Folland Reading

me

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4.1 Theorem 4.1

WTS. *Suppose that A is a subset of X, let $\operatorname{acc} A$ be the set of accumulation points of A, then*

$$\overline{A} = A \cup \operatorname{acc}(A) \tag{1}$$

and A is closed if and only if $\operatorname{acc}(A) \subseteq A$.

Proof. □

4.2 Theorem 4.2

WTS.

4.3 Theorem 4.3

WTS.

Proof. $\square$

4.4 Theorem 4.4

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4.14 Theorem 4.14

WTS. *Suppose that A and B are disjoint closed subsets of the normal space X, and let $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that*

1. *$A \subseteq U_r \subseteq B^c$ for every $r \in \Delta$ and*
2. *$\overline{U_r} \subseteq U_s$ for $r < s$.*

Proof. □

4.15 Theorem 4.15

WTS. *Urysohn's Lemma. Let X be a normal space, if A and B are disjoint closed subsets of X, then there exists a $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.*

Proof. Let $r \in \Delta$ be as in Lemma 4.14, and set U_r accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0, 1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic ε argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since A and B are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that x is not a member of any U_r, but we set $U_1 = X$. Since none of the $r \in (0, 1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The ε argument follows: suppose for every $\varepsilon > 0$, $(1 - \varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannnot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in X. So $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of Δ, there exists an r, $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r<\alpha} U_r$. So $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r<\alpha} U_r$.

Fix an element $x \in \bigcup_{r<\alpha} U_r$, this induces an r such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r<\alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$.

For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an r (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) — we have $x \notin \overline{U_s}$, but $\left(\overline{U_s}\right)^c$ is open in X. It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq \left(\overline{U_s}\right)^c \subseteq \bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$. To show the reverse, fix an element x in the union, then this induces some $x \in \left(\overline{U_s}\right)^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of W. This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen s is a lower bound for W, and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in X, using Theorem 4.9 finishes the proof. $\square$

4.16 Theorem 4.16

WTS. *The Tietze's Extension Theorem. Let X be a normal space, and for any closed subset $A \subseteq X$, and $f \in C(A, [a,b])$, there exists an $F \in C(X, [a,b])$ which extends f.*

Proof. We begin with an important lemma that will serve as a 'black box' for the induction.

Lemma 4.1. *For every $f \in C(A, [0,1])$, there exists a $g \in C(X, [0,1/3])$ such that*

$$0 \leq f - g \leq 2/3 \quad \textit{pointwise on } A \tag{2}$$

Proof. Since f is continuous, $B = f^{-1}([0,1/3])$, and $C = f^{-1}([2/3,1])$ are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function $g \in C(X, [0,1])$ such that $g|_B = 0$ and $g|_C = 1$. Relabel $g = g/3$ then $g \in C(X, [0,1/3])$ (multiplication is continuous).

To show that (2) holds, suppose $x \in B$, then $f(x) \in [0,1/3]$ and $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$. Now suppose that $x \in C$, then $f(x) \in [2/3, 1]$ and $g(x) = 1/3$ (recall that we relabelled g). So we have $0 \leq 1/3 \leq f - g \leq 2/3$. Lastly, for the case where $x \notin (B \cup C)$, then $f(x) \in (1/3, 2/3)$, and $g(x) \in [0, 1/3]$ implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 \qquad &\implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 \qquad &\implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore $0 \leq f(x) - g(x) \leq 2/3$. ☐

We can assume that $f \in C(A, [0,1])$, since we can relabel $f = (f-a)/(b-a)$. The main part of this proof consists of constructing a sequence of $\{g_n\} \subseteq C(X, \mathbb{R})$ where $0 \leq g_n \leq (2/3)^n(1/2)$, and $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$ on A. Let us begin with the base case with $n = 1$. We can apply Lemma 4.1 to get $g_1 \in C(X, [0,1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that $\{g_j\}_{j \leq n}$ has been chosen, we will find our g_{n+1} by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by $(2/3)^{-n}$ and we obtain a new function in $C(A, [0, 1])$.

