

# Chapter 8

**Theorem 8.1**

Proposition 1.1

*Proof.*



**Theorem 8.2**

Proposition 2.1

*Proof.*



**Theorem 8.3****Proposition 3.1**

If  $f \in C^\infty$ , then  $f \in \mathcal{S}$  if and only if  $x^\beta \partial^\alpha f$  is bounded for all multi-indices  $\alpha, \beta$

*Proof.* ■

**Theorem 8.4**

Proposition 4.1

*Proof.*



**Theorem 8.5**

Proposition 5.1

*Proof.*



**Theorem 8.6**

Proposition 6.1

*Proof.*



**Theorem 8.7**

Proposition 7.1

*Proof.*





**Theorem 8.8**

Proposition 8.1

*Proof.*



**Theorem 8.9**

Proposition 9.1

*Proof.*



**Theorem 8.10**

Proposition 10.1

*Proof.*



**Theorem 8.11**

**Proposition 11.1**

*Proof.*



**Theorem 8.12**

**Proposition 12.1**

*Proof.*



**Theorem 8.13**

Proposition 13.1

*Proof.*



**Theorem 8.14****Proposition 14.1**

Suppose  $\phi \in L^1$ , and  $\int \phi(x)dx = a$ .

- (a) If  $f \in L^p$ ,  $p \in [1, +\infty]$ , then  $f * \phi_t \rightarrow af$  in the  $L^p$  norm as  $t \rightarrow 0$ .
- (b) If  $f$  is bounded and uniformly continuous, then  $f * \phi_t \rightarrow af$  uniformly as  $t \rightarrow 0$ .
- (c) If  $f \in L^\infty$  and  $f$  is continuous on an open set  $U$ , then  $f * \phi_t \rightarrow af$  uniformly on compact subsets of  $U$  as  $t \rightarrow 0$

*Proof of Part A.* First, the convolution  $f * \phi_t$  is in  $L^p$  by Young's Inequality (Theorem 8.7). Furthermore,

$$f * \phi_t - af = \int_{y \in \mathbb{R}^n} f(x-y)t^{-n}\phi(t^{-1}y)dy - \int_{y \in \mathbb{R}^n} f(x)\phi(y)dy \quad (1)$$

Now apply Theorem 2.44, with  $y \mapsto y/t$ , and denote this invertible map by  $T \in GL(n, \mathbb{R})$ , so that  $|\det(T)| = t^{-n}$ , then  $y = T(y)t$  for every  $t > 0$ . It follows that

$$\begin{aligned} (f * \phi_t)(x) &= |\det(T)| \cdot \int_{y \in \mathbb{R}^n} f(x - t \cdot Ty)\phi(T(y))dy \\ &= \int_{z \in \mathbb{R}^n} f(x - tz)\phi(z)dz \\ &= \int_{z \in \mathbb{R}^n} \tau_{tz}f(x)\phi(z)dz \end{aligned} \quad (2)$$

Next,  $a = \int \phi$  so  $af = \int f(x)\phi(z)dz$ . Using Equations (1) and (2) we get

$$(f * \phi_t - af)(x) = \int_{z \in \mathbb{R}^n} (\tau_{tz}f - f)\phi(z)dz \quad (3)$$

We wish to apply Minkowski's Inequality for integrals, which states, roughly speaking:

The norm of an integral is less than the integral of the norm.

to Equation (3), and

$$\|f * \phi_t - af\|_p \leq \int_{z \in \mathbb{R}^n} \|(\tau_{tz}f - f)\phi(z)\|_p dz \quad (4)$$

The assumptions for Theorem 6.19 are satisfied by

1. Notice for every  $z \in \mathbb{R}^{n'}$ ,

$$\|(\tau_{tz}f - f)\phi(z)\|_p = \left( \int_{x \in \mathbb{R}^n} |(\tau_{tz}f(x) - f(x))\phi(z)|^p dx \right)^{1/p} \leq |\phi(z)| \left( 2\|f\|_p \right) < +\infty$$

Since  $\|\phi\|_1 < +\infty$ ,  $|\phi(z)| < +\infty$  almost everywhere.

