Folland Reading

me

September 25, 2022

Contents

1 Chapter 4

1.1 Theorem 4.1

WTS. *Suppose that A is a subset of X, let* $\operatorname{acc} A$ *be the set of accumulation points of A, then*

$$\overline{A} = A \cup \operatorname{acc}(A) \tag{1}$$

and A is closed if and only if $\operatorname{acc}(A) \subseteq A$.

Proof. Suppose that $x \notin \overline{A}$, then $x \in (\overline{A})^c = A^{co}$, then $A^c \in \mathcal{N}_B(x)$. But this means that $x \notin \operatorname{acc}(A)$, since there exists a neighbourhood of x (in the form of A^c), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \varnothing$$

Also, $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$ which means that

$$x \notin \overline{A} \implies x \notin A$$

Since $x \notin \overline{A} \implies x \notin A$ and $x \notin \operatorname{acc}(A)$,

$$(\overline{A})^c \subseteq A^c \cap \operatorname{acc}(A)^c = (A \cup \operatorname{acc}(A))^c$$

Now, if $x \notin \operatorname{acc}(A) \cup A$, then $x \notin \operatorname{acc}(A)$, therefore there exists some $U \in \mathcal{N}_B(x)$ such that

$$A \cap U \setminus \{x\} = A \cap U = \varnothing$$

Where for the second last equality we used the fact that $x \notin A \implies A \backslash \{x\} = A$, and taking complements gives us

$$U \subseteq A^c$$

And since $U \in \mathcal{N}_B(x)$, then $x \in U^o \subseteq A^{co}$ (since U^o is an open subset of A^c). then

$$x \in A^{co} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore $(A \cup \operatorname{acc}(A))^c \subseteq (\overline{A})^c$. $\square$

1.2 Theorem 4.2

WTS. *If $\mathcal{T}_X$ is a topology on X and $\mathcal{E} \subseteq \mathcal{T}_X$ then $\mathcal{E}$ is a base for $\mathcal{T}_X$ if and only if for every*

$$\forall U \in \mathcal{T}_X,\ U \neq \varnothing,\ \implies\ U = \bigcup_{V \in B} V$$

Where B is a subset of $\mathcal{E}$.

Proof. Suppose that $\mathcal{E}$ is a base, then fix any non-empty $U \in \mathcal{T}_X$, then for every $x \in U$, there exists a neighbourhood base for this x and a member $V \in \mathcal{E}$ such that $x \in V_x \subseteq U$. Take the union over all V_x and

$$U \subseteq \bigcup_{x \in U} V_x$$

But each $V_x \subseteq U$, so $U = \bigcup_{x \in U} V_x$, where $\{V_x\} \subseteq \mathcal{E}$.

Conversely, if every non-empty U is a union of members in $\mathcal{E}$ then fix any $x \in X$, we claim that we have a neighbourhood base in

$$\{V \in \mathcal{E},\ x \in V\}$$

The reason is as follows

- x belongs to every $E \in \{V \in \mathcal{E},\ x \in V\}$ and
- For every open U, if $x \in U$ then there exists a union of members of $\mathcal{E}$ such that $U = \bigcup E_\alpha$, then $x \in U \iff \exists E_\alpha \in \{V \in \mathcal{E},\ x \in V\}$ and
- Using this particular $E_\alpha \in \mathcal{E}$ that we just found, $x \in E_\alpha \subseteq U$, and we are done.

$\square$

1.3 Theorem 4.3

WTS. *For every $\mathcal{E} \subseteq \mathbb{P}(X)$, $\mathcal{E}$ is base for a topology on X if and only if*

(a) each $x \in X$ is contained in some $V \in \mathcal{E}$, and

(b) if $U, V \in \mathcal{E}$, and $x \in U \cap V$, then there must exist some $W \in \mathcal{E}$ with $x \in W \subseteq U \cap V$.

Proof. Suppose that $\mathcal{E}$ is a base, then we get a), and b) follows since for every $U, V \in \mathcal{E} \subseteq \mathcal{T}_X$, and by closure over finite intersections, $U \cap V \in \mathcal{T}_X$ implies that there exists some $W \in \mathcal{E}$ with

$$x \in W \subseteq U \cap V$$

Now, suppose both a) and b) hold, then we claim that this $\mathcal{E} \subseteq \mathbb{P}(X)$ induces a topology on X

$$\mathcal{T} = \{U \subseteq X,\ \forall x \in U,\ \exists V \in \mathcal{E},\ \text{with } x \in V \subseteq U\}$$

Intuitively speaking, this means that $\mathcal{T}$ is just fine (and not too fine) to satisfy the conditions for $E \subseteq \mathcal{T}$ to be a base of $\mathcal{T}$.

We first show that $\mathcal{T}$ is a topology.

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$, the first is trivial and the second is from a)
- Closure under unions: fix $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, and $U = \bigcup U_\alpha$, and for every $x \in U$ there exists some $V_\alpha \in \mathcal{E}$ such that $x \in V_\alpha \subseteq U_\alpha \subseteq U$, therefore $U \in Tau$.
- Closure under finite intersections, fix any U_1, U_2 as elements in $\mathcal{T}$, then suppose that they are not disjoint (if they are disjoint then their intersection is the empty set, which is also contained in $\mathcal{T}$). If $U_1 \cap U_2 \neq \varnothing$, then for every $x \in U_1 \cap U_2$ induces two sets $V_1, V_2 \in \mathcal{E}$ with $x \in V_1 \subseteq U_2$ and $x \in V_2 \subseteq U_2$, taking their intersection and applying b) gives us some $V \subseteq V_1 \cap V_2$ with $V \in \mathcal{E}$ therefore $x \in V \subseteq U_1 \cap U_2$, and $\mathcal{T}$ is closed under finite intersections.

Now to show that $\mathcal{E}$ is a base for $\mathcal{T}$, $\mathcal{E} \subseteq \mathcal{T}$ is obvious since very $V \in \mathcal{E}$ satisfies the properties laid out by $\mathcal{T}$ by simply choosing V again for any

$x \in V$. Now fix any member $U \in Tau$, then for every $x \in U$, there exists some $V \in \mathcal{E}$ with

$$x \in V \subseteq U$$

(This is an immediate consequence of how we defined $\mathcal{T}$). And we can conclude that $\mathcal{E}$ is a basef or this induced topology $\mathcal{T}$. $\square$

1.4 Theorem 4.4

WTS. *If $\mathcal{E} \subseteq \mathbb{P}(X)$, the topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ consists of $\varnothing, X$ and all unions of finite intersections of $\mathcal{E}$, in symbols*

$$\mathcal{T}(\mathcal{E}) = \{\varnothing, X\} \cup \left\{ \bigcup W_\alpha,\ W_\alpha = \bigcap E_{j \leq n},\ E_j \in \mathcal{E} \right\}$$

1.5 Theorem 4.5

WTS. *Every second countable space is separable. (Countable dense subset).*

Proof. ∎

1.6 Theorem 4.6

WTS. *If X is first countable, then for every $A \subseteq X$, $x \in \overline{A} \iff$ there exists some sequence $\{x_j\}_{j \geq 1} \subseteq A$ such that $x_j \to x$.*

1.7 Theorem 4.7

WTS. *X is a T_1 space $\iff$ $\{x\}$ is closed for every $x \in X$.*

Proof. If X is T_1 and $x \in X$, then for every $y \neq x$ there exists some open U_y that contains y but not x. Following Folland's argument closely, every $y \neq x$ is is in $\cup U_{y\neq x}$. Hence $\{x\}^c \subseteq \cup U_{y\neq x}$. To show the converse, for every $z \in \cup U_{y\neq x}$ that is open, there exists a $y \neq x$ such that $z \in U_y$. But every U_y does not contain x as an element, so $z \neq x$ implies that $z \notin \{x\}$. And $z \in \{x\}^c$. Hence $\cup U_{y\neq x} = \{x\}^c$.

Now conversely if every $x \in X$ satisfies the fact that $\{x\}^c$ is open, then $\{x\}^c$ is an open set that contains every $y \neq x$. Now fix some $y \neq x$, since $\{y\}$ is also closed, we have $X \cap \{x\}^c$ is an open set that contains x but not y. Also, $\{x\}^c$ is an open set that contains y but not x. And therefore X is T_1. $\square$

1.8 Theorem 4.8

WTS.

