

# Chapter A: Review of Topology

## Set Operations

This section is meant for reference.

### Proposition 1.1: Direct and Inverse Images of Maps

Let  $f : \mathbf{X} \rightarrow \mathbf{Y}$ , where  $\mathbf{X}$  and  $\mathbf{Y}$  are sets. If  $A \subseteq \mathbf{X}$ ,  $B \subseteq \mathbf{Y}$ , and  $\{E_\alpha\}$  is an indexed collection of subsets of  $\mathbf{X}$ ,  $\{G_\beta\}$  is an indexed collection of subsets of  $\mathbf{Y}$ , then

Direct images

$$f\left(\bigcap E_\alpha\right) \subseteq \bigcap f(E_\alpha) \quad \text{equality if injective} \quad (1)$$

$$f\left(\bigcup E_\alpha\right) = \bigcup f(E_\alpha) \quad (2)$$

Estimates

$$f\left(f^{-1}(B)\right) \subseteq B \quad \text{equality if surjective} \quad (3)$$

$$A \subseteq f^{-1}(f(A)) \quad \text{equality if injective} \quad (4)$$

Inverse images

$$f^{-1}\left(\bigcup G_\beta\right) = \bigcup f^{-1}(G_\beta) \quad (5)$$

$$f^{-1}\left(\bigcap G_\beta\right) = \bigcap f^{-1}(G_\beta) \quad (6)$$

$$f^{-1}(B^c) = \left(f^{-1}(B)\right)^c \quad (7)$$

### Proposition 1.2: Composition of Maps

Let  $h = g \circ f$ , we assume this composition is well defined.

- If  $h$  is a surjection, then  $g$  is a surjection,
- If  $h$  is an injection, then  $f$  is an injection.

*Proof.* Take the contrapositive. ■

### Proposition 1.3: Left and Right inverses

Let  $F : \mathbf{X} \rightarrow \mathbf{Y}$ ,

- $F$  is surjective if and only if there exists right inverse  $G : \mathbf{Y} \rightarrow \mathbf{X}$ ,

$$F \circ G = \text{id}_{\mathbf{Y}}$$

if  $A \subseteq \mathbf{X}$ ,

$$G^{-1}(A) \subseteq F(A)$$

- $F$  is injective if and only if there exists a left inverse  $H : F(\mathbf{X}) \rightarrow \mathbf{X}$

$$H \circ F = \text{id}_{\mathbf{X}}$$

and if  $B \subseteq Y$ ,

$$F^{-1}(B) \subseteq H(B)$$

## Topological Spaces

This section will roughly follow Munkres text on General Topology, in particular we hope to cover Chapters 2, 3, 4 and 9. The rest of the Chapters should be covered proper by the subsequent section.

### Definition 2.1: Topology

Let  $\mathbf{X}$  be a non-empty set. A topology  $\mathcal{T}$  on  $\mathbf{X}$ , sometimes denoted by  $\mathcal{T}_{\mathbf{X}}$  is a family of subsets of  $\mathbf{X}$ ,

- $\{\emptyset, \mathbf{X}\} \subseteq \mathcal{T}$ ,
- If  $U_1$  and  $U_2$  are elements of  $\mathcal{T}$ , so is their intersection.
- If  $\{U_\alpha\}$  is an arbitrary family of sets in  $\mathcal{T}$ , their union is also contained in  $\mathcal{T}$  as an element.

We call the elements of  $\mathcal{T}$  open sets. The complements of elements in  $\mathcal{T}$  are closed sets.

## Basis of a Topology

### Definition 3.1: Basis of a topology

A basis  $\mathbb{B}$  is a family of subsets of  $\mathbf{X}$ , that satisfies:

- Every  $x \in \mathbf{X}$  belongs (as an element) in some  $V \in \mathbb{B}$ .
- If  $B_1$  and  $B_2$  are basis elements, such that their intersection is non-empty. Then every  $x \in B_1 \cap B_2$  induces a  $B_3 \in \mathbb{B}$  with

$$x \in B_3 \subseteq B_1 \cap B_2$$

This roughly means a basis is 'finitely' fine at every point in  $x$ .

If  $\mathbb{B}$  is a basis, it 'generates' a topology  $\mathcal{T}$  through

$$\mathcal{T} = \left\{ U \subseteq \mathbf{X}, \forall x \in U, x \in B \subseteq U \text{ for some } B \in \mathbb{B} \right\} \quad (8)$$

Notice this is equivalent to  $\mathcal{T}$  is the collection of all unions of basis elements in  $\mathbb{B}$ .

### Proposition 3.1

Let  $\mathbb{B}$  be a basis as defined in Definition 3.1, then  $\mathcal{T}$  as defined in Equation (8) is a valid topology on  $\mathbf{X}$ . And every member of  $\mathcal{T}$  is and is precisely the union of elements in  $\mathbb{B}$ .

*Proof.* Every point in  $\mathbf{X}$  belongs in some basis element, so  $\mathbf{X} \in \mathcal{T}$ , so does  $\emptyset$ . Next, if  $U_1$  and  $U_2$  are in  $\mathcal{T}$ , then

$$\begin{cases} x \in U_1 \rightarrow x \in B_1 \subseteq U_1 \\ x \in U_2 \rightarrow x \in B_2 \subseteq U_2 \end{cases} \implies x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

for some  $B_3 \in \mathbb{B}$ , so  $\mathcal{T}$  is closed under finite intersections (perhaps after a standard induction argument).

If  $\{U_\alpha\} \subseteq \mathcal{T}$ , and  $x$  belongs in the union of all  $U_\alpha$ , then  $x \in B_\alpha \subseteq U_\alpha$ , which is a subset of the entire union. So the union over  $U_\alpha$  is again contained in  $\mathcal{T}$ , and  $\mathcal{T}$  is a topology on  $\mathbf{X}$ .

It is worth noting that  $\mathbb{B} \subseteq \mathcal{T}$ . Finally, if  $U \in \mathcal{T}$ ,

$$U = \bigcup_{x \in U} B_x$$

where  $B_x$  is the basis element taken to satisfy  $x \in B_x \subseteq U$ . Every point in  $U$  is included in some  $B_x$ , and hence is included in the union. For the reverse inclusion, notice the union of subsets of  $U$  is again a subset of  $U$ .

Now, if  $E \subseteq \mathbf{X}$  is the union of basis elements in  $\mathbb{B}$ , if  $E$  is non-empty, then every point  $x \in E$  belongs in some  $B_x$ . Recycling the previous argument, and we see that  $E$  is open in  $\mathcal{T}$ . If  $E$  is empty, we define the 'union' of no sets as the empty set. So  $\mathcal{T}$  is precisely the collection of all unions of basis elements  $\mathbb{B}$ . ■

We are now in a position to compare the relative 'fineness' of topologies.

