AER 540 - Intermediate Dynamics
Dual-Spin Spacecraft Equations of Motion
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1 Dual-Spin Spacecraft

Spin stabilization is useful, but problematic for two reasons: 1) major axis rule dictates the body must be oblate; 2) the entire spacecraft must spin. To circumvent these issues we add another spinning body.

Consider a spacecraft composed of a platform, denoted $\mathcal{P}$, and a wheel, denote $\mathcal{W}$, shown in Fig. 1. The wheel is spinning relative to the platform. Let $\mathcal{B}$ denote the $\mathcal{P}+\mathcal{W}$ system.

Let $\mathcal{F}_a$ be an inertial frame (which is not shown in Fig. 1).

Let $\mathcal{F}_p$ be the platform frame that is fixed in the platform.

Let $\mathcal{F}_w$ be the wheel frame that is fixed in the wheel. Let the origin of $\mathcal{F}_w$ coincide with the center of mass of the wheel.

Let O^w denote the origin of $\mathcal{F}_w$.

Let c denote the center of mass of the $\mathcal{P}+\mathcal{W}$ system.

Let the inertia of the platform relative to c resolved in $\mathcal{F}_p$ be $\mathbf{J}_p^{\mathcal{P}c}$.

Let the inertia of the wheel relative to w resolved in $\mathcal{F}_w$ be $\mathbf{I}_w^{\mathcal{W}w}$.

Let the inertia of the wheel relative to c resolved in $\mathcal{F}_w$ be $\mathbf{J}_w^{\mathcal{W}c}$.

Let $\underrightarrow{a}$ be the unit-length spin axis of the wheel.

The approach to deriving the equations of motion of a dual-aping spacecraft are as follows. First, we'll find the angular momentum of $\mathcal{P}$ relative to c w.r.t. $\mathcal{F}_a$ expressed in $\mathcal{F}_p$. Next, we'll find the angular momentum of $\mathcal{W}$ relative to w w.r.t. $\mathcal{F}_a$ expressed in $\mathcal{F}_w$. Along the way we'll discuss second order tensors. The angular momentum of $\mathcal{W}$ relative to c w.r.t. $\mathcal{F}_a$ expressed in $\mathcal{F}_w$ will then be computed. We will combine the expressions for angular momentum to get the angular momentum of $\mathcal{B}$ relative to c w.r.t. $\mathcal{F}_a$ expressed in $\mathcal{F}_p$; this is the total angular momentum of the system. To fine the equations of motion the time-rate-of-change of the total angular momentum w.r.t. $\mathcal{F}_a$ will be found.

2 Angular Momentum of the Platform

The angular momentum of $\mathcal{P}$ relative to c w.r.t. $\mathcal{F}_a$ is

$$\underrightarrow{h}^{\mathcal{P}c/a} = \underrightarrow{\boldsymbol{\mathcal{F}}}_p^{\mathsf{T}} \mathbf{h}_p^{\mathcal{P}c} = \underrightarrow{\boldsymbol{\mathcal{F}}}_p^{\mathsf{T}} \mathbf{J}_p^{\mathcal{P}c} \boldsymbol{\omega}_p^{pa}.$$

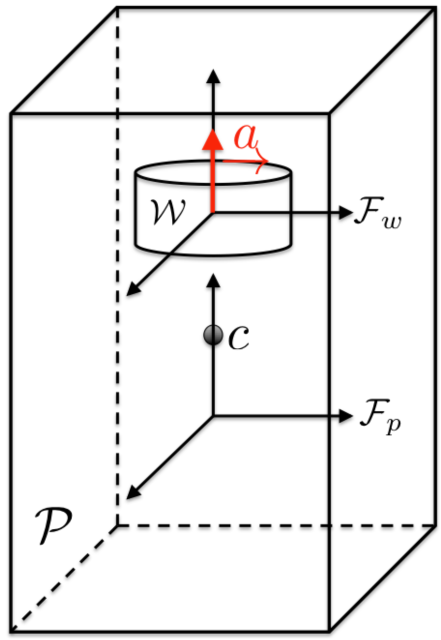

Figure 1: Dual spin spacecraft.