1.9 Theorem 4.9

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1.10 Theorem 4.10

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1.11 Theorem 4.11

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1.12 Theorem 4.12

WTS.

1.13 Theorem 4.13

WTS.

1.14 Theorem 4.14

WTS. *Suppose that A and B are disjoint closed subsets of the normal space X, and let $\Delta = \{k2^{-n} : n \geq 1$ and $0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that*

1. *$A \subseteq U_r \subseteq B^c$ for every $r \in \Delta$,*
2. *$\overline{U_r} \subseteq U_s$ for $r < s$, and*
3. *For every $r < s$, $\overline{U}_r \subseteq U_s$*

Proof. The goal of this proof is to show that for every $r \in \Delta$, there exists a open U_r that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some $N \geq 1$ that all previous U_r in previous layers have been constructed properly, meaning if $r = k/2^n$, then for every $1 \leq n \leq N - 1$, we have

$$r = \frac{k}{2^n},\ 1 \leq n \leq N - 1,\ 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each U_r,

- $A \subseteq U_r \subseteq B^c$ and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer $N \geq 1$, we only have to construct the $U_{k/2^N}$ for every odd k, this is because if k is an even number, then $k = 2j$ and $r = 2j/2^N = j/2^{N-1}$ and for this particular U_r is already constructed. So for every odd $k = 2j + 1$, the sets of the form $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every $k - 1 \neq 0$ and $k + 1 \neq 1$. (We will consider these cases later). We claim that for every pair of open sets, $E_1, E_2 \in \mathcal{T}_X$, then there exists some open set $G \in \mathcal{T}_X$ such that if $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$ where H is defined as the set

$$H = \left\{(E_1, E_2) \subseteq (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \varnothing\right\}$$

Then there exists some $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$ such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any $(E_1, E_2) \in H$, then this pair induces a pair of disjoint sets $\overline{E_1}$ and E_2^c since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \varnothing$$

And by normality, there exists disjoint open sets G_1, G_2 such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \varnothing \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since G_2^c is a closed set that contains G_1 as a subset, $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of G_2 (so we may discard the notion of G_2 in our minds). Let us now replace G with G_1 then it is an easy task to verify that $G = G_1 = \mathcal{J}(E_1, E_2)$ has the required properties.

Now define for every odd k, since $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (we note in passing that $\mathcal{J}$ is not a function as the set G may not be unique).

$$U_{k/2^N} = \mathcal{J}\left(U_{(k-1)/2^N}, U_{(k+1)/2^N}\right)$$

Then, if $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$ sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$ and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed' $U_{k/2^N}$ for every $k - 1 \neq 0$ and $k + 1 \neq 1$. Now let us deal with the remaining pathological cases.

If $k-1$ so happens to be 0, then no $r \in \Delta$ satisfies $r = 0/2^N$, and we substitute

$$\overline{U}_0 = A, \quad \text{or alternatively, } U_0 = A^o$$

Then $U_0 \in \mathcal{T}_X$, $\overline{U}_0 = A \subseteq B^c$. It is at this point that we must mention that $0, 1 \notin \Delta$, so U_0 and U_1 do not have to obey the rules we have laid out for $U_{r\in\Delta}$.

Now if $k+1$ is equal to 2^N (this makes $r = (k+1)/2^N = 1$) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every $0 \leq m \leq 2^N - 1, U_{m/2^N}$ must staisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (even for when $N = 1$, since $A = \overline{U}_0 \subseteq U_1 = B^c$) and a corresponding $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$ such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when $N = 1$. We would only have to construct for $k = 1$, since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. $\square$

1.15 Theorem 4.15

WTS. *Urysohn's Lemma. Let X be a normal space, if A and B are disjoint closed subsets of X, then there exists a $f \in C(X,[0,1])$ such that $f = 0$ on A and $f = 1$ on B.*

Proof. Let $r \in \Delta$ be as in Lemma 4.14, and set U_r accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0,1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic ε argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since A and B are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that x is not a member of any U_r, but we set $U_1 = X$. Since none of the $r \in (0,1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The ε argument follows: suppose for every $\varepsilon > 0$, $(1-\varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannnot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in X. So $f^{-1}((-\infty,\alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha,+\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of Δ, there exists an r, $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r<\alpha} U_r$. So $f^{-1}((-\infty,\alpha)) \subseteq \bigcup_{r<\alpha} U_r$.

Fix an element $x \in \bigcup_{r<\alpha} U_r$, this induces an r such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r<\alpha} U_r \subseteq f^{-1}((-\infty,\alpha))$.

For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an r (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) — we have $x \notin \overline{U_s}$, but $\left(\overline{U_s}\right)^c$ is open in X. It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq \left(\overline{U_s}\right)^c \subseteq \bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$. To show the reverse, fix an element x in the union, then this induces some $x \in \left(\overline{U_s}\right)^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of W. This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen s is a lower bound for W, and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in X, using Theorem 4.9 finishes the proof. $\square$

1.16 Theorem 4.16

WTS. *The Tietze's Extension Theorem. Let X be a normal space, and for any closed subset $A \subseteq X$, and $f \in C(A, [a, b])$, there exists an $F \in C(X, [a, b])$ which extends f.*

Proof. We begin with an important lemma that will serve as a 'black box' for the induction.

Lemma 1.1. *For every $f \in C(A, [0, 1])$, there exists a $g \in C(X, [0, 1/3])$ such that*

$$0 \leq f - g \leq 2/3 \quad \textit{pointwise on } A \tag{2}$$

Proof. Since f is continuous, $B = f^{-1}([0, 1/3])$, and $C = f^{-1}([2/3, 1])$ are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function $g \in C(X, [0, 1])$ such that $g|_B = 0$ and $g|_C = 1$. Relabel $g = g/3$ then $g \in C(X, [0, 1/3])$ (multiplication is continuous).

To show that (2) holds, suppose $x \in B$, then $f(x) \in [0, 1/3]$ and $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$. Now suppose that $x \in C$, then $f(x) \in [2/3, 1]$ and $g(x) = 1/3$ (recall that we relabelled g). So we have $0 \leq 1/3 \leq f - g \leq 2/3$. Lastly, for the case where $x \notin (B \cup C)$, then $f(x) \in (1/3, 2/3)$, and $g(x) \in [0, 1/3]$ implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 \qquad &\implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 \qquad &\implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore $0 \leq f(x) - g(x) \leq 2/3$. □

We can assume that $f \in C(A, [0, 1])$, since we can relabel $f = (f - a)/(b - a)$. The main part of this proof consists of constructing a sequence of $\{g_n\} \subseteq C(X, \mathbb{R})$ where $0 \leq g_n \leq (2/3)^n(1/2)$, and $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$ on A. Let us begin with the base case with $n = 1$. We can apply Lemma 1.1 to get $g_1 \in C(X, [0, 1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that $\{g_j\}_{j \leq n}$ has been chosen, we will find our g_{n+1} by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by $(2/3)^{-n}$ and we obtain a new function in $C(A, [0,1])$.

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) \left(\frac{3}{2} \right)^n \leq 1$$

Applying the Lemma 1.1, we get a function $h \in C(X, [0, 1/3])$ such that, for every $x \in A$

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) \left(\frac{3}{2} \right)^n - h \leq 2/3$$

Multiplying across gives

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) - h \left(\frac{2}{3} \right)^n \leq \left(\frac{2}{3} \right)^{n+1}$$

Set $g_{n+1} = h \left(\frac{2}{3} \right)^n$ and $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$. Furthermore, the sum of all g_j pointwise converges uniformly, as

$$\sum_{j \geq 1} \|g_j\|_u \leq \sum_{j \geq 1} \left(\frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum $F = \sum g_j$, then this $F \in BC(X)$ (by Theorem 4.9), since every $g_j \in BC(X)$. And

$$\left\| f - \sum_{j \leq n} g_j \right\|_u \leq \left(\frac{2}{3} \right)^n \longrightarrow 0$$

So $F = f$ on A, now if we want to obtain our F on $[a, b]$ we simply relabel $F = F(b - a) + a$. This finishes the proof. $\square$

1.17 Theorem 4.17

WTS. *If X is a normal space, and A is a closed subspace of X, and $f \in C(A)$, then there exists an $F \in C(X)$ such that F extends f.*

Proof. First we suppose that f is real valued, so $f \in C(X, \mathbb{R})$. And define a $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$, using

$$g = \frac{f}{1 + |f|}$$

Since g satisfies the assumption of Theorem 4.16 (note that we do not require g to be injective), there exists a $G \in C(X, [-1, +1])$ such that $G|_A = g$. Since the set $\{-1, +1\}$ is closed in $\mathbb{R}$, $G^{-1}(\{-1, +1\})$ is closed as well. Since $G^{-1}((-1, +1)) \subseteq A$, this makes A and $B =^{-1} (\{-1, +1\})$ disjoint closed sets in X.

