Simulating Branching Processes in Python

1. Probability Distributions

$$p_n = \mathbb{P}(Z_{i,t} = n) \qquad n = 0, 1, 2, ..$$

p_n is the probability that case i infects n individuals in generation t
Z is the number of secondary cases - $Z_{i,t}$ number of people infected by individual i in generation t

As an example we will use the **Poisson distribution**, which is a discrete probability distribution

$$Z \sim Poisson(\lambda)$$

$$\mathbb{P}(Z_{i,t=n}) = \frac{e^{-\lambda}\lambda^n}{n!}$$

Mean = λ
Variance = λ
Therefore, for our model $\lambda = R_0$

Probability distributions can be used via `np.random`
To generate random numbers that follow a poisson distribution, we can use:
`np.random.poisson(lam,size)`
`lam` is in our case R_0
`size` is the amount of random numbers generated
The output is an array with the numbers generated

In [1]:
```
#Import numpy
import numpy as np
#Generate random numbers Poisson distribution
print(np.random.poisson(1.5,1))
print(np.random.poisson(1.5,3))
```

```
[0]
[2 1 1]
```

Every time we run the code, we get different random numbers

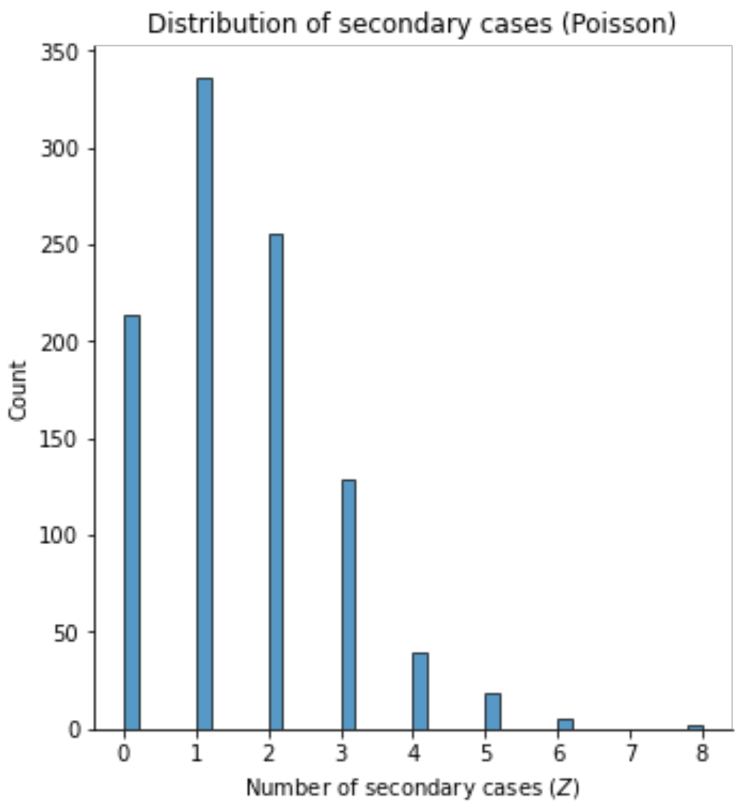
To visualize the output of random number generators, we will use the library `sns.displot`

In [2]:
```python
#Import matplolib
import matplotlib.pyplot as plt
#Import seaborn
import seaborn as sns
```

In [3]:
```python
#Generate random numbers Poisson distribution
Ro = 1.5
size = 1000
random_numbers = np.random.poisson(Ro,size)

#Display numbers
sns.displot(random_numbers)
#sns.displot(random_numbers, kind = "kde")

#Aesthetics of the graph
plt.xlabel("Number of secondary cases ($Z$)")
plt.title("Distribution of secondary cases (Poisson)")
plt.show()
```



