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CONSTITUTIVE MODELLING OF ORTHOTROPIC PLASTICITY IN SHEET METALS

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ABSTRACT

THE CLASSICAL quadratic yield criterion for orthotropic metals is known not to be sufficiently flexible in practice. By the simple expedient of admitting non-integer exponents, however, an improved criterion was devised for sheet with in-plane isotropy (so-called normal anisotropy). On the other hand, an acceptable proposal has not been forthcoming for sheet with in-plane anisotropy (so-called planar anisotropy). It is suggested here that improvement should be sought by incorporating a compatible dependence on orientation in a homogeneous yield function of arbitrary degree. In so doing, the practicalities of forming technology are respected by keeping the number of arbitrary parameters as small as possible. A new criterion is constructed along these lines and its implications are explored in detail. Additionally, a simple means of representing anisotropic yield criteria of any kind is presented with supporting general theorems.

1. INTRODUCTION

THE ORTHOTROPIC plastic response of rolled sheet is commonly modelled by a homogeneous yield function of degree two, in association with the classical flow-rule. This minimal framework was proposed long ago by the writer (HILL, 1948), since when its range of validity has been explored by numerous experiments, beginning with BOURNE and HILL (1950). The consensus is that the model is well suited to specific metals and textures, but is too inflexible generally; for an overall assessment see MELLOR (1982). However, no practical suggestion seems to have been forthcoming as to how the yield criterion might be improved, save when the loading is co-directional with the orthotropy.

In that case there is a proposal to try arbitrary linear combinations of equal *non-integer* powers of the principal values of the deviatoric stress, together with the same powers of their differences (HILL, 1979). To date, only one such combination has been considered seriously, subsequent to a persuasive investigation by PARMAR and MELLOR (1978). These and all later tests have been restricted to transversely isotropic sheet, albeit in a variety of contexts. Significant improvements on the 1948 quadratic have been reported unanimously. The evidence is thus strongly indicative that the 1979 proposal, in one form or another, offers sufficient flexibility when the orthotropy and the loading are co-directional. More to the point, it is not so complex that it cannot be implemented with the data ordinarily available in forming technology.

Similar basic ingredients will accordingly be adopted here in devising a new framework that caters for full orthotropy and any loading orientation. That is, the yield function will be required to be (i) homogeneous overall and (ii) in the form of a sum of terms, each of which involves a non-integer power either on its own or as a factor. Such qualitative features are analytically advantageous in themselves, as will appear. Nevertheless it is not easy to incorporate within this algebraic structure a dependence on orientation that is both simple and realistic. During the investigation several types of dependence were examined in detail; all but one were rejected in the end because of some unacceptable implication.

It happened that the eventual choice was prompted by a version of the original quadratic which is probably little known. Partly for that reason, but also for general background and perspective, we begin with an extended account of the standard framework.

2. The Standard Theory of Sheet Anisotropy

Relative to coordinates along the principal directions of orthotropy, the quadratic yield criterion was originally cast in the form

$$(G+H)\sigma_x^2 - 2H\sigma_x\sigma_y + (F+H)\sigma_y^2 + 2N\sigma_{xy}^2 = 1, \tag{2.1}$$

where $(\sigma_x, \sigma_y, \sigma_{xy})$ are the in-plane components of Cauchy stress (the out-of-plane components are considered to be zero). The material parameters F, G, H, N specify the current state of anisotropy and may depend in any manner on the strain history. Bauschinger effects are ignored, on the presumption that yield is determined at an appropriate offset or by backward extrapolation after sufficient hardening.

The current yield stresses in uniaxial tension along the x and y axes will be denoted by σ_0 and σ_{90} respectively. For algebraic convenience the labels x, y are assigned in accordance with the convention that

$$\sigma_0 \geqslant \sigma_{90}, \tag{2.2}$$

and so the subscript zero is not necessarily associated with the rolling direction. The ordering $F \geqslant G$ is an immediate consequence by virtue of

$$\sigma_0 = (G+H)^{-1/2}, \quad \sigma_{90} = (F+H)^{-1/2}. \tag{2.3}$$

Yielding under equibiaxial tension occurs when σ_x and σ_y are both equal to

$$\sigma = (F+G)^{-1/2}. \tag{2.4}$$

This is also the compressive yield stress perpendicular to the sheet, on the usual assumption that all-round tension or pressure can be superimposed without affecting plastic response. At another extreme, yielding under pure shear parallel to the orthotropic axes occurs when σ_{xy} is equal to

$$\tau = (2N)^{-1/2}. \tag{2.5}$$

Also of interest are a particular pair of combined loadings co-directional with the orthotropic axes and such that

$$\left.\begin{aligned} \sigma_x &= \frac{(F+H)}{H}\sigma_y = \left(\frac{FG+GH+HF}{F+H}\right)^{-1/2}, \\ \sigma_y &= \frac{(G+H)}{H}\sigma_x = \left(\frac{FG+GH+HF}{G+H}\right)^{-1/2}. \end{aligned}\right\} \tag{2.6}$$

Under the classical flow-rule each such combination produces extension along one in-plane axis while suppressing contraction along the other.