By Urysohn's Lemma, there exists a continuous function $h \in C(X, [0, 1])$ such that $h|_B = 0$ and $h|_A = 1$, so that the product $|hG| < 1$ for all $x \in X$. We can think of this h as a continuous indicator function that filters out the parts we do not want, namely $G^{-1}\{-1, +1\}$. Now define F in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that $F|_A = g/(1 - |g|) = f$ indeed. Since $|g| = \frac{|f|}{1+|f|}$, and $g(1 + |f|) = f$ implies that $g/(1 - |g|) = f$, because $g \in C(A, (-1, +1))$ This completes the proof for any $f \in \mathbb{R}$ if $f \in C(A)$, then

1. $\mathrm{Re}(f) = f_1 \in C(A, \mathbb{R})$
2. $\mathrm{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in $C(X, \mathbb{R})$ that extends f_1 and f_2, and $F_1 + iF_2 = f$ on A and $F_1 + iF_2 \in C(X)$, and the proof is complete. □

1.18 Theorem 4.18

WTS. *If X is a topological space, and $E \subseteq X$ and $x \in X$, then $x \in \operatorname{acc} E \iff$ there exists a net in $E \setminus \{x\}$ that converges to x, and $x \in \overline{E} \iff$ there exists a net in E that converges to x.*

Proof. Suppose that $x \in \operatorname{acc} E$, then for every neighbourhood $U \in \mathcal{N}(x)$, $E \cap U \setminus \{x\} \neq \varnothing$, then choose $\mathcal{N}(x)$ as the set of neighbourhoods directed by reverse inclusion (and this makes $(\mathcal{N}(x), \lesssim)$ a directed set), and we will define the net as follows.

Map each $U \in \mathcal{N}(x)$ to some $x_U \in E \cap U \setminus \{x\}$, then this net converges to x. Suppose that we fix a neighbourhood, $V \in \mathcal{N}(x)$, then for every $U \gtrsim V$ we have $x_u \in U \subseteq V$. So $\langle x_U \rangle$ is eventually in V.

Conversely, if $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$, and $x_\alpha \to x$, then every $U \in \mathcal{N}(x)$ there exists a $x_\alpha \in E \cap U \setminus \{x\}$ that makes

$$E \cap U \neq \varnothing \quad \forall U \in \mathcal{N}(x)$$

Hence $x \in \operatorname{acc} E$.

Now for the second part of the Theorem, suppose that $x \in \overline{E}$, if $x \notin E$ then $E = E \setminus \{x\}$ and $x \in \operatorname{acc} E$, so there exists a net in $E \setminus \{x\} \subseteq E$ such that $x_\alpha \to x$. If $x \in E$ then simply choose $\langle x_\alpha \rangle = x$ for every $\alpha \in A$.

Now, suppose that there is a net that converges to x, and this net $\langle x_\alpha \rangle \subseteq E$, if $x \in E$ then there is nothing to prove, since $E \subseteq \overline{E}$, so suppose that $x \notin E$, then there exists a net in $E \setminus \{x\} = E$ such that

$$x_\alpha \to x \implies x \in \operatorname{acc} E \subseteq \overline{E}$$

$\square$

1.19 Theorem 4.19

WTS. *Let X and Y be topological spaces, then every $f : X \to Y$ is continuous at a point $x \in X \iff$ every net $\langle x_\alpha \rangle$ that converges to x implies that $\langle f(x_\alpha) \rangle$ converges to $f(x)$.*

Proof. If f is continuous at a point $x \in X$, then $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$, then for every net $\langle x_\alpha \rangle$ that converges to this x, there there exists an α_0 such that for every $\alpha \gtrsim \alpha_0$ implies that $x_\alpha \in f^{-1}(V)$. Hence

$$f(x_\alpha) \in f\left(f^{-1}(V)\right) \subseteq V$$

And this is equivalent to saying that for every $V \in \mathcal{N}(f(x))$, $\langle f(x_\alpha) \rangle$ is eventually in V, and this proves convergence.

Now suppose that f is not continuous at some x, then there exists a $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$, so

$$x \notin \left(f^{-1}(V)\right)^o \implies x \in \left(f^{-1}(V)\right)^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our $x \in \overline{f^{-1}(V^c)}$ induces a net $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$ that converges to x. But every element in the net is contained within $f^{-1}(V^c)$, and for every $\alpha \in A$

$$f(x_\alpha) \in f\left(f^{-1}(V^c)\right) \subseteq V^c$$

gives $f(x_\alpha) \notin V$, but V is a neighbourhood of $f(x)$, hence there exists some $x_\alpha \to x$ and $f(x_\alpha) \not\to f(x)$. $\square$

1.20 Theorem 4.20

WTS. *If $\langle x_\alpha \rangle$ is a net in X, and $x \in X$ is a cluster point of $\langle x_\alpha \rangle \iff$ there exists a subnet of $\langle x_\alpha \rangle$ that converges to x.*

Proof. Suppose that $\langle y_\beta \rangle_{\beta \in B}$ is a subnet of $\langle x_\alpha \rangle$ that converges to x, then for every neighbourhood $U \in \mathcal{N}(x)$, there exists a β_1 such that for every $\beta \gtrsim \beta_1$ we get $y_\beta = x_{\alpha_\beta} \in U$.

Furthermore, let us fix a $\alpha_0 \in A$ to attempt to show that $\langle x_\alpha \rangle$ is frequently in U, then by the subnet property of $\langle y_\beta \rangle$, there exists some $\beta_2 \in B$ such that for every $\beta \gtrsim \beta_2$, $\alpha_\beta \gtrsim \alpha_0$. (Intuitively this property means that the directed set of B 'grows' as much as the directed set of A, so we can always find elements that are greater than any fixed α_0.)

Since $\langle y_\beta \rangle$ is a net, we there exists some $\beta \in B$ such that $\beta \gtrsim \beta_1$ and $\beta \gtrsim \beta_2$, we then apply the $\beta \mapsto \alpha_\beta$ map and we obtain some $\alpha = \alpha_\beta$ that satisfies:

- $\alpha = \alpha_\beta \gtrsim \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that $\beta \gtrsim \beta_1$ so that y_β falls into U.

Conversely, suppose that x is a cluster point of $\langle x_\alpha \rangle$, then by definition

$$\forall U \in \mathcal{N}(x),\ \forall \alpha_0 \in A,\ \exists \alpha \gtrsim \alpha_0,\ x_\alpha \in U$$

Denote the directed neighbourhoods of x by $\mathcal{N}(x)$, and construct our directed set B for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every $(U, \gamma) \in B$ we can map it to some $\alpha_{(U,\gamma)} \in A$, if we choose some $\alpha_{(U,\gamma)} \gtrsim \gamma$ and $\alpha_{(U,\gamma)} \in U$.

To show that B is a directed set, we say that $(U, \gamma) \gtrsim (U', \gamma')$ if and only if $U \subseteq U'$ and $\gamma \gtrsim \gamma'$. And to show that $\langle y_\beta \rangle = \langle x_{\alpha_{(U,\gamma)}} \rangle$ is indeed a subnet of $\langle x_\alpha \rangle$, fix any $\alpha_0 \in A$, then simply take any neighbourhood U of x (we always have $X \in \mathcal{N}(x)$) — and therefore $(U, \alpha_0) \in B$.