2. Simulating discrete-time branching processes

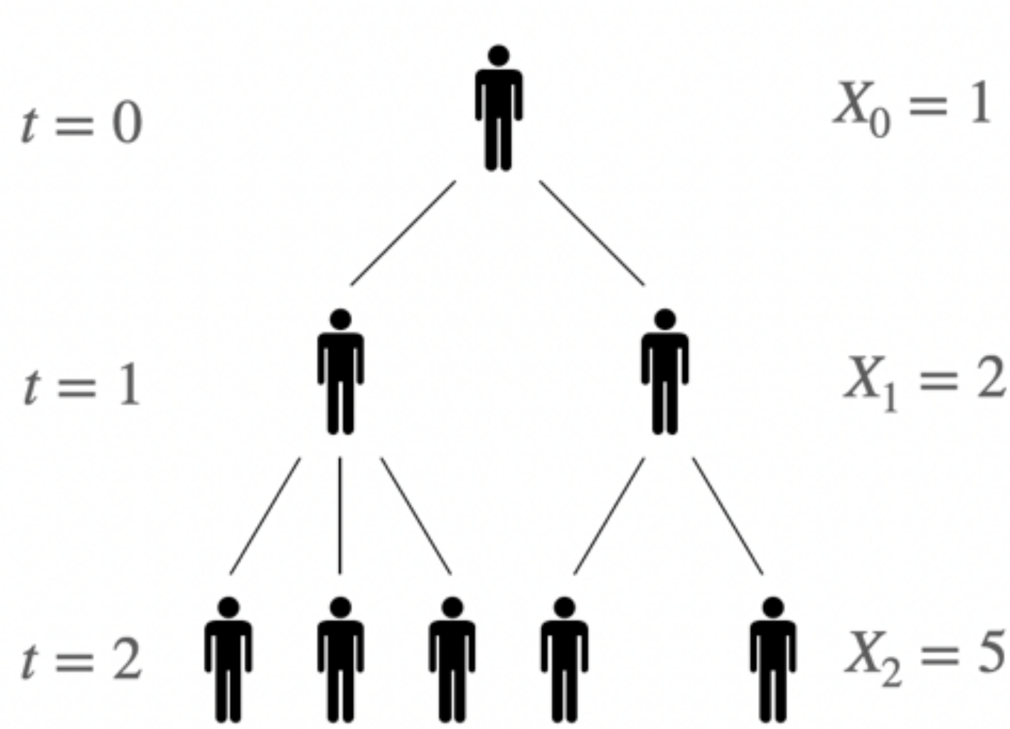


Figure 1

To computationally model this process we will use an array:

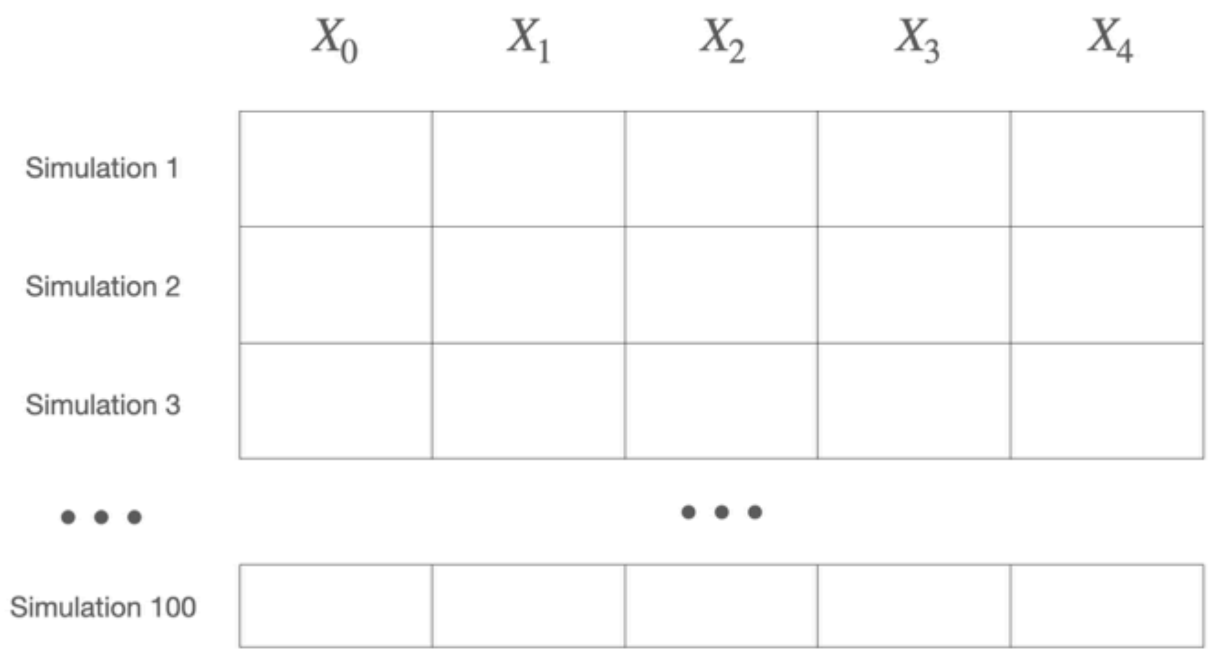


Figure 2

In [4]:
```
#Initial parameters
number_simulations = 100
number_generations = 5
R0 = 1.5

#Initialize information array
array_info = np.zeros((number_simulations, number_generations))
#Verify the shape of the array
# output --> (rows,columns)
print(np.shape(array_info))
```

```
(100, 5)
```

In [5]:
```
#Fill the array
#Loop that goes over each simulation
for i in range(number_simulations):

    #X_0 = 1 for all simulations
    number_info = 1
    array_info[i,0] = number_info

    #Loop that goes over each generation
    #Starts in 1, because X_0 is already filled
    for j in range(1,number_generations):

        #Z for every individual in X_t-1
        number_secondary_cases = np.random.poisson(R0,number_info)
        #Sum of secondary cases in t to obtain Xt
        number_info = np.sum(number_secondary_cases)
        #Fill the array element
        array_info[i,j]=number_info
```

The amount of secondary cases caused by each individual (Z) is drawn from the same probability distribution, in this example we use the Poisson distribution

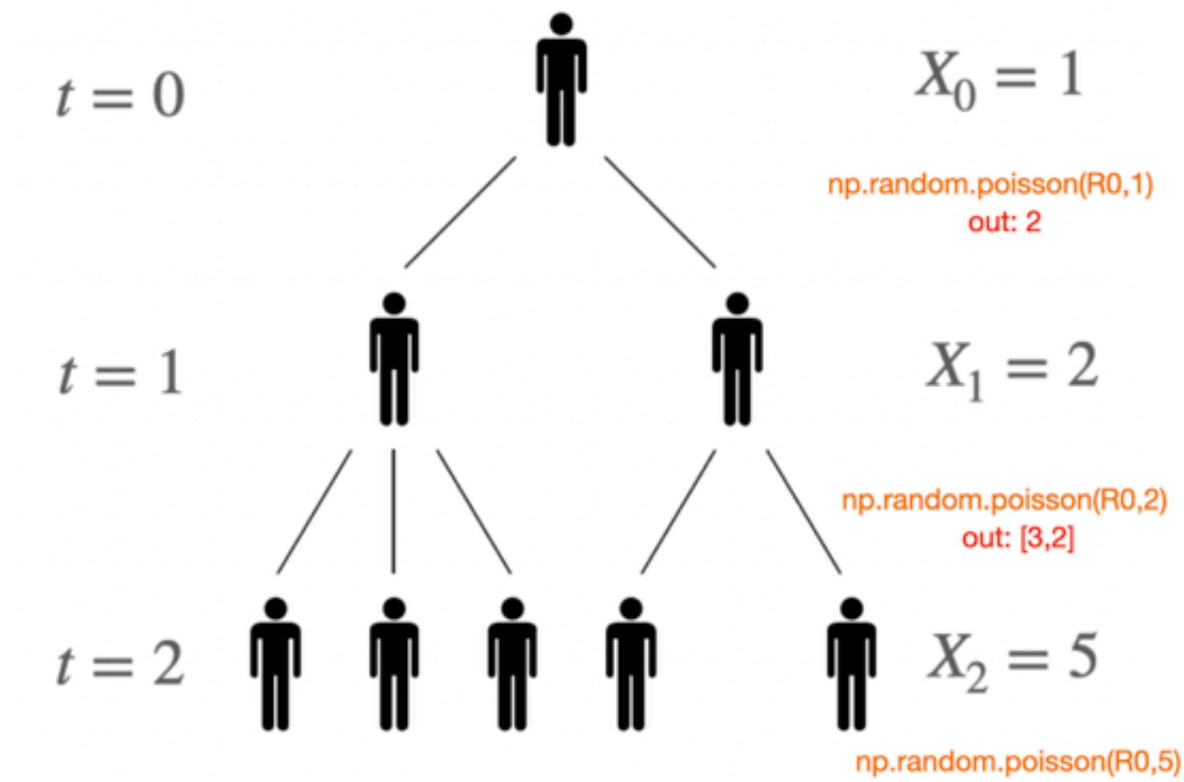
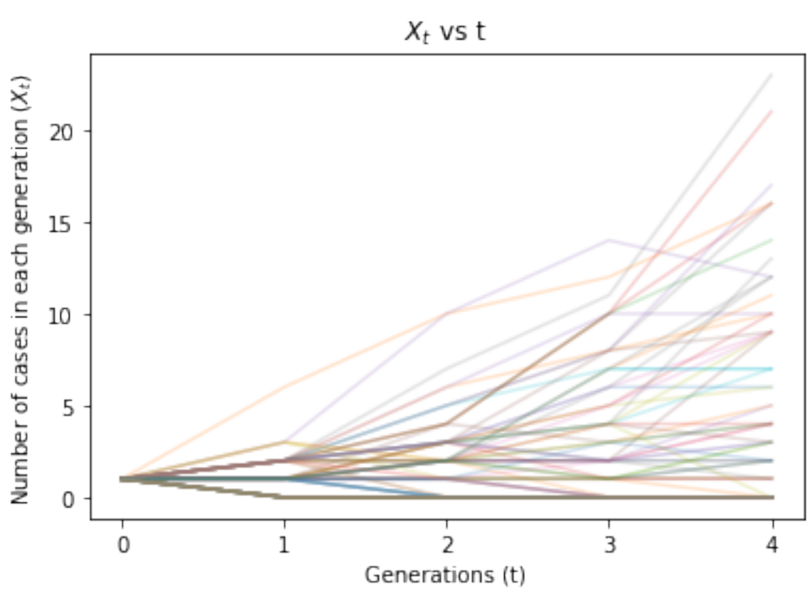