**Definition 3.2: Fineness of topologies**

If  $\mathcal{T}'$  and  $\mathcal{T}$  are both topologies on some non-empty set  $\mathbf{X}$ . We say  $\mathcal{T}'$  is finer than  $\mathcal{T}$ , or  $\mathcal{T}$  is coarser than  $\mathcal{T}'$  if

$$\mathcal{T}' \supseteq \mathcal{T}$$

**Proposition 3.2**

If  $\mathbb{B}$  and  $\mathbb{B}'$  are bases for  $\mathcal{T}'$  and  $\mathcal{T}$ , the following are equivalent:

- $\mathcal{T}'$  is finer than  $\mathcal{T}$ ,
- If  $B$  is an arbitrary basis element in  $\mathbb{B}$ , then every point  $x \in B$  induces a basis element in  $\mathbb{B}'$  with

$$x \in B' \subseteq B$$

*Proof.* Suppose  $\mathcal{T}'$  is finer than  $\mathcal{T}$ . Notice  $\mathbb{B} \subseteq \mathcal{T}'$  as well. By Equation (8), each  $x \in B$  induces a  $B' \in \mathbb{B}'$

$$x \in B' \subseteq B$$

Conversely, fix any open set  $U \in \mathcal{T}$ , and for each  $x \in U$ ,

$$x \in B' \subseteq B \subseteq U$$

Applying Definition 3.1 tells us  $U$  is open in  $\mathcal{T}'$ . ■

The last of the big three 'generating' definitions for topologies will be the sub-basis. It simply means the first condition (but not necessarily) the second, is satisfied in Definition 3.1

**Definition 3.3: Sub-basis of a topology**

A sub-basis  $\mathcal{S} \in \mathbb{P}(\mathbf{X})$  is a family of subsets of  $\mathbf{X}$  that satisfies one property. Any point  $x$  in  $\mathbf{X}$  belongs to at least one member of  $\mathcal{S}$ .

A sub-basis can be upgraded to a basis by collecting all of its finite intersections.

**Proposition 3.3**

Let  $\mathcal{S}$  be a sub-basis of  $\mathbf{X}$ , then the collection of all finite intersections of  $\mathcal{S}$  forms a basis  $\mathbb{B}$  of  $\mathbf{X}$ .

*Proof.* Every point in  $\mathbf{X}$  lies in some element of  $\mathcal{S}$ , hence in some element of  $\mathbb{B}$ . The second basis property is immediate, since  $\mathbb{B}$  is closed under finite intersections. ■

## Product Topology

We will start with products of a finite collection of topological spaces.

**Definition 4.1: Finite Product of Topological Spaces**

Let  $(\mathbf{X}, \mathcal{T}_{\mathbf{X}})$  and  $(\mathbf{Y}, \mathcal{T}_{\mathbf{Y}})$  be topological spaces. The product topology (denoted by  $\mathcal{T}_{\mathbf{X} \times \mathbf{Y}}$ ) on  $\mathbf{X} \times \mathbf{Y}$  is defined as the topology generated by the basis

$$\mathbb{B}_{\mathbf{X} \times \mathbf{Y}} = \left\{ U \times V, (U, V) \in \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}} \right\} \quad (9)$$

Since bases are easier to describe than topologies, we have the following statement concerning the basis of the product topology.

**Proposition 4.1**

If  $\mathbb{B}_{\mathbf{X}}$  and  $\mathbb{B}_{\mathbf{Y}}$  are bases for  $\mathcal{T}_{\mathbf{X}}$  and  $\mathcal{T}_{\mathbf{Y}}$ , then the product topology (as described in Definition 4.1) is also generated by

$$\mathcal{M} = \left\{ U \times V, (U, V) \in \mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \right\} \quad (10)$$

*Proof.* We will introduce (and use) the technique of 'double inclusion' by proving that the topologies generated are both finer than the other. Let us denote the topology generated by  $\mathcal{M}$  in Equation (10) by  $\mathcal{T}_{\mathcal{M}}$ .

Since  $\mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \subseteq \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}$ , if  $U \times V \in \mathcal{M}$  as in Equation (10), then we can pick the same 'open rectangle' again. We trivially have

$$x \in \underbrace{U \times V}_{\text{member of } \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}} \subseteq U \times V$$

and by Proposition 3.2,  $\mathcal{T}_{\mathbf{X} \times \mathbf{Y}}$  is finer than  $\mathcal{T}_{\mathcal{M}}$ .

Fix any set  $U \times V \in \mathbb{B}_{\mathbf{X} \times \mathbf{Y}}$ , and if  $(p, q) \in U \times V$ , each coordinate induces basis elements from  $\mathbb{B}_{\mathbf{X}}$  and  $\mathbb{B}_{\mathbf{Y}}$ , more precisely:

$$\begin{cases} p \in U \implies p \in \text{Basis element of } \mathbb{B}_{\mathbf{X}} \subseteq U \\ q \in V \implies q \in \text{Basis element of } \mathbb{B}_{\mathbf{Y}} \subseteq V \end{cases} \implies (p, q) \in \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{X}}} \times \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{Y}}} \subseteq U \times V$$

by Proposition 3.2,  $\mathcal{T}_M$  is finer than  $\mathcal{T}_{X \times Y}$  and  $\mathcal{T}_{X \times Y} = \mathcal{T}_M$ . ■

## Continuity

### Definition 5.1: Continuous maps $C(X, Y)$

Let  $f$  be a map from  $X$  to  $Y$ . It is called *continuous* if  $f^{-1}(U)$  is open in  $X$  for every open set  $U$  in  $Y$ . We denote the set of continuous functions from  $X$  to  $Y$  by  $C(X, Y)$ .

### Proposition 5.1: Continuity preserving operations

The composition of continuous functions is again continuous, and the product of continuous functions is again continuous.

