{22}^2 + b_{33}^2\right)\right]}, \tag{26}$$

where σ_{45} is the yield stresses under the uniaxial tension in the 45° direction, and R_{45} is the ratio of transverse to through thickness increments of logarithmic strain under the uniaxial tension in the 45° direction.

From our hypothesis and Eq. (10) we know that there are 13 totally pending coefficients. The standard experiments on rolled sheet metals can establish only nine formulas. In order to obtain a general 3-D yield criterion without the need for additional experiments, some assumptions are required. By assuming that the parameters specifying the effects of shearing stresses σ_{12}, σ_{13} and σ_{23} are the same with respect to each yield system, we have

$$\begin{aligned} a_{12} &= a_{23} = a_{31}, \\ b_{12} &= b_{23} = b_{31} \end{aligned} \tag{27}$$

Thus, we can determine all pending coefficients from Eqs. (27). In general, we can obtain only a numerical solution.

With the goal of obtaining a simpler and analytical form of the yield criterion, we need to perform some additional mathematical work. Expansion of Eq. (10) as

$$\begin{aligned} F = {} & X_1(\sigma_{11} - \sigma_{33})^4 + X_2(\sigma_{11} - \sigma_{33})^3(\sigma_{22} - \sigma_{33}) + X_3(\sigma_{11} - \sigma_{33})^2(\sigma_{22} - \sigma_{33})^2 \\ & + X_4(\sigma_{11} - \sigma_{33})(\sigma_{22} - \sigma_{33})^3 + X_5(\sigma_{22} - \sigma_{33})^4 + (\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \\ & \times \left[C_1(\sigma_{11} - \sigma_{33})^2 + C_2(\sigma_{22} - \sigma_{33})^2 - C_3(\sigma_{11} - \sigma_{33})(\sigma_{22} - \sigma_{33})\right] \\ & + X_7\left(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2\right)^2 - 1 = 0, \end{aligned} \tag{28}$$

where the parameters can be expressed

$$X_1 = \frac{1}{\sigma_0^4}, \tag{29}$$

$$X_2 = -\frac{4R_0}{(1 + R_0)\sigma_0^4}, \tag{30}$$

$$X_3 = \frac{1}{\sigma_b^4} - \frac{1}{\sigma_0^4} - \frac{1}{\sigma_{90}^4} + \frac{4R_0}{(1+R_0)\sigma_0^4} + \frac{4R_{90}}{(1+R_{90})\sigma_{90}^4}, \tag{31}$$

$$X_4 = -\frac{4R_{90}}{(1+R_{90})\sigma_{90}^4}, \tag{32}$$

$$X_5 = \frac{1}{\sigma_{90}^4}, \tag{33}$$

$$X_6 = C_1 + C_2 - C_3 = \frac{16}{(1+R_{45})\sigma_{45}^4} - \frac{2}{\sigma_b^4}, \tag{34}$$

$$X_7 = \frac{1}{\sigma_b^4} + \frac{16R_{45}}{(1+R_{45})\sigma_{45}^4}. \tag{35}$$

Representations of Eqs. (29)–(35) imply that all coefficients can be represented analytically except C_1, C_2 and C_3. However, when the principal-stress axes are in two diagonal directions, we can obtain a relation

$$\begin{aligned} &C_1(\sigma_{11}-\sigma_{33})^2 + C_2(\sigma_{22}-\sigma_{33})^2 - C_3(\sigma_{11}-\sigma_{33})(\sigma_{22}-\sigma_{33}) \\ &\quad = X_6\left[\frac{1}{2}\left(\sigma_{11}^* + \sigma_{22}^*\right) - \sigma_{33}^*\right]^2, \end{aligned} \tag{36}$$

where $\sigma_{11}^*, \sigma_{22}^*, \sigma_{33}^*$ are the principal stresses at principal axes in two diagonal directions to the rolling.

This relation can be adapted to the yield function. This modification does not affect the prediction of yield surface fully reflecting all experimental results. As a result, the yield criterion is analytically expressed as

$$\begin{aligned} F = {} & X_1(\sigma_{11}-\sigma_{33})^4 + X_2(\sigma_{11}-\sigma_{33})^3(\sigma_{22}-\sigma_{33}) + X_3(\sigma_{11}-\sigma_{33})^2(\sigma_{22}-\sigma_{33})^2 \\ & + X_4(\sigma_{11}-\sigma_{33})(\sigma_{22}-\sigma_{33})^3 + X_5(\sigma_{22}-\sigma_{33})^4 + X_6\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right) \\ & \times \left[(\sigma_{11}-\sigma_{33})^2 + (\sigma_{22}-\sigma_{33})^2 - (\sigma_{11}-\sigma_{33})(\sigma_{22}-\sigma_{33})\right] \\ & + X_7\left(\sigma_{12}^2+\sigma_{23}^2+\sigma_{31}^2\right)^2 - 1 = 0. \end{aligned} \tag{37}$$

When the z-axis through thickness is fixed as a principal axis, we have

$$\sigma_{23} = \sigma_{31} = 0. \tag{38}$$

The transformed form is valid for application to thin shells. When a plane stress state $\sigma_{33} = 0$ is assumed, the yield criterion is further transformed back to the form of Hu's (2003) anisotropic yield criterion. If the applied material is isotropic, all anisotropic coefficients are equal to 1, the yield criterion reduces to the form of Mises's yield criterion as

$$f(\sigma_{ij}) = 3J_2'^2 - \frac{1}{C^2} = 3J_2'^2 - \frac{1}{3}\bar{\sigma}^4 = 0, \tag{39}$$

where $\bar{\sigma}$ is the effective stress of the Mises's yield function.

