

# Chapter A: Review of Topology

## Properties of Compact Spaces

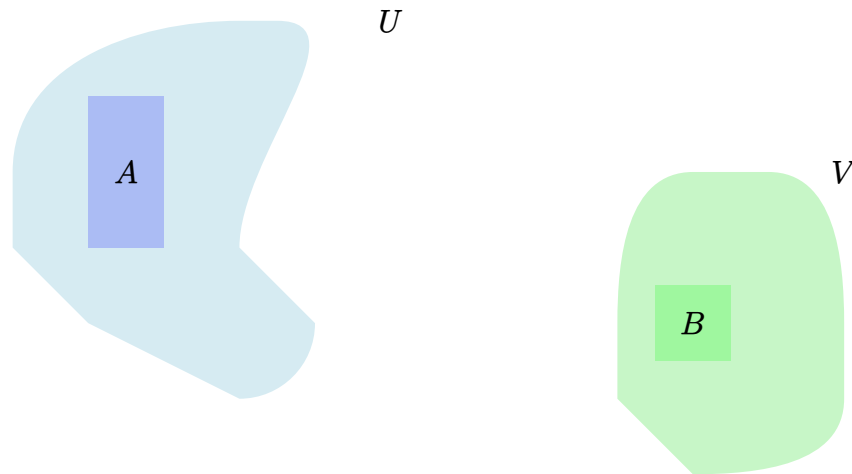


Figure 1: Closed sets  $A$  and  $B$  within open sets  $U$  and  $V$ , respectively.

### Proposition 1.1

Let  $\mathbf{X}$  and  $\mathbf{Y}$  be topological spaces.

- (a) If  $F \in C(\mathbf{X}, \mathbf{Y})$ , and  $\mathbf{X}$  is compact, then  $F(\mathbf{X})$  is compact.
- (b) If  $\mathbf{X}$  is compact and  $F \in C(\mathbf{X}, \mathbb{R})$ , then  $F(\mathbf{X})$  is bounded, and  $F$  attains its supremum and infimum on  $\mathbf{X}$ .
- (c) A finite union of compact subspaces of  $\mathbf{X}$  is again compact.
- (d) If  $\mathbf{X}$  is Hausdorff, and  $A, B$  are disjoint, compact subspaces of  $\mathbf{X}$ , there exists open  $U$  and  $V$ , (see fig. 1).
- (e) Every closed subset of a compact space is compact.
- (f) Every compact subset of a Hausdorff space is closed.
- (g) Every compact subset of a metric space is bounded.
- (h) Every finite product of compact spaces is compact.
- (i) Every quotient of a compact space is compact.

*Proof of Proposition 1.1 Part A.* Let  $f \in C(\mathbf{X}, \mathbf{Y})$  with  $\mathbf{X}$  compact. Fix an open cover of  $f(\mathbf{X})$  in the relative topology,

$$\{U_\alpha \cap f(\mathbf{X})\}_{\alpha \in A} \text{ covers } \mathbf{X}, U_\alpha \text{ open in } \mathbf{Y}$$

So that  $\bigcup f^{-1}(U_\alpha) = \bigcup f^{-1}(U_\alpha \cap f(\mathbf{X})) = \mathbf{X}$ . Since  $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$  is an open cover for  $\mathbf{X}$ , this induces a finite subcollection of indices  $\{\alpha_1, \dots, \alpha_n\}$  with

$$\bigcup_{j=1}^n f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^n f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^n U_{\alpha_j}$$

■

*Proof of Proposition 1.1 Part B.* Let  $\mathbf{X}$  be compact, and  $f \in C(\mathbf{X}, \mathbb{R})$ , so that  $f(\mathbf{X}) \subseteq \mathbb{R}$  is compact. Compact subsets are closed and bounded in  $\mathbb{R}$ , let  $A = \sup f(\mathbf{X})$  and  $B = \inf f(\mathbf{X})$ . Both  $A$  and  $B$  are accumulation points of  $f(\mathbf{X})$ , so  $A = f(x)$  and  $B = f(y)$  for some  $x, y$  in  $\mathbf{X}$ . ■

