Chapter 22: Symplectic Manifold

Manifolds Symplectic Tensors

Symplectic Tensors

Definition 1.1: Billinear forms

Let V be a vector space, a billinear form $\omega: V \times V \to \mathbb{R}$ is a 2-tensor on V.

Definition 1.2: Characterization of billinear forms

Let ω be a billinear form on V, it is

• symmetric if

$$\omega(x,y) = \omega(y,x)$$

• skew-symmetric or anti-symmetric if

$$\omega(x,y) = (-1)\omega(y,x)$$

• alternating if

$$\omega(x,x)=0$$

If V is a vector space over the field F and $\operatorname{char}(F) \neq 2$, then the last two conditions are equivalent. Moreover,

- V is called an orthogonal geometry if ω is symmetric.
- V is called a symplectic geometry if ω is alternating.

Definition 1.3: Metric vector space

A vector space (not necessarily finite dimensional) is called a *metric vector space* if it is a orthogonal or symplectic geometry.

Matrices and billinear forms

Definition 2.1: Matrix of billinear form

If $B = (b_1, \ldots, b_n)$ is an ordered basis for V, we define the matrix representation of ω by

$$\mathcal{M}(\omega) = (a_{ij}) = (\omega(b_i,b_j))$$

Proposition 2.1: Matrix induces a billinear form

Let $A = (a_{ij})$ be a matrix on V with respect to some basis $B = (b_{\underline{n}})$ it is clear that A induces a billinear form, on V through $A(x,y) = [x]_B^T A[y]_B$, where $[\cdot]_B$ denotes the canonical isomorphism $V \cong \mathbb{R}^n$ with respect to the basis B.

$$[x]_B^T A[y]_B = egin{bmatrix} x^1 & \dots & x^n \end{bmatrix} A egin{bmatrix} y^1 \ dots \ y^n \end{bmatrix}$$

for $x = x^i b_i$ and $y = y^j b_j$.

Moreover,

$$A[x]_B = egin{bmatrix} A(b_1,x) \ dots \ A(b_n,x) \end{bmatrix} egin{array}{ll} ext{is a } column ext{ vector} \ ext{whose entries are given} \ ext{by applying } x ext{ on the} \ ext{second coordinate} \end{array}$$

and

$$[x]_B^T A = \begin{bmatrix} A(x,b_1) & \cdots & A(x,b_n) \end{bmatrix}$$
 is a row vector whose entries are given by applying x on the first coordinate

Let A_B be the matrix representation of ω with respect to the B, if C is another basis on V, then how do we compute A_C ? The answer is simple, recall for any vector $x \in V$, $x = x_B^i b_i$ and $x = x_C^j c_j$, then

$$[x]_B = M_{C,B}[x]_C$$
 for some matrix of an automorphism $M_{C,B}$

$$\omega(x,y) = [x]_B^T A_B[y]_B = ([x]_C^T M_{C,B}^T) A_B(M_{C,B}[y]_C) = [x]_C^T A_C[y]_C$$
, then

$$M_{C,B}^T A_B M_{C,B} = A_C (1)$$

We can describe this relation between the two matrices A_B and A_C by the following

Definition 2.2: Congruent matrices

Two matrices M and N are said to be *congruent*, if there exists an invertible matrix P for which

$$P^TMP = N$$

Congruence is an equivalence relation on the space of matrices, and the equivalence classes over congruence are called *congruence classes*.

Proposition 2.2: Characterization of matrices using congruence

Let A_1 and A_2 be matrix representations of two billinear forms with respect to the basis B.

$$A_1 = (A_1(b_i, b_j))_{ij}$$
 $A_2 = (A_2(b_i, b_j))_{ij}$

They induce the same billinear form if and only if they are congruent.

Definition 2.3: Alternate matrices

Let M be a matrix with F-coefficients, it is alternate if it is skew symmetric and is hollow; meaning it

Manifolds Orthogonality

has 0s on the main diagonal. If $F = \mathbb{R}$ or $\operatorname{char}(F) \neq 2$, then alternate matrices are and are precisely the skew-symmetric matrices.

Orthogonality

For this section, (V, ω) will denote a metric vector space, not necessarily finite-dimensional unless we are using matrix representations.

