

# Chapter 4: Point-set Topology

## Topological Spaces

This section will roughly follow Munkres text on General Topology, in particular we hope to cover Chapters 2, 3, 4 and 9. The rest of the Chapters should be covered properly by the subsequent section.

**Definition 1.1: Topology**

Let  $\mathbf{X}$  be a non-empty set. A topology  $\mathcal{T}$  on  $\mathbf{X}$ , sometimes denoted by  $\mathcal{T}_{\mathbf{X}}$  is a family of subsets of  $\mathbf{X}$ ,

- $\{\emptyset, \mathbf{X}\} \subseteq \mathcal{T}$ ,
- If  $U_1$  and  $U_2$  are elements of  $\mathcal{T}$ , so is their intersection.
- If  $\{U_\alpha\}$  is an arbitrary family of sets in  $\mathcal{T}$ , their union is also contained in  $\mathcal{T}$  as an element.

We call the elements of  $\mathcal{T}$  open sets. The complements of elements in  $\mathcal{T}$  are closed sets.

## Basis of a Topology

### Definition 2.1: Basis of a topology

A basis  $\mathbb{B}$  is a family of subsets of  $\mathbf{X}$ , that satisfies:

- Every  $x \in \mathbf{X}$  belongs (as an element) in some  $V \in \mathbb{B}$ .
- If  $B_1$  and  $B_2$  are basis elements, such that their intersection is non-empty. Then every  $x \in B_1 \cap B_2$  induces a  $B_3 \in \mathbb{B}$  with

$$x \in B_3 \subseteq B_1 \cap B_2$$

This roughly means a basis is 'finitely' fine at every point in  $x$ .

If  $\mathbb{B}$  is a basis, it 'generates' a topology  $\mathcal{T}$  through

$$\mathcal{T} = \left\{ U \subseteq \mathbf{X}, \forall x \in U, x \in B \subseteq U \text{ for some } B \in \mathbb{B} \right\} \quad (1)$$

Notice this is equivalent to  $\mathcal{T}$  is the collection of all unions of basis elements in  $\mathbb{B}$ .

### Proposition 2.1

Let  $\mathbb{B}$  be a basis as defined in Definition 2.1, then  $\mathcal{T}$  as defined in Equation (1) is a valid topology on  $\mathbf{X}$ . And every member of  $\mathcal{T}$  is and is precisely the union of elements in  $\mathbb{B}$ .

*Proof.* Every point in  $\mathbf{X}$  belongs in some basis element, so  $\mathbf{X} \in \mathcal{T}$ , so does  $\emptyset$ . Next, if  $U_1$  and  $U_2$  are in  $\mathcal{T}$ , then

$$\begin{cases} x \in U_1 \leftrightarrow x \in B_1 \subseteq U_1 \\ x \in U_2 \leftrightarrow x \in B_2 \subseteq U_2 \end{cases} \implies x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

for some  $B_3 \in \mathbb{B}$ , so  $\mathcal{T}$  is closed under finite intersections (perhaps after a standard induction argument).

If  $\{U_\alpha\} \subseteq \mathcal{T}$ , and  $x$  belongs in the union of all  $U_\alpha$ , then  $x \in B_\alpha \subseteq U_\alpha$ , which is a subset of the entire union. So the union over  $U_\alpha$  is again contained in  $\mathcal{T}$ , and  $\mathcal{T}$  is a topology on  $\mathbf{X}$ .

It is worth noting that  $\mathbb{B} \subseteq \mathcal{T}$ . Finally, if  $U \in \mathcal{T}$ ,

$$U = \bigcup_{x \in U} B_x$$

where  $B_x$  is the basis element taken to satisfy  $x \in B_x \subseteq U$ . Every point in  $U$  is included in some  $B_x$ , and hence is included in the union. For the reverse inclusion, notice the

union of subsets of  $U$  is again a subset of  $U$ .

Now, if  $E \subseteq X$  is the union of basis elements in  $\mathbb{B}$ , if  $E$  is non-empty, then every point  $x \in E$  belongs in some  $B_x$ . Recycling the previous argument, and we see that  $E$  is open in  $\mathcal{T}$ . If  $E$  is empty, we define the 'union' of no sets as the empty set. So  $\mathcal{T}$  is precisely the collection of all unions of basis elements  $\mathbb{B}$ . ■

We are now in a position to compare the relative 'fineness' of topologies.

**Definition 2.2: Fineness of topologies**

If  $\mathcal{T}'$  and  $\mathcal{T}$  are both topologies on some non-empty set  $\mathbf{X}$ . We say  $\mathcal{T}'$  is finer than  $\mathcal{T}$ , or  $\mathcal{T}$  is coarser than  $\mathcal{T}'$  if

$$\mathcal{T}' \supseteq \mathcal{T}$$

**Proposition 2.2**

If  $\mathbb{B}$  and  $\mathbb{B}'$  are bases for  $\mathcal{T}'$  and  $\mathcal{T}$ , the following are equivalent:

- $\mathcal{T}'$  is finer than  $\mathcal{T}$ ,
- If  $B$  is an arbitrary basis element in  $\mathbb{B}$ , then every point  $x \in B$  induces a basis element in  $\mathbb{B}'$  with

$$x \in B' \subseteq B$$

*Proof.* Suppose  $\mathcal{T}'$  is finer than  $\mathcal{T}$ . Notice  $\mathbb{B} \subseteq \mathcal{T}'$  as well. By Equation (1), each  $x \in B$  induces a  $B' \in \mathbb{B}'$

$$x \in B' \subseteq B$$

Conversely, fix any open set  $U \in \mathcal{T}$ , and for each  $x \in U$ ,

$$x \in B' \subseteq B \subseteq U$$

Applying Definition 2.1 tells us  $U$  is open in  $\mathcal{T}'$ . ■

The last of the big three 'generating' definitions for topologies will be the sub-basis. It simply means the first condition (but not necessarily) the second, is satisfied in Definition 2.1

**Definition 2.3: Sub-basis of a topology**

A sub-basis  $\mathcal{S} \in \mathbb{P}(\mathbf{X})$  is a family of subsets of  $\mathbf{X}$  that satisfies one property. Any point  $x$  in  $\mathbf{X}$  belongs to at least one member of  $\mathcal{S}$ .

A sub-basis can be upgraded to a basis by collecting all of its finite intersections.

**Proposition 2.3**

Let  $\mathcal{S}$  be a sub-basis of  $\mathbf{X}$ , then the collection of all finite intersections of  $\mathcal{S}$  forms a basis  $\mathbb{B}$  of  $\mathbf{X}$ .

*Proof.* Every point in  $\mathbf{X}$  lies in some element of  $\mathcal{S}$ , hence in some element of  $\mathbb{B}$ . The second basis property is immediate, since  $\mathbb{B}$  is closed under finite intersections. ■

## Product Topology

We will start with products of a finite collection of topological spaces.

### Definition 3.1: Finite Product of Topological Spaces

Let  $(\mathbf{X}, \mathcal{T}_{\mathbf{X}})$  and  $(\mathbf{Y}, \mathcal{T}_{\mathbf{Y}})$  be topological spaces. The product topology (denoted by  $\mathcal{T}_{\mathbf{X} \times \mathbf{Y}}$ ) on  $X \times Y$  is defined as the topology generated by the basis

$$\mathbb{B}_{\mathbf{X} \times \mathbf{Y}} = \left\{ U \times V, (U, V) \in \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}} \right\} \quad (2)$$

Since bases are easier to describe than topologies, we have the following statement concerning the basis of the product topology.

### Proposition 3.1

If  $\mathbb{B}_{\mathbf{X}}$  and  $\mathbb{B}_{\mathbf{Y}}$  are bases for  $\mathcal{T}_{\mathbf{X}}$  and  $\mathcal{T}_{\mathbf{Y}}$ , then the product topology (as described in Definition 3.1) is also generated by

$$\mathcal{M} = \left\{ U \times V, (U, V) \in \mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \right\} \quad (3)$$

*Proof.* We will introduce (and use) the technique of 'double inclusion' by proving that the topologies generated are both finer than the other. Let us denote the topology generated by  $\mathcal{M}$  in Equation (3) by  $\mathcal{T}_{\mathcal{M}}$ .

Since  $\mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \subseteq \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}$ , if  $U \times V \in \mathcal{M}$  as in Equation (3), then we can pick the same 'open rectangle' again. We trivially have

$$x \in \underbrace{U \times V}_{\text{member of } \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}} \subseteq U \times V$$

and by Proposition 2.2,  $\mathcal{T}_{\mathbf{X} \times \mathbf{Y}}$  is finer than  $\mathcal{T}_{\mathcal{M}}$ .

Fix any set  $U \times V \in \mathbb{B}_{\mathbf{X} \times \mathbf{Y}}$ , and if  $(p, q) \in U \times V$ , each coordinate induces basis elements from  $\mathbb{B}_{\mathbf{X}}$  and  $\mathbb{B}_{\mathbf{Y}}$ , more precisely:

$$\begin{cases} p \in U \implies p \in \text{Basis element of } \mathbb{B}_{\mathbf{X}} \subseteq U \\ q \in V \implies q \in \text{Basis element of } \mathbb{B}_{\mathbf{Y}} \subseteq V \end{cases} \implies (p, q) \in \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{X}}} \times \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{Y}}} \subseteq U \times V$$

by Proposition 2.2,  $\mathcal{T}_{\mathcal{M}}$  is finer than  $\mathcal{T}_{\mathbf{X} \times \mathbf{Y}}$  and  $\mathcal{T}_{\mathbf{X} \times \mathbf{Y}} = \mathcal{T}_{\mathcal{M}}$ . ■

## Quotient Topology



## Product Topology

The Cartesian Product of an arbitrary family of topological spaces, if equipped with the product topology, preserves a lot of the structure. If  $\{X_\alpha\}_{\alpha \in A}$  is a family of topological spaces which are \_\_\_\_\_, then  $\prod X_\alpha$  is \_\_\_\_\_. Replace \_\_\_\_\_ with:

1. Hausdorff, (Folland)
2. Regular,
3. Connected, (Munkres chp23, exercise 10)
4. First countable, if  $A$  is countable,
5. Second countable, if  $A$  is countable,
6. Compact (Tychonoff's Theorem, Folland)

## Connectedness

### Definition 6.1: Connectedness

A topological space  $\mathbf{X}$  is connected if  $U$  and  $V$  are disjoint open subsets whose union is  $\mathbf{X}$ , then at least one of  $U$  or  $V$  is empty.

See Folland Exercise 4.10 for more properties.

### Definition 6.2: Path-connectedness

A topological space  $\mathbf{X}$  is path-connected if for any two pair of points  $x, y \in \mathbf{X}$ . There exists a continuous function  $f : [a, b] \rightarrow \mathbf{X}$ , with  $f(a) = x$  and  $f(b) = y$ .

### Definition 6.3: Connected component

The connected components of  $\mathbf{X}$  is the family of equivalence classes on  $\mathbf{X}$ , where  $x \sim y$  if there is a connected subspace of  $\mathbf{X}$  that contains both of them.

### Proposition 6.1

Continuous functions map connected spaces to connected spaces (in the subspace topology).

*Proof.* Let  $\mathbf{X}$  and  $\mathbf{Y}$  be topological spaces and  $f : \mathbf{X} \rightarrow \mathbf{Y}$  be continuous. If  $f(\mathbf{X})$  is disconnected, then we can find  $U$  and  $V$ , open and disjoint in  $\mathcal{T}_{f(\mathbf{X})}$  such that

$$U \cup V = f(\mathbf{X}) \implies f^{-1}(U) \cup f^{-1}(V) = \mathbf{X}$$

where  $f^{-1}(f(\mathbf{X})) = \mathbf{X}$ . Both  $f^{-1}(U)$  and  $f^{-1}(V)$  are open, non-empty, and are pairwise disjoint. So  $\mathbf{X}$  is separated. ■

### Proposition 6.2

Let  $(\mathbf{X}_\alpha, \mathcal{T}_\alpha)$  be a family of connected topological spaces indexed by  $\alpha \in A$ . Then  $\prod_{\alpha \in A} \mathbf{X}_\alpha$  is disconnected in the product topology.

*Proof.* We will attempt the contrapositive. Suppose  $\prod_{\alpha \in A} \mathbf{X}_\alpha$  is disconnected, then ■

## Topology in Analysis

### Interiors and closures

**Definition 7.1: Interior of a set**

$A^\circ$  is defined to be the largest open subset of  $A$ ,

$$A^\circ = \bigcup_{\substack{U \text{ open,} \\ U \subseteq A}} U$$

**Corollary 7.1**

The union of subsets of  $A$  is again a subset of  $A$ , therefore Corollary 7.1 implies  $A^\circ \subseteq A$  for any  $A \subseteq X$ .

**Definition 7.2: Closure of a set**

and  $\bar{A}$  is the smallest closed superset of  $A$ ,

$$\bar{A} = \bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K$$

**Proposition 7.1**

The complement of the closure is the interior of the complement, or equivalently:  
 $(\bar{A})^c = A^{\circ c}$

*Proof.* Taking complements, and the substitution  $U = K^c$  reads

$$\begin{aligned} (\bar{A})^c &= \left[ \bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K \right]^c \\ &= \bigcup_{\substack{K \text{ closed,} \\ K^c \subseteq A^c}} K^c \\ &= \bigcup_{\substack{U \text{ open,} \\ U \subseteq A^c}} U \\ &= A^{\circ c} \end{aligned}$$

■

**Remark 7.1**

Personally, I remember this as pushing the complement inside and flipping the bar to a  $c$ !

## Neighbourhoods

The concept of a neighbourhood allows us to characterize the interior of a set 'locally'.

### Definition 8.1: Neighbourhood (not necessarily open)

A neighbourhood of  $x \in \mathbf{X}$  is a set  $U \subseteq \mathbf{X}$  where  $x \in U^\circ$ . The set of neighbourhoods for a point  $x \in \mathbf{X}$  will sometimes be denoted by  $\mathcal{N}(\cdot)(x)$ .

### Proposition 8.1: Characterization of the interior

If  $W = \left\{ x \in \mathbf{X}, \text{ there exists a neighbourhood } U \text{ of } x, U \subseteq A \right\}$ , then  $W = A^\circ$ .

*Proof.* If  $x \in A^\circ$ , then  $A$  is a neighbourhood of  $x$ , and  $A \subseteq A$ , so  $x \in W$ . Conversely, if  $x$  is a member of  $W$ , it has a neighbourhood  $U \subseteq A$  (not necessarily open). By monotonicity of the interior,

$$x \in U^\circ \subseteq A^\circ$$

and  $x \in A^\circ$ . ■

It is easy to see that  $A$  is open  $\iff A^\circ = A \iff A$  is a neighbourhood of itself.

- The first equivalence follows from:

$$E \subseteq \mathbf{X} \implies E^\circ \subseteq E$$

and if  $A$  is an open set, it is an open subset of itself, by Corollary 7.1  $A \subseteq A^\circ$ . If  $A^\circ = A$ , then it suffices to show that  $A^\circ$  is open. Which it is, since it is the arbitrary union of open sets.

- To prove the second equivalence: suppose  $A^\circ = A$ , then each  $x \in A$  has a neighbourhood contained (as a subset) in  $A$ , namely  $A$  itself. (This statement is hard to parse, the reader is encouraged to really work through this and be honest).

$$x \in A^\circ \subseteq A \implies A \subseteq A^\circ$$

so  $A$  is a neighbourhood of itself. Conversely, if  $A \subseteq A^\circ$ , then  $A = A^\circ$ , since the reverse inclusion follows immediately from Corollary 7.1.

## Adherent points

Similar to the neighbourhood, the concept of an adherent point of a set allows us to speak of the closure in more concrete terms. The following definition is key in understanding the relationship between the closure, interior, and the boundary.

### Definition 9.1: Adherent point of a set

Let  $A \subseteq X$ ,  $x \in X$  is an adherent point of  $A$  if every neighbourhood  $U$  of  $x$  intersects  $A$ . In symbols,

$$U \cap A \neq \emptyset, \quad \forall U \in \mathcal{N}(x)$$

### Proposition 9.1: Characterization of the closure

Let  $A \subseteq X$ , and let  $W$  be the set of adherent points of  $A$ , then  $\overline{A} = W$

*Proof.* Suppose  $x \notin W$ , then there exists a neighbourhood  $U$  of  $x$  where

$$U \cap A = \emptyset \iff U \subseteq A^c$$

this is exactly the definition of the interior of  $A^c$ , so  $x \in A^{\circ c}$  and recall (from Proposition 7.1) that  $(\overline{A})^c = A^{\circ c}$ , so  $x \notin \overline{A}$ . For the reverse inclusion, read the proof backwards, by flipping  $\forall \rightarrow \exists$  within the set, and we see that

$$W^c = A^{\circ c} = (\overline{A})^c$$

■

## Dense and nowhere dense subsets

### Definition 10.1: Dense subset

A subset of a topological space  $E \subseteq \mathbf{X}$  is dense if  $\overline{E} = \mathbf{X}$ .

### Definition 10.2: Nowhere dense subset

A subset of a topological space  $E \subseteq \mathbf{X}$  is nowhere dense if  $\overline{E}^\circ = \emptyset$ .  
This means  $E$  is dense in none of the (non-trivial) open subspaces of  $\mathbf{X}$ .

### Proposition 10.1

$E$  is dense in  $\mathbf{X}$  iff for every non-empty, open set  $U \subseteq \mathbf{X}$ ,  $U \cap E \neq \emptyset$ .

