

Contents

Chapter 1	6
Theorem 1.1	7
Theorem 1.2	8
Theorem 1.3	10
Theorem 1.4	11
Theorem 1.5	12
Theorem 1.6	13
Theorem 1.7	14
Theorem 1.8	15
Theorem 1.9	16
Theorem 1.10	17
Theorem 1.11	18
Theorem 1.12	19
Theorem 1.13	20
Theorem 1.14	21
Theorem 1.15	22
Theorem 1.16	23
Theorem 1.17	24
Theorem 1.18	25
Exercise 1.4	26
Exercise 1.5	27
Exercise 1.7	28
Exercise 1.8	29
Exercise 1.12	30
Exercise 1.13	32
Exercise 1.14	33
Exercise 1.15	34

Chapter 2	37
Theorem 2.1	38
Chapter 3	39
Theorem 3.1	42
Theorem 3.2	43
Theorem 3.3	44
Theorem 3.4	45
Theorem 3.5	46
Theorem 3.6	47
Theorem 3.7	48
Theorem 3.8	49
Theorem 3.9	50
Theorem 3.10	51
Theorem 3.11	52
Theorem 3.12	53
Theorem 3.13	54
Theorem 3.14	55
Theorem 3.15	56
Theorem 3.16	57
Theorem 3.17	58
Theorem 3.18	61
Theorem 3.19	62
Theorem 3.20	63
Theorem 3.21	64
Theorem 3.22	66
Theorem 3.23	67
Theorem 3.24	68
Theorem 3.25	69
Theorem 3.26	70
Theorem 3.27	71
Theorem 3.28	72
Theorem 3.29	73
Theorem 3.30	74
Theorem 3.31	75
Theorem 3.32	76
Theorem 3.33	77
Theorem 3.34	78

Theorem 3.35	79
Theorem 3.36	80
Chapter 4	81
Theorem 4.1	82
Theorem 4.2	83
Theorem 4.3	84
Theorem 4.4	86
Theorem 4.5	88
Theorem 4.6	89
Theorem 4.7	90
Theorem 4.8	91
Theorem 4.9	92
Theorem 4.10	93
Theorem 4.11	94
Theorem 4.12	95
Theorem 4.13	97
Theorem 4.14	99
Theorem 4.15	102
Theorem 4.16	104
Theorem 4.17	106
Theorem 4.18	107
Theorem 4.19	108
Theorem 4.20	109
Theorem 4.21	111
Theorem 4.22	112
Theorem 4.23	113
Theorem 4.24	114
Theorem 4.25	115
Theorem 4.26	116
Theorem 4.27	117
Theorem 4.28	118
Theorem 4.29	119
Theorem 4.30	121
Theorem 4.31	123
Theorem 4.32	125
Theorem 4.33	127
Theorem 4.34	128

Theorem 4.35	129
Theorem 4.36	131
Theorem 4.37	132
Theorem 4.38	133
Theorem 4.39	134
Theorem 4.40	135
Theorem 4.41	136
Exercise 39	137
Chapter 5	138
Theorem 5.1	139
Theorem 5.2	140
Theorem 5.3	141
Theorem 5.4	142
Theorem 5.5	143
Theorem 5.6	144
Theorem 5.7	145
Theorem 5.8	146
Chapter 6	147
Theorem 6.1	148
Theorem 6.2	149
Theorem 6.3	150
Theorem 6.4	151
Theorem 6.5	152
Theorem 6.6	153
Theorem 6.7	154
Theorem 6.8	155
Theorem 6.9	156
Theorem 6.10	157
Theorem 6.11	158
Theorem 6.12	159
Theorem 6.13	160
Theorem 6.14	161
Theorem 6.15	162

Chapter 7	169
Theorem 7.1	170
Theorem 7.2	171
Theorem 7.3	185
Theorem 7.4	185
Theorem 7.5	186
Theorem 7.6	187
Theorem 7.7	188
Theorem 7.8	189
Theorem 7.9	190
Theorem 7.10	191
Theorem 7.11	192
 Chapter 8	 193
Theorem 8.1	194
Theorem 8.2	195
Theorem 8.3	196
Theorem 8.4	197
Theorem 8.5	198
Theorem 8.6	199
Theorem 8.7	200
Theorem 8.8	201
Theorem 8.9	202
Theorem 8.10	203
Theorem 8.11	204
Theorem 8.12	205
Theorem 8.13	206
Theorem 8.14	207
Theorem 8.15	213
Theorem 8.16	220
Theorem 8.17	221
Theorem 8.18	222
Theorem 8.19	223
Theorem 8.20	224

Chapter 1

Theorem 1.1

WTS. Let $\mathcal{M}(\mathcal{F})$ be the σ -algebra generated by \mathcal{F} , if \mathcal{E} is a subset of $\mathbb{P}(X)$, with $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

Proof. Notice that because $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$,

$$\mathcal{M}(\mathcal{F}) \in \{\mathcal{M}, \mathcal{E} \subseteq \mathcal{M}, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

Taking the intersection, noting that $\mathcal{M}(\mathcal{E})$ is the intersection of all σ -algebras containing \mathcal{E} as a subset, we have

$$\bigcap \{\mathcal{M}(\mathcal{F})\} \supseteq \bigcap \{\mathcal{M}, \mathcal{E} \subseteq \mathcal{M}, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

And

$$\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$$

■

Theorem 1.2

WTS. *The Borel σ -algebra of \mathbb{R} , \mathbb{B} is generated by the following*

- *The family of open intervals $\mathcal{E}_1 = \{(a, b), a < b\}$,*
- *The family of closed intervals $\mathcal{E}_2 = \{[a, b], a < b\}$,*
- *The family of half-open intervals $\mathcal{E}_3 = \{(a, b], a < b\}$ or $\mathcal{E}_4 = \{[a, b), a < b\}$*
- *The open rays $\mathcal{E}_5 = \{(a, +\infty), a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a), a \in \mathbb{R}\}$*
- *The closed rays $\mathcal{E}_7 = \{[a, +\infty), a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a], a \in \mathbb{R}\}$*

Proof. By definition, \mathbb{B} is generated by the family of all open sets in \mathbb{R} , but every open set is a countable union of open intervals. Therefore

$$\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_1) \implies \mathbb{B} \subseteq \mathcal{M}(\mathcal{E}_1)$$

Conversely, every open interval is an open set, hence

$$\mathcal{E}_1 \subseteq \mathcal{T}_{\mathbb{R}} \subseteq \mathbb{B} \implies \mathcal{M}(\mathcal{E}_1) \subseteq \mathbb{B}$$

Every closed interval can also be written as a countable intersection of open intervals, for every $[a, b]$, with $a < b$, we have

$$[a, b] = \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1}) \quad (1)$$

Indeed, fix any $x \in [a, b]$ then for every $n \geq 1$,

$$a - n^{-1} < a \leq x \leq b < b + n^{-1}$$

So $x \in \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1})$. If x an element of the left member, then for every $n \geq 1$,

$$a - n^{-1} < x \implies a - x < n^{-1}$$

Similarly for $x \leq b$, therefore equation (1) is valid, and $\mathcal{E}_2 \subseteq \mathbb{B} = \mathcal{M}(\mathcal{E}_1)$. To show the reverse estimate, every open interval can be written as a countable union of closed intervals,

$$(a, b) = \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \quad (2)$$

To show that the above estimate is indeed true, fix any $x \in (a, b)$, then

$$\begin{aligned} a < x < b &\iff a < a + n^{-1} \leq x \leq b - n^{-1} < b \\ &\iff x \in \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \end{aligned}$$

So that equation (2) holds. By similar argumentation we have $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_2) \implies \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_1)$.

For $\mathcal{E}_3, \mathcal{E}_4$

- $(a, b] = \bigcap_{n \geq 1} (a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1)$,
- $(a, b) = \bigcup_{n \geq 1} (a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3)$,
- $[a, b) = \bigcup_{n \geq 1} [a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_2)$,
- $[a, b] = \bigcap_{n \geq 1} [a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_2) \subseteq \mathcal{M}(\mathcal{E}_4)$

So that $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_4) = \mathbb{B}$. By taking complements of each element we get $\mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$ and $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$. Notice also that

- $(a, b] = (a, +\infty) \cap (-\infty, b]$, proves $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_5)$, and $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$.
- $(a, +\infty) = \bigcup_{n \geq 1} (a, a + n]$, proves $\mathcal{E}_5 \subseteq \mathcal{M}(\mathcal{E}_3)$, and $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_3)$.
- $[a, b) = [a, +\infty) \cap (-\infty, b)$, proves $\mathcal{E}_4 \subseteq \mathcal{M}(\mathcal{E}_6)$, and $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_7)$,
- $[a, +\infty) = \bigcup_{n \geq 1} [a, a + n)$, proves $\mathcal{E}_7 \subseteq \mathcal{M}(\mathcal{E}_4)$, and $\mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{M}(\mathcal{E}_4)$.

Finally, $\mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8) = \mathbb{B}$ and $\mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7) = \mathbb{B}$. ■

Theorem 1.3

WTS. *If A is countable, then $\otimes_{\alpha \in A} \mathcal{M}_\alpha$ is the σ -algebra generated by*

$$W := \left\{ \prod_{\alpha \in A} E_\alpha, E_\alpha \in \mathcal{M}_\alpha \right\}$$

Proof. We agree to define

$$V := \left\{ \pi_\alpha^{-1}(E_\alpha), E_\alpha \in \mathcal{M}_\alpha \right\}$$

By definition, V generates $\otimes_{\alpha \in A} \mathcal{M}_\alpha$. Fix any element in $x = \pi_\alpha^{-1}(E_\alpha) \in V$, then

$$\pi_\alpha(x) \in E_\alpha, \pi_{\beta \neq \alpha}(x) \in X_\beta$$

Then $x \in W$ if we choose $x = \prod_{c \in A} E_c$, for $E_c = E_\alpha$ if $c = \alpha$, and $E_c = X_c$ if $c \neq \alpha$. ■

Theorem 1.4

WTS.

Proof.



Theorem 1.5

WTS.

Proof.



Theorem 1.6

WTS.

Proof.



Theorem 1.7

WTS.

Proof.



Theorem 1.8

WTS.

Proof.



Theorem 1.9

WTS.

Proof.



Theorem 1.10

WTS.

Proof.



Theorem 1.11

WTS.

Proof.



Theorem 1.12

WTS.

Proof.



Theorem 1.13

WTS.

Proof.



Theorem 1.14

WTS.

Proof.



Theorem 1.15

WTS.

Proof.



Theorem 1.16

WTS.

Proof.



Theorem 1.17

WTS.

Proof.



Theorem 1.18

WTS.

Proof.



Exercise 1.4

WTS. *An algebra \mathcal{A} is a σ -algebra \iff it is closed under countable increasing unions.*

Proof. \Leftarrow is trivial. And it suffices to show that \mathcal{A} is closed under countable disjoint unions. Indeed, if $\{E_j\}_{j \geq 1} \subseteq \mathcal{A}$ is a countable disjoint sequence of sets, write

$$F_n = \bigcup E_{j \leq n}$$

Clearly, F_j is increasing, and denote $F = \bigcup E_{j \geq 1}$, which is a member of \mathcal{A} . We claim that

$$\bigcup F_{n \geq 1} = \bigcup E_{j \geq 1}$$

Fix any $x \in \bigcup E_{j \geq 1}$, then x belongs in some $E_j \subseteq F_j$, and \supseteq is proven. Also, if $x \in \bigcup F_{n \geq 1}$, then there exists some F_n for which x is a member of. For this particular F_n , means that $x \in E_j$ where $j \leq n$ and $x \in \bigcup E_{j \geq 1}$. \blacksquare

Exercise 1.5

WTS. Let $\mathcal{M}(\mathcal{E})$ be the σ -algebra generated by $\mathcal{E} \subseteq X$, and

$$\mathcal{N} = \left\{ \mathcal{M}(\mathcal{F}), \mathcal{F} \subseteq \mathcal{E}, \mathcal{F} \text{ is countable} \right\}$$

Show that $\mathcal{M}(\mathcal{E}) = \mathcal{N}$.

Proof. The outline of the proof is as follows,

1. Prove that $\mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$,
2. Show that \mathcal{N} is a σ -algebra,
3. Show that \mathcal{N} contains \mathcal{E} as a subset, and hence $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{N}$.

First, for any $\mathcal{F} \subseteq \mathcal{E}$, where \mathcal{F} is countable, it follows from Lemma 1.1 that $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E})$. Taking the union over all of such \mathcal{F} , we get $\bigcup \mathcal{M}(\mathcal{F}) = \mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$.

To show that \mathcal{N} is a σ -algebra, fix any $A \in \mathcal{N}$, and A belongs to $\mathcal{M}(\mathcal{F})$, therefore $A^c \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{N}$. To show closure under countable unions, fix a sequence $\{E_j\} \subseteq \mathcal{N}$, then each of these E_j belongs to a corresponding $\mathcal{M}(\mathcal{F}_j)$, for $j \in \{1, 2, \dots\}$. Now define

$$\overline{\mathcal{F}} = \bigcup \mathcal{F}_{j \geq 1} \subseteq \mathcal{E}$$

and $\overline{\mathcal{F}}$ is obviously countable. Hence for every $j \geq 1$, $\mathcal{M}(\mathcal{F}_j) \subseteq \mathcal{M}(\overline{\mathcal{F}})$ and taking the union yields

$$\bigcup \mathcal{M}(\mathcal{F}_{j \geq 1}) \subseteq \mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$$

It is also clear that our sequence $\{E_j\}$ is contained in $\mathcal{M}(\overline{\mathcal{F}})$, and $E = \bigcup E_j$ belongs to $\mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$ as an element. Therefore \mathcal{N} is a σ -algebra.

Let $\alpha \in A$ index the family of sets in \mathcal{E} , (so that $E_\alpha \in \mathcal{E}$) and the singleton set of a set $\{E_\alpha\}$ is a countable subset of \mathcal{E} . For every $\alpha \in A$, we have

$$E_\alpha \in \mathcal{M}(\{E_\alpha\}) \subseteq \mathcal{N} \implies \mathcal{E} \subseteq \mathcal{N}$$

And one final application of Lemma 1.1 finishes the proof. ■

Exercise 1.7

WTS. If μ_1, \dots, μ_n are measures on (X, \mathcal{M}) , and $a_1, \dots, a_n \in [0, +\infty)$, then $\mu = \sum_{j=1}^n \mu_j$ is a measure on (X, \mathcal{M}) .

