Theorem 1.2

WTS. *The Borel σ-algebra of $\mathbb{R}$, $\mathbb{B}$ is generated by the following*

- *The family of open intervals* $\mathcal{E}_1 = \{(a,b),\ a < b\}$,
- *The family of closed intervals* $\mathcal{E}_2 = \{[a,b],\ a < b\}$,
- *The family of half-open intervals* $\mathcal{E}_3 = \{(a,b], a < b\}$ *or* $\mathcal{E}_4 = \{[a,b), a < b\}$
- *The open rays* $\mathcal{E}_5 = \{(a,+\infty),\ a \in \mathbb{R}\}$ *or* $\mathcal{E}_6 = \{(-\infty,a),\ a \in \mathbb{R}\}$
- *The closed rays* $\mathcal{E}_7 = \{[a,+\infty),\ a \in \mathbb{R}\}$ *or* $\mathcal{E}_8 = \{(-\infty,a],\ a \in \mathbb{R}\}$

Proof. By definition, $\mathbb{B}$ is generated by the family of all open sets in $\mathbb{R}$, but every open set is a countable union of open intervals. Therefore

$$\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_1) \implies \mathbb{B} \subseteq \mathcal{M}(\mathcal{E}_1)$$

Conversely, every open interval is an open set, hence

$$\mathcal{E}_1 \subseteq \mathcal{T}_{\mathbb{R}} \subseteq \mathbb{B} \implies \mathcal{M}(\mathcal{E}_1) \subseteq \mathbb{B}$$

Every closed interval can also be written as a countable intersection of open intervals, for every $[a,b]$, with $a < b$, we have

$$[a,b] = \bigcap_{n\geq 1} (a - n^{-1}, b + n^{-1}) \tag{1}$$

Indeed, fix any $x \in [a,b]$ then for every $n \geq 1$,

$$a - n^{-1} < a \leq x \leq b < b + n^{-1}$$

So $x \in \bigcap_{n\geq 1}(a - n^{-1}, b + n^{-1})$. If x an element of the left member, then for every $n \geq 1$,

$$a - n^{-1} < x \implies a - x \leq 0$$

Similarly for $x \leq b$, therefore equation (1) is valid, and $\mathcal{E}_2 \subseteq \mathbb{B} = \mathcal{M}(\mathcal{E}_1)$. To show the reverse estimate, every open interval can be written as a countable union of closed intervals,

$$(a,b) = \bigcup_{n\geq 1} [a + n^{-1}, b - n^{-1}] \tag{2}$$

To show that the above estimate is indeed true, fix any $x \in (a, b)$, then

$$\begin{aligned} a < x < b &\iff a < a + n^{-1} \le x \le b - n^{-1} < b \\ &\iff x \in \bigcup_{n \ge 1} [a + n^{-1}, b - n^{-1}] \end{aligned}$$

So that equation (2) holds. By similar argumentation we have $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_2) \implies \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_1)$.

For $\mathcal{E}_3$, $\mathcal{E}_4$

- $(a, b] = \bigcap_{n \ge 1} (a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1)$,
- $(a, b) = \bigcup_{n \ge 1} (a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3)$,
- $[a, b) = \bigcup_{n \ge 1} [a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_2)$,
- $[a, b] = \bigcap_{n \ge 1} [a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_2) \subseteq \mathcal{M}(\mathcal{E}_4)$

So that $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_4) = \mathbb{B}$. By taking complements of each element we get $\mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$ and $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$. Notice also that

- $(a, b] = (a, +\infty) \cap (-\infty, b]$, proves $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_5)$, and $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$.
- $(a, +\infty) = \cup_{n \ge 1} (a, a + n]$, proves $\mathcal{E}_5 \subseteq \mathcal{M}(\mathcal{E}_3)$, and $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_3)$.
- $[a, b) = [a, +\infty) \cap (-\infty, b)$, proves $\mathcal{E}_4 \subseteq \mathcal{M}(\mathcal{E}_6)$, and $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_7)$,
- $[a, +\infty) = \bigcup_{n \ge 1} [a, a + n)$, proves $\mathcal{E}_7 \subseteq \mathcal{M}(\mathcal{E}_4)$, and $\mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{M}(\mathcal{E}_4)$.

Finally, $\mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8) = \mathbb{B}$ and $\mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7) = \mathbb{B}$. $\square$