Chapter 9: Integral Curves and Flows

Flowouts from Boundary

This will be a draft of Problem 9.11. We wish to modify the proof for Theorem 9.20 (Flowout Theorem). We need to modify the following aspects

- M is now a smooth manifold with boundary,
- The original embedded submanifold of the flow $\mathcal{D}$, is $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D}$. We need to change it to

  $$\mathcal{O} = ([0, +\infty) \times \partial M) \cap \mathcal{D}$$

  where $\mathcal{D}$ is the flow of the smooth vector field N, which is everywhere inward pointing on the boundary of M.
- The induced smooth embedding $\Phi : P_\delta \to M$ (and thus diffeomorphism, since ∂M has codimension 1), is defined on

  $$P_\delta = \left\{(t, p),\ p \in \partial M,\ 0 \leq t < \delta(p)\right\}$$

  where $\delta : \partial M \to \mathbb{R}^+$ is a smooth, strictly positive function.
- For each $p \in \partial M$, the map $t \mapsto \Phi(t, p)$ is an integral curve of N starting at p.

Chapter 12: Tensors

Tensor Products

Definition 1.1: Covariant Tensor

Let V be a finite-dimensional vector space. Covariant tensors are multi-linear maps $\alpha : V \times \cdots \times V \to \mathbb{R}$. If α sends k-copies of V into $\mathbb{R}$, it is called a *k-tensor*, or a *covariant tensor of rank k*. It is clear that k-tensors form a vector space over $\mathbb{R}$ for any $k \geq 0$. The convention is that 0-tensors are real numbers as they have 0 arguments in V.

$$T^k(V^*) = \left\{ \alpha : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \to \mathbb{R},\ \alpha \text{ is multi-linear.} \right\}$$

Definition 1.2: Contravariant Tensor

Contravariant tensors are multi-linear maps $\alpha : V^* \times \cdots \times V^* \to \mathbb{R}$, similar to covariant tensors. Denoted by

$$T^k(V) = \left\{ \alpha : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \to \mathbb{R},\ \alpha \text{ is multi-linear.} \right\}$$

Definition 1.3: Mixed Tensors

A (k, l)-mixed tensor, or just (k, l)-tensor is a multi-linear map that takes k copies of V^*, and l copies of V into $\mathbb{R}$. In symbols,

$$T^{(k,l)}(V) = \left\{ \alpha : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \to \mathbb{R},\ \alpha \text{ is multi-linear.} \right\}$$

The space of k-covariant, k-contravariant, and (k, l)-tensors form a vector space over $\mathbb{R}$. We will rarely use contravariant tensors, so we will use the word *tensor* to refer to covariant tensors only.

Definition 1.4: Tensor Product

Let F and G be k and l-tensors on V respectively, we define a bi-linear map,

$$F \otimes G : \underbrace{V \times \cdots \times V}_{k+l \text{ copies}} \to \mathbb{R}$$

by

$$(F \otimes G)(v_{\underline{k}}, w_{\underline{l}}) = \underbrace{F(v_{\underline{k}})G(w_{\underline{l}})}_{\substack{\text{scalar} \\ \text{multiplication} \\ \text{in } \mathbb{R}}}$$

Tensor product is clearly associative, and commutative (although we will not use this fact).

Proposition 1.1: Basis of $T^k(V^*)$

Let V be a n-dimensional vector space with basis (E_i), and dual basis (ε^i). $T^K(V^*)$ has basis

$$\mathbb{B} = \left\{ \bigotimes \varepsilon^{i_{\underline{k}}},\ i_{\underline{k}} \in \{\underline{n}\} \right\} = \left\{ \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},\ 1 \leq i_j \leq n,\ 1 \leq j \leq k \right\}$$

This means, if F is a k-tensor, and for every multi-index $I = (i_{\underline{k}})$, define

$$F_I = F_{i_1,\dots i_k} = F_{i_{\underline{k}}} = F(E_{i_{\underline{k}}}) = F(E_{i_1}, \cdots, E_{i_k}) = F(E_I)$$

are precisely the *basis coefficients* of F with respect the basis $\mathbb{B}$,

$$F = F_{i_1,\dots,i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \tag{1}$$

Tensor Fields on Manifolds

Similar to the tangent bundle $TM = \coprod_{p\in M} T_pM = \bigcup_{p\in M}\{p\} \times T_pM$, is endowed with a unique smooth structure that makes the canonical projection $\pi : TM \to M$, $\pi(p, v) = p$ a smooth embedding. We can use the algebraic machinery we have built up from Chapter 10 to construct abstract tensor bundles on M. For completeness, we will state the definitions for covariant and mixed bundles as well.

