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PLASTIC BEHAVIOR AND STRETCHABILITY OF SHEET METALS. PART I: A YIELD FUNCTION FOR ORTHOTROPIC SHEETS UNDER PLANE STRESS CONDITIONS

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Abstract — A yield function that describes the behavior of orthotropic sheet metals exhibiting planar anisotropy and subjected to plane stress conditions is proposed. It is shown to give a reasonable approximation to plastic potentials calculated with the Taylor/Bishop and Hill theory of polycrystalline plasticity for plane stress states. Therefore, this formulation should be particularly useful to study, at a low degree of complexity, the influence of polycrystalline textures on the forming performances of metal sheets. In part II, this yield function will be used to study the influence of the yield surface shape on failure behavior of sheet metals.

I. INTRODUCTION

In order to simulate forming processes, the mechanics of continuua offers several general equations that apply to any medium. However, the constitutive equations are valid only for a particular class of materials. These equations account for the intrinsic nature of the materials and can be applied only under given conditions. Often, in sheet metal forming problems at room temperature, the classical flow theory of plasticity is employed. This theory assumes the existence of a plastic potential, usually identified with the yield surface, which must be convex. Therefore, strains are related to stresses through the normality rule. However, the yield surface of sheet metals can be quite different according to the nature of the material and its degree of anisotropy.

Phenomenological yield functions have been postulated in order to represent the behavior of metals. The Von Mises and Tresca criteria have been most commonly used in isotropic cases, but there is an infinite number of other isotropic criteria. For instance, using the TAYLOR [1938]/BISHOP & HILL [1951] model based on polycrystalline plasticity, the yield surface of isotropic bcc and fcc metals, calculated with the hypotheses of pencil glide and restricted glide respectively, are different from each other and lie between the Tresca and the Von Mises yield surfaces. This indicates that the stress/strain relations are different for isotropic bcc, fcc, Tresca, and Von Mises materials. If this difference has not a drastic consequence for certain applications, it is primordial in others, particularly in problems dealing with instabilities (BARLAT [1987]).

It has been shown recently that the predicted limit strains before necking for biaxially stretched sheets are strongly dependent on the yield surface shape (BARLAT [1987]).

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Limit strains computed with Tresca or Von Mises yield functions are unrealistic whereas those computed with the bcc and the fcc isotropic yield surfaces are consistent with the general experimental trends. In addition, though these materials are isotropic, their nature (bcc or fcc), discriminated by the types of slip systems, leads to a significant difference in the biaxial stretching limits. According to BARLAT [1987], when the work-hardening ability is identical, isotropic bcc sheet metals have a better stretchability than fcc sheets.

It has to be mentioned that the previously described simulations have been carried out with the assumption of isotropic work-hardening, that is, with an expansion of the yield surface without distortion throughout straining. Since this hypothesis is the simplest possible and since no other assumption has been proven to be much better, at least for moderate strains like those involved in forming of sheet parts, this assumption will be adopted here. Several anisotropic yield functions associated with the isotropic work-hardening expansion rule have been postulated: HILL [1948] and [1979], HERSHEY [1954], HOSFORD [1972], BASSANI [1977], GOTOH [1977], LOGAN & HOSFORD [1980], JONES & GILLIS [1984], BUDIANSKI [1984]. They have been reviewed by BARLAT [1987] and have been shown to be incomplete or inaccurate for representing the general case of sheet exhibiting planar anisotropy, even in plane stress conditions. These yield functions are usually expressed in the axes of orthotropic symmetry (three mutually orthogonal planes of symmetry which are the rolling, the transverse and the normal directions in rolled sheets). Since anisotropic properties are mainly due to the existence of preferred grain orientations, some of these postulated yield functions have been obtained as best fits of yield surfaces for textured sheets calculated with polycrystalline plasticity models. This approach has been taken recently by LEQUEU *et al.* [1987] and it gives predictions of plastic flow properties in good agreement with experimental results. However, the yield functions are expressed in the local axes of the main crystal orientations and, in order to express the yield surface of the whole aggregate, it is necessary to transform each locally calculated yield surface into the global axes of orthotropic symmetry.

So far, because of their simplicity, Von Mises' criterion for isotropic materials and HILL's (1948) criterion for anisotropic orthotropic materials have been the ones most used to describe the plastic behavior of sheet metals in analytical or numerical simulations of forming processes. Moreover, Hill's formulation is able to describe the full behavior of orthotropic materials, in contrast to many others. In plane stress conditions and for the case of in-plane anisotropy, this yield function, expressed in the axes of orthotropic symmetry, does contain a shear stress component. This is important since without the shear component, a yield function is restricted to described only planar isotropy or cases where principal stress and anisotropy axes are superimposed.

In a previous work (BARLAT [1987], BARLAT & RICHMOND [1987]), the yield surfaces of textured polycrystalline sheets were calculated with the Taylor/Bishop and Hill model and used in conjunction with a MARCINIAK & KUCZYNSKY [1967] analysis to predict forming limits and necking directions in thin sheets. The predicted results were found to be in much better agreement with experimental trends than those predicted with current postulated yield functions. Therefore, in the present article, we propose a simple yield condition which is able to represent the full plane stress state (two normal stresses and one shear stress), which is expressed in the axes of orthotropic symmetry and which is able to closely describe the yield surfaces calculated from the Bishop and Hill model. First, the formulation is established for the isotropic case, using stress tensor invariants. Then, it is extended for the cases of planar isotropy and planar anisotropy.