*Proof of Proposition 10.1.* Suppose  $E$  is dense, then  $\overline{E} = \mathbf{X}$ . Every point of  $\mathbf{X}$  is an adherent point of  $E$ . Let  $U \subseteq \mathbf{X}$  be a non-empty open set. If  $x \in U$  then  $U$  is a neighbourhood of  $x$ , thus  $U$  intersects  $E$ . Conversely, suppose every non-empty open set  $U$  intersects  $E$ . Fix any point  $x \in \mathbf{X}$ , and any neighbourhood  $U$  of  $x$ .  $U$  has a non-empty interior (because it must contain  $x$ ). But  $U^\circ$  is a non-empty open set, therefore  $\emptyset \neq U^\circ \cap E \subseteq U \cap E$  ■

### Proposition 10.2

Let  $f : \mathbf{X} \rightarrow \mathbf{X}$  be a homeomorphism.  $E$  is nowhere dense iff  $f(E)$  is nowhere dense.

*Proof.* Since  $f^{-1}$  is a homeomorphism, suppose  $\overline{f^{-1}(E)}^\circ \neq \emptyset$ , there exists a non-empty, open subset  $U \subseteq \mathbf{X}$  with

$$\overline{f^{-1}(E)} \cap U = U$$

The direct image yields

$$f\left(\overline{f^{-1}(E)} \cap U\right) = f(U)$$

since  $f$  is a bijection (injectivity is necessary here), it commutes with intersections.

$$f(\overline{f^{-1}(E)}) \cap f(U) = f\left(\overline{f^{-1}(E)} \cap U\right) = f(U) \quad (4)$$

and  $f$  is continuous, so  $f(\overline{A}) \subseteq \overline{f(A)}$  for any  $A \subseteq \mathbf{X}$ . For the reverse inclusion,  $f$  is a closed map, so  $f(\overline{A})$  is a closed superset of  $f(A)$  so

$$f(\overline{A}) = \overline{f(A)}$$

Take  $A = f^{-1}(E)$ , and  $f(\overline{f^{-1}(E)}) = \overline{f(f^{-1}(E))} = \overline{E}$ . From eq. (4), we see that

$$\overline{E} \cap f(U) = f(U)$$

$f(U)$  is a non-empty open subset of  $\mathbf{X}$ , since  $f$  is an open map, so  $E$  is not no-where dense. The reverse implication can be proven by replacing  $f$  with  $f^{-1}$ . ■



## Urysohn's Lemma Notes

Notes on the construction of the countable 'onion' sequence within a normal space  $\mathbf{X}$ .

If  $\mathbf{X}$  is a normal space, and  $A$  and  $B$  are disjoint closed subsets, then we can easily find an open  $U$  with

$$A \subseteq U \subseteq \bar{U} \subseteq B^c \quad (5)$$

We say that  $U$  hides in  $B^c$  if the closure of  $U$  is contained in  $B^c$ . Define  $\Delta_n = \left\{ k2^{-n}, 1 < k < 2^n \right\}$ , so that  $\Delta_n \subseteq (0, 1)$  for all  $n \geq 1$ . Notice

$$\Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

and the even indices for  $\Delta_{n+1}$  are contained in  $\Delta_n$ . Suppose  $\Delta_n$  is well defined, it suffices to choose the odd indices for  $\Delta_{n+1}$ . If  $r = j2^{-(n+1)}$ , where  $j$  is odd, then  $r$  sits in between precisely two elements in  $\Delta_n \cup \{0, 1\}$ . If  $r$  sits between an endpoint, then define  $\bar{U}_0 = A$ , and  $B^c = U_1$ . And denote the closest left and neighbours by  $s, t$  respectively. If  $s < r < t$ , it is clear that  $\bar{U}_s$  and  $U_t^c$  are disjoint closed sets.

Use the 'normal space' construction to obtain an superset of  $\bar{U}_s$  that hides in  $U_t$ , denote this open set by  $U_r$ , and similar to Equation (5)

$$\bar{U}_s \subseteq U_r \subseteq \bar{U}_r \subseteq U_t$$

Now that the construction of this sequence is complete, we wish to prove Urysohn's Lemma. Let  $A$  and  $B$  be disjoint closed sets. And define

$$f(x) = \inf \left\{ r \in \Delta \cup \{1\}, x \in U_r \right\}$$

where  $U_1 = \mathbf{X}$ . So that  $0 \leq f(x) \leq 1$  is immediate. If  $x \in A$ , then  $x$  is in all  $U_r$ , and by density of  $\Delta \subseteq (0, 1)$ , we have  $f(x) = 0$ . Conversely, if  $x \in B$  then  $x \notin U_r$  for all  $r \in \Delta$ , if  $E$  denotes the indices in  $\Delta$  where  $x \in U_s$  when  $s \in E$ ,

$$(-\infty, r) \subseteq E^c \iff E \subseteq [r, +\infty) \iff \inf(E) \geq r \quad (6)$$

Send  $r \rightarrow 1$  and  $f(x) = 1$ . Thus  $f|_A = 0$  and  $f|_B = 1$ .

To show continuity, it suffices to show that the inverse images of the open half  $\left\{ (x > \alpha), (x < \alpha) \right\}_{\alpha \in \mathbb{R}}$  lines are indeed open in  $\mathbf{X}$ . Let  $\alpha$  be fixed. And if  $x \in \{f < \alpha\}$ , we can 'wiggle' the infimum towards the right (towards  $\alpha$ ), and using density of  $\Delta$  within  $(0, 1)$ , there exists a  $r \in E$  that satisfies  $f(x) < r < \alpha$ . This is equivalent to

$$x \in \bigcup_{r < \alpha} U_r$$

If there exists an  $r < \alpha$  st  $x$  belongs to  $U_r$  as an element, then  $f(x) \leq r < \alpha$ .

If  $f(x) > \alpha$ , then  $(-\infty, \alpha) \subseteq E^c$ , by Equation (6). Suppose  $\alpha < 1$ , otherwise  $\{f > \alpha\} = \emptyset$ . Wiggle  $f(x)$  to the left and obtain an  $r \in \Delta$ ,  $\alpha < r < f(x)$  with  $x \notin U_r$ . By density again, take any  $s < r$  by a small amount (st  $s > \alpha$ ,  $s \in \Delta$ ), and

$$\overline{U}_s \subseteq U_r \iff U_r^c \subseteq \overline{U}_s$$

so that  $x \in \overline{U}_s^c$  for some  $s > \alpha$ . This is equivalent to

$$x \in \bigcup_{s > \alpha} \overline{U}_s^c$$

Conversely, if  $x \notin \overline{U}_s^c$  for some  $s > \alpha$ , since  $\{U_r\}$  (thus  $\{\overline{U}_r\}$ ) is increasing, and  $x \notin U_r$  for every  $r \leq s$ . Hence,

$$(-\infty, s] \subseteq E^c \iff E \subseteq (s, +\infty) \iff f(x) \geq s > \alpha$$

## Compactness

Compactness is one of the most important concepts in topology and analysis.

### Definition 12.1: Compact topological space

A topological space  $\mathbf{X}$  is compact if every open covering  $\{U_\alpha\}$  contains a finite subcover. That is, if  $\{U_\alpha\}$  is an arbitrary collection of open sets, then

$$\mathbf{X} = \bigcup_{\alpha \in A} U_\alpha \implies \bigcup_{j \leq n} U_{\alpha_j}$$

### Definition 12.2: Compact set

$E \subseteq \mathbf{X}$  is compact if it is compact in the subspace topology.

### Definition 12.3: Precompact set

$E \subseteq \mathbf{X}$  is precompact if its closure is compact (as a subset).

### Definition 12.4: Paracompact space

A topological space  $\mathbf{X}$  is paracompact if every open covering of  $\mathbf{X}$  has a locally finite open refinement that covers  $\mathbf{X}$ .

### Definition 12.5: Locally finite collection of sets

Let  $\mathcal{A}$  be a collection of subsets of  $\mathbf{X}$ . It is called locally finite, if at every point  $p \in \mathbf{X}$ , we can find a neighbourhood  $U$  of  $p$  (not necessarily open), that intersects only finitely many members of  $\mathcal{A}$ . In symbols,

$$U \cap E = \emptyset \quad \text{for all but finitely many } E \in \mathcal{A}$$

We do not require  $\mathcal{A}$  to be a cover of  $\mathbf{X}$ , nor do we require  $\mathcal{A}$  to be a collection of open sets.

### Definition 12.6: Countably locally finite

A collection  $\mathbb{B}$  is countably locally finite if it is the countable union of locally finite

families.

$$\mathbb{B} = \bigcup_{\mathbb{N}}^{\text{countable union}} \mathbb{B}_n, \quad \text{where each } \mathbb{B}_n \text{ is a locally finite collection}$$

### Definition 12.7: Refinement

If  $\mathcal{A}$  is a collection of sets,  $\mathbb{B}$  is a refinement of  $\mathcal{A}$  if every element  $B \in \mathbb{B}$ , induces an element  $A \in \mathcal{A}$ , such that  $B \subseteq A$ .

### Remark 12.1: Intuition for refinements

If  $\mathbb{B}$  is a refinement of  $\mathcal{A}$ , we can use the 'absolute continuity' muscle. For each element in  $\mathbb{B}$  is dominated by some element (through subset inclusion) in  $\mathcal{A}$ . Recall, if  $\nu$  and  $\mu$  are non-negative measures, then  $\nu \ll \mu$  if for every measurable set  $E \in \mathcal{M}$ ,  $\mu(E) = 0 \implies \nu(E) = 0$ .

A refinement of a family of sets is another family of sets, whose elements are dominated by some other element in the un-refined family. *Refining families makes them 'smaller', cover less area.*

### Proposition 12.1

Compact Hausdorff spaces are normal, compact subsets of Hausdorff spaces are closed, and closed subsets of compact sets are again compact.

## Locally Compact Hausdorff Spaces

Compactness is an intrinsic topological property (in the subspace topology). We see from Proposition 4.25 that compact Hausdorff spaces are normal, which gives a sufficient condition for us to approximate and extend any continuous function; and allows us to extend certain 'local' properties to 'global' properties.

If given a Hausdorff space, not necessarily compact, the natural question is to ask 1) whether a topological space has 'enough' compact subsets to work with, and 2) whether we can embed a given topological space in a larger one to force it to be compact.

### Definition 13.1: LCH space

Let  $\mathbf{X}$  be a Hausdorff space. We call  $\mathbf{X}$  a LCH space if every point  $p \in \mathbf{X}$  admits a compact neighbourhood. That is, a compact set  $K$  whose interior contains  $p$ .

We note in passing that the above definition differs slightly from the usual 'local' definitions.

### Definition 13.2: Locally connected

Let  $\mathbf{X}$  be a topological space, it is locally connected if for every  $x \in \mathbf{X}$ , and open neighbourhood  $U$  containing  $x$ , there exists a connected, open neighbourhood  $V$  of  $x$  such that  $x \in V \subseteq U$ .

### Definition 13.3: Locally path-connected

Let  $\mathbf{X}$  be a topological space, it is locally path-connected if for every  $x \in \mathbf{X}$ , and open neighbourhood  $U$  containing  $x$ , there exists a path-connected, open neighbourhood  $V$  of  $x$  such that  $x \in V \subseteq U$ .

### Definition 13.4: Local homeomorphism

$\mathbf{X}$  locally homeomorphic to  $\mathbb{R}^n$  if every point  $x \in \mathbf{X}$  belongs to a coordinate chart  $(U, \phi)$ , where  $U$  is an open neighbourhood of  $x$  and  $\phi$  is a homeomorphism from  $U \rightarrow \phi(U) \subseteq \mathbb{R}^n$ .

### Definition 13.5: Local diffeomorphism

Let  $M$  be a smooth manifold and  $F \in C^\infty(M, N)$ .  $F$  is a local diffeomorphism if every  $p \in M$  in its domain induces a neighbourhood  $U \subseteq M$  with  $F|_U : U \rightarrow F(U)$  is a diffeomorphism (in the sense of two open sub-manifolds).

**Theorem 4.1****Proposition 1.1**

Suppose that  $A$  is a subset of  $X$ , let  $\text{acc } A$  be the set of accumulation points of  $A$ , then

$$\overline{A} = A \cup \text{acc } (A) \quad (7)$$

and  $A$  is closed if and only if  $\text{acc } (A) \subseteq A$ .

*Proof.* Suppose that  $x \notin \overline{A}$ , then  $x \in (\overline{A})^c = A^{\circ\circ}$ , then  $A^c \in \mathcal{N}_B(x)$ . But this means that  $x \notin \text{acc } (A)$ , since there exists a neighbourhood of  $x$  (in the form of  $A^c$ ), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \emptyset$$

Also,  $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$  which means that

$$x \notin \overline{A} \implies x \notin A$$

Since  $x \notin \overline{A} \implies x \notin A$  and  $x \notin \text{acc } (A)$ ,

$$(\overline{A})^c \subseteq A^c \cap \text{acc } (A)^c = (A \cup \text{acc } (A))^c$$

Now, if  $x \notin \text{acc } (A) \cup A$ , then  $x \notin \text{acc } (A)$ , therefore there exists some  $U \in \mathcal{N}_B(x)$  such that

$$A \cap U \setminus \{x\} = A \cap U = \emptyset$$

Where for the second last equality we used the fact that  $x \notin A \implies A \setminus \{x\} = A$ , and taking complements gives us

$$U \subseteq A^c$$

And since  $U \in \mathcal{N}_B(x)$ , then  $x \in U^{\circ} \subseteq A^{\circ\circ}$  (since  $U^{\circ}$  is an open subset of  $A^c$ ). then

$$x \in A^{\circ\circ} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore  $(A \cup \text{acc } (A))^c \subseteq (\overline{A})^c$ . ■

**Theorem 4.2****Proposition 2.1**

If  $\mathcal{T}_X$  is a topology on  $X$  and  $\mathcal{E} \subseteq \mathcal{T}_X$  then  $\mathcal{E}$  is a base for  $\mathcal{T}_X$  if and only if for every

$$\forall U \in \mathcal{T}_X, U \neq \emptyset, \implies U = \bigcup_{V \in B} V$$

Where  $B$  is a subset of  $\mathcal{E}$ .

*Proof.* Suppose that  $\mathcal{E}$  is a base, then fix any non-empty  $U \in \mathcal{T}_X$ , then for every  $x \in U$ , there exists a neighbourhood base for this  $x$  and a member  $V \in \mathcal{E}$  such that  $x \in V_x \subseteq U$ . Take the union over all  $V_x$  and

$$U \subseteq \bigcup_{x \in U} V_x$$

But each  $V_x \subseteq U$ , so  $U = \bigcup_{x \in U} V_x$ , where  $\{V_x\} \subseteq \mathcal{E}$ .

Conversely, if every non-empty  $U$  is a union of members in  $\mathcal{E}$  then fix any  $x \in X$ , we claim that we have a neighbourhood base in

$$\{V \in \mathcal{E}, x \in V\}$$

The reason is as follows

- $x$  belongs to every  $E \in \{V \in \mathcal{E}, x \in V\}$  and
- For every open  $U$ , if  $x \in U$  then there exists a union of members of  $\mathcal{E}$  such that  $U = \bigcup E_\alpha$ , then  $x \in U \iff \exists E_\alpha \in \{V \in \mathcal{E}, x \in V\}$  and
- Using this particular  $E_\alpha \in \mathcal{E}$  that we just found,  $x \in E_\alpha \subseteq U$ , and we are done.

■

**Theorem 4.3****Proposition 3.1**

For every  $\mathcal{E} \subseteq \mathbb{P}(X)$ ,  $\mathcal{E}$  is base for a topology on  $X$  if and only if

- (a) each  $x \in X$  is contained in some  $V \in \mathcal{E}$ , and
- (b) if  $U, V \in \mathcal{E}$ , and  $x \in U \cap V$ , then there must exist some  $W \in \mathcal{E}$  with  $x \in W \subseteq U \cap V$ .

*Proof.* Suppose that  $\mathcal{E}$  is a base, then we get a), and b) follows since for every  $U, V \in \mathcal{E} \subseteq \mathcal{T}_X$ , and by closure over finite intersections,  $U \cap V \in \mathcal{T}_X$  implies that there exists some  $W \in \mathcal{E}$  with

$$x \in W \subseteq U \cap V$$

Now, suppose both a) and b) hold, then we claim that this  $\mathcal{E} \subseteq \mathbb{P}(X)$  induces a topology on  $X$

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, \text{ with } x \in V \subseteq U\}$$

Intuitively speaking, this means that  $\mathcal{T}$  is just fine (and not too fine) to satisfy the conditions for  $\mathcal{E} \subseteq \mathcal{T}$  to be a base of  $\mathcal{T}$ .

We first show that  $\mathcal{T}$  is a topology.

- $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$ , the first is trivial and the second is from a)
- Closure under unions: fix  $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ , and  $U = \bigcup U_\alpha$ , and for every  $x \in U$  there exists some  $V_\alpha \in \mathcal{E}$  such that  $x \in V_\alpha \subseteq U_\alpha \subseteq U$ , therefore  $U \in \mathcal{T}$ .
- Closure under finite intersections, fix any  $U_1, U_2$  as elements in  $\mathcal{T}$ , then suppose that they are not disjoint (if they are disjoint then their intersection is the empty set, which is also contained in  $\mathcal{T}$ ). If  $U_1 \cap U_2 \neq \emptyset$ , then for every  $x \in U_1 \cap U_2$  induces two sets  $V_1, V_2 \in \mathcal{E}$  with  $x \in V_1 \subseteq U_1$  and  $x \in V_2 \subseteq U_2$ , taking their intersection and applying b) gives us some  $V \subseteq V_1 \cap V_2$  with  $V \in \mathcal{E}$  therefore  $x \in V \subseteq U_1 \cap U_2$ , and  $\mathcal{T}$  is closed under finite intersections.