Definition 2.1: Tensor Bundles

Let M be a smooth manifold with or without boundary, define

- Covariant k-bundle over M by
$$T^kT^*M = \coprod_{p\in M} T^k(T_p^*M)$$
- Contravariant k-bundle over M by
$$T^kTM = \coprod_{p\in M} T^k(T_pM)$$
- (k, l)-mixed bundle over M by
$$T^{(k,l)}TM = \coprod_{p\in M} T^{(k,l)}(T_pM)$$

Definition 2.2: Tensor Field on M

A *k-covariant tensor field*, or just a *k-tensor field* over a smooth manifold (with or without boundary) M, is a smooth section of the vector bundle as defined in Definition 2.1. As with vector fields $\mathfrak{X}(M)$, they form a R-module over $C^\infty(M)$ by pointwise multiplication.

We denote the space of k-tensor fields over M by $\mathcal{T}^k(M)$. Moreover, if $A \in \mathcal{T}^k(M)$, we can write A in local coordinates (x^i) by

$$A = A_{i_1,\dots,i_k} dx^{i_1} \otimes \cdots \otimes dx^{i_k}$$

with summation convention in effect, since $(dx^1, \dots, dx^n)$, when evaluated pointwise in local coordinates, form a dual basis of T_pM. Where each dx^i is the Chapter 11 differential of the i-th coordinate function.

Technicalities for Tensor Fields

We will skip the technicalities for now. Recall k-tensors on a vector space act on a k-tuple of vectors, the same is true for k-tensor fields over a manifold M. If $A \in \mathcal{T}^k(M)$, and $(X_1, \dots, X_k) \subseteq \mathfrak{X}(M)$, then

$$A(X_{\underline{k}}) = A(X_1, \dots, X_k) \quad \text{such that} \quad A(X_{\underline{k}})(p) = A_p(X_{\underline{k}}|_p) = A_p(X_1|_p, \dots, X_k|_p) \in \mathbb{R}$$

We should expect, that if A is a rough k-tensor field over M, it is smooth if and only if for every k-tuple of smooth vector fields, the resulting function (as defined pointwise above), is $C^\infty(M)$. See Lee Proposition 12.19 for more details.

Pullbacks of Tensor Fields

Definition 4.1: Pointwise pullback $dF_p^*: T^k(T_{F(p)}N) \to T^k(T_pM)$

Let M and N be smooth manifolds with or without boundary. Let $F: M \to N$ be a smooth map. For every $p \in M$, if α is a k-tensor on $T_{F(p)}N$, (so that $\alpha \in T^k(T^*_{F(p)}N)$), the *pointwise pullback* of α through dF_p is a k-tensor on T_pM. Denoted by $dF_p^*(\alpha)$, if $(v_{\underline{k}})$ are tangent vectors in T_pM, then

$$dF_p^*(\alpha)(v_{\underline{k}}) = \alpha\Big(dF_p(v_{\underline{k}})\Big)$$

Definition 4.2: Tensor Pullback $F^*: T^kN \to T^kM$

Let $F: M \to N$ be a smooth map, between smooth manifolds with or without boundary, if A is a k-tensor field on N, we define

$$(F^*A)_p(v_{\underline{k}}) = dF_p^*(A_{F(p)}) = A_{F(p)}(dF_p(v_{\underline{k}}))$$

the result is a k-tensor field (which is smooth, by the next Proposition).

Proposition 4.1: Proposition 12.25, Properties of Tensor Pullbacks

Let $F: M \to N$ and $G: N \to P$ be smooth maps between smooth manifold with or without boundary, and A and B are k-tensor fields on N, then