Figure 3

In [6]:
```
#Plot results (X_t vs t)
#Plot every simulation
for i in range(number_simulations):
    plt.plot(range(number_generations),array_info[i,:],alpha=0.2)

#Aesthetics of the graph
plt.title("$X_t$ vs t")
plt.xlabel("Generations (t)")
plt.ylabel("Number of cases in each generation ($X_t$)")
plt.xticks([0,1,2,3,4],["0","1","2","3","4"])
plt.show()
```



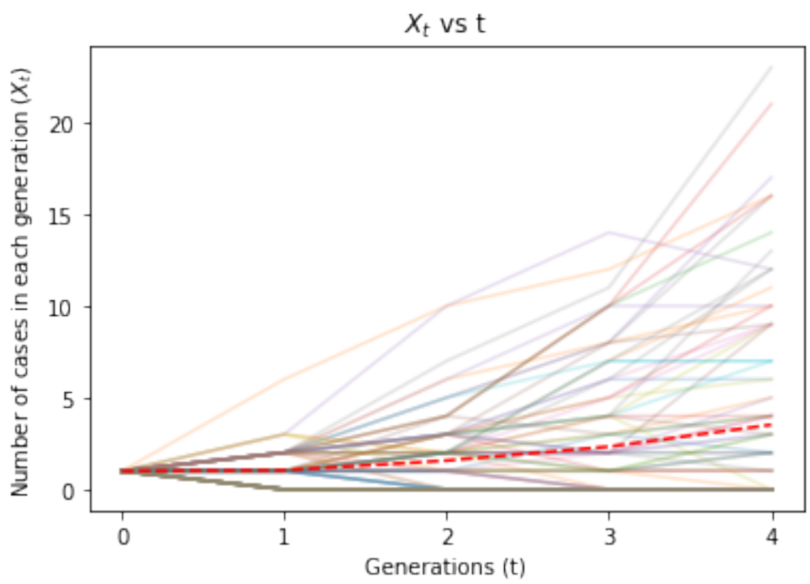
To calculate the mean of X_t for all generations, we will use `np.mean(array,axis)`
`array` is the array for which we want to calculate the mean
`axis=0` calculates the mean by columns
`axis=1` calculates the mean by rows