2. Next, to show  $z \mapsto \|\phi(z)(\tau_{tz}f - f)\|_p$  is in  $L^1\mathbb{R}^n$ ,  $z$ . Reuse the last estimate in the previous bullet point, and

$$\|\phi(z)(\tau_{tz}f - f)\|_p \leq |\phi(z)| \left( 2\|f\|_p \right)$$

Taking the integral in  $L^+$  with respect to  $z$ , we get

$$\left\| \left( \|\phi(z)(\tau_{tz}f - f)\|_p \right) \right\|_1 < +\infty$$

so both assumptions are satisfied.

Therefore Equation (4) holds. Next, fix any sequence of  $t_n > 0$  with  $t_n \rightarrow 0$ . The Dominated Convergence Theorem gives, since  $|\phi(z)|\|\tau_{t_n z}f - f\|_p$  is dominated by  $|\phi(z)| \cdot 2\|f\|_p \in L^1 \cap L^+$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{z \in \mathbb{R}^n} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz &= \int_{z \in \mathbb{R}^n} \lim_{n \rightarrow \infty} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz \\ &= \int_{z \in \mathbb{R}^n} 0 dz \\ &= 0 \end{aligned}$$

The second last equality is from Lemma 8.4, as translation is continuous in the  $L^p$  norm for  $p \in [1, +\infty)$ . So almost every  $z \in \mathbb{R}^n$  (since again,  $|\phi(z)|$  can be infinite on a null set),

$$\|\tau_{t_n z}f - f\|_p \rightarrow 0 \implies \|\tau_{t_n z}f - f\|_p |\phi(z)| \rightarrow 0$$

as  $n \rightarrow +\infty$ . It follows that

$$\lim_{n \rightarrow \infty} \|f * \phi_{t_n} - af\|_p = \lim_{n \rightarrow \infty} \left\| \int_{z \in \mathbb{R}^n} [\tau_{t_n z}f(x) - f(x)] \phi(z) dz \right\|_p = 0$$

Since the sequence  $t_n \rightarrow 0$  is arbitrary, we conclude that the function  $t \mapsto \|f * \phi_t - af\|_p$  has a limit of 0 as  $t \rightarrow 0$ . ■

*Proof of Part B.* Suppose  $f \in \text{UBC}(\mathbb{R}^n)$ , so that  $f$  is uniformly continuous and bounded. We wish to show  $f * \phi_t \rightarrow af$  uniformly as  $t \rightarrow 0$ . In symbols,

$$g : t \mapsto \|f * \phi_t - af\|_u, \quad g \rightarrow 0, \text{ as } t \rightarrow 0$$

The convolution between  $f$  and  $\phi_t$  makes sense at every  $x \in \mathbb{R}^n$ , as

$$\int |\tau_y f(x)| |\phi(y)| dy \leq \|f\|_u \cdot \|\phi\|_1 < +\infty$$



Taking the supremum norm on both sides of Equation (3), we get

$$\begin{aligned}
 \|f * \phi_t - af\|_u &= \sup_{x \in \mathbb{R}^n} \left| \int_{z \in \mathbb{R}^n} (\tau_{tz}f - f) \cdot \phi(z) dz \right| \\
 &\leq \sup_{x \in \mathbb{R}^n} \int_{z \in \mathbb{R}^n} |\tau_{tz}f - f| \cdot |\phi(z)| dz \\
 &\leq \int_{z \in \mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |\tau_{tz}f - f| \cdot |\phi(z)| dz \\
 &= \int_{z \in \mathbb{R}^n} \|\tau_{tz}f - f\|_u \cdot |\phi(z)| dz
 \end{aligned} \tag{5}$$

the last equality is a simple consequence of the monotonicity of the integral in  $L^+$ , indeed. For every  $x \in \mathbb{R}^n$ , the following holds pointwise for almost every  $z$

$$|\tau_{tz}f - f| \leq \|\tau_{tz}f - f\|_u \implies \sup_{x \in \mathbb{R}^n} |\tau_{tz}f - f| \leq \|\tau_{tz}f - f\|_u$$