3 Angular Momentum of the Wheel

3.1 Angular Momentum of the Wheel Relative to O^w w.r.t. $\mathcal{F}_a$

The angular momentum of $\mathcal{W}$ relative to O^w w.r.t. $\mathcal{F}_a$ is

$$
\begin{aligned}
\underset{\rightarrow}{h}^{\mathcal{W}w/a} &= \int_{\mathcal{V}} \underset{\rightarrow}{\rho}^{dm\,w} \times \underset{\rightarrow}{\rho}^{dm\,w^{\bullet}}\, dm \\
&= \int_{\mathcal{V}} -\underset{\rightarrow}{\rho}^{dm\,w} \times (\underset{\rightarrow}{\rho}^{dm\,w} \times \underset{\rightarrow}{\omega}^{wa})\, dm \\
&= \underset{\rightarrow}{\mathcal{F}}_w^{\mathsf{T}} \mathbf{I}_w^{\mathcal{W}w} \boldsymbol{\omega}_w^{wa} \\
&= \underset{\rightarrow}{\mathcal{F}}_w^{\mathsf{T}} \mathbf{h}_w^{\mathcal{W}w/a},
\end{aligned}
$$

where $\underset{\rightarrow}{\rho}^{dm\,w^{\bullet}} = \underset{\rightarrow}{\omega}^{wa} \times \underset{\rightarrow}{\rho}^{dm\,w} = -\underset{\rightarrow}{\rho}^{dm\,w} \times \underset{\rightarrow}{\omega}^{wa}$. Note that

$$
\begin{aligned}
\underset{\rightarrow}{h}^{\mathcal{W}w/a} &= \underset{\rightarrow}{\mathcal{F}}_w^{\mathsf{T}} \mathbf{I}_w^{\mathcal{W}w} \underbrace{\underset{\rightarrow}{\mathcal{F}}_w \cdot \underset{\rightarrow}{\mathcal{F}}_w^{\mathsf{T}}}_{\mathbf{1}} \boldsymbol{\omega}_w^{wa} \\
&= \underbrace{\underset{\rightarrow}{\mathcal{F}}_w^{\mathsf{T}} \mathbf{I}_w^{\mathcal{W}w} \underset{\rightarrow}{\mathcal{F}}_w}_{\underset{\rightarrow}{I}^{\mathcal{W}w}} \cdot \underbrace{\underset{\rightarrow}{\mathcal{F}}_w^{\mathsf{T}} \boldsymbol{\omega}_w^{wa}}_{\underset{\rightarrow}{\omega}^{wa}} \\
&= \underset{\rightarrow}{I}^{\mathcal{W}w} \cdot \underset{\rightarrow}{\omega}^{wa}
\end{aligned}
$$

where $\underset{\rightarrow}{I}^{\mathcal{W}w}$ is the inertia tensor of $\mathcal{W}$ relative to w.

3.2 Second Order Tensors

An inertia tensor is a second order tensor. A second order tensor, also called a dyadic, maps a first order tensor (i.e., a physical vector) to another first order tensor. Consider $\underrightarrow{h}^{\mathcal{W}w} = \underrightarrow{I}^{\mathcal{W}w} \cdot \underrightarrow{\omega}^{wa}$; $\underrightarrow{I}^{\mathcal{W}w}$ maps physical vectors to physical vectors. Just like a physical vector, a second order tensor can be resolved in any frame:

$$\begin{aligned}\underrightarrow{I}^{\mathcal{W}w} &= \underrightarrow{\mathcal{F}}_w^{\mathsf{T}} \mathbf{I}_w^{\mathcal{W}w} \underrightarrow{\mathcal{F}}_w \\ &= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \mathbf{C}_{pw} \mathbf{I}_w^{\mathcal{W}w} \mathbf{C}_{wp} \underrightarrow{\mathcal{F}}_p \\ &= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \mathbf{I}_p^{\mathcal{W}w} \underrightarrow{\mathcal{F}}_p,\end{aligned}$$

where $\mathbf{I}_p^{\mathcal{W}w} = \mathbf{C}_{pw}\mathbf{I}_w^{\mathcal{W}w}\mathbf{C}_{wp}$ and $\underrightarrow{\mathcal{F}}_w^{\mathsf{T}} = \underrightarrow{\mathcal{F}}_p^{\mathsf{T}}\mathbf{C}_{pw}$. Note that $\mathbf{I}_w^{\mathcal{W}w}$ is the inertia matrix of $\mathcal{W}$ relative to O^w resolved in $\mathcal{F}_w$, while $\mathbf{I}_p^{\mathcal{W}w}$ is the inertia matrix of $\mathcal{W}$ relative to O^w resolved in $\mathcal{F}_p$.

