

Notation

We will use the following notation to simplify computations with multilinear maps. Let E and F be sets, and $v_1, \dots, v_k \in E$. $f : E \rightarrow F$.

- Listing individual elements: $v_{\underline{k}}$ means v_1, \dots, v_k as separate elements.
- Creating a k -list: $(v_{\underline{k}}) = (v_1, \dots, v_k) \in \prod E_{j \leq k}$ if $v_i \in E_i$ for $i = \underline{k}$.
- Double indices: $(v_{\underline{n_k}}) = (v_{n_k}) = (v_{n_1}, \dots, v_{n_k})$, and

$$(v_{\underline{n_k}}) \neq (v_{n_{(1, \dots, k)}})$$

- Closest bracket convention:

$$(v_{(n_k)}) = (v_{(n_1, \dots, n_k)}) \quad \text{and} \quad (v_{n_{(k)}}) = (v_{n_{(1, \dots, k)}})$$

- Underlining 0 means it is iterated 0 times:

$$(v_{\underline{0}}, a, b, c) = (a, b, c)$$

- Skipping an index:

$$(v_{\underline{i-1}}, v_{i+\underline{k-i}}) = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k) \quad (1)$$

for $i = \underline{k}$.

- Applying f to a particular index:

$$(v_{\underline{i-1}}, f(v_i), v_{i+\underline{k-i}}) = (v_1, \dots, v_{i-1}, f(v_i), v_{i+1}, \dots, v_k) \quad (2)$$

Of course, if $i = 1$, then the above expression reads $(f(v_1), v_2, \dots, v_k)$ by the $\underline{0}$ interpretation.

- If $\wedge : E \times E \rightarrow F$ is any associative binary operation,

$$\bigcircled{\wedge}(v_{\underline{k}}) = v_1 \wedge \dots \wedge v_k$$

- In any list using this 'underline' notation, we can find the size of a list by summing over all the underlined numbers and the number of terms without an underline. We see eq. (1), eq. (2) have $k-1$, k terms respectively.

Remark 1.1: Preview of exterior calculus

We can write the cofactor expansion formula of the determinant of a $\mathbb{R}^{k \times k}$ matrix in this notation. Suppose $a_i \in \mathbb{R}$, and $b_i \in \mathbb{R}^{k-1}$ for $i = \underline{k}$.

$$M = \begin{bmatrix} a_1 & \dots & a_k \\ | & & | \\ b_1 & \dots & b_k \\ | & & | \end{bmatrix}$$

The determinant of M , can then be written as

$$\det(M) = \sum_{i=\underline{k}} (-1)^{i-1} a_i \det(b_{\underline{i-1}}, b_{\underline{i+k-i}})$$

k -linear maps

Definition 2.1: k -linear maps

Let $E_{\underline{k}}$, F be Banach spaces. A map $\varphi : \prod E_{\underline{k}}$ is k -linear if for every $i = \underline{k}$, $v_i \in E_i$,

$$\varphi(\cdot, v_i, \cdot) : \bigoplus (E_{\underline{i-1}}, E_{\underline{i+k-i}}) \rightarrow F \quad \text{is } (k-1)\text{-linear}$$

The following theorem should give confidence to the notation we have adopted to use.

Proposition 2.1

Let $E_{\underline{k}}$ and F be Banach spaces, a k -linear map $\varphi : \prod E_{\underline{k}} \rightarrow F$ is continuous iff there exists a $C > 0$, such that for every $x_i \in E_i$, $i = \underline{k}$

$$|\varphi(x_{\underline{k}})| \leq C \prod |x_{\underline{k}}|$$

Proof. Suppose φ is continuous, then it is continuous at the origin. Picking $\varepsilon = 1$ induces a $\delta > 0$ such that for $|(x_{\underline{k}})| \leq \delta$, $|\varphi(x_{\underline{k}})| \leq 1$. The usual trick of normalizing an arbitrary vector $(x_{\underline{k}}) \in \prod E_{\underline{k}}$ does the job:

$$|\varphi(x_{\underline{k}} \cdot |x_{\underline{k}}|^{-1} \cdot \delta)| \leq 1 \implies |\varphi(x_{\underline{k}})| \leq \delta^{-k} \prod |x_{\underline{k}}|$$