Apply the Dominated Theorem to the right member of (5), noting that it is dominated by  $|\phi(z)| \cdot 2\|f\|_u \in L^1 \cap L^+$  as we have done for Part A of the proof. Since this holds for every sequence  $t_n \rightarrow 0$ , the proof is complete.  $\blacksquare$

*Proof of Part C.* Next, suppose that  $f \in L^\infty$ , and  $f \in C(U)$ , where  $U$  is open in  $\mathbb{R}^n$ . We claim that

$$f * \phi_t \rightarrow af$$

within the topology of uniform convergence on compact subsets of  $U$ . So that for every  $K \in \mathfrak{A}$ ,  $K \subseteq U$

$$\sup_{x \in K} |f * \phi_t - af| \rightarrow 0, \text{ as } t \rightarrow 0$$

First, a small technical Lemma.

**Lemma 14.1**

If  $\phi \in L^1(\mathbb{R}^n)$ , then for every  $\varepsilon > 0$ , there exists  $E \in \mathfrak{A}$ , with

$$\int_{E^c} |\phi| = \|\phi \chi_{E^c}\|_1 < +\varepsilon$$

*Proof.* Assume that  $\phi \geq 0$ , if not, replace  $\phi$  by  $|\phi|$ . Since  $C_c(\mathbb{R}^n)$  is dense in  $L^1$  for every  $\varepsilon 2^{-1} > 0$  there exists some  $\psi \in C_c(\mathbb{R}^n)$  with  $\|\psi - \phi\|_1 < \varepsilon^{-1}$ , and denote  $E = \text{supp}(\psi) \in \mathfrak{A}$ , then

$$\|\psi - \phi\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

So we can assume  $\psi \geq 0$  as well, perhaps by relabelling  $\psi$  by  $|\psi|$ . Then,

$$\|\psi - \chi_E \phi\|_1 = \|\chi_E(\psi - \phi)\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

by monotonicity in  $L^+$ . The Triangle Inequality in  $L^1$  gives

$$\|\chi_{E^c} \phi\|_1 = \|\phi - \chi_E \phi\|_1 = \|\phi(1 - \chi_E)\|_1 \leq \|\phi - \psi\|_1 + \|\psi - \chi_E \phi\|_1 < \varepsilon$$

■

Back to the main proof of Part C, fix any  $\varepsilon > 0$ , then by Lemma 14.1,  $\phi$  induces some  $E \in \mathcal{J}$  with  $\|\chi_{E^c} \phi\|_1 < +\varepsilon$ . By Lemma 8.4,  $\chi_K f \in C_c(\mathbb{R}^n) \subseteq \text{UBC}(\mathbb{R}^n)$ . Uniform continuity of  $\chi_K f$  gives us the continuity of translations. Now for the same  $\varepsilon > 0$ , there exists  $r > 0$ , for every  $w \in \mathbb{R}^n$ ,

$$|w| < r \implies \|\tau_w \chi_K f - \chi_K f\|_u < +\varepsilon \quad (6)$$

Since  $E \in \mathcal{J}$ , it is bounded, and let  $t$  be a small positive number such that for every  $z \in E$ ,

$$|tz| < t \cdot (1 + \sup_{z \in E} |z|) < r$$

There exists such a  $t$ , namely  $t = r 2^{-1} (1 + \sup_{z \in E} |z|)^{-1}$ . And for this  $t > 0$ , it follows that for every  $z \in E$ ,

$$\sup_{x \in K} |\tau_{tz} f - f| < +\varepsilon$$

Since this holds for every  $z \in E$ , we write

$$\sup_{x \in K, z \in E} |\tau_{tz} f - f| < +\varepsilon$$

And

$$|\phi(z)| \left[ \sup_{x \in K, z \in E} |\tau_{tz} f - f| \right] < |\phi(z)| \varepsilon$$

Monotonicity in  $L^+(E, z)$  reads, for every  $x \in K$ ,

$$\int_{z \in E} |\phi(z)(\tau_{tz} f - f)| dz \leq \int_{z \in E} |\phi| \varepsilon dz = \varepsilon \|\chi_E \phi\|_1 \leq \varepsilon \|\phi\|_1$$