Now for every $(U', \alpha_0') \gtrsim (U, \alpha_0)$ implies that $\alpha_0' \gtrsim \alpha_0$, therefore we have

$$\alpha_{(U', \alpha_0')} \gtrsim \alpha_0' \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that $\langle y_\beta \rangle$ indeed converges to x, fix any $V \in \mathcal{N}(x)$, then with any $\alpha_0 \in A$, and for every $(V', \alpha_0') \gtrsim (V, \alpha_0) \in B$, we have

$$x_{\alpha_{(V', \alpha_0')}} \in V' \subseteq V$$

So $\langle x_{\alpha_{(U,\gamma)}} \rangle$ converges to x. $\square$

1.21 Theorem 4.21

WTS. *A topological space X is compact $\iff$ every family of closed sets, $\{F_\alpha\}_{\alpha\in A}$ that has the finite intersection property, implies that*

$$\bigcap_{\alpha\in A} F_\alpha \neq \varnothing$$

Proof. We first examine the assertion, Theorem 4.21 proposes for any family of closed sets $\{F_\alpha\}_{\alpha\in A}$, and for every finite subset $B \subseteq A$ then,

$$\bigcap_{\alpha\in B} F_\alpha \neq \varnothing \implies \bigcap_{\alpha\in A} F_\alpha \neq \varnothing$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha\in A} F_\alpha = \varnothing \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha\in B} F_\alpha = \varnothing$$

Applying DeMorgan's theorem, and since every $\{F_\alpha\}_{\alpha\in A}$ induces a family of open sets (and vice versa), where $U_\alpha = F_\alpha^c$, so for any familiy of open sets $\{U_\alpha\}_{\alpha\in A}$ we have

$$\bigcup_{\alpha\in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha\in B} U_\alpha = X$$

Which is equivalent to saying that X is compact. $\square$

1.22 Theorem 4.22

WTS. *A closed subset of a compact space X is compact.*

Proof. Suppose $F \subseteq X$ and F is open, then fix an open cover for F, so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since F^c is an open set, we can obtain a valid open cover for X, then we pick out a finite subcover, for some finite $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left(\bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with F on both sides yields

$$\begin{aligned}
&F = X \cap F \subseteq (F^c \cap F) \cup \left(F \cap \left(\bigcup_{\alpha \in B} U_\alpha \right) \right) \\
&F = \left(F \cap \left(\bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\
&F \subseteq \bigcup_{\alpha \in B} U_\alpha
\end{aligned}$$

Therefore every open cover of F has a finite subcover, and F is compact. $\square$

1.23 Theorem 4.23

WTS. *If F is a compact subset of a Hausdorff space X, and $x \notin F$, there are disjoint open sets U, V such that $x \in U$ and $F \subseteq V$.*

Proof. Since $x \in F^c$, for every $y \in F$, $x \neq y$ induces two sets U_y, V_y (because X is T_2).

- $U_y \cap V_y = \varnothing$
- $x \in U_y$
- $y \in V_y$

But $\{V_y\}_{y \in F}$ is an open cover for the compact set F, then there exists a finite subcollection $H \subseteq F$ such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since H is finite, $U = \bigcap_{y \in H} U_y$ is an open set that contains x, also define $V = \bigcup_{y \in H} V_y$. If for every $y \in H$, $U_y \cap V_y = \varnothing$, then $U \cap V_y = U \cap V = \varnothing$. This completes the proof. $\square$

Remark. *Every metric space (X, d) is first countable, and T_2 (it is actually T_4, but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element $x \in X$ and we notice that $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$ is a countable neighbourhood base for every x. To show that (X, d) is T_2, for every pair of elements $x \neq y$, we can take $r = d(x, y)/2$ and there exists disjoint open sets $V_r(x)$ and $V_r(y)$ such that $x \in V_r(x)$ and $y \in V_r(y)$.*

1.24 Theorem 4.24

WTS. *Every compact subset of a Hausdorff (T_2) space is closed.*

Proof. If F is compact, then for every $x \in F^c$, by Theorem 4.23, there exists two disjoint open sets such that $x \in U$ and $F \subseteq V$, but

$$U \cap V = \varnothing \implies U \cap F = \varnothing \implies U \subseteq F^c$$

But since $x \in F^c$ is arbitrary, and U is an open subset of F^c,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that F^c is open and F is closed. □

1.25 Theorem 4.25

WTS. *Every compact Hausdorff (T_2) space is normal (T_4).*

Proof. Fix A, B which are disjoint closed subsets of X, by Theorem 4.22, we know that these two sets are compact. Hence for every $y \in B$ there exists two disjoint open sets U, V_y (by Theorem 4.23)

$A \subseteq U_y$ and $y \in V_y$. But the family $\{V_y\}_{y \in B}$ is s a valid open cover for the compact set B, hence there exists a finite subcollection $H \subseteq B$ such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \varnothing$$

The second equality holds for every $y \in H$ so that $U_y \cap (\cup V_{y \in H}) = \varnothing$. Define $U = \cap U_{y \in H}$ and $V = \cup V_{y \in H}$, where both of these are disjoint open sets that that contain A and B as subsets, since for each $y \in H$, $A \subseteq U_y$ hence the intersection of all U_y also contains A as a subset. Therefore X is normal. $\square$

1.26 Theorem 4.26

WTS. *If X is compact, and $f : X \to Y$ is continuous, then $f(X)$ is compact.*

A small lemma.

Lemma 1.2. *For every $\{E_j\} \subseteq X, f(\cup E_j) = \cup f(E_j)$.*

The proof is trivial.

Proof. If $\{V_{\alpha\in A}\}$ is an open cover for $f(X)$, then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha\in A} V_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(V_\alpha) \subseteq X$$

Since f is continuous, we have an open cover in the form of $\{f^{-1}(V_\alpha)\}$ for X, then there exists a finite subset $B \subset A$ such that

$$X \subseteq \bigcup_{\alpha\in B} f^{-1}(V_\alpha)$$

Then we wish to show that for this $B \subseteq A$, $\{V_{\alpha\in B}\}$ is a finite open cover for $f(X)$. Fix any element $y \in f(X)$, then this induces a $x \in X$ such that $y = f(x)$, but because $\{f^{-1}(V_{\alpha\in B})\}$ is an open cover for X, there exists some $\alpha \in B$ such that $x \in f^{-1}(V_\alpha)$, hence by definition of the inverse image

$$f(x) \in V_\alpha \implies f(X) \subseteq \bigcup_{\alpha\in B} V_\alpha$$

Therefore $f(X)$ is compact and this completes the proof. $\square$

1.27 Theorem 4.27

WTS. *If X is compact, then $C(X) = BC(X)$.*

Proof. Notice that $BC(X) \subseteq C(X)$, so we only have to show the reverse estimate. Fix any $f \in C(X)$, since X is compact, by Theorem 4.26 we know that $f(X)$ is also compact. Since $\mathbb{C} = \mathbb{R}^2$ is a complete metric space, $f(X)$ is bounded and $f \in BC(X)$. $\square$

1.28 Theorem 4.28

WTS. *If X is compact, and if Y is Hausdorff, then any continuous bijection $f : X \to Y$ is a homeomorphism.*

Proof. If $E \in X$ is closed, then since X is compact, E is compact as well. By continuity of f, $f(X)$ is a compact set in Y, but compact subsets of Y are closed, so f is continuous.

We used the fact that the inverse of f^{-1} is f, since it suffices to check that every inverse image of a closed set is also closed, f^{-1} is continuous. And by definition of a homeomorphism (f has to be bijective and both f and f^{-1} hav eto be continuous), f is a homeomorphism. □

1.29 Theorem 4.29

WTS. *If X is any topological space, the following are equivalent.*

(a) *X is compact.*

(b) *Every net has a cluster point.*

(c) *Every net in X has a convergent subnet.*

Proof. By Theorem 4.20, every net in X has a cluster point $\iff$ there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose a) holds, then X is compact, and fix an arbitrary net $\langle x_\alpha \rangle$ in X. and define the 'tail' of the net

$$E_\alpha := \{x_\beta,\ \beta \gtrsim \alpha\}$$

We wish to show that the arbitrary intersection of $\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \varnothing$. Where $\overline{E}_\alpha$ is closed, so it suffices to check that every finite $B \subseteq A$, the intersection over $\overline{E}_\alpha$ is non-empty.