3.3 Angular Momentum of the Wheel Relative to the Center of Mass w.r.t. $\mathcal{F}_a$

The angular momentum of $\mathcal{W}$ relative to c w.r.t. $\mathcal{F}_a$ is

$$\begin{aligned}\underrightarrow{h}^{\mathcal{W}c/a} &= \int_{\mathcal{V}} (\underrightarrow{r}^{wc} + \underrightarrow{\rho}^{dm\,w}) \times (\underrightarrow{r}^{wc\bullet} + \underrightarrow{\rho}^{dm\,w\bullet})\, dm \\ &= \int_{\mathcal{V}} \left(\underrightarrow{r}^{wc} \times \underrightarrow{r}^{wc\bullet} + \underrightarrow{r}^{wc} \times \underrightarrow{\rho}^{dm\,w\bullet} + \underrightarrow{\rho}^{dm\,w} \times \underrightarrow{r}^{wc\bullet} + \underrightarrow{\rho}^{dm\,w} \times \underrightarrow{\rho}^{dm\,w\bullet}\right) dm \\ &= m\,\underrightarrow{r}^{wc} \times \underrightarrow{r}^{wc\bullet} + \int_{\mathcal{V}} \underrightarrow{r}^{wc} \times \underrightarrow{\rho}^{dm\,w\bullet}\, dm + \int_{\mathcal{V}} \underrightarrow{\rho}^{dm\,w} \times \underrightarrow{r}^{wc\bullet}\, dm + \int_{\mathcal{V}} \underrightarrow{\rho}^{dm\,w} \times \underrightarrow{\rho}^{dm\,w\bullet}\, dm \\ &= m\,\underrightarrow{r}^{wc} \times \underrightarrow{r}^{wc\bullet} + \int_{\mathcal{V}} -\underrightarrow{r}^{wc} \times (\underrightarrow{\rho}^{dm\,w} \times \underrightarrow{\omega}^{wa})\, dm \\ &\quad + \int_{\mathcal{V}} \underrightarrow{\rho}^{dm\,w} \times \underrightarrow{r}^{wc\bullet}\, dm + \int_{\mathcal{V}} -\underrightarrow{\rho}^{dm\,w} \times (\underrightarrow{\rho}^{dm\,w} \times \underrightarrow{\omega}^{wa})\, dm \\ &= m\,\underrightarrow{r}^{wc} \times \underrightarrow{r}^{wc\bullet} + \underrightarrow{h}^{\mathcal{W}w/a}.\end{aligned}$$

3.4 Expressing the Angular Momentum of the Wheel in the Platform Frame

Consider the angular velocity of $\mathcal{F}_w$ relative to $\mathcal{F}_a$:

$$\begin{aligned}\underrightarrow{\omega}^{wa} &= \underrightarrow{\omega}^{wp} + \underrightarrow{\omega}^{pa} \\ &= \underrightarrow{\mathcal{F}}_w^{\mathsf{T}} \mathbf{a}_w \omega_{w,s}^{wp} + \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \boldsymbol{\omega}_p^{pa} \\ &= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} (\mathbf{C}_{pw}\mathbf{a}_w \omega_{w,s}^{wp} + \boldsymbol{\omega}_p^{pa}) \\ &= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} (\mathbf{a}_p \omega_{p,s}^{wp} + \boldsymbol{\omega}_p^{pa}) \\ &= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \boldsymbol{\omega}_p^{wa}\end{aligned}$$

