

Chapter 1

Theorem 1.1

Proposition 1.1. *Let $\mathcal{M}(\mathcal{F})$ be the σ -algebra generated by \mathcal{F} , if \mathcal{E} is a subset of $\mathbb{P}(X)$, with $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.*

Proof. Notice that because $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$,

$$\mathcal{M}(\mathcal{F}) \in \{\mathcal{M}, \mathcal{E} \subseteq \mathcal{M}, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

Taking the intersection, noting that $\mathcal{M}(\mathcal{E})$ is the intersection of all σ -algebras containing \mathcal{E} as a subset, we have

$$\bigcap \{\mathcal{M}(\mathcal{F})\} \supseteq \bigcap \{\mathcal{M}, \mathcal{E} \subseteq \mathcal{M}, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

And

$$\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$$

■

Theorem 1.2

Proposition 2.1. *The Borel σ -algebra of \mathbb{R} , \mathbb{B} is generated by the following*

- The family of open intervals $\mathcal{E}_\infty = \{(-\infty, b), a < \infty\}$,
- The family of closed intervals $\mathcal{E}_\epsilon = \{[a, b], a < b\}$,
- The family of half-open intervals $\mathcal{E}_\sup = \{(-\infty, b), a < \infty\}$ or $\mathcal{E}_\Delta = \{[a, b], a < b\}$
- The open rays $\mathcal{E}_\nabla = \{(-\infty, +\infty), a \in \mathbb{R}\}$ or $\mathcal{E}_\nabla = \{(-\infty, a), a \in \mathbb{R}\}$
- The closed rays $\mathcal{E}_\lrcorner = \{[a, +\infty), a \in \mathbb{R}\}$ or $\mathcal{E}_\lrcorner = \{(-\infty, a], a \in \mathbb{R}\}$

Proof. By definition, \mathbb{B} is generated by the family of all open sets in \mathbb{R} , but every open set is a countable union of open intervals. Therefore

$$\mathcal{T}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E}_\infty) \implies \mathbb{B} \subseteq \mathcal{M}(\mathcal{E}_\infty)$$

Conversely, every open interval is an open set, hence

$$\mathcal{E}_\infty \subseteq \mathcal{T}_\mathbb{R} \subseteq \mathbb{B} \implies \mathcal{M}(\mathcal{E}_\infty) \subseteq \mathbb{B}$$

Every closed interval can also be written as a countable intersection of open intervals, for every $[a, b]$, with $a < b$, we have

$$[a, b] = \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1}) \quad (1)$$

Indeed, fix any $x \in [a, b]$ then for every $n \geq 1$,

$$a - n^{-1} < a \leq x \leq b < b + n^{-1}$$

So $x \in \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1})$. If x an element of the left member, then for every $n \geq 1$,

$$a - n^{-1} < x \implies a - x < n^{-1}$$

Similarly for $x \leq b$, therefore equation (1) is valid, and $\mathcal{E}_\epsilon \subseteq \mathbb{B} = \mathcal{M}(\mathcal{E}_\infty)$. To show the reverse estimate, every open interval can be written as a countable union of closed intervals,

$$(a, b) = \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \quad (2)$$

To show that the above estimate is indeed true, fix any $x \in (a, b)$, then

$$\begin{aligned} a < x < b &\iff a < a + n^{-1} \leq x \leq b - n^{-1} < b \\ &\iff x \in \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \end{aligned}$$

So that equation (2) holds. By similar argumentation we have $\mathcal{E}_\infty \subseteq \mathcal{M}(\mathcal{E}_\epsilon) \implies \mathcal{M}(\mathcal{E}_\epsilon) = \mathcal{M}(\mathcal{E}_\infty)$.

For \mathcal{E}_\exists , \mathcal{E}_Δ

- $(a, b] = \bigcap_{n \geq 1} (a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1)$,
- $(a, b) = \bigcup_{n \geq 1} (a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3)$,
- $[a, b) = \bigcup_{n \geq 1} [a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_2)$,
- $[a, b] = \bigcap_{n \geq 1} [a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_2) \subseteq \mathcal{M}(\mathcal{E}_4)$

So that $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_4) = \mathbb{B}$. By taking complements of each element we get $\mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$ and $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$. Notice also that

- $(a, b] = (a, +\infty) \cap (-\infty, b]$, proves $\mathcal{E}_\exists \subseteq \mathcal{M}(\mathcal{E}_\nabla)$, and $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$.
- $(a, +\infty) = \bigcup_{n \geq 1} (a, a + n]$, proves $\mathcal{E}_\nabla \subseteq \mathcal{M}(\mathcal{E}_\exists)$, and $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_3)$.
- $[a, b) = [a, +\infty) \cap (-\infty, b)$, proves $\mathcal{E}_\Delta \subseteq \mathcal{M}(\mathcal{E})$, and $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_7)$,
- $[a, +\infty) = \bigcup_{n \geq 1} [a, a + n)$, proves $\mathcal{E}_\Delta \subseteq \mathcal{M}(\mathcal{E}_\Delta)$, and $\mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{M}(\mathcal{E}_4)$.