6. Predictions of yield surface

In using the characterized criterion to predict yield surface, we need to understand how the yielding behaviors vary with the different stress states and the loading directions. We will now focus on the general illustrations of anisotropic properties of materials. Yield surfaces are predicted from a 3-D general loading model to the commonly assumed plane-stress model. Three specified directions are referred when considering the directional dependence of yield surfaces. These are the rolling, transverse and diagonal directions, labeled RD, TD and DD, respectively. In considering the 3-D general stress state, it is assumed that the frictional force acting on the rolled sheet plane induces the change of principal stress axes out of the normal and tangent directions of the sheet plane. The yielding behaviors arising from the change of principal stress directions are of interest because they are often disregarded in the normal analysis. The 3-D general yield surfaces are presented through the use of the π plane, in which yielding under the effects of frictional force is more practical for engineering applications. In the discussion of the plane stress state, our attention focuses on the relations between the yielding characteristic and the direction of principal-axis in the sheet plane.

6.1. Yielding characteristic on the π plane

Since the predicted yield stress does not deal with the effect of the hydrostatic stress (mean normal stress component), the anisotropic property of the 3-D yield surface can be accounted for completely by a plane, called the π plane as shown in Fig. 3. To draw the yield locus on the π plane, we need to define a coordinate $x-y$ on the π plane, in which y axis is identical to the direction of the principal stress σ^*_{22}

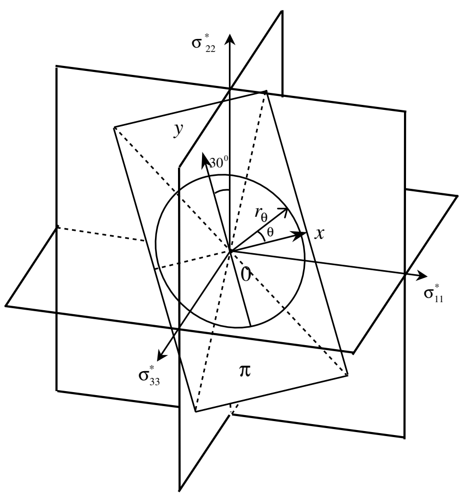


Fig. 3. π plane and $x-y$ coordinate.

projecting on the π plane (Fig. 3). Therefore, a yielding point in the 3-D principal stress space can be transformed to the π plane using

$$x = \frac{\sqrt{2}}{2}\left(\sigma_{11}^* - \sigma_{33}^*\right),\qquad y = \frac{2\sigma_{22}^* - \sigma_{11}^* - \sigma_{33}^*}{\sqrt{6}}, \tag{40}$$

where σ_{11}^*, σ_{22}^* and σ_{33}^* are the principal stresses.

When considering the influence of the tooling force, we may simplify the material property to planar isotropy, and R is defined as the ratio of transverse to through thickness increments of logarithmic strain under the uniaxial tension σ_{u} in the sheet plane. Thus, the frictional shearing stress τ_{p} can be expressed by

$$\sigma_{13}^2 + \sigma_{23}^2 = \tau_p^2 = (\mu p)^2, \tag{41}$$

where μ is the frictional coefficient between the tool surface and the sheet surface, and p is the contact pressure. With the assumption of planar isotropy, the yield criterion of Eq. (37) becomes

$$\begin{aligned} F = {} & \frac{1}{\sigma_u^4}\left[(\sigma_{11}-\sigma_{33})^4 + (\sigma_{22}-\sigma_{33})^4\right] \\ & - \frac{4R}{(1+R)\sigma_u^4}\left[(\sigma_{11}-\sigma_{33})^3(\sigma_{22}-\sigma_{33}) + (\sigma_{11}-\sigma_{33})(\sigma_{22}-\sigma_{33})^3\right] \\ & + \left(\frac{1}{\sigma_b^4} - \frac{2}{\sigma_u^4} + \frac{8R}{(1+R)\sigma_u^4}\right)(\sigma_{11}-\sigma_{33})^2(\sigma_{22}-\sigma_{33})^2 \\ & + \left(\frac{16}{(1+R)\sigma_u^4} - \frac{2}{\sigma_b^4}\right)\left(\tau_p^2 + \sigma_{12}^2\right)\Big[(\sigma_{11}-\sigma_{33})^2 + (\sigma_{22}-\sigma_{33})^2 \\ & - (\sigma_{11}-\sigma_{33})(\sigma_{22}-\sigma_{33})\Big] + \left(\frac{1}{\sigma_b^4} + \frac{16R}{(1+R)\sigma_u^4}\right)\left(\tau_p^2 + \sigma_{12}^2\right)^2 - 1 = 0. \end{aligned} \tag{42}$$

