MATH 263: Section 003, Tutorial 6

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October 14^{th} 2021

1 Complex Numbers, Euler and DeMoivre's Formulas

The ODE $ay" + by' + cy = 0$ gives the characteristic equation $ak^2 + bk + c = 0$, which does not have solutions in $\mathbb{R}$ when $b^2 < 4ac$. Therefore, we can define a number, called i (the imaginary unit), such that $i^2 = -1$ ($i = \sqrt{-1}$). This lets us work with a new kind of numbers, called the **complex numbers**, denoted as $\mathbb{C}$.

$$\mathbb{C} = \{z : z = a + bi, a \in \mathbb{R}, b \in \mathbb{R}\}$$

Where a is the **real part** of z, and b is the **imaginary part** of z.

$$Re(z) = a, \ Im(z) = b$$

Note: In Electrical Engineering, j is used instead as a complex unit. This is to not confuse the imaginary unit i with the current variable i. Complex numbers can be added, subtracted, multiplied, and divided the same way as if i were an algebraic variable, while keeping in mind that $i^2 = -1$. Examples:

$$(3 + 2i) + (2 - 4i) = (3 + 2) + (2 - 4)i = 5 - 2i$$

$$(3 + 2i) - (2 - 4i) = (3 - 2) + (2 + 4)i = 1 + 6i$$

$$(3 + 2i)(2 - 4i) = 6 - 12i + 4i - 8^2 = 6 - 12i + 4i - 8(-1) = 14 - 8i$$

$$\frac{3 + 2i}{2 - 4i} = \frac{-2 + 16i}{2^2 - (4i)^2} = \frac{-2 + 16i}{2^2 + 4^2} = \frac{-2 + 16i}{20} = \frac{1}{10}(-1 + 8i).$$

Note that the division process consists of multiplying the numerator and denominator by the complex conjugate of the denominator. In general, the complex conjugate of $z = a + bi$, often denoted as $z^* = a - bi$. Complex numbers can also be written in a polar form, $z = a + bi = r[\cos(\theta) + i\sin(\theta)]$, where $r = \sqrt{a^2 + b^2}$ is the norm, and $\theta = \arctan(\frac{b}{a})$ is the argument. Those two representations can be illustrated using the **complex plane**.

Defining exponentiation for complex numbers as $e^z = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we can let:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{even\ n} \frac{i^n x^n}{n!} + \sum_{odd\ n} \frac{i^n x^n}{n!}$$

For even numbers, let $n = 2k \Rightarrow i^n = i^{2k} = (-1)^k$. For odd numbers, $n = 2k + 1 \Rightarrow i^n = i^{2k+1} = i \cdot i^{2k} = i(-1)^k$.

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$e^{ix} = \cos x + i \sin x,$$

which is **Euler's Formula**. Therefore, the polar form can be written as

$$z = a + bi = r[\cos(\theta) + i\sin(\theta)] = re^{i\theta}.$$

Similarly, **DeMoivre's Formula** is:

$$(e^{ix})^n = e^{inx} = \cos(nx) + i\sin(nx).$$

This formula can be used to multiply and find powers of complex numbers:

$$z^n = re^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)].$$

Problem 1a. Using Euler's Theorem, compute:

$$(1-i)^{12}$$

Solution: $(1-i)^{12} = re^{i\theta}$ where $r = \sqrt{1^2+1^2} = \sqrt{2}$ and $\theta = \arctan(\frac{-1}{1}) = -\frac{\pi}{4}$

$$(1-i)^{12} = \sqrt{2}^{12} e^{12i(-\frac{\pi}{4})} = 2^{\frac{12}{2}} e^{-3\pi} = 2^6(\cos(-3\pi) + i\sin(-3\pi)) = 2^6(-1 + i\cdot 0) = -64.$$

Problem 1b. Find the general solution of:

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + 6\frac{\mathrm{d}x}{\mathrm{d}t} + 13x = 0$$

Solution: Let $x(t) = e^{kt}$, then the characteristic polynomial is:

$$k^2 + 6k + 13 = 0$$

$$k = \frac{1}{2}(-6 \pm \sqrt{6^2 - 4\cdot 1\cdot 13})$$

$$k = \frac{1}{2}(-6 \pm \sqrt{36-52})$$

$$k = \frac{1}{2}(-6 \pm \sqrt{-16})$$

$$k = \frac{1}{2}(-6 \pm 4i)$$

$$k_1 = (-3+2i),\ k_2 = (-3-2i)$$

Note that the two roots are conjugates of each other, ($k_2 = k_1^*$). In general, it is always true that complex roots of a polynomial with real coefficient always come in conjugate pairs. To show that, it can be shown that for real a and b, $ax_1^* + bx_2^*$, and that $(x^*)^n = (x^n)^*$. Knowing that, showing that x^* solves a real coefficient polynomial given that x is a solution is straightforward. Now, the general solution is:

$$x(t) = c_1e^{(-3+2i)t} + c_2e^{(-3-2i)t}$$

$$x(t) = e^{-3t}(c_1e^{2it} + c_2e^{-2it})$$

Using Euler's formula:

$$x(t) = e^{-3t}[c_1\cos(2t) + ic_1\sin(2t) + c_2\cos(-2t) + ic_2\sin(-2t)]$$

$$x(t) = e^{-3t}[(c_1\cos(2t) + ic_1\sin(2t) + c_2\cos(2t) - ic_2\sin(2t)]$$

$$x(t) = e^{-3t}[(c_1+c_2)\cos(2t) + i(c_1-c_2)\sin(2t)]$$

Restricting ourselves to real solutions, let $c_3 = c_1 + c_2,\ c_4 = i(c_1 - c_2)$ be real:

$$x(t) = e^{-3t}[c_3\cos(2t) + c_4\sin(2t)]$$

2 The Wronskian and Abel's Theorem

Given two solutions of a second order linear ODE, $y_1(x),\ y_2(x)$, they are independent if their **Wronskian** is not 0, which is given by:

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

Given a differential equation of the form:

$$y" + p(x)y' + q(x)y = 0$$

Abel's Theorem states that:

$$W = c\ \exp[-\int p(x)\ dx]$$

where the constant can be found with a initial condition on the Wronskian.

Problem 2a. From Boyce and DiPrima, 10th edition (3.2, exercise 29, p.157):
Given the ODE:

$$t^2y" - t(t+2)y' + (t+2)y = 0$$

Find the general form of the Wronskian.

Solution: First divide by t^2:

$$y" - \frac{(t+2)}{t}y' + \frac{(t+2)}{t^2}y = 0$$
$$y" - \frac{(t+2)}{t}y' + \frac{(t+2)}{t^2}y = 0$$
$$p(t) = -\frac{(t+2)}{t} = -1 - \frac{2}{t}$$
$$W = c\ \exp[\int 1 + \frac{2}{t}\ dt]$$
$$W = c\ \exp[t + 2\ln t]$$
$$W = ct^2e^t$$

3 Euler's Equation

Euler's Equations are of the form:

$$ax^2y" + bxy' + cy = 0$$

this is solved by making the substitution $y = x^r$, $x > 0$. The characteristic polynomial becomes:

$$ar^2 + (b-a)r + c = 0.$$

For complex roots, the solution would be of the form:

$$y(x) = c_1x^{r_1} + c_2x^{r_2},\ x > 0$$

In the case where $x < 0$, one can make the substitution $t = -x > 0$ and $y(x) = u(t)$, which would give the same ODE. Therefore, for all $x \neq 0$, the solution is:

$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_2}$$

For complex roots $r_{1,2} = \lambda \pm i\mu$, the solution would be of the form:

$$y(x) = c_1x^{\lambda+i\mu} + c_2x^{\lambda-i\mu},\ x > 0$$

Knowing that $x^{i\mu} = e^{i\mu \ln x} = \cos(\mu \ln x) + i \sin(\mu \ln x)$, the final real solution would be:

$$y(x) = x^{\lambda}[k_1 \cos(\mu \ln |x|) + k_2 \sin(\mu \ln |x|)]$$

Given a double root r_1, reduction of order gives us a solution of:

$$y(x) = (c_1 + c_2 \ln |x|)|x|^{r_1}$$

Problem 3. Find the general solution of:

$$x^2 y" - xy' + y = 0$$

Show the two solutions are linearly independent for $x > 0$ and solve the IVP : $y(1) = 2,\ y'(1) = 0$.

Solution: The characteristic equation is:

$$r^2 - 2r + 1 = (r-1)^2 = 0$$

$$r_{1,2} = 1$$

Then the solution would be

$$y(x) = c_1|x| + c_2|x| \ln |x|$$

Then the Wronskian for $x > 0$ is:

$$W = \begin{vmatrix} x & x \ln x \\ 1 & \ln x + 1 \end{vmatrix} = x(\ln x + 1) - x \ln x = x$$

which is not 0 for $x \neq 1$. Now solve the IVP $y(1) = 2,\ y'(1) = 0$:

$$y(1) = c_1|1| + c_2|1| \ln |x| = 2$$

$$c_1 = 2$$

For $x > 0$,

$$y'(1) = c_1 + c_2(1 + \ln 1) = 0$$

$$c_2 = -2$$

$$y(x) = 2|x|(1 - \ln |x|).$$

Problem 4. Find the general solution of:

$$4x^2 y" + 8xy' + 17y = 0$$

Solution: Let $y = x^r$. The characteristic equation is:

$$4r^2 + 4r + 17 = 0$$

$$r = \frac{1}{8}(-4 \pm \sqrt{16 - 16 \cdot 17})$$

$$r = \frac{1}{8}(-4 \pm \sqrt{-16^2})$$

$$r = \frac{1}{8}(-4 \pm 16i)$$

$$r_1 = \frac{-1}{2} + 2i,\ r_2 = \frac{-1}{2} - 2i.$$

Therefore,

$$y(x) = \frac{1}{\sqrt{x}}[k_1 \cos(2 \ln |x|) + k_2 \sin(2 \ln |x|)].$$