Proof. If $\{E_j\}$ is a disjoint sequence in \mathcal{M} , and denote $E = \bigcup (E_j)$. If for each $k \leq n$, $\mu_k(E) < +\infty$,

$$\mu_k(E) = \sum \mu_k(E_j) \implies a_k \mu_k(E) = \sum a_k \mu_k(E_j)$$

Then,

$$\mu(E) = \sum_{k \leq n} a_k \mu_k(E) = \sum_{k \leq n} \sum_{j \geq 1} a_k \mu_k(E_j) = \sum_{j \geq 1} \sum_{k \leq n} a_k \mu_k(E_j) = \sum_{j \geq 1} \mu(E_j)$$

If there exists some μ_k such that $\mu_k(E) = +\infty$, then

$$\mu(E) = \sum_{k \leq n} \sum_{j \geq 1} a_k \mu_k(E_j)$$

Now if there exists some $\mu_{k'}$ with $\mu_{k'}(E) = +\infty$, then $\mu(E) = \sum_{k \leq n} \mu_k(E) = +\infty$, and

$$\sum_{j \geq 1} \mu(E_j) = \sup_N \sum_{j \leq N} \sum_{k \leq n} a_k \mu_k(E_j) \geq \mu_{k'}(E)$$

Therefore $\mu(E) = \sum_{j \geq 1} \mu(E_j)$, and μ is a measure. ■

Exercise 1.8

WTS. If (X, \mathcal{M}, μ) is a measure space, and $\{E_j\} \subseteq \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\bigcup E_{j \geq 1}) < +\infty$

Proof. If $\{E_j\}_{j \geq 1}$ is a sequence in \mathcal{M} , and define $F_m = \bigcap_{j \geq m} E_j$

$$\liminf E_j = \bigcup_{m \geq 1} \bigcap_{j \geq m} E_j = \bigcup_{m \geq 1} F_m$$

Also, for every $m \geq 1$, $F_m \subseteq F_{m+1}$, and F_m is an increasing sequence, because

$$[m, +\infty) \supseteq [m+1, +\infty) \implies F_m \subseteq F_{m+1}$$

Using continuity above, and writing $F = \bigcup F_{m \geq 1} = \liminf E_j$, we have

$$\begin{aligned} \mu(\liminf E_j) &= \mu(F) \\ &= \liminf \mu(F_m) \\ &\leq \liminf \mu(E_m) \end{aligned}$$

The second part of the proof is similar, if $G_m = \bigcup_{j \geq m} E_j$, then

$$\limsup E_j = \bigcap_{m \geq 1} \bigcup_{j \geq m} E_j = \bigcap_{m \geq 1} G_m$$

Similarly, G_m is a decreasing sequence, and since $\mu(\bigcup E_{j \geq 1}) = \mu(G_1)$ is finite, we can use continuity from above in the same manner, and the proof is complete. ■

Exercise 1.12

WTS. Let (X, \mathcal{M}, μ) be a finite measure space,

- If $E, F \in \mathcal{M}$, and $\mu(E \Delta F) = 0$, then $\mu(E) = \mu(F)$,
- Say that $E \sim F$ if $\mu(E \Delta F) = 0$, then \sim is an equivalence relation on \mathcal{M} ,
- For every $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \Delta F)$. Show that ρ defines a metric on the space of \mathcal{M}/\sim equivalence classes.

Proof of Part A. Use the fact that $\mu(F) = \mu(E \cap F) + \mu(F \cap E^c)$, and by monotonicity,

$$\mu(F \cap E^c) \leq \mu(E \Delta F) = 0$$

And $\mu(F) = \mu(E \cap F) = \mu(E)$, the last equality follows after a simple modification. ■

Proof of Part B. Suppose that $\mu(E \Delta F) = \mu(F \Delta G) = 0$, then

- $\mu(E \cap F^c) = \mu(F \cap E^c) \leq \mu(E \Delta F) = 0$ by monotonicity,
- Similarly, we have $\mu(F \cap G^c) = \mu(G \cap F^c) = 0$, and
- By subadditivity,
 - $\mu(E \cap G^c) = \mu(E \cap F^c \cap G^c) + \mu(E \cap F \cap G^c) \leq 0$, and $\mu(E \cap G^c) = 0$,
and
 - $\mu(G \cap E^c) = 0$
- Therefore $\mu(E \Delta G) = \mu(E \cap G^c) + \mu(G \cap E^c) = 0$

It is clear that the relation is reflexive, since $E \Delta E = \emptyset$, and symmetry is trivial. ■

Proof of Part C. Since $\rho(E, F) = \rho(F, E)$, and $\rho(E, F) \geq 0$ for every $E, F \in \mathcal{M}$, and $\rho(E, F) = 0 \iff E \sim F$. We only have to prove the Triangle Inequality. Notice that

$$\begin{aligned} \mu(E \setminus F) &= \mu(E \cap F^c \cap G) + \mu(E \cap F^c \cap G^c) \\ &\leq \mu(F^c \cap G) + \mu(E \cap F^c) \end{aligned}$$

and in the same fashion,

$$\mu(F \setminus E) \leq \mu(F \cap G^c) + \mu(E^c \cap F)$$

Combining the two inequalities, and applying additivity finishes the proof. ■

Exercise 1.13

WTS. *Every σ -finite measure is semi-finite*

Proof. Suppose μ is σ -finite then there exists an increasing sequence of sets $E_j \nearrow X$ with $\mu(E_j) < +\infty$. Now for every $W \in \mathcal{M}$, if $\mu(W) = +\infty$ then $\mu(W) = \lim_{j \rightarrow \infty} \mu(E_j \cap W) = +\infty$. Since this real-valued limit converges to its supremum $+\infty$, there exists a non-null subset $E_j \cap W$ of positive and finite measure. ■

Exercise 1.14

WTS. If μ is a semi-finite measure, and if $\mu(E) = +\infty$, for every $C > 0$, there exists an $F \subseteq E$ with $0 < \mu(F) < +\infty$.

Proof. Suppose by contradiction that there exists a $C > 0$ so for every $F \subseteq E$, if F is of finite measure, then $0 \leq \mu(F) \leq C$. Let $s = \sup\{\mu(F), F \subseteq E, 0 < \mu(F) < +\infty\}$, and for any $n^{-1} > 0$, this induces a F_n with measure

$$\mu(F_n) > s - n^{-1}$$

and take $A_n = \bigcup_{j \leq n} F_j$. A simple induction will show that $\mu(A_n) \leq \sum_{j \leq n} \mu(F_j) < +\infty$, therefore $\mu(A_n) \leq s$ for every $n \geq 1$. By continuity from below

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{j \geq 1} F_j\right) \leq s$$

Next, by monotonicity, denoting the union over A_n by A , for every $n^{-1} > 0$

$$s - n^{-1} \leq \mu(A_n) \leq \mu(A) \leq s \implies \mu(A) = s$$

Now, $E \setminus A$ is a set of infinite measure, and by semi-finiteness. Find a set $B \subseteq E \setminus A$ with strictly positive measure, so that

$$\mu(A \cup B) = \mu(A) + \mu(B) > s$$

And this finishes the proof. ■

Exercise 1.15

WTS. Given a measure μ on $(\mathbf{X}, \mathcal{M})$, and define $\mu_0 = \sup\{\mu(F), F \subseteq E, \mu(F) < +\infty\}$. Show μ_0 is semi-finite. Then, show that if μ is semi-finite, $\mu = \mu_0$. Lastly, there exists a measure ν on $(\mathbf{X}, \mathcal{M})$, with $\mu = \nu + \mu_0$, where ν only assumes the values 0 or $+\infty$.

Proof. First, a small Lemma. We claim that $\mu_0 = \mu$ on finite sets. Let $E \in \mathcal{M}$, and $\mu(E) < +\infty$, since

$$\mu(E) \in \{\mu(F), F \subseteq E, \mu(F) < +\infty\} \implies \mu(E) \leq \mu_0(E)$$

Next, for every $W \subseteq E$, $\mu(W) \leq \mu(E)$, so $\mu_0(E) \leq \mu(E)$. This proves the equality.

If E is any measurable subset of \mathbf{X} , and suppose also $\mu_0(E) = +\infty$, one can easily find subsets of E , $\{E_n\}_{n \geq 1}$ with

$$n \geq \mu(E_n) < +\infty$$

But E_n is a subset of finite measure, so $0 < \mu(E_n) = \mu_0(E_n) < +\infty$. This proves the semi-finiteness of μ_0 .

Next, suppose μ is semi-finite, and fix any measurable set E . If E is of finite measure, then $\mu(E) = \mu_0(E)$, and if $\mu(E) = +\infty$, apply Exercise 14, so there exists a sequence of subsets of finite measure $E_n \subseteq E$ for every $n \geq 1$, with $\mu(E_n) \rightarrow \mu(E)$. Therefore $\mu_0(E) = \mu(E)$.

For the last part of the proof, let μ be an arbitrary measure. And let $E \in \mathcal{M}$. If $\mu(E) < +\infty$, then $\nu(E) = 0$ would suffice (this proves the first property of the measure). If $\mu(E) = +\infty$, and if $\mu(E)$ is not semi-finite, then set $\nu(E) = +\infty$. So that $\mu_0(E) + \nu(E) = 0 + \infty = \infty = \mu(E)$. The additivity of ν is immediate, since ν can only assume two values. This finishes the proof. ■

Problem 1.24

WTS. If μ is a finite measure on (X, \mathcal{M}) , and let μ^* be the outer measure. Suppose that $E \subseteq X$ satisfies $\mu^*(E) = \mu^*(X)$ (but $E \notin \mathcal{M}$ necessarily). Show that

- (a) For any $A, B \in \mathcal{M}$, and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.
- (b) Let $\mathcal{M}_E = \{A \cap E, A \in \mathcal{M}\}$, and define ν on \mathcal{M} with $\nu(A \cap E) = \mu(A)$. Then \mathcal{M}_E is a σ -algebra, and ν is a measure on \mathcal{M}_E .

Proof of Part A.

$$\mu^*(E) = \mu^*(X) \implies \mu^*(X \setminus E) = 0$$

This is a simple consequence of the μ^* -measurability of X , since $X \in \mathcal{M}$, and the μ is a pre-measure on \mathcal{M} , and by monotonicity,

$$\begin{cases} A \cap (X \setminus E) \subseteq (X \setminus E) \\ B \cap (X \setminus E) \subseteq (X \setminus E) \end{cases} \implies \begin{cases} \mu^*(A \cap (X \setminus E)) = 0 \\ \mu^*(B \cap (X \setminus E)) = 0 \end{cases}$$

Write $A \cap X = (A \cap E) \cup (A \cap X \setminus E)$, and by subadditivity of μ^* ,

$$\begin{aligned} \mu(A) &= \mu^*(A \cap X) \\ &\leq \mu^*(A \cap E) + \mu^*(X \setminus E) \\ &= \mu^*(B \cap E) \\ &\leq \mu^*(B \cap X) \\ &= \mu(B) \end{aligned}$$

Therefore $\mu(A) \leq \mu(B)$, and $\mu(B) \leq \mu(A)$ is trivial. ■

Proof of Part B. We want to show \mathcal{M}_E is a σ -algebra.

- Closure under complements,

$$\forall A \cap E \in \mathcal{M}_E, A \in \mathcal{M} \implies (E \setminus A^c) \in \mathcal{M}_E$$

Therefore $(E \setminus A^c) \cap E \in \mathcal{M}_E$. Note that the question mentions that \mathcal{M}_E is a σ -algebra on E , therefore we take complements relative to E .

- Closure under countable unions. Fix any countable sequence $\{A_j \cap E\} \subseteq \mathcal{M}_E$ where $\{A_j\} \subseteq \mathcal{M}$. It is obvious that $A = \cup A_j \in \mathcal{M}$, therefore $\cup(A_j \cap E) = E \cap A \in \mathcal{M}_E$ as well.

Since $\nu(\emptyset) = \mu(\emptyset \cap E) = 0$, and for countable additivity, fix any disjoint sequence $\{A_j \cap E\}_{j \geq 1} \subseteq \mathcal{M}_E$, where $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$, and let $A = \cup A_{j \geq 1}$

$$\begin{aligned}\nu(A \cap E) &= \mu(A) \\ &= \sum \mu(A_{j \geq 1}) \\ &= \sum \nu(A_{j \geq 1} \cap E)\end{aligned}$$

■

Chapter 2

Theorem 2.1

WTS.

Proof.



Chapter 3

Notes on Chapter 3

WTS. *Prove two things,*

1. $\limsup_{r \rightarrow R} \phi(r) = \lim_{\varepsilon \rightarrow 0} \sup_{0 < |r-R| < \varepsilon} \phi(r) = \inf_{\varepsilon > 0} \sup_{0 < |r-R| < \varepsilon} \phi(r),$
2. $\lim_{r \rightarrow R} \phi(r) = c \iff \limsup_{r \rightarrow R} |\phi(r) - c| = 0$

Proof. ■

WTS. If $U \subseteq B(1, 0) = \{|x| < 1\}$, and $U \in \mathbb{B}$, and if $m(U) > 0$, then the family of sets

$$E_r = \left\{ x + ry, y \in U \right\}$$

shrinks nicely to $x \in \mathbb{R}^n$.

Proof. Let $r > 0$ be fixed then $\forall z \in E_r \ni z = x + ry$. Hence,

$$\begin{aligned} d(x, z) &= d(x, x + ry) \\ &= |r|d(0, y) < |r| \end{aligned}$$

by translation invariance. ■

Theorem 3.1**WTS.**

Proof. Let ν be a signed measure, and fix any increasing sequence $E_j \nearrow E = \bigcup E_{j \geq 1}$ of sets. This induces a disjoint sequence in $\{F_n\}$. Define $F_1 = E_1$, and if $n \geq 2$,

$$F_n = E_n \setminus \bigcup E_{j \leq n-1}$$

and from this, the finite It is clear that $\bigcup F_{n \geq 1} = E$, and let us assume $\nu(E)$ is of finite measure.

By countable additivity, and the absolute convergence of the series $\sum_{j \leq n} \nu(F_j)$

$$\begin{aligned} \nu\left(\bigcup E_{j \geq 1}\right) &= \sum_{j \geq 1} \nu(F_j) \\ &= \lim_n \sum_{j \leq n} \nu(F_j) \\ &= \lim \nu(E_n) \end{aligned}$$

■

Theorem 3.2

WTS.

Proof.



Theorem 3.3

WTS.

Proof.



Theorem 3.4

WTS.

Proof.



Theorem 3.5

WTS.

Proof.



Theorem 3.6

WTS.

Proof.



Theorem 3.7

WTS.

Proof.



Theorem 3.8

WTS.

Proof.



Theorem 3.9

WTS.

Proof.



Theorem 3.10

WTS.

Proof.



Theorem 3.11

WTS.

Proof.



Theorem 3.12

WTS.

Proof.



Theorem 3.13

WTS.

Proof.



Theorem 3.14

WTS.

Proof.



Theorem 3.15

WTS.

Proof.



Theorem 3.16

WTS.

Proof.



Theorem 3.17

WTS. Let the maximal function of any measurable $f \in \mathbb{B}_{\mathbb{R}^n}$ be denoted by $Hf(x)$, more precisely,

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

where $A_r|f|$ is the average of $|f|$ on a ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$. In symbols,

$$A_r|f| = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

The maximal theorem makes two claims:

1. $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, and Hf is measurable for every $f \in L^1_{loc}$.
2. There exists a $C > 0$, for every $f \in L^1$

$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1$$

for every $\alpha > 0$.