Definition 3.1: Orthogonal complements

A vector $x \in V$ is orthogonal to another vector $y \in V$, written $x \perp y$, if $\omega(x,y) = 0$.

If V is an orthogonal or symplectic geometry then \bot is a symmetric relation. If E is a subset of V, we denote the *orthogonal complement of* E by

$$E^{\perp} \stackrel{\Delta}{=} \left\{ v \in V, \ v \perp E
ight\}$$

Definition 3.2: Characterization of metric vector spaces

- A nonzero vector $x \in V$ is *isotropic*, or null if $\omega(x,x) = 0$
- ullet V is isotropic if it contains at least one isotropic vector.
- V is anisotropic or nonisotropic if for every $x \in V$, $\omega(x,x) = 0 \implies x = 0$,
- V is totally isotropic (that is, symplectic if $\operatorname{char}(F) \neq 2$) if $\omega(x,x) = 0$ for every vector $x \in V$. The first bullet point above is about vectors in V, while the others are properties of V.
 - A vector $x \in V$ is called degenerate if $x \perp V$, that is,

$$\forall y \in V, \ \omega(x,y) = 0$$

• The radical of V, denoted by rad(V) is the set of all degenerate vectors in V,

$$\mathrm{rad}(V) \stackrel{\Delta}{=} V^{\perp}$$

- V is singular or degenerate if $rad(V) \neq \{0\}$,
- V is non-singular or non-degenerate if $rad(V) = \{0\},\$
- V is totally singular, if rad(V) = V.

To summarize,

- V is isotropic if there exists a non-zero isotropic vector, meaning $\omega(x,x)=0$, for some $x\neq 0$,
- V is degenerate if there exists a degenerate vector, $x \perp V$.

Proposition 3.1: Matrix invariants under congruence

Non-singularity, symmetry, and skew-symmetry are invariants under congruence.

Proof.

Proposition 3.2: Characterization of non-degeneracy

V is non-degenerate if and only if every matrix representation A of ω is non-singular.

Proof. Suppose V is non-degenerate, then let $B = (b_{\underline{n}})$ be a basis for V, if A is the matrix representation of ω with respect to B, let x be a non-zero vector in V, so $x \notin \operatorname{rad}(V)$

$$b_i^T A[x]_B = \omega(b_i, x) \neq 0 \implies A[x]_B \neq 0$$

so A is non-singular. If A' is another matrix representation with respect to another basis C, by Equation (1) A' is non-singular as well.

Conversely, if every matrix representation of ω is non-singular, let x be a non-zero vector in V, then $A[x]_B \neq 0$ is a non-zero vector so there exists some basis component $(A[x]_B)^j$ that is non zero, and

$$[b_j]_B^T A[x]_B = \omega(b_j, x) \neq 0$$

therefore V is non-degenerate.

Proposition 3.3: Characterisation of billinear forms from matrix representations

Let ω be a billinear form on V, if $\mathcal{M}(\omega)$ the induced matrix representation relative to any basis. Assume V is a vector space over \mathbb{R} , then

- it is symmetric iff $\mathcal{M}(\omega)$ symmetric as a matrix,
- it is skew-symmetric, iff alternating iff $\mathcal{M}(\omega)$ is skew-symmetric as a matrix.

Corollary 3.1: Characterisation of non-singular symplectic form

Let (V, ω) be a finite dimensional vector space over \mathbb{R} , equipped with a billinear form ω . (V, ω) is a non-singular symplectic vector space iff the matrix representation of ω with respect to every basis is non-singular and skew-symmetric.

Riesz Representation Theorems

Proposition 4.1

Let (V,ω) be a nonsingular metric vector space, the map $x\mapsto x\lrcorner\omega\in V^*$ defined by

$$x\lrcorner\omega=\omega(x,\cdot),\quad ext{and}\quad (x\lrcorner\omega)(y)=\omega(x,y),\quad \forall y\in V$$

Manifolds Isometries

is a linear isomorphism from V to V^* .