From the preceding identifications it follows that the values of the anisotropic parameters must conform to

$$F+G > 0, \quad N > 0, \quad FG+GH+HF > 0. \tag{2.7}$$

These inequalities evidently imply that $F+H > 0$ and $G+H > 0$ also, but leave open the possibility that one of F, G and H may be negative. They are in fact necessary and sufficient conditions for the yield surface in $(\sigma_x, \sigma_y, \sigma_{xy})$ space to be strictly convex (an ellipsoid in this case).

To prepare the ground for an improved yield criterion we first rearrange (2.1) as

$$\tfrac{1}{4}(F+G)(\sigma_x+\sigma_y)^2+\tfrac{1}{4}(F+G+4H)(\sigma_x-\sigma_y)^2-\tfrac{1}{2}(F-G)(\sigma_x^2-\sigma_y^2)+2N\sigma_{xy}^2 = 1,$$

and then substitute

$$\sigma_x+\sigma_y = \sigma_1+\sigma_2, \quad \sigma_x-\sigma_y = (\sigma_1-\sigma_2)\cos 2\alpha, \quad 2\sigma_{xy} = (\sigma_1-\sigma_2)\sin 2\alpha. \tag{2.8}$$

Here σ_1 and σ_2 are the principal components of stress and are typically directed at an anticlockwise angle α to x and y respectively. Having regard to (2.4) and (2.5) the outcome is expressible as

$$\left.\begin{aligned} &(\sigma_1+\sigma_2)^2+(\sigma^2/\tau^2)(\sigma_1-\sigma_2)^2-2a(\sigma_1^2-\sigma_2^2)\cos 2\alpha+b(\sigma_1-\sigma_2)^2\cos^2 2\alpha = (2\sigma)^2 \\ &\text{where} \\ &\qquad a = (F-G)/(F+G) \geqslant 0, \quad b = (F+G+4H-2N)/(F+G) \end{aligned}\right\} \tag{2.9}$$

are dimensionless parameters that characterize the anisotropy and vanish with it. Together with the yield stresses σ and τ they replace F, G, H and N in defining the current state of the material. Equivalent to (2.7) we now have

$$\sigma^2 > 0, \quad \tau^2 > 0, \quad b > a^2-\sigma^2/\tau^2 \tag{2.10}$$

which ensure that, for each α, the corresponding yield locus in (σ_1, σ_2) space is strictly convex (an ellipse).

The change from cartesian to intrinsic variables in (2.8) was motivated by the wish to make explicit the dependence on loading orientation. Moreover the terms not involving α in (2.9) have been structured like those in the currently favoured class of yield functions for in-plane isotropy:

$$|\sigma_1+\sigma_2|^m+(\sigma^m/\tau^m)|\sigma_1-\sigma_2|^m = (2\sigma)^m \tag{2.11}$$

where m can have any value greater than unity (HILL, 1979). A comparable improvement for in-plane anisotropy will therefore be sought by adding to (2.11) a compatible dependence on orientation which maintains the homogeneity and reduces to the a, b terms in (2.9) when $m = 2$.

Before coming to that, however, some consequences of the reformulation (2.9) must be reviewed. By substituting $(\sigma_\alpha, 0)$ for (σ_1, σ_2) the tensile yield stress at any orientation α is obtained in the form

$$(2\sigma/\sigma_\alpha)^2 = 1+\sigma^2/\tau^2-2a\cos 2\alpha+b\cos^2 2\alpha. \tag{2.12}$$

An equivalent expression in the original parameters was given by HILL (1948, 1950). In particular

$$\left.\begin{aligned}(2\sigma/\sigma_0)^2 &= 1+\sigma^2/\tau^2-2a+b,\\ (2\sigma/\sigma_{90})^2 &= 1+\sigma^2/\tau^2+2a+b,\\ (2\sigma/\sigma_{45})^2 &= 1+\sigma^2/\tau^2.\end{aligned}\right\} \tag{2.13}$$

In principle, therefore, the anisotropic parameters could be determined from

$$a = (\sigma/\sigma_{90})^2-(\sigma/\sigma_0)^2, \quad b = \tfrac{1}{2}\{(2\sigma/\sigma_0)^2+(2\sigma/\sigma_{90})^2\}-(2\sigma/\sigma_{45})^2 \tag{2.14}$$

by measuring the four yield stresses involved. The particular role of b is further illustrated by the connection

$$b = (\sigma/\tau')^2-(\sigma/\tau)^2 \tag{2.15}$$

where τ' is the yield stress in shear at 45° to the orthotropic axes ($\sigma_1 = -\sigma_2 = \tau'$ with $\alpha = 0$).

The magnitude of σ_α, regarded as a function of orientation, is stationary at $\alpha = 0$ and $\frac{1}{2}\pi$ (trivially because of the material symmetry) and falls monotonically from σ_0 to σ_{90} when $|b| \leqslant a$ ($G+2H \leqslant N \leqslant F+2H$). On the other hand, when $|b| > a$ ($N < G+2H$ or $N > F+2H$), σ_α has a non-trivial extremum at $\alpha = \frac{1}{2}\cos^{-1}(a/b)$; this value is a maximum (exceeding σ_0) when $b > a$ ($N < G+2H$) and is a minimum (less than σ_{90}) when $b < -a$ ($N > F+2H$). The likely relevance of these categories to different modes of earing in deep-drawn cups was remarked by HILL (1948, 1950) and by BOURNE and HILL (1950). It was noted also that a principal axis of strain-rate coincides with the applied tension if and only if σ_α is oriented in one of the directions for which it is stationary.