Finally, $\mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8) = \mathbb{B}$ and $\mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7) = \mathbb{B}$. ■

Theorem 1.3

Proposition 3.1. *If A is countable, then $\otimes_{\alpha \in A} \mathcal{M}_\alpha$ is the σ -algebra generated by*

$$W := \left\{ \prod_{\alpha \in A} E_\alpha, E_\alpha \in \mathcal{M}_\alpha \right\}$$

Proof. We agree to define

$$V := \left\{ \pi_\alpha^{-1}(E_\alpha), E_\alpha \in \mathcal{M}_\alpha \right\}$$

By definition, V generates $\otimes_{\alpha \in A} \mathcal{M}_\alpha$. Fix any element in $x = \pi_\alpha^{-1}(E_\alpha) \in V$, then

$$\pi_\alpha(x) \in E_\alpha, \pi_{\beta \neq \alpha}(x) \in X_\beta$$

Then $x \in W$ if we choose $x = \prod_{c \in A} E_c$, for $E_c = E_\alpha$ if $c = \alpha$, and $E_c = X_c$ if $c \neq \alpha$. ■

Theorem 1.4

Proposition 4.1.

Proof.



Theorem 1.5

Proposition 5.1.

Proof.



Theorem 1.6

Proposition 6.1.

Proof.



Theorem 1.7

Proposition 7.1.

Proof.



Theorem 1.8

Proposition 8.1.

Proof.



Theorem 1.9

Proposition 9.1.

Proof.



Theorem 1.10

Proposition 10.1.

Proof.



Theorem 1.11

Proposition 11.1 (Caratheodory's Theorem). *If μ^* is an outer measure on X , the collection \mathcal{M} of μ^* -measurable sets is a σ -algebra, and the restriction of μ^* to \mathcal{M} is a complete measure.*

Proof. We quote the definition for a set $A \subseteq X$ to be μ^* measurable. For any $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \quad (3)$$

- Show \mathcal{M} is an algebra.
- μ^* is finitely additive on \mathcal{M} .
- \mathcal{M} is closed under countable disjoint (this makes \mathcal{M} a sigma algebra, since it is an algebra that is closed under countable disjoint unions)

Lemma 11.1 *The family of μ^* -measurable sets is an algebra.*

Proof of Lemma 11.1. Clearly \mathcal{M} is closed under complements. To show that it is a σ -algebra, and if $A, B \in \mathcal{M}$, then $\left\{ \underbrace{E \cap A}_1, \underbrace{E \setminus A}_2 \right\} \subseteq \mathbb{P}(X)$. And because B is μ^* -measurable,

$$\mu^*(E) = \underbrace{\mu^*(E \cap A \cap B) + \mu^*(E \cap A \setminus B)}_1 + \underbrace{\mu^*(E \cap B \setminus A) + \mu^*(E \setminus (A \cup B))}_2$$

By subadditivity of μ^* , $A \cup B = A \cap B + A \setminus B + B \setminus A$ with $+$ denoting the disjoint union, hence

$$\mu^*(E \cap (A \cap B)) \leq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \setminus B)) + \mu^*(E \cap (B \setminus A))$$

and

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B))$$

■

Lemma 11.2 *μ^* is finitely additive on \mathcal{M} , the family of μ^* -measurable sets.*

Proof of Lemma 11.2. Let A, B be disjoint μ^* -measurable sets. It suffices to show $\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B)$, as the reverse estimate follows from subadditivity. From Lemma 11.1, $A \cup B \in \mathcal{M}$, so

$$\begin{aligned}\mu^*(A \cup B) &= \mu^*(A \cup B \cap A) + \mu^*(A \cup B \setminus A) \\ &= \mu^*(A \cup \emptyset) + \mu^*(A \setminus A \cup B \setminus A) \\ &= \mu^*(A) + \mu^*(B)\end{aligned}$$

■

Corollary 11.1 *If $\{A_j\}_{j \geq N} \subseteq \mathcal{M}$ is a finite disjoint family, then*

$$\mu^*\left(\bigcup A_{j \leq N}\right) = \sum \mu^*(A_{j \leq N})$$

Lemma 11.3 *Let $\{A_j\}_{j \geq 1}$ be a countable disjoint sequence in \mathcal{M} , and denote $B_n = \bigcup A_{j \leq n} \in \mathcal{M}$ by Lemma 11.1. For every $E \subseteq X$,*

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j \leq n})$$

Proof of Lemma 11.3. We will proceed by induction. If $n = 1$ then we have equality, suppose the result holds for $n \in \mathbb{N}^+$, and $A_{n+1} \in \mathcal{M}$ so

$$\begin{aligned}\mu^*(E \cap B_{n+1}) &= \mu^*(E \cap B_{n+1} \cap A_{n+1}) = \mu^*(E \cap B_{n+1} \setminus A_{n+1}) \\ &= \mu^*(E \cap A_{n+1}) + \mu^*(E \cap B_n) \\ &= \sum_{j \leq n+1} \mu^*(E \cap A_j)\end{aligned}$$

as $A_j \cap A_{n+1} = \emptyset \iff A_j \setminus A_{n+1} = A_j$, and $B_n \cap A_n = A_n \iff A_n \subseteq B_n$. ■

To show \mathcal{M} is a sigma-algebra, fix any disjoint sequence $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$, and denote B_n as in lemma 11.3. Define $B = \bigcup A_{j \geq 1} \supseteq B_n$ and for every $n \geq 1$,

we have

$$\begin{aligned}
 \mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \setminus B_n) \\
 &= \sum \mu^*(E \cap A_{j \leq n}) + \mu^*(E \setminus B_n) \\
 &\geq \sum \mu^*(E \cap A_{j \leq n}) + \mu^*(E \setminus B) \quad \text{since } B_n \subseteq B \iff B^c \subseteq B_n^c \\
 &\geq \sup_n \left[\sum \mu^*(E \cap A_{j \leq n}) \right] + \mu^*(E \setminus B)
 \end{aligned}$$