Now to show that  $\mathcal{E}$  is a base for  $\mathcal{T}$ ,  $\mathcal{E} \subseteq \mathcal{T}$  is obvious since every  $V \in \mathcal{E}$  satisfies the properties laid out by  $\mathcal{T}$  by simply choosing  $V$  again for any  $x \in V$ . Now fix any member  $U \in \mathcal{T}$ , then for every  $x \in U$ , there exists some  $V \in \mathcal{E}$  with

$$x \in V \subseteq U$$

(This is an immediate consequence of how we defined  $\mathcal{T}$ ). And we can conclude that  $\mathcal{E}$  is a base for this induced topology  $\mathcal{T}$ . ■



**Theorem 4.4****Proposition 4.1**

If  $\mathcal{E} \subseteq \mathbb{P}(X)$ , the topology  $\mathcal{T}(\mathcal{E})$  generated by  $\mathcal{E}$  consists of  $\emptyset, X$  and all unions of finite intersections of  $\mathcal{E}$ , in symbols

$$\mathcal{T}(\mathcal{E}) = \{\emptyset, X\} \cup \left\{ \bigcup W_\alpha, W_\alpha = \bigcap E_{j \leq n}, E_j \in \mathcal{E} \right\}$$

*Proof.* Denote the set

$$W = \{X\} \cup \left\{ \bigcap V_{j \leq n}, V_j \in \mathcal{E} \right\}$$

We claim this set  $W$  satisfies Theorem 4.3. Since 4.3a) is satisfied with  $X \in W$ . 4.3b) follows since the right member in  $W$  is closed under intersections.

And if we are taking an element from each member,  $E_1 \in \{\emptyset, X\}$  and  $E_2$  is an element in the right member, then it is trivial to verify that their intersection is always contained within  $W$ . Therefore  $W$  induces a topology by Theorem 4.2, and we call this topology  $\mathcal{T}$  — and for the sake of completeness

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, x \in V \subseteq U\}$$

We so claim that if we define  $\overline{W}$  as the union of all members  $w \in W$ , together with the empty set, is equal to the set  $\mathcal{T}$ .

$$\overline{W} = \left\{ \bigcup_{w \in W} w \right\} \cup \{\emptyset\}$$

- We want to show  $\mathcal{T} \subseteq \overline{W}$ , since  $W$  is a base for the topology  $\mathcal{T}$ , every (non-empty)  $U \in \mathcal{T}$  is the union of members in  $W$  (Theorem 4.2), and there exists some  $B \subseteq W$  with

$$U = \bigcup E_{\alpha \in B} \in \overline{W}$$

Now if  $U$  is the empty set then it is trivially contained within  $\overline{W}$ .

- Next, we show that  $\overline{W} \subseteq \mathcal{T}$ , fix any element  $E \in \overline{W}$ , if  $E = \emptyset$  then there is nothing to prove since  $\mathcal{T}$  is a topology. Now for every  $x \in E$ ,

$$x \in E = \bigcup_{w \in W} w \implies x \in w$$

Therefore  $E \in \mathcal{T}$  by definition. This proves that  $\mathcal{T} = \overline{W}$ .

Now that  $\overline{W}$  is a topology, that contains  $\mathcal{E}$  as a subset, and by definition of  $\mathcal{T}(\mathcal{E})$

$$\mathcal{T}(\mathcal{E}) = \bigcap \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Tells us

$$\mathcal{T}(\mathcal{E}) \subseteq \overline{W}, \quad \text{since } \overline{W} \in \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Conversely, fix any member  $E \in \overline{W}$ , if  $E = \emptyset$  then  $E \in \mathcal{T}(\mathcal{E})$ , if not, then there exists some subset  $B \subseteq W$  such that

$$E = \bigcup_{w \in B} w = \bigcup_{w \in B} \bigcap_{j \leq n} V_{j \leq n}^w V_j \in \mathcal{E} \cup \{X\}$$

Since  $\mathcal{T}(\mathcal{E})$  is closed under finite intersections and unions, and it contains  $\mathcal{E}$  as a subset,  $\overline{W} = \mathcal{T}(\mathcal{E})$  and we are done. ■

**Theorem 4.5****Proposition 5.1**

Every second countable space is separable. (Countable dense subset).

*Proof.* What we wish to prove is that if a space  $X$  has a countable base, then it has a countable dense subset. Denote this base of  $X$  by  $\mathcal{E}$  as usual, then we claim that

$$W = \{x_u, U \in \mathcal{E}\}$$

Is a dense subset in  $X$ . Note that  $(\overline{W})^c = W^{co} \in \mathcal{T}_X$ . If  $W^{co} = \emptyset$  then we simply take complements and we get  $\overline{W} = X$ . So suppose that  $W^{co}$  is non-empty, then for each  $x \in W^{co}$  (by definition of a base), it should induce some  $V_x \in \mathcal{E}$  with

$$x \in V_x \subseteq W^{co}$$

But clearly, for every element in  $\mathcal{E}$ , the second estimate can never be satisfied, since for every  $U \in \mathcal{E}$ ,  $x_U \notin W^{co}$  for this particular set  $W^{co}$ . Therefore  $W^{co}$  must be empty, and this completes the proof.  $\blacksquare$

## Theorem 4.6

### Proposition 6.1

If  $X$  is first countable, then for every  $A \subseteq X$ ,  $x \in \overline{A} \iff$  there exists some sequence  $\{x_j\}_{j \geq 1} \subseteq A$  such that  $x_j \rightarrow x$ .

*Proof.* Suppose that  $X$  is first countable, and  $A \subseteq X$ , and fix any element  $x \in \overline{A}$ . Since  $X$  is first countable, there is a sequence of descending neighbourhoods of  $\{U_j\}_{j \geq 1}$  of  $x$  such that

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_j \supseteq U_{j+1}$$

If  $x \in A$ , take  $x_n = x$  for all  $n \geq 1$ . If  $x \in \text{acc}(A)$ , then take  $x_n \in U_n \cap A \setminus \{x\} = U_n \cap A$ , which is not empty. Then it remains to show that this sequence converges to  $x$ . Fix any neighbourhood  $U \in \mathcal{N}_B(x)$  then there exists some  $N$ , for every  $n \geq N$

$$x \in U^o \implies \exists N \in \mathbb{N}^+, x \in U_N \subseteq U^o$$

Then every  $x_n \in A \cap U_N \subseteq A \cap U^o \subseteq U^o$ . And this establishes  $\implies$ .

Now suppose that  $x \notin \overline{A}$ , so that  $x \notin A$  and  $x \notin \text{acc}(A)$ , then fix any sequence  $\{x_j\} \subseteq A$ . We wish to show that  $x_j \not\rightarrow x$ .

Since  $x \notin \text{acc}(A)$ , there exists some  $V \in \mathcal{N}_B(X)$  with

$$A \cap V \setminus \{x\} = \emptyset \implies V \subseteq A^c$$

Since  $\{x_j\}_{j \geq 1} \subseteq A \implies x_j \notin A^c$  for every  $j \geq 1$ , then choose  $V$  as the neighbourhood around  $x$ , and  $x_j \not\rightarrow x$  for any arbitrary sequence  $x_j$  in  $A$ . ■

### Remark 6.1

To truly understand what is going on one should recall that all metric space spaces are first countable.

**Theorem 4.7****Proposition 7.1**

$X$  is a  $T_1$  space  $\iff \{x\}$  is closed for every  $x \in X$ .

*Proof.* If  $X$  is  $T_1$  and  $x \in X$ , then for every  $y \neq x$  there exists some open  $U_y$  that contains  $y$  but not  $x$ . Following Folland's argument closely, every  $y \neq x$  is in  $\cup U_{y \neq x}$ . Hence  $\{x\}^c \subseteq \cup U_{y \neq x}$ . To show the converse, for every  $z \in \cup U_{y \neq x}$  that is open, there exists a  $y \neq x$  such that  $z \in U_y$ . But every  $U_y$  does not contain  $x$  as an element, so  $z \neq x$  implies that  $z \notin \{x\}$ . And  $z \in \{x\}^c$ . Hence  $\cup U_{y \neq x} = \{x\}^c$ .

Now conversely if every  $x \in X$  satisfies the fact that  $\{x\}^c$  is open, then  $\{x\}^c$  is an open set that contains every  $y \neq x$ . Now fix some  $y \neq x$ , since  $\{y\}$  is also closed, we have  $X \cap \{x\}^c$  is an open set that contains  $x$  but not  $y$ . Also,  $\{x\}^c$  is an open set that contains  $y$  but not  $x$ . And therefore  $X$  is  $T_1$ . ■

**Theorem 4.8****Proposition 8.1**

The map  $f : X \rightarrow Y$  is continuous if and only if  $f$  is continuous at every  $x \in X$ .

*Proof.* Suppose that  $f$  is continuous, then fix any  $f(x) \in Y$  and any of its neighbourhood  $V \in \mathcal{N}_B(f(x))$ ,

$$f(x) \in V^o \implies f^{-1}(V^o) \in \mathcal{N}_B(x)$$

But by continuity,  $f^{-1}(V^o)$  is an open set that contains  $x$ , with

$$f(f^{-1}(V^o)) \subseteq V^o$$

Therefore  $f$  is continuous at  $x$ . Now suppose that  $f$  is continuous at every  $x \in X$ , then for every open subset  $V \subseteq Y$ , and for every point  $f(x) \in V = V^o$  means that  $V \in \mathcal{N}_B(f(x))$  for all such points  $f(x)$ . By continuity, for every  $x$  in  $f^{-1}(V)$ , implies that  $f^{-1}(V)$  is a neighbourhood of all of its elements, therefore  $f^{-1}(V) \subseteq (f^{-1}(V))^o$ , and  $f^{-1}(V)$  is open. ■

**Theorem 4.9****Proposition 9.1**

If  $\mathcal{E}_Y$  generates the topology on  $Y$ , and  $f$  is a mapping from  $X \rightarrow Y$ , then  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(V) \in \mathcal{T}_X$  for every  $V \in \mathcal{E}_Y$ .

*Proof.* The inverse image commutes with intersections, complements, and unions. To prove  $\Leftarrow$ , use Theorem 4.4, since every  $U \in \mathcal{T}_Y$  can be represented the union of finite intersections of elements  $\mathcal{E}_Y$ , and use the fact that  $\mathcal{T}_X$  is closed under arbitrary unions and finite intersections.

To show  $\Rightarrow$ , since  $\mathcal{E}_Y \subseteq \mathcal{T}_Y$ , if  $f^{-1}$  is open for every  $U \in \mathcal{T}_Y$ , then it is open for every  $U \in \mathcal{E}_Y$  as well. ■

**Theorem 4.10****Proposition 10.1**

If  $X_\alpha$  is Hausdorff for each  $\alpha \in A$ , then  $X = \prod_{\alpha \in A} X_\alpha$  is Hausdorff.

*Proof.* If two elements in  $X$ ,  $x \neq y$  then there exists some  $\alpha \in A$  such that  $\pi_\alpha(x) \neq \pi_\alpha(y) \in X_\alpha$ , but this  $X_\alpha$  is Hausdorff, then there exists two open, disjoint sets  $V_x, V_y \subseteq X_\alpha$  such that

- $x \in \pi_\alpha^{-1}(V_x)$ , and  $y \in \pi_\alpha^{-1}(V_y)$
- $\pi_\alpha^{-1}(V_x) \cap \pi_\alpha^{-1}(V_y) = \pi_\alpha^{-1}(V_x \cap V_y) = \emptyset$
- $\pi_\alpha^{-1}(V_x), \pi_\alpha^{-1}(V_y) \in \mathcal{T}_X$

Where for the last bullet point we used the fact that the product topology makes all the projection maps continuous. This proves that  $X$  is Hausdorff. ■



**Theorem 4.11****Proposition 11.1**

If  $X_\alpha$  and  $Y$  are topological spaces, and  $X = \prod_{\alpha \in A} X_\alpha$ , and  $f : Y \rightarrow X$  is a mapping. Then  $f$  is continuous if and only if  $\pi_\alpha \circ f$  is continuous for each  $\alpha \in A$ .

*Proof.* If  $\pi_\alpha \circ f$  is continuous at each  $\alpha$ , this means that

$$\forall \alpha \in A, \forall E_\alpha \in \mathcal{T}_\alpha, f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{T}_Y$$

But it is exactly sets of the form  $\pi_\alpha^{-1}(E_\alpha)$  which generate the weak topology for  $\mathcal{T}_X$ . Therefore  $f$  is continuous.

Now, suppose that  $f$  is continuous, by definition of the weak topology (as it is generated by the set of inverse projections), for every  $\alpha \in A$ ,  $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{T}_X$  and by continuity of  $f$ , its inverse image is open in  $Y$  as well. ■

**Remark 11.1**

The take-away intuition here is that if the range space is generated by some  $\mathcal{E}$ , then a function is continuous if and only if all inverse images of sets in  $\mathcal{E}$  are open in the domain space. Furthermore, if the range space is endowed with the product topology (which is generated by sets of the form  $\pi_\alpha^{-1}(E_\alpha)$ , where  $E_\alpha \in \mathcal{T}_\alpha$ ), then it suffices to check all inverse images of those. And this is equivalent to checking that  $\pi_\alpha(\cdot) \circ f$  is continuous at each  $\alpha$ .

**Theorem 4.12****Proposition 12.1**

If  $X$  is a topological space, and  $A$  is any non-empty set,  $\{f_n\} \subseteq X^A$  is a sequence, then  $f_n \rightarrow f$  with respect to the product topology if and only if  $f_n \rightarrow f$  pointwise.

*Proof.* Suppose that  $f_n \rightarrow f$  pointwise. Since the product topology  $\mathcal{T}_X$  is generated from sets of the form

$$\pi_\alpha^{-1}(E_\alpha), \quad E_\alpha \in \mathcal{T}_\alpha$$

And by Theorem 4.4,  $\mathcal{T}_X$  consists of  $\emptyset, X$  and unions of finite intersections of  $\pi_\alpha^{-1}(E_\alpha)$ . We claim that for every  $f \in X^A$ , the following is a valid neighbourhood base for  $f$

$$\left\{ \bigcap_{j \leq n} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j} \cap \mathcal{N}_B(\pi_{\alpha_j}(f)) \right\}$$

A couple things to note

- Each  $E_{\alpha_j}$  is open in  $X_{\alpha_j}$ , so that its inverse image is also open (in  $X$ ). Since any neighbourhood base has to be a subset of  $\mathcal{T}_X$ .
- Only finitely many intersections are involved, so each element in the above set is open in  $X$ .
- Each  $E_{\alpha_j}$  is a neighbourhood of  $\pi_{\alpha_j}(f)$ , meaning  $f \in E_{\alpha_j}^\circ = E_{\alpha_j}$ .
- Last and perhaps most importantly for intuition, fix any non-empty open set  $U \in \mathcal{T}_X$  then by Theorem 4.4 (or my reading of it),  $U$  can be written as the union of sets like

$$\bigcap_{j \leq m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}$$

Then applying Theorem 4.2, the family of finite intersections of  $\pi_\alpha^{-1}(E_\alpha)$  is a base for  $\mathcal{T}_X$ . Then,

$$N_{base}(f) = \left\{ V = \bigcap_{j \leq m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}, \quad f \in V \right\}$$

Has to be a neighbourhood base for any  $f \in X$ .

Now to show that  $f_n \rightarrow f$  in the product topology, fix any neighbourhood  $U \in \mathcal{N}_B(f)$ , then  $f \in U^\circ$ , and by definition of a neighbourhood base, there exists some  $E \in N_{base}(f)$  such that  $f \in E \subseteq U^\circ$ , but this  $E$  is just the finite intersection of  $\pi_{\alpha_j}^{-1}(E_{\alpha_j})$ , then at every  $\alpha_j$

- Let  $N_j$  be an integer such that for every  $n \geq N_j$ ,  $\pi_{\alpha_j}(f_n) \in E_{\alpha_j}$

- Set  $N = \sum_{j \leq m} N_j \geq N_j$  for every  $j \leq m$ .

Then for every  $n \geq N$ ,  $f_n \in E \subseteq U^o \subseteq U$  for any arbitrary neighbourhood  $U$  of  $f$ . So  $f_n \rightarrow f$  in the product topology.

Conversely, suppose that  $f_n \rightarrow f$  in the product topology, then fix any  $\alpha \in A$ , and for every neighbourhood  $E_\alpha$  of  $\pi_\alpha(f)$ ,  $\pi_\alpha^{-1}(E_\alpha)$  is a neighbourhood of  $f$ . Hence for every  $\alpha \in A$ , and for every neighbourhood  $E_\alpha$  of  $\pi_\alpha(f)$ ,  $p_{i_\alpha}(f_n)$  is eventually in  $E_\alpha$ . This completes the proof. ■

**Theorem 4.13****Proposition 13.1**

If  $X$  is a topological space then  $BC(X)$  is a closed subspace of  $B(X)$  in the uniform metric, and  $BC(X)$  is complete.

*Proof.* We will prove four things, the last two are just book-keeping. Parts (b, d) imply Part (c), as the closure of any set under a complete metric space is again complete.