Next some formulae are recorded for the strain-ratios in tension tests. The method of derivation will be given later, see (3.10), in the context of deformation under arbitrary loading. Let r_α be the ratio (in-plane transverse strain-rate)/(through-thickness strain-rate) under tension at orientation α. Then

$$\left.\begin{aligned}1+2r_\alpha &= \frac{\sigma^2/\tau^2-a\cos 2\alpha+b\cos^2 2\alpha}{1-a\cos 2\alpha},\\ 2(1+r_\alpha) &= (2\sigma/\sigma_\alpha)^2/(1-a\cos 2\alpha),\end{aligned}\right\} \tag{2.16}$$

which are variants of an expression given by HILL (1950). In particular

$$\left.\begin{aligned} 2r_0 &= (\sigma^2/\tau^2-1+b)/(1-a) \equiv H/G, \\ 2r_{90} &= (\sigma^2/\tau^2-1+b)/(1+a) \equiv H/F, \\ 2r_{45} &= \sigma^2/\tau^2-1 \equiv 2N/(F+G)-1. \end{aligned}\right\} \tag{2.17}$$

It follows that

$$r_0/r_{90} = (1+a)/(1-a), \quad a = (r_0-r_{90})/(r_0+r_{90}), \tag{2.18}$$

which provides another means of determining a experimentally.

Finally, under compression perpendicular to the sheet, the ratio (x component of strain-rate)/(y component of strain-rate) is

$$\dot{\varepsilon}_x/\dot{\varepsilon}_y = (1-a)/(1+a). \tag{2.19}$$

This offers yet another way to measure a and hence to check the internal consistency of the theory.

3. Some Preliminary Formulae

A possible approach to an improved yield criterion for in-plane anisotropy has just been formulated in terms of the principal stresses and their orientations (in preference to the more usual components relative to the orthotropic directions). Such intrinsic variables are preferable also when the flow-rule is used to generate normal and shearing components of strain-rate on the principal axes of stress. This type of decomposition is needed in particular when the deformation is viewed from the standpoint of a rectangular specimen cut from the sheet at any angle. This is the natural standpoint when analysing strain-ratios or localized necking under uniaxial or biaxial loads.

Accordingly, in preparation for a construction based on intrinsic variables, the relevant general formulae will be derived first. For this purpose the yield criterion is written non-committally as

$$\Phi(\sigma_1, \sigma_2, \alpha) = 1, \tag{3.1}$$

omitting explicit reference to material parameters. Orthotropy without Bauschinger effects requires the symmetries

$$\Phi(\sigma_1, \sigma_2, \alpha) \equiv \Phi(-\sigma_1, -\sigma_2, \alpha) \equiv \Phi(\sigma_1, \sigma_2, -\alpha) \equiv \Phi(\sigma_2, \sigma_1, \tfrac{1}{2}\pi \pm \alpha) \tag{3.2}$$

together with periodicity π in α. In particular there is symmetry in σ_1 and σ_2 when $\alpha = \frac{1}{4}\pi$. If the yield surface in $(\sigma_x, \sigma_y, \sigma_{xy})$ space is strictly convex, so is any plane section $\sigma_{xy}/(\sigma_x-\sigma_y) =$ constant; it follows from (2.8) that function Φ is necessarily strictly convex in σ_1 and σ_2 for each α. Finally the proposed homogeneity in the cartesian components implies a similar homogeneity in the principal components, whence

$$\sigma_1 \frac{\partial \Phi}{\partial \sigma_1} + \sigma_2 \frac{\partial \Phi}{\partial \sigma_2} \equiv m\Phi \tag{3.3}$$

identically for some $m > 1$ by Euler's theorem.

Components of strain-rate on the principal axes of stress are denoted by $(\dot{\varepsilon}_{11}, \dot{\varepsilon}_{22}, \dot{\varepsilon}_{12})$. With a similar decomposition of any virtual stress, the classical flow-rule asserts that

$$\dot{\varepsilon}_{11}\,\mathrm{d}\sigma_{11} + \dot{\varepsilon}_{22}\,\mathrm{d}\sigma_{22} + 2\dot{\varepsilon}_{12}\,\mathrm{d}\sigma_{12} = 0$$

for all differential increments tangential to the yield surface in $(\sigma_{11}, \sigma_{22}, \sigma_{12})$ space. Such increments are generated by

$$\mathrm{d}\sigma_{11} = \mathrm{d}\sigma_1, \quad \mathrm{d}\sigma_{22} = \mathrm{d}\sigma_2, \quad \mathrm{d}\sigma_{12} = (\sigma_1 - \sigma_2)\,\mathrm{d}\alpha$$

subject to

$$\frac{\partial \Phi}{\partial \sigma_1}\mathrm{d}\sigma_1 + \frac{\partial \Phi}{\partial \sigma_2}\mathrm{d}\sigma_2 + \frac{\partial \Phi}{\partial \alpha}\mathrm{d}\alpha = 0.$$

It follows that

$$\dot{\varepsilon}_{11}\Big/\frac{\partial \Phi}{\partial \sigma_1} = \dot{\varepsilon}_{22}\Big/\frac{\partial \Phi}{\partial \sigma_2} = 2(\sigma_1 - \sigma_2)\dot{\varepsilon}_{12}\Big/\frac{\partial \Phi}{\partial \alpha}. \tag{3.4}$$

This formulation of the flow-rule in terms of intrinsic variables, together with the corollaries (3.6) and (3.8) below, is taken from HILL (1980).