*Proof.* Suppose  $f \in C(X, Y)$  and  $g \in C(Y, Z)$ . Fix an open set  $U \subseteq Z$ . Then  $g^{-1}(U)$  is open in  $Y$ , hence

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \quad \text{is open in } X$$

Next, let  $\{f_\alpha\}_{\alpha \in A}$  be a collection of continuous functions, where each  $f_\alpha \in C(X_\alpha, Y_\alpha)$ . Let us write

$$X \triangleq \prod X_\alpha \quad \text{and} \quad Y \triangleq \prod Y_\alpha$$

and the projection maps:

$$\pi_\alpha^X : X \rightarrow X_\alpha, \quad \text{For every } x \in X, \pi_\alpha^X(x) = x(\alpha) \in X_\alpha$$

similarly for  $\pi_\alpha^Y : Y \rightarrow Y_\alpha$ . The product function  $F = \prod f_\alpha$ , is defined through its behaviour 'on each coordinate'

$$\pi_\alpha^Y \circ F = f_\alpha \circ \pi_\alpha^X \tag{11}$$

A function  $F : X \rightarrow \prod Y_\alpha$  is continuous iff  $\pi_\alpha^Y \circ F$  is continuous for each  $\alpha \in A$ . By Equation (11), it is clear that each  $\pi_\alpha^Y \circ F$  is continuous, since the right member is the composition of two continuous functions, which is again continuous by the first part of this proof, therefore  $F$  is continuous. ■

### Definition 5.2: Open/Closed Maps

Let  $f : X \rightarrow Y$  be a map (not necessarily continuous), it is called *open* (resp. *closed*) if for every open (resp. closed) set  $E \subseteq X$ ,  $f(E)$  is open (resp. closed).

Clearly, the composition of open (resp. closed) maps is again open (resp. closed).

## Quotient Topology

## Product Topology

The Cartesian Product of an arbitrary family of topological spaces, if equipped with the product topology, preserves a lot of the structure. If  $\{X_\alpha\}_{\alpha \in A}$  is a family of topological spaces which are \_\_\_\_\_, then  $\prod X_\alpha$  is \_\_\_\_\_. Replace \_\_\_\_\_ with:

1. Hausdorff, (Folland)
2. Regular,
3. Connected, (Munkres chp23, exercise 10)
4. First countable, if  $A$  is countable,
5. Second countable, if  $A$  is countable,
6. Compact (Tynchonoff's Theorem, Folland)

**Proposition 7.1: Product of Closed sets again Closed**

The product of closed sets is again closed. More concretely, if  $\{E_\alpha\}_{\alpha \in A}$  is a family of sets such that  $E_\alpha \subseteq X_\alpha$ , then

$$\prod \overline{E_\alpha} = \overline{\prod E_\alpha}$$

**Connectedness**

**Definition 8.1: Connectedness**

A topological space  $X$  is connected if  $U$  and  $V$  are disjoint open subsets whose union is  $X$ , then at least one of  $U$  or  $V$  is empty.

See Folland Exercise 4.10 for more properties.

**Definition 8.2: Path-connectedness**

A topological space  $X$  is path-connected if for any two pair of points  $x, y \in X$ . There exists a continuous function  $f : [a, b] \rightarrow X$ , with  $f(a) = x$  and  $f(b) = y$ .

**Definition 8.3: Connected component**

The connected components of  $X$  is the family of equivalence classes on  $X$ , where  $x \sim y$  if there is a connected subspace of  $X$  that contains both of them.

**Proposition 8.1**

Continuous functions map connected spaces to connected spaces (in the subspace topology).

*Proof.* Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be continuous. If  $f(X)$  is disconnected, then we can find  $U$  and  $V$ , open and disjoint in  $\mathcal{T}_{f(X)}$  such that

$$U \cup V = f(X) \implies f^{-1}(U) \cup f^{-1}(V) = X$$

where  $f^{-1}(f(X)) = X$ . Both  $f^{-1}(U)$  and  $f^{-1}(V)$  are open, non-empty, and are pairwise disjoint. So  $X$  is separated. ■

**Proposition 8.2**

Let  $(X_\alpha, \mathcal{T}_\alpha)$  be a family of connected topological spaces indexed by  $\alpha \in A$ . Then  $\prod_{\alpha \in A} X_\alpha$  is disconnected in the product topology.

*Proof.* We will attempt the contrapositive. Suppose  $\prod_{\alpha \in A} X_\alpha$  is disconnected, then ■

**Interiors and closures**

**Definition 9.1: Interior of a set**

$A^\circ$  is defined to be the largest open subset of  $A$ ,

$$A^\circ = \bigcup_{\substack{U \text{ open,} \\ U \subseteq A}} U$$

**Corollary 9.1**

The union of subsets of  $A$  is again a subset of  $A$ , therefore Corollary 9.1 implies  $A^\circ \subseteq A$  for any  $A \subseteq X$ .

**Definition 9.2: Closure of a set**

and  $\overline{A}$  is the smallest closed superset of  $A$ ,

$$\overline{A} = \bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K$$

**Proposition 9.1**

The complement of the closure is the interior of the complement, or equivalently:  $(\overline{A})^c = A^{co}$

*Proof.* Taking complements, and the substitution  $U = K^c$  reads

$$\begin{aligned} (\overline{A})^c &= \left[ \bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K \right]^c \\ &= \bigcup_{\substack{K \text{ closed,} \\ K^c \subseteq A^c}} K^c \\ &= \bigcap_{\substack{U \text{ open,} \\ U \subseteq A^c}} U \\ &= A^{co} \end{aligned}$$

■



**Remark 9.1**

Personally, I remember this as pushing the complement inside and flipping the bar to a c!

## Neighbourhoods

The concept of a neighbourhood allows us to characterize the interior of a set 'locally'.

**Definition 10.1: Neighbourhood (not necessarily open)**

A neighbourhood of  $x \in \mathbf{X}$  is a set  $U \subseteq \mathbf{X}$  where  $x \in U^\circ$ . The set of neighbourhoods for a point  $x \in \mathbf{X}$  will sometimes be denoted by  $\mathcal{N}(x)$ .

**Proposition 10.1: Characterization of the interior**

If  $W = \left\{ x \in \mathbf{X}, \text{ there exists a neighbourhood } U \text{ of } x, U \subseteq A \right\}$ , then  $W = A^\circ$ .

*Proof.* If  $x \in A^\circ$ , then  $A$  is a neighbourhood of  $x$ , and  $A \subseteq A$ , so  $x \in W$ . Conversely, if  $x$  is a member of  $W$ , it has a neighbourhood  $U \subseteq A$  (not necessarily open). By monotonicity of the interior,

$$x \in U^\circ \subseteq A^\circ$$

and  $x \in A^\circ$ . ■

It is easy to see that  $A$  is open  $\iff A^\circ = A \iff A$  is a neighbourhood of itself.

- The first equivalence follows from:

$$E \subseteq \mathbf{X} \implies E^\circ \subseteq E$$

and if  $A$  is an open set, it is an open subset of itself, by Corollary 9.1  $A \subseteq A^\circ$ . If  $A^\circ = A$ , then it suffices to show that  $A^\circ$  is open. Which it is, since it is the arbitrary union of open sets.

- To prove the second equivalence: suppose  $A^\circ = A$ , then each  $x \in A$  has a neighbourhood contained (as a subset) in  $A$ , namely  $A$  itself. (This statement is hard to parse, the reader is encouraged to really work through this and be honest).

$$x \in A^\circ \subseteq A \implies A \subseteq A^\circ$$

so  $A$  is a neighbourhood of itself. Conversely, if  $A \subseteq A^\circ$ , then  $A = A^\circ$ , since the reverse inclusion follows immediately from Corollary 9.1.