Proof. Let $\alpha > 0$ and fix $z \in (Hf)^{-1}((\alpha, +\infty))$, so $Hf(z) > \alpha$ and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with $Hf(z) - \alpha > 0$, we get some $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some $r_0 > 0$. Since $A_r|f|$ (a function indexed by $r > 0$) is continuous in $x \in \mathbb{R}^n$, $(A_r|f|)^{-1}((\alpha, +\infty))$ is open, and Hf is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{ Hf > \alpha \right\} = \bigcup_{r>0} \{A_r |f| > \alpha\}$$

Let $x \in E_\alpha$, this induces a $r_x > 0$ where $x \in \left\{ A_{r_x} |f| > \alpha \right\}$. Rearranging gives

$$\left(\frac{1}{\alpha} \int_{B(r,x)} |f| dz \right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each $x \in E_\alpha \mapsto r_x > 0 \mapsto A_{r_x} |f|$,
- If $U = \bigcup_{x \in E_\alpha} B(r_x, x)$, then $E_\alpha \subseteq U$,
- Choose $c < m(E_\alpha) \leq m(U)$ (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

$$x_1, \dots, x_N \in E_\alpha$$

so that $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$, and $c < 3^n \sum_{j \leq k} m(B_j)$

- Define $B_j = B(r_{x_j}, x_j)$ for all $j \leq k$, and

$$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

by finite additivity,

$$c 3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

and finally

$$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of m on \mathbb{B} , since

$$m(E_\alpha) = \sup \left\{ m(K), K \in \mathcal{J}_{\mathbb{R}^n}, K \subseteq E_\alpha \right\}$$

for any $K \in \mathcal{J}_{\mathbb{R}^n}$, $K \subseteq E_\alpha$, we have $m(K) < +\infty$, $m(K) \leq m(E_\alpha)$ and

$$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

Remark 17.1 *We used the properties of a Radon Measure here, without relying on the phrase ‘sending $c \rightarrow E_\alpha$ ’, which would require us to deal with two cases $m(E_\alpha) < +\infty$ and $m(E_\alpha) = +\infty$.*

■

Theorem 3.18**WTS.***Proof.*

Theorem 3.19

WTS.

Proof.



Theorem 3.20

WTS.

Proof.



Theorem 3.21

WTS. *The Lebesgue Differentiation Theorem. Suppose $f \in L^1_{loc}$, and for every $x \in \mathcal{L}_f$, (so that $x \in \mathbb{R}^n$ a.e). We have*

1. $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0,$
2. $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x),$

For every family $\{E_r\}_{r>0}$ that shrinks nicely to $x \in \mathbb{R}^{n'}$.

Proof. Since the family $\{E_r\}_{r>0}$ shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r, x)) \implies m(E_r) > \alpha \cdot m(B(r, x))$$

for some $\alpha > 0$, independent on r . Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r, x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r, x)} |f(y) - f(x)| dy$$

Combining the last two results, for every $\varepsilon > 0$, if $0 < r < \varepsilon$, then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0 < r < \varepsilon} m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

and sending $\varepsilon \rightarrow 0$, proves the first claim. The second claim is immediate upon applying the L^1 inequality.

Fix any $\varepsilon > 0$, and

$$\begin{aligned}
 \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} f(y) dy = f(x) &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y) dy - f(x) \right| \\
 &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)] dy \right| \\
 &\leq \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
 &= \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
 &= 0
 \end{aligned}$$

■

Theorem 3.22

WTS.

Proof.



Theorem 3.23

WTS.

Proof.



Theorem 3.24

WTS.

Proof.



Theorem 3.25

WTS.

Proof.



Theorem 3.26

WTS.

Proof.



Theorem 3.27

WTS.

Proof.



Theorem 3.28

WTS.

Proof.



Theorem 3.29

WTS.

Proof.



Theorem 3.30

WTS.

Proof.



Theorem 3.31

WTS.

Proof.



Theorem 3.32

WTS.

Proof.



Theorem 3.33

WTS.

Proof.



Theorem 3.34

WTS.

Proof.



Theorem 3.35

WTS.

Proof.



Theorem 3.36

WTS.

Proof.



Chapter 4

Theorem 4.1

WTS. Suppose that A is a subset of X , let $\text{acc } A$ be the set of accumulation points of A , then

$$\overline{A} = A \cup \text{acc } (A) \quad (3)$$

and A is closed if and only if $\text{acc } (A) \subseteq A$.

Proof. Suppose that $x \notin \overline{A}$, then $x \in (\overline{A})^c = A^{\circ}$, then $A^c \in \mathcal{N}_B(x)$. But this means that $x \notin \text{acc } (A)$, since there exists a neighbourhood of x (in the form of A^c), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \emptyset$$

Also, $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$ which means that

$$x \notin \overline{A} \implies x \notin A$$

Since $x \notin \overline{A} \implies x \notin A$ and $x \notin \text{acc } (A)$,

$$(\overline{A})^c \subseteq A^c \cap \text{acc } (A)^c = (A \cup \text{acc } (A))^c$$

Now, if $x \notin \text{acc } (A) \cup A$, then $x \notin \text{acc } (A)$, therefore there exists some $U \in \mathcal{N}_B(x)$ such that

$$A \cap U \setminus \{x\} = A \cap U = \emptyset$$

Where for the second last equality we used the fact that $x \notin A \implies A \setminus \{x\} = A$, and taking complements gives us

$$U \subseteq A^c$$

And since $U \in \mathcal{N}_B(x)$, then $x \in U^{\circ} \subseteq A^{\circ}$ (since U° is an open subset of A^c). then

$$x \in A^{\circ} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore $(A \cup \text{acc } (A))^c \subseteq (\overline{A})^c$. ■

Theorem 4.2

WTS. If \mathcal{T}_X is a topology on X and $\mathcal{E} \subseteq \mathcal{T}_X$ then \mathcal{E} is a base for \mathcal{T}_X if and only if for every

$$\forall U \in \mathcal{T}_X, U \neq \emptyset, \implies U = \bigcup_{V \in B} V$$

Where B is a subset of \mathcal{E} .

Proof. Suppose that \mathcal{E} is a base, then fix any non-empty $U \in \mathcal{T}_X$, then for every $x \in U$, there exists a neighbourhood base for this x and a member $V \in \mathcal{E}$ such that $x \in V_x \subseteq U$. Take the union over all V_x and

$$U \subseteq \bigcup_{x \in U} V_x$$

But each $V_x \subseteq U$, so $U = \bigcup_{x \in U} V_x$, where $\{V_x\} \subseteq \mathcal{E}$.

Conversely, if every non-empty U is a union of members in \mathcal{E} then fix any $x \in X$, we claim that we have a neighbourhood base in

$$\{V \in \mathcal{E}, x \in V\}$$

The reason is as follows

- x belongs to every $E \in \{V \in \mathcal{E}, x \in V\}$ and
- For every open U , if $x \in U$ then there exists a union of members of \mathcal{E} such that $U = \bigcup E_\alpha$, then $x \in U \iff \exists E_\alpha \in \{V \in \mathcal{E}, x \in V\}$ and
- Using this particular $E_\alpha \in \mathcal{E}$ that we just found, $x \in E_\alpha \subseteq U$, and we are done.

■

Theorem 4.3

WTS. For every $\mathcal{E} \subseteq \mathbb{P}(X)$, \mathcal{E} is base for a topology on X if and only if

- (a) each $x \in X$ is contained in some $V \in \mathcal{E}$, and
- (b) if $U, V \in \mathcal{E}$, and $x \in U \cap V$, then there must exist some $W \in \mathcal{E}$ with $x \in W \subseteq U \cap V$.

Proof. Suppose that \mathcal{E} is a base, then we get a), and b) follows since for every $U, V \in \mathcal{E} \subseteq \mathcal{T}_X$, and by closure over finite intersections, $U \cap V \in \mathcal{T}_X$ implies that there exists some $W \in \mathcal{E}$ with

$$x \in W \subseteq U \cap V$$

Now, suppose both a) and b) hold, then we claim that this $\mathcal{E} \subseteq \mathbb{P}(X)$ induces a topology on X

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, \text{ with } x \in V \subseteq U\}$$

Intuitively speaking, this means that \mathcal{T} is just fine (and not too fine) to satisfy the conditions for $\mathcal{E} \subseteq \mathcal{T}$ to be a base of \mathcal{T} .

We first show that \mathcal{T} is a topology.

- $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$, the first is trivial and the second is from a)
- Closure under unions: fix $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, and $U = \bigcup U_\alpha$, and for every $x \in U$ there exists some $V_\alpha \in \mathcal{E}$ such that $x \in V_\alpha \subseteq U_\alpha \subseteq U$, therefore $U \in \mathcal{T}$.
- Closure under finite intersections, fix any U_1, U_2 as elements in \mathcal{T} , then suppose that they are not disjoint (if they are disjoint then their intersection is the empty set, which is also contained in \mathcal{T}). If $U_1 \cap U_2 \neq \emptyset$, then for every $x \in U_1 \cap U_2$ induces two sets $V_1, V_2 \in \mathcal{E}$ with $x \in V_1 \subseteq U_1$ and $x \in V_2 \subseteq U_2$, taking their intersection and applying b) gives us some $V \subseteq V_1 \cap V_2$ with $V \in \mathcal{E}$ therefore $x \in V \subseteq U_1 \cap U_2$, and \mathcal{T} is closed under finite intersections.

Now to show that \mathcal{E} is a base for \mathcal{T} , $\mathcal{E} \subseteq \mathcal{T}$ is obvious since every $V \in \mathcal{E}$ satisfies the properties laid out by \mathcal{T} by simply choosing V again for any

$x \in V$. Now fix any member $U \in \mathcal{T}$, then for every $x \in U$, there exists some $V \in \mathcal{E}$ with

$$x \in V \subseteq U$$

(This is an immediate consequence of how we defined \mathcal{T}). And we can conclude that \mathcal{E} is a base for this induced topology \mathcal{T} . ■

Theorem 4.4

WTS. If $\mathcal{E} \subseteq \mathbb{P}(X)$, the topology $\mathcal{T}(\mathcal{E})$ generated by \mathcal{E} consists of \emptyset, X and all unions of finite intersections of \mathcal{E} , in symbols

$$\mathcal{T}(\mathcal{E}) = \{\emptyset, X\} \cup \left\{ \bigcup W_\alpha, W_\alpha = \bigcap E_{j \leq n}, E_j \in \mathcal{E} \right\}$$

Proof. Denote the set

$$W = \{X\} \cup \left\{ \bigcap V_{j \leq n}, V_j \in \mathcal{E} \right\}$$

We claim this set W satisfies Theorem 4.3. Since 4.3a) is satisfied with $X \in W$. 4.3b) follows since the right member in W is closed under intersections.

And if we are taking an element from each member, $E_1 \in \{\emptyset, X\}$ and E_2 is an element in the right member, then it is trivial to verify that their intersection is always contained within W . Therefore W induces a topology by Theorem 4.2, and we call this topology \mathcal{T} — and for the sake of completeness

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, x \in V \subseteq U\}$$

We so claim that if we define \overline{W} as the union of all members $w \in W$, together with the empty set, is equal to the set \mathcal{T} .

$$\overline{W} = \left\{ \bigcup_{w \in W} w \right\} \cup \{\emptyset\}$$

- We want to show $\mathcal{T} \subseteq \overline{W}$, since W is a base for the topology \mathcal{T} , every (non-empty) $U \in \mathcal{T}$ is the union of members in W (Theorem 4.2), and there exists some $B \subseteq W$ with

$$U = \bigcup E_{\alpha \in B} \in \overline{W}$$

Now if U is the empty set then it is trivially contained within \overline{W} .

- Next, we show that $\overline{W} \subseteq \mathcal{T}$, fix any element $E \in \overline{W}$, if $E = \emptyset$ then there is nothing to prove since \mathcal{T} is a topology. Now for every $x \in E$,

$$x \in E = \bigcup_{w \in W} w \implies x \in w$$

Therefore $E \in \mathcal{T}$ by definition. This proves that $\mathcal{T} = \overline{W}$.

Now that \overline{W} is a topology, that contains \mathcal{E} as a subset, and by definition of $\mathcal{T}(\mathcal{E})$

$$\mathcal{T}(\mathcal{E}) = \bigcap \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Tells us

$$\mathcal{T}(\mathcal{E}) \subseteq \overline{W}, \quad \text{since } \overline{W} \in \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Conversely, fix any member $E \in \overline{W}$, if $E = \emptyset$ then $E \in \mathcal{T}(\mathcal{E})$, if not, then there exists some subset $B \subseteq W$ such that

$$E = \bigcup_{w \in B} w = \bigcup_{w \in B} \bigcap_{j \leq n} V_{j \leq n}^w V_j \in \mathcal{E} \cup \{X\}$$

Since $\mathcal{T}(\mathcal{E})$ is closed under finite intersections and unions, and it contains \mathcal{E} as a subset, $\overline{W} = \mathcal{T}(\mathcal{E})$ and we are done. ■

Theorem 4.5

WTS. *Every second countable space is separable. (Countable dense subset).*

Proof. What we wish to prove is that if a space X has a countable base, then it has a countable dense subset. Denote this base of X by \mathcal{E} as usual, then we claim that

$$W = \{x_u, U \in \mathcal{E}\}$$

Is a dense subset in X . Note that $(\overline{W})^c = W^{\circ} \in \mathcal{T}_X$. If $W^{\circ} = \emptyset$ then we simply take complements and we get $\overline{W} = X$. So suppose that W° is non-empty, then for each $x \in W^{\circ}$ (by definition of a base), it should induce some $V_x \in \mathcal{E}$ with

$$x \in V_x \subseteq W^{\circ}$$

But clearly, for every element in \mathcal{E} , the second estimate can never be satisfied, since for every $U \in \mathcal{E}$, $x_U \notin W^{\circ}$ for this particular set W° . Therefore W° must be empty, and this completes the proof. ■

Theorem 4.6

WTS. *If X is first countable, then for every $A \subseteq X$, $x \in \overline{A} \iff$ there exists some sequence $\{x_j\}_{j \geq 1} \subseteq A$ such that $x_j \rightarrow x$.*

Proof. Suppose that X is first countable, and $A \subseteq X$, and fix any element $x \in \overline{A}$. Since X is first countable, there is a sequence of descending neighbourhoods of $\{U_j\}_{j \geq 1}$ of x such that

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_j \supseteq U_{j+1}$$

If $x \in A$, take $x_n = x$ for all $n \geq 1$. If $x \in \text{acc}(A)$, then take $x_n \in U_n \cap A \setminus \{x\} = U_n \cap A$, which is not empty. Then it remains to show that this sequence converges to x . Fix any neighbourhood $U \in \mathcal{N}_B(x)$ then there exists some N , for every $n \geq N$

$$x \in U^o \implies \exists N \in \mathbb{N}^+, x \in U_N \subseteq U^o$$

Then every $x_n \in A \cap U_N \subseteq A \cap U^o \subseteq U^o$. And this establishes \implies .

Now suppose that $x \notin \overline{A}$, so that $x \notin A$ and $x \notin \text{acc}(A)$, then fix any sequence $\{x_j\} \subseteq A$. We wish to show that $x_j \not\rightarrow x$.