The mutual orientation of the principal axes of stress and strain-rate can now be calculated directly. Letting β denote the anticlockwise rotation from σ_1, σ_2 to $\dot{\varepsilon}_1$, $\dot{\varepsilon}_2$ respectively, we have

$$\dot{\varepsilon}_{11} + \dot{\varepsilon}_{22} = \dot{\varepsilon}_1 + \dot{\varepsilon}_2, \quad \dot{\varepsilon}_{11} - \dot{\varepsilon}_{22} = (\dot{\varepsilon}_1 - \dot{\varepsilon}_2)\cos 2\beta, \quad 2\dot{\varepsilon}_{12} = (\dot{\varepsilon}_1 - \dot{\varepsilon}_2)\sin 2\beta. \tag{3.5}$$

Then

$$\tan 2\beta = 2\dot{\varepsilon}_{12}/(\dot{\varepsilon}_{11} - \dot{\varepsilon}_{22}) = \frac{\partial \Phi}{\partial \alpha}\Big/\left\{(\sigma_1 - \sigma_2)\left(\frac{\partial \Phi}{\partial \sigma_1} - \frac{\partial \Phi}{\partial \sigma_2}\right)\right\}. \tag{3.6}$$

If Φ is strictly convex in σ_1 and σ_2, the last denominator is positive when $\sigma_1 \neq \sigma_2$. The strain-rate is coaxial with the stress when and only when

$$\frac{\partial \Phi}{\partial \alpha} = 0. \tag{3.7}$$

There is trivial coaxiality when $\sigma_1 = \sigma_2$ (Φ being unaffected by α while $\beta = -\alpha$), or when $\alpha = 0$ or $\frac{1}{2}\pi$ with any $\sigma_1 \neq \sigma_2$ (Φ being even in α and $\alpha - \frac{1}{2}\pi$). With certain types of anisotropy, by contrast, coaxiality occurs also in specific non-trivial orientations dependent on the ratio σ_1/σ_2. These will be calculated later for the proposed yield function. Further implications of (3.7) will also be mentioned in regard to a diagrammatic representation of anisotropic yield criteria generally.

Directions of zero component strain-rate are of interest as potential nuclei of localized deformation. They are present when the principal strain-rates have opposite signs (or one is zero), in which case their orientations are given by $\tan^2(\psi-\beta) = -\dot{\varepsilon}_1/\dot{\varepsilon}_2$ or

$$\cos 2(\psi-\beta) = -(\dot{\varepsilon}_1+\dot{\varepsilon}_2)/(\dot{\varepsilon}_1-\dot{\varepsilon}_2),$$

where ψ is reckoned anticlockwise from the σ_1 axis. Equivalently

$$\cos 2(\psi-\beta)/\cos 2\beta = -\left(\frac{\partial\Phi}{\partial\sigma_1}+\frac{\partial\Phi}{\partial\sigma_2}\right)\Big/\left(\frac{\partial\Phi}{\partial\sigma_1}-\frac{\partial\Phi}{\partial\sigma_2}\right), \tag{3.8}$$

having regard to (3.4) and (3.5).

In practice the sides of a rectangular test specimen are aligned with the principal stresses, but not with the principal strain-rates unless $\partial\Phi/\partial\alpha = 0$. Nevertheless it is customary to measure the in-plane and through-thickness components of strain-rate transverse to the major stress (σ_1 say), and to accord a special status to their ratio

$$r = -\dot{\varepsilon}_{22}/(\dot{\varepsilon}_{11}+\dot{\varepsilon}_{22}). \tag{3.9}$$

According to theory this is calculable from

$$1+2r = (\dot{\varepsilon}_{11}-\dot{\varepsilon}_{22})/(\dot{\varepsilon}_{11}+\dot{\varepsilon}_{22}) = \left(\frac{\partial\Phi}{\partial\sigma_1}-\frac{\partial\Phi}{\partial\sigma_2}\right)\Big/\left(\frac{\partial\Phi}{\partial\sigma_1}+\frac{\partial\Phi}{\partial\sigma_2}\right). \tag{3.10}$$

In uniaxial tests the ratio is basically dependent on α alone and was previously denoted by r_α in (2.16).

4. An Improved Yield Criterion

In terms of intrinsic variables the particular yield criterion proposed here for sheet with in-plane anisotropy is

$$|\sigma_1+\sigma_2|^m+(\sigma^m/\tau^m)|\sigma_1-\sigma_2|^m$$
$$+|\sigma_1^2+\sigma_2^2|^{(m/2)-1}\{-2a(\sigma_1^2-\sigma_2^2)+b(\sigma_1-\sigma_2)^2\cos 2\alpha\}\cos 2\alpha = (2\sigma)^m, \tag{4.1}$$

where $m > 1$. The state of the material is characterized by five material parameters: the yield stress σ in equibiaxial tension ($\sigma_1 = \sigma_2$), the yield stress τ in pure shear parallel to the orthotropic axes ($\sigma_1 = -\sigma_2$, $\alpha = \frac{1}{4}\pi$), and the dimensionless constants a, b and m. In conformity with what was envisaged, the new criterion reduces to (2.9) when $m = 2$ and to (2.11) when a and b both vanish (or $\alpha = \frac{1}{4}\pi$). Moreover, by (2.8), it is equivalent to a homogeneous function of degree m in the stress components on the orthotropic axes:

$$|\sigma_x+\sigma_y|^m+(\sigma^m/\tau^m)|(\sigma_x-\sigma_y)^2+4\sigma_{xy}^2|^{m/2}$$
$$+|\sigma_x^2+\sigma_y^2+2\sigma_{xy}^2|^{(m/2)-1}\{-2a(\sigma_x^2-\sigma_y^2)+b(\sigma_x-\sigma_y)^2\} = (2\sigma)^m. \tag{4.2}$$