## Adherent points

Similar to the neighbourhood, the concept of an adherent point of a set allows us to speak of the closure in more concrete terms. The following definition is key in understanding the relationship between the closure, interior, and the boundary.

**Definition 11.1: Adherent point of a set**

Let  $A \subseteq X$ ,  $x \in X$  is an adherent point of  $A$  if every neighbourhood  $U$  of  $x$  intersects  $A$ . In symbols,

$$U \cap A \neq \emptyset, \quad \forall U \in \mathcal{N}(x)$$

**Proposition 11.1: Characterization of the closure**

Let  $A \subseteq X$ , and let  $W$  be the set of adherent points of  $A$ , then  $\overline{A} = W$

*Proof.* Suppose  $x \notin W$ , then there exists a neighbourhood  $U$  of  $x$  where

$$U \cap A = \emptyset \iff U \subseteq A^c$$

this is exactly the definition of the interior of  $A^c$ , so  $x \in A^{co}$  and recall (from Proposition 9.1) that  $(\overline{A})^c = A^{co}$ , so  $x \notin \overline{A}$ . For the reverse inclusion, read the proof backwards, by flipping  $\forall \rightarrow \exists$  within the set, and we see that

$$W^c = A^{co} = (\overline{A})^c$$

■

## Dense and nowhere dense subsets

**Definition 12.1: Dense subset**

A subset of a topological space  $E \subseteq X$  is dense if  $\overline{E} = X$ .

**Definition 12.2: Nowhere dense subset**

A subset of a topological space  $E \subseteq X$  is nowhere dense if  $\overline{E}^o = \emptyset$ .

This means  $E$  is dense in none of the (non-trivial) open subspaces of  $X$ .

**Proposition 12.1**

$E$  is dense in  $X$  iff for every non-empty, open set  $U \subseteq X$ ,  $U \cap E \neq \emptyset$ .

*Proof of Proposition 12.1.* Suppose  $E$  is dense, then  $\overline{E} = X$ . Every point of  $X$  is an adherent point of  $E$ . Let  $U \subseteq X$  be a non-empty open set. If  $x \in U$  then  $U$  is a neighbourhood of  $x$ , thus  $U$  intersects  $E$ . Conversely, suppose every non-empty open set  $U$  intersects  $E$ . Fix any point  $x \in X$ , and any neighbourhood  $U$  of  $x$ .  $U$  has a non-empty interior (because it must contain  $x$ ). But  $U^o$  is a non-empty open set, therefore  $\emptyset \neq U^o \cap E \subseteq U \cap E$  ■

**Proposition 12.2**

Let  $f : X \rightarrow X$  be a homeomorphism.  $E$  is nowhere dense iff  $f(E)$  is nowhere dense.

*Proof.* Since  $f^{-1}$  is a homeomorphism, suppose  $\overline{f^{-1}(E)}^o \neq \emptyset$ , there exists a non-empty, open subset  $U \subseteq \mathbf{X}$  with

$$\overline{f^{-1}(E)} \cap U = U$$

The direct image yields

$$f\left(\overline{f^{-1}(E)} \cap U\right) = f(U)$$

since  $f$  is a bijection (injectivity is necessary here), it commutes with intersections.

$$f(\overline{f^{-1}(E)}) \cap f(U) = f\left(\overline{f^{-1}(E)} \cap U\right) = f(U) \quad (12)$$

and  $f$  is continuous, so  $f(\overline{A}) \subseteq \overline{f(A)}$  for any  $A \subseteq \mathbf{X}$ . For the reverse inclusion,  $f$  is a closed map, so  $f(\overline{A})$  is a closed superset of  $f(A)$  so

$$f(\overline{A}) = \overline{f(A)}$$

Take  $A = f^{-1}(E)$ , and  $f(\overline{f^{-1}(E)}) = \overline{f(f^{-1}(E))} = \overline{E}$ . From eq. (12), we see that

$$\overline{E} \cap f(U) = f(U)$$

$f(U)$  is a non-empty open subset of  $\mathbf{X}$ , since  $f$  is an open map, so  $E$  is not no-where dense. The reverse implication can be proven by replacing  $f$  with  $f^{-1}$ . ■

## Urysohn's Lemma

### Proposition 13.1: Folland Theorem 4.14

Suppose that  $A$  and  $B$  are disjoint closed subsets of the normal space  $X$ , and let  $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$  be the set of dyadic rationals in  $(0, 1)$ . There is a family  $\{U_r : r \in \Delta\}$  of open sets such that

1.  $A \subseteq U_r \subseteq B^c$  for every  $r \in \Delta$ ,
2.  $\overline{U_r} \subseteq U_s$  for  $r < s$ , and
3. For every  $r < s$ ,  $\overline{U_r} \subseteq U_s$

*Proof.* The goal of this proof is to show that for every  $r \in \Delta$ , there exists a open  $U_r$  that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some  $N \geq 1$  that all previous  $U_r$  in previous layers have been constructed properly, meaning if  $r = k/2^n$ , then for every  $1 \leq n \leq N - 1$ , we have

$$r = \frac{k}{2^n}, 1 \leq n \leq N - 1, 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each  $U_r$ ,

- $A \subseteq U_r \subseteq B^c$  and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer  $N \geq 1$ , we only have to construct the  $U_{k/2^N}$  for every odd  $k$ , this is because if  $k$  is an even number, then  $k = 2j$  and  $r = 2j/2^N = j/2^{N-1}$  and for this particular  $U_r$  is already constructed. So for every odd  $k = 2j + 1$ , the sets of the form  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every  $k - 1 \neq 0$  and  $k + 1 \neq 1$ . (We will consider these cases later). We claim that for every pair of open sets,  $E_1, E_2 \in \mathcal{T}_X$ , then there exists some open set  $G \in \mathcal{T}_X$  such that if  $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$  where  $H$  is defined as the set

$$H = \left\{ (E_1, E_2) \subseteq (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \emptyset \right\}$$

Then there exists some  $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$  such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any  $(E_1, E_2) \in H$ , then this pair induces a pair of disjoint sets  $\overline{E_1}$  and  $E_2^c$  since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \emptyset$$

And by normality, there exists disjoint open sets  $G_1, G_2$  such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \emptyset \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since  $G_2^c$  is a closed set that contains  $G_1$  as a subset,  $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of  $G_2$  (so we may discard the notion of  $G_2$  in our minds). Let us now replace  $G$  with  $G_1$  then it is an easy task to verify that  $G = G_1 = \mathcal{J}(E_1, E_2)$  has the required properties.