(i) For every $f \in C^\infty(N)$,

$$F^*(fB) = \underbrace{(f \circ F)}_{C^\infty(M)} F^*B$$

(ii) F^* commutes with tensor products,

$$F^*(A \otimes B) = F^*(A) \otimes F^*(B)$$

(iii) F^* is linear over $\mathbb{R}$,

$$F^*(aA + bB) = aF^*(A) + bF^*(B)$$

(iv) $F^*(B)$ is smooth,

(v) The tensor pullback satisfies the following co-functoral properties

- The (tensor) pullback of the composition is the (tensor) pullback of the *pre*-composition,

$$(G \circ F)^* = F^* \circ G^*$$

- The tensor pullback of the identity map on M is the identity map on tensor fields over M,

$$\mathrm{id}_M^* = \mathrm{id}_{\mathcal{T}^k(M)}$$

Chapter 14: Differential Forms

Multi-linear algebra

Definition 1.1: Wedge Product

Let $\omega \in \Lambda^k(V^*)$, $\eta \in \Lambda^l(V^*)$ the wedge product of ω and η is defined

$$\omega \wedge \eta \triangleq \frac{(k+l)!}{k!l!} \operatorname{Alt}(\omega \otimes \eta) \tag{2}$$

with $\operatorname{Alt}(\omega \otimes \eta) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma)^{\sigma}(\omega \otimes \eta)$, and if $(v_1 \dots v_k, v_{k+1}, \dots v_{k+l})$ are vectors in V, then

$$\operatorname{Alt}(\omega \otimes \eta)(v_1 \dots v_k, v_{k+1}, \dots v_{k+l}) \triangleq \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \tag{3}$$

To simplify the expressions, we will use the following shorthand,

$$(v_1, \dots v_k) = (v_{\underline{k}}) \tag{4}$$
$$(v_{k+1}, \dots v_{k+l}) = (v_{k+1, \underline{k+l}}) \tag{5}$$

and Equation (3) becomes

$$\operatorname{Alt}(\omega \otimes \eta)(v_{\underline{k+l}}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\operatorname{sgn} \sigma) \omega(v_{\underline{\sigma(k)}}) \eta(v_{\sigma(k+1), \underline{\sigma(k+l)}}) \tag{6}$$

Definition 1.2: Elementary k-covectors

Let V be a finite dimensional vector space with basis (E_i), and if (ε^i) is a dual basis for V^*, and if I is a k-index, where $I = (i_{\underline{k}})$, then

$$\varepsilon^I(v_{\underline{k}}) = \det\left(\begin{bmatrix} \varepsilon^{i_1}(v_1) & \cdots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \dots \\ \varepsilon^{i_k}(v_1) & \cdots & \varepsilon^{i_k}(v_k) \end{bmatrix}\right) = \det\left(\begin{bmatrix} v_1^{i_1} & \cdots & v_k^{i_1} \\ \vdots & \ddots & \dots \\ v_1^{i_k} & \cdots & v_k^{i_k} \end{bmatrix}\right) = \det\big((\varepsilon^{i_j})v_l\big)_{jl}$$

where $\varepsilon^{i_j}(v_l)$ is the projection of v_l onto the i_j basis vector E_{i_j}. The space of all ε^I where I is a k-index are called *elementary k-covectors*. It is clear that ε^I are alternating k-covectors.

Definition 1.3: Concatenating multi-indices

Let I and J be multi-indices, $I = (i_{\underline{k}})$ and $J = (j_{\underline{l}})$, then

$$IJ = (i_{\underline{k}}, j_{\underline{l}})$$

is a $k+l$ multi-index obtained by concatenating I and J.

Proposition 1.1: Proposition 14.8

Let V be a n-dimensional vector space with basis (E_i), and if (ε^i) is a dual basis for V^*, then $\Lambda^k(V^*)$ has the basis:

$$\overline{\mathcal{E}} = \left\{ \varepsilon^I,\ I \text{ is an increasing } k\text{-index} \right\}$$

In particular, this means the vector space (algebra) of alternating k-covectors have dimension $\binom{n}{k}$. If $k = n$, then it is spanned by

$$\varepsilon^{(1,\dots,n)} = \varepsilon^{(\underline{n})}$$

and all alternating n-covectors are of the form $A\varepsilon^{(\underline{n})}$, where $A \in \mathbb{R}$. This roughly means there is, up to a scalar multiple of $\varepsilon^{\underline{n}}$, only one (oriented) way of measuring volume in a basis-independent, and dimension-independent manner.

Proposition 1.2: Lemma 14.10

Let I and J be multi-indices, $I = (i_{\underline{k}})$ and $J = (j_{\underline{l}})$, then

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} \tag{7}$$

Proposition 1.3: Lemma 14.11, Properties of the Wedge Product

The wedge product satisfies the following properties

- It is bi-linear over $\mathbb{R}$,
- It is associative,
- It is anti-commutative, for $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, then

  $$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$

- Formula for elementary covectors, let $\varepsilon^{i_{\underline{k}}}$ be covectors (covectors are assumed to have rank one), then

  $$\varepsilon^I \triangleq \varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}$$

  is an alternating k-covector.
- Determinant Law

  $$\omega^1 \wedge \cdots \wedge \omega^k(v_{\underline{k}}) = \det\!\left(\omega^j(v_i)\right)_{ji}$$

  where $(A^j_i)_{ji}$ denotes the matrix with entries A^j_i in the jth row and ith column.