- (a)  $B(X)$  endowed with the uniform norm of an  $f \in B(X)$

$$\|f\|_u = \sup\{|f(x)|, x \in X\}$$

Is indeed a normed vector space.

- (b)  $B(X)$  with its norm (and induced metric), is a complete metric space. So that our  $\{f_n\} \rightarrow f$  at worst, converges to  $f \in B(X)$ .
- (c) If  $\{f_n\}_{n \geq 1} \subseteq BC(X)$  is a uniformly Cauchy sequence, and  $f_n \rightarrow f$ , then  $f \in BC(X)$ .
- (d) If  $f$  is an adherent point of  $BC(X)$ , then  $f \in BC(X)$ .

To show that  $B(X)$  is a normed vector space, for any  $k \in \mathbb{C}$ ,  $f_1, f_2 \in B(X)$ , then at every  $x \in X$

$$|f_1(x) + kf_2(x)| \leq |f_1(x)| + |k| \cdot |f_2(x)| \leq \|f_1\|_u + |k| \|f_2\|_u$$

And to show absolute homogeneity, note that  $\sup |kA| = |k| \cdot \sup A$  for any non-empty bounded above set of reals  $A$ . This proves (a).

To show (b), fix any Cauchy sequence in  $B(X)$  (with respect to the uniform metric), then for every  $\varepsilon > 0$ , there exists an  $N$  so large that for every  $n, m \geq N$  we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_u < \varepsilon$$

This shows that  $\{f_n(x)\}_{n \geq 1} \subseteq \mathbb{C}$  is Cauchy, and it makes sense to call its limit  $f(x) = \lim f_n(x)$ . To show that for this  $f$ ,

- $f_n \rightarrow f$  uniformly, and
- $f \in B(X)$

Fix an  $\varepsilon > 0$ , and there exists an  $N$  so large that for every  $m, n \geq N$  implies that

$$\|f_n(x) - f_m(x)\|_u < \varepsilon$$

Since  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ , this means that

$$\lim_{n \rightarrow \infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| \leq \varepsilon$$

The above holds for any  $x$ , hence

$$\|f_m - f\|_u \leq \varepsilon \implies \|f\|_u \leq \|f_m - f\|_u + \|f_m\|_u < +\infty$$

This proves both bullet points.

Proof of (c): Now we will prove Theorem 4.13, for any sequence  $\{f_n\} \subseteq \text{BC}(X)$ , if it does converge to some  $f$  uniformly, then we claim that  $f \in \text{BC}(X)$ . Note that  $f \in B(X)$ , so it suffices to show continuity at every  $x_0 \in X$ .

Fix any ball with radius  $\varepsilon > 0$  at  $f(x_0) \in \mathbb{C}$ , and since

- $\varepsilon/3 > 0$  induces some  $N$  such that for every  $n \geq N$ , at every point  $x \in X$

$$|f_n(x_0) - f(x_0)| \leq \|f_n - f\|_u < \varepsilon/3$$

- Another  $\varepsilon/3$  gives us an open ball around  $f_n(x_0)$  in  $\mathbb{C}$  (using the same point  $x_0 \in X$ ). Continuity of  $f_n$  gives us

$$f_n^{-1}(B(\varepsilon/3, f_n(x_0))) = U \in \mathcal{T}_X$$

- If  $x$  is a point in  $U$ ,

$$|f_n(x) - f(x)| \leq \|f_n - f\|_i < \varepsilon/3$$

this gives us the last  $\varepsilon/3$ .

Combining these three,

$$|f(x) - f(x_0)| \leq \underbrace{|f(x) - f_n(x)| + |f(x_0) - f_n(x_0)|}_{\text{uniform convergence}} + \underbrace{|f_n(x_0) - f_n(x)|}_{\text{continuity of } f_n} < \varepsilon$$

So there exists some open set  $U \in \mathcal{T}_X$  (and hence neighbourhood of every  $x$ ), for every open ball of radius  $\varepsilon > 0$ , around every  $f(x) \in \mathbb{C}$ , such that

$$f(U) \subseteq B \in \mathcal{T}_{\mathbb{C}}$$

Since the open balls are a neighbourhood base at every point in  $\mathbb{C}$ , and  $f$  is continuous at every point  $x \in X$ , we must conclude that  $f \in \text{BC}(X)$ .

Part (d): Let  $f \in \overline{\text{BC}(X)}$ . Notice  $\text{BC}(X)$  is a metric space, hence first countable. There exists a sequence  $\{f_n\} \subseteq \text{BC}(X)$  that converges to  $f$ . Convergent sequences in any metric space is Cauchy, apply Part (c) finishes the proof. ■

**Theorem 4.14****Proposition 14.1**

Suppose that  $A$  and  $B$  are disjoint closed subsets of the normal space  $X$ , and let  $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$  be the set of dyadic rationals in  $(0, 1)$ . There is a family  $\{U_r : r \in \Delta\}$  of open sets such that

1.  $A \subseteq U_r \subseteq B^c$  for every  $r \in \Delta$ ,
2.  $\overline{U_r} \subseteq U_s$  for  $r < s$ , and
3. For every  $r < s$ ,  $\overline{U_r} \subseteq U_s$

*Proof.* The goal of this proof is to show that for every  $r \in \Delta$ , there exists a open  $U_r$  that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some  $N \geq 1$  that all previous  $U_r$  in previous layers have been constructed properly, meaning if  $r = k/2^n$ , then for every  $1 \leq n \leq N - 1$ , we have

$$r = \frac{k}{2^n}, 1 \leq n \leq N - 1, 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each  $U_r$ ,

- $A \subseteq U_r \subseteq B^c$  and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer  $N \geq 1$ , we only have to construct the  $U_{k/2^N}$  for every odd  $k$ , this is because if  $k$  is an even number, then  $k = 2j$  and  $r = 2j/2^N = j/2^{N-1}$  and for this particular  $U_r$  is already constructed. So for every odd  $k = 2j + 1$ , the sets of the form  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  are already defined, and satisfy

$$A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every  $k - 1 \neq 0$  and  $k + 1 \neq 1$ . (We will consider these cases later). We claim that for every pair of open sets,  $E_1, E_2 \in \mathcal{T}_X$ , then there exists some open set  $G \in \mathcal{T}_X$  such that if  $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$  where  $H$  is defined as the set

$$H = \{(E_1, E_2) \in (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \emptyset\}$$

Then there exists some  $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$  such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any  $(E_1, E_2) \in H$ , then this pair induces a pair of disjoint sets  $\overline{E_1}$  and  $E_2^c$  since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \emptyset$$

And by normality, there exists disjoint open sets  $G_1, G_2$  such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \emptyset \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since  $G_2^c$  is a closed set that contains  $G_1$  as a subset,  $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of  $G_2$  (so we may discard the notion of  $G_2$  in our minds). Let us now replace  $G$  with  $G_1$  then it is an easy task to verify that  $G = G_1 = \mathcal{J}(E_1, E_2)$  has the required properties.

Now define for every odd  $k$ , since  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (we note in passing that  $\mathcal{J}$  is not a function as the set  $G$  may not be unique).

$$U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$$

Then, if  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  is 'well constructed' we have

$$A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore  $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$  sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{k/2^N}$  and
- $\overline{U_{k/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed'  $U_{k/2^N}$  for every  $k-1 \neq 0$  and  $k+1 \neq 1$ . Now let us deal with the remaining pathological cases.

If  $k-1$  so happens to be 0, then no  $r \in \Delta$  satisfies  $r = 0/2^N$ , and we substitute

$$\overline{U_0} = A, \quad \text{or alternatively, } U_0 = A^o$$

Then  $U_0 \in \mathcal{T}_X$ ,  $\overline{U_0} = A \subseteq B^c$ . It is at this point that we must mention that  $0, 1 \notin \Delta$ , so  $U_0$  and  $U_1$  do not have to obey the rules we have laid out for  $U_{r \in \Delta}$ .

Now if  $k+1$  is equal to  $2^N$  (this makes  $r = (k+1)/2^N = 1$ ) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every  $0 \leq m \leq 2^N - 1$ ,  $U_{m/2^N}$  must satisfy

$$\overline{U_{m/2^N}} \subseteq B^c = U_1$$

And the pair  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (even for when  $N = 1$ , since  $A = \overline{U_0} \subseteq U_1 = B^c$ ) and a corresponding  $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$  such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when  $N = 1$ . We would only have to construct for  $k = 1$ , since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■



**Theorem 4.15****Proposition 15.1**

Urysohn's Lemma. Let  $X$  be a normal space, if  $A$  and  $B$  are disjoint closed subsets of  $X$ , then there exists a  $f \in C(X, [0, 1])$  such that  $f = 0$  on  $A$  and  $f = 1$  on  $B$ .

*Proof.* Let  $r \in \Delta$  be as in Lemma 4.14, and set  $U_r$  accordingly except for  $U_1 = X$ . Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write  $W = \{k : x \in U_k\}$ . Then for every  $x \in A$  we have  $f(x) = 0$ , since by the construction of the 'onion' function in Lemma 4.14, for each  $r \in \Delta \cap (0, 1)$ ,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since  $r > 0$  is arbitrary, and  $0 \in W$ , we can use a classic  $\varepsilon$  argument. If  $f(x) > 0$  then there exists some  $0 < r < f(x)$  by density of the dyadic rationals on the line, if  $f(x) < 0$  then this implies that there exists some  $f(x) < r < 0$  such that  $x \in U_r$ , but no  $r \in \Delta$  can be negative, hence  $f(x) = 0$ .

Now, for every  $x \in B$ , since  $A$  and  $B$  are disjoint, and  $A \subseteq U_r \subseteq B^c$ , then for every  $x \in B$  means that  $x$  is not a member of any  $U_r$ , but we set  $U_1 = X$ . Since none of the  $r \in (0, 1)$  is a member of the set we are taking the infimum, and  $x \in U_1 = X$ . The  $\varepsilon$  argument follows: suppose for every  $\varepsilon > 0$ ,  $(1 - \varepsilon) \notin W$ , and  $1 \in W$ , then  $f(x) = 1$ .

Since  $x \in U_1 = X$ , for every  $x \in X$ ,  $f(x) \leq 1$ , and  $f(x)$  cannot be negative as  $r > 0$  for every  $r \in \Delta$ . So  $0 \leq f(x) \leq 1$ . Now we have to show that this  $f(x)$  is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in  $X$ . So  $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$  and  $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$ .

Suppose that  $f(x) < \alpha$ , so  $\inf W < \alpha$ , and using the density of  $\Delta$ , there exists an  $r$ ,  $f(x) < r < \alpha$  such that  $x \in U_r$  such that  $x \in \bigcup_{r < \alpha} U_r$ . So  $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r < \alpha} U_r$ .

Fix an element  $x \in \bigcup_{r < \alpha} U_r$ , this induces an  $r$  such that  $\inf W \leq r < \alpha$  therefore  $f(x) < \alpha$ , and  $\bigcup_{r < \alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$ .

For the second case, suppose that  $f(x) > \alpha$ , then  $\inf W > \alpha$ , and there exists an  $r$  (by density) such that  $\inf W > r > \alpha$  such that for every  $k \in W$ ,  $k \neq r$ . Therefore  $x \notin U_r$ , but by density again, and using the property of the onion function: for every  $s < r$  we get  $\overline{U_s} \subseteq U_r$ , taking complements (which reverses the estimate) — we have  $x \notin \overline{U_s}$ , but  $(\overline{U_s})^c$  is open in  $X$ . It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq (\overline{U_s})^c \subseteq \bigcup_{s > \alpha} (\overline{U_s})^c$$

So  $f^{-1}((\alpha, +\infty))$  is a subset of  $\bigcup_{s > \alpha} (\overline{U_s})^c$ . To show the reverse, fix an element  $x$  in the union, then this induces some  $x \in (\overline{U_s})^c \subseteq (U_s)^c$ . Then for this  $s > \alpha$ ,  $(-\infty, s)$

contains no elements of  $W$ . This is because for every  $p < s$  implies that  $(U_s)^c \subseteq (U_p)^c$ , so  $p \notin W$ . Our chosen  $s$  is a lower bound for  $W$ , and  $\alpha < s \leq \inf W = f(x)$ .

Since all of the inverse images from the generating set of  $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$  are open in  $X$ , using Theorem 4.9 finishes the proof. ■

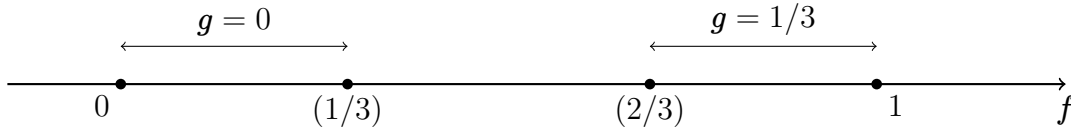


Figure 1: Lemma 16.1 for Theorem 4.16: Separate the range of  $f \in C(A, [0, 1])$  into three parts. Subtract an additional  $g$  that reduces the error even further.

## Theorem 4.16

### Proposition 16.1

The Tietze's Extension Theorem. Let  $X$  be a normal space, and for any closed subset  $A \subseteq X$ , and  $f \in C(A, [a, b])$ , there exists an  $F \in C(X, [a, b])$  which extends  $f$ .

*Proof.* We begin with an important lemma that will serve as a 'black box' for the induction.

### Lemma 16.1

For every  $f \in C(A, [0, 1])$ , there exists a  $g \in C(X, [0, 1/3])$  such that

$$0 \leq f - g \leq 2/3 \quad \text{pointwise on } A \quad (8)$$

*Proof.* Since  $f$  is continuous,  $B = f^{-1}([0, 1/3])$ , and  $C = f^{-1}([2/3, 1])$  are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function  $g \in C(X, [0, 1])$  such that  $g|_B = 0$  and  $g|_C = 1$ . Rescale  $g$  by a factor of  $1/3$ , and  $g \in C(X, [0, 1/3])$ .

To show that Equation (8) holds, suppose  $x \in B$ , then  $f(x) \in [0, 1/3]$  and  $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$ . Now suppose that  $x \in C$ , then  $f(x) \in [2/3, 1]$  and  $g(x) = 1/3$  (recall that we relabelled  $g$ ). So we have  $0 \leq 1/3 \leq f - g \leq 2/3$ . Lastly, for the case where  $x \notin (B \cup C)$ , then  $f(x) \in (1/3, 2/3)$ , and  $g(x) \in [0, 1/3]$  implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 &\implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 &\implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore  $0 \leq f(x) - g(x) \leq 2/3$ . See Figure 1. ■

We can assume that  $f \in C(A, [0, 1])$ , since we can relabel  $f = (f - a)/(b - a)$ . The main part of this proof consists of constructing a sequence of  $\{g_n\} \subseteq C(X, \mathbb{R})$  where  $0 \leq g_n \leq (2/3)^n(1/2)$ , and  $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$  on  $A$ . Let us begin with the base case with  $n = 1$ . We can apply Lemma 16.1 to get  $g_1 \in C(X, [0, 1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that  $\{g_j\}_{j \leq n}$  has been chosen, we will find our  $g_{n+1}$  by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by  $(2/3)^{-n}$  and we obtain a new function in  $C(A, [0, 1])$ .

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) \left( \frac{3}{2} \right)^n \leq 1$$

Applying Lemma 16.1, we get a function  $h \in C(X, [0, 1/3])$  that reduces the error between  $f$  and the partial sums of  $g_{j \leq n-1}$ . For every  $x \in A$

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) \left( \frac{3}{2} \right)^n - h \leq 2/3$$

Multiplying across gives

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) - h \left( \frac{2}{3} \right)^n \leq \left( \frac{2}{3} \right)^{n+1}$$

Set  $g_{n+1} = h \left( \frac{2}{3} \right)^n$  and  $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$ . Furthermore, the sum of all  $g_j$  pointwise converges uniformly, as

$$\sum_{j \geq 1} \|g_j\|_u \leq \sum_{j \geq 1} \left( \frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum  $F = \sum g_j$ , then this  $F \in BC(X)$  (by Theorem 4.13 and 5.1). And

$$\left\| f - \sum_{j \leq n} g_j \right\|_u \leq \left( \frac{2}{3} \right)^n \rightarrow 0$$

So  $F = f$  on  $A$ , now if we want to obtain our  $F$  on  $[a, b]$  we simply relabel  $F = F(b-a) + a$ . This finishes the proof.  $\blacksquare$

**Theorem 4.17****Proposition 17.1**

If  $X$  is a normal space, and  $A$  is a closed subspace of  $X$ , and  $f \in C(A)$ , then there exists an  $F \in C(X)$  such that  $F$  extends  $f$ .

*Proof.* First we suppose that  $f$  is real valued, so  $f \in C(X, \mathbb{R})$ . And define a  $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$ , using

$$g = \frac{f}{1 + |f|}$$

Since  $g$  satisfies the assumption of Theorem 4.16 (note that we do not require  $g$  to be injective), there exists a  $G \in C(X, [-1, +1])$  such that  $G|_A = g$ . Since the set  $\{-1, +1\}$  is closed in  $\mathbb{R}$ ,  $G^{-1}(\{-1, +1\})$  is closed as well. Since  $G^{-1}((-1, +1)) \subseteq A$ , this makes  $A$  and  $B = X \setminus G^{-1}((-1, +1))$  disjoint closed sets in  $X$ .