Let $J \subseteq \mathbb{N}^+$ be a finite non-empty set. And $\sup J \in \mathbb{N}^+$, $\sup J < +\infty$. By the Archimedean Property we can find a large $N \in \mathbb{N}^+$, with $N > J$ so that

$$\sum_{j \in J} \mu^*(E \cap A_j) \leq \sum_{j \leq N} \mu^*(E \cap A_j)$$

Applying the estimate $\sup_n \left[\sum \mu^*(E \cap A_{j \leq n}) \right] + \mu^*(E \setminus B) \leq \mu^*(E)$ reads

$$\left[\sum_{j \in J} \mu^*(E \cap A_j) \right] + \mu^*(E \setminus B) \leq \mu^*(E)$$

Now by Chapter 0, the infinite sum

$$\sum_{j \geq 1} \mu^*(E \cap A_j) = \sup \left\{ \sum_{j \in J} \mu^*(E \cap A_j), J \subseteq \mathbb{N}^+, 0 < |J| < +\infty \right\}$$

and $\bigcup A_{j \geq 1} = B$ is μ^* -measurable. Since $\mu^*(\emptyset) = 0$, and μ^* is countably additive on \mathcal{M} , (perhaps by replacing E with the union over all disjoint sets), μ^* is a measure on \mathcal{M} . To show μ^* is a complete measure, fix $A \in \mathcal{M}$ where $\mu^*(A) = 0$. Then any $B \subseteq A$ is also null, and for $E \subseteq X$,

$$\mu^*(E) \geq \underbrace{\mu^*(E \cap B)}_0 + \mu^*(E \setminus B) \implies B \in \mathcal{M}$$

■

Theorem 1.12

Proposition 12.1.

Proof.



Theorem 1.13

Proposition 13.1.

Proof.



Theorem 1.14

Proposition 14.1.

Proof.



Theorem 1.15**Proposition 15.1.**

Proof. If $\{E_j\}_{j \geq 1} \subseteq \mathcal{A}$ such that each $E_j = FDU(I_{ji})$ over finitely many i , and suppose E_j are disjoint, and that $DU(E_j) \in \mathcal{A}$. So that $DU(E_j) = FDU(I_\alpha)$ for some finite collection of half-intervals $\{I_\alpha\}$.

We will first prove the simpler case. Suppose we have already proven:

$$\{E_j\}_{j \geq 1} \subseteq \mathcal{A}, DU(E_j) = I_\alpha \in \mathcal{A} \implies \mu_0\left(DU(E_j)\right) = \sum \mu_0(E_j) = \mu_0(I_\alpha) \quad (4)$$

but each E_j is a FDU of I_{ji} , and for every $j \geq 1$, $E_j \cap I_\alpha \in \mathcal{A}$ (closure under intersections, because the family of FDU of h-intervals is an algebra).

Thus we have a disjoint sequence whose union is one h-interval. In symbols:

$$DU(E_j) = FDU(I_\alpha) \implies I_\alpha = DU(E_j \cap I_\alpha)$$

$$\forall j \geq 1, E_j \cap I_\alpha \in \mathcal{A} \implies$$

$$\begin{aligned} \mu_0(FDU(I_\alpha)) &= \sum_{\alpha < +\infty} \mu_0(I_\alpha) \\ &= \sum_{\alpha < +\infty} \sum_{j \geq 1} \mu_0(E_j \cap I_\alpha) \\ &= \sum_{j \geq 1} \sum_{\alpha < +\infty} \mu_0(E_j \cap I_\alpha) \\ &= \sum_{j \geq 1} \mu_0(E_j) \end{aligned}$$

It is permissible to swap the two summations because we are using the supremum definition for a sum of non-negative terms. And we applied finite-additivity (see earlier), to conclude that $\sum_{j \geq 1} \sum_{\alpha} \mu_0(E_j \cap I_\alpha) = \sum_{j \geq 1} \mu_0(E_j)$. ■

Define

- $\mathcal{H}_1 = \left\{ (a, b], -\infty \leq a < b < +\infty \right\},$
- $\mathcal{H}_2 = \left\{ (a, +\infty), a \in \mathbb{R} \cup \{-\infty\} \right\},$
- $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \{\emptyset\}$. Where $+$ denotes the disjoint union.
- DU : disjoint union, FDU : finite disjoint union.

Steps:

1. Show that \mathcal{H} is an elementary family.
2. Show that if $I_\alpha \in \mathcal{H}_1$, then for every $I_\beta \in \mathcal{H}_1 \cup \mathcal{H}_2$, $I_\alpha \cap I_\beta \in \mathcal{H}_1$. We write this as

$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1, I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_1$$

3. Show that if $I_\alpha \in \mathcal{H}_2$, then

$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1, I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_2$$

4. Show that $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$ is well defined. (modify the proof in Folland to check for $a = -\infty$ with

$$\overline{F} : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}, \quad \begin{cases} \overline{F}|_{\mathbb{R}} &= F \\ \overline{F}(+\infty) &= \sup_x F(x), \\ \overline{F}(-\infty) &= \inf_x F(x) \end{cases}$$

5. Show that $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$ is well defined for $b < +\infty$. If $E = (a, b] \in \mathcal{A}$, then E is an FDU of \mathcal{H}_1 , and \mathcal{H}_2 . So we write

$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2) = FDU(\mathcal{H}_1)$$

since E is bounded above, the \mathcal{H}_2 part of the FDU must be null. Now fix $E = FDU_{\mathcal{H}_1}(I_j) = FDU_{\mathcal{H}_1}(I_2)$. And follow the proof in Folland to see the 'well-definedness' of μ_0 if $E \in \mathcal{H}_1$.