II. A YIELD FUNCTION FOR ISOTROPY AND PLANAR ISOTROPY

LOGAN & HOSFORD [1980] have shown that the yield function:

$$f = |\sigma_1 - \sigma_3|^M + |\sigma_3 - \sigma_2|^M + |\sigma_2 - \sigma_1|^M = 2\bar{\sigma}^M \tag{1}$$

proposed by HERSHEY [1954], HOSFORD [1972), and HILL [1979] is able to closely represent the yield surface of isotropic bcc and fcc sheet metals, calculated with the Bishop and Hill model when $M = 6$ or 8, respectively. M is a material parameter and $\bar{\sigma}$ is the effective stress identified with the uniaxial flow stress. A yield function is convex if the quadratic form defined by its Hessian matrix:

$$H_{ij} = \frac{\partial^2 f}{\partial\sigma_i \partial\sigma_j} \tag{2}$$

is positive, semi-definite (EGGLESTON [1985], ROCKAFELLAR [1972]). It has been shown that the yield function defined by eqn (1) is convex (LEQUEU *et al.* [1987]). Now, expressing this yield function in a different reference frame x, y, z and using the plane stress assumption, it follows (BARLAT & RICHMOND [1987]):

$$f = |K_1 + K_2|^M + |K_1 - K_2|^M + |2K_2|^M = 2\bar{\sigma}^M$$

$$K_1 = \frac{\sigma_{xx} + \sigma_{yy}}{2} \tag{3}$$

$$K_2 = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \sigma_{xy}^2}$$

where K_1 and K_2 are stress tensor invariants. Though the function given by eqn (1) is convex in principal stress space, the convexity of the function given by eqn (3) is not obvious in the $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ space. Therefore, the convexity condition has been verified (appendix I).

HOSFORD [1985] has unsuccessfully tried to add a shear stress component $|\sigma_{12}|^M$ in eqn (1) in order to describe a full plane stress state, but since the formulation was not written in terms of stress tensor invariants, he was only able to obtain an isotropic formulation for the particular case $M = 2$ (Von Mises criterion). In addition, there was no coupling between shear and normal stress components. However, BARLAT [1987] and BARLAT & RICHMOND [1987] have calculated the tricomponent plane stress yield surface for isotropic fcc metals with the Bishop and Hill model and it appears that a coupling should exist between shear and normal components. This is illustrated in Fig. 1 which shows the section of the yield surface with planes parallel to the $\sigma_{xx}/\bar{\sigma}, \sigma_{yy}/\bar{\sigma}$ plane for different values of $S = \sigma_{xy}/\bar{\sigma}$. Since, these curves (full lines) do not exhibit the same shape, a coupling between shear and normal components must exist in any formulation attempting to describe this yield surface, like formulation (3).

Figure 2 shows the tricomponent yield surface calculated from eqn (3) for $M = 8$. It appears that the yield surfaces of Figs. 1 and 2 are perfectly coincident, which is particularly interesting since the first one has been calculated on the basis of polycrystalline plasticity while the second has been established on the basis of a phenomenological

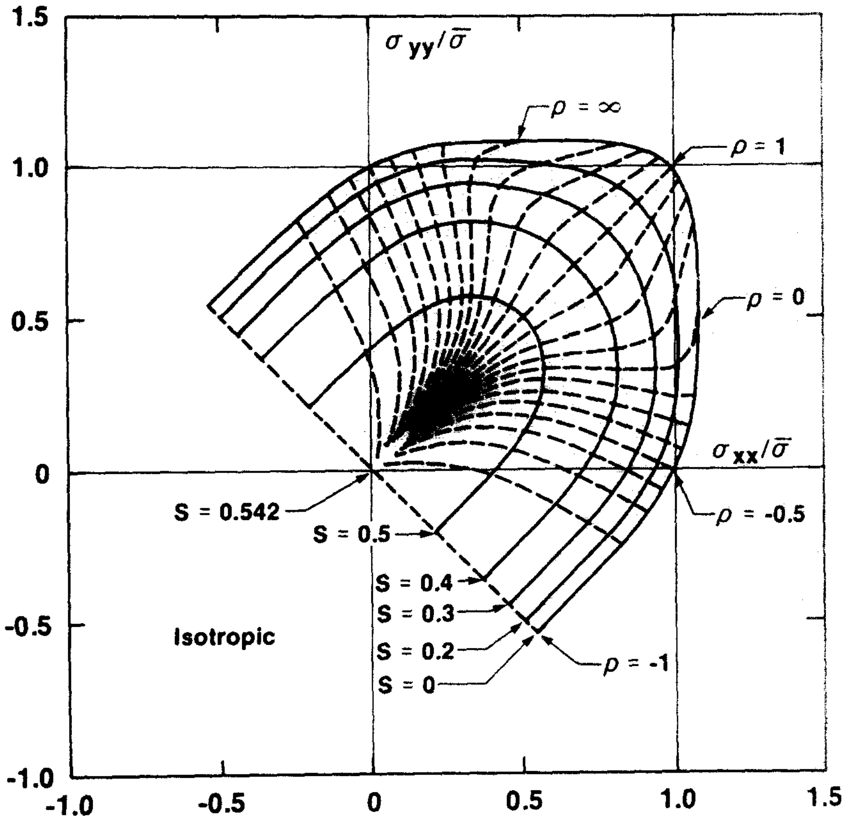

Fig. 1. Tricomponent plane stress yield surface $(\sigma_{xx}/\bar{\sigma},\sigma_{yy}/\bar{\sigma},\sigma_{xy}/\bar{\sigma})$ for isotropic fcc sheet calculated with the Bishop and Hill model. The solid line represents the section of the yield surface by a plane parallel to $(\sigma_{xx}/\bar{\sigma},\sigma_{yy}/\bar{\sigma})$ for different value $S = \sigma_{xy}/\bar{\sigma}$ and projected onto the $(\sigma_{xx}/\bar{\sigma},\sigma_{yy}/\bar{\sigma})$ plane. Dashed lines represent the projection onto $(\sigma_{xx}/\bar{\sigma},\sigma_{yy}/\bar{\sigma})$ of curves defined by points having the same strain ratio $\rho(\dot{\epsilon}_{yy}/\dot{\epsilon}_{xx})$.