By Urysohn's Lemma, there exists a continuous function  $h \in C(X, [0, 1])$  such that  $h|_B = 0$  and  $h|_A = 1$ , so that the product  $|hG| < 1$  for all  $x \in X$ . We can think of this  $h$  as a continuous indicator function that filters out the parts we do not want, namely  $G^{-1}\{-1, +1\}$ . Now define  $F$  in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that  $F|_A = g/(1 - |g|) = f$  indeed. Since  $|g| = \frac{|f|}{1+|f|}$ , and  $g(1 + |f|) = f$  implies that  $g/(1 - |g|) = f$ , because  $g \in C(A, (-1, +1))$ . This completes the proof for any  $f \in \mathbb{R}$  if  $f \in C(A)$ , then

1.  $\operatorname{Re}(f) = f_1 \in C(A, \mathbb{R})$
2.  $\operatorname{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in  $C(X, \mathbb{R})$  that extends  $f_1$  and  $f_2$ , and  $F_1 + iF_2 = f$  on  $A$  and  $F_1 + iF_2 \in C(X)$ , and the proof is complete. ■

**Theorem 4.18****Proposition 18.1**

If  $X$  is a topological space, and  $E \subseteq X$  and  $x \in X$ , then  $x \in \text{acc } E \iff$  there exists a net in  $E \setminus \{x\}$  that converges to  $x$ , and  $x \in \overline{E} \iff$  there exists a net in  $E$  that converges to  $x$ .

*Proof.* Suppose that  $x \in \text{acc } E$ , then for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $E \cap U \setminus \{x\} \neq \emptyset$ , then choose  $\mathcal{N}(x)$  as the set of neighbourhoods directed by reverse inclusion (and this makes  $(\mathcal{N}(x), \lesssim)$  a directed set), and we will define the net as follows.

Map each  $U \in \mathcal{N}(x)$  to some  $x_U \in E \cap U \setminus \{x\}$ , then this net converges to  $x$ . Suppose that we fix a neighbourhood,  $V \in \mathcal{N}(x)$ , then for every  $U \gtrsim V$  we have  $x_U \in U \subseteq V$ . So  $\langle x_U \rangle$  is eventually in  $V$ .

Conversely, if  $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$ , and  $x_\alpha \rightarrow x$ , then every  $U \in \mathcal{N}(x)$  there exists a  $x_\alpha \in E \cap U \setminus \{x\}$  that makes

$$E \cap U \neq \emptyset \quad \forall U \in \mathcal{N}(x)$$

Hence  $x \in \text{acc } E$ .

Now for the second part of the Theorem, suppose that  $x \in \overline{E}$ , if  $x \notin E$  then  $E = E \setminus \{x\}$  and  $x \in \text{acc } E$ , so there exists a net in  $E \setminus \{x\} \subseteq E$  such that  $x_\alpha \rightarrow x$ . If  $x \in E$  then simply choose  $\langle x_\alpha \rangle = x$  for every  $\alpha \in A$ .

Now, suppose that there is a net that converges to  $x$ , and this net  $\langle x_\alpha \rangle \subseteq E$ , if  $x \in E$  then there is nothing to prove, since  $E \subseteq \overline{E}$ , so suppose that  $x \notin E$ , then there exists a net in  $E \setminus \{x\} = E$  such that

$$x_\alpha \rightarrow x \implies x \in \text{acc } E \subseteq \overline{E}$$

■

**Theorem 4.19****Proposition 19.1**

Let  $X$  and  $Y$  be topological spaces, then every  $f : X \rightarrow Y$  is continuous at a point  $x \in X \iff$  every net  $\langle x_\alpha \rangle$  that converges to  $x$  implies that  $\langle f(x_\alpha) \rangle$  converges to  $f(x)$ .

*Proof.* If  $f$  is continuous at a point  $x \in X$ , then  $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$ , then for every net  $\langle x_\alpha \rangle$  that converges to this  $x$ , there exists an  $\alpha_0$  such that for every  $\alpha \succ \alpha_0$  implies that  $x_\alpha \in f^{-1}(V)$ . Hence

$$f(x_\alpha) \in f(f^{-1}(V)) \subseteq V$$

And this is equivalent to saying that for every  $V \in \mathcal{N}(f(x))$ ,  $\langle f(x_\alpha) \rangle$  is eventually in  $V$ , and this proves convergence.

Now suppose that  $f$  is not continuous at some  $x$ , then there exists a  $V \in \mathcal{N}(f(x))$  such that  $f^{-1}(V) \notin \mathcal{N}(x)$ , so

$$x \notin (f^{-1}(V))^o \implies x \in (f^{-1}(V))^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our  $x \in \overline{f^{-1}(V^c)}$  induces a net  $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$  that converges to  $x$ . But every element in the net is contained within  $f^{-1}(V^c)$ , and for every  $\alpha \in A$

$$f(x_\alpha) \in f(f^{-1}(V^c)) \subseteq V^c$$

gives  $f(x_\alpha) \notin V$ , but  $V$  is a neighbourhood of  $f(x)$ , hence there exists some  $x_\alpha \rightarrow x$  and  $f(x_\alpha) \not\rightarrow f(x)$ . ■

**Theorem 4.20****Proposition 20.1**

If  $\langle x_\alpha \rangle$  is a net in  $X$ , and  $x \in X$  is a cluster point of  $\langle x_\alpha \rangle \iff$  there exists a subnet of  $\langle x_\alpha \rangle$  that converges to  $x$ .

*Proof.* Suppose that  $\langle y_\beta \rangle_{\beta \in B}$  is a subnet of  $\langle x_\alpha \rangle$  that converges to  $x$ , then for every neighbourhood  $U \in \mathcal{N}(x)$ , there exists a  $\beta_1$  such that for every  $\beta \succeq \beta_1$  we get  $y_\beta = x_{\alpha_\beta} \in U$ .

Furthermore, let us fix a  $\alpha_0 \in A$  to attempt to show that  $\langle x_\alpha \rangle$  is frequently in  $U$ , then by the subnet property of  $\langle y_\beta \rangle$ , there exists some  $\beta_2 \in B$  such that for every  $\beta \succeq \beta_2$ ,  $\alpha_\beta \succeq \alpha_0$ . (Intuitively this property means that the directed set of  $B$  'grows' as much as the directed set of  $A$ , so we can always find elements that are greater than any fixed  $\alpha_0$ .)

Since  $\langle y_\beta \rangle$  is a net, we there exists some  $\beta \in B$  such that  $\beta \succeq \beta_1$  and  $\beta \succeq \beta_2$ , we then apply the  $\beta \mapsto \alpha_\beta$  map and we obtain some  $\alpha = \alpha_\beta$  that satisfies:

- $\alpha = \alpha_\beta \succeq \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that  $\beta \succeq \beta_1$  so that  $y_\beta$  falls into  $U$ .

Conversely, suppose that  $x$  is a cluster point of  $\langle x_\alpha \rangle$ , then by definition

$$\forall U \in \mathcal{N}(x), \forall \alpha_0 \in A, \exists \alpha \succeq \alpha_0, x_\alpha \in U$$

Denote the directed neighbourhoods of  $x$  by  $\mathcal{N}(x)$ , and construct our directed set  $B$  for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every  $(U, \gamma) \in B$  we can map it to some  $\alpha_{(U, \gamma)} \in A$ , if we choose some  $\alpha_{(U, \gamma)} \succeq \gamma$  and  $\alpha_{(U, \gamma)} \in U$ .

To show that  $B$  is a directed set, we say that  $(U, \gamma) \succeq (U', \gamma')$  if and only if  $U \subseteq U'$  and  $\gamma \succeq \gamma'$ . And to show that  $\langle y_\beta \rangle = \langle x_{\alpha_{(U, \gamma)}} \rangle$  is indeed a subnet of  $\langle x_\alpha \rangle$ , fix any  $\alpha_0 \in A$ , then simply take any neighbourhood  $U$  of  $x$  (we always have  $x \in \mathcal{N}(x)$ ) — and therefore  $(U, \alpha_0) \in B$ .

Now for every  $(U', \alpha'_0) \succeq (U, \alpha_0)$  implies that  $\alpha'_0 \succeq \alpha_0$ , therefore we have

$$\alpha_{(U', \alpha'_0)} \succeq \alpha'_0 \succeq \alpha_0$$



And this satisfies the subnet property. Now to show that  $\langle y_\beta \rangle$  indeed converges to  $x$ , fix any  $V \in \mathcal{N}(x)$ , then with any  $\alpha_0 \in A$ , and for every  $(V', \alpha'_0) \succeq (V, \alpha_0) \in B$ , we have

$$x_{\alpha_{(V', \alpha'_0)}} \in V' \subseteq V$$

So  $\langle x_{\alpha_{(U, \gamma)}} \rangle$  converges to  $x$ . ■

**Theorem 4.21****Proposition 21.1**

A topological space  $X$  is compact  $\iff$  every family of closed sets,  $\{F_\alpha\}_{\alpha \in A}$  that has the finite intersection property, implies that

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

*Proof.* We first examine the assertion, Theorem 4.21 proposes for any family of closed sets  $\{F_\alpha\}_{\alpha \in A}$ , and for every finite subset  $B \subseteq A$  then,

$$\bigcap_{\alpha \in B} F_\alpha \neq \emptyset \implies \bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha \in A} F_\alpha = \emptyset \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha \in B} F_\alpha = \emptyset$$

Applying DeMorgan's theorem, and since every  $\{F_\alpha\}_{\alpha \in A}$  induces a family of open sets (and vice versa), where  $U_\alpha = F_\alpha^c$ , so for any family of open sets  $\{U_\alpha\}_{\alpha \in A}$  we have

$$\bigcup_{\alpha \in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha \in B} U_\alpha = X$$

Which is equivalent to saying that  $X$  is compact. ■

**Theorem 4.22****Proposition 22.1**

A closed subset of a compact space  $X$  is compact.

*Proof.* Suppose  $F \subseteq X$  and  $F$  is open, then fix an open cover for  $F$ , so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since  $F^c$  is an open set, we can obtain a valid open cover for  $X$ , then we pick out a finite subcover, for some finite  $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left( \bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with  $F$  on both sides yields

$$\begin{aligned} F &= X \cap F \subseteq (F^c \cap F) \cup \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \\ F &= \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\ F &\subseteq \bigcup_{\alpha \in B} U_\alpha \end{aligned}$$

Therefore every open cover of  $F$  has a finite subcover, and  $F$  is compact. ■

**Theorem 4.23****Proposition 23.1**

If  $F$  is a compact subset of a Hausdorff space  $X$ , and  $x \notin F$ , there are disjoint open sets  $U, V$  such that  $x \in U$  and  $F \subseteq V$ .

*Proof.* Since  $x \in F^c$ , for every  $y \in F$ ,  $x \neq y$  induces two sets  $U_y, V_y$  (because  $X$  is  $T_2$ ).

- $U_y \cap V_y = \emptyset$
- $x \in U_y$
- $y \in V_y$

But  $\{V_y\}_{y \in F}$  is an open cover for the compact set  $F$ , then there exists a finite subcollection  $H \subseteq F$  such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since  $H$  is finite,  $U = \bigcap_{y \in H} U_y$  is an open set that contains  $x$ , also define  $V = \bigcup_{y \in H} V_y$ . If for every  $y \in H$ ,  $U_y \cap V_y = \emptyset$ , then  $U \cap V_y = U \cap V = \emptyset$ . This completes the proof. ■

**Remark 23.1**

Every metric space  $(X, d)$  is first countable, and  $T_2$  (it is actually  $T_4$ , but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element  $x \in X$  and we notice that  $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$  is a countable neighbourhood base for every  $x$ . To show that  $(X, d)$  is  $T_2$ , for every pair of elements  $x \neq y$ , we can take  $r = d(x, y)/2$  and there exists disjoint open sets  $V_r(x)$  and  $V_r(y)$  such that  $x \in V_r(x)$  and  $y \in V_r(y)$ .

**Theorem 4.24****Proposition 24.1**

Every compact subset of a Hausdorff ( $T_2$ ) space is closed.

*Proof.* If  $F$  is compact, then for every  $x \in F^c$ , by Theorem 4.23, there exists two disjoint open sets such that  $x \in U$  and  $F \subseteq V$ , but

$$U \cap V = \emptyset \implies U \cap F = \emptyset \implies U \subseteq F^c$$

But since  $x \in F^c$  is arbitrary, and  $U$  is an open subset of  $F^c$ ,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that  $F^c$  is open and  $F$  is closed. ■

**Theorem 4.25****Proposition 25.1**

Every compact Hausdorff  $(T_2)$  space is normal  $(T_4)$ .

*Proof.* Fix  $A, B$  which are disjoint closed subsets of  $X$ , by Theorem 4.22, we know that these two sets are compact. Hence for every  $y \in B$  there exists two disjoint open sets  $U, V_y$  (by Theorem 4.23)

$A \subseteq U_y$  and  $y \in V_y$ . But the family  $\{V_y\}_{y \in B}$  is a valid open cover for the compact set  $B$ , hence there exists a finite subcollection  $H \subseteq B$  such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \emptyset$$

The second equality holds for every  $y \in H$  so that  $U_y \cap (\bigcup V_{y \in H}) = \emptyset$ . Define  $U = \bigcap U_{y \in H}$  and  $V = \bigcup V_{y \in H}$ , where both of these are disjoint open sets that contain  $A$  and  $B$  as subsets, since for each  $y \in H$ ,  $A \subseteq U_y$  hence the intersection of all  $U_y$  also contains  $A$  as a subset. Therefore  $X$  is normal. ■

**Theorem 4.26****Proposition 26.1**

If  $X$  is compact, and  $f : X \rightarrow Y$  is continuous, then  $f(X)$  is compact.

A small lemma.

**Lemma 26.1**

For every  $\{E_j\} \subseteq X$ ,  $f(\cup E_j) = \cup f(E_j)$ .

The proof is trivial.

*Proof.* If  $\{V_{\alpha \in A}\}$  is an open cover for  $f(X)$ , then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha \in A} V_{\alpha}\right) = \bigcup_{\alpha \in A} f^{-1}(V_{\alpha}) \subseteq X$$

Since  $f$  is continuous, we have an open cover in the form of  $\{f^{-1}(V_{\alpha})\}$  for  $X$ , then there exists a finite subset  $B \subset A$  such that

$$X \subseteq \bigcup_{\alpha \in B} f^{-1}(V_{\alpha})$$

Then we wish to show that for this  $B \subseteq A$ ,  $\{V_{\alpha \in B}\}$  is a finite open cover for  $f(X)$ . Fix any element  $y \in f(X)$ , then this induces a  $x \in X$  such that  $y = f(x)$ , but because  $\{f^{-1}(V_{\alpha \in B})\}$  is an open cover for  $X$ , there exists some  $\alpha \in B$  such that  $x \in f^{-1}(V_{\alpha})$ , hence by definition of the inverse image

$$f(x) \in V_{\alpha} \implies f(X) \subseteq \bigcup_{\alpha \in B} V_{\alpha}$$

Therefore  $f(X)$  is compact and this completes the proof. ■

**Theorem 4.27****Proposition 27.1**

If  $X$  is compact, then  $C(X) = BC(X)$ .

*Proof.* Notice that  $BC(X) \subseteq C(X)$ , so we only have to show the reverse estimate. Fix any  $f \in C(X)$ , since  $X$  is compact, by Theorem 4.26 we know that  $f(X)$  is also compact. Since  $\mathbb{C} = \mathbb{R}^2$  is a complete metric space,  $f(X)$  is bounded and  $f \in BC(X)$ . ■



**Theorem 4.28****Proposition 28.1**

If  $X$  is compact, and if  $Y$  is Hausdorff, then any continuous bijection  $f : X \rightarrow Y$  is a homeomorphism.

*Proof.* If  $E \subset X$  is closed, then since  $X$  is compact,  $E$  is compact as well. By continuity of  $f$ ,  $f(E)$  is a compact set in  $Y$ , but compact subsets of  $Y$  are closed, so  $f$  is continuous.

We used the fact that the inverse of  $f^{-1}$  is  $f$ , since it suffices to check that every inverse image of a closed set is also closed,  $f^{-1}$  is continuous. And by definition of a homeomorphism ( $f$  has to be bijective and both  $f$  and  $f^{-1}$  have to be continuous),  $f$  is a homeomorphism. ■

**Theorem 4.29****Proposition 29.1**

If  $X$  is any topological space, the following are equivalent.

- (a)  $X$  is compact.
- (b) Every net has a cluster point.
- (c) Every net in  $X$  has a convergent subnet.

*Proof.* By Theorem 4.20, every net in  $X$  has a cluster point  $\iff$  there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose **a**) holds, then  $X$  is compact, and fix an arbitrary net  $\langle x_\alpha \rangle$  in  $X$ . and define the 'tail' of the net

$$E_\alpha \triangleq \{x_\beta, \beta \succeq \alpha\}$$

We wish to show that the arbitrary intersection of  $\bigcap_{\alpha \in A} \overline{E_\alpha} \neq \emptyset$ . Where  $\overline{E_\alpha}$  is closed, so it suffices to check that every finite  $B \subseteq A$ , the intersection over  $\overline{E_\alpha}$  is non-empty.