Conditions on a and b sufficient to ensure strict convexity at any $m > 1$ are left for future investigation. However, several necessary conditions will be derived; some of these are also sufficient when $m = 2$, so are plausibly critical over a range of neighbouring values of m.

In uniaxial tension at orientation α the predicted yield stress σ_α is such that

$$(2\sigma/\sigma_\alpha)^m = 1+\sigma^m/\tau^m - 2a\cos 2\alpha + b\cos^2 2\alpha. \tag{4.3}$$

The expression closely resembles (2.12) and is equally explicit. This last feature is attributable to the imposed homogeneity, without which σ_α would have to be extracted as the root of some equation. Analogously to (2.13) the yield stresses in particular orientations are given by

$$\left.\begin{aligned}(2\sigma/\sigma_0)^m &= 1+\sigma^m/\tau^m - 2a+b,\\ (2\sigma/\sigma_{90})^m &= 1+\sigma^m/\tau^m + 2a+b,\\ (2\sigma/\sigma_{45})^m &= 1+\sigma^m/\tau^m\end{aligned}\right\} \tag{4.4}$$

as special cases of (4.3) or direct from (4.2). It is deduced, as in (2.14), that

$$\left.\begin{aligned}a &= \tfrac{1}{4}\{(2\sigma/\sigma_{90})^m - (2\sigma/\sigma_0)^m\},\\ b &= \tfrac{1}{2}\{(2\sigma/\sigma_0)^m + (2\sigma/\sigma_{90})^m\} - (2\sigma/\sigma_{45})^m,\end{aligned}\right\} \tag{4.5}$$

where $a \geqslant 0$ but b may be positive or negative.

The sign of a is pre-determined by the same convention as before. The labels (x, y) are bestowed so that $\sigma_0 \geqslant \sigma_{90}$ always, regardless of which orthotropic axis may be the direction of rolling. We also have

$$b = 2^{(m/2)-1}\{(\sigma/\tau')^m - (\sigma/\tau)^m\} \tag{4.6}$$

in analogy to (2.15) with the same definition of τ'. This indicates a lower bound

$$b > -2^{(m/2)-1}(\sigma/\tau)^m \tag{4.7}$$

on the allowable range of b. Typical values of $2^{(m/2)-1}$ are 0.841 when $m = 1.5$, 1 when $m = 2$, and 1.414 when $m = 3$. A different bound is prompted by (4.3), namely

$$b > a^2 - (\sigma/\tau)^m, \tag{4.8}$$

which makes

$$(2\sigma/\sigma_\alpha)^m > (1-a\cos 2\alpha)^2 + (\sigma/\tau)^m \sin^2 2\alpha \geqslant 0$$

for all α (and any a). When $m \geqslant 2$ it is evident that (4.8) implies (4.7) as $2^{(m/2)-1} \geqslant 1$. On the other hand, when $m < 2$, both (4.7) and (4.8) must be observed unless the range of a is suitably restricted.

The possible dependences of σ_α on orientation may be categorized in exactly the way mentioned for the quadratic criterion. When $|b| \leqslant a$, σ_α falls monotonically from σ_0 to σ_{90}. By contrast σ_α has an analytic maximum at $\alpha = \frac{1}{2}\cos^{-1}(a/b)$ when $b > a$, and an analytic minimum when $b < -a$.

The flow-rule (3.4) will now be implemented in a rearranged and slightly expanded form. This is

$$\frac{\dot\varepsilon_{11}+\dot\varepsilon_{22}}{\partial\phi/\partial\sigma_1+\partial\phi/\partial\sigma_2} = \frac{\dot\varepsilon_{11}-\dot\varepsilon_{22}}{\partial\phi/\partial\sigma_1-\partial\phi/\partial\sigma_2} = \frac{2(\sigma_1-\sigma_2)\dot\varepsilon_{12}}{\partial\phi/\partial\alpha} = \frac{\sigma\dot\varepsilon}{m(2\sigma)^m} \tag{4.9}$$

where the yield criterion is written as $\phi(\sigma_1, \sigma_2, \alpha) = (2\sigma)^m$ to match (4.1). The common value of the ratios has been expressed in a coordinate-invariant format by means of (3.3) and a work-equivalent strain-rate $\dot{\varepsilon}$ conjugate to σ. That is,

$$\sigma\dot{\varepsilon} = \sigma_1\dot{\varepsilon}_{11} + \sigma_2\dot{\varepsilon}_{22}. \tag{4.10}$$

The derivatives needed in (4.9) and elsewhere are given in an Appendix when function ϕ is defined by (4.1).