representation using stress tensor invariants. The formulation described by the set of eqn (3) is extended to the anisotropic case by adding some coefficients a, b, and c that characterize the degree of anisotropy:

$$f = a|K_1 + K_2|^M + b|K_1 - K_2|^M + c|2K_2|^M = 2\bar{\sigma}^M. \tag{4}$$

K_1 and K_2 being unchanged, this equation can only describe planar isotropy unless a, b, and c are functions themselves of the three stress components. In this case, formulation (4) would lose a relative simplicity and the convexity requirement would not be easy to check. Therefore, a, b, and c will be taken as constant parameters. The continuity of the derivatives of this function for equal biaxial stresses is obtained when:

$$a = b. \tag{5}$$

Therefore, f can be split into two parts:

$$f = ag_1 + cg_2, \tag{6}$$

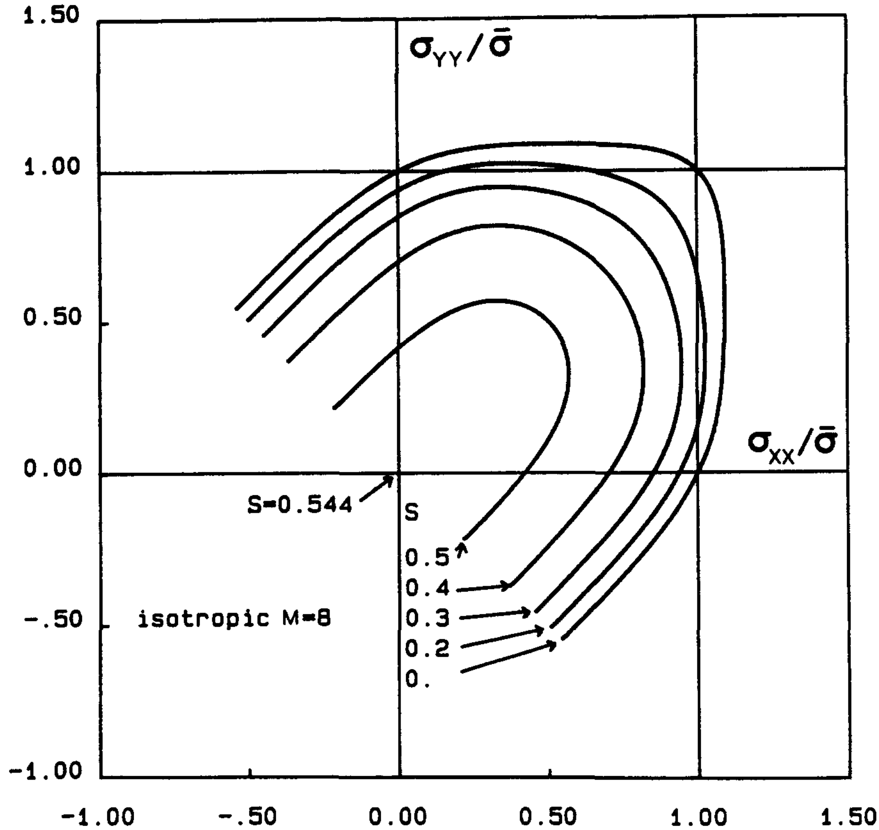

Fig. 2. Tricomponent plane stress yield surface obtained with eqn (3) for $M = 8$.

where it can be shown using the Hessian matrix that g_1 and g_2 are convex functions when M is larger than 1 (appendix I). Therefore, f also is a convex function if a and c are positive numbers (EGGLESTON [1958]; ROCKAFELLAR [1972]). Actually, a simple calculation leads to:

$$a = b = 2 - c = \frac{2}{1 + R}, \tag{7}$$

where the R value (width to thickness strain rate ratio in uniaxial tension) is a positive number. Consequently, a and c are positive and f is convex. So far, formulations (3) and (4) do not provide a substantial improvement compared to formulation (1) since for these cases, it is always possible to choose a reference frame where the shear stress vanishes. However, by introducing coefficients in K_1 and K_2, it is possible to obtain a yield function for planar anisotropy that includes a shear stress term.

III. A YIELD FUNCTION FOR PLANAR ANISOTROPY

The tricomponent plane stress yield surfaces of fcc sheet metals have been calculated for various textures (BARLAT & RICHMOND [1987]). It appears that those surfaces can be roughly approximated by a dilatation of the normalized isotropic surface in one or both directions $\sigma_{yy}/\bar{\sigma}$ and $\sigma_{xy}/\bar{\sigma}$. Therefore, a simple yield function which describes planar anisotropy for a full plane stress state can be written in the following manner:

$$f = a|K_1 + K_2|^M + a|K_1 - K_2|^M + c|2K_2|^M = 2\bar{\sigma}^M$$

$$K_1 = \frac{\sigma_{xx} + h\sigma_{yy}}{2} \tag{8}$$

$$K_2 = \sqrt{\left(\frac{\sigma_{xx} - h\sigma_{yy}}{2}\right)^2 + p^2\sigma_{xy}^2}$$

where a, c, h, and p are material constants. This new function is obtained from eqn (3) after a linear transformation of the components σ_{xx}, σ_{yy}, and σ_{xy}. Therefore, this function still obeys the convexity requirement (appendix I). When M equals 2 eqn (8) reduces to Hill's (1948) criterion.

Actually, in eqn (8) a relationship between a and c is needed in order to identify $\bar{\sigma}$ with the yield stress in the rolling direction. So, if M is assumed to be known, only three parameters are independent. Two simple methods can be used to determine the value of a, c, h, and p. In the first, yield stresses calculated by the Bishop and Hill model for different loading conditions are used. For instance, if σ_{90}, τ_{S2}, and τ_{S1} are the yield stresses for uniaxial tension in the transverse direction, shear such that $\sigma_{yy} = -\sigma_{xx} = \tau_{S2}$, $\sigma_{xy} = 0$ and shear such that $\sigma_{xx} = \sigma_{yy} = 0$, $\sigma_{xy} = \tau_{S1}$, then:

$$a = 2 - c = \frac{2\left(\dfrac{\bar{\sigma}}{\tau_{S2}}\right)^M - 2\left(1 + \dfrac{\bar{\sigma}}{\sigma_{90}}\right)^M}{1 + \left(\dfrac{\bar{\sigma}}{\sigma_{90}}\right)^M - \left(1 + \dfrac{\bar{\sigma}}{\sigma_{90}}\right)^M}$$

$$h = \frac{\bar{\sigma}}{\sigma_{90}} \tag{9}$$

$$p = \frac{\bar{\sigma}}{\tau_{S1}}\left(\frac{2}{2a + 2^M c}\right)^{1/M}.$$

Of course, if the yield function has to be utilized mainly in the biaxial range, it would be better to use a biaxial flow stress instead a shear flow stress for the calculation of the coefficients.