Now define for every odd  $k$ , since  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (we note in passing that  $\mathcal{J}$  is not a function as the set  $G$  may not be unique).

$$U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$$

Then, if  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore  $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$  sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$  and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed'  $U_{k/2^N}$  for every  $k-1 \neq 0$  and  $k+1 \neq 1$ . Now let us deal with the remaining pathological cases.

If  $k-1$  so happens to be 0, then no  $r \in \Delta$  satisfies  $r = 0/2^N$ , and we substitute

$$\bar{U}_0 = A, \quad \text{or alternatively, } U_0 = A^c$$

Then  $U_0 \in \mathcal{T}_X$ ,  $\bar{U}_0 = A \subseteq B^c$ . It is at this point that we must mention that  $0, 1 \notin \Delta$ , so  $U_0$  and  $U_1$  do not have to obey the rules we have laid out for  $U_{r \in \Delta}$ .

Now if  $k+1$  is equal to  $2^N$  (this makes  $r = (k+1)/2^N = 1$ ) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every  $0 \leq m \leq 2^N - 1$ ,  $U_{m/2^N}$  must satisfy

$$\bar{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (even for when  $N = 1$ , since  $A = \bar{U}_0 \subseteq U_1 = B^c$ ) and a corresponding  $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$  such that

- $A \subseteq \bar{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\bar{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when  $N = 1$ . We would only have to construct for  $k = 1$ , since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■

**Proposition 13.2: Folland Theorem 4.15: Urysohn's Lemma**

Urysohn's Lemma. Let  $X$  be a normal space, if  $A$  and  $B$  are disjoint closed subsets of  $X$ , then there exists a  $f \in C(X, [0, 1])$  such that  $f = 0$  on  $A$  and  $f = 1$  on  $B$ .

*Proof.* Let  $r \in \Delta$  be as in Lemma 4.14, and set  $U_r$  accordingly except for  $U_1 = X$ . Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write  $W = \{k : x \in U_k\}$ , Then for every  $x \in A$  we have  $f(x) = 0$ , since by the construction of the 'union' function in Lemma 4.14, for each  $r \in \Delta \cap (0, 1)$ ,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since  $r > 0$  is arbitrary, and  $0 \in W$ , we can use a classic  $\varepsilon$  argument. If  $f(x) > 0$  then there exists some  $0 < r < f(x)$  by density of the dyadic rationals on the line, if  $f(x) < 0$  then this implies that there exists some  $f(x) < r < 0$  such that  $x \in U_r$ , but no  $r \in \Delta$  can be negative, hence  $f(x) = 0$ .

Now, for every  $x \in B$ , since  $A$  and  $B$  are disjoint, and  $A \subseteq U_r \subseteq B^c$ , then for every  $x \in B$  means that  $x$  is not a member of any  $U_r$ , but we set  $U_1 = X$ . Since none of the  $r \in (0, 1)$  is a member of the set we

are taking the infimum, and  $x \in U_1 = X$ . The  $\varepsilon$  argument follows: suppose for every  $\varepsilon > 0$ ,  $(1 - \varepsilon) \notin W$ , and  $1 \in W$ , then  $f(x) = 1$ .

Since  $x \in U_1 = X$ , for every  $x \in X$ ,  $f(x) \leq 1$ , and  $f(x)$  cannot be negative as  $r > 0$  for every  $r \in \Delta$ . So  $0 \leq f(x) \leq 1$ . Now we have to show that this  $f(x)$  is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in  $X$ . So  $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$  and  $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$ .

Suppose that  $f(x) < \alpha$ , so  $\inf W < \alpha$ , and using the density of  $\Delta$ , there exists an  $r$ ,  $f(x) < r < \alpha$  such that  $x \in U_r$  such that  $x \in \bigcup_{r < \alpha} U_r$ . So  $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r < \alpha} U_r$ .

Fix an element  $x \in \bigcup_{r < \alpha} U_r$ , this induces an  $r$  such that  $\inf W \leq r < \alpha$  therefore  $f(x) < \alpha$ , and  $\bigcup_{r < \alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$ .

For the second case, suppose that  $f(x) > \alpha$ , then  $\inf W > \alpha$ , and there exists an  $r$  (by density) such that  $\inf W > r > \alpha$  such that for every  $k \in W$ ,  $k \neq r$ . Therefore  $x \notin U_r$ , but by density again, and using the property of the onion function: for every  $s < r$  we get  $\overline{U_s} \subseteq U_r$ , taking complements (which reverses the estimate) — we have  $x \notin \overline{U_s}$ , but  $(\overline{U_s})^c$  is open in  $X$ . It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq (\overline{U_s})^c \subseteq \bigcup_{s > \alpha} (\overline{U_s})^c$$

So  $f^{-1}((\alpha, +\infty))$  is a subset of  $\bigcup_{s > \alpha} (\overline{U_s})^c$ . To show the reverse, fix an element  $x$  in the union, then this induces some  $x \in (\overline{U_s})^c \subseteq (U_s)^c$ . Then for this  $s > \alpha$ ,  $(-\infty, s)$  contains no elements of  $W$ . This is because for every  $p < s$  implies that  $(U_s)^c \subseteq (U_p)^c$ , so  $p \notin W$ . Our chosen  $s$  is a lower bound for  $W$ , and  $\alpha < s \leq \inf W = f(x)$ .

Since all of the inverse images from the generating set of  $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$  are open in  $X$ , using Theorem 4.9 finishes the proof. ■

Notes on the construction of the countable 'onion' sequence within a normal space  $\mathbf{X}$ .

If  $\mathbf{X}$  is a normal space, and  $A$  and  $B$  are disjoint closed subsets, then we can easily find an open  $U$  with

$$A \subseteq U \subseteq \overline{U} \subseteq B^c \tag{13}$$

We say that  $U$  hides in  $B^c$  if the closure of  $U$  is contained in  $B^c$ . Define  $\Delta_n = \left\{ k2^{-n}, 1 < k < 2^n \right\}$ , so that  $\Delta_n \subseteq (0, 1)$  for all  $n \geq 1$ . Notice

$$\Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

and the even indices for  $\Delta_{n+1}$  are contained in  $\Delta_n$ . Suppose  $\Delta_n$  is well defined, it suffices to choose the odd indices for  $\Delta_{n+1}$ . If  $r = j2^{-(n+1)}$ , where  $j$  is odd, then  $r$  sits in between precisely two elements in  $\Delta_n \cup \{0, 1\}$ . If  $r$  sits between an endpoint, then define  $\overline{U_0} = A$ , and  $B^c = U_1$ . And denote the closest left and neighbours by  $s, t$  respectively. If  $s < r < t$ , it is clear that  $\overline{U_s}$  and  $U_t^c$  are disjoint closed sets.

