MATH 263: Section 003, Tutorial 7

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1 Higher Order Homogeneous Equations and the Wronskian

Given n solutions to an n^{th} order linear ODE, showing their independence can also be shown by their **Wronskian**, which must be nonzero for linear independence. In general, it is of the form:

$$W(y_1,\ y_2,\ y_3,\ \dots\ y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1" & y_2" & y_3" & \cdots & y_n" \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Problem 1. From Boyce and DiPrima, 10th edition (4.2, exercise 32, p.234):
Find the general solution to:

$$y''' - y'' + y' - y = 0$$

Find the Wronskian of the fundamental solutions: are the fundamental solutions independent? Then, solve the IVP: for $y(0) = 2,\ y'(0) = -1,\ y''(0) = -2$.

Note: **Abel's Theorem** also can show the Wronskian for higher order ODEs. For an n^{th} order linear ODE of the form:

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0$$

Given n fundamental solutions $y_1,\ y_2,\ y_3,\ \dots\ y_n$, **Abel's Theorem** states that:

$$W(y_1,\ y_2,\ y_3,\ \dots\ y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1" & y_2" & y_3" & \cdots & y_n" \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = C\ \exp[-\int p_{n-1}(x)\ dx]$$

2 Existence and Uniqueness Theorem

Given the IVP:

$$y" + p(x)y' + q(x)y = g(x),\ y(x_0) = y_0,\ y'(x_0) = y_0$$

if p, q, and g are continuous on the open interval I that contains the point x_0, then the solution to the IVP is **unique, differentiable, and exists** on the interval I.

Problem 2. From Boyce and DiPrima, 10th edition (3.2, exercise 10, p.155):
Determine the longest interval in which the initial value problem:

$$y''(x) + \cos x \; y'(x) + 3\ln|x| \; y(x) = 0, \; y(2) = 3, \; y'(2) = 1$$

is certain to have a unique twice-differentiable solution.

3 Reduction of Order

Recall in **Tutorial 5** that given the ODE:

$$y'' + p(x)y'(x) + q(x)y(x) = 0$$

and one solution $y_1(x)$, we can find **a general solution of the form** $y(x) = v(x)y_1(x)$. Then, find y"s derivatives and substitute them in the ODE to find v and y:

$$y'(x) = v'(x)y_1(x) + v(x)y_1'(x)$$
$$y_1v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0$$
$$y_1v'' + (2y_1' + py_1)v' = 0$$

Problem 3. From Boyce and DiPrima, 10th edition (3.4, exercise 27, p.174):
Use reduction of order to find a general solution to:

$$xy'' - y' + 4x^3y = 0, \; x > 0$$

given one solution $y_1(x) = \sin x^2$.

4 Euler's Equations: Change of Variables

More general Euler Equations of the form:

$$a(x - x_0)^2y'' + b(x - x_0)y' + cy = 0$$

can solved by simply letting $t = x - x_0, \; dt = dx$.
Then, the solution is solved the same way as done in **Tutorial 6** to find in general:

$$y = y(t) = y(x - x_0), \; x \neq x_0.$$

Problem 4. From Boyce and DiPrima, 10th edition (5.3, exercise 10, p.280):
Find the general solution of:

$$(x - 2)^2y'' + 5(x - 2)y' + 8y = 0, \; x \neq 2.$$