Since $x \notin \text{acc}(A)$, there exists some $V \in \mathcal{N}_B(X)$ with

$$A \cap V \setminus \{x\} = \emptyset \implies V \subseteq A^c$$

Since $\{x_j\}_{j \geq 1} \subseteq A \implies x_j \notin A^c$ for every $j \geq 1$, then choose V as the neighbourhood around x , and $x_j \not\rightarrow x$ for any arbitrary sequence x_j in A . ■

Remark 6.1 *To truly understand what is going on one should recall that all metric space spaces are first countable.*

Theorem 4.7

WTS. X is a T_1 space $\iff \{x\}$ is closed for every $x \in X$.

Proof. If X is T_1 and $x \in X$, then for every $y \neq x$ there exists some open U_y that contains y but not x . Following Folland's argument closely, every $y \neq x$ is in $\cup U_{y \neq x}$. Hence $\{x\}^c \subseteq \cup U_{y \neq x}$. To show the converse, for every $z \in \cup U_{y \neq x}$ that is open, there exists a $y \neq x$ such that $z \in U_y$. But every U_y does not contain x as an element, so $z \neq x$ implies that $z \notin \{x\}$. And $z \in \{x\}^c$. Hence $\cup U_{y \neq x} = \{x\}^c$.

Now conversely if every $x \in X$ satisfies the fact that $\{x\}^c$ is open, then $\{x\}^c$ is an open set that contains every $y \neq x$. Now fix some $y \neq x$, since $\{y\}$ is also closed, we have $X \cap \{y\}^c$ is an open set that contains x but not y . Also, $\{x\}^c$ is an open set that contains y but not x . And therefore X is T_1 . ■

Theorem 4.8

WTS. *The map $f : X \rightarrow Y$ is continuous if and only if f is continuous at every $x \in X$.*

Proof. Suppose that f is continuous, then fix any $f(x) \in Y$ and any of its neighbourhood $V \in \mathcal{N}_B(f(x))$,

$$f(x) \in V^o \implies f^{-1}(V^o) \in \mathcal{N}_B(x)$$

But by continuity, $f^{-1}(V^o)$ is an open set that contains x , with

$$f\left(f^{-1}(V^o)\right) \subseteq V^o$$

Therefore f is continuous at x . Now suppose that f is continuous at every $x \in X$, then for every open subset $V \subseteq Y$, and for every point $f(x) \in V = V^o$ means that $V \in \mathcal{N}_B(f(x))$ for all such points $f(x)$. By continuity, for every x in $f^{-1}(V)$, implies that $f^{-1}(V)$ is a neighbourhood of all of its elements, therefore $f^{-1}(V) \subseteq (f^{-1}(V))^o$, and $f^{-1}(V)$ is open. ■

Theorem 4.9

WTS. *If \mathcal{E}_Y generates the topology on Y , and f is a mapping from $X \rightarrow Y$, then $f : X \rightarrow Y$ is continuous if and only if $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{E}_Y$.*

Proof. The inverse image commutes with intersections, complements, and unions. To prove \Leftarrow , use Theorem 4.4, since every $U \in \mathcal{T}_Y$ can be represented the union of finite intersections of elements \mathcal{E}_Y , and use the fact that \mathcal{T}_X is closed under arbitrary unions and finite intersections.

To show \Rightarrow , since $\mathcal{E}_Y \subseteq \mathcal{T}_Y$, if f^{-1} is open for every $U \in \mathcal{T}_Y$, then it is open for every $U \in \mathcal{E}_Y$ as well. ■

Theorem 4.10

WTS. *If X_α is Hausdorff for each $\alpha \in A$, then $X = \prod_{\alpha \in A} X_\alpha$ is Hausdorff.*

Proof. If two elements in X , $x \neq y$ then there exists some $\alpha \in A$ such that $\pi_\alpha(x) \neq \pi_\alpha(y) \in X_\alpha$, but this X_α is Hausdorff, then there exists two open, disjoint sets $V_x, V_y \subseteq X_\alpha$ such that

- $x \in \pi_\alpha^{-1}(V_x)$, and $y \in \pi_\alpha^{-1}(V_y)$
- $\pi_\alpha^{-1}(V_x) \cap \pi_\alpha^{-1}(V_y) = \pi_\alpha^{-1}(V_x \cap V_y) = \emptyset$
- $\pi_\alpha^{-1}(V_x), \pi_\alpha^{-1}(V_y) \in \mathcal{T}_X$

Where for the last bullet point we used the fact that the product topology makes all the projection maps continuous. This proves that X is Hausdorff. ■

Theorem 4.11

WTS. If X_α and Y are topological spaces, and $X = \prod_{\alpha \in A} X_\alpha$, and $f : Y \rightarrow X$ is a mapping. Then f is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in A$.

Proof. If $\pi_\alpha \circ f$ is continuous at each α , this means that

$$\forall \alpha \in A, \forall E_\alpha \in \mathcal{T}_\alpha, f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{T}_Y$$

But it is exactly sets of the form $\pi_\alpha^{-1}(E_\alpha)$ which generate the weak topology for \mathcal{T}_X . Therefore f is continuous.

Now, suppose that f is continuous, by definition of the weak topology (as it is generated by the set of inverse projections), for every $\alpha \in A$, $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{T}_X$ and by continuity of f , its inverse image is open in Y as well. ■

Remark 11.1 *The take-away intuition here is that if the range space is generated by some \mathcal{E} , then a function is continuous if and only if all inverse images of sets in \mathcal{E} are open in the domain space. Furthermore, if the range space is endowed with the product topology (which is generated by sets of the form $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{T}_\alpha$), then it suffices to check all inverse images of those. And this is equivalent to checking that $\pi_\alpha(\cdot) \circ f$ is continuous at each α .*

Theorem 4.12

WTS. *If X is a topological space, and A is any non-empty set, $\{f_n\} \subseteq X^A$ is a sequence, then $f_n \rightarrow f$ with respect to the product topology if and only if $f_n \rightarrow f$ pointwise.*

Proof. Suppose that $f_n \rightarrow f$ pointwise. Since the product topology \mathcal{T}_X is generated from sets of the form

$$\pi_\alpha^{-1}(E_\alpha), E_\alpha \in \mathcal{T}_\alpha$$

And by Theorem 4.4, \mathcal{T}_X consists of \emptyset, X and unions of finite intersections of $\pi_\alpha^{-1}(E_\alpha)$. We claim that for every $f \in X^A$, the following is a valid neighbourhood base for f

$$\left\{ \bigcap_{j \leq n} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), E_{\alpha_j} \in \mathcal{T}_{\alpha_j} \cap \mathcal{N}_B(\pi_{\alpha_j}(f)) \right\}$$

A couple things to note

- Each E_{α_j} is open in X_{α_j} , so that its inverse image is also open (in X). Since any neighbourhood base has to be a subset of \mathcal{T}_X .
- Only finitely many intersections are involved, so each element in the above set is open in X .
- Each E_{α_j} is a neighbourhood of $\pi_{\alpha_j}(f)$, meaning $f \in E_{\alpha_j}^\circ = E_{\alpha_j}$.
- Last and perhaps most importantly for intuition, fix any non-empty open set $U \in \mathcal{T}_X$ then by Theorem 4.4 (or my reading of it), U can be written as the union of sets like

$$\bigcap_{j \leq m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}$$

Then applying Theorem 4.2, the family of finite intersections of $\pi_\alpha^{-1}(E_\alpha)$ is a base for \mathcal{T}_X . Then,

$$N_{base}(f) = \left\{ V = \bigcap_{j \leq m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}, \quad f \in V \right\}$$

Has to be a neighbourhood base for any $f \in X$.

Now to show that $f_n \rightarrow f$ in the product topology, fix any neighbourhood $U \in \mathcal{N}_B(f)$, then $f \in U^o$, and by definition of a neighbourhood base, there exists some $E \in N_{base}(f)$ such that $f \in E \subseteq U^o$, but this E is just the finite intersection of $\pi_{\alpha_j}^{-1}(E_{\alpha_j})$, then at every α_j

- Let N_j be an integer such that for every $n \geq N_j$, $\pi_{\alpha_j}(f_n) \in E_{\alpha_j}$
- Set $N = \sum_{j \leq m} N_j \geq N_j$ for every $j \leq m$.

Then for every $n \geq N$, $f_n \in E \subseteq U^o \subseteq U$ for any arbitrary neighbourhood U of f . So $f_n \rightarrow f$ in the product topology.

Conversely, suppose that $f_n \rightarrow f$ in the product topology, then fix any $\alpha \in A$, and for every neighbourhood E_α of $\pi_\alpha(f)$, $\pi_\alpha^{-1}(E_\alpha)$ is a neighbourhood of f . Hence for every $\alpha \in A$, and for every neighbourhood E_α of $\pi_\alpha(f)$, $\pi_\alpha(f_n)$ is eventually in E_α . This completes the proof. ■

Theorem 4.13

WTS. *If X is a topological space then $BC(X)$ is a closed subspace of $B(X)$ in the uniform metric, and $BC(X)$ is complete.*

Proof. Suppose that $\{f_n\} \subseteq BC(X)$ converges to some f . There are a couple things that we need to show prior to tackling the main proof.

- (a) $B(X)$ endowed with the uniform norm of an $f \in B(X)$

$$\|f\|_u = \sup\{|f(x)|, x \in X\}$$

Is indeed a normed vector space.

- (b) $B(X)$ with its norm (and induced metric), is a complete metric space.
So that our $\{f_n\} \rightarrow f$ at worst, converges to $f \in B(X)$.

To show that $B(X)$ is a normed vector space, for any $k \in \mathbb{C}$, $f_1, f_2 \in B(X)$, then at every $x \in X$

$$|f_1(x) + kf_2(x)| \leq |f_1(x)| + |k| \cdot |f_2(x)| \leq \|f_1\|_u + |k|\|f_2\|_u$$

And to show absolute homogeneity, note that $\sup |kA| = |k| \cdot \sup A$ for any non-empty bounded above set of reals A . This proves (a).

To show (b), fix any Cauchy sequence (with respect to the uniform metric), then for every $\varepsilon > 0$, there exists an N so large that for every $n, m \geq N$ we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_u < \varepsilon$$

This shows that $\{f_n(x)\}_{n \geq 1} \subseteq \mathbb{C}$ is a Cauchy, and it makes sense to call its limit $f(x) = \lim f_n(x)$. To show that for this f ,

- $f_n \rightarrow f$ uniformly, and
- $f \in B(X)$

Fix an $\varepsilon > 0$, and there exists an N so large that for every $m, n \geq N$ implies that

$$\|f_n(x) - f_m(x)\|_u < \varepsilon$$

Since $\lim_{n \rightarrow \infty} f_n(x) = f(x)$, this means that

$$\lim_{n \rightarrow \infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| \leq \varepsilon$$

In the above we replaced the strict inequality with an inequality since the sequence may converge to its supremum. Since this holds for any $x \in X$, we have

$$\|f_m - f\|_u \leq \varepsilon$$

One can easily replace all the ε with $\varepsilon/2$ to obtain strict inequalities, to finish the proof, simply send $m \rightarrow \infty$ (since $f_m \rightarrow f$ pointwise everywhere, the uniform norm goes to zero as well). This proves both bullet points.

Now we will prove Theorem 4.13, for any sequence $\{f_n\} \subseteq BC(X)$, if it does converge to some f uniformly, then we claim that $f \in BC(X)$. Note that $f \in B(X)$, so it suffices for us to show that f is continuous at every point $x \in X$.

Fix any ball with radius $\varepsilon > 0$ at $f(x) \in \mathbb{C}$, and since

- $\varepsilon/3 > 0$ induces some N such that for every $n \geq N$, at every point $x \in X$

$$|f_n(x) - f(x)| \leq \|f_n - f\|_u < \varepsilon/3$$

- Another $\varepsilon/3$ ball around $f_n(x)$ (using the same point $x \in X$), such that its inverse image is an open set $U \in \mathcal{T}_X$, because $f_n \in BC(X)$

$$f_n^{-1}(V_{\varepsilon/3} f_n(x)) = U \in \mathcal{T}_X$$

- The last $\varepsilon/3$ comes from the fact that $y \in U \subseteq X$ so it satisfies

$$|f_n(y) - f(y)| \leq \|f_n - f\|_i < \varepsilon/3$$

Combining these three,

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f(x) - f_n(x)| + |f_n(x) - f_n(y)| < \varepsilon$$

So there exists some open set $U \in \mathcal{T}_X$ (and hence neighbourhood of every x), for every open ball of radius $\varepsilon > 0$, around every $f(x) \in \mathbb{C}$, such that

$$f(U) \subseteq B \in \mathcal{T}_{\mathbb{C}}$$

Since the open balls are a neighbourhood base at every point in \mathbb{C} , and f is continuous at every point $x \in X$, we must conclude that $f \in BC(X)$. ■

Theorem 4.14

WTS. Suppose that A and B are disjoint closed subsets of the normal space X , and let $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that

1. $A \subseteq U_r \subseteq B^c$ for every $r \in \Delta$,
2. $\overline{U_r} \subseteq U_s$ for $r < s$, and
3. For every $r < s$, $\overline{U_r} \subseteq U_s$

Proof. The goal of this proof is to show that for every $r \in \Delta$, there exists an open U_r that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some $N \geq 1$ that all previous U_r in previous layers have been constructed properly, meaning if $r = k/2^n$, then for every $1 \leq n \leq N - 1$, we have

$$r = \frac{k}{2^n}, 1 \leq n \leq N - 1, 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each U_r ,

- $A \subseteq U_r \subseteq B^c$ and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer $N \geq 1$, we only have to construct the $U_{k/2^N}$ for every odd k , this is because if k is an even number, then $k = 2j$ and $r = 2j/2^N = j/2^{N-1}$ and for this particular U_r is already constructed. So for every odd $k = 2j + 1$, the sets of the form $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ are already defined, and satisfy

$$A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every $k - 1 \neq 0$ and $k + 1 \neq 1$. (We will consider these cases later). We claim that for every pair of open sets, $E_1, E_2 \in \mathcal{T}_X$, then there exists some open set $G \in \mathcal{T}_X$ such that if $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$ where H is defined as the set

$$H = \{(E_1, E_2) \in (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \emptyset\}$$

Then there exists some $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$ such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any $(E_1, E_2) \in H$, then this pair induces a pair of disjoint sets $\overline{E_1}$ and E_2^c since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \emptyset$$

And by normality, there exists disjoint open sets G_1, G_2 such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \emptyset \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since G_2^c is a closed set that contains G_1 as a subset, $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of G_2 (so we may discard the notion of G_2 in our minds). Let us now replace G with G_1 then it is an easy task to verify that $G = G_1 = \mathcal{J}(E_1, E_2)$ has the required properties.

Now define for every odd k , since $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (we note in passing that \mathcal{J} is not a function as the set G may not be unique).

$$U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$$

Then, if $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ is 'well constructed' we have

$$A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$ sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{k/2^N}$ and
- $\overline{U_{k/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed' $U_{k/2^N}$ for every $k - 1 \neq 0$ and $k + 1 \neq 1$. Now let us deal with the remaining pathological cases.