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) \left(\frac{3}{2} \right)^n \leq 1$$

Applying the Lemma 4.1, we get a function $h \in C(X, [0, 1/3])$ such that, for every $x \in A$

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) \left(\frac{3}{2} \right)^n - h \leq 2/3$$

Multiplying across gives

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) - h \left(\frac{2}{3} \right)^n \leq \left(\frac{2}{3} \right)^{n+1}$$

Set $g_{n+1} = h \left(\frac{2}{3} \right)^n$ and $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$. Furthermore, the sum of all g_j pointwise converges uniformly, as

$$\sum_{j \geq 1} \|g_j\|_u \leq \sum_{j \geq 1} \left(\frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum $F = \sum g_j$, then this $F \in BC(X)$ (by Theorem 4.9), since every $g_j \in BC(X)$. And

$$\left\| f - \sum_{j \leq n} g_j \right\|_u \leq \left(\frac{2}{3} \right)^n \longrightarrow 0$$

So $F = f$ on A, now if we want to obtain our F on $[a, b]$ we simply relabel $F = F(b - a) + a$. This finishes the proof. $\square$

4.17 Theorem 4.17

WTS. *If X is a normal space, and A is a closed subspace of X, and $f \in C(A)$, then there exists an $F \in C(X)$ such that F extends f.*

Proof. First we suppose that f is real valued, so $f \in C(X, \mathbb{R})$. And define a $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$, using

$$g = \frac{f}{1 + |f|}$$

Since g satisfies the assumption of Theorem 4.16 (note that we do not require g to be injective), there exists a $G \in C(X, [-1, +1])$ such that $G|_A = g$. Since the set $\{-1, +1\}$ is closed in $\mathbb{R}$, $G^{-1}(\{-1, +1\})$ is closed as well. Since $G^{-1}((-1, +1)) \subseteq A$, this makes A and $B =^{-1} (\{-1, +1\})$ disjoint closed sets in X.

By Urysohn's Lemma, there exists a continuous function $h \in C(X, [0, 1])$ such that $h|_B = 0$ and $h|_A = 1$, so that the product $|hG| < 1$ for all $x \in X$. We can think of this h as a continuous indicator function that filters out the parts we do not want, namely $G^{-1}\{-1, +1\}$. Now define F in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that $F|_A = g/(1 - |g|) = f$ indeed. Since $|g| = \frac{|f|}{1+|f|}$, and $g(1 + |f|) = f$ implies that $g/(1 - |g|) = f$, because $g \in C(A, (-1, +1))$ This completes the proof for any $f \in \mathbb{R}$ if $f \in C(A)$, then

1. $\mathrm{Re}(f) = f_1 \in C(A, \mathbb{R})$
2. $\mathrm{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in $C(X, \mathbb{R})$ that extends f_1 and f_2, and $F_1 + iF_2 = f$ on A and $F_1 + iF_2 \in C(X)$, and the proof is complete. $\square$

4.18 Theorem 4.18

WTS. *If X is a topological space, and $E \subseteq X$ and $x \in X$, then $x \in \operatorname{acc} E \iff$ there exists a net in $E \setminus \{x\}$ that converges to x, and $x \in \overline{E} \iff$ there exists a net in E that converges to x.*

Proof. Suppose that $x \in \operatorname{acc} E$, then for every neighbourhood $U \in \mathcal{N}(x)$, $E \cap U \setminus \{x\} \neq \varnothing$, then choose $\mathcal{N}(x)$ as the set of neighbourhoods directed by reverse inclusion (and this makes $(\mathcal{N}(x), \lesssim)$ a directed set), and we will define the net as follows.

Map each $U \in \mathcal{N}(x)$ to some $x_U \in E \cap U \setminus \{x\}$, then this net converges to x. Suppose that we fix a neighbourhood, $V \in \mathcal{N}(x)$, then for every $U \gtrsim V$ we have $x_u \in U \subseteq V$. So $\langle x_U \rangle$ is eventually in V.

Conversely, if $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$, and $x_\alpha \to x$, then every $U \in \mathcal{N}(x)$ there exists a $x_\alpha \in E \cap U \setminus \{x\}$ that makes

$$E \cap U \neq \varnothing \quad \forall U \in \mathcal{N}(x)$$

Hence $x \in \operatorname{acc} E$.