Conversely, fix a sequence (indexed by n , in k elements in the product space $\prod E_{\underline{k}}$), so

$$(x_n^{\underline{k}}) \rightarrow (x^{\underline{k}}) \quad \text{as } n \rightarrow +\infty \tag{3}$$

To proceed any further, we need to prove eq. (4) that expresses the difference of two values of φ in terms its arguments.

$$\varphi(b^{\underline{k}}) - \varphi(a^{\underline{k}}) = \sum_{i=\underline{k}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{\underline{i+k-i}}) \tag{4}$$

where $(b^{\underline{k}})$ and $(a^{\underline{k}})$ are elements in $\prod E_{\underline{k}}$, and $\Delta_i = b^i - a^i$ for $i = \underline{k}$. The proof is contained in the following note, which is in more detail than usual - to help the reader ease into the new notation.

Note 2.1

We proceed by induction, and eq. (4) follows by setting $m = k$ in

$$\varphi(a^{\underline{k}}) = \varphi(b^{\underline{m}}, a^{\underline{m+k-m}}) - \sum_{i=\underline{m}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{\underline{i+k-i}}) \tag{5}$$

Base case: set $m = 1$, by definition of k -linearity (definition 2.1) of φ . Since $a^1 = b^1 - \Delta_1$,

$$\varphi(a^{\underline{k}}) = \varphi(b^1 - \Delta_1, a^{\underline{1+k-1}}) = \varphi(b^1, a^{\underline{1+k-1}}) - \varphi(\Delta_1, a^{\underline{1+k-1}})$$

Induction hypothesis: suppose eq. (5) holds for a fixed m . Since $a^{m+1} = b^{m+1} - \Delta_{m+1}$,

$$\begin{aligned}\varphi(a^{\underline{k}}) &= \varphi(b^{\underline{m}}, a^{m+\underline{k}-m}) - \sum_{i=\underline{m}} \varphi(b^{i-1}, \Delta_i, a^{i+\underline{k}-i}) \\ &= \varphi(b^{\underline{m}}, a^{m+1}, a^{(m+1)+\underline{k}-(m+1)}) - \sum_{i=\underline{m}} \varphi(b^{i-1}, \Delta_i, a^{i+\underline{k}-i}) \\ &= \varphi(b^{m+1}, a^{(m+1)+\underline{k}-(m+1)}) - \varphi(b^{m+1}, \Delta_{m+1}, a^{(m+1)+\underline{k}-(m+1)}) - \sum_{i=\underline{m}} \varphi(b^{i-1}, \Delta_i, a^{i+\underline{k}-i})\end{aligned}$$

and this proves eq. (4)

We substitute $a^i = x^i$, and $b^i = x_n^i$ for $i = \underline{k}$, and eq. (4) becomes eq. (6)

$$\varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) = \sum_{i=\underline{k}} \varphi(x_n^{i-1}, x_n^i - x^i, x^{i+\underline{k}-i}) \quad (6)$$

Then the triangle inequality reads

$$\begin{aligned}\left| \varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) \right| &\leq \sum_{i=\underline{k}} \left| \varphi(x_n^{i-1}, x_n^i - x^i, x^{i+\underline{k}-i}) \right| \\ &\leq \sum_{i=\underline{k}} |\varphi| \cdot \bigoplus \left(x_n^{i-1}, \Delta_i, x^{i+\underline{k}-i} \right) \\ &\leq \sum_{i=\underline{k}} |\varphi| \cdot \left| x_n^i - x^i \right| \bigoplus \left(x_n^{i-1}, x^{i+\underline{k}-i} \right) \\ &\lesssim_n |\varphi| \sup_{i=\underline{k}} |x_n^i - x^i| \rightarrow 0\end{aligned}$$

where we identify the product $\bigoplus(v^{\underline{k}})$ with the product of their norms $\bigoplus(|v^{\underline{k}}|)$. ■