Suppose we are given a finite $B \subseteq A$, then fix any two elements α and $\beta \in B$, by the definition of a net there exists a $\gamma \in A$ such that $\gamma \gtrsim \alpha$ and $\gamma \gtrsim \beta$, and

$$\varnothing \neq \subseteq E_\alpha \cap E_\beta \implies \overline{E}_\alpha \cap \overline{E}_\beta \neq \varnothing$$

Therefore for any finite collection of $\{\overline{E}_{\alpha \in B}\}$, then

$$\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \varnothing$$

Now fix an element $x \in \bigcap_{\alpha \in A} \overline{E}_\alpha$. Then for every $\alpha \in A$, $x \in \overline{E}_\alpha$, and for every neighbourhood $U \in \mathcal{N}(x)$, $U \cap E_\alpha \neq \varnothing$. This is because if $x \in E_\alpha$, then $U \cap E_\alpha$ contains at least $\{x\}$, if $x \in \operatorname{acc} E_\alpha$, then by definition of an accumulation point, $U \cap E_\alpha \setminus \{x\} \neq \varnothing$, so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net, E_α, if for every $\alpha \in A$, $x \in E_\alpha$ if and only if there exists some $\gamma \gtrsim \alpha$, $x_\gamma \in U \cap E_\alpha$, this is equivalent to saying that x is a cluster point of $\langle x_\alpha \rangle$. So $a) \implies b)$.

Now let us suppose that X is not compact, then there exists an open cover $\{U_{\alpha \in A}\}$ of X that has no finite subcover. Let $\mathbb{B}$ be the collection of all finite subsets of A, directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every $B \in \mathbb{B}$, find some $x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c$. So we have a net in X. Now we will show that no $x \in X$ can be a cluster point of this net. Suppose not, then take a neighbourhood U_β with $\beta \in A$ such that U_β belongs to the open cover we first discussed. Then for any $B \in \mathbb{B}$ such that $B \gtrsim \{\beta\}$ (meaning that $\{\beta\} \subseteq B$, where B is a finite set), then

$$x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c \implies x_B \notin \left(\bigcup_{\alpha \in \{\beta\}} U_\beta\right) \implies x_B \in U_\beta^c$$

Hence no point in X can be a cluster point for this net, and the proof is complete. $\square$

1.30 Theorem 4.30

WTS. *If X is a LCH space, and for every $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$, there exists a compact $N \subseteq U$ where $N \in \mathcal{N}_B(x)$.*

Proof. For every $U \in \mathcal{N}_B(x) \cap \mathcal{T}_x$, we can find an E open subset of U that has a compact closure, since every $x \in X$ induces some compact $F \in \mathcal{N}_B(x)$, therefore

$$E \coloneqq U \cap F^o \implies \overline{E} \subseteq F$$

Since closed subsets of compact sets are compact (by Theorem 4.22), $\overline{E}$ is compact. More is true, since E is open,

$$x \in U \cap F^o \implies x \in E^o \implies E \in \mathcal{N}_B(x)$$

Now it suffices to show that there exists some compact $N \subseteq E \subseteq U$ such that $N \in \mathcal{N}_B(x)$. Since $\overline{E}$ is compact, the closed subset $\partial E = \overline{E} \cap \overline{E^c}$ of $\overline{E}$ is also compact.

Since $\partial E \cap E^o = \varnothing$, $x \in E^o = E$ means that $x \notin \partial E$. Applying Theorem 4.23 to the compact set ∂E and $x \notin \partial E$ gives us two disjoint open sets V' and W'. We list their properties

1. $V', W' \in \mathcal{T}_X$
2. $x \in V'$
3. $\partial E \subseteq W'$
4. $V' \cap W' = \varnothing$

The two disjoint pairs induce another pair of open sets relative to $\overline{E}$, recall the definition of the topology relative to $\overline{E}$,

$$\mathcal{T}_{\overline{E}} = \left\{A \cap \overline{E} : A \in \mathcal{T}_X\right\}$$

We now agree to define

- $V = V' \cap \overline{E}$
- $W = W' \cap \overline{E}$

Then evidently $V, W \in \mathcal{T}_{\overline{E}}$ and

1. $x \in V' \cap \overline{E} \implies x \in V$
2. $\partial E \subseteq \overline{E} \implies \partial E \subseteq W$
3. $V' \cap W' = \varnothing \implies V \cap W = \varnothing$

Furthermore,

$$\partial E \subseteq W \implies W^c \subseteq (\partial E)^c = E^o \cup E^{co}$$

Taking the intersection over $\overline{E}$ gives us

$$\overline{E} \setminus W \subseteq \overline{E} \cap (E^o \cup E^{co})$$

Note that $E^{co} = (\overline{E})^c$, since $(E^c)^{oc} = \overline{(E^{cc})} = \overline{E}$ therefore $\overline{E} \cap E^{oc} = \varnothing$, hence

$$\overline{E} \setminus W \subseteq \overline{E} \cap E^o = E^o$$

Using the fact from 3, $V \subseteq W^c$ and $V \subseteq \overline{E}$ and $V \subseteq W^c$ implies that $V \subseteq \overline{E} \setminus W$. Compiling everything, we have

$$V \subseteq \overline{E} \setminus W \subseteq E$$

Note that the set $\overline{E} \setminus W$ is closed in $\mathcal{T}_X$ (and hence closed in $\overline{E}$) by closure over intersections, $\overline{V}$ is therefore a closed subset of $\overline{E} \setminus W$, and $\overline{V}$ is compact. Also

$$\overline{V} \subseteq \overline{E} \setminus W \subseteq E$$

To check that $\overline{V} \in \mathcal{N}_B(x)$, note that

$$x \in V^o \subseteq (\overline{V})^o \implies \overline{V} \in \mathcal{N}_B(x)$$

The subset relation $V^o \subseteq \overline{V}^o$ comes from the fact that V^o is an open subset of $\overline{V}$, and hence is contained in $(\overline{V})^o$ as a subset. Now let us define $N = \overline{V}$, and N satisfies the assertions in the Theorem, since

- $N \in \mathcal{N}_B(x)$
- N is compact
- $N \subseteq E \subseteq U$

And this completes the proof. □

Remark. *Intuitively speaking, this means that if X is any LCH space, then for every open neighbourhood $U \in \mathcal{N}_B(x)$, there exists a compact $E \in \mathcal{N}_B(x)$ such that $x \in E \subseteq U^o$. This property is indeed a very strong one as it allows us to have effectively 'infinite' descending compact neighbourhoods of x.*

1.31 Theorem 4.31

WTS. *X is a LCH space, and $K \subseteq U \subseteq X$ where K is compact, and U is open, then there exists some precompact, open V with*

$$K \subseteq V \subseteq \overline{V} \subseteq U$$

Proof. For every $x \in K$, we can apply Proposition 4.30, since $x \in K \subseteq U$, this induces some compact $F_x \subseteq U$ where $F_x \in \mathcal{N}_B(x)$. Then we can obtain an open cover of U in the form of $\{F_x^o\}_{x \in K}$. By compactness of K, there exists a finite $B \subseteq K$ such that

$$K \subseteq \bigcup_{x \in B} F_x^o$$

Let $V = \bigcup_{x \in B} F_x^o$, then clearly V is open, and $K \subseteq V$. Since each F_x is closed (compact sets are closed in any Hausdorff Space), we have

$$V \subseteq \bigcup_{x \in B} F_x \implies \overline{V} \subseteq \bigcup_{x \in B} F_x$$

Since $\bigcup_{x \in B} F_x$ is a finite union of compact sets, we claim that it is also compact. Consider two compact sets E_1 and E_2, then if $\{U_\alpha\}_{\alpha \in A}$ is any open cover of $E_1 \cup E_2$, it must be an open cover for E_1 and E_2 as well, because

$$E_1, E_2 \subseteq E_1 \cup E_2 \bigcup_{\alpha \in A} U_\alpha$$

Since E_1 and E_2 are both compact sets, they each induce two finite subsets of B_1, B_2 of A whose union $B = B_1 \cup B_2$ is also compact. Therefore

$$E_1 \cup E_2 \subseteq \bigcup_{\alpha \in B} U_\alpha$$

Then a simple proof by induction will show that if $\{E_{j \leq n}\}$ is a family of compact sets, then $E = \bigcup E_{j \leq n}$ is also compact.