Suppose we are given a finite  $B \subseteq A$ , then fix any two elements  $\alpha$  and  $\beta \in B$ , by the definition of a net there exists a  $\gamma \in A$  such that  $\gamma \succeq \alpha$  and  $\gamma \succeq \beta$ , and

$$\emptyset \neq E_\alpha \cap E_\beta \implies \overline{E_\alpha} \cap \overline{E_\beta} \neq \emptyset$$

Therefore for any finite collection of  $\{\overline{E_{\alpha \in B}}\}$ , then

$$\bigcap_{\alpha \in A} \overline{E_\alpha} \neq \emptyset$$

Now fix an element  $x \in \bigcap_{\alpha \in A} \overline{E_\alpha}$ . Then for every  $\alpha \in A$ ,  $x \in \overline{E_\alpha}$ , and for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $U \cap E_\alpha \neq \emptyset$ . This is because if  $x \in E_\alpha$ , then  $U \cap E_\alpha$  contains at least  $\{x\}$ , if  $x \in \text{acc } E_\alpha$ , then by definition of an accumulation point,  $U \cap E_\alpha \setminus \{x\} \neq \emptyset$ , so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net,  $E_\alpha$ , if for every  $\alpha \in A$ ,  $x \in E_\alpha$  if and only if there exists some  $\gamma \succeq \alpha$ ,  $x_\gamma \in U \cap E_\alpha$ , this is equivalent to saying that  $x$  is a cluster point of  $\langle x_\alpha \rangle$ . So **a**)  $\implies$  **b**).

Now let us suppose that  $X$  is not compact, then there exists an open cover  $\{U_{\alpha \in A}\}$  of  $X$  that has no finite subcover. Let  $\mathbb{B}$  be the collection of all finite subsets of  $A$ , directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every  $B \in \mathbb{B}$ , find some  $x_B \in (\bigcup_{\alpha \in B} U_\alpha)^c$ . So we have a net in  $X$ . Now we will show that no  $x \in X$  can be a cluster point of this net. Suppose not, then take a

neighbourhood  $U_\beta$  with  $\beta \in A$  such that  $U_\beta$  belongs to the open cover we first discussed. Then for any  $B \in \mathbb{B}$  such that  $B \gtrsim \{\beta\}$  (meaning that  $\{\beta\} \subseteq B$ , where  $B$  is a finite set), then

$$x_B \in \left( \bigcup_{\alpha \in B} U_\alpha \right)^c \implies x_B \notin \left( \bigcup_{\alpha \in \{\beta\}} U_\alpha \right) \implies x_B \in U_\beta^c$$

Hence no point in  $X$  can be a cluster point for this net, and the proof is complete. ■

**Theorem 4.30****Proposition 30.1**

If  $X$  is a LCH space, and for every  $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$ , there exists a compact  $N \subseteq U$  where  $N \in \mathcal{N}_B(x)$ .

*Proof.* For every  $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$ , we can find an  $E$  open subset of  $U$  that has a compact closure, since every  $x \in X$  induces some compact  $F \in \mathcal{N}_B(x)$ , therefore

$$E \triangleq U \cap F^\circ \implies \overline{E} \subseteq F$$

Since closed subsets of compact sets are compact (by Theorem 4.22),  $\overline{E}$  is compact. More is true, since  $E$  is open,

$$x \in U \cap F^\circ \implies x \in E^\circ \implies E \in \mathcal{N}_B(x)$$

Now it suffices to show that there exists some compact  $N \subseteq E \subseteq U$  such that  $N \in \mathcal{N}_B(x)$ . Since  $\overline{E}$  is compact, the closed subset  $\partial E = \overline{E} \cap \overline{E}^c$  of  $\overline{E}$  is also compact.

Since  $\partial E \cap E^\circ = \emptyset$ ,  $x \in E^\circ = E$  means that  $x \notin \partial E$ . Applying Theorem 4.23 to the compact set  $\partial E$  and  $x \notin \partial E$  gives us two disjoint open sets  $V'$  and  $W'$ . We list their properties

1.  $V', W' \in \mathcal{T}_X$
2.  $x \in V'$
3.  $\partial E \subseteq W'$
4.  $V' \cap W' = \emptyset$

The two disjoint pairs induce another pair of open sets relative to  $\overline{E}$ , recall the definition of the topology relative to  $\overline{E}$ ,

$$\mathcal{T}_{\overline{E}} = \{A \cap \overline{E} : A \in \mathcal{T}_X\}$$

We now agree to define

- $V = V' \cap \overline{E}$
- $W = W' \cap \overline{E}$

Then evidently  $V, W \in \mathcal{T}_{\overline{E}}$  and

1.  $x \in V' \cap \overline{E} \implies x \in V$
2.  $\partial E \subseteq \overline{E} \implies \partial E \subseteq W$

$$3. V' \cap W' = \emptyset \implies V \cap W = \emptyset$$

Furthermore,

$$\partial E \subseteq W \implies W^c \subseteq (\partial E)^c = E^\circ \cup E^{\circ\circ}$$

Taking the intersection over  $\overline{E}$  gives us

$$\overline{E} \setminus W \subseteq \overline{E} \cap (E^\circ \cup E^{\circ\circ})$$

Note that  $E^{\circ\circ} = (\overline{E})^c$ , since  $(E^c)^{\circ\circ} = \overline{(E^c)} = \overline{E}$  therefore  $\overline{E} \cap E^{\circ\circ} = \emptyset$ , hence

$$\overline{E} \setminus W \subseteq \overline{E} \cap E^\circ = E^\circ$$

Using the fact from 3,  $V \subseteq W^c$  and  $V \subseteq \overline{E}$  and  $V \subseteq W^c$  implies that  $V \subseteq \overline{E} \setminus W$ . Compiling everything, we have

$$V \subseteq \overline{E} \setminus W \subseteq E$$

Note that the set  $\overline{E} \setminus W$  is closed in  $\mathcal{T}_X$  (and hence closed in  $\overline{E}$ ) by closure over intersections,  $\overline{V}$  is therefore a closed subset of  $\overline{E} \setminus W$ , and  $\overline{V}$  is compact. Also

$$\overline{V} \subseteq \overline{E} \setminus W \subseteq E$$

To check that  $\overline{V} \in \mathcal{N}_B(x)$ , note that

$$x \in V^\circ \subseteq (\overline{V})^\circ \implies \overline{V} \in \mathcal{N}_B(x)$$

The subset relation  $V^\circ \subseteq \overline{V}^\circ$  comes from the fact that  $V^\circ$  is an open subset of  $\overline{V}$ , and hence is contained in  $(\overline{V})^\circ$  as a subset. Now let us define  $N = \overline{V}$ , and  $N$  satisfies the assertions in the Theorem, since

- $N \in \mathcal{N}_B(x)$
- $N$  is compact
- $N \subseteq E \subseteq U$

And this completes the proof. ■

#### Remark 30.1

Intuitively speaking, this means that if  $X$  is any LCH space, then for every open neighbourhood  $U \in \mathcal{N}_B(x)$ , there exists a compact  $E \in \mathcal{N}_B(x)$  such that  $x \in E \subseteq U^\circ$ . This property is indeed a very strong one as it allows us to have effectively 'infinite' descending compact neighbourhoods of  $x$ .

**Theorem 4.31****Proposition 31.1**

$X$  is a LCH space, and  $K \subseteq U \subseteq X$  where  $K$  is compact, and  $U$  is open, then there exists some precompact, open  $V$  with

$$K \subseteq V \subseteq \bar{V} \subseteq U$$

*Proof.* For every  $x \in K$ , we can apply Proposition 4.30, since  $x \in K \subseteq U$ , this induces some compact  $F_x \subseteq U$  where  $F_x \in \mathcal{N}_B(x)$ . Then we can obtain an open cover of  $U$  in the form of  $\{F_x^o\}_{x \in K}$ . By compactness of  $K$ , there exists a finite  $B \subseteq K$  such that

$$K \subseteq \bigcup_{x \in B} F_x^o$$

Let  $V = \bigcup_{x \in B} F_x^o$ , then clearly  $V$  is open, and  $K \subseteq V$ . Since each  $F_x$  is closed (compact sets are closed in any Hausdorff Space), we have

$$V \subseteq \bigcup_{x \in B} F_x \implies \bar{V} \subseteq \bigcup_{x \in B} F_x$$

Since  $\bigcup_{x \in B} F_x$  is a finite union of compact sets, we claim that it is also compact. Consider two compact sets  $E_1$  and  $E_2$ , then if  $\{U_\alpha\}_{\alpha \in A}$  is any open cover of  $E_1 \cup E_2$ , it must be an open cover for  $E_1$  and  $E_2$  as well, because

$$E_1, E_2 \subseteq E_1 \cup E_2 \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since  $E_1$  and  $E_2$  are both compact sets, they each induce two finite subsets of  $B_1, B_2$  of  $A$  whose union  $B = B_1 \cup B_2$  is also compact. Therefore

$$E_1 \cup E_2 \subseteq \bigcup_{\alpha \in B} U_\alpha$$

Then a simple proof by induction will show that if  $\{E_{j \leq n}\}$  is a family of compact sets, then  $E = \bigcup E_{j \leq n}$  is also compact.

Returning to the main part of the proof,  $\bigcup_{x \in B} F_x$  is a compact set, therefore  $\bar{V}$  is also compact. Moreover

$$\forall x \in K, F_x \subseteq U \implies \bar{V} \subseteq \bigcup_{x \in B} F_x \subseteq U$$

Combining, we have

- $K \subseteq V \subseteq \bar{V}$ ,
- $V$  is open and  $\bar{V}$  is compact, and
- $\bar{V} \subseteq U$

This completes the proof. ■

**Theorem 4.32****Proposition 32.1**

Urysohn's Lemma, Locally Compact Version. For any LCH space  $X$ , and if  $K \subseteq U \subseteq X$  where  $K$  is compact and  $U$  is open, then there exists some  $f \in C(X, [0, 1])$  with

- $f = 1$  on  $K$
- $f = 0$  outside some compact  $\bar{V} \subseteq U$

*Proof.* Let  $V$  be as in Theorem 4.31, for our fixed  $K \subseteq U \subseteq X$ , there exists a pre-compact, open  $V$  that satisfies

$$K \subseteq V \subseteq \bar{V} \subseteq X$$

It follows that this  $(\bar{V}, \mathcal{T}_{\bar{V}})$  is a normal space by Theorem 4.25 (compact Hausdorff spaces are normal), and by Urysohn's Lemma (Theorem 4.15) on normal spaces, since we can easily find two disjoint closed subsets of  $\bar{V}$  in the form of

- $K \subseteq V^o = V \subseteq \bar{V}$  (compact sets in Hausdorff spaces are closed)
- $\partial V = \bar{V} \cap \bar{V}^c$  (closed sets in compact spaces are compact)
- $K \subseteq V^o$  implies that  $K \cap \partial V = K \cap (\bar{V} \setminus V^o) = \emptyset$

Then there exists a continuous  $f|_{\bar{V}} \in C(\bar{V}, [0, 1])$  that evaluates to

- $f|_{\bar{V}} = 1$  on closed  $K$
- $f|_{\bar{V}} = 0$  on closed  $\partial V$

Now let us extend  $f|_{\bar{V}}$  to  $f$  by defining

$$f|_{(\bar{V})^c} = 0$$

We will show that this extension of  $f$  is indeed continuous. Indeed, for every closed set  $E \subseteq [0, 1]$  that does not contain 0, we have:

$$0 \notin E \implies \{0\} \cap E = \emptyset \implies f^{-1}(\{0\}) \cap f^{-1}(E) = \emptyset$$

But  $(\bar{V})^c \subseteq f^{-1}(\{0\})$  therefore

$$(\bar{V})^c \cap f^{-1}(\{0\}) \cap f^{-1}(E) = (\bar{V})^c \cap f^{-1}(E) = \emptyset \implies f^{-1}(E) \subseteq \bar{V}$$

We can write

$$f^{-1}(E) = f|_{\bar{V}}^{-1}(E)$$

But we know that  $f|_{\bar{V}}$  is continuous, so  $f|_{\bar{V}}^{-1}(E)$  must be closed (with respect to  $\bar{V}$ ), and therefore is closed wrt  $X$ , since  $\bar{V}$  is closed wrt  $X$ .

For the case where  $0 \in E$ , note that

$$f^{-1}(E) = (f^{-1}(E) \cap \bar{V}) \cup (f^{-1}(E) \cap (\bar{V})^c) = (f|_{\bar{V}})^{-1}(E) \cup (f|_{\bar{V}^c})^{-1}(E)$$

The above equalities are messy in print. They are but a simple consequence of disjoint decomposition of the pre-images, since

$$\bar{V} \cap f^{-1}(E) = \{x \in \bar{V} : f(x) \in E\} = f|_{\bar{V}}^{-1}(E)$$

Back to our main discussion, recall that for every  $x \in \partial V$

$$f(x) = 0 \in f^{-1}(\{0\}) \subseteq f^{-1}|_{\bar{V}}(E)$$

Therefore  $\partial V \subseteq f^{-1}|_{\bar{V}}(E)$ , and  $(\bar{V})^c = f^{-1}|_{(\bar{V})^c}(E)$  gives us (since  $V^c$  is closed),

$$\begin{aligned} f^{-1}(E) &= f^{-1}|_{\bar{V}}(E) \cup \partial V \cup (\bar{V})^c \\ &= f^{-1}|_{\bar{V}}(E) \cup \overline{(V^c)} \cup (\bar{V})^c \\ &= f^{-1}|_{\bar{V}}(E) \cup (V^c \cup V^{\text{co}}) \\ &= f^{-1}|_{\bar{V}}(E) \cup V^c \end{aligned}$$

Since  $f^{-1}|_{\bar{V}}(E)$  and  $V^c$  are closed subsets of  $X$ , then  $f^{-1}(E)$  is also closed, and  $f \in C(X, [0, 1])$ . ■



**Theorem 4.33****Proposition 33.1**

Every LCH space is completely regular (or  $T_{3.5}$ ).

*Proof.* Recall that a space  $X$  is completely regular if it is  $T_1$  and every closed subset  $A$  and every  $x \notin A$  there exists some

$$f \in C(X, [0, 1]), \quad f(x) = 1, \quad f|_A = 0$$

Fix a closed set  $A \subseteq X$ , then for every  $x \in A^c$ , there exists a compact  $E_x \in \mathcal{N}_B(x)$  with  $E_x \subseteq A^c$  (by Theorem 4.30).

Note that  $E_x \subseteq A^c$  where  $E_x$  is compact and  $A^c$  is closed, then an application of Theorem 4.31 tell us that there exists an  $f \in C(X, [0, 1])$  such that for every  $x \in E_x$ ,  $f(x) = 1$  and for points  $y \notin A^c$  (which means that  $y \in A$ ),  $f(y) = 0$ . Therefore  $X$  is completely regular.  $\blacksquare$

**Theorem 4.34**

**Proposition 34.1**

*Proof.*



**Theorem 4.35****Proposition 35.1**

If  $X$  is a LCH space, we claim that

$$\overline{C_c(X)} = C_0(X)$$

*Proof.* We begin by proving several things that are mentioned before this Theorem, namely

$$C_c(X) \subseteq C_0(X) \subseteq BC(X)$$

Fix an  $f \in C_c(X)$ , and for every  $\varepsilon > 0$ ,

$$x \in |f|^{-1}([\varepsilon, +\infty)) \implies |f(x)| \geq \varepsilon > 0$$

Therefore  $|f|^{-1}([\varepsilon, +\infty))$  is a closed subset of  $\text{supp}(f)$ , since  $(-\infty, \varepsilon)$  is open in  $\mathbb{R}$ , then  $[\varepsilon, +\infty)$  is a closed set. And by continuity of  $|\cdot| \circ f$  (a composition of two continuous functions),  $|f|^{-1}([\varepsilon, +\infty))$  is closed. Using the fact that closed subsets of compact  $\text{supp}(f)$  are also compact, we get  $f \in C_0(X)$ .

Next, we show that  $C_0(X) \subseteq BC(X)$ . Fix any element  $f$  of  $C_0(X)$  with an arbitrary  $\varepsilon > 0$ , then  $E_\varepsilon = \{x \in X : |f(x)| \geq \varepsilon\}$  is compact. The continuity of  $f$  guarantees that the direct image of a compact set is another compact set (Theorem 4.26)

$$|f|(E_\varepsilon) \text{ is a compact subset of } \mathbb{R}$$

And therefore for every  $x \in E_\varepsilon \implies |f(x)| \in |f|(E_\varepsilon)$ , then by Heine-Borel, there exists some  $M \geq 0$  such that  $|f(x)| \leq M$ . If  $x \notin E_\varepsilon$ , then by definition of  $E_\varepsilon$ , implies that  $|f(x)| < \varepsilon$ . Then  $|f(x)| \leq M + \varepsilon$  for every  $x \in X$ . Hence  $f \in BC(X)$ .

Here I wish to offer an alternate proof for  $C_0(X) \subseteq BC(X)$ , we begin by constructing an open cover for  $\text{supp}(f)$  such that

$$\{U_n\}_{n>0} = \{x \in X : |f(x)| < n\}$$

Then there exists a finite subcollection of  $\{U_n\}_{n \in B}$  where  $B$  is a finite set, then define  $M = 1 + \sum_{n \in B} n$  and for every  $x \in \text{supp}(f)$  we have  $|f(x)| < n$  and since  $n > 0$  this holds for every  $x \in X$  too. Therefore  $f \in BC(X)$ .

For the main proof of Theorem 4.35, since  $BC(X)$  is endowed with the uniform metric, it is also first countable, and therefore by Theorem 4.6, it suffices to show that every sequence  $\{f_n\}_{n \geq 1} \subseteq C_c(X)$  converges in  $C_0(X)$ . And every element  $f \in C_0(X)$  has a convergence sequence in  $C_c(X)$ .

