MATH 263: Section 003, Tutorial 3

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1 Bernoulli Equations (review)

A **Bernoulli equation** is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

When $n \notin \{0, 1\}$, we can let $v = y^{1-n}$, making the ODE linear for v.

Problem 1. Use the substitution $v = y^{1-n}$ to turn the general Bernoulli equation into a linear ODE for v. You do **not** have to solve for v.

2 Exact ODEs

An **exact ODE** is of the form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$
$$M(x, y)\ dx + N(x, y)\ dy = 0$$

where

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Then, we can define some $F(x, y)$ such that:

$$d(F(x, y)) = M(x, y)\ dx + N(x, y)\ dy = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$$

Then, find $F(x, y)$, and the relation $F(x, y) = \int 0\ dx = C$ is the solution.

Problem 2.2a. From Boyce and DiPrima, 10th edition (2.6, exercise 14, p.101):
Check whether the differential equation:

$$(9x^2 + y - 1)\ dx + (x - 4y)\ dy = 0$$

is exact. If so, solve the IVP $y(1) = 0$.

Note: when the ODE is not exact, an integrating factor may make it exact. Let it be $\mu(x, y)$:

$$(\mu(x, y)M(x, y)) + (\mu(x, y)N(x, y))\frac{dy}{dx} = 0$$

To make the ODE exact,

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$$

$$\frac{\partial N}{\partial x}\mu + N\frac{\partial \mu}{\partial x} = \frac{\partial M}{\partial y}\mu + M\frac{\partial \mu}{\partial y}$$

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu + N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = 0.$$

Instead of solving a PDE, consider two specific cases:

1. $\mu(x, y) = \mu(x)$:

$$(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\mu = N\frac{d\mu}{dx}$$

$$\mu(x) = \exp[\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\ dx]$$

2. $\mu(x, y) = \mu(y)$:

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu = M\frac{d\mu}{dy}$$

$$\mu(y) = \exp[\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\ dy.]$$

Try both cases if needed, your integrating factor must be single-variable for it to work. In the general case, both or none may work.

Problem 2.2b. From Boyce and DiPrima, 10th edition (2.6, exercise 28, p.102):
Find the general solution of:

$$y\ dx + (2xy - e^{-2y})\ dy = 0.$$

3 Change of Variables

As explained in tutorial 2, various substitutions can be used, such as $v = y'(x)$ *or* $v = ax + by$.

Problem 2.3. Find the general solution of:

$$y'' = x^2 + 2xy' + (y')^2.$$

Hint: Use the substitution $v(x) = x + y'(x)$.

4 Slope Fields

(From Tutorial 2): A **slope field** is a graphical representation of a family of functions satisfying $y' = f(x, y)$. For some point (x, y), one draws the slope $y' = f(x, y)$ to qualitatively represent solutions. Given a slope field, starting at an initial condition and tracing along the field sketches the particular solution.

Problem 2.4. Draw the slope field of

$$y' = x(y^2 + 2y - 3).$$