Now for the second part of the Theorem, suppose that $x \in \overline{E}$, if $x \notin E$ then $E = E \setminus \{x\}$ and $x \in \operatorname{acc} E$, so there exists a net in $E \setminus \{x\} \subseteq E$ such that $x_\alpha \to x$. If $x \in E$ then simply choose $\langle x_\alpha \rangle = x$ for every $\alpha \in A$.

Now, suppose that there is a net that converges to x, and this net $\langle x_\alpha \rangle \subseteq E$, if $x \in E$ then there is nothing to prove, since $E \subseteq \overline{E}$, so suppose that $x \notin E$, then there exists a net in $E \setminus \{x\} = E$ such that

$$x_\alpha \to x \implies x \in \operatorname{acc} E \subseteq \overline{E}$$

□

4.19 Theorem 4.19

WTS. *Let X and Y be topological spaces, then every $f : X \to Y$ is continuous at a point $x \in X \iff$ every net $\langle x_\alpha \rangle$ that converges to x implies that $\langle f(x_\alpha) \rangle$ converges to $f(x)$.*

Proof. If f is continuous at a point $x \in X$, then $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$, then for every net $\langle x_\alpha \rangle$ that converges to this x, there there exists an α_0 such that for every $\alpha \gtrsim \alpha_0$ implies that $x_\alpha \in f^{-1}(V)$. Hence

$$f(x_\alpha) \in f\left(f^{-1}(V)\right) \subseteq V$$

And this is equivalent to saying that for every $V \in \mathcal{N}(f(x))$, $\langle f(x_\alpha) \rangle$ is eventually in V, and this proves convergence.

Now suppose that f is not continuous at some x, then there exists a $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$, so

$$x \notin \left(f^{-1}(V)\right)^o \implies x \in \left(f^{-1}(V)\right)^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our $x \in \overline{f^{-1}(V^c)}$ induces a net $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$ that converges to x. But every element in the net is contained within $f^{-1}(V^c)$, and for every $\alpha \in A$

$$f(x_\alpha) \in f\left(f^{-1}(V^c)\right) \subseteq V^c$$

gives $f(x_\alpha) \notin V$, but V is a neighbourhood of $f(x)$, hence there exists some $x_\alpha \to x$ and $f(x_\alpha) \not\to f(x)$. $\square$

4.20 Theorem 4.20

WTS. *If $\langle x_\alpha \rangle$ is a net in X, and $x \in X$ is a cluster point of $\langle x_\alpha \rangle \iff$ there exists a subnet of $\langle x_\alpha \rangle$ that converges to x.*

Proof. Suppose that $\langle y_\beta \rangle_{\beta \in B}$ is a subnet of $\langle x_\alpha \rangle$ that converges to x, then for every neighbourhood $U \in \mathcal{N}(x)$, there exists a β_1 such that for every $\beta \gtrsim \beta_1$ we get $y_\beta = x_{\alpha_\beta} \in U$.

Furthermore, let us fix a $\alpha_0 \in A$ to attempt to show that $\langle x_\alpha \rangle$ is frequently in U, then by the subnet property of $\langle y_\beta \rangle$, there exists some $\beta_2 \in B$ such that for every $\beta \gtrsim \beta_2$, $\alpha_\beta \gtrsim \alpha_0$. (Intuitively this property means that the directed set of B 'grows' as much as the directed set of A, so we can always find elements that are greater than any fixed α_0.)

Since $\langle y_\beta \rangle$ is a net, we there exists some $\beta \in B$ such that $\beta \gtrsim \beta_1$ and $\beta \gtrsim \beta_2$, we then apply the $\beta \mapsto \alpha_\beta$ map and we obtain some $\alpha = \alpha_\beta$ that satisfies:

- $\alpha = \alpha_\beta \gtrsim \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that $\beta \gtrsim \beta_1$ so that y_β falls into U.

Conversely, suppose that x is a cluster point of $\langle x_\alpha \rangle$, then by definition

$$\forall U \in \mathcal{N}(x),\ \forall \alpha_0 \in A,\ \exists \alpha \gtrsim \alpha_0,\ x_\alpha \in U$$

Denote the directed neighbourhoods of x by $\mathcal{N}(x)$, and construct our directed set B for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every $(U, \gamma) \in B$ we can map it to some $\alpha_{(U,\gamma)} \in A$, if we choose some $\alpha_{(U,\gamma)} \gtrsim \gamma$ and $\alpha_{(U,\gamma)} \in U$.