Because of the influence of tooling frictional force, the principal axis σ_{33}^* (original through thickness) is no longer perpendicular to the sheet surface. The other two principal axes may no longer lie on the sheet plane either. Suppose $\sigma_{23} = 0$, $\tau_p = \sigma_{13}$. The principal stress axis σ_{22}^* remains parallel to the sheet surface, and another principal stress axis σ_{11}^* together with the principal stress axis σ_{33}^* will rotate to an angle ϖas shown in Fig. 4. The inclined angle ϖ can be calculated through equations of transformation

$$\left.\begin{aligned} \sigma_{11} &= \sigma_{11}^*\cos^2\varpi + \sigma_{33}^*\sin^2\varpi \\ \sigma_{33} &= \sigma_{11}^*\sin^2\varpi + \sigma_{33}^*\cos^2\varpi \\ \tau_p &= \left(\sigma_{33}^* - \sigma_{11}^*\right)\sin\varpi\cos\varpi \end{aligned}\right\}. \tag{43}$$

The yielding characteristic with the influence of strain ratio R is shown in Fig. 5. For $R > 1$, the yield stresses increase in the areas of **I, II, IV, V** and decrease in the areas of **III, VI**; the maximum and minimum points are both at the plane-strain

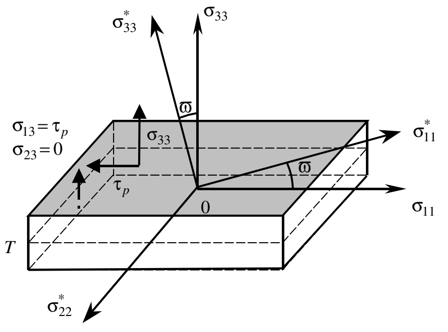


Fig. 4. The direction change of principal-stress coordinate due to the friction force arising from the tooling contact.

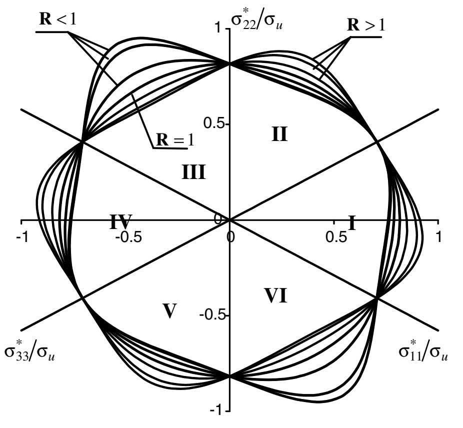


Fig. 5. Profiles of yielding loci on the π plane with different R-values.

state. For $R < 1$, it shows just the opposite result. Fig. 6 shows the anisotropic yield behavior with the effect of frictional force, the larger the frictional force (larger inclined angle ϖ), the larger the anisotropic influence of yield surface. In order to understand the influence of the change of principal axes more clearly, Figs. 7 and 8 show the relative conditions with different anisotropic parameters of material. All results show that the stronger the anisotropy of material, either of the anisotropic variables R and σ_b, the larger the influence of yield surface due to the change of principal axes.

6.2. Thin shell assumption

A thin shell assumption is often defined in the context of sheet metal forming simulations, which covers two specified stress states: a. three-stress state but assuming the direction through thickness as a fixed principal axis, i.e., $\sigma_{23} = \sigma_{31} = 0$; b. plane

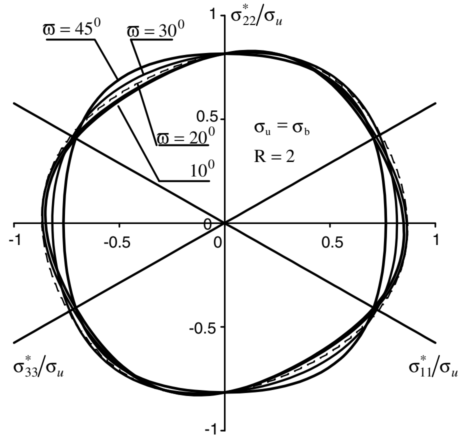


Fig. 6. Yielding features with the effect of tooling friction force.

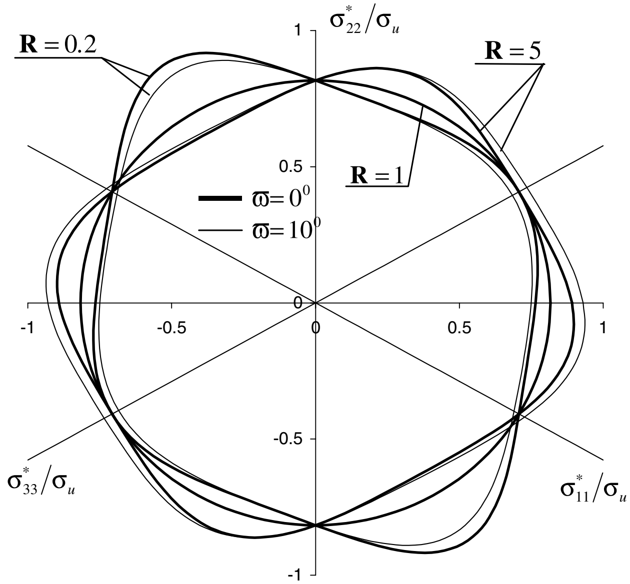


Fig. 7. Yielding behaviors with the influence of anisotropic parameter *R*.

stress state $\sigma_{23} = \sigma_{31} = \sigma_{33} = 0$. With the assumed stress state, the anisotropic behaviors of the yield surface only deal with the change of principal axes in the sheet plane. To see the influence more clearly, suppose that a rolled sheet metal sample shows its anisotropic variables as $R_0 = R_{90} = 0.2$, $R_{45} = 10$ and $\sigma_0 = \sigma_{45} = \sigma_{90} = \sigma_b$. From the illustration as shown in Fig. 9, the change of principal axes out of the rolling and transverse directions varies the yield stress at any possible stress state except in the