As an illustration the directions of the principal axes of strain-rate will be calculated for uniaxial tension at any orientation α. With β defined generally as in (3.6), but now denoted specifically by β_α it is found that

$$\frac{\sin 2(\alpha+\beta_\alpha)}{\sin 2\beta_\alpha} = \frac{m\sigma^m/\tau^m - (m-2)(a-\frac{1}{2}b\cos 2\alpha)\cos 2\alpha}{2(a-b\cos 2\alpha)}. \tag{4.11}$$

This formula is more compact than the evaluation of $\tan 2\beta_\alpha$ (albeit not explicit). It is seen that $\beta_\alpha = 0$ when $\alpha = 0$, $\frac{1}{2}\pi$, or $\frac{1}{2}\cos^{-1}(a/b)$, which are precisely the directions for which the yield stress in (4.3) has an absolute or local extremum. The particular directions, moreover, are the same for any m; this is because the angular dependence in (4.1) does not involve m directly (only through a factor in σ_1 and σ_2 which is immaterial in the present context).

The transverse strain-ratio r_α in uniaxial tests is calculable from (3.10) in the first instance. Afterwards the formula can be rearranged with the help of (4.3):

$$\left.\begin{aligned} 1+2r_\alpha &= \frac{\sigma^m/\tau^m - a\cos 2\alpha + \{(m+2)/2m\}b\cos^2 2\alpha}{1-a\cos 2\alpha + \{(m-2)/2m\}b\cos^2 2\alpha}, \\ 2(1+r_\alpha) &= (2\sigma/\sigma_\alpha)^m/[1-a\cos 2\alpha + \{(m-2)/2m\}b\cos^2 2\alpha]. \end{aligned}\right\} \tag{4.12}$$

These variants duly reduce to their counterparts in (2.16) when $m = 2$. The strain-ratio in tension at 45° is of especial interest because of the relation

$$1+2r_{45} = \sigma^m/\tau^m \tag{4.13}$$

connecting it directly with the coefficient of the second term in (4.1) or (4.2). As to the potential directions ψ of localized necks, it is seen from (3.8) and (3.10) that these can be calculated for any loading by

$$\cos 2(\psi-\beta) = -\cos 2\beta/(1+2r) \tag{4.14}$$

in terms of r and β. The values of the latter pair have just been obtained in (4.11) and (4.12) for uniaxial tension in particular.

Finally we consider compression perpendicular to the sheet or equibiaxial tension in its plane. The ratio of the extensions along the orthotropic axes is found most easily from (4.2) and the flow-rule in cartesian variables, leading to

$$\dot{\varepsilon}_x/\dot{\varepsilon}_y = (2^{(m/2)-1} - 2a/m)/(2^{(m/2)-1} + 2a/m), \tag{4.15}$$

which reduces to (2.19) when $m = 2$. It follows that the anisotropic parameter a must not exceed $2^{(m/2)-1} \times \frac{1}{2}m$ if the ratio is to be positive. Typical values of this bound are 0.631, 1, and 2.121 when $m = 1.5$, 2, and 3.

5. Representation in Principal Stress Space

The new yield criterion has been constructed with the aid of the intrinsic variables σ_1, σ_2 and α. The choice was motivated at the outset by the nature of the immediate problem, as mentioned in connection with (2.11). Beyond that, there is a distinct advantage in applications: implementation of the flow-rule in these variables is automatically in terms that closely match the specimen geometry and loading in standard laboratory tests. Finally the formulation itself suggests a useful perspective in visualizing the effects of in-plane anisotropy on yielding. This will be described now.

For sheet with transverse isotropy it is commonplace to depict a yield criterion by an ellipse or some oval in (σ_1, σ_2) space. It is hardly less common to assemble typical members of a continuous family of such loci, when parametrized by notional values of a material constant (for example the exponent m). It is rare, on the other hand, to visualize a yield criterion asssociated with in-plane anisotropy from this standpoint. For that purpose the independent variable α must be regarded ambivalently as a parameter. Then, to each value of α in $(0, \frac{1}{2}\pi)$, there corresponds a (σ_1, σ_2) locus that represents yielding under biaxial loads with arbitrary magnitudes but fixed orientations (α and $\frac{1}{2}\pi+\alpha$ respectively). With this in mind, formulation of the flow-rule in terms of σ_1, σ_2 and α prompts consideration of all (σ_1, σ_2) loci parametrized by α, and especially of the annular domain spanned by the family.

The radial width of this annulus is non-uniform and depends strongly on the stress ratio (for instance it is trivially zero in an equibiaxial state, where α is without significance). The width depends also, of course, on the type and degree of anisotropy. A quantitative appraisal of all these factors is necessarily a long-term objective which calls for extensive computation. This preliminary account aims only at a qualitative understanding of some general features of the representation in (σ_1, σ_2) space. Prominent among these is the precise nature of the boundaries of the annulus.