where $\underrightarrow{a} = \underrightarrow{\mathcal{F}}_w^{\mathsf{T}}\mathbf{a}_w = \underrightarrow{\mathcal{F}}_p^{\mathsf{T}}\mathbf{a}_p$ is the unit-length (i.e., $\underrightarrow{a} \cdot \underrightarrow{a} = 1$) spin axis of the wheel, $\mathbf{a}_p = \mathbf{C}_{pw}\mathbf{a}_w$, and $\omega_{w,s}^{wp} = \omega_{p,s}^{wp}$ is the wheel speed about the spin axis relative to $\mathcal{F}_p$ expressed in $\mathcal{F}_w$. Then, the angular momentum of $\mathcal{W}$ relative to O^w w.r.t. $\mathcal{F}_a$ is

$$\begin{aligned}
\underrightarrow{h}^{\mathcal{W}w/a} &= \underrightarrow{\mathcal{F}}_w^{\mathsf{T}} \mathbf{I}_w^{\mathcal{W}w} \boldsymbol{\omega}_w^{wa} \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \mathbf{C}_{pw} \mathbf{I}_w^{\mathcal{W}w} \mathbf{C}_{wp} \mathbf{C}_{pw} \boldsymbol{\omega}_w^{wa} \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \mathbf{I}_p^{\mathcal{W}w} \boldsymbol{\omega}_p^{wa} \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(\mathbf{I}_p^{\mathcal{W}w} \mathbf{a}_p \omega_{w,s}^{wp} + \mathbf{I}_p^{\mathcal{W}w} \boldsymbol{\omega}_p^{pa} \right) \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(\mathbf{I}_p^{\mathcal{W}w} \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s} \right)
\end{aligned}$$

where $\mathbf{I}_p^{\mathcal{W}w} = \mathbf{C}_{pw} \mathbf{I}_w^{\mathcal{W}w} \mathbf{C}_{wp}$, $\mathbf{C}_{wp} \mathbf{C}_{pw} = \mathbf{1}$, and $\boldsymbol{\omega}_p^{wa} = \mathbf{a}_p \omega_{p,s}^{wp} + \boldsymbol{\omega}_p^{pa}$. Additionally, $h_{p,s} = I_{p,s}^{\mathcal{W}w} \omega_{p,s}^{wp}$ where and $I_{p,s}^{\mathcal{W}w}$ is the inertia of the wheel relative to O^w resolved in $\mathcal{F}_p$ along the spin axis defined by $\underrightarrow{a}$. It follows that

$$\begin{aligned}
\underrightarrow{h}^{\mathcal{W}c/a} &= m \, \underrightarrow{r}^{wc} \times \underrightarrow{r}^{wc\bullet} + \underrightarrow{h}^{\mathcal{W}w/a} \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(-m \mathbf{r}_p^{wc\times} \mathbf{r}_p^{wc\times} \boldsymbol{\omega}_p^{pa} \right) + \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(\mathbf{I}_p^{\mathcal{W}w} \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s} \right) \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left((\mathbf{I}_p^{\mathcal{W}w} - m \mathbf{r}_p^{wc\times} \mathbf{r}_p^{wc\times}) \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s} \right) \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(\mathbf{J}_p^{\mathcal{W}c} \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s} \right)
\end{aligned}$$

where $\underrightarrow{r}^{wc\bullet} = -\underrightarrow{r}^{wc} \times \underrightarrow{\omega}^{pa}$ and $\mathbf{J}_p^{\mathcal{W}c} = \mathbf{I}_p^{\mathcal{W}w} - m \mathbf{r}_p^{wc\times} \mathbf{r}_p^{wc\times}$.

4 The Total Angular Momentum of the Body

Consider the angular momentum of $\mathcal{B}$ relative to c w.r.t. $\mathcal{F}_a$:

$$\begin{aligned}
\underrightarrow{h}^{\mathcal{B}c/a} &= \underrightarrow{h}^{\mathcal{P}c} + \underrightarrow{h}^{\mathcal{W}c} \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \mathbf{J}_p^{\mathcal{P}c} \boldsymbol{\omega}_p^{pa} + \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(\mathbf{J}_p^{\mathcal{W}c} \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s} \right) \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left((\mathbf{J}_p^{\mathcal{P}c} + \mathbf{J}_p^{\mathcal{W}c}) \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s} \right) \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(\mathbf{I}_p^{\mathcal{B}c} \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s} \right)
\end{aligned}$$

where $\mathbf{I}_p^{\mathcal{B}c} = \mathbf{J}_p^{\mathcal{P}c} + \mathbf{J}_p^{\mathcal{W}c}$ is the moment of inertia of $\mathcal{B}$ relative to c resolved in $\mathcal{F}_p$.