Definition 1.4: Interior multiplication

Let V be a finite dimensional vector space. If $x \in V$ is a vector, it induces a linear map $i_x : \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ such that for every $\omega \in \Lambda^k(V^*)$, and $v_{2,\underline{k}} \in V$,

$$i_x(\omega)(v_{2,\underline{k}}) \triangleq \omega(x, v_{2,\underline{k}})$$

by fixing x in the first argument of ω. The result is an alternating $k-1$ covector, and it is clearly linear in ω over $\mathbb{R}$. We also write

$$i_x(\omega) = x \lrcorner \omega$$

Definition 1.5: Graded algebra

An algebra A is said to be *graded* if it has a direct sum decomposition $A = \bigoplus_{k\in\mathbb{Z}} A^k$ such that the algebra-product $(\cdot,\cdot) : A \times A \to A$ satisfies

$$(a,b) \in A^{k+l} \quad \text{for every } a \in A^k,\ b \in A^l$$

A graded algebra is said to be *anti-commutative*, if the algebra product satisfies

$$(a,b) = (-1)^{kl}(b,a)$$

We see that $\bigoplus_{k\in\mathbb{Z}} \Lambda^k(V^*)$ is a graded algebra, if we define $\Lambda^k(V^*) = \{0\}$ for $k \le -1$, or $k \ge n+1$.

Proposition 1.4: Lemma 14.13, Properties of interior multiplication

Let V be a finite dimensional vector space and fix $x \in V$,

(i) $i_x \circ i_x \equiv 0$, where we interpret i_v as a linear map on the entire graded algebra.

(ii) i_x satisfies some kind of product rule. For every $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,

$$i_x(\omega \wedge \eta) = (i_x\omega) \wedge \eta + (-1)^k \omega \wedge (i_x\eta)$$

Differential Forms on Manifolds

Definition 2.1: Differential k-form on M

Let M be a smooth manifold with or without boundary, and $p \in M$ be fixed, $\lambda^k(T_p^*M)$ is a linear subspace of $T^k(T_p^*M)$ of dimension $\binom{n}{k}$, and

$$\Lambda^k T^*M \triangleq \coprod_{p\in M} \Lambda^k(T_p^*M)$$

is a smooth vector bundle of rank $\binom{n}{k}$ over M.

A differential k-form on M is a k-tensor field (or a smooth global section of the k-tensor bundle T^kT^*M) that is *pointwise alternating*, or equivalently: it is a smooth global section of the vector bundle $\Lambda^k T^*M$. The space of differential k-forms on M is denoted by

$$\Omega^k(M) = \Gamma(\Lambda^k T^*M)$$

Definition 2.2: Graded algebra of all k-forms

$\Omega^k(M)$ is a vector space over $\mathbb{R}$, and a left $C^\infty(M)$ module. If we stick with scalars over $\mathbb{R}$, and define the wedge product of two tensor fields ω, $\eta \in \Omega^k(M)$ (recall these are smooth global sections of some vector bundle),

$$(\omega \wedge \eta)_p \triangleq \omega_p \wedge \eta_p$$

We can inherit all the properties of the pointwise wedge product (see Proposition 1.3), namely

- Bi-linearity over $\mathbb{R}$,
- Associativity,

- Anti-commutativity

With this, the direct sum of all $\Omega^k(M)$ forms an associative, anticommutative graded algebra over $\mathbb{R}$, with the tensor wedge product as the algebra product.

$$\Omega^*(M) = \bigoplus_{k\geq 0} \Omega^k(M)$$

Definition 2.3: $\Omega^k(M)$ in local coordinates

Let $\omega \in \Omega^k(M)$, and (x^i) be the local coordinates on some open subset U of M. If

$$\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right) \quad \text{is a coordinate frame on } TU \triangleq \pi^{-1}(U) \tag{8}$$

where $\pi : TM \to M$ is the canonical projection of the tangent bundle. Then,

$$\left(dx^1, \dots, dx^n\right) \quad \text{is the dual co-frame to the frame in Equation (8).}$$

by Proposition 1.1, we can write ω uniquely as a linear combination (over $C^\infty(M)$) of k-tensors which are *increasing elementary k-covectors* pointwise.