Returning to the main part of the proof, $\bigcup_{x \in B} F_x$ is a compact set, therefore $\overline{V}$ is also compact. Moreover

$$\forall x \in K,\ F_x \subseteq U \implies \overline{V} \subseteq \bigcup_{x \in B} F_x \subseteq U$$

Combining, we have

- $K \subseteq V \subseteq \overline{V}$,
- V is open and $\overline{V}$ is compact, and
- $\overline{V} \subseteq U$

This completes the proof. $\square$

1.32 Theorem 4.32

WTS. *Urysohn's Lemma, Locally Compact Version. For any LCH space X, and if $K \subseteq U \subseteq X$ where K is compact and U is open, then there exists some $f \in C(X, [0, 1])$ with*

- $f = 1$ *on* K
- $f = 0$ *outside some compact* $\overline{V} \subseteq U$

Proof. Let V be as in Theorem 4.31, for our fixed $K \subseteq U \subseteq X$, there exists a pre-compact, open V that satisfies

$$K \subseteq V \subseteq \overline{V} \subseteq X$$

It follows that this $(\overline{V}, \mathcal{T}_{\overline{V}})$ is a normal space by Theorem 4.25 (compact Hausdorff spaces are normal), and by Urysohn's Lemma (Theorem 4.15) on normal spaces, since we can easily find two disjoint closed subsets of $\overline{V}$ in the form of

- $K \subseteq V^o = V \subseteq \overline{V}$ (compact sets in Hausdorff spaces are closed)
- $\partial V = \overline{V} \cap \overline{V^c}$ (closed sets in compact spaces are compact)
- $K \subseteq V^o$ implies that $K \cap \partial V = K \cap (\overline{V} \setminus V^o) = \varnothing$

Then there exists a continuous $f|_{\overline{V}} \in C(\overline{V}, [0, 1])$ that evaluates to

- $f|_{\overline{V}} = 1$ on closed K
- $f|_{\overline{V}} = 0$ on closed ∂V

Now let us extend $f|_{\overline{V}}$ to f by defining

$$f|_{(\overline{V})^c} = 0$$

We will show that this extension of f is indeed continuous. Indeed, for every closed set $E \subseteq [0, 1]$ that does not contain 0, we have:

$$0 \notin E \implies \{0\} \cap E = \varnothing \implies f^{-1}(\{0\}) \cap f^{-1}(E) = \varnothing$$

But $(\overline{V})^c \subseteq f^{-1}(\{0\})$ therefore

$$(\overline{V})^c \cap f^{-1}(\{0\}) \cap f^{-1}(E) = (\overline{V})^c \cap f^{-1}(E) = \varnothing \implies f^{-1}(E) \subseteq \overline{V}$$

We can write
$$f^{-1}(E) = f|_{\overline{V}}^{-1}(E)$$
But we know that $f|_{\overline{V}}$ is continuous, so $f|_{\overline{V}}^{-1}(E)$ must be closed (with respect to $\overline{V}$), and therefore is closed wrt X, since $\overline{V}$ is closed wrt X.

For the case where $0 \in E$, note that
$$f^{-1}(E) = \left(f^{-1}(E) \cap \overline{V}\right) \cup \left(f^{-1}(E) \cap (\overline{V})^c\right) = \left(f|_{\overline{V}}\right)^{-1}(E) \cup \left(f|_{\overline{V}^c}\right)^{-1}(E)$$
The above equalities are messy in print. They are but a simple consequence of disjoint decomposition of the pre-images, since
$$\overline{V} \cap f^{-1}(E) = \{x \in \overline{V} : f(x) \in E\} = f|_{\overline{V}}^{-1}(E)$$
Back to our main discussion, recall that for every $x \in \partial V$
$$f(x) = 0 \in f^{-1}(\{0\}) \subseteq f^{-1}|_{\overline{V}}(E)$$
Therefore $\partial V \subseteq f^{-1}|_{\overline{V}}(E)$, and $(\overline{V})^c = f^{-1}|_{(\overline{V})^c}(E)$ gives us (since V^c is closed),
$$\begin{aligned} f^{-1}(E) &= f^{-1}|_{\overline{V}}(E) \cup \partial V \cup (\overline{V})^c \\ &= f^{-1}|_{\overline{V}}(E) \cup \overline{(V^c)} \cup (\overline{V})^c \\ &= f^{-1}|_{\overline{V}}(E) \cup (V^c \cup V^{co}) \\ &= f^{-1}|_{\overline{V}}(E) \cup V^c \end{aligned}$$

Since $f^{-1}|_{\overline{V}}(E)$ and V^c are closed subsets of X, then $f^{-1}(E)$ is also closed, and $f \in C(X, [0, 1])$. $\square$

1.33 Theorem 4.33

WTS. *Every LCH space is completely regular (or $T_{3.5}$).*

Proof. Recall that a space X is completely regular if it is T_1 and every closed subset A and every $x \notin A$ there exists some

$$f \in C(X, [0,1]), \quad f(x) = 1,\ f|_A = 0$$

Fix a closed set $A \subseteq X$, then for every $x \in A^c$, there exists a compact $E_x \in \mathcal{N}_B(x)$ with $E_x \subseteq A^c$ (by Theorem 4.30).

Note that $E_x \subseteq A^c$ where E_x is compact and A^c is closed, then an application of Theorem 4.31 tell us that there exists an $f \in C(X, [0,1])$ such that for every $x \in E_x$, $f(x) = 1$ and for points $y \notin A^c$ (which means that $y \in A$), $f(y) = 0$. Therefore X is completely regular. $\square$

1.34 Theorem 4.34

WTS.

Proof. □

1.35 Theorem 4.35

WTS. *If X is a LCH space, we claim that*

$$\overline{\mathrm{C}_c(X)} = \mathrm{C}_0(X)$$

Proof. We begin by proving several things that are mentioned before this Theorem, namely

$$\mathrm{C}_c(X) \subseteq \mathrm{C}_0(X) \subseteq \mathrm{BC}(X)$$

Fix an $f \in \mathrm{C}_c(X)$, and for every $\varepsilon > 0$,

$$x \in |f|^{-1}([\varepsilon, +\infty)) \implies |f(x)| \geq \varepsilon > 0$$

Therefore $|f|^{-1}([\varepsilon, +\infty))$ is a closed subset of $\operatorname{supp}(f)$, since $(-\infty, \varepsilon)$ is open in $\mathbb{R}$, then $[\varepsilon, +\infty)$ is a closed set. And by continuity of $|\cdot| \circ f$ (a composition of two continuous functions), $|f|^{-1}([\varepsilon, +\infty))$ is closed. Using the fact that closed subsets of compact $\operatorname{supp}(f)$ are also compact, we get $f \in \mathrm{C}_0(X)$.

Next, we show that $\mathrm{C}_0(X) \subseteq \mathrm{BC}(X)$. Fix any element f of $\mathrm{C}_0(X)$ with an arbitrary $\varepsilon > 0$, then $E_\varepsilon = \{x \in X : |f(x)| \geq \varepsilon\}$ is compact. The continuity of f guarantees that the direct image of a compact set is another compact set (Theorem 4.26)

$$|f|(E_\varepsilon) \text{ is a compact subset of } \mathbb{R}$$

And therefore for every $x \in E_\varepsilon \implies |f(x)| \in f(E\varepsilon)$, then by Heine-Borel, there exists some $M \geq 0$ such that $|f(x)| \leq M$. If $x \notin E_\varepsilon$, then by definition of E_ε, implies that $|f(x)| < \varepsilon$. Then $|f(x)| \leq M + \varepsilon$ for every $x \in X$. Hence $f \in \mathrm{BC}(X)$.

Here I wish to offer an alternate proof for $\mathrm{C}_0(X) \subseteq \mathrm{BC}(X)$, we begin by constructing an open cover for $\operatorname{supp}(f)$ such that

$$\{U_n\}_{n>0} = \{x \in X | f(x)| < n\}$$

Then there exists a finite subcollection of $\{U_n\}_{n \in B}$ where B is a finite set, then define $M = 1 + \sum_{n \in B} n$ and for every $x \in \operatorname{supp}(f)$ we have $|f(x)| < n$ and since $n > 0$ this holds for every $x \in X$ too. Therefore $f \in \mathrm{BC}(X)$.

For the main proof of Theorem 4.35, since $\mathrm{BC}(X)$ is endowed with the uniform metric, it is also first countable, and therefore by Theorem 4.6, it suffices to show that every sequence $\{f_n\}_{n\geq 1} \subseteq \mathrm{C}_c(X)$ converges in $\mathrm{C}_0(X)$. And every element $f \in \mathrm{C}_0(X)$ has a convergence sequence in $\mathrm{C}_c(X)$.

