MATH 263: Section 003, Tutorial 12

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1 The Laplace Transform

The **Laplace Transform** is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt$$

where the integral is defined.

Note that the **Laplace Transform** is an **integral Transform**, meaning that it is linear (due to the properties of the integral). Therefore, for all $a, b \in \mathbb{R}$:

$$\mathcal{L}\{a\ f(t) + b\ g(t)\} = a\ \mathcal{L}\{f(t)\} + b\ \mathcal{L}\{g(t)\}$$

Problem 1a. From Boyce and DiPrima, 10th edition (6.1, exercise 9, p.315):
Find the Laplace Transform of:

$$f(t) = e^{at}\cosh bt$$

Hint: Recall that

$$\cosh bt = \frac{1}{2}(e^{bt} + e^{-bt})$$

$$\sinh bt = \frac{1}{2}(e^{bt} - e^{-bt})$$

Problem 1b. From Boyce and DiPrima, 10th edition (6.1, exercise 22, p.315):
Find the Laplace Transform of:

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 0 & t \ge 1 \end{cases}$$

2 Solution of Initial Value Problems using The Laplace Transform

Using integration by parts, we can find the Laplace Transform of a derivative:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

note that this Laplace Transform is also in terms of the function's initial conditions. Now, from induction, we can find the Laplace Transform of the n'th derivative of a function:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

This Laplace Transform has $n-1$ initial conditions. Note that this way, we can solve first order constant coefficient ODE of the form:

$$ay"(t) + by'(t) + cy(t) = g(t)$$

By taking the Laplace Transform on both sides, using the formula to take the Laplace Transform of a derivative, solve for the solution $Y(s)$, and apply the Inverse Laplace Transform ($\mathcal{L}^{-1}\{F(s)\}$) to find the solution to the IVP. To find inverse Laplace Transforms in this course, one should decompose $Y(s)$ into simpler terms (usually using partial fractions) and use a Laplace Transform table to find the solution $y(t)$.

Problem 2a. From Boyce and DiPrima, 10th edition (6.2, exercise 4, p.324):
Find the inverse Laplace Transform of:

$$F(s) = \frac{3s}{s^2 - s - 6}.$$

Problem 2b. From Boyce and DiPrima, 10th edition (6.2, exercise 14, p.325):
Solve this IVP using the Laplace Transform:

$$y\text{''} - 4y' + 4y = 0;\ y(0) = 1,\ y'(0) = 1$$