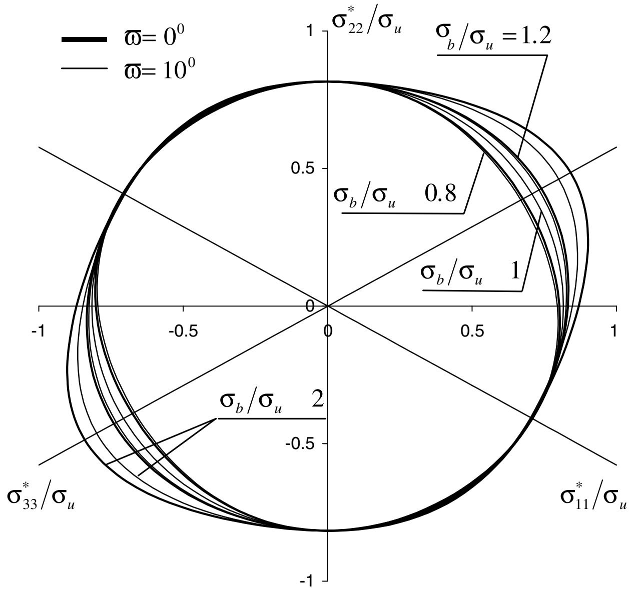


Fig. 8. Yielding behaviors with the influence of equibiaxial-tension yield stress.

equibiaxial tension state. With this defined material properties, it shows that if the principal axes rotate toward the direction with the larger strain ratio R (e.g., from the rolling approaching to the diagonal direction), the yield stress increases at the

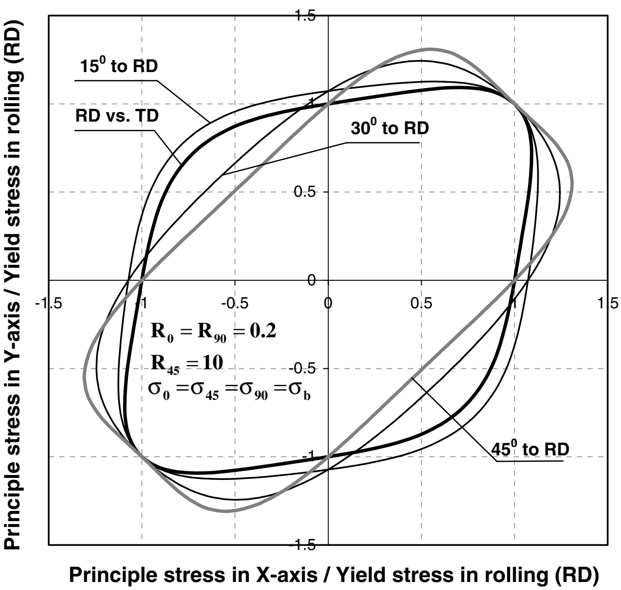


Fig. 9. Yielding behavior with the change of principal stress axes.

uniaxial tension state. The principal axis at 15° to the rolling direction shows the largest yield stresses under the whole stress states. This yielding behavior predicted using the proposed yield function will be discussed further in the experimental comparison section of this paper.

This illustration also shows that the strain-hardening model used to predict subsequent yield surface should be carefully considered, particularly in the use of the isotropic-hardening model. We observe that the increases of experimental stresses after passing the original yield point do not retain identical proportions under each equivalent yield-increment. How to define an applicable equivalent stress will require additionally study beyond the scope of this research. Because of the change of principal axes, the predicted yield surface is quite different, and the strain-increment predicted will also be quite different when the associated flow rule is addressed.

7. Comparisons with experiments

In order to assure that the characterized yield criterion is both useful and practical for use in engineering applications, we will now compare experimental data with the predicted results for commonly used materials. The comparisons include the yield stress under the uniaxial tension state in different loading directions to the rolling, the corresponding ratios of the transverse to through-thickness increments of logarithmic strain, and results obtained under other stress states in order to verify the predicted yield surface in stress space.

7.1. Yielding behavior in uniaxial tension state

By using the stated process in characterizing the yield criterion, we find that the yield criterion fully reflects the experimental results in uniaxial tension at the rolling, transverse and diagonal directions with both the yield stress and the ratio of increment of logarithmic strain. In order to determine whether the yield criterion can represent the uniaxial tension behavior in other directions against the rolling direction effectively, some additional experiments were carried out. Materials used in the experiments were Y350 MPa high strength steel (HSS) with different coatings: cold rolled (CR) and hot dipped galvanized (GA). The typically chemical compositions of the steels are listed in Table 1. Tables 2 and 3 are the typical mechanical properties of CR and GA steels, respectively. Since the Hill's (1948) yield criterion is the most extensively used in commercial finite element programs, the calibration also includes this model for comparison with the proposed yield criterion where its associated

Table 1
Chemical compositions (wt.%) of steel specimen

Material	C	Mn	P	S	Si	Cu	Ni	Cr	Mo	Sn	Al	Cb	Ti	V
CR	0.053	0.47	0.011	0.006	0.015	0.016	0.011	0.029	0.006	0.008	0.066	0.040	0.029	–
GA	0.060	0.39	0.011	0.008	0.016	0.051	0.010	0.029	0.007	0.004	0.039	0.029	0.006	0.001