Fix a convergent sequence $\{f_n\}_{n\geq 1} \subseteq \mathrm{C}_c(X)$ that converges uniformly to some $f \in \mathrm{BC}(X)$ (since $\mathrm{BC}(X)$ is a closed subset of $C(X)$ with respect to the uniform norm), then for every $\varepsilon > 0$, there exists some $n \geq 1$ with

$$\|f_n - f\|_u < \varepsilon$$

We aim to show that $(\operatorname{supp}(f_n))^c \subseteq |f|^{-1}((-\infty, \varepsilon))$, so fix any $x \notin \operatorname{supp}(f_n)$, then

$$|f(x) - f_n(x)| = |f(x)| \leq \|f - f_n\|_u < \varepsilon$$

This establishes the estimate, and taking complements

$$|f|^{-1}([\varepsilon, +\infty)) \subseteq \operatorname{supp}(f_n)$$

Therefore for any arbitrary $\varepsilon > 0, \{x \in X,\ |f(x)| \geq \varepsilon\}$ is compact, and $\overline{\mathrm{C}_c(X)} \subseteq \mathrm{C}_0(X)$. Conversely, fix any $f \in \mathrm{C}_0(X)$, and for every $n \geq 1$, define

$$K_n = \{x \in X,\ |f(x)| \geq 1/n\}$$

Using Urysohn's Lemma for our LCH space X, there exists some g_n that has a compact support, and $g_n(x) = 1$ for every $x \in K_n$. We then write $f_n = g_n \cdot f \in \mathrm{C}_c(X)$. We wish to show that $f_n \to f$ uniformly. Notice that for any fixed $n \geq 1$, if $x \in K_n$ then

$$f_n(x) = f(x) \implies |f_n - f|(x) = 0$$

If $x \notin K_n$, $|f(x)| < 1/n$ (recall what K_n does), and $f_n = g_n \cdot f \in [0, 1]$ by definition of g_n from Theorem 4.32, hence

$$|f_n(x) - f(x)| = |f(x)| \cdot |1 - g_n| \leq |f(x)| < 1/n$$

Taking the supremum over $x \in X$, we have

$$\|f_n - f\|_u < 1/n \to 0$$

As we send n to $+\infty$, and $f_n \to f$ uniformly. This completes the proof. $\square$

1.36 Theorem 4.36

WTS.

Proof. □

1.37 Theorem 4.37

WTS. *If X is an LCH space and $E \subseteq X$. E is closed if and only if $E \cap K$ is closed for every compact $K \subseteq X$.*

Proof. Suppose that E is closed, then $E \cap K$ is closed, since compact subsets of Hausdorff spaces are closed, and $E \cap K \subseteq K$ tells us that $E \cap K$ is indeed compact.

Now suppose that E is not closed, by Theorem 4.1, $E \neq \overline{E}$, so pick some $x \in (\overline{E} \setminus E) = \operatorname{acc}(E) \cap E^c$, since X is locally compact, let K_x be a compact neighbourhood of x, then for every neighbourhood $U \in \mathcal{N}_B(x)$, we have

$$x \in U^o,\, x \in K_x^o,\, \implies x \in (U^o \cap K_x^o) \subseteq (U \cap K_x)^o$$

Since $(U^o \cap K_x^o)$ is an open subset of $(U \cap K_x)$, then $(U \cap K_x) \in \mathcal{N}_B(x)$, and recall that $x \in \operatorname{acc}(E)$, therefore

$$(U \cap K_x) \cap E \setminus \{x\} = U \cap (K_x \cap E) \neq \varnothing$$

But $x \notin E \implies x \notin E \cap K_x$. So x is an accumulation point of $E \cap K_x$ that is not in $E \cap K_x$. Therefore there exists some $E \cap K_x$ which is not closed. $\square$

1.38 Theorem 4.38

WTS.

Proof. $\square$

1.39 Theorem 4.39

WTS.

Proof. $\square$

1.40 Theorem 4.40

WTS.

Proof. ∎

1.41 Theorem 4.41

WTS.

Proof. □

2 Chapter 6

2.1 Theorem 6.1

WTS. *For every* a*,* $b \geq 0$*, and* $0 < \lambda < 1$*, then*

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

2.2 Theorem 6.2

WTS.

Proof. $\square$

2.3 Theorem 6.3

WTS.

Proof. $\square$

2.4 Theorem 6.4

WTS.

Proof. □

2.5 Theorem 6.5

WTS.

Proof. $\square$

2.6 Theorem 6.6

WTS.

2.7 Theorem 6.7

WTS.

2.8 Theorem 6.8

WTS.

2.9 Theorem 6.9

WTS.

Proof. $\square$

2.10 Theorem 6.10

WTS.

Proof. $\square$

2.11 Theorem 6.11

WTS.

Proof. □

2.12 Theorem 6.12

WTS.

Proof. □

2.13 Theorem 6.13

WTS.

2.14 Theorem 6.14

WTS.

2.15 Theorem 6.15

WTS.

First suppose that $(X, \mathcal{M}, \mu)$ is finite measure space. If $\mu(X) < +\infty$, then for every $E \in \mathcal{M}$, by monotonicity $E \subseteq X$ yields $\mu(E) \leq \mu(X) < +\infty$. Next, for any $p < +\infty$, $\|\chi_E\|_p^p < +\infty$ and $\|\chi_E\|_{+\infty} \leq 1 < +\infty$. So all indicator functions are in L^p.

It follows that every simple function is also in L^p, since it is a finite linear combination of indicators. We now define $\nu(E) = \phi(\chi_E)$, we wish to show that $\nu : \mathcal{M} \longrightarrow \mathbb{C}$ is a complex measure which is absolutely continuous with respect to μ.

To show σ-additivity, fix any disjoint sequence $\{E_j\}_{j\geq 1} \subseteq \mathcal{M}$. Where we also note that $\mu(E) = \mu(\cup E_j) < +\infty$. Now suppose that $p < +\infty$, then the following converges in the p-norm

$$\chi_E = \sum_{j\geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left(\bigcup E_{j\leq n}\right) = \left(\bigcup E_{j\geq 1}\right) \setminus \left(\bigcup E_{j\leq n}\right) = \bigcup E_{j\geq n+1}$$

and define F_{n+1} as the rightmost member above. Then $\{F_{n\geq 1}\}$ is a decreasing sequence of sets. All sets are of finite measure, hence $\mu(E) - \mu(\cup E_{j\leq n}) = \mu(F_{n+1}) \to 0$.

Now, for any fixed $n \geq 1$,

$$\left|\chi_E - \sum \chi_{E_{j\leq n}}\right| = \left|\sum \chi_{E_{j\geq n+1}}\right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the pth power does not change pointwise, and

$$\left|\sum \chi_{E_{j\geq n+1}}\right|^p = \left|\sum \chi_{E_{j\geq n+1}}\right| = \sum \chi_{E_{j\geq n+1}}$$

Convergence in p-norm is given by

$$\left\|\chi_E - \sum \chi_{E_{j\le n}}\right\| = \left\|\sum \chi_{E_{j\ge n+1}}\right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our $\phi \in L^{p*}$

$$\begin{aligned}
\nu(E) &= \phi(\chi_E) \\
&= \phi\left(\lim_{n\to\infty} \sum \chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \phi\left(\sum \chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \sum \phi\left(\chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \sum \nu(E_{j\le n})
\end{aligned}$$

To show absolute convergence, recall that for any $\phi(\chi_{E_j}) \in \mathbb{C}$, define $\beta_j = \overline{\operatorname{sgn}(\|\phi(\chi_{E_j})\|)}$ then multiplication yields

$$\|\phi(\chi_{E_j})\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the p-norm.

$$\left\|\sum_{j\ge 1} \beta_j \chi_{E_j} - \sum_{j\le n} \beta_j \chi_{E_j}\right\|_p = \left\|\sum_{j\ge n+1} \beta_j \chi_{E_j}\right\|_p$$