Since this holds for every  $x \in \mathbb{R}^n$ ,

$$\sup_{x \in K} \left\{ \int_{z \in E} |\phi(z)| \cdot |\tau_{tz} f - f| dz \right\} \leq \varepsilon \|\phi\|_1 \quad (7)$$

Next, notice for every  $t, z$ , we have

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f\|_u + \|f\|_u \leq 2 \cdot \|f\|_u$$

And the following holds  $z \in E^c$  a.e,

$$|\phi(z)| \cdot |\tau_{tz} f - f| \leq |\phi(z)| \cdot 2 \|f\|_u$$

Taking the integral, and applying the condition we imposed on  $E$  from Lemma (14.1), so that

$$\int_{z \in E^c} |\phi(z)| \cdot |\tau_{tz}f - f| dz \leq 2\|f\|_u \int_{z \in E^c} |\phi(z)| dz \leq 2\|f\|_u \varepsilon$$

Taking the supremum of the above estimate, so

$$\sup_{x \in K} \left\{ \int_{z \in E^c} |\phi(z)(\tau_{tz}f - f)| dz \right\} \leq 2\|f\|_u \varepsilon \quad (8)$$

Combining Equations (7) and (8). Applying the additivity of the supremum (of  $x \in K$ ), since both members are finite,

$$\sup_{x \in K} \left\{ \int_E |\phi(z)|(\tau_{tz}f - f) dz + \int_{E^c} |\phi(z)|(\tau_{tz}f - f) dz \right\} < \varepsilon(2\|f\|_u + \|\phi\|_1)$$

The left member above is equal to  $\sup_{x \in K} |f * \phi_t - af|$ . Since  $\varepsilon > 0$  is arbitrary, this completes the proof of Part C. ■

**Theorem 8.15****Proposition 15.1**

If  $|\phi(x)| \leq C(1 + |x|)^{-n-\varepsilon}$ , where  $\varepsilon > 0$ , and if  $f \in L^p$ , for  $p \in [1, +\infty)$ , then

$$f * \phi_t \rightarrow af$$

pointwise for every  $x$  in the Lebesgue set of  $f$ ,

$$\mathcal{L}_f = \left\{ x \in \mathbb{R}^n, \quad \lim_{r \rightarrow 0} \frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy = 0 \right\}$$

We also claim that  $m(\mathcal{L}_f^c) = 0$ , and  $x \in \mathcal{L}_f$  at every continuous  $f(x)$ .

The proof is long, and will be divided into several parts. Let us start with a couple of Lemmas about the Lebesgue Set of  $f$ , and several pointwise estimates that will be of use.

**Lemma 15.1**

If  $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$ , and

$$|\phi(x)| \leq C(1 + |x|)^{n-\varepsilon}, \quad \varepsilon > 0 \tag{9}$$

then  $\phi \in L^1$ . Furthermore,  $\phi_t \in L^1$  for every  $t > 0$ .

*Proof of 15.1.* If  $x \neq 0$ , then

$$|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)} \leq C \cdot |x|^{-(n+\varepsilon)}$$

on some  $B^c$  as defined in Theorem 2.52, so  $\phi \in L^1(B^c)$ . Next,

$$n + \varepsilon > n > n/2 = a$$

and by monotonicity,

$$|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)} \leq C \cdot (1 + |x|)^{-(n/2)}$$

so  $\phi \in L^1(\mathbb{R}^n)$ . Next, if  $\phi \in L^1$ , then

$$|\phi_t(x)| = t^{-n} |\phi(t^{-1}x)|$$

taking the integral in  $L^+$ , and applying Theorem 2.44, with  $T : x \mapsto t^{-1}x$ , and  $\det(T) = t^{-n}$ , so that

$$\int |\phi_t|(x) dx = |\det(T)| \int |\phi| \circ T(x) dx = \int |\phi|(x) dx < +\infty$$

This completes the Lemma. ■

**Lemma 15.2**

If  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , and if  $f \in C(\mathbb{R}^n)$ , then  $\mathcal{L}_f = \mathbb{R}^n$ .