To show that B is a directed set, we say that $(U, \gamma) \gtrsim (U', \gamma')$ if and only if $U \subseteq U'$ and $\gamma \gtrsim \gamma'$. And to show that $\langle y_\beta \rangle = \langle x_{\alpha_{(U,\gamma)}} \rangle$ is indeed a subnet of $\langle x_\alpha \rangle$, fix any $\alpha_0 \in A$, then simply take any neighbourhood U of x (we always have $X \in \mathcal{N}(x)$) — and therefore $(U, \alpha_0) \in B$.

Now for every $(U', \alpha_0') \gtrsim (U, \alpha_0)$ implies that $\alpha_0' \gtrsim \alpha_0$, therefore we have

$$\alpha_{(U',\alpha_0')} \gtrsim \alpha_0' \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that $\langle y_\beta \rangle$ indeed converges to x, fix any $V \in \mathcal{N}(x)$, then with any $\alpha_0 \in A$, and for every $(V', \alpha_0') \gtrsim (V, \alpha_0) \in B$, we have

$$x_{\alpha_{(V',\alpha_0')}} \in V' \subseteq V$$

So $\langle x_{\alpha_{(U,\gamma)}} \rangle$ converges to x. $\square$

5 Chapter 5

6 Chapter 6

6.1 Theorem 6.15

WTS.

First suppose that $(X, \mathcal{M}, \mu)$ is finite measure space. If $\mu(X) < +\infty$, then for every $E \in \mathcal{M}$, by monotonicity $E \subseteq X$ yields $\mu(E) \leq \mu(X) < +\infty$. Next, for any $p < +\infty$, $\|\chi_E\|_p^p < +\infty$ and $\|\chi_E\|_{+\infty} \leq 1 < +\infty$. So all indicator functions are in L^p.

It follows that every simple function is also in L^p, since it is a finite linear combination of indicators. We now define $\nu(E) = \phi(\chi_E)$, we wish to show that $\nu : \mathcal{M} \longrightarrow \mathbb{C}$ is a complex measure which is absolutely continuous with respect to μ.

To show σ-additivity, fix any disjoint sequence $\{E_j\}_{j\geq 1} \subseteq \mathcal{M}$. Where we also note that $\mu(E) = \mu(\cup E_j) < +\infty$. Now suppose that $p < +\infty$, then the following converges in the p-norm

$$\chi_E = \sum_{j\geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left(\bigcup E_{j\leq n}\right) = \left(\bigcup E_{j\geq 1}\right) \setminus \left(\bigcup E_{j\leq n}\right) = \bigcup E_{j\geq n+1}$$

and define F_{n+1} as the rightmost member above. Then $\{F_{n\geq 1}\}$ is a decreasing sequence of sets. All sets are of finite measure, hence $\mu(E) - \mu(\cup E_{j\leq n}) = \mu(F_{n+1}) \to 0$.

Now, for any fixed $n \geq 1$,

$$\left|\chi_E - \sum \chi_{E_{j\leq n}}\right| = \left|\sum \chi_{E_{j\geq n+1}}\right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the pth power does not change pointwise, and

$$\left|\sum \chi_{E_{j\geq n+1}}\right|^p = \left|\sum \chi_{E_{j\geq n+1}}\right| = \sum \chi_{E_{j\geq n+1}}$$

Convergence in p-norm is given by

$$\left\|\chi_E - \sum \chi_{E_{j\leq n}}\right\| = \left\|\sum \chi_{E_{j\geq n+1}}\right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our $\phi \in L^{p*}$

$$\begin{aligned}
\nu(E) &= \phi(\chi_E) \\
&= \phi\left(\lim_{n\to\infty} \sum \chi_{E_{j\leq n}}\right) \\
&= \lim_{n\to\infty} \phi\left(\sum \chi_{E_{j\leq n}}\right) \\
&= \lim_{n\to\infty} \sum \phi\left(\chi_{E_{j\leq n}}\right) \\
&= \lim_{n\to\infty} \sum \nu(E_{j\leq n})
\end{aligned}$$