If $k - 1$ so happens to be 0, then no $r \in \Delta$ satisfies $r = 0/2^N$, and we substitute

$$\overline{U}_0 = A, \quad \text{or alternatively, } U_0 = A^c$$

Then $U_0 \in \mathcal{T}_X$, $\overline{U}_0 = A \subseteq B^c$. It is at this point that we must mention that $0, 1 \notin \Delta$, so U_0 and U_1 do not have to obey the rules we have laid out for $U_{r \in \Delta}$.

Now if $k + 1$ is equal to 2^N (this makes $r = (k + 1)/2^N = 1$) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every $0 \leq m \leq 2^N - 1$, $U_{m/2^N}$ must satisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (even for when $N = 1$, since $A = \overline{U}_0 \subseteq U_1 = B^c$) and a corresponding $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$ such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when $N = 1$. We would only have to construct for $k = 1$, since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■

Theorem 4.15

WTS. *Urysohn's Lemma.* Let X be a normal space, if A and B are disjoint closed subsets of X , then there exists a $f \in C(X, [0, 1])$ such that $f = 0$ on A and $f = 1$ on B .

Proof. Let $r \in \Delta$ be as in Lemma 4.14, and set U_r accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0, 1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic ε argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since A and B are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that x is not a member of any U_r , but we set $U_1 = X$. Since none of the $r \in (0, 1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The ε argument follows: suppose for every $\varepsilon > 0$, $(1 - \varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in X . So $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of Δ , there exists an r , $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r < \alpha} U_r$. So $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r < \alpha} U_r$.

Fix an element $x \in \bigcup_{r < \alpha} U_r$, this induces an r such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r < \alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$.

For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an r (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) — we have $x \notin \overline{U_s}$, but $(\overline{U_s})^c$ is open in X . It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq (\overline{U_s})^c \subseteq \bigcup_{s > \alpha} (\overline{U_s})^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s > \alpha} (\overline{U_s})^c$. To show the reverse, fix an element x in the union, then this induces some $x \in (\overline{U_s})^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of W . This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen s is a lower bound for W , and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in X , using Theorem 4.9 finishes the proof. ■

Theorem 4.16

WTS. *The Tietze's Extension Theorem. Let X be a normal space, and for any closed subset $A \subseteq X$, and $f \in C(A, [a, b])$, there exists an $F \in C(X, [a, b])$ which extends f .*

Proof. We begin with an important lemma that will serve as a 'black box' for the induction.

Lemma 16.1 *For every $f \in C(A, [0, 1])$, there exists a $g \in C(X, [0, 1/3])$ such that*

$$0 \leq f - g \leq 2/3 \quad \text{pointwise on } A \quad (4)$$

Proof. Since f is continuous, $B = f^{-1}([0, 1/3])$, and $C = f^{-1}([2/3, 1])$ are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function $g \in C(X, [0, 1])$ such that $g|_B = 0$ and $g|_C = 1$. Relabel $g = g/3$ then $g \in C(X, [0, 1/3])$ (multiplication is continuous).

To show that (4) holds, suppose $x \in B$, then $f(x) \in [0, 1/3]$ and $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$. Now suppose that $x \in C$, then $f(x) \in [2/3, 1]$ and $g(x) = 1/3$ (recall that we relabelled g). So we have $0 \leq 1/3 \leq f - g \leq 2/3$. Lastly, for the case where $x \notin (B \cup C)$, then $f(x) \in (1/3, 2/3)$, and $g(x) \in [0, 1/3]$ implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 & \implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 & \implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore $0 \leq f(x) - g(x) \leq 2/3$. ■

We can assume that $f \in C(A, [0, 1])$, since we can relabel $f = (f - a)/(b - a)$. The main part of this proof consists of constructing a sequence of $\{g_n\} \subseteq C(X, \mathbb{R})$ where $0 \leq g_n \leq (2/3)^n(1/2)$, and $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$ on A . Let us begin with the base case with $n = 1$. We can apply Lemma 16.1 to get $g_1 \in C(X, [0, 1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that $\{g_j\}_{j \leq n}$ has been chosen, we will find our g_{n+1} by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by $(2/3)^{-n}$ and we obtain a new function in $C(A, [0, 1])$.

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) \left(\frac{3}{2} \right)^n \leq 1$$

Applying the Lemma 16.1, we get a function $h \in C(X, [0, 1/3])$ such that, for every $x \in A$

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) \left(\frac{3}{2} \right)^n - h \leq 2/3$$

Multiplying across gives

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) - h \left(\frac{2}{3} \right)^n \leq \left(\frac{2}{3} \right)^{n+1}$$

Set $g_{n+1} = h \left(\frac{2}{3} \right)^n$ and $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$. Furthermore, the sum of all g_j pointwise converges uniformly, as

$$\sum_{j \geq 1} \|g_j\|_u \leq \sum_{j \geq 1} \left(\frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum $F = \sum g_j$, then this $F \in BC(X)$ (by Theorem 4.9), since every $g_j \in BC(X)$. And

$$\left\| f - \sum_{j \leq n} g_j \right\|_u \leq \left(\frac{2}{3} \right)^n \rightarrow 0$$

So $F = f$ on A , now if we want to obtain our F on $[a, b]$ we simply relabel $F = F(b - a) + a$. This finishes the proof. \blacksquare

Theorem 4.17

WTS. If X is a normal space, and A is a closed subspace of X , and $f \in C(A)$, then there exists an $F \in C(X)$ such that F extends f .

Proof. First we suppose that f is real valued, so $f \in C(X, \mathbb{R})$. And define a $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$, using

$$g = \frac{f}{1 + |f|}$$

Since g satisfies the assumption of Theorem 4.16 (note that we do not require g to be injective), there exists a $G \in C(X, [-1, +1])$ such that $G|_A = g$. Since the set $\{-1, +1\}$ is closed in \mathbb{R} , $G^{-1}(\{-1, +1\})$ is closed as well. Since $G^{-1}((-1, +1)) \subseteq A$, this makes A and $B = G^{-1}(\{-1, +1\})$ disjoint closed sets in X .

By Urysohn's Lemma, there exists a continuous function $h \in C(X, [0, 1])$ such that $h|_B = 0$ and $h|_A = 1$, so that the product $|hG| < 1$ for all $x \in X$. We can think of this h as a continuous indicator function that filters out the parts we do not want, namely $G^{-1}\{-1, +1\}$. Now define F in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that $F|_A = g/(1 - |g|) = f$ indeed. Since $|g| = \frac{|f|}{1+|f|}$, and $g(1 + |f|) = f$ implies that $g/(1 - |g|) = f$, because $g \in C(A, (-1, +1))$. This completes the proof for any $f \in \mathbb{R}$ if $f \in C(A)$, then

1. $\operatorname{Re}(f) = f_1 \in C(A, \mathbb{R})$
2. $\operatorname{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in $C(X, \mathbb{R})$ that extends f_1 and f_2 , and $F_1 + iF_2 = f$ on A and $F_1 + iF_2 \in C(X)$, and the proof is complete. ■

Theorem 4.18

WTS. *If X is a topological space, and $E \subseteq X$ and $x \in X$, then $x \in \text{acc } E \iff$ there exists a net in $E \setminus \{x\}$ that converges to x , and $x \in \overline{E} \iff$ there exists a net in E that converges to x .*

Proof. Suppose that $x \in \text{acc } E$, then for every neighbourhood $U \in \mathcal{N}(x)$, $E \cap U \setminus \{x\} \neq \emptyset$, then choose $\mathcal{N}(x)$ as the set of neighbourhoods directed by reverse inclusion (and this makes $(\mathcal{N}(x), \supseteq)$ a directed set), and we will define the net as follows.

Map each $U \in \mathcal{N}(x)$ to some $x_U \in E \cap U \setminus \{x\}$, then this net converges to x . Suppose that we fix a neighbourhood, $V \in \mathcal{N}(x)$, then for every $U \supseteq V$ we have $x_U \in U \subseteq V$. So $\langle x_U \rangle$ is eventually in V .

Conversely, if $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$, and $x_\alpha \rightarrow x$, then every $U \in \mathcal{N}(x)$ there exists a $x_\alpha \in E \cap U \setminus \{x\}$ that makes

$$E \cap U \neq \emptyset \quad \forall U \in \mathcal{N}(x)$$

Hence $x \in \text{acc } E$.

Now for the second part of the Theorem, suppose that $x \in \overline{E}$, if $x \notin E$ then $E = E \setminus \{x\}$ and $x \in \text{acc } E$, so there exists a net in $E \setminus \{x\} \subseteq E$ such that $x_\alpha \rightarrow x$. If $x \in E$ then simply choose $\langle x_\alpha \rangle = x$ for every $\alpha \in A$.

Now, suppose that there is a net that converges to x , and this net $\langle x_\alpha \rangle \subseteq E$, if $x \in E$ then there is nothing to prove, since $E \subseteq \overline{E}$, so suppose that $x \notin E$, then there exists a net in $E \setminus \{x\} = E$ such that

$$x_\alpha \rightarrow x \implies x \in \text{acc } E \subseteq \overline{E}$$

■

Theorem 4.19

WTS. Let X and Y be topological spaces, then every $f : X \rightarrow Y$ is continuous at a point $x \in X \iff$ every net $\langle x_\alpha \rangle$ that converges to x implies that $\langle f(x_\alpha) \rangle$ converges to $f(x)$.

Proof. If f is continuous at a point $x \in X$, then $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$, then for every net $\langle x_\alpha \rangle$ that converges to this x , there exists an α_0 such that for every $\alpha \gtrsim \alpha_0$ implies that $x_\alpha \in f^{-1}(V)$. Hence

$$f(x_\alpha) \in f(f^{-1}(V)) \subseteq V$$

And this is equivalent to saying that for every $V \in \mathcal{N}(f(x))$, $\langle f(x_\alpha) \rangle$ is eventually in V , and this proves convergence.

Now suppose that f is not continuous at some x , then there exists a $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$, so

$$x \notin (f^{-1}(V))^o \implies x \in (f^{-1}(V))^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our $x \in \overline{f^{-1}(V^c)}$ induces a net $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$ that converges to x . But every element in the net is contained within $f^{-1}(V^c)$, and for every $\alpha \in A$

$$f(x_\alpha) \in f(f^{-1}(V^c)) \subseteq V^c$$

gives $f(x_\alpha) \notin V$, but V is a neighbourhood of $f(x)$, hence there exists some $x_\alpha \rightarrow x$ and $f(x_\alpha) \not\rightarrow f(x)$. ■

Theorem 4.20

WTS. If $\langle x_\alpha \rangle$ is a net in X , and $x \in X$ is a cluster point of $\langle x_\alpha \rangle \iff$ there exists a subnet of $\langle x_\alpha \rangle$ that converges to x .

Proof. Suppose that $\langle y_\beta \rangle_{\beta \in B}$ is a subnet of $\langle x_\alpha \rangle$ that converges to x , then for every neighbourhood $U \in \mathcal{N}(x)$, there exists a β_1 such that for every $\beta \gtrsim \beta_1$ we get $y_\beta = x_{\alpha_\beta} \in U$.

Furthermore, let us fix a $\alpha_0 \in A$ to attempt to show that $\langle x_\alpha \rangle$ is frequently in U , then by the subnet property of $\langle y_\beta \rangle$, there exists some $\beta_2 \in B$ such that for every $\beta \gtrsim \beta_2$, $\alpha_\beta \gtrsim \alpha_0$. (Intuitively this property means that the directed set of B 'grows' as much as the directed set of A , so we can always find elements that are greater than any fixed α_0 .)

Since $\langle y_\beta \rangle$ is a net, we there exists some $\beta \in B$ such that $\beta \gtrsim \beta_1$ and $\beta \gtrsim \beta_2$, we then apply the $\beta \mapsto \alpha_\beta$ map and we obtain some $\alpha = \alpha_\beta$ that satisfies:

- $\alpha = \alpha_\beta \gtrsim \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that $\beta \gtrsim \beta_1$ so that y_β falls into U .

Conversely, suppose that x is a cluster point of $\langle x_\alpha \rangle$, then by definition

$$\forall U \in \mathcal{N}(x), \forall \alpha_0 \in A, \exists \alpha \gtrsim \alpha_0, x_\alpha \in U$$

Denote the directed neighbourhoods of x by $\mathcal{N}(x)$, and construct our directed set B for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every $(U, \gamma) \in B$ we can map it to some $\alpha_{(U, \gamma)} \in A$, if we choose some $\alpha_{(U, \gamma)} \gtrsim \gamma$ and $\alpha_{(U, \gamma)} \in U$.

To show that B is a directed set, we say that $(U, \gamma) \gtrsim (U', \gamma')$ if and only if $U \subseteq U'$ and $\gamma \gtrsim \gamma'$. And to show that $\langle y_\beta \rangle = \langle x_{\alpha_{(U, \gamma)}} \rangle$ is indeed a subnet of $\langle x_\alpha \rangle$, fix any $\alpha_0 \in A$, then simply take any neighbourhood U of x (we always

have $X \in \mathcal{N}(x)$ — and therefore $(U, \alpha_0) \in B$.

Now for every $(U', \alpha'_0) \gtrsim (U, \alpha_0)$ implies that $\alpha'_0 \gtrsim \alpha_0$, therefore we have

$$\alpha_{(U', \alpha'_0)} \gtrsim \alpha'_0 \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that $\langle y_\beta \rangle$ indeed converges to x , fix any $V \in \mathcal{N}(x)$, then with any $\alpha_0 \in A$, and for every $(V', \alpha'_0) \gtrsim (V, \alpha_0) \in B$, we have

$$x_{\alpha_{(V', \alpha'_0)}} \in V' \subseteq V$$

So $\langle x_{\alpha_{(U, \gamma)}} \rangle$ converges to x . ■

Theorem 4.21

WTS. *A topological space X is compact \iff every family of closed sets, $\{F_\alpha\}_{\alpha \in A}$ that has the finite intersection property, implies that*

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

Proof. We first examine the assertion, Theorem 4.21 proposes for any family of closed sets $\{F_\alpha\}_{\alpha \in A}$, and for every finite subset $B \subseteq A$ then,

$$\bigcap_{\alpha \in B} F_\alpha \neq \emptyset \implies \bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha \in A} F_\alpha = \emptyset \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha \in B} F_\alpha = \emptyset$$

Applying DeMorgan's theorem, and since every $\{F_\alpha\}_{\alpha \in A}$ induces a family of open sets (and vice versa), where $U_\alpha = F_\alpha^c$, so for any family of open sets $\{U_\alpha\}_{\alpha \in A}$ we have

$$\bigcup_{\alpha \in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha \in B} U_\alpha = X$$

Which is equivalent to saying that X is compact. ■

Theorem 4.22

WTS. *A closed subset of a compact space X is compact.*

Proof. Suppose $F \subseteq X$ and F is open, then fix an open cover for F , so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since F^c is a closed set, we can obtain a valid open cover for X , then we pick out a finite subcover, for some finite $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left(\bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with F on both sides yields

$$\begin{aligned} F &= X \cap F \subseteq (F^c \cap F) \cup \left(F \cap \left(\bigcup_{\alpha \in B} U_\alpha \right) \right) \\ F &= \left(F \cap \left(\bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\ F &\subseteq \bigcup_{\alpha \in B} U_\alpha \end{aligned}$$

Therefore every open cover of F has a finite subcover, and F is compact. ■

Theorem 4.23

WTS. *If F is a compact subset of a Hausdorff space X , and $x \notin F$, there are disjoint open sets U, V such that $x \in U$ and $F \subseteq V$.*

Proof. Since $x \in F^c$, for every $y \in F$, $x \neq y$ induces two sets U_y, V_y (because X is T_2).