Use the 'normal space' construction to obtain an superset of  $\overline{U_s}$  that hides in  $U_t$ , denote this open set by  $U_r$ , and similar to Equation (13)

$$\overline{U_s} \subseteq U_r \subseteq \overline{U_r} \subseteq U_t$$

Now that the construction of this sequence is complete, we wish to prove Urysohn's Lemma. Let  $A$  and  $B$  be disjoint closed sets. And define

$$f(x) = \inf \left\{ r \in \Delta \cup \{1\}, x \in U_r \right\}$$

where  $U_1 = \mathbf{X}$ . So that  $0 \leq f(x) \leq 1$  is immediate. If  $x \in A$ , then  $x$  is in all  $U_r$ , and by density of  $\Delta \subseteq (0, 1)$ , we have  $f(x) = 0$ . Conversely, if  $x \in B$  then  $x \notin U_r$  for all  $r \in \Delta$ , if  $E$  denotes the indices in  $\Delta$  where  $x \in U_s$  when  $s \in E$ ,

$$(-\infty, r) \subseteq E^c \iff E \subseteq [r, +\infty) \iff \inf(E) \geq r \quad (14)$$

Send  $r \rightarrow 1$  and  $f(x) = 1$ . Thus  $f|_A = 0$  and  $f|_B = 1$ .

To show continuity, it suffices to show that the inverse images of the open half  $\left\{ (x > \alpha), (x < \alpha) \right\}_{\alpha \in \mathbb{R}}$  lines are indeed open in  $\mathbf{X}$ . Let  $\alpha$  be fixed. And if  $x \in \{f < \alpha\}$ , we can 'wiggle' the infimum towards the right (towards  $\alpha$ ), and using density of  $\Delta$  within  $(0, 1)$ , there exists a  $r \in E$  that satisfies  $f(x) < r < \alpha$ . This is equivalent to

$$x \in \bigcup_{r < \alpha} U_r$$

If there exists an  $r < \alpha$  st  $x$  belongs to  $U_r$  as an element, then  $f(x) \leq r < \alpha$ .

If  $f(x) > \alpha$ , then  $(-\infty, \alpha) \subseteq E^c$ , by Equation (14). Suppose  $\alpha < 1$ , otherwise  $\{f > \alpha\} = \emptyset$ . Wiggle  $f(x)$  to the left and obtain an  $r \in \Delta$ ,  $\alpha < r < f(x)$  with  $x \notin U_r$ . By density again, take any  $s < r$  by a small amount (st  $s > \alpha$ ,  $s \in \Delta$ ), and

$$\overline{U}_s \subseteq U_r \iff U_r^c \subseteq \overline{U}_s$$

so that  $x \in \overline{U}_s^c$  for some  $s > \alpha$ . This is equivalent to

$$x \in \bigcup_{s > \alpha} \overline{U}_s^c$$

Conversely, if  $x \notin \overline{U}_s^c$  for some  $s > \alpha$ , since  $\{U_r\}$  (thus  $\{\overline{U}_r\}$ ) is increasing, and  $x \notin U_r$  for every  $r \leq s$ . Hence,

$$(-\infty, s] \subseteq E^c \iff E \subseteq (s, +\infty) \iff f(x) \geq s > \alpha$$

## Compactness

Compactness is one of the most important concepts in topology and analysis.

### Definition 14.1: Compact topological space

A topological space  $\mathbf{X}$  is compact if every open covering  $\{U_\alpha\}$  contains a finite subcover. That is, if  $\{U_\alpha\}$  is an arbitrary collection of open sets, then

$$\mathbf{X} = \bigcup_{\alpha \in A} U_\alpha \implies \bigcup_{j \leq n} U_{\alpha_j}$$

**Definition 14.2: Compact set**

$E \subseteq \mathbf{X}$  is compact if it is compact in the subspace topology.

**Definition 14.3: Precompact set**

$E \subseteq \mathbf{X}$  is precompact if its closure is compact (as a subset).

**Definition 14.4: Paracompact space**

A topological space  $\mathbf{X}$  is paracompact if every open covering of  $\mathbf{X}$  has a locally finite open refinement that covers  $\mathbf{X}$ .

**Definition 14.5: Locally finite collection of sets**

Let  $\mathcal{A}$  be a collection of subsets of  $\mathbf{X}$ . It is called locally finite, if at every point  $p \in \mathbf{X}$ , we can find a neighbourhood  $U$  of  $p$  (not necessarily open), that intersects only finitely many members of  $\mathcal{A}$ . In symbols,

$$U \cap E = \emptyset \quad \text{for all but finitely many } E \in \mathcal{A}$$

We do not require  $\mathcal{A}$  to be a cover of  $\mathbf{X}$ , nor do we require  $\mathcal{A}$  to be a collection of open sets.

**Definition 14.6: Countably locally finite**

A collection  $\mathbb{B}$  is countably locally finite if it is the countable union of locally finite families.

$$\mathbb{B} = \bigcup_{\mathbb{N}}^{\text{countable union}} \mathbb{B}_n, \quad \text{where each } \mathbb{B}_n \text{ is a locally finite collection}$$

**Definition 14.7: Refinement**

If  $\mathcal{A}$  is a collection of sets,  $\mathbb{B}$  is a refinement of  $\mathcal{A}$  if every element  $B \in \mathbb{B}$ , induces an element  $A \in \mathcal{A}$ , such that  $B \subseteq A$ .

**Remark 14.1: Intuition for refinements**

If  $\mathbb{B}$  is a refinement of  $\mathcal{A}$ , we can use the 'absolute continuity' muscle. For each element in  $\mathbb{B}$  is dominated by some element (through subset inclusion) in  $\mathcal{A}$ . Recall, if  $\nu$  and  $\mu$  are non-negative measures, then  $\nu \ll \mu$  if for every measurable set  $E \in \mathcal{M}$ ,  $\mu(E) = 0 \implies \nu(E) = 0$ .

A refinement of a family of sets is another family of sets, whose elements are dominated by some other element in the un-refined family. *Refining families makes them 'smaller', cover less area.*



**Proposition 14.1**

Compact Hausdorff spaces are normal, compact subsets of Hausdorff spaces are closed, and closed subsets of compact sets are again compact.