Fix a convergent sequence  $\{f_n\}_{n \geq 1} \subseteq C_c(X)$  that converges uniformly to some  $f \in BC(X)$  (since  $BC(X)$  is a closed subset of  $C(X)$  with respect to the uniform norm), then for every  $\varepsilon > 0$ , there exists some  $n \geq 1$  with

$$\|f_n - f\|_u < \varepsilon$$

We aim to show that  $(\text{supp}(f_n))^c \subseteq |f|^{-1}((-\infty, \varepsilon))$ , so fix any  $x \notin \text{supp}(f_n)$ , then

$$|f(x) - f_n(x)| = |f(x)| \leq \|f - f_n\|_u < \varepsilon$$

This establishes the estimate, and taking complements

$$|f|^{-1}([\varepsilon, +\infty)) \subseteq \text{supp}(f_n)$$

Therefore for any arbitrary  $\varepsilon > 0$ ,  $\{x \in X, |f(x)| \geq \varepsilon\}$  is compact, and  $\overline{C_c(X)} \subseteq C_0(X)$ . Conversely, fix any  $f \in C_0(X)$ , and for every  $n \geq 1$ , define

$$K_n = \{x \in X, |f(x)| \geq 1/n\}$$

Using Urysohn's Lemma for our LCH space  $X$ , there exists some  $g_n$  that has a compact support, and  $g_n(x) = 1$  for every  $x \in K_n$ . We then write  $f_n = g_n \cdot f \in C_c(X)$ . We wish to show that  $f_n \rightarrow f$  uniformly. Notice that for any fixed  $n \geq 1$ , if  $x \in K_n$  then

$$f_n(x) = f(x) \implies |f_n - f|(x) = 0$$

If  $x \notin K_n$ ,  $|f(x)| < 1/n$  (recall what  $K_n$  does), and  $f_n = g_n \cdot f \in [0, 1]$  by definition of  $g_n$  from Theorem 4.32, hence

$$|f_n(x) - f(x)| = |f(x)| \cdot |1 - g_n| \leq |f(x)| < 1/n$$

Taking the supremum over  $x \in X$ , we have

$$\|f_n - f\|_u < 1/n \rightarrow 0$$

As we send  $n$  to  $+\infty$ , and  $f_n \rightarrow f$  uniformly. This completes the proof. ■

**Theorem 4.36****Proposition 36.1***Proof.*

**Theorem 4.37****Proposition 37.1**

If  $X$  is an LCH space and  $E \subseteq X$ .  $E$  is closed if and only if  $E \cap K$  is closed for every compact  $K \subseteq X$ .

*Proof.* Suppose that  $E$  is closed, then  $E \cap K$  is closed, since compact subsets of Hausdorff spaces are closed, and  $E \cap K \subseteq K$  tells us that  $E \cap K$  is indeed compact.

Now suppose that  $E$  is not closed, by Theorem 4.1,  $E \neq \overline{E}$ , so pick some  $x \in (\overline{E} \setminus E) = \text{acc}(E) \cap E^c$ , since  $X$  is locally compact, let  $K_x$  be a compact neighbourhood of  $x$ , then for every neighbourhood  $U \in \mathcal{N}_B(x)$ , we have

$$x \in U^o, x \in K_x^o \implies x \in (U^o \cap K_x^o) \subseteq (U \cap K_x)^o$$

Since  $(U^o \cap K_x^o)$  is an open subset of  $(U \cap K_x)$ , then  $(U \cap K_x) \in \mathcal{N}_B(x)$ , and recall that  $x \in \text{acc}(E)$ , therefore

$$(U \cap K_x) \cap E \setminus \{x\} = U \cap (K_x \cap E) \neq \emptyset$$

But  $x \notin E \implies x \notin E \cap K_x$ . So  $x$  is an accumulation point of  $E \cap K_x$  that is not in  $E \cap K_x$ . Therefore there exists some  $E \cap K_x$  (with  $K_x$  compact) that is not closed. ■

**Theorem 4.38****Proposition 38.1**

If  $\mathbf{X}$  is an LCH space,  $C(\mathbf{X})$  is a closed subspace of  $\mathbb{C}^{\mathbf{X}}$  in the topology of uniform convergence on compact sets. (We will sometimes refer to this as the topology of compact convergence.)

*Proof.* Let  $E \subseteq \mathbf{X}$  be closed and endowed with the subspace topology.

$$\mathcal{T}_E = \left\{ U \cap E, U \in \mathcal{T}_{\mathbf{X}} \right\}$$

Then  $A \cap E$  is closed relative to  $E$  iff it is closed relative to  $\mathbf{X}$ . The proof for this can be found in the Notes.

Let  $f$  be an adherent point of  $C(\mathbf{X})$  endowed with the topology of compact convergence. If  $W$  is closed in  $\mathbb{C}$ , let  $K$  range through compact sets of  $\mathbf{X}$ .  $f|K$  is in the closure of  $C(K, \mathbb{C})$ , therefore continuous by Proposition 4.13, as  $C_c(K) \subseteq BC(K)$ . So  $f|K$  is continuous, and  $(f|K)^{-1}(W)$  is closed Rel.  $K$ . Notice

$$(f|K)^{-1}(W) = f^{-1}(W) \cap K \tag{9}$$

since we can write  $(f|K)(x) = (f \circ \iota_K)(x)$ . Where  $\iota_K : K \rightarrow \mathbf{X}$  is the inclusion map, which is an embedding. Equation (9) follows immediately. Therefore  $f^{-1}(W) \cap K = (f|K)^{-1}(W)$  is closed Rel.  $K$ , so it is closed Rel.  $\mathbf{X}$ . This holds for every compact  $K$ , so  $f^{-1}(W)$  is closed for any closed  $W \subseteq \mathbb{C}$ , and  $f$  is continuous. ■

**Theorem 4.39**

**Proposition 39.1**

*Proof.*





**Theorem 4.40****Proposition 40.1***Proof.*

**Theorem 4.41****Proposition 41.1***Proof.*

## Exercises

### Exercise 4.1

#### Proposition 1.1

If  $\text{card } \mathbf{X} \geq 2$ , there is a topology on  $\mathbf{X}$  that is  $T_0$  but not  $T_1$ .

*Proof.* Let  $\mathcal{T}_{\mathbf{X}} = \{\emptyset\} \cup \{\{x\} \cup B, B \subseteq \mathbf{X}\}$ , where  $x \in \mathbf{X}$  is any point in  $\mathbf{X}$ . Suppose  $U_1$ , and  $U_2$  are open sets in  $\mathcal{T}_{\mathbf{X}}$ , if either is empty then their intersection must be contained in  $\mathcal{T}_{\mathbf{X}}$ . Otherwise  $U_1 = \{x\} \cup B_1$ , and  $U_2 = \{x\} \cup B_2$ , where  $B_1$  and  $B_2$  are subsets of  $\mathbf{X}$ .

$$U_1 \cap U_2 = \{x\} \cup (B_1 \cap B_2) \in \mathcal{T}_{\mathbf{X}}$$

Notice also  $\{\emptyset, \mathbf{X}\} \subseteq \mathcal{T}_{\mathbf{X}}$ . Fix an arbitrary family of open sets  $\{U_\alpha\}_{\alpha \in A}$ , in similar fashion we have  $\bigcup U_\alpha = \{x\} \cup \left(\bigcup B_{\alpha \in A}\right)$  so their union is contained in  $\mathcal{T}_{\mathbf{X}}$  as well.

This topology is  $T_0$ . Fix  $y \neq z$  in  $\mathbf{X}$ , if either  $y$  or  $z$  is  $x$ , then choosing  $\{x\}$  does the job. So assume  $x \neq y \neq z \neq x$ , and  $\{y\} \cup \{x\}$  is an open set that does not contain  $z$ . This topology cannot be  $T_1$ , as  $x$  sticks onto every open set, so there are no open sets which separate  $x$  from the other points in  $\mathbf{X}$ . ■

## Exercise 4.2

**Proposition 2.1**

If  $\mathbf{X}$  is an infinite set, the cofinite topology on  $\mathbf{X}$  is  $T_1$  but not  $T_2$ , and is first countable iff  $\mathbf{X}$  is countable.

*Proof.* We will first verify that the cofinite topology  $\mathcal{T}_{\mathbf{X}}$  is a topology.

$$\mathcal{T}_{\mathbf{X}} = \left\{ U, \quad U^c \text{ is finite.} \right\} \cup \{\emptyset\}$$

So that  $\{\emptyset, \mathbf{X}\} \subseteq \mathcal{T}_{\mathbf{X}}$ . Let  $U_1$  and  $U_2$  be a pair of open sets, assuming if neither of them are empty, then  $U_2^c$  and  $U_1^c$  are finite sets, so that  $U_1^c \cup U_2^c$  is finite as well. Use DeMorgan to see that  $U_1 \cap U_2 \in \mathcal{T}_{\mathbf{X}}$ .

If  $\{U_\alpha\}_{\alpha \in A}$  is an arbitrary collection of open sets, then

$$\bigcap_{\alpha \in A} U_\alpha^c \subseteq U_\beta^c$$

where  $\beta \in A$  is arbitrary, so  $U_\beta^c$  is finite. And the union  $\bigcup U_\alpha$  is contained in  $\mathcal{T}_{\mathbf{X}}$ .

To show that  $\mathcal{T}_{\mathbf{X}}$  is  $T_1$ , every singleton set is closed. To show that  $\mathcal{T}_{\mathbf{X}}$  is not  $T_2$ , fix  $x \neq y$ . If  $B_x$  and  $B_y$  are open sets that contain  $x$  and  $y$  respectively. If  $B_x$  and  $B_y$  disjoint, then

$$B_x \subseteq B_y^c$$

Which means  $B_x$  is an open, finite subset. But the only open and finite subset of  $\mathbf{X}$  is the empty set. This contradicts  $x \in B_x$ .

If  $\mathbf{X}$  is countable, we will find a neighbourhood base  $\mathcal{N}_B(x)$  for any  $x \in \mathbf{X}$  as follows:

- We can index  $\mathbf{X}$  using  $\mathbb{N}^+ \cup \{0\}$ , so without loss of generality, let  $x_0 = x$ , and
- Define  $U_1 = \{x_1\}^c$ , and  $U_n = \bigcap_{j=1}^n \{x_j\}^c$  are open sets that contain  $x$ . Equivalently,

$$U_n = \left\{ x_j, j \geq n+1 \right\} \cup \{x_0\}$$

- If  $V$  is an open set that contains  $x_0$ , then  $V^c$  is finite, let  $M \in \mathbb{N}^+$  be the largest index of  $x_j \notin V$  (the negation of this is that if  $j \geq M+1$ , then  $x_j \in V$ ) then  $U_{M+1} \subseteq V$  as needed, and  $\mathbf{X}$  is first countable.

Conversely, if  $\mathbf{X}$  is first countable, we can find a descending sequence of neighbourhoods which form a neighbourhood base,  $\{U_j\}_{j \geq 1} \subseteq \mathcal{T}_{\mathbf{X}}$ . And each  $U_j^c$  is finite, so  $\bigcup U_j^c$  is countable. Assume for contradiction that  $\mathbf{X}$  is uncountable, then

$$\bigcup U_j^c = \left( \bigcap U_j \right)^c$$

is countable, hence the intersection  $\bigcap U_j$  must be uncountable (hence infinite). Pick  $y \neq x$ , where  $y$  belongs in the intersection of all neighbourhoods  $U_j$ . This contradicts the fact that  $\{U_j\}$  is a neighbourhood base, as  $x$  is an element in the open set  $\{y\}^c$  therefore there must be a  $U_k$

$$x \in U_k \subseteq \{y\}^c$$

But  $y \in U_k$  for each  $U_k$  and the proof is complete. ■

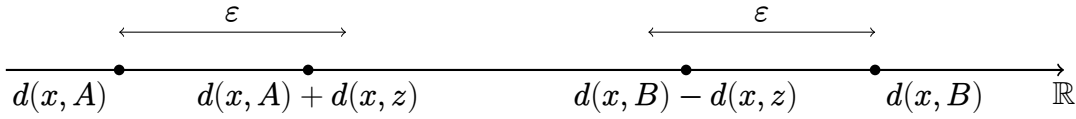


Figure 2: Exercise 4.3: Finding a  $\varepsilon$  small enough that fits within  $d(x, A) < d(x, B)$

### Exercise 4.3

#### Proposition 3.1

Every metric space is normal. (If  $A, B$  are disjoint closed sets in the metric space, consider the set of points  $x$  where  $d(x, A) < d(x, B)$  or  $d(x, A) > d(x, B)$ ).

*Proof.* First, we show that if  $A$  is closed, then  $d(x, A) = 0 \iff x \in A$ . If  $x \in A$ , then  $d(x, A) \leq d(x, x) = 0$ . If  $x \notin A$ , then there exists a ball of radius  $\varepsilon > 0$  where  $B(\varepsilon, x) \cap A = \emptyset$ . Hence,  $\varepsilon$  is a lower bound for the set  $\{d(x, y), y \in A\}$ , taking the infimum over this set we see that  $d(x, A) \geq \varepsilon > 0$ .

Fix some  $x \in \Phi_A$  where  $\Phi_A = \left\{y \in X, d(y, A) < d(y, B)\right\}$ . We wish to find an open ball about  $x$  that is contained in  $\Phi_A$ . The Triangle Inequality works for this definition of distance as well, as

$$f(a) \leq g(a), \forall a \in A \implies \inf_{a \in A} f(a) \leq \inf_{a \in A} g(a) \quad (10)$$

If  $a \in A$ , then  $d(z, A) \leq d(x, z) + d(x, A)$ , using Equation (10) yields

$$\begin{cases} d(z, A) \leq d(x, A) + d(x, z) \\ d(x, B) - d(x, z) \leq d(z, B) \end{cases}$$

where  $z \in B(\varepsilon, x)$  so  $d(x, z) \lesssim \varepsilon$ . The second estimate above is found by 'flipping an upper bound to become a lower bound'. We can choose  $d(x, z)$  sufficiently small that

$$d(x, A) + d(x, z) < d(x, B) - d(x, z)$$

in order to 'pipe' the two inequalities, so

$$2d(x, z) < d(x, B) - d(x, A) \quad (11)$$

Take  $\varepsilon = [d(x, B) - d(x, A)]3^{-1}$ , then  $z \in B(\varepsilon, x)$  implies  $d(x, z) < \varepsilon$ , and Equation (11) holds. See Figure 2 for details. ■

**Exercise 4.4**

**Proposition 4.1**

*Proof.*



## Exercise 4.5

**Proposition 5.1**

Every separable metric space is second countable.

*Proof.* We wish to show that if  $\mathbf{X}$  is a metric space, then

$$\text{second countable} \iff \text{separable}$$

Suppose  $\mathbf{X}$  is separable, where  $A$  is a countable dense subset in  $\mathbf{X}$ , and  $x \in \mathbf{X}$ . Let  $U$  be an open set that contains  $x$ , so  $B(\varepsilon, x) \subseteq U$  for some  $\varepsilon > 0$ .  $B(\varepsilon/2, x)$  is a non-empty open set, therefore contains some  $y \in A$  (this follows from the definition of density). If we choose  $r \in \mathbb{Q}$  wisely,

$$d(x, y) < r < \varepsilon/2$$

So that  $x \in B(r, y)$ , and if  $z \in B(r, y)$ , then

$$d(x, z) \leq d(x, y) + d(z, y) < r + r < \varepsilon$$

So  $x \in B(r, y) \subseteq U$ . But  $\{B(r, y), r \in \mathbb{Q}^+, y \in A\}$  is countable. Therefore  $\mathbf{X}$  is second countable.

Conversely, Proposition 4.5 gives us the  $\Leftarrow$  direction. But we will repeat anyway, if  $\mathbf{X}$  is second-countable with  $\mathcal{E}$  as a countable base, then take

$$W = \left\{ x_\alpha \in U, \quad U \in \mathcal{E} \right\}$$

by picking a point from each set, we claim  $W$  is dense in  $\mathbf{X}$ , so  $\overline{W} = \mathbf{X}$ . If not, then  $\overline{W}^c \neq \emptyset$ , and

$$\overline{W}^c = (W^c)^o \neq \emptyset$$

Pick a point  $x \in W^c$ , which is an open set containing  $x$ . But the way we chose  $W$  does not allow for any open set  $U \in \mathcal{E}$  with  $x \in U \subseteq W^c$ , since

By picking one point from each of the base sets, grouping these points and call it  $W$ , and flipping to the complement. Each  $U \in \mathcal{E}$  admits a point that escapes  $W^c$ . Therefore we can ensure no  $U \in \mathcal{E}$  can be a subset of  $W^c$ .

■



**Exercise 4.6****Proposition 6.1***Proof.*

## Exercise 4.7

**Proposition 7.1**

If  $\mathbf{X}$  is a topological space, a point  $x \in \mathbf{X}$  is called a cluster point of the sequence  $\{x_j\}$  if for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $x_j \in U$  for infinitely many  $j$ . If  $\mathbf{X}$  is first countable,  $x$  is a cluster point of  $\{x_j\}$  iff some subsequence of  $\{x_j\}$  converges to  $x$ .

*Proof.* Suppose  $\{x_n\}$  has a cluster point in  $z \in \mathbf{X}$ . Fix a descending sequence of neighbourhoods  $U_k \subseteq \mathcal{N}(z)$ , where

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k$$

Define  $n_k = \text{least} \left\{ j \in \mathbb{N}^+, j > n_{k-1}, x_j \in U_k \right\}$  with  $n_0 = 0$ , so that for every  $m \geq k$ ,  $x_{n_m} \in U_k$  eventually. And  $\{x_{n_j}\}_{j \geq 1}$  is a subsequence which converges to  $z$ . This proves  $(\implies)$ .