- $U_y \cap V_y = \emptyset$
- $x \in U_y$
- $y \in V_y$

But $\{V_y\}_{y \in F}$ is an open cover for the compact set F , then there exists a finite subcollection $H \subseteq F$ such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since H is finite, $U = \bigcap_{y \in H} U_y$ is an open set that contains x , also define $V = \bigcup_{y \in H} V_y$. If for every $y \in H$, $U_y \cap V_y = \emptyset$, then $U \cap V_y = U \cap V = \emptyset$. This completes the proof. ■

Remark 23.1 *Every metric space (X, d) is first countable, and T_2 (it is actually T_4 , but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element $x \in X$ and we notice that $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$ is a countable neighbourhood base for every x . To show that (X, d) is T_2 , for every pair of elements $x \neq y$, we can take $r = d(x, y)/2$ and there exists disjoint open sets $V_r(x)$ and $V_r(y)$ such that $x \in V_r(x)$ and $y \in V_r(y)$.*

Theorem 4.24

WTS. *Every compact subset of a Hausdorff (T_2) space is closed.*

Proof. If F is compact, then for every $x \in F^c$, by Theorem 4.23, there exists two disjoint open sets such that $x \in U$ and $F \subseteq V$, but

$$U \cap V = \emptyset \implies U \cap F = \emptyset \implies U \subseteq F^c$$

But since $x \in F^c$ is arbitrary, and U is an open subset of F^c ,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that F^c is open and F is closed. ■

Theorem 4.25

WTS. *Every compact Hausdorff (T_2) space is normal (T_4).*

Proof. Fix A, B which are disjoint closed subsets of X , by Theorem 4.22, we know that these two sets are compact. Hence for every $y \in B$ there exists two disjoint open sets U, V_y (by Theorem 4.23)

$A \subseteq U_y$ and $y \in V_y$. But the family $\{V_y\}_{y \in B}$ is a valid open cover for the compact set B , hence there exists a finite subcollection $H \subseteq B$ such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \emptyset$$

The second equality holds for every $y \in H$ so that $U_y \cap (\bigcup_{y \in H} V_y) = \emptyset$. Define $U = \bigcap_{y \in H} U_y$ and $V = \bigcup_{y \in H} V_y$, where both of these are disjoint open sets that contain A and B as subsets, since for each $y \in H$, $A \subseteq U_y$ hence the intersection of all U_y also contains A as a subset. Therefore X is normal. ■

Theorem 4.26

WTS. *If X is compact, and $f : X \rightarrow Y$ is continuous, then $f(X)$ is compact.*

A small lemma.

Lemma 26.1 *For every $\{E_j\} \subseteq X$, $f(\cup E_j) = \cup f(E_j)$.*

The proof is trivial.

Proof. If $\{V_{\alpha \in A}\}$ is an open cover for $f(X)$, then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha \in A} V_{\alpha}\right) = \bigcup_{\alpha \in A} f^{-1}(V_{\alpha}) \subseteq X$$

Since f is continuous, we have an open cover in the form of $\{f^{-1}(V_{\alpha})\}$ for X , then there exists a finite subset $B \subset A$ such that

$$X \subseteq \bigcup_{\alpha \in B} f^{-1}(V_{\alpha})$$

Then we wish to show that for this $B \subseteq A$, $\{V_{\alpha \in B}\}$ is a finite open cover for $f(X)$. Fix any element $y \in f(X)$, then this induces a $x \in X$ such that $y = f(x)$, but because $\{f^{-1}(V_{\alpha \in B})\}$ is an open cover for X , there exists some $\alpha \in B$ such that $x \in f^{-1}(V_{\alpha})$, hence by definition of the inverse image

$$f(x) \in V_{\alpha} \implies f(X) \subseteq \bigcup_{\alpha \in B} V_{\alpha}$$

Therefore $f(X)$ is compact and this completes the proof. ■

Theorem 4.27

WTS. *If X is compact, then $C(X) = BC(X)$.*

Proof. Notice that $BC(X) \subseteq C(X)$, so we only have to show the reverse estimate. Fix any $f \in C(X)$, since X is compact, by Theorem 4.26 we know that $f(X)$ is also compact. Since $\mathbb{C} = \mathbb{R}^2$ is a complete metric space, $f(X)$ is bounded and $f \in BC(X)$. ■

Theorem 4.28

WTS. *If X is compact, and if Y is Hausdorff, then any continuous bijection $f : X \rightarrow Y$ is a homeomorphism.*

Proof. If $E \subset X$ is closed, then since X is compact, E is compact as well. By continuity of f , $f(E)$ is a compact set in Y , but compact subsets of Y are closed, so f is continuous.

We used the fact that the inverse of f^{-1} is f , since it suffices to check that every inverse image of a closed set is also closed, f^{-1} is continuous. And by definition of a homeomorphism (f has to be bijective and both f and f^{-1} have to be continuous), f is a homeomorphism. ■

Theorem 4.29

WTS. *If X is any topological space, the following are equivalent.*

- (a) X is compact.
- (b) Every net has a cluster point.
- (c) Every net in X has a convergent subnet.

Proof. By Theorem 4.20, every net in X has a cluster point \iff there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose a) holds, then X is compact, and fix an arbitrary net $\langle x_\alpha \rangle$ in X . and define the 'tail' of the net

$$E_\alpha := \{x_\beta, \beta \succeq \alpha\}$$

We wish to show that the arbitrary intersection of $\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \emptyset$. Where \overline{E}_α is closed, so it suffices to check that every finite $B \subseteq A$, the intersection over \overline{E}_α is non-empty.

Suppose we are given a finite $B \subseteq A$, then fix any two elements α and $\beta \in B$, by the definition of a net there exists a $\gamma \in A$ such that $\gamma \succeq \alpha$ and $\gamma \succeq \beta$, and

$$\emptyset \neq E_\alpha \cap E_\beta \implies \overline{E}_\alpha \cap \overline{E}_\beta \neq \emptyset$$

Therefore for any finite collection of $\{\overline{E}_{\alpha \in B}\}$, then

$$\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \emptyset$$

Now fix an element $x \in \bigcap_{\alpha \in A} \overline{E}_\alpha$. Then for every $\alpha \in A$, $x \in \overline{E}_\alpha$, and for every neighbourhood $U \in \mathcal{N}(x)$, $U \cap E_\alpha \neq \emptyset$. This is because if $x \in E_\alpha$, then $U \cap E_\alpha$ contains at least $\{x\}$, if $x \in \text{acc } E_\alpha$, then by definition of an accumulation point, $U \cap E_\alpha \setminus \{x\} \neq \emptyset$, so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net, E_α , if for every $\alpha \in A$, $x \in E_\alpha$ if and only if there exists some $\gamma \succeq \alpha$, $x_\gamma \in U \cap E_\alpha$,

this is equivalent to saying that x is a cluster point of $\langle x_\alpha \rangle$. So $a) \implies b)$.

Now let us suppose that X is not compact, then there exists an open cover $\{U_{\alpha \in A}\}$ of X that has no finite subcover. Let \mathbb{B} be the collection of all finite subsets of A , directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every $B \in \mathbb{B}$, find some $x_B \in (\bigcup_{\alpha \in B} U_\alpha)^c$. So we have a net in X . Now we will show that no $x \in X$ can be a cluster point of this net. Suppose not, then take a neighbourhood U_β with $\beta \in A$ such that U_β belongs to the open cover we first discussed. Then for any $B \in \mathbb{B}$ such that $B \gtrsim \{\beta\}$ (meaning that $\{\beta\} \subseteq B$, where B is a finite set), then

$$x_B \in \left(\bigcup_{\alpha \in B} U_\alpha \right)^c \implies x_B \notin \left(\bigcup_{\alpha \in \{\beta\}} U_\beta \right) \implies x_B \in U_\beta^c$$

Hence no point in X can be a cluster point for this net, and the proof is complete. ■

Theorem 4.30

WTS. *If X is a LCH space, and for every $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$, there exists a compact $N \subseteq U$ where $N \in \mathcal{N}_B(x)$.*

Proof. For every $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$, we can find an E open subset of U that has a compact closure, since every $x \in X$ induces some compact $F \in \mathcal{N}_B(x)$, therefore

$$E := U \cap F^o \implies \overline{E} \subseteq F$$

Since closed subsets of compact sets are compact (by Theorem 4.22), \overline{E} is compact. More is true, since E is open,

$$x \in U \cap F^o \implies x \in E^o \implies E \in \mathcal{N}_B(x)$$

Now it suffices to show that there exists some compact $N \subseteq E \subseteq U$ such that $N \in \mathcal{N}_B(x)$. Since \overline{E} is compact, the closed subset $\partial E = \overline{E} \cap \overline{E}^c$ of \overline{E} is also compact.

Since $\partial E \cap E^o = \emptyset$, $x \in E^o = E$ means that $x \notin \partial E$. Applying Theorem 4.23 to the compact set ∂E and $x \notin \partial E$ gives us two disjoint open sets V' and W' . We list their properties

1. $V', W' \in \mathcal{T}_X$
2. $x \in V'$
3. $\partial E \subseteq W'$
4. $V' \cap W' = \emptyset$

The two disjoint pairs induce another pair of open sets relative to \overline{E} , recall the definition of the topology relative to \overline{E} ,

$$\mathcal{T}_{\overline{E}} = \{A \cap \overline{E} : A \in \mathcal{T}_X\}$$

We now agree to define

- $V = V' \cap \overline{E}$
- $W = W' \cap \overline{E}$

Then evidently $V, W \in \mathcal{T}_{\overline{E}}$ and

1. $x \in V' \cap \overline{E} \implies x \in V$
2. $\partial E \subseteq \overline{E} \implies \partial E \subseteq W$
3. $V' \cap W' = \emptyset \implies V \cap W = \emptyset$

Furthermore,

$$\partial E \subseteq W \implies W^c \subseteq (\partial E)^c = E^o \cup E^{co}$$

Taking the intersection over \overline{E} gives us

$$\overline{E} \setminus W \subseteq \overline{E} \cap (E^o \cup E^{co})$$

Note that $E^{co} = (\overline{E})^c$, since $(E^c)^{oc} = \overline{(E^{cc})} = \overline{E}$ therefore $\overline{E} \cap E^{oc} = \emptyset$, hence

$$\overline{E} \setminus W \subseteq \overline{E} \cap E^o = E^o$$

Using the fact from 3, $V \subseteq W^c$ and $V \subseteq \overline{E}$ and $V \subseteq W^c$ implies that $V \subseteq \overline{E} \setminus W$. Compiling everything, we have

$$V \subseteq \overline{E} \setminus W \subseteq E$$

Note that the set $\overline{E} \setminus W$ is closed in \mathcal{T}_X (and hence closed in \overline{E}) by closure over intersections, \overline{V} is therefore a closed subset of $\overline{E} \setminus W$, and \overline{V} is compact. Also

$$\overline{V} \subseteq \overline{E} \setminus W \subseteq E$$

To check that $\overline{V} \in \mathcal{N}_B(x)$, note that

$$x \in V^o \subseteq (\overline{V})^o \implies \overline{V} \in \mathcal{N}_B(x)$$

The subset relation $V^o \subseteq \overline{V}^o$ comes from the fact that V^o is an open subset of \overline{V} , and hence is contained in $(\overline{V})^o$ as a subset. Now let us define $N = \overline{V}$, and N satisfies the assertions in the Theorem, since

- $N \in \mathcal{N}_B(x)$
- N is compact
- $N \subseteq E \subseteq U$

And this completes the proof. ■

Remark 30.1 *Intuitively speaking, this means that if X is any LCH space, then for every open neighbourhood $U \in \mathcal{N}_B(x)$, there exists a compact $E \in \mathcal{N}_B(x)$ such that $x \in E \subseteq U^o$. This property is indeed a very strong one as it allows us to have effectively 'infinite' descending compact neighbourhoods of x .*

Theorem 4.31

WTS. X is a LCH space, and $K \subseteq U \subseteq X$ where K is compact, and U is open, then there exists some precompact, open V with

$$K \subseteq V \subseteq \bar{V} \subseteq U$$

Proof. For every $x \in K$, we can apply Proposition 4.30, since $x \in K \subseteq U$, this induces some compact $F_x \subseteq U$ where $F_x \in \mathcal{N}_B(x)$. Then we can obtain an open cover of U in the form of $\{F_x^o\}_{x \in K}$. By compactness of K , there exists a finite $B \subseteq K$ such that

$$K \subseteq \bigcup_{x \in B} F_x^o$$

Let $V = \bigcup_{x \in B} F_x^o$, then clearly V is open, and $K \subseteq V$. Since each F_x is closed (compact sets are closed in any Hausdorff Space), we have

$$V \subseteq \bigcup_{x \in B} F_x \implies \bar{V} \subseteq \bigcup_{x \in B} F_x$$

Since $\bigcup_{x \in B} F_x$ is a finite union of compact sets, we claim that it is also compact. Consider two compact sets E_1 and E_2 , then if $\{U_\alpha\}_{\alpha \in A}$ is any open cover of $E_1 \cup E_2$, it must be an open cover for E_1 and E_2 as well, because

$$E_1, E_2 \subseteq E_1 \cup E_2 \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since E_1 and E_2 are both compact sets, they each induce two finite subsets of B_1, B_2 of A whose union $B = B_1 \cup B_2$ is also compact. Therefore

$$E_1 \cup E_2 \subseteq \bigcup_{\alpha \in B} U_\alpha$$

Then a simple proof by induction will show that if $\{E_{j \leq n}\}$ is a family of compact sets, then $E = \bigcup E_{j \leq n}$ is also compact.