**Properties of Compact Spaces**

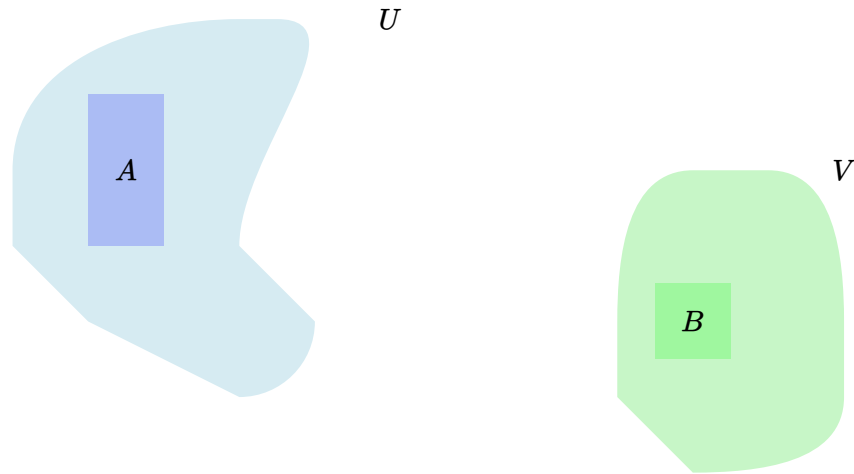


Figure 1: Closed sets  $A$  and  $B$  within open sets  $U$  and  $V$ , respectively.

**Proposition 15.1**

Let  $\mathbf{X}$  and  $\mathbf{Y}$  be topological spaces.

- (a) If  $F \in C(\mathbf{X}, \mathbf{Y})$ , and  $\mathbf{X}$  is compact, then  $F(\mathbf{X})$  is compact.
- (b) If  $\mathbf{X}$  is compact and  $F \in C(\mathbf{X}, \mathbb{R})$ , then  $F(\mathbf{X})$  is bounded, and  $F$  attains its supremum and infimum on  $\mathbf{X}$ .
- (c) A finite union of compact subspaces of  $\mathbf{X}$  is again compact.
- (d) If  $\mathbf{X}$  is Hausdorff, and  $A, B$  are disjoint, compact subspaces of  $\mathbf{X}$ , there exists open  $U$  and  $V$ , (see fig. 1).
- (e) Every closed subset of a compact space is compact.
- (f) Every compact subset of a Hausdorff space is closed.
- (g) Every compact subset of a metric space is bounded.
- (h) Every finite product of compact spaces is compact.
- (i) Every quotient of a compact space is compact.

*Proof of Proposition 15.1 Part A.* Let  $f \in C(\mathbf{X}, \mathbf{Y})$  with  $\mathbf{X}$  compact. Fix an open cover of  $f(\mathbf{X})$  in the

relative topology,

$$\{U_\alpha \cap f(\mathbf{X})\}_{\alpha \in A} \text{ covers } \mathbf{X}, U_\alpha \text{ open in } \mathbf{Y}$$

So that  $\bigcup f^{-1}(U_\alpha) = \bigcup f^{-1}(U_\alpha \cap f(\mathbf{X})) = \mathbf{X}$ . Since  $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$  is an open cover for  $\mathbf{X}$ , this induces a finite subcollection of indices  $\{\alpha_1, \dots, \alpha_n\}$  with

$$\bigcup_{j=1}^n f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^n f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^n U_{\alpha_j}$$

■

*Proof of Proposition 15.1 Part B.* Let  $\mathbf{X}$  be compact, and  $f \in C(\mathbf{X}, \mathbb{R})$ , so that  $f(\mathbf{X}) \subseteq \mathbb{R}$  is compact. Compact subsets are closed and bounded in  $\mathbb{R}$ , let  $A = \sup f(\mathbf{X})$  and  $B = \inf f(\mathbf{X})$ . Both  $A$  and  $B$  are accumulation points of  $f(\mathbf{X})$ , so  $A = f(x)$  and  $B = f(y)$  for some  $x, y$  in  $\mathbf{X}$ . ■

*Proof of Proposition 15.1 Part C.* Let  $\mathbf{X}$  be a topological space, and  $K_1, \dots, K_n$  be compact subspaces. Denote  $K = \bigcup_{j=1}^n K_j$ . Let  $\{U_\alpha \cap K\}_{\alpha \in A}$  be an open cover for  $K$ , where  $U_\alpha$  is open in  $\mathbf{X}$ . We can pass the argument to each individual  $K_j$  as follows. Let  $1 \leq j \leq n$ , then  $\{U_\alpha \cap K_j\}_{\alpha \in A}$  is an open cover for  $K_j$ , so there exists a finite subcollection of indices  $I_j \subseteq A$ , (a finite subset of  $A$ ) whose open sets cover  $K_j$ . Repeat this process for each  $j$  and

$$I = \bigcup_{j=1}^n I_j \text{ is a finite subset of } A$$

with  $K_j \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K)$ . Taking the union over all  $K_j$  reads

$$K = \bigcup_{j=1}^n K_j \subseteq \bigcup_{j=1}^n \bigcup_{\alpha \in I_j} (U_\alpha \cap K) = \bigcup_{\alpha \in I} U_\alpha \cap K$$

■

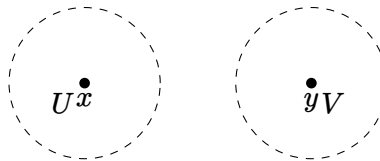


Figure 2: In a Hausdorff space, any two distinct points  $x$  and  $y$  can be separated by disjoint open neighbourhoods  $U$  and  $V$ .

*Proof of Proposition 15.1 Part D.* Let  $\mathbf{X}$  be Hausdorff. We first prove that compact subspaces of  $\mathbf{X}$  are closed. Indeed, if  $K$  is compact in  $\mathbf{X}$ , fix any  $x \in K^c$ . Let  $y$  range through the elements of  $K$ , then  $x \neq y$  induces a pair of disjoint open sets  $U_y$  and  $V_y$ , such that

- $x \in U_y$
- $y \in V_y$
- $U_y \cap V_y = \emptyset$
- See fig. 2

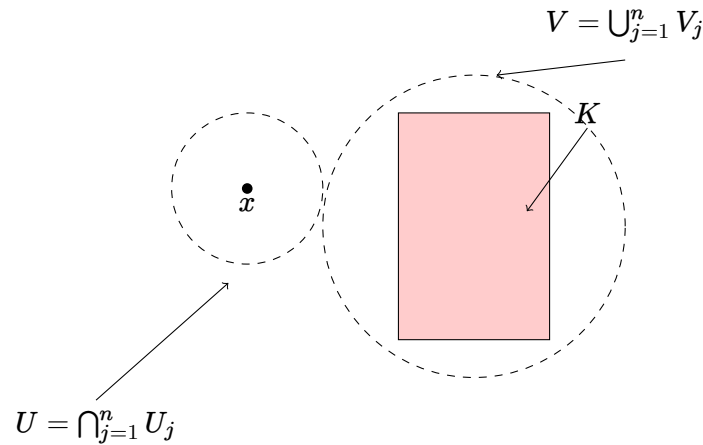


Figure 3: Compact sets are closed in Hausdorff spaces

Let  $V_y$  range through all possible  $y \in K$ , So that  $\{V_y\}_{y \in K}$  is an open cover. There exists a finite subcollection of 'anchor points' of  $K$ ,  $y_1, \dots, y_n$  that corresponds with  $\{V_{y_j}\}_{j=1}^n$ . A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^n U_{y_j} \text{ is open}$$

Define  $V = \bigcup_{j=1}^n V_{y_j}$ , so  $V \subseteq K$  and  $U \cap V = \emptyset$  and  $x \in U \subseteq K^c$  (see fig. 3). Therefore  $K$  is closed.