Table 2
Typical mechanical properties of cold rolled (CR) HSS Y350 MPa

Rolling direction	0°	15°	30°	45°	60°	75°	90°
Gauge (mm)	1.19	1.19	1.19	1.19	1.19	1.19	1.19
Yield strength (MPa)	358.4	352.9	349.9	344.3	342.7	337.2	330.7
Tensile strength (MPa)	461.6	454.5	446.7	441.6	444.9	449.6	446.8
YPE (%)	12.7	12.6	12.6	13.2	14.0	15.8	13.0
Total elongation (%)	25.5	25.0	25.6	26.8	27.2	28.8	27.1
Strength coefficient (MPa)	709.0	691.5	685.1	678.9	698.4	702.2	705.4
n value	0.146	0.141	0.145	0.147	0.157	0.155	0.159
R value	0.827	0.926	0.830	0.838	0.930	1.119	1.255

Table 3
Typical mechanical properties of hot dipped galvanized (GA) HSS Y350 MPa

Rolling direction	0°	15°	30°	45°	60°	75°	90°
Gauge (mm)	1.50	1.50	1.50	1.50	1.50	1.50	1.50
Yield strength (MPa)	383.0	382.6	382.6	384.3	370.0	369.0	366.4
Tensile strength (MPa)	442.8	446.2	446.2	448.3	441.9	440.4	436.6
YPE (%)	11.6	12.5	13.9	13.1	14.1	14.5	13.8
Total elongation (%)	18.6	12.5	27.6	28.7	28.5	29.6	30.0
Strength coefficient (MPa)	681.3	671.2	665.7	681.6	676.3	677.5	675.4
n value	0.145	0.136	0.133	0.141	0.145	0.147	0.150
R value	0.892	0.931	0.972	0.995	1.064	1.046	1.082

parameters were determined by using the ratios of increments of logarithmic strain in uniaxial tension at 0°, 45°, 90° and the yield stress at 0° to the rolling direction.

From calculations presented in Fig. 10 it can be seen that the proposed yield criterion gives reasonable predictions of both the yield stress and the strain ratio, while the Hill's (1948) criterion matches only the strain ratio. Whereas the associated parameters in the Hill's (1948) yield criterion were determined by the yield stresses in the 0°, 45° and 90° directions and the strain ratio in the 0° direction to the rolling, it shows a poor description of strain ratio behavior except in the rolling direction (Lademo et al., 1999).

Because the Hill's (1948) yield criterion is often invalid for analyzing aluminum alloy sheets, in addition to the comparisons of HSS, a similar experiment on an aluminum alloy sheet, AA2090-T3, reported by Barlat et al. (2003) was referenced to further calibrate the proposed yield criterion. Two recently proposed yield functions Yid96 (Barlat et al., 1997a,b) and Yid2000-2d (Barlat et al., 2003), specified for the aluminum alloy sheet, were also compared against the experimental results. Fig. 11 presents the calculated results. As shown, the proposed yield criterion showed excellent correlation for both the yield stress and the strain ratio yielding similar results to the yield functions of Yid96 and Yid2000-2d in the predictions of aluminum property, but with greater simplicity from an application standpoint.

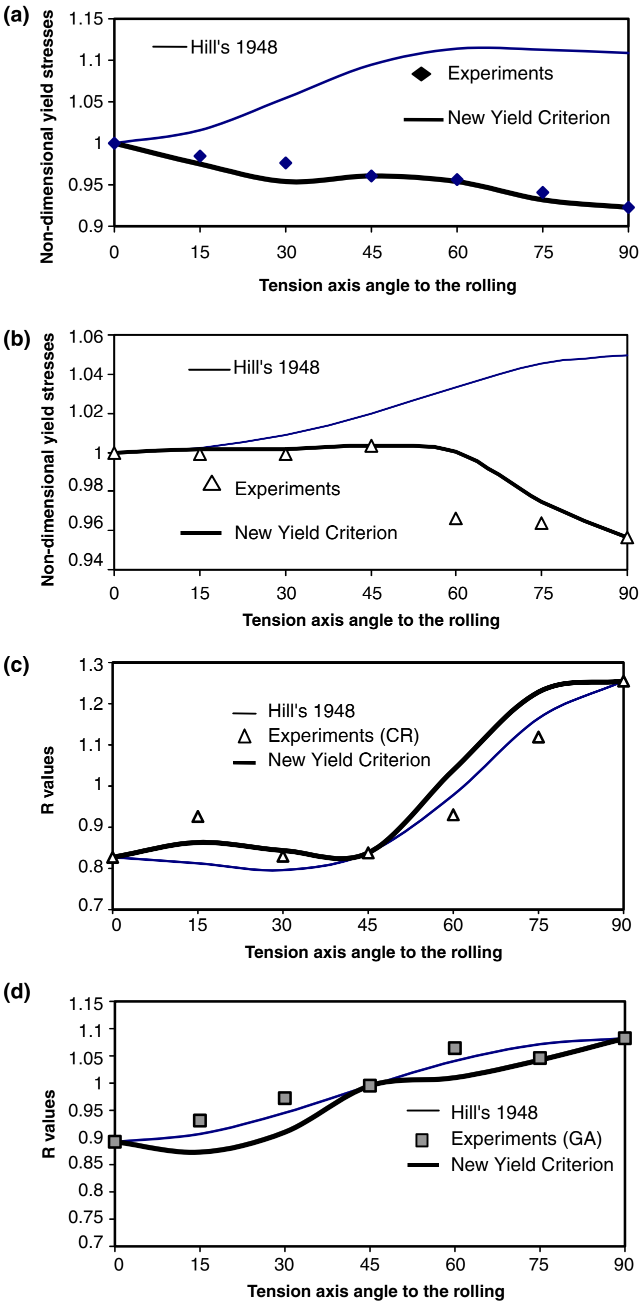


Fig. 10. Anisotropic behavior of yield stress in uniaxial tension from 0° to 90° against the direction of rolling for high strength steels (HSS).

