Chapter 22: Symplectic Manifold

Symplectic Tensors

Definition 1.1: Billinear forms

Let V be a vector space, a *billinear form* $\omega : V \times V \to \mathbb{R}$ is a 2-tensor on V.

Definition 1.2: Characterization of billinear forms

Let ω be a billinear form on V, it is

- *symmetric* if
$$\omega(x, y) = \omega(y, x)$$
- *skew-symmetric* or *anti-symmetric* if
$$\omega(x, y) = (-1)\omega(y, x)$$
- *alternating* if
$$\omega(x, x) = 0$$

If V is a vector space over the field F and $\mathrm{char}(F) \neq 2$, then the last two conditions are equivalent. Moreover,

- V is called an *orthogonal geometry* if ω is symmetric.
- V is called a *symplectic geometry* if ω is alternating.

Definition 1.3: Metric vector space

A vector space (not necessarily finite dimensional) is called a *metric vector space* if it is a orthogonal or symplectic geometry.

Matrices and billinear forms

Definition 2.1: Matrix of billinear form

If $B = (b_1, \ldots, b_n)$ is an ordered basis for V, we define the *matrix representation of* ω by

$$\mathcal{M}(\omega) = (a_{ij}) = (\omega(b_i, b_j))$$

Proposition 2.1: Matrix induces a billinear form

Let $A = (a_{ij})$ be a matrix on V with respect to some basis $B = (b_{\underline{n}})$ it is clear that A induces a billinear form, on V through $A(x, y) = [x]_B^T A[y]_B$, where $[\cdot]_B$ denotes the canonical isomorphism $V \cong \mathbb{R}^n$ with respect to the basis B.

$$[x]_B^T A[y]_B = \begin{bmatrix} x^1 & \ldots & x^n \end{bmatrix} A \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}$$

for $x = x^i b_i$ and $y = y^j b_j$.

Moreover,

$$A[x]_B = \begin{bmatrix} A(b_1, x) \\ \vdots \\ A(b_n, x) \end{bmatrix}$$ is a *column* vector whose entries are given by applying x on the second coordinate

and

$$[x]_B^T A = \begin{bmatrix} A(x, b_1) & \cdots & A(x, b_n) \end{bmatrix}$$ is a *row* vector whose entries are given by applying x on the first coordinate

Let A_B be the matrix representation of ω with respect to the B, if C is another basis on V, then how do we compute A_C? The answer is simple, recall for any vector $x \in V$, $x = x_B^i b_i$ and $x = x_C^j c_j$, then

$$[x]_B = M_{C,B}[x]_C \quad \text{for some matrix of an automorphism } M_{C,B}$$

$\omega(x, y) = [x]_B^T A_B [y]_B = ([x]_C^T M_{C,B}^T) A_B (M_{C,B}[y]_C) = [x]_C^T A_C [y]_C$, then

$$M_{C,B}^T A_B M_{C,B} = A_C \tag{1}$$

We can describe this relation between the two matrices A_B and A_C by the following

Definition 2.2: Congruent matrices

Two matrices M and N are said to be *congruent*, if there exists an invertible matrix P for which

$$P^T M P = N$$

Congruence is an equivalence relation on the space of matrices, and the equivalence classes over congruence are called *congruence classes*.

Proposition 2.2: Characterization of matrices using congruence

Let A_1 and A_2 be matrix representations of two billinear forms with respect to the basis B.

$$A_1 = (A_1(b_i, b_j))_{ij} \quad A_2 = (A_2(b_i, b_j))_{ij}$$

They induce the same billinear form if and only if they are congruent.

Definition 2.3: Alternate matrices

Let M be a matrix with real coefficients, it is *alternate* if it is skew symmetric and is *hollow*; meaning it has 0s on the main diagonal.

Orthogonality

For this section, (V, ω) will denote a metric vector space, not necessarily finite-dimensional unless we are using matrix representations.

Definition 3.1: Orthogonal complements

A vector $x \in V$ is orthogonal to another vector $y \in V$, written $x \perp y$, if $\omega(x, y) = 0$.

If V is an orthogonal or symplectic geometry then $\perp$ is a symmetric relation. If E is a subset of V, we denote the *orthogonal complement of* E by

$$E^{\perp} \triangleq \left\{ v \in V,\, v \perp E \right\}$$

Definition 3.2: Characterization of metric vector spaces

- A nonzero vector $x \in V$ is *isotropic*, or *null* if $\omega(x, x) = 0$
- V is *isotropic* if it contains at least one isotropic vector.
- V is *anisotropic* or *nonisotropic* if for every $x \in V$, $\omega(x, x) = 0 \implies x = 0$,
- V is *totally isotropic* (that is, symplectic if $\mathrm{char}(F) \neq 2$) if $\omega(x, x) = 0$ for every vector $x \in V$.

The first bullet point above is about vectors in V, while the others are properties of V.

- A vector $x \in V$ is called *degenerate* if $x \perp V$, that is,

$$\forall y \in V,\, \omega(x, y) = 0$$

- The *radical* of V, denoted by $\mathrm{rad}(V)$ is the set of all degenerate vectors in V,

$$\mathrm{rad}(V) \triangleq V^{\perp}$$

- V is *singular* or *degenerate* if $\mathrm{rad}(V) \neq \{0\}$,
- V is *non-singular* or *non-degenerate* if $\mathrm{rad}(V) = \{0\}$,
- V is *totally singular*, if $\mathrm{rad}(V) = V$.

To summarize,

- V is isotropic if there exists a non-zero isotropic vector, meaning $\omega(x, x) = 0$, for some $x \neq 0$,
- V is degenerate if there exists a degenerate vector, $x \perp V$.

Proposition 3.1: Characterization of non-degeneracy

V is non-degenerate if and only if every matrix representation A of ω is non-singular.

Proof. Suppose V is non-degenerate, then let $B = (b_{\underline{n}})$ be a basis for V, if A is the matrix representation of ω with respect to B, let x be a non-zero vector in V, so $x \notin \mathrm{rad}(V)$

$$b_i^T A[x]_B = \omega(b_i, x) \neq 0 \implies A[x]_B \neq 0$$

so A is non-singular. If A' is another matrix representation with respect to another basis C, by Equation (1) A' is non-singular as well.

Conversely, if every matrix representation of ω is non-singular, let x be a non-zero vector in V, then $A[x]_B \neq 0$ is a non-zero vector so there exists some basis component $(A[x]_B)^j$ that is non zero, and

$$[b_j]_B^T A[x]_B = \omega(b_j, x) \neq 0$$

therefore V is non-degenerate. ■

Riesz Representation Theorems

Isometries

Definition 5.1: Isometry between MVS

Let (V, ω) and (W, η) be metric vector spaces. An *isometry* $\tau \in L(V, W)$ is a linear isomorphism that preserves the billinear form.

$$\omega(u, v) = \eta(\tau u, \tau v)$$

Definition 5.2: Orthogonal, symplectic groups

Let V be a nonsingular metric vector space. If V is an orthogonal (resp. symplectic) geometry, the set of all isometries on V is called the *orthogonal (resp. symplectic) group on* V. It is a group under composition, and is denoted by $\mathcal{O}(V)$ (resp. $\mathrm{Sp}(V)$).

Hyperbolic spaces, nonsingular completions

Symplectic transvections