MATH 263: Section 003, Tutorial 10

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1 (Review) Series Solutions Near an Ordinary Point, Part I

Consider the general second order linear ODE:

$$P(x)y\text{"}(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$

Where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are analytical around $x = x_0$. Such a point where $P(x_0) \neq 0$ is called an **ordinary point**. We can divide both sides by and get: $P(x)$ and get:

$$y\text{"}(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

In that case, one can solve it by plugging in the power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

and finding the coefficients a_n, usually through a recurrence relation. It usually cannot be solved and one may only give the few first terms of the solution.

Problem 1. From Boyce and DiPrima, 10th edition (5.2, exercise 9, p.263):
Find the general solution of:

$$(1 + x^2)y\text{"} - 4xy' + 6y = 0.$$

Solution: Choosing $x_0 = 0$, let the solution be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Take the first two derivatives (term by term):

$$y'(x) = \sum_{n=1}^{\infty} n a_n\ x^{n-1}$$

$$y\text{"}(x) = \sum_{n=2}^{\infty} n(n-1)a_n\ x^{n-2}$$

Note that taking derivatives changes the initial index and that reindexing is required. Now, plug these series into the ODE:

$$(1 + x^2)\sum_{n=2}^{\infty} n(n-1)a_n\ x^{n-2} - 4x\sum_{n=1}^{\infty} n a_n\ x^{n-1} + 6\sum_{n=0}^{\infty} a_n x^n = 0$$

Distribute the x and the $1 + x^2$:

$$\sum_{n=2}^{\infty} n(n-1)a_n\ x^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n\ x^n - \sum_{n=1}^{\infty} 4na_n\ x^n + \sum_{n=0}^{\infty} 6a_n x^n = 0$$

Now we want all the sums to have the same coefficients and start at the same index. Therefore, we take out one term from the first and third series and reindex the second one:

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\ x^n + \sum_{n=2}^{\infty}n(n-1)a_n\ x^n - \sum_{n=1}^{\infty}4na_n\ x^n + \sum_{n=0}^{\infty}6a_nx^n = 0$$

$$2a_2+6a_3x+\sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}\ x^n+\sum_{n=2}^{\infty}n(n-1)a_n\ x^n-4a_1x-\sum_{n=1}^{\infty}4na_n\ x^n+6a_0+6a_1x+\sum_{n=0}^{\infty}6a_nx^n=0$$

Collect terms:

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x + \sum_{n=2}^{\infty}(n+1)(n+2)a_{n+2}\ x^n + \sum_{n=2}^{\infty}n(n-1)a_n\ x^n - \sum_{n=2}^{\infty}4na_n\ x^n + \sum_{n=2}^{\infty}6a_nx^n = 0$$

$$6a_0 + 2a_2 + (2a_1 + 6a_3)x + \sum_{n=2}^{\infty}[((n+1)(n+2)a_{n+2}\ + n(n-1)a_n\ - 4na_n\ + 6a_n)x^n] = 0$$

Now, since $1, x, x^2, x^3, \ldots$ are linearly independent from each other we can check the constants and x's coefficients alone, which need to amount to 0:

$$6a_0 + 2a_2 = 0$$

$$a_2 = -3a_0$$

Also,

$$2a_1 + 6a_3 = 0$$

$$a_3 = -\frac{1}{3}a_1$$

Furthermore, for all other x^n:

$$(n+1)(n+2)a_{n+2}\ + n(n-1)a_n\ - 4na_n\ + 6a_n = 0$$

$$(n+1)(n+2)a_{n+2}\ + (n^2-n)a_n\ - 4na_n\ + 6a_n = 0$$

$$(n+1)(n+2)a_{n+2}\ + (n^2-5n+6)a_n = 0$$

$$(n+1)(n+2)a_{n+2}\ + (n-2)(n-3)a_n = 0$$

$$a_{n+2} = \frac{-(n-2)(n-3)}{(n+1)(n+2)}a_n$$

For even coefficients we have:

$$a_4 = \frac{-(0)(-1)}{(3)(4)}a_n = 0$$

Since $a_4 = 0$, all subsequent even coefficients are zero, since they are all written as a multiple of a_4. Indeed,

$$a_6 = \frac{-(2)(1)}{(5)(6)} \cdot 0 = 0$$

$$a_8 = \frac{-(4)(3)}{(7)(8)} \cdot 0 = 0$$

$$a_4 = a_6 = a_8 = a_{10} = \cdots = 0.$$

For odd coefficients we have:

$$a_5 = \frac{-(1)(0)}{(4)(5)}a_3 = 0$$

Since $a_3 = 0$, all other subsequent even coefficients are zero, since they are all written as a multiple of a_4. Indeed,

$$a_5 = a_7 = a_9 = a_{11} = \cdots = 0.$$

By superposition, the solution is:

$$y(x) = a_0(1-3x^2) + a_1(x - \frac{1}{3}x^3)$$

The solution is then a finite polynomial!

2 Series Solutions Near an Ordinary Point, Part II

Consider the general second order linear ODE:

$$P(x)y''(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$
$$y''(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

where P, Q, and R are polynomials and where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are analytical in the neighborhood of an ordinary point x_0. Assuming the ODE has an analytical solution of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

From the definition of the Taylor series at $x = x_0$, we get:

$$n!\ a_n = y^{(n)}(x_0)$$

Using that, we get

$$y''(x) = -p(x)\ y'(x) - q(x)\ y(x)$$

By taking successive derivatives and evaluating them at $x = x_0$, we can get the coefficients of the Taylor Series corresponding to the ODE's solution.

Problem 2. From Boyce and DiPrima, 10th edition (5.3, exercise 2, p.269):
Find the first terms (up to a_4) of the solution of:

$$y''(x) + \sin x\ y'(x) + \cos x\ y(x) = 0;\ y(0) = 0,\ y'(0) = 1.$$

Solution:

$$y(0) = a_0 = 0,\ y'(0) = a_1 = 1$$
$$y''(x) = -\sin x\ y'(x) - \cos x\ y(x)$$
$$y''(0) = -\sin 0\ y'(0) - \cos 0\ y(0) = -y(0) = 0$$
$$a_2 = \frac{y''(0)}{2!} = 0$$

Take the derivative:

$$y'''(x) = -\sin x\ y''(x) - \cos x\ y'(x) - \cos x\ y'(x) + \sin x\ y(x)$$
$$y'''(x) = -\sin x\ y''(x) - 2\cos x\ y'(x) + \sin x\ y(x)$$
$$a_3 = \frac{y'''(0)}{3!} = \frac{1}{3!}(-\sin 0\ y''(0) - 2\cos 0\ y'(0) + \sin 0\ y(0))$$
$$a_3 = \frac{1}{6}(-2\ y'(0)) = \frac{-1}{3}$$

Take the derivative:

$$y^{(4)}(x) = -\sin x\ y'''(x) - 3\cos x\ y''(x) + 3\sin x\ y'(x) + \cos x\ y(x)$$
$$a_4 = \frac{y^{(4)}(0)}{4!} = \frac{1}{4!}(-\sin 0\ y'''(0) - 3\cos 0\ y''(0) + 3\sin 0\ y'(0) + \cos 0\ y(0))$$
$$a_4 = 0$$

Therefore, the first terms of the solution to this IVP is:

$$y(x) = 0 + x + 0x^2 - \frac{1}{3}x^3 + 0x^4 + \dots$$
$$y(x) = x - \frac{1}{3}x^3 + O(x^5)$$

3 Series Solutions Near a Regular Singular Point, Part I

Consider the general second order linear ODE:

$$P(x)y"(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$

Where $P(x_0) = 0$, meaning that $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are not analytical at $x = x_0$. $x = x_0$ is then a **singular point**.