The second method for determining a, c, h, and p is to use R values obtained from uniaxial tension tests in three different directions, for instance R_0, R_{45}, R_{90}. This method has the advantage of taking R values which are widely used parameters and which are obtained from a simple test on real materials or readily calculated from polycrystalline models.

Associated flow rule and R value calculations are written in Appendix II. From those, it is shown that a, c, and h depend on R_0 and R_{90} only.

$$a = 2 - c = 2 - 2\sqrt{\frac{R_0}{1 + R_0}\,\frac{R_{90}}{1 + R_{90}}}$$
$$h = \sqrt{\frac{R_0}{1 + R_0}\,\frac{1 + R_{90}}{R_{90}}}\,. \tag{10}$$

p can not be calculated analytically. However, when a, c, and h are known, a relationship between R value for uniaxial tension in a direction making an angle ϕ with the rolling direction and p does exist. It is difficult to prove that this relationship is a monotonic function but the many examples simulated so far when $\phi = 45°$ have shown that the R value is always an increasing function of p. Hence, to a given R_{45} value, determined experimentally or calculated from a polycrystalline yield surface, there is a corresponding unique value of p. A graphical or a numerical method is used to obtain p. Also, p could be calculated from the shear flow stress $\tau_{S1}/\bar{\sigma}$ (eqn 9) since a, c, and h are calculated first.

Figure 3 presents the tricomponent plane stress yield surface calculated with the Taylor/Bishop and Hill model for a generated texture consisting of 50% of a gaussian distribution of grains around the $\{1\bar{1}0\}\langle 112\rangle$ ideal orientation (brass texture) and 50% of

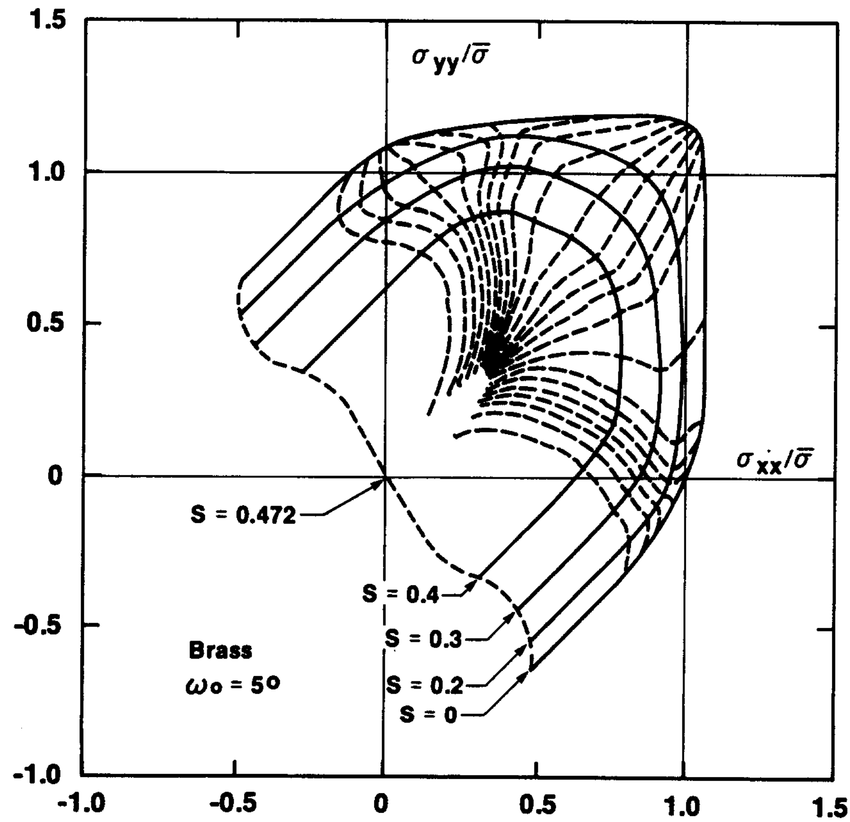

Fig. 3. Tricomponent plane stress yield surface calculated with the Bishop and Hill model for a material containing 50% brass texture and 50% randomly distributed grains (Barlat & Richmond [1987]).

randomly distributed grains (BARLAT & RICHMOND [1987]). The scatter width of the distribution is small and consequently the texture is quite strong. Therefore, the yield surface of Fig. 3 is quite anisotropic. It is elongated in the $\sigma_{yy}/\bar{\sigma}$ direction whereas it is contracted in the $\sigma_{xy}/\bar{\sigma}$ direction when compared to the isotropic fcc yield surface of Fig. 1. Moreover, near balanced biaxial stress conditions, the surface presents a small radius of curvature. In order to calculate the coefficients a, c, h, and p, a relatively high M value is first selected ($M = 14$) to obtain this small radius of curvature. a, c, and h are calculated from eqns (10) with R_0 and R_{90} equal to 0.7 and 1.0, respectively.