*Proof of 15.2.* Let  $x \notin \mathcal{L}_f$ , and there exists a sequence  $r_k \rightarrow 0$  and  $\varepsilon_0 > 0$  but

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy \geq \varepsilon_0$$

We claim that for every  $k \geq 1$ , we can find a  $y_k \in B(r_k, x) \setminus \{x\}$  with

$$|f(x) - f(y_k)| \geq \varepsilon_0$$

Indeed, suppose by contradiction that no such  $y_k$  exists, and by monotonicity,

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy < \frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} \varepsilon_0 dy = \varepsilon_0$$

So choose  $y_k$  as above, and it is clear that  $y_k \rightarrow x$  as  $k \rightarrow \infty$ , but  $f(y_k) \not\rightarrow f(x)$ . Therefore  $f$  is not continuous at  $x$ . ■

**Lemma 15.3**

If  $x \in \mathcal{L}_f$ , then for every  $\delta > 0$  there exists a  $\eta > 0$ , with

$$r \leq \eta \implies \int_{|y| < r} |f(x - y) - f(x)| dy \leq \delta \cdot r^n$$

*Proof of 15.3.* We will start with something trivial.

$$m(B(r)) = r^n m(B(1)) \tag{10}$$

where  $B(r) = \{x \in \mathbb{R}^n, |x| < r\}$ . By Theorem 2.44,

$$\begin{aligned} m(B(r)) &= \int \chi_{B(r)}(x) dx \\ &= |\det(T)|^{-1} \int \chi_B(x) dx \\ &= r^n m(B(1)) \end{aligned}$$

where  $T : x \mapsto x/r$  and  $\det(T) = r^{-n}$ . Fix  $x \in \mathcal{L}_f$ , and take  $\varepsilon = \delta/m(B(1)) > 0$ , and by definition this induces some  $\eta > 0$ , and for every  $r \leq \eta$

$$\frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy \leq \varepsilon$$

By translation invariance of  $m$ ,

$$m(B(r, x)) = m(B(r)) = r^n \cdot m(B(1))$$

and apply the map  $y \mapsto x - y$ , which is a composition a rotation by  $|-1|$  and a translation by  $x \in \mathbb{R}^n$ . By Theorems 2.44 and 2.42,

$$\int_{|y| \in B(r)} |f(x) - f(x - y)| dy = \int_{y \in B(r, x)} |f(x) - f(y)| dy < \varepsilon m(B(1)) \cdot r^n = \delta r^n$$

where we used the fact that

$$\begin{aligned} d(x - y, x) < r &\iff d(-y, 0) < r \\ &\iff d(y, 0) < r \end{aligned}$$

hence

$$\chi_{B(r, x)}(x - y) = \chi_{B(r, 0)}(y)$$

■

#### Lemma 15.4

Let  $A_j = \left\{ |y| \in [2^{-j}\eta, 2^{1-j}\eta] \right\}$ , and if Equation (9) holds for  $\phi$  then  $\phi_t$  satisfies

$$|\phi_t| \leq C \cdot t^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \quad (11)$$

on  $A_j$  for every  $t > 0$ , where  $\alpha = t^{-1}\eta$  for some  $\eta > 0$ .

Moreover, if  $A_0 = \left\{ |y| < 2^{-K}\eta \right\}$ , where  $K \geq 0$ , then

$$|\phi_t(y)| \leq C \cdot t^{-n} \quad (12)$$

on  $A_0$

*Proof of 15.4.* Notice that

$$t^{-1}y \in [2^{-j} \cdot \eta/t, 2^{1-j} \cdot \eta/t] = [2^{-j} \cdot \alpha, 2^{1-j} \cdot \alpha]$$

And

$$1 + |t^{-1}y| \geq |t^{-1}y| \geq 2^{-j}\alpha$$

Therefore

$$C \cdot t^{-n} (1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)}$$

and applying Equation (9) establishes the first claim.