Returning to the main part of the proof, $\bigcup_{x \in B} F_x$ is a compact set, therefore \bar{V} is also compact. Moreover

$$\forall x \in K, F_x \subseteq U \implies \bar{V} \subseteq \bigcup_{x \in B} F_x \subseteq U$$

Combining, we have

- $K \subseteq V \subseteq \overline{V}$,
- V is open and \overline{V} is compact, and
- $\overline{V} \subseteq U$

This completes the proof. ■

Theorem 4.32

WTS. *Urysohn's Lemma, Locally Compact Version.* For any LCH space X , and if $K \subseteq U \subseteq X$ where K is compact and U is open, then there exists some $f \in C(X, [0, 1])$ with

- $f = 1$ on K
- $f = 0$ outside some compact $\bar{V} \subseteq U$

Proof. Let V be as in Theorem 4.31, for our fixed $K \subseteq U \subseteq X$, there exists a pre-compact, open V that satisfies

$$K \subseteq V \subseteq \bar{V} \subseteq X$$

It follows that this $(\bar{V}, \mathcal{T}_{\bar{V}})$ is a normal space by Theorem 4.25 (compact Hausdorff spaces are normal), and by Urysohn's Lemma (Theorem 4.15) on normal spaces, since we can easily find two disjoint closed subsets of \bar{V} in the form of

- $K \subseteq V^\circ = V \subseteq \bar{V}$ (compact sets in Hausdorff spaces are closed)
- $\partial V = \bar{V} \cap \bar{V}^c$ (closed sets in compact spaces are compact)
- $K \subseteq V^\circ$ implies that $K \cap \partial V = K \cap (\bar{V} \setminus V^\circ) = \emptyset$

Then there exists a continuous $f|_{\bar{V}} \in C(\bar{V}, [0, 1])$ that evaluates to

- $f|_{\bar{V}} = 1$ on closed K
- $f|_{\bar{V}} = 0$ on closed ∂V

Now let us extend $f|_{\bar{V}}$ to f by defining

$$f|_{(\bar{V})^c} = 0$$

We will show that this extension of f is indeed continuous. Indeed, for every closed set $E \subseteq [0, 1]$ that does not contain 0, we have:

$$0 \notin E \implies \{0\} \cap E = \emptyset \implies f^{-1}(\{0\}) \cap f^{-1}(E) = \emptyset$$

But $(\bar{V})^c \subseteq f^{-1}(\{0\})$ therefore

$$(\bar{V})^c \cap f^{-1}(\{0\}) \cap f^{-1}(E) = (\bar{V})^c \cap f^{-1}(E) = \emptyset \implies f^{-1}(E) \subseteq \bar{V}$$

We can write

$$f^{-1}(E) = f|_{\bar{V}}^{-1}(E)$$

But we know that $f|_{\bar{V}}$ is continuous, so $f|_{\bar{V}}^{-1}(E)$ must be closed (with respect to \bar{V}), and therefore is closed wrt X , since \bar{V} is closed wrt X .

For the case where $0 \in E$, note that

$$f^{-1}(E) = (f^{-1}(E) \cap \bar{V}) \cup (f^{-1}(E) \cap (\bar{V})^c) = (f|_{\bar{V}})^{-1}(E) \cup (f|_{\bar{V}^c})^{-1}(E)$$

The above equalities are messy in print. They are but a simple consequence of disjoint decomposition of the pre-images, since

$$\bar{V} \cap f^{-1}(E) = \{x \in \bar{V} : f(x) \in E\} = f|_{\bar{V}}^{-1}(E)$$

Back to our main discussion, recall that for every $x \in \partial V$

$$f(x) = 0 \in f^{-1}(\{0\}) \subseteq f^{-1}|_{\bar{V}}(E)$$

Therefore $\partial V \subseteq f^{-1}|_{\bar{V}}(E)$, and $(\bar{V})^c = f^{-1}|_{(\bar{V})^c}(E)$ gives us (since V^c is closed),

$$\begin{aligned} f^{-1}(E) &= f^{-1}|_{\bar{V}}(E) \cup \partial V \cup (\bar{V})^c \\ &= f^{-1}|_{\bar{V}}(E) \cup \overline{(V^c)} \cup (\bar{V})^c \\ &= f^{-1}|_{\bar{V}}(E) \cup (V^c \cup V^{\text{co}}) \\ &= f^{-1}|_{\bar{V}}(E) \cup V^c \end{aligned}$$

Since $f^{-1}|_{\bar{V}}(E)$ and V^c are closed subsets of X , then $f^{-1}(E)$ is also closed, and $f \in C(X, [0, 1])$. ■

Theorem 4.33

WTS. *Every LCH space is completely regular (or $T_{3.5}$).*

Proof. Recall that a space X is completely regular if it is T_1 and every closed subset A and every $x \notin A$ there exists some

$$f \in C(X, [0, 1]), \quad f(x) = 1, \quad f|_A = 0$$

Fix a closed set $A \subseteq X$, then for every $x \in A^c$, there exists a compact $E_x \in \mathcal{N}_B(x)$ with $E_x \subseteq A^c$ (by Theorem 4.30).

Note that $E_x \subseteq A^c$ where E_x is compact and A^c is closed, then an application of Theorem 4.31 tell us that there exists an $f \in C(X, [0, 1])$ such that for every $x \in E_x$, $f(x) = 1$ and for points $y \notin A^c$ (which means that $y \in A$), $f(y) = 0$. Therefore X is completely regular. ■

Theorem 4.34

WTS.

Proof.



Theorem 4.35

WTS. *If X is a LCH space, we claim that*

$$\overline{C_c(X)} = C_0(X)$$

Proof. We begin by proving several things that are mentioned before this Theorem, namely

$$C_c(X) \subseteq C_0(X) \subseteq BC(X)$$

Fix an $f \in C_c(X)$, and for every $\varepsilon > 0$,

$$x \in |f|^{-1}([\varepsilon, +\infty)) \implies |f(x)| \geq \varepsilon > 0$$

Therefore $|f|^{-1}([\varepsilon, +\infty))$ is a closed subset of $\text{supp}(f)$, since $(-\infty, \varepsilon)$ is open in \mathbb{R} , then $[\varepsilon, +\infty)$ is a closed set. And by continuity of $|\cdot| \circ f$ (a composition of two continuous functions), $|f|^{-1}([\varepsilon, +\infty))$ is closed. Using the fact that closed subsets of compact $\text{supp}(f)$ are also compact, we get $f \in C_0(X)$.

Next, we show that $C_0(X) \subseteq BC(X)$. Fix any element f of $C_0(X)$ with an arbitrary $\varepsilon > 0$, then $E_\varepsilon = \{x \in X : |f(x)| \geq \varepsilon\}$ is compact. The continuity of f guarantees that the direct image of a compact set is another compact set (Theorem 4.26)

$$|f|(E_\varepsilon) \text{ is a compact subset of } \mathbb{R}$$

And therefore for every $x \in E_\varepsilon \implies |f(x)| \in |f|(E_\varepsilon)$, then by Heine-Borel, there exists some $M \geq 0$ such that $|f(x)| \leq M$. If $x \notin E_\varepsilon$, then by definition of E_ε , implies that $|f(x)| < \varepsilon$. Then $|f(x)| \leq M + \varepsilon$ for every $x \in X$. Hence $f \in BC(X)$.

Here I wish to offer an alternate proof for $C_0(X) \subseteq BC(X)$, we begin by constructing an open cover for $\text{supp}(f)$ such that

$$\{U_n\}_{n>0} = \{x \in X : |f(x)| < n\}$$

Then there exists a finite subcollection of $\{U_n\}_{n \in B}$ where B is a finite set, then define $M = 1 + \sum_{n \in B} n$ and for every $x \in \text{supp}(f)$ we have $|f(x)| < n$ and since $n > 0$ this holds for every $x \in X$ too. Therefore $f \in BC(X)$.

For the main proof of Theorem 4.35, since $BC(X)$ is endowed with the uniform metric, it is also first countable, and therefore by Theorem 4.6, it suffices to show that every sequence $\{f_n\}_{n \geq 1} \subseteq C_c(X)$ converges in $C_0(X)$. And every element $f \in C_0(X)$ has a convergence sequence in $C_c(X)$.

Fix a convergent sequence $\{f_n\}_{n \geq 1} \subseteq C_c(X)$ that converges uniformly to some $f \in BC(X)$ (since $BC(X)$ is a closed subset of $C(X)$ with respect to the uniform norm), then for every $\varepsilon > 0$, there exists some $n \geq 1$ with

$$\|f_n - f\|_u < \varepsilon$$

We aim to show that $(\text{supp}(f_n))^c \subseteq |f|^{-1}((-\infty, \varepsilon))$, so fix any $x \notin \text{supp}(f_n)$, then

$$|f(x) - f_n(x)| = |f(x)| \leq \|f - f_n\|_u < \varepsilon$$

This establishes the estimate, and taking complements

$$|f|^{-1}([\varepsilon, +\infty)) \subseteq \text{supp}(f_n)$$

Therefore for any arbitrary $\varepsilon > 0$, $\{x \in X, |f(x)| \geq \varepsilon\}$ is compact, and $\overline{C_c(X)} \subseteq C_0(X)$. Conversely, fix any $f \in C_0(X)$, and for every $n \geq 1$, define

$$K_n = \{x \in X, |f(x)| \geq 1/n\}$$

Using Urysohn's Lemma for our LCH space X , there exists some g_n that has a compact support, and $g_n(x) = 1$ for every $x \in K_n$. We then write $f_n = g_n \cdot f \in C_c(X)$. We wish to show that $f_n \rightarrow f$ uniformly. Notice that for any fixed $n \geq 1$, if $x \in K_n$ then

$$f_n(x) = f(x) \implies |f_n - f|(x) = 0$$

If $x \notin K_n$, $|f(x)| < 1/n$ (recall what K_n does), and $f_n = g_n \cdot f \in [0, 1]$ by definition of g_n from Theorem 4.32, hence

$$|f_n(x) - f(x)| = |f(x)| \cdot |1 - g_n| \leq |f(x)| < 1/n$$

Taking the supremum over $x \in X$, we have

$$\|f_n - f\|_u < 1/n \rightarrow 0$$

As we send n to $+\infty$, and $f_n \rightarrow f$ uniformly. This completes the proof. ■

Theorem 4.36

WTS.

Proof.



Theorem 4.37

WTS. *If X is an LCH space and $E \subseteq X$. E is closed if and only if $E \cap K$ is closed for every compact $K \subseteq X$.*

Proof. Suppose that E is closed, then $E \cap K$ is closed, since compact subsets of Hausdorff spaces are closed, and $E \cap K \subseteq K$ tells us that $E \cap K$ is indeed compact.

Now suppose that E is not closed, by Theorem 4.1, $E \neq \overline{E}$, so pick some $x \in (\overline{E} \setminus E) = \text{acc}(E) \cap E^c$, since X is locally compact, let K_x be a compact neighbourhood of x , then for every neighbourhood $U \in \mathcal{N}_B(x)$, we have

$$x \in U^\circ, x \in K_x^\circ \implies x \in (U^\circ \cap K_x^\circ) \subseteq (U \cap K_x)^\circ$$

Since $(U^\circ \cap K_x^\circ)$ is an open subset of $(U \cap K_x)$, then $(U \cap K_x) \in \mathcal{N}_B(x)$, and recall that $x \in \text{acc}(E)$, therefore

$$(U \cap K_x) \cap E \setminus \{x\} = U \cap (K_x \cap E) \neq \emptyset$$

But $x \notin E \implies x \notin E \cap K_x$. So x is an accumulation point of $E \cap K_x$ that is not in $E \cap K_x$. Therefore there exists some $E \cap K_x$ (with K_x compact) that is not closed. ■

Theorem 4.38

WTS.

Proof.



Theorem 4.39

WTS.

Proof.



Theorem 4.40

WTS.

Proof.



Theorem 4.41

WTS.

Proof.



Notes on Chapter 4

Exercise 39

WTS. *Show that every sequentially compact space is countably compact. With the following definitions:*

- *Sequential compactness: every sequence has a convergent subsequence.*
- *Countably Compact: every countable open cover has a finite subcover.*

Proof. We will proceed by proving the contrapositive. Suppose \mathbf{X} admits a countable open cover where no finite subcover exists. Intuitively, we want to produce a sequence that 'leads to nowhere'.

Denote this 'bad' open cover by $\{U_n\}_{n \geq 1}$. Since $\{1, 2, \dots, n\}$ is finite, we can choose a sequence

$$x_n \in U_n \setminus \left(\bigcup_{j \leq n-1} U_j \right) \neq \emptyset$$

where $U_0 = \emptyset$ for convenience. If U_j is empty for any $j \geq 1$, we can discard this U_j from our open cover, and if what remains is a finite ■

Chapter 5

Theorem 5.1

WTS.

Proof.



Theorem 5.2

WTS.

Proof.



Theorem 5.3

WTS.

Proof.



Theorem 5.4

WTS.

Proof.



Theorem 5.5

WTS.

Proof.



Theorem 5.6

WTS.

Proof.



Theorem 5.7

WTS.

Proof.



Theorem 5.8

WTS.

Proof.



Chapter 6

Theorem 6.1

WTS. *For every $a, b \geq 0$, and $0 < \lambda < 1$, then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b$$

Theorem 6.2

WTS.

Proof.



Theorem 6.3

WTS.

Proof.



Theorem 6.4

WTS.

Proof.



Theorem 6.5

WTS.

Proof.



Theorem 6.6

WTS.

Theorem 6.7

WTS.

Theorem 6.8

WTS.

Theorem 6.9

WTS.

Proof.



Theorem 6.10

WTS.

Proof.



Theorem 6.11

WTS.

Proof.



Theorem 6.12

WTS.

Proof.



Theorem 6.13

WTS.

Theorem 6.14

WTS.

Theorem 6.15**WTS.**

Proof. First suppose that (X, \mathcal{M}, μ) is finite measure space. If $\mu(X) < +\infty$, then for every $E \in \mathcal{M}$, by monotonicity $E \subseteq X$ yields $\mu(E) \leq \mu(X) < +\infty$. Next, for any $p < +\infty$, $\|\chi_E\|_p^p < +\infty$ and $\|\chi_E\|_{+\infty} \leq 1 < +\infty$. So all indicator functions are in L^p .

It follows that every simple function is also in L^p , since it is a finite linear combination of indicators. We now define $\nu(E) = \phi(\chi_E)$, we wish to show that $\nu : \mathcal{M} \rightarrow \mathbb{C}$ is a complex measure which is absolutely continuous with respect to μ .

To show σ -additivity, fix any disjoint sequence $\{E_j\}_{j \geq 1} \subseteq \mathcal{M}$. Where we also note that $\mu(E) = \mu(\cup E_j) < +\infty$. Now suppose that $p < +\infty$, then the following converges in the p -norm

$$\chi_E = \sum_{j \geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left(\bigcup_{j \leq n} E_j \right) = \left(\bigcup_{j \geq 1} E_j \right) \setminus \left(\bigcup_{j \leq n} E_j \right) = \bigcup_{j \geq n+1} E_j$$

and define F_{n+1} as the rightmost member above. Then $\{F_{n \geq 1}\}$ is a decreasing sequence of sets. All sets are of finite measure, hence $\mu(E) - \mu(\cup E_{j \leq n}) = \mu(F_{n+1}) \rightarrow 0$.