In order to obtain p, a compromise between yield surface approximation and R value variation according to the tension angle is adopted. This leads to take $R_{45} = 5$ though the value obtained from the Bishop and Hill yield surface is about 7. However, such a high value is not compatible with the contraction of the yield surface in the $\sigma_{xy}/\bar{\sigma}$ direction. This underlines the fact that the use of R values or yield stresses to calculate a, c, h, and p is not equivalent. Figure 4 shows the yield surface for the brass texture approximated by the proposed yield function. The similarity of this surface with the Bishop and Hill yield surface (Fig. 3) is quite good though some discrepancy still exists. Figure 5 represents the Hill's (1948) yield surface ($M = 2$) whose parameters have been

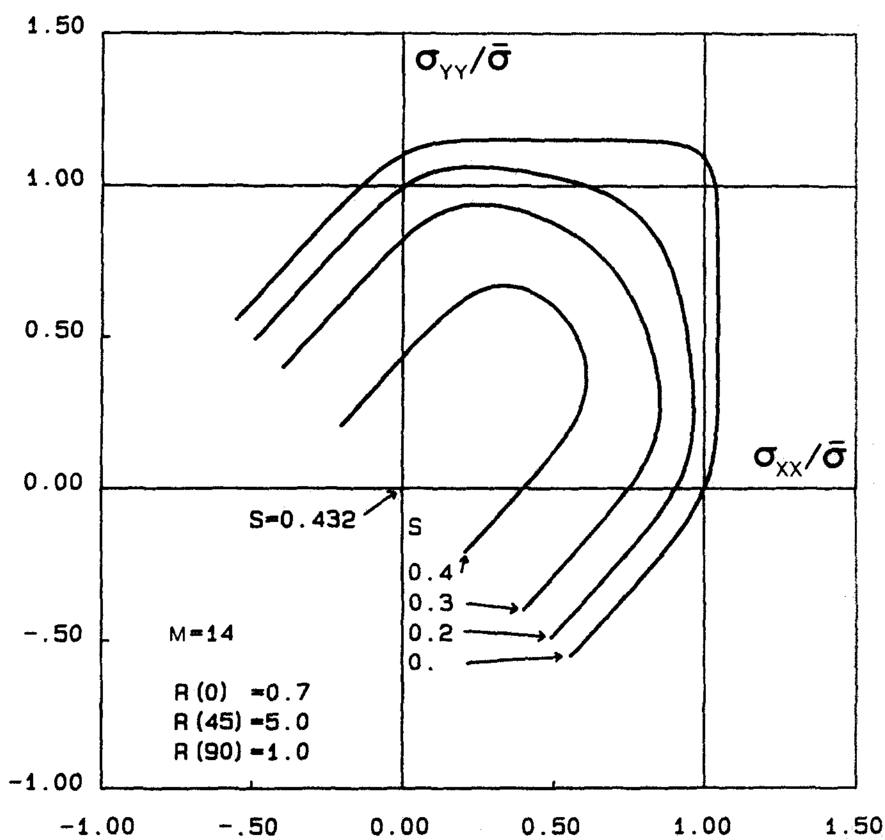

Fig. 4. Tricomponent plane stress yield surface obtained with eqn (8) with $M = 14$. a, c, h, and p are calculated with $R_0 = 0.7$, $R_{45} = 5$, $R_{90} = 1$.

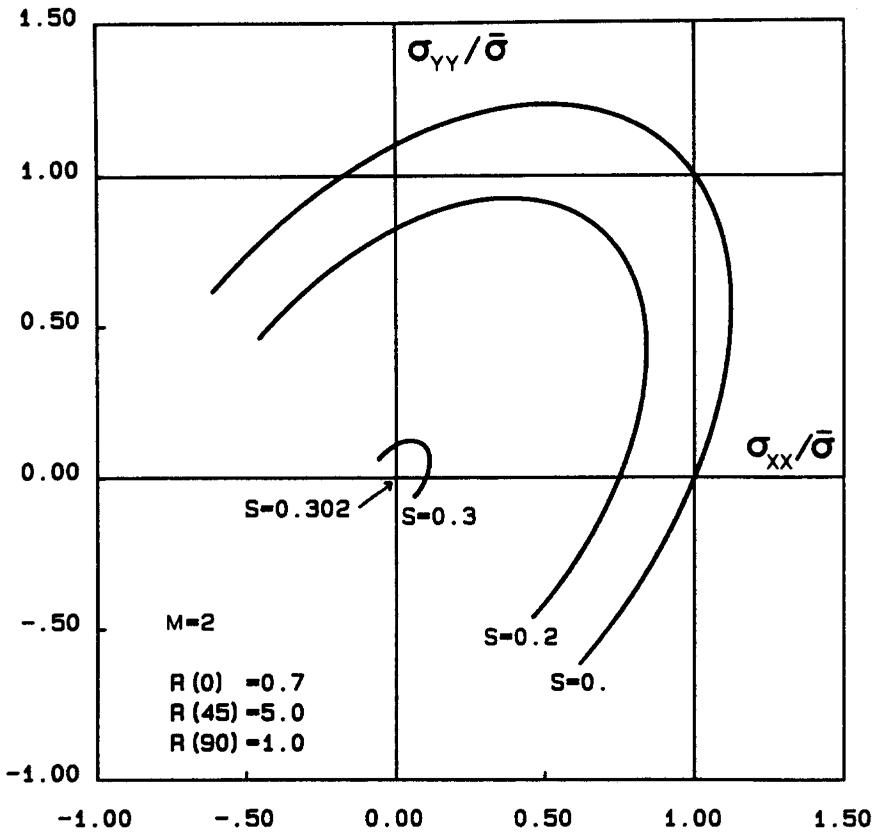

Fig. 5. Tricomponent plane stress yield surface obtained with eqn (8) with $M = 2$ (corresponds to Hill's 1948 yield function). a, c, h, and p are calculated with $R_0 = 0.7$, $R_{45} = 5$, $R_{90} = 1$.

identified with the same R values. In this case, the yield surface does not exhibit a rounded vertex near the equal biaxial stress range and the maximum value of the shear flow stress $S = \sigma_{xy}/\bar{\sigma}$ is only 0.302. Similarities with the Bishop and Hill yield surface are far less evident.