Consider the case of **regular singular points**, where $(x - x_0)p(x) = (x - x_0)\frac{Q(x)}{P(x)}$ and $(x - x_0)^2 q(x) = (x-x_0)^2 \frac{R(x)}{P(x)}$ are analytic at $x = x_0$. We can write them as: $(x-x_0)p(x) = (x-x_0)\frac{Q(x)}{P(x)}$ and $(x-x_0)^2 q(x) = (x - x_0)^2 \frac{R(x)}{P(x)}$ are

$$(x - x_0)p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n,$$

and

$$(x - x_0)^2 q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n,$$

Plugging them in the ODE, we get

$$(x - x_0)^2 y" + (x - x_0)[(x - x_0)p(x)]y' + [(x - x_0)^2 q(x)]y = 0$$

$$(x-x_0)^2 y" + (x-x_0)(p_0 + p_1(x-x_0) + \cdots + p_n(x-x_0)^n + \dots)y' + (q_0 + q_1(x-x_0) + \cdots + q_n(x-x_0)^n + \dots)y = 0.$$

As x approaches x_0, the ODE behaves as an Euler equation as such:

$$(x - x_0)^2 y" + p_0(x - x_0)y' + q_0 y = 0.$$

where $p_0 = \lim_{x \to x_0} \frac{Q(x)}{P(x)}$ and $q_0 = \lim_{x \to x_0} \frac{R(x)}{P(x)}$

The solutions will be of the form of Euler solutions times a power series as such:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n (x - x_0)^n,\ a_0 \neq 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{r+n}$$

Then plug it in the ODE. The Euler characteristic equation will arise along with a recurrence relation for a_n that depends on r.

Problem 3. From Boyce and DiPrima, 10th edition (5.5, exercise 3, p.286):
Find one fundamental solution of:

$$xy" + y = 0.$$

Solution: Let $x_0 = 0$: $P(x_0) = 0$, making it a singular point.

$$\lim_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)} = \lim_{x \to 0} x\frac{0}{x} = 0$$

and

$$\lim_{x \to x_0} (x - x_0)\frac{R(x)}{P(x)} = x^2 \frac{1}{x} = 0$$

So $x_0 = 0$ is a regular singular point.

Let:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$$

$$y'(x) = \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}$$

$$y"(x) = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2}$$

Plug in the ODE:

$$x\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

$$\sum_{n=-1}^{\infty}(r+n+1)(r+n)a_{n+1} x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

Take out the $n = -1$ term and combine the sums:

$$r(r-1)a_0 x^{r-1} + \sum_{n=0}^{\infty}[(r+n+1)(r+n)a_{n+1} + a_n]x^{r+n} = 0.$$

Therefore, as before, we have:

$$r(r-1) = 0 \Rightarrow r_1 = 1,\ r_2 = 0$$

and

$$(r+n+1)(r+n)a_{n+1} + a_n = 0.$$

$$a_{n+1} = -\frac{a_n}{(r+n+1)(r+n)}$$

Note that when the Euler characteristic equation roots are the same or differ by an integer, like here, we can only find one fundamental solution. Else, the two roots correspond to two fundamental solutions with their respective recurrence relations for a_n. This issue will be discussed later, for now let's work with $r_1 = 1$, to find one of the solutions and plug it in the recurrence relation:

$$a_{n+1} = -\frac{a_n}{(n+1)(n+2)}$$

$$a_1 = -\frac{a_0}{1\cdot 2}$$

$$a_2 = -\frac{a_1}{2\cdot 3} = \frac{a_0}{1\cdot 2\cdot 2\cdot 3} = \frac{a_0}{2!\cdot 3!}$$

$$a_3 = -\frac{a_2}{3\cdot 4} = -\frac{a_0}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3\cdot 4} = -\frac{a_0}{3!\cdot 4!}$$

From looking at the pattern, we get

$$a_n = (-1)^n\frac{a_0}{n!(n+1)!}$$

and letting $a_0 = 1$ we get **one** of the fundamental solutions, which is:

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{r+n} = x^r\sum_{n=0}^{\infty} a_n x^n$$

$$y_1(x) = x\ (1 - \frac{1}{1\cdot 2}x + \frac{1}{2!\cdot 3!}x^2 + \dots) = x[\sum_{n=0}^{\infty}(-1)^n\frac{a_0}{n!(n+1)!}x^n]$$