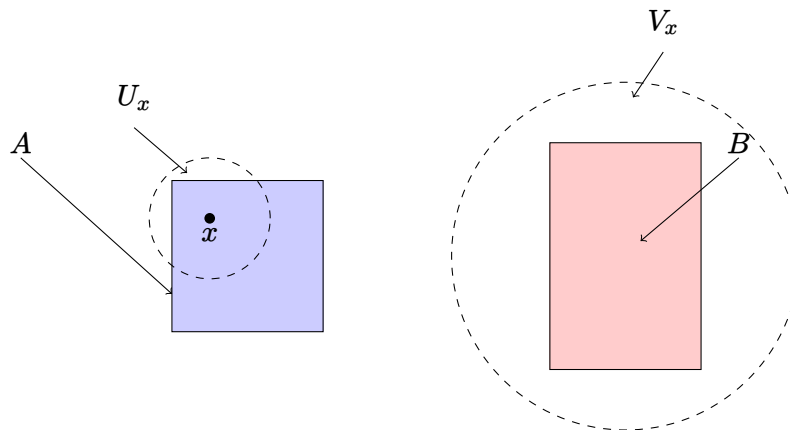


Figure 4: Closed sets  $A$  and  $B$ , point  $x$  in  $A$ , and disjoint neighbourhoods  $U$  around  $x$  and  $V$  around  $B$ .

Finally, if  $A$  and  $B$  are disjoint compact sets, then each  $x \in A \subseteq B^c$  induces neighbourhoods  $x \in U_x$ , and  $B \subseteq V_x$  (see fig. 4), let  $x$  range through all the elements of  $A$ . By compactness of  $A$ , this produces a finite subcover, and

$$U = \bigcup_{j=1}^n U_{x_j} \quad V = \bigcap_{j=1}^n V_{x_j}$$

are disjoint open sets that contain  $A$  and  $B$  respectively. ■

*Proof of Proposition 15.1 Part E.* Let  $K \subseteq \mathbf{X}$  be a closed set of a compact space. Let  $\{U_\alpha \cap K\}$  be an open cover for  $K$ , where each  $U_\alpha$  is open in  $\mathbf{X}$ . We can append an extra set  $K^c$  which is open in  $\mathbf{X}$ . The collection

$$W = \{U_\alpha\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of  $W_1, \dots, W_n$  that cover  $\mathbf{X}$  (since  $\mathbf{X}$  is compact by itself). Remove  $K^c$  from this finite subcollection if it exists, and take the intersection with  $K$  for each element  $W_j$ , and

$$\{W_1 \cap K, \dots, W_n \cap K\} = \{U_1 \cap K, \dots, U_n \cap K\} \text{ covers } K$$

so  $K$  is compact. ■

*Proof of Proposition 15.1 Part F.* Proven in Part D. ■

*Proof of Proposition 15.1 Part G.* let  $K \subseteq \mathbf{X}$  be a compact subset of the metric space  $(\mathbf{X}, d)$ . Compact subsets of  $\mathbf{X}$  are totally bounded, and hence bounded. ■

*Proof of Proposition 15.1 Part H.* See Tychonoff's Theorem in Folland Chapter 4. ■

*Proof of Proposition 15.1 Part I.* Let  $\mathbf{X}$  and  $\mathbf{Y}$  be topological spaces and  $\pi : \mathbf{X} \rightarrow \mathbf{Y}$  be a quotient map. So that  $\mathbf{Y}$  is endowed with the quotient topology. So that  $\pi$  is a surjective continuous map. and  $\pi(\mathbf{X}) = \mathbf{Y}$ . Apply Part A, and we see that  $\mathbf{Y}$  is compact. ■

## Locally Compact Hausdorff Spaces

Compactness is an intrinsic topological property (in the subspace topology). We see from Proposition 4.25 that compact Hausdorff spaces are normal, which gives a sufficient condition for us to approximate and extend any continuous function; and allows us to extend certain 'local' properties to 'global' properties.

If given a Hausdorff space, not necessarily compact, the natural question is to ask 1) whether a topological space has 'enough' compact subsets to work with, and 2) whether we can embed a given topological space in a larger one to force it to be compact.

### Definition 16.1: LCH space

Let  $\mathbf{X}$  be a Hausdorff space. We call  $\mathbf{X}$  a LCH space if every point  $p \in \mathbf{X}$  admits a compact neighbourhood. That is, a compact set  $K$  whose interior contains  $p$ .

We note in passing that the above definition differs slightly from the usual 'local' definitions.

**Definition 16.2: Locally connected**

Let  $\mathbf{X}$  be a topological space, it is locally connected if for every  $x \in \mathbf{X}$ , and open neighbourhood  $U$  containing  $x$ , there exists a connected, open neighbourhood  $V$  of  $x$  such that  $x \in V \subseteq U$ .

**Definition 16.3: Locally path-connected**

Let  $\mathbf{X}$  be a topological space, it is locally connected if for every  $x \in \mathbf{X}$ , and open neighbourhood  $U$  containing  $x$ , there exists a path-connected, open neighbourhood  $V$  of  $x$  such that  $x \in V \subseteq U$ .

**Definition 16.4: Local homeomorphism**

$\mathbf{X}$  locally homeomorphic to  $\mathbb{R}^n$  if every point  $x \in \mathbf{X}$  belongs to a coordinate chart  $(U, \phi)$ , where  $U$  is an open neighbourhood of  $x$  and  $\phi$  is a homeomorphism from  $U \rightarrow \phi(U) \subseteq \mathbb{R}^n$ .

**Definition 16.5: Local diffeomorphism**

Let  $M$  be a smooth manifold and  $F \in C^\infty(M, N)$ .  $F$  is a local diffeomorphism if every  $p \in M$  in its domain induces a neighbourhood  $U \subseteq M$  with  $F|_U : U \rightarrow F(U)$  is a diffeomorphism (in the sense of two open sub-manifolds).