Figure 6 shows the variation of the R value according to the tension direction calculated with the three yield surfaces (Figs. 3–5); as mentioned previously, the R value variation has a larger amplitude when it is predicted with the Bishop and Hill model. Nevertheless, the trends given by the yield function for $M = 2$ and $M = 14$ is good. It is worth noting that the R value variation calculated with the yield function is not affected very much by a variation of M. However, Fig. 7 shows that the uniaxial yield stress variation is largely dependent on M and that it is in better agreement with the yield stress variation predicted by the Bishop and Hill model when M equals 14.

IV. CONCLUSION

The yield function proposed in this article is able to describe the behavior of orthotropic sheet metals under a full plane stress state (two normal stresses and one shear

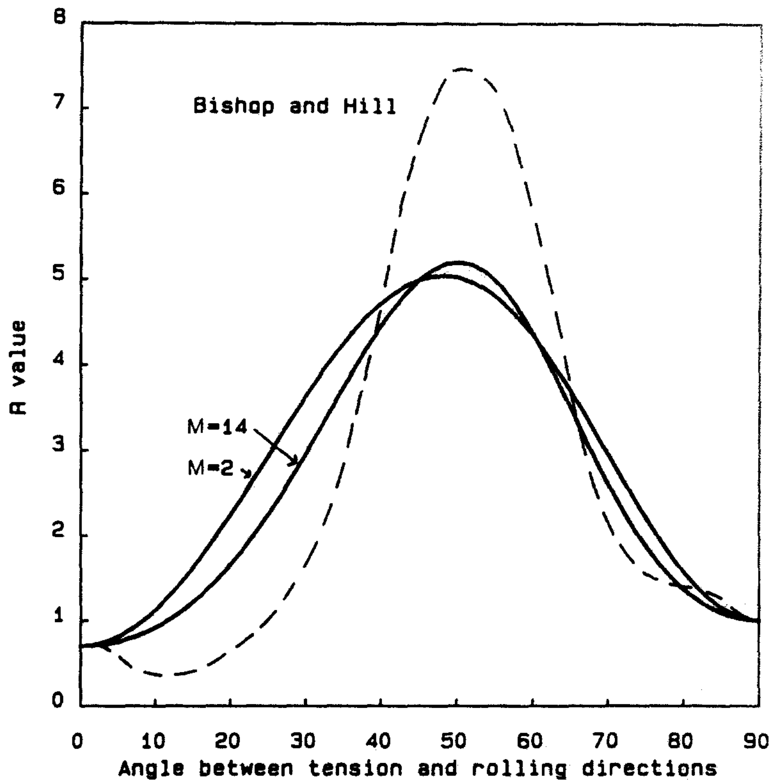

Fig. 6. R value variation according to the tension direction predicted with the yield surfaces of Figs. 3, 4, and 5.

stress). When this function reduces to a particular case of isotropy ($M = 8$), it describes very well the yield surface for fcc isotropic sheet metals under plane stress calculated by the Bishop and Hill model of polycrystalline plasticity. In the case of strongly textured sheets, the proposed yield function is not as accurate as in the isotropic case though the general shape of the postulated yield surface is in good agreement with polycrystalline calculations. The advantage of the proposed yield function is that it takes a full plane stress state and planar anisotropy into account. In addition, it gives an approximate representation of polycrystalline yield surfaces, particularly in a region with a small radius of curvature near the balanced biaxial stress range.

Therefore, this function could be used in plane stress problems, like the bulging of planar anisotropic sheets (YANG & KIM [1987]), to evaluate the effect of yield surface shapes and more particularly, the influence of rounded vertices on the behavior of materials during forming processes. In part II, the influence of the shape of tricomponent yield surfaces on the forming limits of thin sheets will be assessed, using the yield function postulated in this work.

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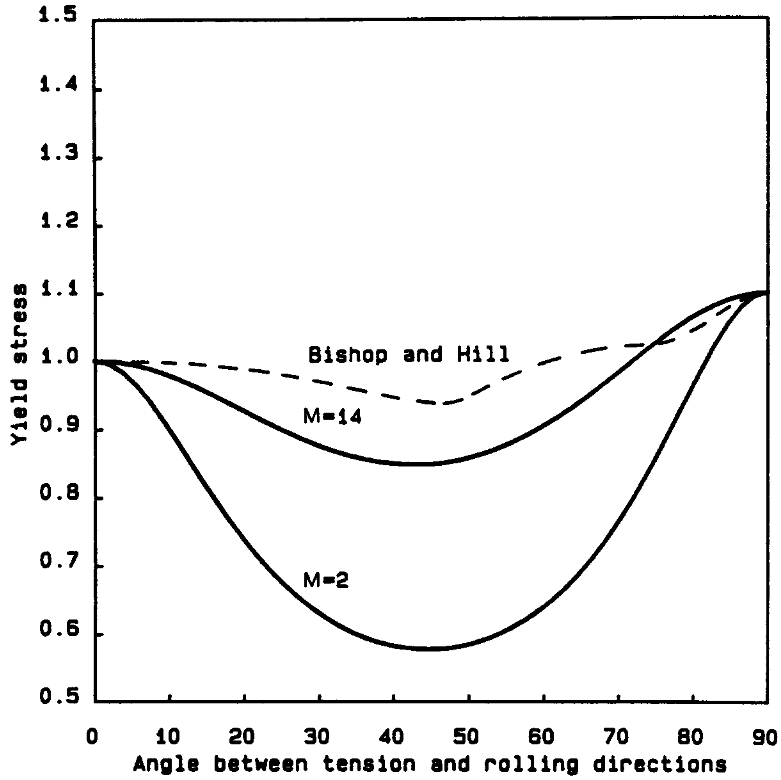

Fig. 7. Yield stress variation according to the tension direction predicted with the yield surfaces of Figs. 3, 4, and 5.