And because $\left|\sum_{j\ge n+1} \beta_j \chi_{E_j}\right|$ is pointwise equal to $\left|\sum_{j\ge n+1} \chi_{E_j}\right|$, since $|\beta_j| = 1$ for every $j \ge 1$. We can reuse the same continuity and linearity argument. We also note that $\sum_{j\ge 1} \beta_j \chi_{E_j} \in L^p$ since its p-norm is equal to $\mu(E)^{1/p}$.

$$\begin{aligned}
\sum_{j\geq 1}|\nu(E_j)| &= \sup_{n\geq 1}\sum_{j\leq n}\|\nu(E_{j\leq n})\| \\
&= \lim_{n\to\infty}\sum_{j\leq n}\|\phi(\chi_{E_j})\| \\
&= \lim_{n\to\infty}\sum_{j\leq n}\beta_j\phi(\chi_{E_j}) \\
&= \lim_{n\to\infty}\phi\left(\sum_{j\leq n}\beta_j\chi_{E_j}\right) \\
&= \phi\left(\lim_{n\to\infty}\sum_{j\leq n}\beta_j\chi_{E_j}\right) \\
&\leq \|\phi\|\left\|\sum_{j\geq 1}\beta_j\chi_{E_j}\right\|_p \\
&< +\infty
\end{aligned}$$

Assuming the above estimate holds, then we only need $\nu(E) = \phi(\chi_E) = \mu(E) = 0$ (ν is now a measure and $\nu \ll \mu$), As the indicator of a null set is equal to the zero element in L^p. Then by Radon-Nikodym we can have some $g \in L^1(\mu)$ such that

$$d\nu = gd\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function g. For every χ_E measurable, $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$, by monotonicity of the integral in L^+. So any simple function, $\alpha = \sum a_j \cdot \chi_{E_j}$ means that αg is in $L^1(\mu)$, and

$$\phi(\alpha) = \int \alpha g d\mu$$

If $\|\alpha\|_p = 1$, then

$$\left|\int \alpha g\right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$$M_q(g) = \sup\left\{\left|\int \alpha \cdot g\right|, \|\alpha\|_p = 1, \text{ and } \alpha \text{ is simple and vanishes outside a set of finite measure.}\right\}$$

Since $S_g = \{x \in X, g(x) \neq 0\}$ is σ-finite, an application of Theorem 6.14 tells us that $g \in L^q$, and $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$. Now that we know g is in L^q we can use the density of α in L^p to show, for every single $f \in L^p$

$$\phi(f) = \int fg d\mu$$

Conjure a sequence of 'α's, and call them $\{f_n\} \to f$ p.w.a.e, then each $f_n \cdot g \in L^1$. An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n\, g d\mu = \int fg d\mu = \phi(f)$$

This completes the proof for when μ is finite.

Let us upgrade our μ into a σ-finite measure. Then there exists an increasing sequence $\{E_n\} \nearrow X$ such that each E_n is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in P_n vanishes outside a set of finite measure and is also in L^p. And Q_n is defined in a similar manner. Now, fix our $\phi \in L^{p*}$, and for each $f \in P_n$, there exists a corresponding $g_n \in Q_n$. Then $p \in [1, +\infty)$ tells us that $q \in (1, +\infty]$, and the assumptions for Theorem 6.13 all hold. Therefore for each $g_n \in Q_n$, there is a corresponding bounded linear operator $\phi_{g_n} \in (P_n)^*$ such that

$$\phi(f) = \phi|_{P_n}(f) = \int fg_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of g_n towards some $g \in L^q$. We claim that this limit makes sense. As for any $n < m$, such that $E_n \subseteq E_m$ then $g_n = g_m$ on E_n pointwise. The proof is simple since each the restriction of our $\phi \in L^{p*}$ onto E_n and E_m spawns two functions g_n and $g_m \in L^1$. To verify, take any subset $Z \subseteq E_n$ then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So $g_n = g_m$ pointwise a.e on E_n. Now we define g measurable such that $g|_{E_n} = g_n$ for every n. And

$$
\begin{aligned}
|g_n| = \chi_{E_n} \cdot |g_m| &\implies \\
|g_n| \leq |g_{n+1}| &\implies \\
&\|g_n\|_q \leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty
\end{aligned}
$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with ($P_n \subseteq P_{n+1}$). If $q = +\infty$ then $g \in L^\infty$ is trivial, but for any $q < +\infty$. We wish to show that $g \in L^q$. Since $|g_n| \leq |g|$ pointwise for every n, and for each $x \in X$, there exists a N, where $n \geq N$ implies $|g(x)| = |g_n(x)|$, so $|g(x)|$ is an upperbound that is actually attained by the sequence $|g_n(x)|$. So, $|g(x)| = \sup_{n \geq 1}\{|g_n(x)|\}$.

Using the Monotone Convergence Theorem on $|g_n|$,

$$
\begin{aligned}
\int \lim_{n\to\infty} |g_n|^q d\mu &= \int \sup_{n\geq 1} |g_n|^q d\mu \\
&= \int |g|^q d\mu \\
&= \lim \int |g_n|^q d\mu
\end{aligned}
$$

Which yields $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$. It follows that $g \in L^q$.

Finally, we will show that $\phi(f) = \int fg$ for every $f \in L^p$. Redefine $f_n = f \cdot \chi_{E_n} \in P_n$ for every $n \geq 1$. We claim that $f_n \to f$ in the p-norm.

$$
\begin{aligned}
|f_n - f| &\leq |f_n| + |f| \\
&\leq |f| + |f| \\
&\leq 2|f|
\end{aligned}
$$

And $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$. Now it is permissiable to apply the Dominated Theorem, and we will do so.

$$\begin{aligned}
\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\
\lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\
&= 0
\end{aligned}$$

And we have $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}
\phi(f) &= \lim \phi|_{P_n}(f_n) \\
&= \lim \int f_n \cdot g_n \\
&= \lim \int f \cdot g \cdot \chi_{E_n} \\
&= \int \lim (fg \cdot \chi_{E_n}) \\
&= \int fg
\end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of $fg\chi_{E_n} \to fg$ pointwise and Holder's Inequality. This completes the proof for when μ is of σ-finite measure, and $p \in [1, +\infty)$.

Suppose now μ is arbitrary, and $p \in (1, +\infty)$, then $q < +\infty$. Now let us agree to define, for every σ-finite $E \subseteq X$

$$P_E = \{L^p, |f| = |f| \cdot \chi_E\}$$

Where Q_E does not hold any surprises. Then for each E we have a $\phi|_E$ which induces a g_E that vanishes outside E. We are ready for the final part of the proof.

First, if $E \subseteq F$ and both E and F are σ-finite, then $\|g_E\|_q \leq \|g_F\|_q$. This is a simple consequence of monotonicity in L^+ if we take $|g_E|^q \leq |g_F|^q$.

Second, we define

$$W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}$$

Let M be the supremum of W, then there exists a sequence of σ-finite sets, $\{E_n\}$ where $\|g_{E_n}\|_q \to M \leq \|\phi\|_{p*}$. Take a set $F = \cup E_{n\geq 1}$, which is also σ-finite, so that $\|g_F\|_q = M$. Now assume there exists another σ-finite superset of F, let us call it A. Then

$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have $g_{A\setminus F} = 0$ pointwise a.e. For any $f \in L^p$, the spots where f does not vanish is σ-finite. This comes from $\int |f|^p < +\infty$. So it suffices to integrate over this σ-finite set. But we already know, even if this set A contains F as a subset, $\int fg_F = \int fg_A$.

We now define $g = g_F$, and the proof is complete. As for every $\phi \in L^{p*}$, there exists a $g \in L^q$ such that the evaluation of any $f \in L^p$ is given by integrating f with g. ■

2.16 Theorem 6.16

WTS.

Proof. $\square$

2.17 Theorem 6.17

WTS.

Proof. □

2.18 Theorem 6.18

WTS. *For every pair of σ-finite measure spaces,*

2.19 Theorem 6.19

WTS.

2.20 Theorem 6.20

WTS.

Proof. $\square$

2.21 Theorem 6.21

WTS.

Proof. □

2.22 Theorem 6.22

WTS.

2.23 Theorem 6.23

WTS.

2.24 Theorem 6.24

WTS.

Proof. □

2.25 Theorem 6.25

WTS.

Proof. $\square$

2.26 Theorem 6.26

WTS.

Proof. □

2.27 Theorem 6.27

WTS.