In [7]:
```python
#Calculate the average
mean_info = np.mean(array_info,axis=0)

#Plot results (X_t vs t)
#Plot every simulation
for i in range(number_simulations):
    plt.plot(range(number_generations),array_info[i,:],alpha=0.2)

#Plot mean
plt.plot(range(number_generations),mean_info,"r--")

#Aesthetics of the graphs
plt.title("$X_t$ vs t")
plt.xlabel("Generations (t)")
plt.ylabel("Number of cases in each generation ($X_t$)")
plt.xticks([0,1,2,3,4],["0","1","2","3","4"])
plt.show()
```



3. Calculate the extinction probability with simulations

We want to know how many of the total amount of simulations have no individuals infected in t_{final}, which in this case is $t = 4$

In [8]:
```python
#We use the information contained in the array
#We check how many of the 100 simulations have X_4 = 0

#Count how many simulations become extinct in t_final
Amount_simu_extincted = 0

#Loop over each simulation
for i in range(number_simulations):
    #Take info of last generation
    last_generation = array_info[i,-1]

    #Check if it is equal to zero
    if last_generation == 0:
        Amount_simu_extincted = Amount_simu_extincted + 1

print(Amount_simu_extincted)

#Probability of extinction
prob_extinction = Amount_simu_extincted/number_simulations
print(prob_extinction)

#Probability of outbreak
prob_outbreak = 1 - prob_extinction
print(prob_outbreak)
```

```
53
0.53
0.47
```

For the case of $Z \sim Poisson(R_0)$, with $R_0 = 1.5$, the probability of extinction is q. Therefore, the probability of outbreak is $1 - q$. For 100 simulations.

4. Calculate the extinction probability with numerical methods

To obtain the extinction probability q, we have to solve the equation $q = f(q)$.
$f(s)$ is the probability generating function, which is specific to each probability distribution.

$$f(s) = \sum_{n=0}^{\infty} p_n s^n$$

It is not possible to find an analytical solution of $q = f(q)$ for every distribution, therefore we need to use numerical methods

The first step to solve $q = f(q)$ is to graph $y = q$ and $y = f(q)$. The point(s) where these graphs intersect each other will give us a first approximation of the solution(s) of $q = f(q)$.

For example, for the case of the Poisson distribution (as it was calculated in the homework)

$$f(s) = e^{\lambda(s-1)}$$

Therefore,

$$q = f(q) \rightarrow q = e^{\lambda(q-1)}$$

So we need to graph $y = q$ and $y = e^{\lambda(q-1)}$

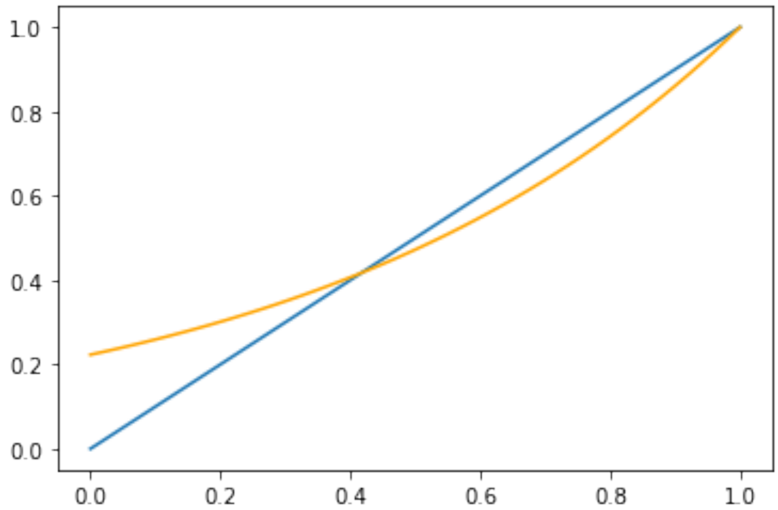
In [9]:

```
#Define functions
#y=q
def func1(x):
    ans = x
    return ans

#y=f(q)
def func2(Ro,x):
    ans = np.exp(Ro*(x-1))
    return ans

#Define function parameters
#q range (0,1) because q is a prob.extinction
q=np.linspace(0,1)
#Ro
Ro=1.5


#Graph functions
plt.plot(q,func1(q))
plt.plot(q,func2(Ro,q),color="orange")
plt.show()
```


1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.2
0.4
0.6
0.8
1.0

At first glance, we can see that there are 2 solutions $q \sim 1$ and $q \sim 0.4$
To have more accurate solutions q=f(q), we will use `fsolve` from `scipy.optimize`. This function is based on the Newton-Raphson method.