$$\omega = {\sum_I}' \omega_I dx^{i_1} \wedge \dots \wedge dx^{i_k} = {\sum_I}' \omega_I dx^I$$

where ${\sum}'_I$ a sum over all increasing k-indices, and dx^I is the shorthand for the wedge product above, and ω_I is a number obtained by evaluating ω at the following

$$\omega_I \triangleq \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right), \quad \text{where} \quad I = (i_1, \dots i_k)$$

Since differential k-forms are (smooth) k-tensor fields, the pullback $F^* : \Omega^k(N) \to \Omega^k(M)$ is well defined, in particular,

Proposition 2.1: Lemma 14.16, Properties of k-form pullback

Let $F : M \to N$ be a smooth map, then

- The k-form pullback is linear over $\mathbb{R}$,
- The k-form pullback commutes with wedge products, or

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

- In any smooth chart, if (y^i) are the local coordinates of N. For any k-form on N $\omega = {\sum}'_I \omega_I dy^{i_1} \wedge \dots \wedge dy^{i_k} = {\sum}'_I \omega_I dy^I$,

$$F^*\left({\sum}' \omega_I dy^{i_1} \wedge \dots \wedge dy^{i_k}\right) = {\sum_I}' (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F)$$

Proof. Linearity of the k-form pullback over $\mathbb{R}$ is an immediate consequence of the linearity of the tensor field pullback. To prove the second claim, notice

$$F^*(A \otimes B) = F^*(A) \otimes F^*(B)$$

and

$$F^*(\mathrm{Alt}(A\otimes B)) = F^*\left(\frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)\sigma(A\otimes B)\right)$$
$$= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)F^*(\sigma(A\otimes B))$$

The tensor pullback commutes with permutations, let C be a k-tensor field on N, and $\sigma \in S_k$, then

$$(\sigma F^*(C))_p(v_{\underline{k}}) = (\sigma C)(dF_p(v_{\underline{k}})) = C(dF_p(v_{\underline{\sigma(k)}})) = F^*(\sigma C)_p(v_{\underline{k}})$$

Hence,

$$F^*(\mathrm{Alt}(A\otimes B)) = \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)\sigma[F^*(A\otimes B)]$$
$$= \frac{1}{(k+l)!}\sum_{\sigma\in S_{k+l}}(\operatorname{sgn}\sigma)\sigma[F^*(A)\otimes F^*(B)]$$
$$= \mathrm{Alt}(F^*(A)\otimes F^*(B))$$

Since the wedge product differs by the alternating product by a constant, F^* commutes with the wedge product. The general case follows upon induction, and associativity of the wedge product.

For the third claim, use linearity, and the pullback of the wedge product is the wedge product of the pullback, noting that each y^{i_j} is a smooth function on N, so $F^*(dy^{i_j}) = d(y^{i_j}\circ F)$ by Proposition 11.25b in Lee, we state this Proposition for completeness:

> Let $F : M \to N$ be a smooth map, and u is a smooth function on N, then
>
> $$F^*du = d(u\circ F)$$

■

Proposition 2.2: Corollary 14.21, Tensor Pullback of Top-Degree Forms

Let $F : M \to N$ be a smooth map between manifolds with or without boundary *of the same rank n*. If (x^i) and (y^j) are local coordinates on open subsets U and V of M and N respectively, for every $u \in C(V,\mathbb{R})$,

$$F^*(u\,dy^1\wedge\cdots\wedge dy^n) = (u\circ F)(\det DF)dx^1\wedge\cdots\wedge dx^n$$

on an open set $U\cap F^{-1}(V)$, where DF is the Jacobian matrix of the coordinate representation of F.

Proof. The proof uses the fact that both manifolds are of dimension n, and any n-form on U is necessarily of the form

$$g\,dx^1\wedge\cdots\wedge dx^n \tag{9}$$

where g is some continuous, real valued function on U. Solving for g pointwise yields the result. Suppose $p \in$