Conversely (this part does not require that  $\mathbf{X}$  be first countable), if  $\{x_{n_k}\}_{k \geq 1}$  is a subsequence that converges to  $z \in \mathbf{X}$ . Every neighbourhood of  $z$  must intersect all but infinitely many  $x_{n_k}$ , therefore  $z$  is a cluster point of  $\{x_n\}$ . ■

## Exercise 4.8

**Proposition 8.1**

If  $\mathbf{X}$  is an infinite set with the cofinite topology and  $\{x_j\}$  is a sequence of distinct points in  $\mathbf{X}$ , then  $x_j \rightarrow x$  for every  $x \in \mathbf{X}$ .

*Proof.* The intuition here is that the cofinite topology does not distinguish between points, so it acts as a type of jelly that hides the points.

Let  $x \in \mathbf{X}$  be arbitrary, if  $U \in \mathcal{N}(x)$  then  $U^o \in \mathcal{N}(x)$ , so that  $\{y_j\}_{j \leq k}$  are the  $k$  points that are required to extend  $U^o$  to  $\mathbf{X}$ . (All but finitely many points are in any open set of  $\mathbf{X}$ ).

There exists a large  $N \in \mathbb{N}^+$  so that for every  $n \geq N$ ,

$$x_j \notin \{y_j\}_{j \leq k} \implies x_j \in U^o$$

eventually. And  $x_j \rightarrow x$ . ■

**Exercise 4.9**

**Proposition 9.1**

*Proof.*



## Exercise 4.10

**Proposition 10.1**

A topological space  $\mathbf{X}$  is called disconnected if there exists non-empty, disjoint open sets  $U, V$  and  $U \cup V = \mathbf{X}$ ; otherwise  $\mathbf{X}$  is connected. When we speak of connected or disconnected subsets of  $\mathbf{X}$ , we refer to the relative topology on them

- (a)  $\mathbf{X}$  is connected iff  $\emptyset$  and  $\mathbf{X}$  are the only two clopen sets.
- (b) If  $\{E_\alpha\}_{\alpha \in A}$  is a collection of connected subsets of  $\mathbf{X}$ , and  $\bigcap E_\alpha$  is non-empty, then  $\bigcup E_\alpha$  is connected.
- (c) If  $A \subseteq \mathbf{X}$  is connected, then  $\overline{A}$  is connected,
- (d) Every point in  $x \in \mathbf{X}$  contained in a unique maximal connected subset of  $\mathbf{X}$ , and this subset is closed. It is called the connected component of  $x$ .

*Proof.* The proof is rather long, so we will split it in several parts. A topological space is disconnected iff it can be written as a disjoint union of two non-empty open sets. Often it is easier to show that a space is disconnected rather than connected.

Part A: Suppose  $\mathbf{X}$  is disconnected, this induces a pair of non-empty open sets,  $A$ , and  $B$  whose union is  $\mathbf{X}$ , and

$$A \cap B = \emptyset \iff A \subseteq B^c$$

their union is  $\mathbf{X}$ , hence

$$A \cup B = \mathbf{X} \iff B^c \subseteq A$$

combining the last two estimates, we see that  $B = A^c$ , so both  $A$  and  $A^c = B$  are closed. This proves ( $\Leftarrow$ ).

Now suppose  $\{A, A^c\} \neq \{\emptyset, \mathbf{X}\}$  are both clopen. Clearly  $A$  is disjoint from its complement, and their union is  $\mathbf{X}$ .

Part B: We will attempt the contrapositive. Suppose  $E = \bigcup E_\alpha$  is disconnected. This induces  $D$  and  $D^c$  which are clopen in the relative topology of  $E$ , (by Part A). More precisely,

$$\bigcup E_\alpha = \underbrace{\bigcup (E_\alpha \cap D)}_{\neq \emptyset} + \underbrace{\bigcup (E_\alpha \setminus D)}_{\neq \emptyset} \quad (12)$$

The intersection  $\bigcap E_\alpha$  is non-trivial, hence

$$\bigcap E_\alpha = \underbrace{\bigcap (E_\alpha \cap D)}_{\neq \emptyset} + \bigcap (E_\alpha \setminus D) \neq \emptyset \quad (13)$$

so at least one of the members on the right are non-empty. Assume without loss of generality that  $\bigcap (E_\alpha \cap D)$  is not empty. This tells us  $E_\alpha \cap D \neq \emptyset$  for each  $\alpha \in A$ .

But by Equation (12), if we concentrate on the right member,

$$\bigcup (E_\alpha \setminus D) \neq \emptyset \implies \exists \beta \in A, E_\beta \setminus D \neq \emptyset$$

And for this particular  $\beta \in A$ , we see that both  $D$  and  $D^c$  are non-trivially open in  $E_\beta$ , and the proof is complete. A poetic way to summarize the proof would be:

If the whole is disconnected, and there exists common ground over which the family of sets covers, and because the common ground (intersection) is non-trivial, either  $D$  or  $D^c$  is non-trivially open in all  $E_\alpha$ . The intersection gives us " $\forall$ ", while the union gives us " $\exists$ " for a non-trivially open  $D$  or  $D^c$ .

There is an alternate way of proving Part B, without using the clopen definition of connectedness. Let  $C$  and  $D$  be non-empty, disjoint, open sets in  $\bigcup E_\alpha$  whose union is  $\bigcup E_\alpha$ .

$$\bigcap E_\alpha = \bigcap [E_\alpha \cap C] + \bigcap [E_\alpha \cap D] \neq \emptyset$$

Pick  $p \in \bigcap E_\alpha$ , without loss of generality, assume  $p \in \bigcap [E_\alpha \cap C]$ , then for every  $\alpha$  we have

$$p \in E_\alpha \cap C \implies E_\alpha \cap C \neq \emptyset$$

Since  $E_\alpha$  is connected,  $E_\alpha \cap D = \emptyset$  for each  $\alpha$ . Taking the union over all  $E_\alpha \cap D$ , we see that

$$\bigcup [E_\alpha \cap D] = \emptyset$$

which contradicts the assumption  $D \neq \emptyset$ .

Part C: Suppose  $\overline{A}$  is disconnected, this induces a non-trivial clopen set  $D$  relative to  $\overline{A}$ .

- Since  $\overline{A} \cap D \neq \emptyset$ , choose any  $y \in \overline{A} \cap D \subseteq \overline{A}$ , since  $D$  is a neighbourhood of  $y$ , and  $y$  is an adherent point of  $A$ . It is immediate that  $A \cap D$  is non-empty.
- Similarly for  $A \setminus D \neq \emptyset$ ,

therefore  $\{D, D^c\}$  is non-trivially clopen in  $A$ , and  $A$  is disconnected.

Part D: The idea here is to use Part B. Let  $x$  be fixed, and  $\{E_\alpha\}_{\alpha \in A}$  be the family of all connected sets containing  $x$ , since their intersection is non-trivial, their union,  $E$  is connected. The closure of their union is then the maximal connected component containing  $x$ . Indeed, if  $G$  is a connected set containing  $x$ , then  $G \subseteq \bigcup E_\alpha = E$ , so  $G \subseteq \overline{E}$ . ■

## Exercise 4.11

**Proposition 11.1**

If  $E_1, \dots, E_n$  are subsets of a topological space, the closure of  $\bigcup_1^n E_j$  is  $\bigcup_1^n \overline{E_j}$

*Proof.* The finite union of closed sets is again closed, so

$$\forall j \leq n, E_j \subseteq \overline{E_j} \implies \overline{\bigcup_1^n E_j} \subseteq \bigcup_1^n \overline{E_j}$$

For the reverse estimate,  $E_j \subseteq \bigcup_1^n E_j \subseteq \overline{\bigcup_1^n E_j}$  is a closed set that contains each  $E_j$ , therefore

$$\forall j \leq n, \overline{E_j} \subseteq \overline{\bigcup_1^n E_j} \implies \bigcup_1^n \overline{E_j} \subseteq \overline{\bigcup_1^n E_j}$$

■

**Corollary 11.1**

The interior operator distributes over intersections. If  $A$  and  $B$  are subsets of  $\mathbf{X}$ , then

$$\begin{aligned} \overline{(A^c \cup B^c)} &= (\overline{A^c} \cap \overline{B^c}) \\ \left( \overline{(A^c \cup B^c)} \right)^c &= A^o \cap B^o \\ \left( A^c \cup B^c \right)^{co} &= A^o \cap B^o \\ (A \cap B)^o &= A^o \cap B^o \end{aligned}$$

## Exercise 4.12

**Proposition 12.1**

Let  $\mathbf{X}$  be a set. A Kuratowski closure operator on  $\mathbf{X}$  is a map  $A \mapsto A^*$  from  $\mathbb{P}(\mathbf{X})$  to itself satisfying

- (i)  $\emptyset^* = \emptyset$  (does nothing to the empty set),
- (ii)  $A \subseteq A^*$  (monotonicity),
- (iii)  $(A^*)^* = A^*$  (idempotence)
- (iv)  $(A \cup B)^* = A^* \cup B^*$  (distributes over finite unions)

Prove

- (a) If  $\mathbf{X}$  is a topological space, the map  $A \mapsto \bar{A}$  is a Kuratowski closure operator. (Use Exercise 11.)
- (b) Conversely, given a Kuratowski closure operator, let  $\mathcal{F} = \{A \subseteq \mathbf{X}, A = A^*\}$  and  $\mathcal{T} = \{U \subseteq \mathbf{X}, U^c \in \mathcal{F}\}$ , then  $\mathcal{T}$  is a topology on  $\mathbf{X}$ , and for any set  $A \subseteq \mathbf{X}$ ,  $A^*$  will be its closure with respect to  $\mathcal{T}$ .

*Proof.* Part A: The empty set is closed, so  $\bar{\emptyset} = \emptyset$ , and  $\bar{A}$  is the smallest closed superset of  $A$ , so  $A \subseteq \bar{A}$  for every  $A \subseteq \mathbf{X}$ .  $A \subseteq \mathbf{X}$  is closed iff  $\bar{A} = A$ , so idempotence holds. Distributivity follows from Exercise 11 directly.

Part B: We first show that  $\mathcal{T}$  is indeed a topology. Fix  $U_1$  and  $U_2$  in  $\mathcal{T}$ , so that  $U_1^c \cup U_2^c = (U_1 \cap U_2)^c$ . The map  $A \mapsto A^*$  distributes over finite unions, hence

$$(U_1^c \cup U_2^c)^* = (U_1^c)^* \cup (U_2^c)^* = U_1^c \cup U_2^c$$

Therefore  $U_1 \cap U_2 \in \mathcal{T}$ . Now suppose  $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ , then

$$\left(\bigcup U_\alpha\right)^c = \bigcap U_\alpha^c$$

by monotonicity (Property ii):  $\bigcap U_\alpha^c \subseteq \left(\bigcap U_\alpha^c\right)^*$ . To prove the reverse inclusion, notice if  $\alpha$  is held fixed,

$$\bigcap U_\alpha^c \subseteq U_\alpha^c \implies \left(\bigcap U_\alpha^c\right)^* \subseteq U_\alpha^{c*}$$

this follows from 'monotonicity' of the closure operator: if  $A$  is a subset of  $B$ , then we can write

$$B = A \cup (B \setminus A) \implies B^* \subseteq A^* \cup (B \setminus A)^* = B^*$$



Take the intersection over all  $\alpha \in A$  on the right member,

$$\left(\bigcap U_\alpha^c\right)^* \subseteq \bigcap U_\alpha^{c*} = \bigcap U_\alpha^c$$

Hence  $\left(\bigcap U_\alpha^c\right)^* = \bigcap U_\alpha^c$ . The empty set and  $\mathbf{X}$  are elements of  $\mathcal{F}$ . Since  $\mathbf{X} \subseteq \mathbf{X}^* \subseteq \mathbf{X}$ , and  $\{\emptyset, \mathbf{X}\} \subseteq \mathcal{T}$ . So  $\mathcal{T}$  is a topology.

Finally,  $A^*$  is a closed superset of  $A$  and suppose  $K$  is another closed superset,

$$A \subseteq K \implies A^* \subseteq K^*$$

So  $A^*$  is the smallest closed superset of  $A$  and this proves the last claim. ■

## Exercise 4.13

**Proposition 13.1**

If  $\mathbf{X}$  is a topological space,  $U$  is open in  $\mathbf{X}$  and  $A$  is dense in  $\mathbf{X}$ , then  $\overline{U} = \overline{U \cap A}$ .

*Proof.* The takeaway here is that if  $A$  is dense in  $\mathbf{X}$ , every point  $z \in U$  can be approximated by points in  $U \cap A$ . And an important technique of 'demoting' the neighbourhood to become the interior of the neighbourhood can yield some nice properties. Since the interior of a neighbourhood is again a neighbourhood. This allows intersection with open sets to inherit the 'neighbourhoodness' of the set.

Let  $z \in \overline{U}$ , and fix a neighbourhood  $V \in \mathcal{N}(z)$ , so that the interior of  $V$  is also a neighbourhood. By the alternate definition of  $\overline{U}$  in terms of adherent points (see Proposition 9.1) of  $\overline{U}$ ,  $V^\circ \cap U \neq \emptyset$ . This is a non-empty open set, therefore it must intersect  $A$  non-trivially.

$$x \in (V^\circ \cap U) \cap A = V^\circ \cap (U \cap A)$$

and  $z \in \overline{U \cap A}$ . ■

**Remark 13.1**

We simply used the fact

$$\overline{E} = \left\{ x \in \mathbf{X}, \forall V \in \mathcal{N}(x), V \cap E \neq \emptyset \right\}$$

and the following equivalent characterization of density

$$E \text{ is dense in } \mathbf{X} \iff \text{For every non-empty open set } U, U \cap E \neq \emptyset$$

## Exercise 4.14

**Proposition 14.1**

If  $\mathbf{X}$  and  $\mathbf{Y}$  are topological spaces,  $f : \mathbf{X} \rightarrow \mathbf{Y}$  is continuous iff  $f(\overline{A}) \subseteq \overline{f(A)}$  for every  $A \subseteq \mathbf{X}$  iff  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$  for all  $B \subseteq \mathbf{Y}$ .

*Proof. First Equivalence:* If  $f$  is continuous, fix any  $A \subseteq \mathbf{X}$ , and  $z \in \overline{A}$ , by Proposition 9.1 (I will spare you the flipping by including):

$$\overline{A} = \{x \in \mathbf{X}, \forall U \in \mathcal{N}(x), U \cap A \neq \emptyset\}$$

Let  $U \in \mathcal{N}(f(z))$ , so that  $f^{-1}(U^\circ)$  is an open set containing  $z$  and  $f^{-1}(U^\circ) \in \mathcal{N}(z)$ , so

$$f^{-1}(U^\circ) \cap A \neq \emptyset \implies U^\circ \cap f(A) \subseteq U \cap f(A)$$

so  $f(\overline{A}) \subseteq \overline{f(A)}$ . Conversely, suppose  $f(\overline{A}) \subseteq \overline{f(A)}$  holds for every  $A \subseteq \mathbf{X}$ . The following is a sequence of symbolic manipulations that I found but have zero intuitive understanding about. First take the inverse image

$$\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right) \subseteq f^{-1}\left(\overline{f(\overline{A})}\right)$$

Next, let  $F$  be a closed set in  $\mathbf{Y}$ , and make the substitution  $A = f^{-1}(F)$ , hence

$$\overline{f^{-1}(F)} \subseteq f^{-1}\left(\overline{f(f^{-1}(F))}\right) \subseteq f^{-1}(\overline{F}) = f^{-1}(F)$$

for the second inclusion we used the monotonicity of the closure, and since  $\overline{f^{-1}(F)} = f^{-1}(F)$ , we are done.

*Second Equivalence:* Suppose  $f \in C(\mathbf{X}, \mathbf{Y})$ , then  $\overline{B} \subseteq \mathbf{Y}$  is a closed set, so  $f^{-1}(\overline{B})$  is closed in  $\mathbf{X}$ . By monotonicity of the inverse image,

$$f^{-1}(B) \subseteq f^{-1}(\overline{B}) \implies \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$$

Conversely, if  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$  for any  $B \subseteq \mathbf{Y}$ , take any closed  $B \subseteq \mathbf{Y}$ , and

$$\overline{f^{-1}(B)} \subseteq f^{-1}(B) \subseteq \overline{f^{-1}(B)}$$

so  $f^{-1}(B)$  is closed, and  $f$  is in  $C(\mathbf{X}, \mathbf{Y})$ . ■

## Exercise 4.16

## Proposition 15.1

*Proof.*

- (a) Let  $x \in \{f \neq g\}$ , then there exists disjoint open subsets of  $\mathbf{Y}$ ,  $f(x) \in U$  and  $g(x) \in V$ ,  $U \cap V = \emptyset$ , but  $f^{-1}(U) \cap g^{-1}(V)$  is an open set in  $\mathbf{X}$  that contains  $x$ . Therefore  $\{f \neq g\}$  is open in  $\mathbf{X}$ .
- (b) Suppose  $\{f = g\} = E$  is dense in  $\mathbf{X}$ . Let  $x \in E$ , induces two disjoint open sets exactly like in part a. This is an open set that contains  $x$ , and  $y \in f^{-1}(U) \cap g^{-1}(V) \cap E$ . Since  $y \in E$ , it follows that  $f(y) = g(y)$ , and

$$\begin{cases} y \in f^{-1}(U) \implies f(y) \in U \\ y \in g^{-1}(V) \implies g(y) \in V \end{cases}$$

■

**Exercise 4.17**