*Proof of Proposition 1.1 Part C.* Let  $\mathbf{X}$  be a topological space, and  $K_1, \dots, K_n$  be compact subspaces. Denote  $K = \bigcup_{j=1}^n K_j$ . Let  $\{U_\alpha \cap K\}_{\alpha \in A}$  be an open cover for  $K$ , where  $U_\alpha$  is open in  $\mathbf{X}$ . We can pass the argument to each individual  $K_j$  as follows. Let  $1 \leq j \leq n$ , then  $\{U_\alpha \cap K_j\}_{\alpha \in A}$  is an open cover for  $K_j$ , so there exists a finite subcollection of indices  $I_j \subseteq A$ , (a finite subset of  $A$ ) whose open sets cover  $K_j$ . Repeat this process for each  $j$  and

$$I = \bigcup_{j=1}^n I_j \text{ is a finite subset of } A$$

with  $K_j \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K)$ . Taking the union over all  $K_j$  reads

$$K = \bigcup_{j=1}^n K_j \subseteq \bigcup_{j=1}^n \bigcup_{\alpha \in I_j} (U_\alpha \cap K) = \bigcup_{\alpha \in I} U_\alpha \cap K$$

■

*Proof of Proposition 1.1 Part D.* Let  $\mathbf{X}$  be Hausdorff. We first prove that compact subspaces of  $\mathbf{X}$  are closed. Indeed, if  $K$  is compact in  $\mathbf{X}$ , fix any  $x \in K^c$ . Let  $y$  range through the elements of  $K$ , then  $x \neq y$  induces a pair of disjoint open sets  $U_y$  and  $V_y$ , such that

- $x \in U_y$

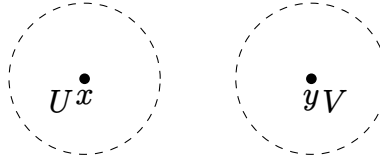


Figure 2: In a Hausdorff space, any two distinct points  $x$  and  $y$  can be separated by disjoint open neighbourhoods  $U$  and  $V$ .

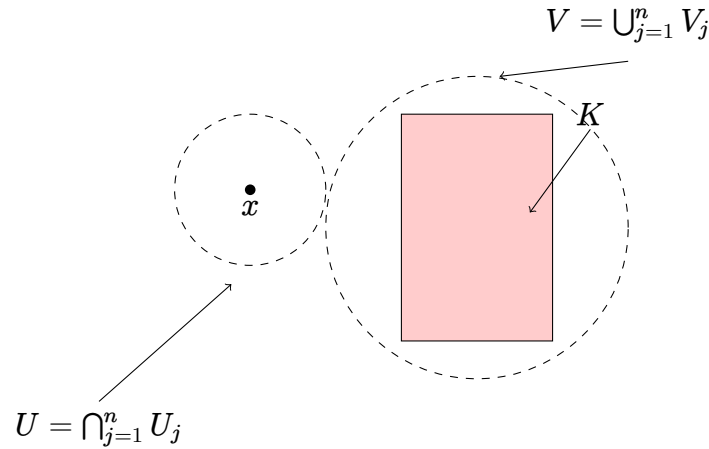


Figure 3: Compact sets are closed in Hausdorff spaces

- $y \in V_y$
- $U_y \cap V_y = \emptyset$
- See fig. 2

Let  $V_y$  range through all possible  $y \in K$ , So that  $\{V_y\}_{y \in K}$  is an open cover. There exists a finite subcollection of 'anchor points' of  $K$ ,  $y_1, \dots, y_n$  that corresponds with  $\{V_{y_j}\}_{j=1}^n$ . A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^n U_{y_j} \text{ is open}$$

Define  $V = \bigcup_{j=1}^n V_{y_j}$ , so  $V \subseteq K$  and  $U \cap V = \emptyset$  and  $x \in U \subseteq K^c$  (see fig. 3). Therefore  $K$  is closed.