6. Next, suppose $E \in \mathcal{H}_2$ and

$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2)$$

Clearly $FDU(\mathcal{H}_2) \neq \emptyset$, since E is unbounded above, and $FDU(\mathcal{H}_2)$ consists of exactly one element, so we write

$$E = FDU(\mathcal{H}_1) + (z, +\infty)$$

7. Show that $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$ is well defined. Hint: use the fact that if $E \in \mathcal{A}$, such that $E = FDU(E, \mathcal{H}_1) + FDU(E, \mathcal{H}_2)$, then $FDU(E, \mathcal{H}_2)$ contains at most one element (after throwing away empty sets), then use this to deduce $E \cap I_\alpha$ has a $FDU(E \cap I_\alpha, \mathcal{H}_2)$ of exactly one \mathcal{H}_2 interval, where I_α participates in $FDU(E, \mathcal{H}_2)$, if E is unbounded above. Then take $E \setminus I_\alpha = FDU(E \setminus I_\alpha, \mathcal{H}_1) = FDU(E, \mathcal{H}_1)$.
8. Now show that μ_0 is well-defined on all $E \in \mathcal{A}$.
9. Continue the proof for Folland until you reach the unbounded intervals, then modify the 'right continuity argument' to add an extra \mathcal{H}_2 interval. Let $I = \mathcal{H}_1 + \mathcal{H}_2 = I_\alpha + I_\beta$, meaning I can be represented by at most one \mathcal{H}_1 and \mathcal{H}_2 interval. If $(I_k) \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$, then $\{I_k \cap I_\alpha\} \subseteq \mathcal{H}_1$, and continue the proof as usual.

Theorem 1.16

Proposition 16.1.

Proof.



Theorem 1.17

Proposition 17.1.

Proof.



Theorem 1.18

Proposition 18.1.

Proof.



Exercises

Exercise 1.1

Proposition 1.1.

Proof.



Exercise 1.2

Proposition 2.1.

Proof.



Exercise 1.3

Proposition 3.1.

Proof.



Exercise 1.4

Proposition 4.1. *An algebra \mathcal{A} is a σ -algebra \iff it is closed under countable increasing unions.*

Proof. \Leftarrow is trivial. And it suffices to show that \mathcal{A} is closed under countable disjoint unions. Indeed, if $\{E_j\}_{j \geq 1} \subseteq \mathcal{A}$ is a countable disjoint sequence of sets, write

$$F_n = \bigcup E_{j \leq n}$$

Clearly, F_j is increasing, and denote $F = \bigcup E_{j \geq 1}$, which is a member of \mathcal{A} . We claim that

$$\bigcup F_{n \geq 1} = \bigcup E_{j \geq 1}$$

Fix any $x \in \bigcup E_{j \geq 1}$, then x belongs in some $E_j \subseteq F_j$, and \supseteq is proven. Also, if $x \in \bigcup F_{n \geq 1}$, then there exists some F_n for which x is a member of. For this particular F_n , means that $x \in E_j$ where $j \leq n$ and $x \in \bigcup E_{j \geq 1}$. \blacksquare

Exercise 1.5

Proposition 5.1. *Let $\mathcal{M}(\mathcal{E})$ be the σ -algebra generated by $\mathcal{E} \subseteq X$, and*

$$\mathcal{N} = \left\{ \mathcal{M}(\mathcal{F}), \mathcal{F} \subseteq \mathcal{E}, \mathcal{F} \text{ is countable} \right\}$$

Show that $\mathcal{M}(\mathcal{E}) = \mathcal{N}$.

Proof. The outline of the proof is as follows,

1. Prove that $\mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$,
2. Show that \mathcal{N} is a σ -algebra,
3. Show that \mathcal{N} contains \mathcal{E} as a subset, and hence $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{N}$.

First, for any $\mathcal{F} \subseteq \mathcal{E}$, where \mathcal{F} is countable, it follows from Lemma 1.1 that $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E})$. Taking the union over all of such \mathcal{F} , we get $\bigcup \mathcal{M}(\mathcal{F}) = \mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$.

To show that \mathcal{N} is a σ -algebra, fix any $A \in \mathcal{N}$, and A belongs to $\mathcal{M}(\mathcal{F})$, therefore $A^c \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{N}$. To show closure under countable unions, fix a sequence $\{E_j\} \subseteq \mathcal{N}$, then each of these E_j belongs to a corresponding $\mathcal{M}(\mathcal{F}_j)$, for $j \in \{1, 2, \dots\}$. Now define

$$\overline{\mathcal{F}} = \bigcup \mathcal{F}_{j \geq 1} \subseteq \mathcal{E}$$

and $\overline{\mathcal{F}}$ is obviously countable. Hence for every $j \geq 1$, $\mathcal{M}(\mathcal{F}_j) \subseteq \mathcal{M}(\overline{\mathcal{F}})$ and taking the union yields

$$\bigcup \mathcal{M}(\mathcal{F}_{j \geq 1}) \subseteq \mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$$

It is also clear that our sequence $\{E_j\}$ is contained in $\mathcal{M}(\overline{\mathcal{F}})$, and $E = \bigcup E_j$ belongs to $\mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$ as an element. Therefore \mathcal{N} is a σ -algebra.