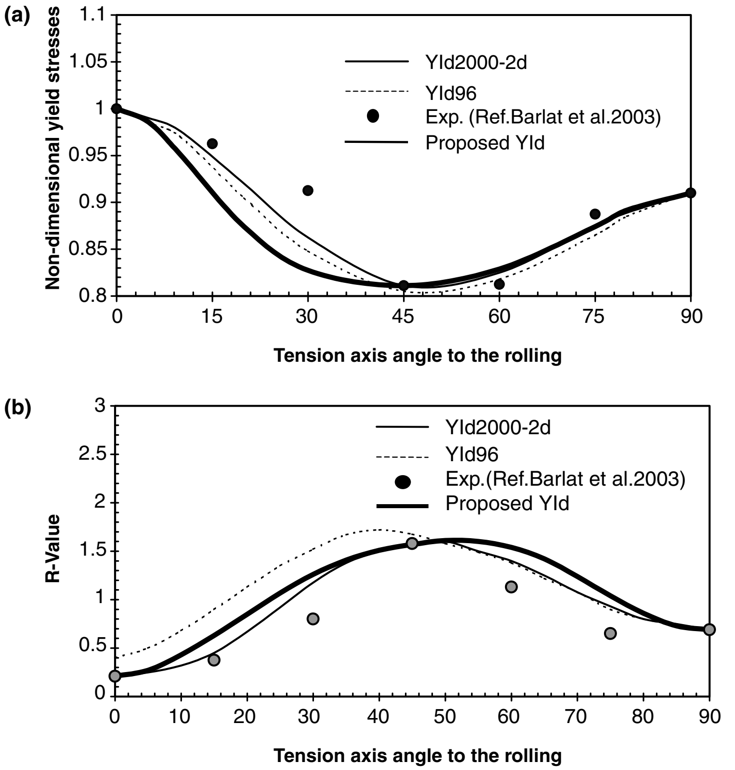


Fig. 11. Anisotropic behavior of yield stress in uniaxial tension from 0° to 90° against the direction of rolling for aluminum alloy 2090-T3.

7.2. Yielding surface in stress space

Because engineering applications may deal with a more general stress state, anisotropic behavior of the predicted yield surface in a whole stress space is of greater significance for study more. In fact, configurations of the predicted yield surface directly impact the level of plastic strain-increments when the constitutive relations are derived from the associated flow rule (Drucker, 1951) and anisotropic work hardening model (Hu, 2004). The yield surfaces were predicted by using the proposed model and the Hill's (1948) function with both methods in determinations of the associated parameters. The predicted yield surfaces were compared with experimental results using the two abovementioned materials. The yield surfaces as shown in Fig. 12 represented the yielding behavior for the high strength steel, in which the proposed criterion again yielded acceptable results in describing the yielding behavior from uniaxial tension to equibiaxial tension and while the Hill's criterion could not be reconciled to the experimental results with either of the determining methods. Fig. 13 shows the yielding property with the aluminum alloy. As shown, the proposed criterion was also able to predict the yielding behavior of the

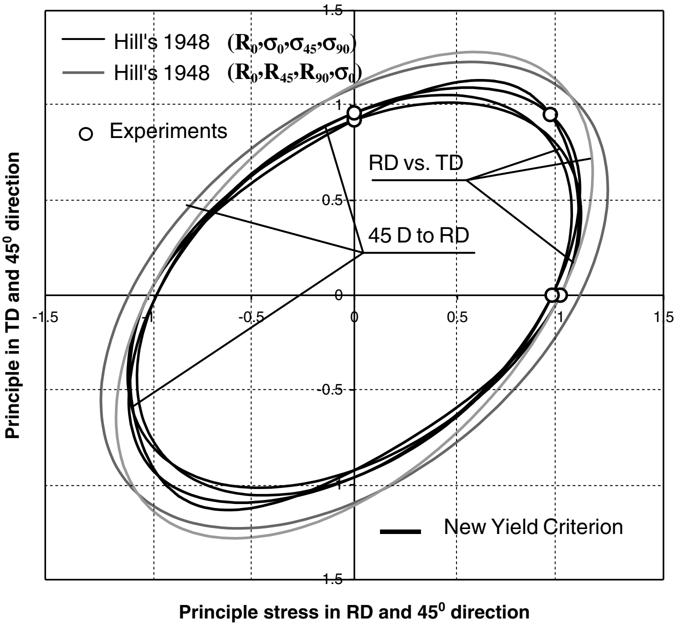
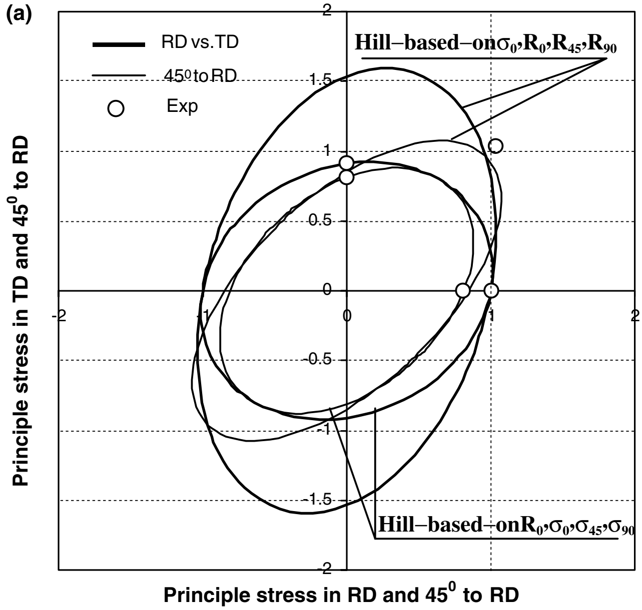