Finally, if  $A$  and  $B$  are disjoint compact sets, then each  $x \in A \subseteq B^c$  induces neighbourhoods  $x \in U_x$ , and  $B \subseteq V_x$  (see fig. 4), let  $x$  range through all the elements of  $A$ . By compactness of  $A$ , this produces a finite subcover, and

$$U = \bigcup_{j=1}^n U_{x_j} \quad V = \bigcap_{j=1}^n V_{x_j}$$

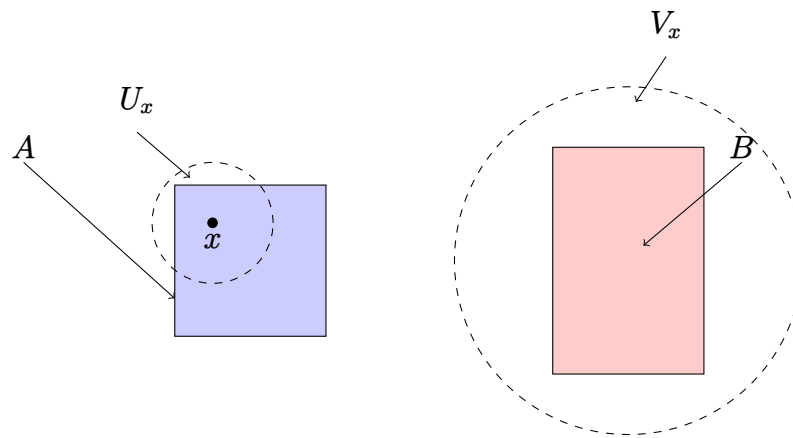


Figure 4: Closed sets  $A$  and  $B$ , point  $x$  in  $A$ , and disjoint neighbourhoods  $U$  around  $x$  and  $V$  around  $B$ .

are disjoint open sets that contain  $A$  and  $B$  respectively.

■

*Proof of Proposition 1.1 Part E.* Let  $K \subseteq \mathbf{X}$  be a closed set of a compact space. Let  $\{U_\alpha \cap K\}$  be an open cover for  $K$ , where each  $U_\alpha$  is open in  $\mathbf{X}$ . We can append an extra set  $K^c$  which is open in  $\mathbf{X}$ . The collection

$$W = \{U_\alpha\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of  $W_1, \dots, W_n$  that cover  $\mathbf{X}$  (since  $\mathbf{X}$  is compact by itself). Remove  $K^c$  from this finite subcollection if it exists, and take the intersection with  $K$  for each element  $W_j$ , and

$$\{W_1 \cap K, \dots, W_n \cap K\} = \{U_1 \cap K, \dots, U_n \cap K\} \text{ covers } K$$

so  $K$  is compact. ■

*Proof of Proposition 1.1 Part F.* Proven in Part D. ■

*Proof of Proposition 1.1 Part G.* let  $K \subseteq \mathbf{X}$  be a compact subset of the metric space  $(\mathbf{X}, d)$ . Compact subsets of  $\mathbf{X}$  are totally bounded, and hence bounded. ■

*Proof of Proposition 1.1 Part H.* See Tynchonoff's Theorem in Folland Chapter 4. ■

*Proof of Proposition 1.1 Part I.* Let  $\mathbf{X}$  and  $\mathbf{Y}$  be topological spaces and  $\pi : \mathbf{X} \rightarrow \mathbf{Y}$  be a quotient map. So that  $\mathbf{Y}$  is endowed with the quotient topology. So that  $\pi$  is a surjective continuous map. and  $\pi(\mathbf{X}) = \mathbf{Y}$ . Apply Part A, and we see that  $\mathbf{Y}$  is compact. ■