Let $\alpha \in A$ index the family of sets in \mathcal{E} , (so that $E_\alpha \in \mathcal{E}$) and the singleton set of a set $\{E_\alpha\}$ is a countable subset of \mathcal{E} . For every $\alpha \in A$, we have

$$E_\alpha \in \mathcal{M}(\{E_\alpha\}) \subseteq \mathcal{N} \implies \mathcal{E} \subseteq \mathcal{N}$$

And one final application of Lemma 1.1 finishes the proof. ■

Exercise 1.6

Proposition 6.1.

Proof.



Exercise 1.7

Proposition 7.1. *If μ_1, \dots, μ_n are measures on (X, \mathcal{M}) , and $a_1, \dots, a_n \in [0, +\infty)$, then $\mu = \sum_1^n \mu_j$ is a measure on (X, \mathcal{M}) .*

Proof. If $\{E_j\}$ is a disjoint sequence in \mathcal{M} , and denote $E = \bigcup (E_j)$. If for each $k \leq n$, $\mu_k(E) < +\infty$,

$$\mu_k(E) = \sum \mu_k(E_j) \implies a_k \mu_k(E) = \sum a_k \mu_k(E_j)$$

Then,

$$\mu(E) = \sum_{k \leq n} a_k \mu_k(E) = \sum_{k \leq n} \sum_{j \geq 1} a_k \mu_k(E_j) = \sum_{j \geq 1} \sum_{k \leq n} a_k \mu_k(E_j) = \sum_{j \geq 1} \mu(E_j)$$

If there exists some μ_k such that $\mu_k(E) = +\infty$, then

$$\mu(E) = \sum_{k \leq n} \sum_{j \geq 1} a_k \mu_k(E_j)$$

Now if there exists some $\mu_{k'}$ with $\mu_{k'}(E) = +\infty$, then $\mu(E) = \sum_{k \leq n} \mu_k(E) = +\infty$, and

$$\sum_{j \geq 1} \mu(E_j) = \sup_N \sum_{j \leq N} \sum_{k \leq n} a_k \mu_k(E_j) \geq \mu_{k'}(E)$$

Therefore $\mu(E) = \sum_{j \geq 1} \mu(E_j)$, and μ is a measure. ■

Exercise 1.8

Proposition 8.1. *If (X, \mathcal{M}, μ) is a measure space, and $\{E_j\} \subseteq \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\bigcup E_{j \geq 1}) < +\infty$*

Proof. If $\{E_j\}_{j \geq 1}$ is a sequence in \mathcal{M} , and define $F_m = \bigcap_{j \geq m} E_j$

$$\liminf E_j = \bigcup_{m \geq 1} \bigcap_{j \geq m} E_j = \bigcup_{m \geq 1} F_m$$

Also, for every $m \geq 1$, $F_m \subseteq E_m$, and F_m is an increasing sequence, because

$$[m, +\infty) \supseteq [m+1, +\infty) \implies F_m \subseteq F_{m+1}$$

Using continuity above, and writing $F = \bigcup F_{m \geq 1} = \liminf E_j$, we have

$$\begin{aligned} \mu(\liminf E_j) &= \mu(F) \\ &= \liminf \mu(F_m) \\ &\leq \liminf \mu(E_m) \end{aligned}$$

The second part of the proof is similar, if $G_m = \bigcup_{j \geq m} E_j$, then

$$\limsup E_j = \bigcap_{m \geq 1} \bigcup_{j \geq m} E_j = \bigcap_{m \geq 1} G_m$$

Similarly, G_m is a decreasing sequence, and since $\mu(\bigcup E_{j \geq 1}) = \mu(G_1)$ is finite, we can use continuity from above in the same manner, and the proof is complete. ■

Exercise 1.9

Proposition 9.1.

Proof.



Exercise 1.10**Proposition 10.1.***Proof.*

Exercise 1.11

Proposition 11.1.

Proof.



Exercise 1.12

Proposition 12.1. *Let (X, \mathcal{M}, μ) be a finite measure space,*

- *If $E, F \in \mathcal{M}$, and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$,*
- *Say that $E \sim F$ if $\mu(E \Delta F) = 0$, then \sim is an equivalence relation on \mathcal{M} ,*
- *For every $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Show that ρ defines a metric on the space of \mathcal{M}/\sim equivalence classes.*

Proof of Part A. Use the fact that $\mu(F) = \mu(E \cap F) + \mu(F \cap E^c)$, and by monotonicity,

$$\mu(F \cap E^c) \leq \mu(E \Delta F) = 0$$

And $\mu(F) = \mu(E \cap F) = \mu(E)$, the last equality follows after a simple modification. ■