Newton-Raphson Method

Root-finding algorithm which produces successively better approximations to the roots (or zeroes) $f(x) = 0$ of a function $y = f(x)$.

Steps:

1. Start with an initial guess x_1 which is reasonably close to the root.
2. Take the tangent line to $y = f(x)$ at x_1.
3. Given that the slope of the tangent line is $m = f'(x_1)$, we can use the **formula of the tangent line** to calcuate the x-intercept of the tangent line i.e. x_2. x_2 is a better approximation to the root than x_1. (See Figure 4)
4. Use x_2 to perform steps 2 and 3.
5. Repeat steps 2,3 and 4 repeatedly to obtain an accurate approximation to the root. (See Figure 5)

Formula of the tangent line
We can write the equation of the tangent line, given that we have two points in the line (x_a, y_a) and (x_b, y_b)

$$m = f'(x_a) = \frac{\Delta Y}{\Delta X} = \frac{y_b - y_a}{x_b - x_a}$$

In this case, we have $(x_b, y_b) = (x_2, 0)$ and $(x_a, y_a) = (x_1, f(x_1))$. Therefore,

$$m = f'(x_1) = \frac{\Delta Y}{\Delta X} = \frac{0 - f(x_1)}{x_2 - x_1}$$

$$f'(x_1) = \frac{-f(x_1)}{x_2 - x_1}$$

Calculation x_2

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Generalization | Newton-Raphson method