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NOMENCLATURE

a, c, h, p	= coefficients in a yield function
a_0, a_1, a_2, a_3	= coefficients of a third order polynomial function
$f(K_1, K_2)$	= yield function
g, g_1, g_2	= functions of K_1 and K_2
H, H_{ij}	= Hessian matrix and components
K_1, K_2	= functions of plane stress components (plane stress invariants in isotropic cases)
M	= coefficient in a yield function (exponent)
$T(t)$	= third order polynomial function
t_1, t_2, t_3	= roots of $T(t) = 0$
R, R_ϕ	= R value: width to thickness strain rate ratio in uniaxial tension in a direction making an angle ϕ with the rolling direction
$S = \sigma_{xy}/\bar{\sigma}$	= normalized shear stress
x, y, z	= orthotropic axes (rolling, transverse and normal directions)
$\dot{\epsilon}_{xx}, \dot{\epsilon}_{yy}, \dot{\epsilon}_{zz}, \dot{\epsilon}_{xy}$	= strain rates expressed in the orthotropic axes and resulting from a plane stress state
$\dot{\epsilon}_{11}, \dot{\epsilon}_{22}, \dot{\epsilon}_{33}, \dot{\epsilon}_{12}$	= strain rates resulting from uniaxial tension in direction 1 (longitudinal, transverse, normal and shear strain rates, respectively)
$\rho = \dot{\epsilon}_{yy}/\dot{\epsilon}_{xx}$	= strain path
$\dot{\lambda}$	= proportionality coefficient in a flow rule
ϕ	= angle between uniaxial tension and rolling directions
$\sigma_1, \sigma_2, \sigma_3$	= principal stresses
$\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$	= plane stress components in orthotropic axes
σ	= uniaxial yield stress
$\bar{\sigma}$	= effective stress – uniaxial yield stress in the rolling direction
σ_{90}	= uniaxial yield stress in the transverse direction
τ_{S1}	= shear yield stress (when $\sigma_{xx} = \sigma_{yy} = 0$)
τ_{S2}	= shear yield stress (when $\sigma_{yy} = -\sigma_{xx}, \sigma_{xy} = 0$)

APPENDIX I: CONVEXITY OF THE YIELD FUNCTION

Let us consider the function $g(K_1, K_2)$ where K_1 and K_2 are defined by:

$$K_1 = X_1$$
$$K_2 = \sqrt{X_2^2 + X_3^2}, \tag{I.1}$$

then g is a convex function in the X_1, X_2, X_3 space if the quadratic form associated with its Hessian matrix H is positive semi-definite (EGGLESTON [1958]; ROCKAFELLAR [1972]), that is, if the eigenvalues of H are non-negative, where:

$$H_{ij} = \frac{\partial^2 g}{\partial X_i \partial X_j}. \tag{I.2}$$

In this particular case, the Hessian matrix is given by:

$$\begin{aligned}
H_{11} &= \frac{\partial^2 g}{\partial K_1^2} \\
H_{21} = H_{12} &= \frac{\partial^2 g}{\partial K_1 \partial K_2} \frac{X_2}{K_2} \\
H_{31} = H_{13} &= \frac{\partial^2 g}{\partial K_1 \partial K_2} \frac{X_3}{K_2} \\
H_{22} &= \frac{\partial^2 g}{\partial K_2^2} \frac{X_2^2}{K_2^2} - \frac{\partial g}{\partial K_2} \frac{X_3^2}{K_2^3} \\
H_{32} = H_{23} &= \frac{\partial^2 g}{\partial K_2^2} \frac{X_2 X_3}{K_2^2} - \frac{\partial g}{\partial K_2} \frac{X_2 X_3}{K_2^3} \\
H_{33} &= \frac{\partial^2 g}{\partial K_2^2} \frac{X_3^2}{K_2^2} + \frac{\partial g}{\partial K_2} \frac{X_2 X_3}{K_2^3}.
\end{aligned} \tag{I.3}$$

The eigenvalues of H are the roots of the characteristic third order polynominal function:

$$T(t) = a_3 t^3 + a_2 t^2 + a_1 t + a_0 = 0 \tag{I.4}$$

where

$$\begin{aligned}
a_3 &= -1 \\
a_2 &= \frac{\partial^2 g}{\partial K_1^2} + \frac{\partial^2 g}{\partial K_2^2} + \frac{1}{K_2} \frac{\partial g}{\partial K_2} \\
a_1 &= \left(\frac{\partial^2 g}{\partial K_1 \partial K_2} \right)^2 - \frac{\partial^2 g}{\partial K_1^2} \frac{\partial^2 g}{\partial K_2^2} - \frac{1}{K_2} \frac{\partial g}{\partial K_2} \left(\frac{\partial^2 g}{\partial K_1^2} + \frac{\partial^2 g}{\partial K_2^2} \right) \\
a_0 &= \frac{1}{K_2} \frac{\partial g}{\partial K_2} \left[\frac{\partial^2 g}{\partial K_1^2} \frac{\partial^2 g}{\partial K_2^2} - \left(\frac{\partial^2 g}{\partial K_1 \partial K_2} \right)^2 \right].
\end{aligned} \tag{I.5}$$

Using the relationship between the roots and the coefficients of equation (I.4), it follows that:

$$t_1 + t_2 + t_3 = -a_2/a_3 = a_2$$
$$t_1 t_2 + t_2 t_3 + t_3 t_1 = a_1/a_3 = -a_1 \tag{I.6}$$
$$t_1 t_2 t_3 = -a_0/a_3 = a_0.$$

In order to get three non-negative roots, the following relationship must exist:

$$a_0 \gtrsim 0$$
$$a_1 \lesssim 0 \tag{I.7}$$
$$a_2 \gtrsim 0.$$

These inequalities are respected for M larger than 1 in the case of the following functions:

$$g = g_1 = |K_1 + K_2|^M + |K_1 - K_2|^M$$
$$g = g_2 = |2K_2|^M. \tag{I.8}$$

Therefore, g_1 and g_2 are convex functions in the space X_1, X_2, and X_3. Moreover, the following linear transformation:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1/2 & h/2 & 0 \\ 1/2 & -h/2 & 0 \\ 0 & 0 & p \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{bmatrix} \tag{I.9}$$

lets g_1 and g_2 still be convex functions in the $\sigma_{xx}, \sigma_{yy}, \sigma_{xy}$ space (EGGLESTON [1958], ROCKAFELLAR [1972]) when M is larger than 1. In conclusion, the proposed anisotropic yield function f (eqn 8):

$$f = ag_1 + cg_2 \tag{I.10}$$

is a convex function if a and c are positive constant coefficients.