Isometries

Definition 5.1: Isometry between MVS

Let (V, ω) and (W, η) be metric vector spaces. An isometry $\tau \in L(V, W)$ is a linear isomorphism that preserves the billinear form.

$$\omega(u,v)=\eta(\tau u,\tau v)$$

Definition 5.2: Orthogonal, symplectic groups

Let V be a nonsingular metric vector space. If V is an orthogonal (resp. symplectic) geometry, the set of all isometries on V is called the *orthogonal (resp. symplectic) group on* V. It is a group under composition, and is denoted by $\mathcal{O}(V)$ (resp. $\mathrm{Sp}(V)$).

Hyperbolic spaces, nonsingular completions

Canonical Forms

Symplectic Manifolds

Darboux's Theorem

Proposition 9.1: Lie Derivatives of Tensor Fields (along time-varying vector fields)

Let M be a smooth manifold. Suppose $V: J \times M \to TM$ is a smooth time-varying vector field on M. Denote the time-varying flow of V by $\psi: \mathcal{E} \to M$. Let $A \in \mathcal{T}^k(M)$ be a smooth time-invariant covariant k-tensor field on M. For every $(t_1, t_0, p) \in \mathcal{E}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1} (\psi_{t,t_0}^* A)_p = (\psi_{t_1,t_0}^* (\mathcal{L}V_{t_1} A))_p$$
(2)

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Manifolds Darboux's Theorem

Definition 0.1: Symplectic vector space

Let V be a finite dimensional vector space over \mathbb{R} . It is a *symplectic vector space* if it admits a non-singular, antisymmetric billinear form $\omega: V \times V \to \mathbb{R}$.

$$\omega(u,v) = -\omega(v,u)$$

for $u,v\in V.$ By the previous section on Riesz Representation, the linear map

$$\hat{\omega}: V \to V^*, \quad v \mapsto \omega(v, \cdot)$$

is a linear isomorphism of V onto its dual vector V^* .

We define the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$, where $n \in \mathbb{N}^+$, where

$$\omega_0(u,v) = < Ju, v> \quad J \stackrel{\Delta}{=} egin{bmatrix} 0 & I_n \ -I_n & 0 \end{bmatrix}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^{2n} .

$$\omega_0(u, v) = \langle Ju, v \rangle = \langle u, J^T v \rangle = u^T J^T v$$
 (3)

 $J^T = -J$ by Corollary 3.1.

We will mainly deal with non-singular symplectic forms because of Riesz isomorphism.

Definition 0.2: Symplectic linear map

Let (V, ω) be a symplectic vector space. A linear map $F \in \text{Hom}(V)$ is *symplectic* if it preserves symplectic form ω . For every $u \in V$,

$$< u, v > = < Au, Av > \stackrel{\triangle}{=} A^*\omega(u, v)$$

where $A^*: \mathcal{T}^*(V) \to \mathcal{T}^*(V)$ denotes the tensor pullback by precomposing any tensor S by A

$$orall S \in \mathcal{T}^k(V), \quad A^*S(v_{\underline{k}}) \stackrel{\Delta}{=} S(Av_{\underline{k}})$$

The set of linear symplectic maps on a 2n-dimensional vector space form a group under composition. It is a Lie Group denoted by Sp(n).

Proposition 0.2: Symplectic Maps are Area-preserving

Let $(\mathbb{R}^{2n}, \omega_0)$ denote the standard symplectic space. If $\varphi \in \operatorname{Sp}(n)$, then det $\varphi = 1$.

Proof. See page 4.

Manifolds Darboux's Theorem

$$(\Lambda_{s-|\alpha|}\partial^{\alpha}f)^{\hat{}} = (1+|\zeta|^2)^{s/2-|\alpha|/2} \cdot (\partial^{\alpha}f)^{\hat{}}$$

$$\tag{4}$$

$$= (1 + |\zeta|^2)^{s/2 - |\alpha|/2} \cdot (2\pi i \zeta)^{\alpha} \cdot \hat{f}$$

$$\tag{5}$$

$$= (2\pi i)^{|\alpha|} (1 + |\zeta|^2)^{(s-|\alpha|)/2} \cdot |\zeta|^{|\alpha|} \cdot \hat{f}$$
 (6)

$$\leq |\alpha|(1+|\zeta|^2)^{s/2}\hat{f} \tag{7}$$