To show absolute convergence, recall that for any $\phi(\chi_{E_j}) \in \mathbb{C}$, define $\beta_j = \overline{\operatorname{sgn}(\|\phi(\chi_{E_j})\|)}$ then multiplication yields

$$\|\phi(\chi_{E_j})\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the p-norm.

$$\left\|\sum_{j\geq 1} \beta_j \chi_{E_j} - \sum_{j\leq n} \beta_j \chi_{E_j}\right\|_p = \left\|\sum_{j\geq n+1} \beta_j \chi_{E_j}\right\|_p$$

And because $\left|\sum_{j\geq n+1} \beta_j \chi_{E_j}\right|$ is pointwise equal to $\left|\sum_{j\geq n+1} \chi_{E_j}\right|$, since $|\beta_j| = 1$ for every $j \geq 1$. We can reuse the same continuity and linearity argument. We also note that $\sum_{j\geq 1} \beta_j \chi_{E_j} \in L^p$ since its p-norm is equal to $\mu(E)^{1/p}$.

$$
\begin{aligned}
\sum_{j\geq 1} |\nu(E_j)| &= \sup_{n\geq 1} \sum_{j\leq n} \|\nu(E_{j\leq n})\| \\
&= \lim_{n\to\infty} \sum_{j\leq n} \|\phi(\chi_{E_j})\| \\
&= \lim_{n\to\infty} \sum_{j\leq n} \beta_j \phi(\chi_{E_j}) \\
&= \lim_{n\to\infty} \phi\left(\sum_{j\leq n} \beta_j \chi_{E_j}\right) \\
&= \phi\left(\lim_{n\to\infty} \sum_{j\leq n} \beta_j \chi_{E_j}\right) \\
&\leq \|\phi\| \left\| \sum_{j\geq 1} \beta_j \chi_{E_j} \right\|_p \\
&< +\infty
\end{aligned}
$$

Assuming the above estimate holds, then we only need $\nu(E) = \phi(\chi_E) = \mu(E) = 0$ (ν is now a measure and $\nu \ll \mu$), As the indicator of a null set is equal to the zero element in L^p. Then by Radon-Nikodym we can have some $g \in L^1(\mu)$ such that

$$d\nu = g d\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function g. For every χ_E measurable, $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$, by monotonicity of the integral in L^+. So any simple function, $\alpha = \sum a_j \cdot \chi_{E_j}$ means that αg is in $L^1(\mu)$, and

$$\phi(\alpha) = \int \alpha g d\mu$$

If $\|\alpha\|_p = 1$, then

$$\left| \int \alpha g \right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$M_q(g) = \sup \left\{ \left| \int \alpha \cdot g \right|, \|\alpha\|_p = 1, \text{ and } \alpha \text{ is simple and vanishes outside a set of finite measure.} \right\}$

Since $S_g = \{x \in X, g(x) \neq 0\}$ is σ-finite, an application of Theorem 6.14 tells us that $g \in L^q$, and $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$. Now that we know g is in L^q we can use the density of α in L^p to show, for every single $f \in L^p$

$$\phi(f) = \int fg d\mu$$

Conjure a sequence of 'α's, and call them $\{f_n\} \to f$ p.w.a.e, then each $f_n \cdot g \in L^1$. An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n \, gd\mu = \int fgd\mu = \phi(f)$$

This completes the proof for when μ is finite.

Let us upgrade our μ into a σ-finite measure. Then there exists an increasing sequence $\{E_n\} \nearrow X$ such that each E_n is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in P_n vanishes outside a set of finite measure and is also in L^p. And Q_n is defined in a similar manner. Now, fix our $\phi \in L^{p*}$, and for each $f \in P_n$, there exists a corresponding $g_n \in Q_n$. Then $p \in [1, +\infty)$ tells us that $q \in (1, +\infty]$, and the assumptions for Theorem 6.13 all hold. Therefore for each $g_n \in Q_n$, there is a corresponding bounded linear operator $\phi_{g_n} \in (P_n)^*$ such that