Proof of Part B. Suppose that $\mu(E \Delta F) = \mu(F \Delta G) = 0$, then

- $\mu(E \cap F^c) = \mu(F \cap E^c) \leq \mu(E \Delta F) = 0$ by monotonicity,
- Similarly, we have $\mu(F \cap G^c) = \mu(G \cap F^c) = 0$, and
- By subadditivity,
 - $\mu(E \cap G^c) = \mu(E \cap F^c \cap G^c) + \mu(E \cap F \cap G^c) \leq 0$, and $\mu(E \cap G^c) = 0$,
and
 - $\mu(G \cap E^c) = 0$
- Therefore $\mu(E \Delta G) = \mu(E \cap G^c) + \mu(G \cap E^c) = 0$

It is clear that the relation is reflexive, since $E \Delta E = \emptyset$, and symmetry is trivial. ■

Proof of Part C. Since $\rho(E, F) = \rho(F, E)$, and $\rho(E, F) \geq 0$ for every $E, F \in \mathcal{M}$, and $\rho(E, F) = 0 \iff E \sim F$. We only have to prove the Triangle Inequality. Notice that

$$\begin{aligned} \mu(E \setminus F) &= \mu(E \cap F^c \cap G) + \mu(E \cap F^c \cap G^c) \\ &\leq \mu(F^c \cap G) + \mu(E \cap F^c) \end{aligned}$$

and in the same fashion,

$$\mu(F \setminus E) \leq \mu(F \cap G^c) + \mu(E^c \cap F)$$

Combining the two inequalities, and applying additivity finishes the proof. ■

Exercise 1.13

Proposition 13.1. *Every σ -finite measure is semi-finite*

Proof. Suppose μ is σ -finite then there exists an increasing sequence of sets $E_j \nearrow X$ with $\mu(E_j) < +\infty$. Now for every $W \in \mathcal{M}$, if $\mu(W) = +\infty$ then $\mu(W) = \lim_{j \rightarrow \infty} \mu(E_j \cap W) = +\infty$. Since this real-valued limit converges to its supremum $+\infty$, there exists a non-null subset $E_j \cap W$ of positive and finite measure. ■

Exercise 1.14

Proposition 14.1. *If μ is a semi-finite measure, and if $\mu(E) = +\infty$, for every $C > 0$, there exists an $F \subseteq E$ with $0 < \mu(F) < +\infty$.*

Proof. Suppose by contradiction that there exists a $C > 0$ so for every $F \subseteq E$, if F is of finite measure, then $0 \leq \mu(F) \leq C$. Let $s = \sup\{\mu(F), F \subseteq E, 0 < \mu(F) < +\infty\}$, and for any $n^{-1} > 0$, this induces a F_n with measure

$$\mu(F_n) > s - n^{-1}$$

and take $A_n = \bigcup_{j \leq n} F_j$. A simple induction will show that $\mu(A_n) \leq \sum_{j \leq n} \mu(F_j) < +\infty$, therefore $\mu(A_n) \leq s$ for every $n \geq 1$. By continuity from below

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{j \geq 1} F_j\right) \leq s$$

Next, by monotonicity, denoting the union over A_n by A , for every $n^{-1} > 0$

$$s - n^{-1} \leq \mu(A_n) \leq \mu(A) \leq s \implies \mu(A) = s$$

Now, $E \setminus A$ is a set of infinite measure, and by semi-finiteness. Find a set $B \subseteq E \setminus A$ with strictly positive measure, so that

$$\mu(A \cup B) = \mu(A) + \mu(B) > s$$

And this finishes the proof. ■

Exercise 1.15

Proposition 15.1. *Given a measure μ on $(\mathbf{X}, \mathcal{M})$, and define $\mu_0 = \sup\{\mu(F), F \subseteq E, \mu(F) < +\infty\}$. Show μ_0 is semi-finite. Then, show that if μ is semi-finite, $\mu = \mu_0$. Lastly, there exists a measure ν on $(\mathbf{X}, \mathcal{M})$, with $\mu = \nu + \mu_0$, where ν only assumes the values 0 or $+\infty$.*

Proof. First, a small Lemma. We claim that $\mu_0 = \mu$ on finite sets. Let $E \in \mathcal{M}$, and $\mu(E) < +\infty$, since

$$\mu(E) \in \{\mu(F), F \subseteq E, \mu(F) < +\infty\} \implies \mu(E) \leq \mu_0(E)$$

Next, for every $W \subseteq E$, $\mu(W) \leq \mu(E)$, so $\mu_0(E) \leq \mu(E)$. This proves the equality.

If E is any measurable subset of \mathbf{X} , and suppose also $\mu_0(E) = +\infty$, one can easily find subsets of E , $\{E_n\}_{n \geq 1}$ with

$$n \geq \mu(E_n) < +\infty$$

But E_n is a subset of finite measure, so $0 < \mu(E_n) = \mu_0(E_n) < +\infty$. This proves the semi-finiteness of μ_0 .

Next, suppose μ is semi-finite, and fix any measurable set E . If E is of finite measure, then $\mu(E) = \mu_0(E)$, and if $\mu(E) = +\infty$, apply Exercise 14, so there exists a sequence of subsets of finite measure $E_n \subseteq E$ for every $n \geq 1$, with $\mu(E_n) \rightarrow \mu(E)$. Therefore $\mu_0(E) = \mu(E)$.