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

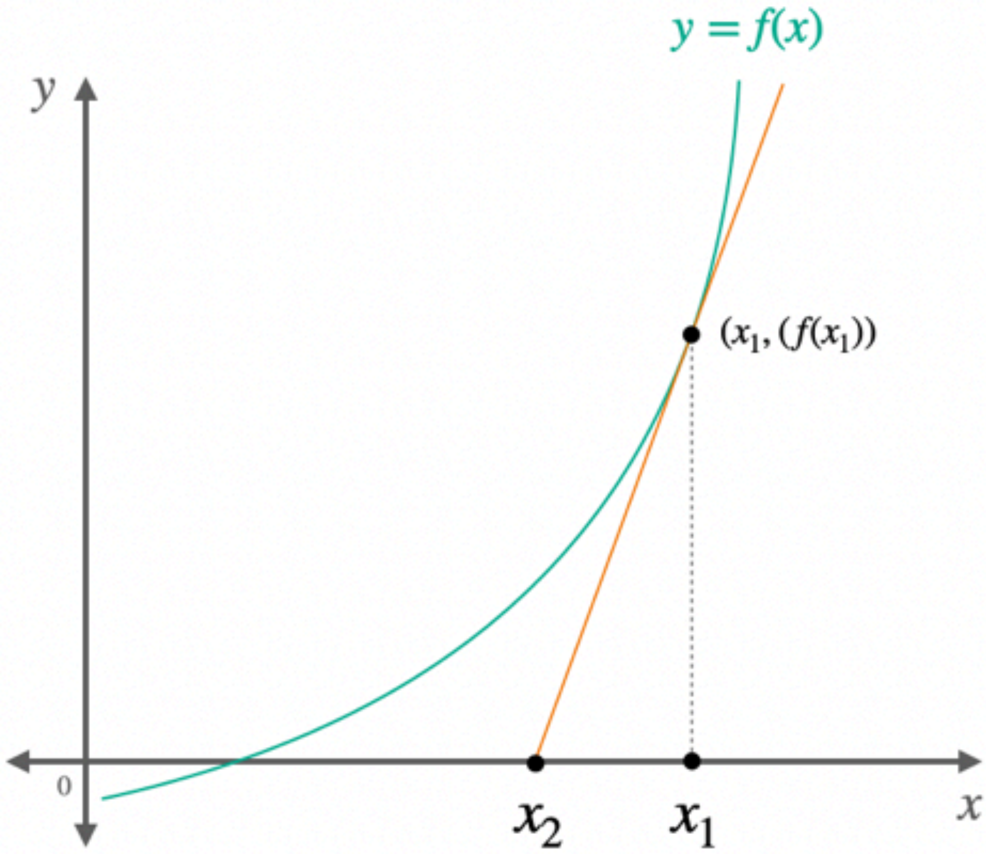


Figure 4

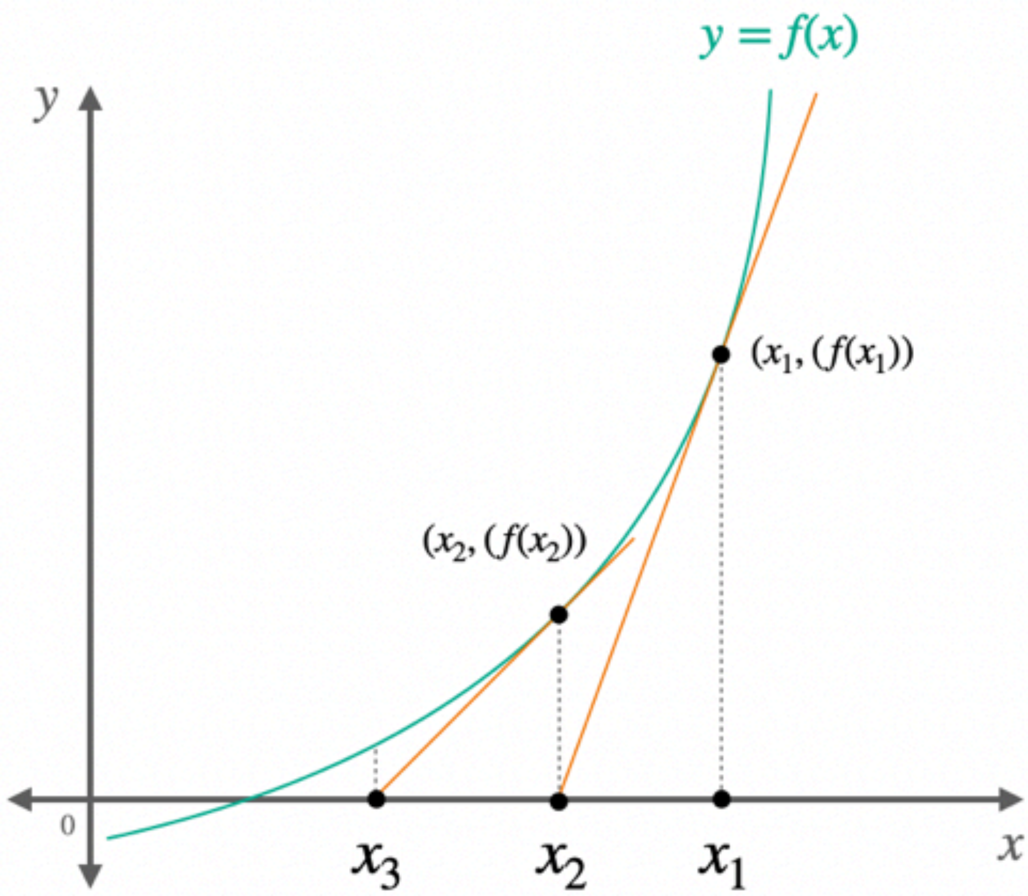


Figure 5

Solve $q = f(q)$

To solve $q = f(q)$ we will find the roots of $0 = f(q) - q$.
For this we will use `fsolve` from `scipy.optimize`, such that `fsolve(func,x1)`
`func` in our case is $f(q) - q$
`x1` is the initial guess of the root (obtained from the graphs)
The output is the solution of solving $f(q) = q$ for q. This is the probability of extinction q.

In [10]:
```
from scipy.optimize import fsolve

#define the function f(q)-q
def func3(x):
    Ro=1.5
    ans1 = x
    ans2 = np.exp(Ro*(x-1))
    ans = ans2-ans1
    return ans

#Initial guess obtained from the graph
x1=0.4

#Use fsolve to obtain the solution
solution = fsolve(func3,x1)
print(solution)

#Initial guess obtained from the graph
x1=1

#Use fsolve to obtain the solution
solution = fsolve(func3,x1)
print(solution)
```

```
[0.41718836]
[1.]
```

The extinction probability is $q = 0.41718836 \to 41.7\%$ and $q = 1$ for a Poisson distribution and $R_0 = 1.5$