$$\phi(f) = \phi|_{P_n}(f) = \int fg_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of g_n towards some $g \in L^q$. We claim that this limit makes sense. As for any $n < m$, such that $E_n \subseteq E_m$ then $g_n = g_m$ on E_n pointwise. The proof is simple since each the restriction of our $\phi \in L^{p*}$ onto E_n and E_m spawns two functions g_n and $g_m \in L^1$. To verify, take any subset $Z \subseteq E_n$ then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So $g_n = g_m$ pointwise a.e on E_n. Now we define g measurable such that $g|_{E_n} = g_n$ for every n. And

$$\begin{aligned}
|g_n| = \chi_{E_n} \cdot |g_m| &\implies \\
|g_n| \leq |g_{n+1}| &\implies \\
&\|g_n\|_q \leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty
\end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with ($P_n \subseteq P_{n+1}$). If $q = +\infty$ then $g \in L^\infty$ is trivial, but for any $q < +\infty$. We wish to show that $g \in L^q$. Since $|g_n| \leq |g|$ pointwise for every n, and for each $x \in X$, there exists a N, where $n \geq N$ implies $|g(x)| = |g_n(x)|$, so $|g(x)|$ is an upperbound that is actually attained by the sequence $|g_n(x)|$. So, $|g(x)| = \sup_{n\geq 1}\{|g_n(x)|\}$.

Using the Monotone Convergence Theorem on $|g_n|$,

$$\begin{aligned}
\int \lim_{n\to\infty} |g_n|^q d\mu &= \int \sup_{n\geq 1} |g_n|^q d\mu \\
&= \int |g|^q d\mu \\
&= \lim \int |g_n|^q d\mu
\end{aligned}$$

Which yields $\|g\|_q^q = \lim\|g_n\|_q^q = \sup\|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$. It follows that $g \in L^q$.

Finally, we will show that $\phi(f) = \int fg$ for every $f \in L^p$. Redefine $f_n = f \cdot \chi_{E_n} \in P_n$ for every $n \geq 1$. We claim that $f_n \to f$ in the p-norm.

$$\begin{aligned}
|f_n - f| &\leq |f_n| + |f| \\
&\leq |f| + |f| \\
&\leq 2|f|
\end{aligned}$$

And $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$. Now it is permissiable to apply the Dominated Theorem, and we will do so.

$$
\begin{aligned}
\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\
\lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\
&= 0
\end{aligned}
$$

And we have $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$
\begin{aligned}
\phi(f) &= \lim \phi|_{P_n}(f_n) \\
&= \lim \int f_n \cdot g_n \\
&= \lim \int f \cdot g \cdot \chi_{E_n} \\
&= \int \lim \left(fg \cdot \chi_{E_n}\right) \\
&= \int fg
\end{aligned}
$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of $fg\chi_{E_n} \to fg$ pointwise and Holder's Inequality. This completes the proof for when μ is of σ-finite measure, and $p \in [1, +\infty)$.

Suppose now μ is arbitrary, and $p \in (1, +\infty)$, then $q < +\infty$. Now let us agree to define, for every σ-finite $E \subseteq X$

$$
P_E = \{L^p, |f| = |f| \cdot \chi_E\}
$$

Where Q_E does not hold any surprises. Then for each E we have a $\phi|_E$ which induces a g_E that vanishes outside E. We are ready for the final part of the proof.

First, if $E \subseteq F$ and both E and F are σ-finite, then $\|g_E\|_q \leq \|g_F\|_q$. This is a simple consequence of monotonicity in L^+ if we take $|g_E|^q \leq |g_F|^q$.

Second, we define

$$
W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}
$$

Let M be the supremum of W, then there exists a sequence of σ-finite sets, $\{E_n\}$ where $\|g_{E_n}\|_q \to M \leq \|\phi\|_{p*}$. Take a set $F = \cup E_{n\geq 1}$, which is also σ-finite, so that $\|g_F\|_q = M$. Now assume there exists another σ-finite superset of F, let us call it A. Then

$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have $g_{A\setminus F} = 0$ pointwise a.e. For any $f \in L^p$, the spots where f does not vanish is σ-finite. This comes from $\int |f|^p < +\infty$. So it suffices to integrate over this σ-finite set. But we already know, even if this set A contains F as a subset, $\int f g_F = \int f g_A$.

We now define $g = g_F$, and the proof is complete. As for every $\phi \in L^{p*}$, there exists a $g \in L^q$ such that the evaluation of any $f \in L^p$ is given by integrating f with g. ■