$U \cap F^{-1(V)}$, and evaluating eq. (9) with the coordinate derivations $\left(\frac{\partial}{\partial x^{\underline{n}}}\Big|_p\right)$ reads

$$
\begin{aligned}
\left(g dx^1 \wedge \cdots \wedge dx^n\right)_p \left(\frac{\partial}{\partial x^{\underline{n}}}\Big|_p\right) &= g(p)(dx^1 \wedge \cdots \wedge dx^n)_p \left(\frac{\partial}{\partial x^{\underline{n}}}\Big|_p\right) && \text{(pw. eval. of tensor field)} \\
&= g(p) dx^1_p \wedge \cdots \wedge dx^n_p \left(\frac{\partial}{\partial x^{\underline{n}}}\Big|_p\right) && \text{(wedge pw. is pw. wedge)} \\
&= g(p) \det\left(dx^i_p\left(\frac{\partial}{\partial x^j}\right)\right)_{ij} && \text{(det. law from Proposition 1.3)} \\
&= g(p) \det\left(\delta^i_j\right)_{ij} \\
&= g(p)
\end{aligned}
$$

Let $F^i = y^i \circ F$ be the ith coordinate component of F (which is a smooth function on M). So that dF^i and $\frac{\partial}{\partial x^j}$ form a covector-vector pair if evaluated pointwise. Apply the determinant law from Proposition 1.3 for every $p \in U$ reads

$$
(dF^1_p \wedge \cdots \wedge dF^n_p)\left(\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p\right) = \det\left(\left[dF^i_p\left(\frac{\partial}{\partial x^j}\Big|_p\right)\right]_{ij}\right) \tag{10}
$$

and

$$
dF^1 \wedge \cdots \wedge dF^n\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right) = \det\left(dF^i\left(\frac{\partial}{\partial x^j}\right)\right)
$$

Using the tensor pullback formula from Chapter 12, and by Lemma 14.16,

$$
\begin{aligned}
F^*(u dy^1 \wedge \cdots \wedge dy^n) &= (u \circ F) d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) \\
&= (u \circ F) dF^1 \wedge \cdots \wedge dF^n
\end{aligned}
$$

Evaluating the last expression using the coordinate frame on $\left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}\right)$ at some $p \in U \cap F^{-1}(V) \subseteq U$,

$$
\begin{aligned}
F^*(u dy^1 \wedge \cdots \wedge dy^n)_p \left(\frac{\partial}{\partial x^{\underline{n}}}\Big|_p\right) &= \underbrace{(u \circ F)(p)}_{\text{scalar}}(dF^1 \wedge \cdots \wedge dF^n)_p \left(\frac{\partial}{\partial x^{\underline{n}}}\Big|_p\right) && \text{(pw. eval. of tensor field)} \\
&= (u \circ F)(p)(dF^1_p \wedge \cdots \wedge dF^n_p)\left(\frac{\partial}{\partial x^{\underline{n}}}\Big| p\right) && \text{(wedge pw. is pw. wedge)} \\
&= (u \circ F)(p) \det\left(dF^i\left(\frac{\partial}{\partial x^j}\right)\right)_p && \text{(Equation (10))}
\end{aligned}
$$

Equating the two expressions we see that

$$
g(p) = (u \circ F)(p) \det\left(dF^i\left(\frac{\partial}{\partial x^j}\right)\right)_p = (u \circ F)(p) \det(DF)_p
$$

■

Exterior Derivatives

Definition 3.1: Exterior Derivative on Euclidean Space

Let U be an open subset of $\mathbb{R}^n$ or $\mathbb{H}^n$, and (x^i) be the local coordinates in U. If $\omega \in \Omega^k(U)$,

$$
\omega = \sum_J{}' \omega_J dx^J
$$

and we define the *exterior derivative of* ω by

$$d\omega = d\left(\sum_J{}' \omega_J dx^J\right) = \sum_J{}' d\omega_J \wedge dx^J$$

where $d\omega_J$ is the Chapter 11 differential of the smooth function ω_J, and $d\omega_J$ is a covector field on U. If $p \in u$ and $v \in T_pU$,

$$d(\omega_J)_p(v) \overset{\Delta}{=} v(\omega_J)$$

Since $\Omega^1(U) = \mathfrak{X}(U)$, the wedge product $d\omega_J \wedge dx^J = d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$ is alternating.

With this, $d : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ (resp. $\mathbb{H}^n$) is a linear map over $\mathbb{R}$. We can rewrite $d\omega_J$ in terms of the coframe on U, suppose

$$d\omega_J = \frac{\partial \omega_J}{\partial x^i} dx^i \implies d\omega_J \wedge dx^J = \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^J = \sum_{i=1}^n \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^J$$

Summing over all increasing k-indices J, we get

$$d\omega = d\left(\sum_J{}' \omega_j dx^J\right) = \sum_J{}' \sum_{i=1}^n \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$