The second claim follows from Equation (9),

$$|\phi_t(y)| \leq C \cdot t^{-n} (1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}$$

■

*Main Proof of Theorem 8.15.* The outline of the proof is as follows,

1.  $|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)}$  for  $\varepsilon > 0$  and
2.  $f \in L^p$  for  $p \in [1, +\infty)$ ,
3. for any  $x \in \mathcal{L}_f$ , we wish to show

$$|f * \phi_t - af|(x) \rightarrow 0, \quad \text{as } t \rightarrow 0$$

4. To prove this, we fix some  $\beta > 0$  and show that

$$|f * \phi_t - af|(x) < \beta$$

since  $\beta$  is arbitrary, the proof will be complete.

5. By Lemma 15.3, for every  $\delta > 0$  there exists a  $\eta > 0$  where  $r \leq \eta$  implies

$$\int_{|y| < r} |f(x) - f(x - y)| dy \leq \delta \cdot r^n$$

and using the  $L^1$  inequality,

$$\begin{aligned} |f * \phi_t - af|(x) &= \left| \int [f(x - y) - f(x)] \cdot \phi_t(y) dy \right| \\ &\leq \int |f(x - y) - f(x)| \cdot |\phi_t(y)| dy \\ &= \int_{|y| < \eta} |f(x - y) - f(y)| \cdot |\phi_t(y)| dy + \int_{|y| \geq \eta} |f(x - y) - f(y)| \cdot |\phi_t(y)| dy \\ &= I_1 + I_2 \end{aligned}$$

6. Let  $\delta = \beta(2A)^{-1}$ , where

$$A = 2^n \cdot C \left[ \frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$$

we make the claim that this choice of  $\delta$  will give us  $I_1 < \beta/2$

7. After choosing  $\delta > 0$ , (which induces  $\eta > 0$ ), we will show that  $I_2 < \beta/2$  (for a fixed  $\eta > 0$ ) for  $t$  sufficiently small, and applying the Triangle Inequality finishes the proof.

Let  $\eta$  be as above, and for  $t > 0$  and suppose we can find a  $K \in \mathbb{N}^+$  with

$$2^K \leq \eta/t \leq 2^{K+1} \tag{13}$$

and define  $\alpha = \eta/t$  for convenience.

Notice for any  $K \geq 1$ , the interval  $[0, 1)$  can be partitioned in the following manner

$$[0, 1) = [0, 2^{-K}) \cup \left( \bigcup_{j=1}^K [2^{-j}, 2^{1-j}) \right)$$

and let us define

$$A_j = \left\{ |y| \in [2^{-j}\eta, 2^{1-j}\eta) \right\}, \quad A_0 = \left\{ |y| \in [0, 2^{-K}\eta) \right\}$$

If no such  $K$  exists, then let  $A_j = \emptyset$  and set  $A_0 = \{|y| \in [0, \eta)\}$ . The disjoint union of all  $A_{j \geq 0}$  is the open ball  $\{|y| \in [0, \eta)\}$ . By Lemma 15.4 and Lemma 15.3 each  $j \geq 0$ ,

$$\begin{aligned} I_1 &= \sum_{j=0}^K \int_{y \in A_j} |f(x-y) - f(y)| |\phi_t(y)| dy \\ &\leq Ct^{-n} \delta(2^{-K}\eta)^n + \sum_{j=1}^K \int_{y \in A_j} |f(x-y) - f(y)| |\phi_t(y)| dy \\ &\leq Ct^{-n} \delta(2^{-K}\eta)^n + \sum_{j=1}^K Ct^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \delta(2^{1-j}\eta)^n \end{aligned}$$

The left member reads,

$$\begin{aligned} Ct^{-n} \delta(2^{-K}\eta)^n &\leq C\delta\alpha^n 2^{-Kn} \\ &\leq C\delta 2^{n(K+1)} 2^{-Kn} \\ &= C\delta 2^n \end{aligned}$$

and termwise for the right,

$$\begin{aligned} Ct^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \delta(2^{1-j}\eta)^n &= C\delta \cdot t^\varepsilon \cdot 2^{j\varepsilon+n} \eta^{-\varepsilon} \\ &= (C\delta 2^n \alpha^{-\varepsilon}) \cdot 2^{j\varepsilon} \end{aligned}$$