For the last part of the proof, let μ be an arbitrary measure. And let $E \in \mathcal{M}$. If $\mu(E) < +\infty$, then $\nu(E) = 0$ would suffice (this proves the first property of the measure). If $\mu(E) = +\infty$, and if $\mu(E)$ is not semi-finite, then set $\nu(E) = +\infty$. So that $\mu_0(E) + \nu(E) = 0 + \infty = \infty = \mu(E)$. The additivity of ν is immediate, since ν can only assume two values. This finishes the proof. ■

Exercise 1.16**Proposition 16.1.***Proof.*

Exercise 1.17

Proposition 17.1. *Let $\{A_j\}_{j \geq 1}$ be a countable disjoint sequence in \mathcal{M} , and denote $B_n = \bigcup A_{j \leq n} \in \mathcal{M}$. For every $E \subseteq X$,*

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j \leq n})$$

Proof. Proven in Theorem 1.11 as a Lemma. ■

Exercise 1.18

Proposition 18.1. Let $\mathcal{A} \subseteq \mathbb{P}(\mathbf{X})$ be an algebra. \mathcal{A}_σ the collection of countable unions of sets in \mathcal{A} , and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersection of sets in \mathcal{A}_σ . Let μ_0 be a premeasure on \mathcal{A} , and μ^* be the induced outer-measure.

- (a) For any $E \subseteq \mathbf{X}$, and $\varepsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.
- (b) If $\mu^*(E) < +\infty$, then E is μ^* -measurable \iff there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.
- (c) If μ_0 is σ -finite, the restriction $\mu^*(E) < +\infty$ in (b) is superfluous.

Proof of Part A. Let $E \subseteq \mathbf{X}$ and $\varepsilon > 0$, then by definition of μ^* ,

$$\mu^*(E) + \varepsilon \geq \sum \mu_0(A_j) = \sum \mu^*(A_j) \geq \mu^*(A)$$

by subadditivity and $A = \bigcup A_j$. ■

Proof of Part B. Suppose E is outer-measurable and of finite outermeasure, then by part A we have a sequence of $A_n \in \mathcal{A}_\sigma$ with

$$\mu^*(E) + n^{-1} \geq \mu^*(A_n) \implies \mu^*(E) = \mu^*(A)$$

if we define $A = \bigcap A_n \supseteq E$. Using the μ^* -measurability of E , we get

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) < +\infty \implies \mu^*(A \setminus E) = 0$$

Conversely, if $\mu^*(A \setminus E) = 0$, for any $V \subseteq \mathbf{X}$, with $\mu^*(V) < +\infty$, we have

$$\begin{aligned} \mu^*(V) &= \mu^*(V \cap A) + \mu^*(V \setminus A) \\ &\geq \mu^*(V \cap E) + \mu^*(V \setminus A) + \mu^*(V \cap [A \setminus E]) \\ &\geq \mu^*(V \cap E) + \mu^*(V \setminus E) \end{aligned}$$
■

Proof of Part C. Suppose μ_0 is σ -finite, then $E \in \mathcal{M}^*$ induces a sequence $E_j \nearrow E$, where each E_j is of finite measure. By part b) we obtain $\{A_j\} \subseteq \mathcal{A}_{\sigma\delta}$ with

$$\mu^*(A_j \setminus E_j) = 0$$

Now define $B = \bigcup A_j$, so that $B \in \mathcal{A}_{\sigma\delta}$. Observe $\bigcup (A_j \setminus E_j) = B \setminus E_1 \supseteq B \setminus E$ (verify these). And $\mu^*(B \setminus E) \leq \sum \mu^*(A_j \setminus E_j) = 0$ by subadditivity. Since $B \supseteq E$, and $B \in \mathcal{A}_{\sigma\delta}$, this proves \implies .

Conversely, suppose $E \subseteq X$ and there exists a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$, $\mu^*(B \setminus E) = 0$. Let $\{X_j\} \nearrow X$ as a sequence of sets of finite measure. Then,

$$(X_j \cap B) \setminus (X_j \cap E) = X_j \cap (B \setminus E) \subseteq B \setminus E$$

$X_j \cap B \in \mathcal{A}_{\sigma\delta}$, and $X_j \cap B \supseteq (X_j \cap E)$. Each $E_j = X_j \cap E$ is μ^* measurable by monotonicity, so is their countable union. \blacksquare

Exercise 1.19

Proposition 19.1. *Let μ^* be an outer measure on \mathbf{X} induced from a finite premeasure μ_0 . If $E \subseteq \mathbf{X}$, define the inner measure of E to be $\mu_*(E) = \mu_0(\mathbf{X}) - \mu^*(E^c)$. Then E is μ^* -measurable iff $\mu^*(E) = \mu_*(E)$.*

Proof. Suppose $E \subseteq \mathbf{X}$ is μ^* -measurable. Then

$$\mu^*(\mathbf{X}) = \mu^*(\mathbf{X} \cap E) + \mu^*(\mathbf{X} \setminus E) = \mu_0(\mathbf{X})$$

Rearranging gives the result, since all quantities are finite.