APPENDIX II: ASSOCIATED FLOW RULE AND R VALUE

The flow rule associated with the anisotropic yield function given by eqn (8) is:

$$\dot{\epsilon}_{xx} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{xx}} = \dot{\lambda}M\left\{a(K_1 - K_2)|K_1 - K_2|^{M-2}\left(\frac{1}{2} - \frac{\sigma_{xx} - h\sigma_{yy}}{4K_2}\right)\right.$$
$$+ a(K_1 + K_2)|K_1 + K_2|^{M-2}\left(\frac{1}{2} + \frac{\sigma_{xx} - h\sigma_{yy}}{4K_2}\right)$$
$$\left. + 2^M c K_2^{M-1}\frac{\sigma_{xx} - h\sigma_{yy}}{4K_2}\right\}$$

$$\dot{\epsilon}_{yy} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{yy}} = \dot{\lambda}M\left\{a(K_1 - K_2)|K_1 - K_2|^{M-2}\left(\frac{h}{2} + h\frac{\sigma_{xx} - h\sigma_{yy}}{4K_2}\right)\right. \tag{II.1}$$

$$+ a(K_1 + K_2)|K_1 + K_2|^{M-2}\left(\frac{h}{2} - h\frac{\sigma_{xx} - h\sigma_{yy}}{4K_2}\right)$$

$$\left. - 2^M c K_2^{M-1} h \frac{\sigma_{xx} - h\sigma_{yy}}{4K_2}\right\}$$

$$\dot{\epsilon}_{xy} = \dot{\lambda}\frac{\partial f}{\partial \sigma_{xy}} = \dot{\lambda}M\{a(K_1 + K_2)|K_1 + K_2|^{M-2} - a(K_1 - K_2)|K_1 - K_2|^{M-2}$$

$$+ 2^M c K_2^{M-1}\} p^2 \frac{\sigma_{xy}}{2K_2}.$$

For uniaxial tension σ in a direction 1 inclined at ϕ from the rolling direction, tensor transformations lead to:

$$\begin{aligned} \sigma_{xx} &= \sigma \cos^2\phi \\ \sigma_{yy} &= \sigma \sin^2\phi \\ \sigma_{xy} &= \sigma \sin\phi \cos\phi \end{aligned} \tag{II.2}$$

and

$$\begin{aligned} \dot{\epsilon}_{11} &= \cos^2\phi\dot{\epsilon}_{xx} + \sin^2\phi\dot{\epsilon}_{yy} + 2\sin\phi\cos\phi\dot{\epsilon}_{xy} \\ \dot{\epsilon}_{22} &= \sin^2\phi\dot{\epsilon}_{xx} + \cos^2\phi\dot{\epsilon}_{yy} - 2\sin\phi\cos\phi\dot{\epsilon}_{xy}, \end{aligned} \tag{II.3}$$

where $\dot{\epsilon}_{11}$ and $\dot{\epsilon}_{22}$ are the longitudinal and width strain rate in uniaxial tension, respectively. Using plastic incompressibility, we can express the R value (width to thickness strain rate) for any angle ϕ:

$$R_\phi = -\frac{\dot{\epsilon}_{22}}{\dot{\epsilon}_{11} + \dot{\epsilon}_{22}} = \frac{\dot{\epsilon}_{11}}{\dot{\epsilon}_{11} + \dot{\epsilon}_{22}} - 1 = \frac{\dot{\epsilon}_{11}}{\dot{\epsilon}_{xx} + \dot{\epsilon}_{yy}} - 1. \tag{II.4}$$

Using Euler's theorem on homogeneous functions, we can write:

$$\frac{\sigma(\cos^2\phi\dot{\epsilon}_{xx} + \sin^2\phi\dot{\epsilon}_{yy} + 2\sin\phi\cos\phi\dot{\epsilon}_{xy})}{\dot{\lambda}} = Mf, \tag{II.5}$$

and consequently:

$$\dot{\epsilon}_{11} = \frac{\dot{\lambda}Mf}{\sigma} = \frac{2\dot{\lambda}M\bar{\sigma}^M}{\sigma}. \tag{II.6}$$

Therefore the R_ϕ value is given by:

$$R_\phi = \frac{\dfrac{\dot{\lambda} M f}{\sigma}}{\left(\dfrac{\partial f}{\partial \sigma_{xx}} + \dfrac{\partial f}{\partial \sigma_{yy}}\right)\dot{\lambda}} - 1 = \frac{2M\bar{\sigma}^{M}}{\left(\dfrac{\partial f}{\partial \sigma_{xx}} + \dfrac{\partial f}{\partial \sigma_{yy}}\right)\sigma} - 1. \tag{II.7}$$

Using this equation together with the yield function (eqn (8)) and the associated flow rule (eqn (II.1), it is possible to calculate the R value for any direction of tension. In particular, this set of equations is used to identify a, c, and h from R_0 and R_{90}.

$$\begin{aligned} a &= 2 - c = 2 - 2\sqrt{\frac{R_0}{1 + R_0}\,\frac{R_{90}}{1 + R_{90}}} \\ h &= \sqrt{\frac{R_0}{1 + R_0}\,\frac{1 + R_{90}}{R_{90}}}. \end{aligned} \tag{II.8}$$

a and c are always positive which is a necessary condition to achieve convexity (appendix I).

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