5 The Equations of Motion

The time-rate-of-change of $\underrightarrow{h}^{\mathcal{B}c}$ w.r.t. $\mathcal{F}_a$ is

$$\begin{aligned}
\underrightarrow{h}^{\mathcal{B}c/a\bullet} &= \underrightarrow{h}^{\mathcal{B}c/a\circ} + \underrightarrow{\omega}^{pa} \times \underrightarrow{h}^{\mathcal{B}c/a} \\
&= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(\mathbf{I}_p^{\mathcal{B}c} \dot{\boldsymbol{\omega}}_p^{pa} + \mathbf{a}_p \dot{h}_{p,s} + \boldsymbol{\omega}_p^{pa\times} (\mathbf{I}_p^{\mathcal{B}c} \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s}) \right) \\
&= \underrightarrow{\tau}^{\mathcal{B}c},
\end{aligned}$$

where $(\ )^\circ$ denotes time differentiation w.r.t. $\mathcal{F}_p$. For a constant wheel speed $\dot{h}_{p,s} = 0$, thus

$$\begin{aligned}
\underrightarrow{h}^{\mathcal{B}c/a\bullet} = \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \left(\mathbf{I}_p^{\mathcal{B}c} \dot{\boldsymbol{\omega}}_p^{pa} + \boldsymbol{\omega}_p^{pa\times} (\mathbf{I}_p^{\mathcal{B}c} \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s}) \right) &= \underrightarrow{\mathcal{F}}_p^{\mathsf{T}} \boldsymbol{\tau}_p^{\mathcal{B}c}, \\
\mathbf{I}_p^{\mathcal{B}c} \dot{\boldsymbol{\omega}}_p^{pa} + \boldsymbol{\omega}_p^{pa\times} (\mathbf{I}_p^{\mathcal{B}c} \boldsymbol{\omega}_p^{pa} + \mathbf{a}_p h_{p,s}) &= \boldsymbol{\tau}_p^{\mathcal{B}c}
\end{aligned}$$

Consider the case where

$$
\begin{aligned}
\boldsymbol{\omega}_p^{pa} &= \operatorname{diag}\left\{\omega_1, \omega_2, \omega_3\right\}, \\
\mathbf{I}_p^{\mathcal{B}c} &= \operatorname{diag}\left\{I_{p,1}^{\mathcal{B}c}, I_{p,2}^{\mathcal{B}c}, I_{p,3}^{\mathcal{B}c}\right\} = \operatorname{diag}\left\{I_1, I_2, I_3\right\}, \\
\mathbf{I}_p^{\mathcal{W}w} &= \operatorname{diag}\left\{I_{p,1}^{\mathcal{W}w}, I_{p,2}^{\mathcal{W}w}, I_{p,3}^{\mathcal{W}w}\right\} = \operatorname{diag}\left\{I_{p,t}^{\mathcal{W}w}, I_{p,t}^{\mathcal{W}w}, I_{p,s}^{\mathcal{W}w}\right\}, \\
\mathbf{a}_p &= [0 \;\; 0 \;\; 1]^{\mathsf{T}}, \\
\boldsymbol{\tau}_p^{\mathcal{B}c} &= \mathbf{0},
\end{aligned}
$$

so that

$$
\mathbf{I}_p^{\mathcal{W}w}\mathbf{a}_p = [0 \;\; 0 \;\; I_{p,3}^{\mathcal{W}w}]^{\mathsf{T}}
$$

and $I_{p,3}^{\mathcal{W}w} = I_{p,s}^{\mathcal{W}w}$. Then, the dual-spin spacecraft equations of motion become

$$
\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_3\omega_2 + h_{p,s}\omega_2 &= 0, \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 - h_{p,s}\omega_1 &= 0, \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0.
\end{aligned}
$$