If $\mu^*(E) = \mu_*(E)$, then $\mu^*(E^c) = \mu_*(E^c)$, since the definition of μ_* is symmetric. Let $B \in \mathcal{A}_{\sigma\delta}$, with $\mu^*(B) = \mu^*(E)$, $E \subseteq B$. We can always find such a B by taking the intersection over all $B_n \in \mathcal{A}_\sigma$,

$$\mu^*(E) + n^{-1} \geq \sum_j \mu^*(B(j, n)) \geq \mu^*\left(\bigcup_j B(j, n) = B_n\right)$$

Notice $E \subseteq B \iff E^c \supseteq B^c \iff E^c \cap B^c = B^c$. Since B is μ^* -measurable, we have

$$\begin{aligned} \mu^*(E^c \cap B) + \mu^*(E^c \setminus B) &= \mu^*(E^c) \\ &= \mu^*(\mathbf{X}) - \mu^*(E) \\ \mu^*(B \setminus E) + \mu^*(B^c) &= \mu^*(\mathbf{X}) - \mu^*(E) \\ &= \mu^*(B) + \mu^*(B^c) - \mu^*(E) \\ \mu^*(B \setminus E) &= \mu^*(B) - \mu^*(E) \\ &= 0 \end{aligned}$$

■

Exercise 1.20**Proposition 20.1.***Proof.*

Exercise 1.21

Proposition 21.1. *Let μ^* be an outermeasure induced from a premeasure, and $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$, where \mathcal{M}^* denotes the family of μ^* -measurable sets. Show that $\bar{\mu}$ is saturated. That is, $\widetilde{\mathcal{M}^*} = \mathcal{M}^*$*

Proof. Suppose E is locally measurable (with respect to $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$). Fix $V \subseteq \mathbf{X}$, with $\mu^*(V) < +\infty$. It suffices to show $\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$.

By 18a), find a $V' \in \mathcal{A}_{\sigma\delta}$, with $V \subseteq V'$, and $\mu^*(V') = \mu^*(V) < +\infty$. so that $E \cap V'$ is μ^* -measurable.

$$\mu^*(V) = \mu^*(V \cap E \cap V') + \mu^*(V \setminus (V \cap (V' \cap E)))$$

therefore

$$\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$$

■

Exercise 1.22

Proposition 22.1.

Proof. To show $\bar{\mu}$ is complete, Fix $U \subseteq F$, where $F \in \mathcal{M}^*$, with $\bar{\mu}(F) = 0$. Let $F' \in \mathcal{A}_{\sigma\delta}$, with $F' \supseteq F$, and

$$\mu^*(F') = \mu^*(F) \geq \mu^*(F' \setminus U)$$

Since $F' \supseteq U$, applying Exercise 18b gives $\overline{\mathcal{M}^*} \subseteq \mathcal{M}^*$. For the other direction,

■

Exercise 1.23**Proposition 23.1.***Proof.*

Exercise 1.24

Proposition 24.1. *If μ is a finite measure on (X, \mathcal{M}) , and let μ^* be the outer measure. Suppose that $E \subseteq X$ satisfies $\mu^*(E) = \mu^*(X)$ (but $E \notin \mathcal{M}$ necessarily). Show that*

- (a) *For any $A, B \in \mathcal{M}$, and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.*
- (b) *Let $\mathcal{M}_E = \{A \cap E, A \in \mathcal{M}\}$, and define ν on \mathcal{M} with $\nu(A \cap E) = \mu(A)$. Then \mathcal{M}_E is a σ -algebra, and ν is a measure on \mathcal{M}_E .*

Proof of Part A.

$$\mu^*(E) = \mu^*(X) \implies \mu^*(X \setminus E) = 0$$

This is a simple consequence of the μ^* -measurability of X , since $X \in \mathcal{M}$, and the μ is a pre-measure on \mathcal{M} , and by monotonicity,

$$\begin{cases} A \cap (X \setminus E) \subseteq (X \setminus E) \\ B \cap (X \setminus E) \subseteq (X \setminus E) \end{cases} \implies \begin{cases} \mu^*(A \cap (X \setminus E)) = 0 \\ \mu^*(B \cap (X \setminus E)) = 0 \end{cases}$$

Write $A \cap X = (A \cap E) \cup (A \cap X \setminus E)$, and by subadditivity of μ^* ,

$$\begin{aligned} \mu(A) &= \mu^*(A \cap X) \\ &\leq \mu^*(A \cap E) + \mu^*(X \setminus E) \\ &= \mu^*(B \cap E) \\ &\leq \mu^*(B \cap X) \\ &= \mu(B) \end{aligned}$$

Therefore $\mu(A) \leq \mu(B)$, and $\mu(B) \leq \mu(A)$ is trivial. ■

Proof of Part B. We want to show \mathcal{M}_E is a σ -algebra.

- Closure under complements,

$$\forall A \cap E \in \mathcal{M}_E, A \in \mathcal{M} \implies (E \setminus A^c) \in \mathcal{M}_E$$

Therefore $(E \setminus A^c) \cap E \in \mathcal{M}_E$. Note that the question mentions that \mathcal{M}_E is a σ -algebra on E , therefore we take complements relative to E .

- Closure under countable unions. Fix any countable sequence $\{A_j \cap E\} \subseteq \mathcal{M}_E$ where $\{A_j\} \subseteq \mathcal{M}$. It is obvious that $A = \cup A_j \in \mathcal{M}$, therefore $\cup(A_j \cap E) = E \cap A \in \mathcal{M}_E$ as well.

Since $\nu(\emptyset) = \mu(\emptyset \cap E) = 0$, and for countable additivity, fix any disjoint sequence $\{A_j \cap E\}_{j \geq 1} \subseteq \mathcal{M}_E$, where $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$, and let $A = \cup A_{j \geq 1}$

$$\begin{aligned}\nu(A \cap E) &= \mu(A) \\ &= \sum \mu(A_{j \geq 1}) \\ &= \sum \nu(A_{j \geq 1} \cap E)\end{aligned}$$

■