Fig. 12. Yield surface of high strength steel.

aluminum alloy when compared to the experimental results. The same experimental result reveals that the Hill's (1948) criterion was invalid for analyzing the aluminum alloy.

To further increase our confidence in the application of our proposed function, a previously specified case with respect to the prediction of yield surface was further studied and results compared experimentally. The case was reported by Barlat et al. (1997a,b, 2003). In their papers, the authors stated that when the uniaxial yield stresses at 0°, 45° and 90° to the rolling direction and the equibiaxial tension yield stress are all assumed to be identical, most proposed anisotropic yield criteria lead to an isotropic response of the material. In fact, when the pending parameters of an anisotropic yield criterion are fewer than seven it is difficult to describe an anisotropic material that exhibits a larger pure shear yield stress relative to other plane strain yield stresses, such criteria often induces the predicted yield surfaces out of the experimental results significantly (Barlat et al., 1997a,b). For this kind of material, the polycrystal models (Taylor, 1938; Bishop and Hill, 1951a,b) can predict its yield surface more accurately. In order to calibrate the proposed criterion, the experimental data reported by Barlat et al. (1997a,b) were compared with the predicted results from our proposed criterion. With the as-received material, its anisotropic variables are: $R_0 = 0.20$, $R_{45} = 0.28$,$R_{90} = 0.20$ and $\sigma_0 = \sigma_{45} = \sigma_{90} = \sigma_b$. The predicted yield surfaces and associated experimental data are shown in Fig. 14. The predicted yield surfaces were found to be in fairly good agreement with the experimental results with the principal axes of the rolling vs. the transverse directions and of that



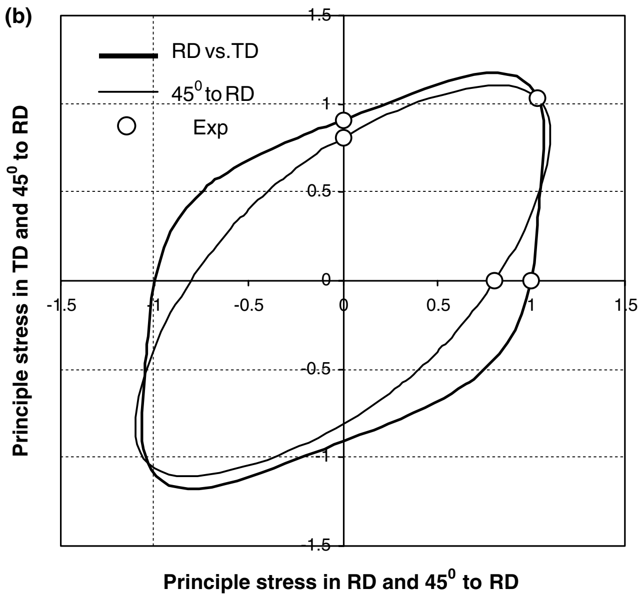


Fig. 13. Yield surface of aluminum alloy 2090-T3.

with the two diagonal directions to the rolling. When using a smaller equibiaxial tension yield stress $\sigma_b = 0.95\sigma_0$ the predicted yield surface produced even greater degrees of correlation to the experimental results.

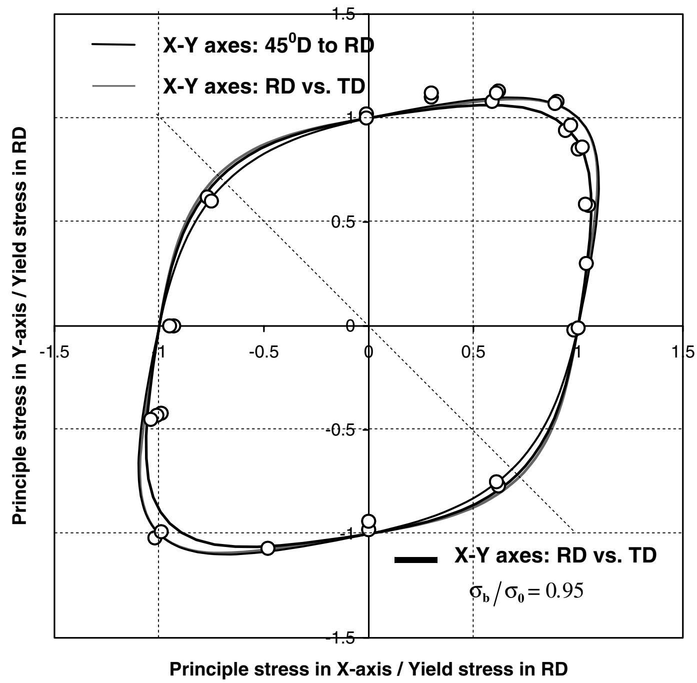


Fig. 14. Yield locos prediction for aluminum alloy Al–2.5 wt.% Mg.