In order to settle this question, consider a section of the annulus by a typical radius, $\sigma_1/\sigma_2 =$ constant. Let $\rho(\alpha)$ denote the magnitude $(\sigma_1^2+\sigma_2^2)^{1/2}$ of the stress vector at the point where the radius is crossed by the yield locus parametrized by α. Increments in ρ and α along the section are subject to

$$\frac{\partial \Phi}{\partial \sigma_1}\delta\sigma_1+\frac{\partial \Phi}{\partial \sigma_2}\delta\sigma_2+\frac{\partial \Phi}{\partial \alpha}\delta\alpha = 0, \quad \delta\sigma_1/\sigma_1 = \delta\sigma_2/\sigma_2 = \delta\rho/\rho.$$

On recalling (3.1) and (3.3) it may be concluded by eliminating $\delta\sigma_1$ and $\delta\sigma_2$ that

$$m\,\mathrm{d}\rho/\mathrm{d}\alpha = -\rho\,\partial\Phi/\partial\alpha \tag{5.1}$$

at constant σ_1/σ_2. Because of the orthotropic symmetry, $\partial\Phi/\partial\alpha$ vanishes when $\alpha = 0$ and $\frac{1}{2}\pi$ regardless of σ_1/σ_2, so $\rho(\alpha)$ is always stationary at these values. On radial sections where $\partial\Phi/\partial\alpha$ has no other zeros, $\rho(\alpha)$ is (i) monotonic increasing or (ii) monotonic decreasing in the open interval $(0, \frac{1}{2}\pi)$ according to whether $\partial\Phi/\partial\alpha$ is (i) negative or (ii) positive there. Consequently, where the annulus is intersected by such radii, its inner and outer boundaries coincide respectively with parts of the loci $\alpha = 0$ and $\frac{1}{2}\pi$ in case (i) and with parts of the loci $\alpha = \frac{1}{2}\pi$ and 0 in case (ii).

By contrast, on radial sections where $\partial\Phi/\partial\alpha$ vanishes again between 0 and $\frac{1}{2}\pi$, say at $\alpha = \bar{\alpha}$, the value of $\rho(\bar{\alpha})$ is an absolute extremum. Evidently $\alpha = \bar{\alpha}$ on the outer or inner boundary according to whether this extremum is a maximum or a minimum. In either case the other boundary is associated with $\alpha = 0$ or $\frac{1}{2}\pi$, depending on the stress ratio and type of anisotropy. The value of $\bar{\alpha}$ necessarily ranges between 0 and $\frac{1}{2}\pi$ over a certain sector of σ_1/σ_2 values. The corresponding segment of boundary (whether outer or inner) is thus a true envelope of the entire family of loci, and is derived by eliminating α between

$$\Phi = 1 \quad \text{and} \quad \partial\Phi/\partial\alpha = 0. \tag{5.2}$$

The proof runs as follows. Let $\Delta\sigma_1$ and $\Delta\sigma_2$ be increments along an arc of yieldpoint states $\Phi(\sigma_1, \sigma_2, \alpha) = 1$ characterized by $\partial\Phi/\partial\alpha = 0$. Then

$$\Delta\sigma_1/\Delta\sigma_2 = -\frac{\partial\Phi}{\partial\sigma_2}\bigg/\frac{\partial\Phi}{\partial\sigma_1}$$

evaluated at $\alpha = \bar{\alpha}$ corresponding to some assigned σ_1/σ_2. But this expression is also the local ratio of increments along an actual yield locus, namely the one defined by $\Phi(\sigma_1, \sigma_2, \bar{\alpha}) = 1$ at constant $\bar{\alpha}$. Therefore the boundary on which $\partial\Phi/\partial\alpha$ vanishes is pointwise tangential to this yield locus, and is likewise so for the entire family parametrized by $\bar{\alpha}$ in $(0, \frac{1}{2}\pi)$.

These results will now be illustrated by the particular criterion (4.1) when represented in principal stress space. Because of the symmetries (3.2) it suffices to deal with the halfspace $\sigma_1 \geqslant \sigma_2$ only. It is hence convenient to take polar coordinates such that

$$(\sigma_1+\sigma_2)/\sqrt{2} = \rho\sin\theta, \quad (\sigma_1-\sigma_2)/\sqrt{2} = \rho\cos\theta \tag{5.3}$$

where $|\theta| \leqslant \frac{1}{2}\pi$. Then (4.1) transforms to

$$(2\sigma/\rho)^m = |\sqrt{2}\sin\theta|^m + (\sigma^m/\tau^m)|\sqrt{2}\cos\theta|^m \\ + 2(-2a\sin\theta + b\cos\theta\cos 2\alpha)\cos\theta\cos 2\alpha. \tag{5.4}$$

This may be regarded as the $\rho(\theta)$ equation of a yield locus at a fixed orientation (α = constant), or as the $\rho(\alpha)$ relation at a fixed section (θ = constant). For instance (4.3) is recovered with ρ identified as σ_α when $\theta = \frac{1}{4}\pi$, while $\rho = \sqrt{2}\sigma$ for all α when $\theta = \frac{1}{2}\pi$. It is observed also that the locus for $\alpha = \frac{1}{4}\pi$ is symmetric about the radii $\theta = 0$ and $\frac{1}{2}\pi$, as remarked before in other terms after (3.2).