Summing over the geometric series,

$$\begin{aligned} \sum_{j=1}^K 2^{j\varepsilon} &= 2^\varepsilon \sum_{j=0}^{K-1} 2^{j\varepsilon} \\ &= \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} \end{aligned}$$

using the estimate for  $\alpha$  in Equation (13)

$$\alpha \in [2^K, 2^K + 1) \implies \alpha^{-\varepsilon} \in [2^{-\varepsilon(K+1)}, 2^{-\varepsilon K})$$

and combining the last few equations, the right member becomes

$$\begin{aligned} (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} &\leq (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)}}{2^\varepsilon - 1} \\ &\leq (C\delta 2^n) \cdot \frac{2^\varepsilon}{2^\varepsilon - 1} \end{aligned}$$



Finally,  $I_1 \leq (C\delta 2^n) \left[ \frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$ , and by Step 6,  $I_1 \leq \beta/2$ .

Obtaining an estimate for  $I_2$  is another laborious enterprise. Let us define  $W = \{|y| \geq \eta\}$ , and

- By Holder's Inequality,

$$I_2 \leq \|f\|_p \|\chi_W \cdot \phi_t\|_q + |f(x)| \|\chi_W \cdot \phi_t\|_1$$

where  $q$  is the conjugate exponent to  $p$ . Since  $p \in [1, +\infty)$ , it suffices to show  $\|\chi_W \cdot \phi_t\|_q \rightarrow 0$  as  $t \rightarrow 0$  for  $q \in [1, +\infty]$ .

- Suppose  $q = +\infty$ ,

$$y \in W \iff |y| \geq \eta \iff |t^{-1}y| \geq \alpha$$

$$\text{then } \|\chi_W \cdot \phi_t\|_\infty \leq Ct^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq Ct^\varepsilon \eta^{-(n+\varepsilon)}$$

- Now suppose  $q \in [1, +\infty)$ , by polar integration and Theorems 2.51, 2.52 (brace yourselves):

$$\begin{aligned} \|\chi_W \cdot \phi_t\|_q^q &= t^{-nq} \cdot \int_{y \in W} C^q \cdot |t^{-1}y|^{-q \cdot (n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \int_{|y| \geq \eta} |y|^{-q \cdot (n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \sigma(S^{n-1}) \int_{r \geq \eta} r^{n-1} \cdot r^{-q \cdot (n+\varepsilon)} dr \\ &= \frac{C^q t^{\varepsilon q}}{n - q \cdot (n + \varepsilon)} r^{n - q \cdot (n + \varepsilon)} \Big|_\eta^\infty \\ &= \frac{C^q t^{\varepsilon q}}{q \cdot (n + \varepsilon) - n} \eta^{n - q \cdot (n + \varepsilon)} \\ \|\chi_W \cdot \phi_t\|_q &= \left[ \frac{C}{(q \cdot (n + \varepsilon) - n)^{1/q}} \left( \eta^{n - q \cdot (n + \varepsilon)} \right)^{1/q} \right] t^\varepsilon \\ &= C_3(q) t^\varepsilon \end{aligned}$$

- Find a  $t$  sufficiently small so that

$$t^\varepsilon < \min \left\{ \beta(4C_3(1)|f(x)|)^{-1}, \beta(4C_3(q)\|f\|_p)^{-1}, \beta(4C \cdot \eta^{-(n+\varepsilon)})^{-1} \right\}$$

- Therefore  $I_2 < \beta/2$ , and the proof is complete upon sending  $\beta \rightarrow 0$ .

■

**Theorem 8.16****Proposition 16.1**

See Theorem 8.15

*Proof.*



**Theorem 8.17**

Proposition 17.1

*Proof.*



**Theorem 8.18**

**Proposition 18.1**

*Proof.*



**Theorem 8.19**

Proposition 19.1

*Proof.*



**Theorem 8.20**

Proposition 20.1

*Proof.*