8. A proposed relation to calculate the equibiaxial-tension yield stress

Although the yield stress under the equibiaxial tension state is an important parameter in describing the anisotropic characteristic of rolled sheet metals, in general, we cannot obtain the experimental data easily. In order to solve this problem, a method was proposed herein, which can be used to calculate the equibiaxial-tension yield stress when the experimental data are not available.

Suppose that a uniaxial tension is loaded in direction through thickness of a rolled sheet metal sample. Strain-increments in the rolling, transverse, and through thickness directions are $\mathrm{d}\varepsilon_r$, $\mathrm{d}\varepsilon_{\mathrm{tr}}$ and $d\varepsilon_{\mathrm{th}}$. Because of the existence of anisotropic properties of the material, the strain-increments $d\varepsilon_r$ and $d\varepsilon_{\mathrm{tr}}$ will be different. However, based on the information obtained from the uniaxial tensions in the rolling and transverse directions we can construct an approximate relation with respect to the strain-increments $\mathrm{d}\varepsilon_r$ and $\mathrm{d}\varepsilon_{\mathrm{tr}}$ as

$$\begin{pmatrix} \mathrm{d}\varepsilon_{\mathrm{r}} \\ \mathrm{d}\varepsilon_{\mathrm{tr}} \end{pmatrix} = \mathrm{d}\varepsilon_{\mathrm{th}} \begin{pmatrix} R_{90} \\ R_0 \end{pmatrix}, \tag{44}$$

where r, tr and th denote the rolling, transverse and through thickness directions, respectively.

Table 4
Comparisons of equibiaxial yield stress between experiment and calculation

Material	Yield stresses (σ/σ_0) σ_0:σ_{90}:σ_b	R value R_0:R_{90}	Calculating value ($\sigma_{b(c)}$)	$\frac{\sigma_{b(c)}-\sigma_b}{\sigma_{b(c)}}$ (%)	References
2090 T3	1; 0.910; 1.035	0.21; 0.69	1.0846	4.573	Yoon et al. (2000)
AA5182-0	1; 0.9855; 0.9859	0.72; 0.84	1.0217	3.499	Wu et al. (2003)
AA3104-H19	1; 1.0388; 0.9088	0.43; 1.26	0.9959	3.513	Wu et al. (2003)

Thus, we can further obtain a relation that states

$$\frac{\mathrm{d}\varepsilon_{\mathrm{r}}}{\mathrm{d}\varepsilon_{\mathrm{tr}}} = \frac{R_{90}}{R_0} \Rightarrow 1. \tag{45}$$

By using the associated flow rule with the function of Eq. (37) as the plastic potential, we have

$$\begin{pmatrix} \mathrm{d}\varepsilon_{\mathrm{r}} \\ \mathrm{d}\varepsilon_{\mathrm{tr}} \end{pmatrix} = -\mathrm{d}\lambda \times \sigma_t^4 \begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix}, \tag{46}$$

where $\mathrm{d}\lambda$ is a constant, and σ_t is the yield stress of the suggested uniaxial tension.

From Eqs. (45) and (46) we obtain the following relation

$$\frac{R_0 - R_{90}}{(R_0 + R_{90})\sigma_b^4} = \frac{1 - R_{90}}{(1 + R_{90})\sigma_{90}^4} - \frac{1 - R_0}{(1 + R_0)\sigma_0^4}. \tag{47}$$

Based on Eq. (47) we can obtain a calculated value of the equibiaxial-tension yield stress while the parameters R_0, R_{90}, σ_0, σ_{90} are known. The main limitation in using this method is that the values of strain ratios must obey, $R_0 \neq R_{90}$. Several calculated results and the associated experimental data are given in Table 4, which shows that the calculated results are very close to the experimental results.

9. Conclusions

To define a general representation of the stress tensor with respect to the anisotropic materials is the primary part in describing the yield state with considering the influence of both the magnitude of loading force and the loading direction. The proposed three-yield-system hypothesis made the representation of the stress tensor simply and practically. For a general anisotropic material, a minimum of twelve anisotropic coefficients needs to be determined by using experimental data and rational relations. By adapting the interpretation of the invariants of the stress tensor, an anisotropic criterion was proposed with a general type (Eq. (10)). The proposed anisotropic yield criterion can be reduced to a relatively simple and analytical form when applied to rolled sheet metals under a 3-D general stress state. Predicted yield surfaces with the specified model can be verified by using experimental results for both the yield stress and the strain ratio in uniaxial tension at 0°, 45°, 90° against the rolling direction and the yield stress in equibiaxial tension. It was also found that the predicted yield surfaces were in fairly good agreement with experimental data obtained under other stress states. In situations where experimental data for equibiaxial tension were not available, a method was introduced that can be used to calculate the yield stress. The proposed method was then compared with several experimental results and showed acceptable correlation.

The characterized yield criterion can be applied to high strength steels, aluminum alloys, and mild steels. Because its convexity is proven, it can be implemented effectively in finite element programs for the analysis and evaluation of product baseline and forming processes. The proposed yield function on its general form can also be characterized by other experimental results and applied to different anisotropic materials.

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