At any fixed θ the function $\rho(\alpha)$ is stationary when $\alpha = 0$, $\frac{1}{2}\pi$, or

$$\cos 2\alpha = (a/b)\tan\theta, \tag{5.5}$$

the same for any m. When these values are introduced in (5.4), the polar equations of two yield loci and the envelope are obtained, respectively

$$(2\sigma/\rho)^m = |\sqrt{2}\sin\theta|^m + (\sigma^m/\tau^m)|\sqrt{2}\cos\theta|^m + 2(-2a\sin\theta + b\cos\theta)\cos\theta \quad (\alpha = 0),$$

$$(2\sigma/\rho)^m = |\sqrt{2}\sin\theta|^m + (\sigma^m/\tau^m)|\sqrt{2}\cos\theta|^m + 2(2a\sin\theta + b\cos\theta)\cos\theta \quad (\alpha = \tfrac{1}{2}\pi),$$

$$(2\sigma/\rho)^m = |\sqrt{2}\sin\theta|^m + (\sigma^m/\tau^m)|\sqrt{2}\cos\theta|^m - 2(a^2/b)\sin^2\theta \quad (|\theta| \leqslant \gamma),$$

where $\tan\gamma = |b|/a$ and $0 \leqslant \gamma < \frac{1}{2}\pi$. It is noted that $(\rho, \theta) = (\sqrt{2}\tau, 0)$ lies on the envelope whatever the values of a and b, and that this is the point of tangency with the locus $\alpha = \frac{1}{4}\pi$. Continuation beyond the sector $|\theta| \leqslant \gamma$ has no practical relevance as it merely represents the formal eliminant in (5.2) when α is complex. The loci $\alpha = 0$ and $\frac{1}{2}\pi$ cross each other on the radii $\theta = 0$ and $\pm\frac{1}{2}\pi$; the locus $\alpha = 0$ is outside where $\theta > 0$ and inside where $\theta < 0$. In the sector $|\theta| \leqslant \gamma$, however, the envelope is the external boundary of the annulus when $b > 0$, and its internal boundary when $b < 0$. (In the degenerate case $b = 0$ all loci pass through $\rho = \sqrt{2}\tau$ on $\theta = 0$, consistent with $\gamma = 0$.) When $\gamma > \frac{1}{4}\pi$, which is when $|b| > a$, the radius for uniaxial tension falls in the sector; this general setting gives perspective to the categories mentioned in connection with (2.12) and (4.3).

For the standard quadratic ($m = 2$) the envelope is

$$2\sigma^2/\rho^2 = (\sigma^2/\tau^2)\cos^2\theta + (1 - a^2/b)\sin^2\theta.$$

In the considered halfspace its arc is part of (i) an ellipse when $b < 0$ or $b > a^2$, (ii) a hyperbola when $0 < b < a^2$, and (iii) a line when $b = a^2$ (namely $\rho\cos\theta = \sqrt{2}\tau$ or $\sigma_1 - \sigma_2 = 2\tau$). In case (ii) the arc forms a *concave* segment of the external boundary of the annular domain, notwithstanding that the individual yield loci are all convex.

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APPENDIX

With $\phi(\sigma_1, \sigma_2, \alpha)$ defined by (4.1) the derivatives needed in the expressions (4.3) for the intrinsic components of strain-rate are

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial\sigma_1} + \frac{\partial\phi}{\partial\sigma_2}\right) = m|\sigma_1 + \sigma_2|^{m-2}(\sigma_1 + \sigma_2) + |\sigma_1^2 + \sigma_2^2|^{(m/2)-1} f(\sigma_1, \sigma_2, \alpha)\cos 2\alpha, \qquad (A.1)$$

$$\frac{1}{2}\left(\frac{\partial\phi}{\partial\sigma_1} - \frac{\partial\phi}{\partial\sigma_2}\right) = m(\sigma^m/\tau^m)|\sigma_1 - \sigma_2|^{m-2}(\sigma_1 - \sigma_2) + |\sigma_1^2 + \sigma_2^2|^{(m/2)-1} g(\sigma_1, \sigma_2, \alpha)\cos 2\alpha, \qquad (A.2)$$

$$\frac{1}{2}\frac{\partial\phi}{\partial\alpha}\Big/(\sigma_1 - \sigma_2) = 2|\sigma_1^2 + \sigma_2^2|^{(m/2)-1}\{a(\sigma_1 + \sigma_2) - b(\sigma_1 - \sigma_2)\cos 2\alpha\}\sin 2\alpha. \qquad (A.3)$$

Here the functions

$$f(\sigma_1,\sigma_2,\alpha) = \tfrac{1}{2}(m-2)\frac{(\sigma_1^2-\sigma_2^2)}{(\sigma_1^2+\sigma_2^2)}\{-2a(\sigma_1+\sigma_2)+b(\sigma_1-\sigma_2)\cos 2\alpha\}-2a(\sigma_1-\sigma_2), \quad \text{(A.4)}$$

$$g(\sigma_1,\sigma_2,\alpha) = \tfrac{1}{2}(m-2)\frac{(\sigma_1-\sigma_2)^2}{(\sigma_1^2+\sigma_2^2)}\{-2a(\sigma_1+\sigma_2)+b(\sigma_1-\sigma_2)\cos 2\alpha\}$$
$$-2a(\sigma_1+\sigma_2)+2b(\sigma_1-\sigma_2)\cos 2\alpha \quad \text{(A.5)}$$

are homogeneous of degree one in σ_1 and σ_2. They vanish with the anisotropy for any m and reduce to their counterparts in the standard quadratic criterion when $m = 2$. The components of strain-rate themselves are homogeneous functions of degree $m-1 > 0$ and are finite on the yield surface.