Now, for any fixed $n \geq 1$,

$$\left| \chi_E - \sum \chi_{E_{j \leq n}} \right| = \left| \sum \chi_{E_{j \geq n+1}} \right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the p th power does not change pointwise, and

$$\left| \sum \chi_{E_{j \geq n+1}} \right|^p = \left| \sum \chi_{E_{j \geq n+1}} \right| = \sum \chi_{E_{j \geq n+1}}$$

Convergence in p -norm is given by

$$\left\| \chi_E - \sum \chi_{E_{j \leq n}} \right\| = \left\| \sum \chi_{E_{j \geq n+1}} \right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our $\phi \in L^{p*}$

$$\begin{aligned} \nu(E) &= \phi(\chi_E) \\ &= \phi\left(\lim_{n \rightarrow \infty} \sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \phi\left(\sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \sum \phi\left(\chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \sum \nu(E_{j \leq n}) \end{aligned}$$

To show absolute convergence, recall that for any $\phi(\chi_{E_j}) \in \mathbb{C}$, define $\beta_j = \frac{\phi(\chi_{E_j})}{\|\phi(\chi_{E_j})\|}$ then multiplication yields

$$\|\phi(\chi_{E_j})\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the p -norm.

$$\left\| \sum_{j \geq 1} \beta_j \chi_{E_j} - \sum_{j \leq n} \beta_j \chi_{E_j} \right\|_p = \left\| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right\|_p$$

And because $\left| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right|$ is pointwise equal to $\left| \sum_{j \geq n+1} \chi_{E_j} \right|$, since $|\beta_j| = 1$ for every $j \geq 1$. We can reuse the same continuity and linearity argument. We also note that $\sum_{j \geq 1} \beta_j \chi_{E_j} \in L^p$ since its p -norm is equal to $\mu(E)^{1/p}$.

$$\begin{aligned}
\sum_{j \geq 1} |\nu(E_j)| &= \sup_{n \geq 1} \sum_{j \leq n} \|\nu(E_{j \leq n})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \|\phi(\chi_{E_j})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \phi(\chi_{E_j}) \\
&= \lim_{n \rightarrow \infty} \phi \left(\sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&= \phi \left(\lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&\leq \|\phi\| \left\| \sum_{j \geq 1} \beta_j \chi_{E_j} \right\|_p \\
&< +\infty
\end{aligned}$$

Assuming the above estimate holds, then we only need $\nu(E) = \phi(\chi_E) = \mu(E) = 0$ (ν is now a measure and $\nu \ll \mu$). As the indicator of a null set is equal to the zero element in L^p . Then by Radon-Nikodym we can have some $g \in L^1(\mu)$ such that

$$d\nu = g d\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function g . For every χ_E measurable, $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$, by monotonicity of the integral in L^+ . So any simple function, $\alpha = \sum a_j \cdot \chi_{E_j}$ means that αg is in $L^1(\mu)$, and

$$\phi(\alpha) = \int \alpha g d\mu$$

If $\|\alpha\|_p = 1$, then

$$\left| \int \alpha g \right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$$M_q(g) = \sup \left\{ \left| \int \alpha \cdot g \right|, \|\alpha\|_p = 1, \text{ and } \alpha \text{ is simple, and vanishes out-} \right\} < \infty$$

side a set of finite measure.

Since $S_g = \{x \in X, g(x) \neq 0\}$ is σ -finite, an application of Theorem 6.14 tells us that $g \in L^q$, and $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$. Now that we know g is in L^q we can use the density of α in L^p to show, for every single $f \in L^p$

$$\phi(f) = \int f g d\mu$$

Conjure a sequence of α 's, and call them $\{f_n\} \rightarrow f$ p.w.a.e, then each $f_n \cdot g \in L^1$. An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n g d\mu = \int f g d\mu = \phi(f)$$

This completes the proof for when μ is finite.

Let us upgrade our μ into a σ -finite measure. Then there exists an increasing sequence $\{E_n\} \nearrow X$ such that each E_n is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in P_n vanishes outside a set of finite measure and is also in L^p . And Q_n is defined in a similar manner. Now, fix our $\phi \in L^{p*}$, and for each $f \in P_n$, there exists a corresponding $g_n \in Q_n$. Then $p \in [1, +\infty)$ tells us that $q \in (1, +\infty]$, and the assumptions for Theorem 6.13 all hold. Therefore for each $g_n \in Q_n$, there is a corresponding bounded linear operator $\phi_{g_n} \in (P_n)^*$ such that

$$\phi(f) = \phi|_{P_n}(f) = \int f g_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of g_n towards some $g \in L^q$. We claim that this limit makes sense. As for any $n < m$, such that $E_n \subseteq E_m$ then $g_n = g_m$ on E_n pointwise. The proof is simple since each the restriction of our $\phi \in L^{p*}$ onto E_n and E_m spawns two functions g_n and $g_m \in L^1$. To verify, take any subset $Z \subseteq E_n$ then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So $g_n = g_m$ pointwise a.e on E_n . Now we define g measurable such that $g|_{E_n} = g_n$ for every n . And

$$\begin{aligned}
|g_n| &= \chi_{E_n} \cdot |g_m| \implies \\
|g_n| &\leq |g_{n+1}| \implies \\
\|g_n\|_q &\leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty
\end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with $(P_n \subseteq P_{n+1})$. If $q = +\infty$ then $g \in L^\infty$ is trivial, but for any $q < +\infty$. We wish to show that $g \in L^q$. Since $|g_n| \leq |g|$ pointwise for every n , and for each $x \in X$, there exists a N , where $n \geq N$ implies $|g(x)| = |g_n(x)|$, so $|g(x)|$ is an upperbound that is actually attained by the sequence $|g_n(x)|$. So, $|g(x)| = \sup_{n \geq 1} \{|g_n(x)|\}$.

Using the Monotone Convergence Theorem on $|g_n|$,

$$\begin{aligned}
\int \lim_{n \rightarrow \infty} |g_n|^q d\mu &= \int \sup_{n \geq 1} |g_n|^q d\mu \\
&= \int |g|^q d\mu \\
&= \lim \int |g_n|^q d\mu
\end{aligned}$$

Which yields $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$. It follows that $g \in L^q$.

Finally, we will show that $\phi(f) = \int fg$ for every $f \in L^p$. Redefine $f_n = f \cdot \chi_{E_n} \in P_n$ for every $n \geq 1$. We claim that $f_n \rightarrow f$ in the p -norm.

$$\begin{aligned}
|f_n - f| &\leq |f_n| + |f| \\
&\leq |f| + |f| \\
&\leq 2|f|
\end{aligned}$$

And $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$. Now it is permissible to apply the Dominated Theorem, and we will do so.

$$\begin{aligned}
\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\
\lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\
&= 0
\end{aligned}$$

And we have $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}
 \phi(f) &= \lim \phi|_{P_n}(f_n) \\
 &= \lim \int f_n \cdot g_n \\
 &= \lim \int f \cdot g \cdot \chi_{E_n} \\
 &= \int \lim (fg \cdot \chi_{E_n}) \\
 &= \int fg
 \end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of $fg\chi_{E_n} \rightarrow fg$ pointwise and Holder's Inequality. This completes the proof for when μ is of σ -finite measure, and $p \in [1, +\infty)$.

Suppose now μ is arbitrary, and $p \in (1, +\infty)$, then $q < +\infty$. Now let us agree to define, for every σ -finite $E \subseteq X$

$$P_E = \{L^p, |f| = |f| \cdot \chi_E\}$$

Where Q_E does not hold any surprises. Then for each E we have a $\phi|_E$ which induces a g_E that vanishes outside E . We are ready for the final part of the proof.

First, if $E \subseteq F$ and both E and F are σ -finite, then $\|g_E\|_q \leq \|g_F\|_q$. This is a simple consequence of monotonicity in L^+ if we take $|g_E|^q \leq |g_F|^q$.

Second, we define

$$W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}$$

Let M be the supremum of W , then there exists a sequence of σ -finite sets, $\{E_n\}$ where $\|g_{E_n}\|_q \rightarrow M \leq \|\phi\|_{p^*}$. Take a set $F = \cup E_{n \geq 1}$, which is also σ -finite, so that $\|g_F\|_q = M$. Now assume there exists another σ -finite superset of F , let us call it A . Then

$$\int |g_F|^q + \int |g_{A \setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have $g_{A \setminus F} = 0$ pointwise a.e. For any $f \in L^p$, the spots where f does not vanish is σ -finite. This comes from $\int |f|^p < +\infty$. So it suffices to integrate over this σ -finite set. But we already know, even if this set A contains F as a subset, $\int f g_F = \int f g_A$.

We now define $g = g_F$, and the proof is complete. As for every $\phi \in L^{p*}$, there exists a $g \in L^q$ such that the evaluation of any $f \in L^p$ is given by integrating f with g . ■

Chapter 7

Theorem 7.1

WTS. *If I is a linear functional on $C_c(X)$, then for every compact $K \subseteq X$, there exists some $C_K \geq 0$ with*

$$|I(f)| \leq C_K \cdot \|f\|_u$$

Proof. Since $\text{supp}(f)$ is compact, by Urysohn's Lemma (Theorem 4.32), there exists a $\phi \in C_c(X, [0, 1])$ such that $\phi = 1$ on K and vanishes outside some compact $\bar{V} \subseteq X$. Then at every x ,

$$-\|f\|_u \leq f(x) \leq +\|f\|_u$$

Implies that

$$(-\|f\|_u)\phi \leq f(x) \leq (+\|f\|_u)\phi$$

So that $f + \|f\|_u\phi \geq 0$ and $+\|f\|_u - f \geq 0$, and by linearity,

$$(-\|f\|_u)I(\phi) \leq I(f) \leq (+\|f\|_u)I(\phi)$$

Therefore $|I(f)| \leq I(\phi)\|f\|_u$, and taking $C_K = I(\phi)$ will suffice. ■

Theorem 7.2

WTS. *The Riesz-Markov-Kakutani Representation Theorem. If (for every) I is a positive linear functional on $C_c(X)$, then there exists a unique Radon measure μ on X , such that*

$$I(f) = \int f d\mu$$

for every $f \in C_c(X)$. μ also satisfies, for every open U , and every compact $K \subseteq X$

$$\mu(U) = \sup \{I(f), f \in C_c(X), f \prec U\} \quad (5)$$

$$\mu(K) = \inf \{I(f), f \in C_c(X), f \geq \chi_K\} \quad (6)$$

For the sake of completeness, we place the definitions for a Radon measure. Let X be a LCH space, and \mathbb{B} be its usual σ -algebra, a measure ν is a Radon measure iff

- (i) $\nu(K) < +\infty$ for every compact K .
- (ii) ν is outer-regular on all Borel sets E ,

$$\nu(E) = \inf \{\nu(U), U \supseteq E, U \in \mathcal{T}\}$$

Intuition: approximation by open supersets.

- (iii) ν is inner-regular on all open sets $U \in \mathcal{T}$,

$$\nu(U) = \sup \{\mu(K), K \subseteq U, K \text{ compact}\}$$

Intuition: approximation by compact subsets

The main proof is extremely long, so we will divide it into several parts. Following Folland's argumentation closely, we will prove (in order)

- (a) If μ_1, μ_2 are Radon measures on X such that for every $f \in C_c(X)$

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

then μ_1, μ_2 must satisfy (5), and $\mu_1 = \mu_2$ on \mathbb{B} .

- (b) If we define, for every open set U , define $\mu : \mathcal{T} \rightarrow [0, +\infty]$ such that

$$\mu(U) = \sup \{I(f), f \in C_c(X), f \prec U\} \quad (7)$$

Then μ is countably subadditive, meaning for every $U \in \mathcal{T}$, $\{U_{j \geq 1}\} \subseteq \mathcal{T}$

$$U = \bigcup U_{j \geq 1} \implies \mu(U) \leq \sum \mu(U_{j \geq 1})$$

- (c) $\mu(\emptyset) = 0$, $\{\emptyset, X\} \subseteq \mathcal{T}$, so that by Theorem 1.10 μ induces an outer-measure μ^*

$$\mu^*(E) = \inf \left\{ \sum \mu(U_{j \geq 1}), U_j \in \mathcal{T}, E \subseteq \bigcup U_{j \geq 1} \right\} \quad (8)$$

- (d) If μ^* is as described above, then if μ is countably subadditive on \mathcal{T} , then

$$\mu^*(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \} \quad (9)$$

Meaning the two definitions in (8) and (9) are equal.

- (e) μ^* and μ agree on all open sets, and $\mu^*|_{\mathcal{T}} = \mu$,
 (f) Using again the definition in (8) and (9), we show that every open set $U \in \mathcal{T}_X$ is μ^* -measurable, meaning for every $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

With this, since the set of all outer-measurable (μ^* -measurable) sets, \mathcal{M}^* form a σ -algebra,

$$\mathcal{T} \subseteq \mathcal{M}^* \implies \mathbb{B} \subseteq \mathcal{M}^*$$

By Theorem 1.1, and define

$$\mu = \mu^*|_{\mathbb{B}} \quad (10)$$

is a Borel measure. And we note in passing that μ is outer-regular on all $E \in \mathbb{B}$,

$$\mu(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \} \quad (11)$$

(g) Using (10) for the definition of μ on \mathbb{B} , we prove that

- μ is outer-regular on all Borel sets, and
- μ satisfies Equation (5).

(h) μ satisfies Equation (6)

(i) μ is finite on all compact sets.

(j) μ is inner-regular on all open sets.

(k) For every $f \in C_c(X, [0, 1])$,

$$I(f) = \int f d\mu \quad (12)$$

(l) For every $f \in C_c(X)$,

$$I(f) = \int f d\mu \quad (13)$$

A small lemma needs to be made before proceeding, that concerns the 'monotonicity' of I on $C_c X$.

Lemma 2.1 *Suppose that $f, g \in C_c(X)$, and $f \geq g \geq 0$ for every X , then $f - g \in C_c(X)$ and $I(f) \geq I(g)$*

Proof. Suppose that $x \in X$ where $f(x) = 0$, then

$$f(x) - g(x) = -g(x) \geq 0 \implies g(x) = 0 \implies f - g = 0$$

Hence

$$\begin{aligned} \{x, f(x) = 0\} &\subseteq \{x, f(x) - g(x) = 0\} \implies \{x, f(x) - g(x) \neq 0\} \subseteq \{x, f(x) \neq 0\} \\ &\implies \text{supp}(f - g) \subseteq \text{supp}(f) \end{aligned}$$

Since $\text{supp}(f)$ is compact, and $\text{supp}(f - g)$ is a closed subset of $\text{supp}(f)$, yields $f - g \in C_c(X)$. And if I is any positive linear functional on $C_c(X)$, then

$$\begin{aligned} f - g \geq 0 &\implies I(f - g) \geq 0 \\ &\implies I(f) \geq I(g) \geq 0 \end{aligned}$$

■

Remark 2.1 If $f \prec U$ and $g \prec U$ for some open subset $U \subseteq X$, then clearly $\text{supp}(f - g) \subseteq \text{supp}(f) \subseteq U$, and $1 \geq f \geq f - g \geq 0$ means that $f - g \prec U$ as well.

Part a

Proof. Suppose that μ_1 and μ_2 are Radon measures on X , and for every $f \in C_c(X)$,

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

We first prove (5). Without loss of generality, by monotonicity of L^+ , if $f \prec U$ for some open U , then $0 \leq f \leq \|f\|_u \chi_U = \chi_U$ for all x and

$$\int f d\mu_1 \leq \int \|f\|_u \chi_U d\mu_1 \leq \mu_1(U)$$

Therefore $\mu_1(U)$ (resp. $\mu_2(U)$) is an upper-bound for the set

$$\{I(f), f \in C_c(X), f \prec U\}$$

Since μ_1 is inner-regular on $U \in \mathcal{T}$, for every $\varepsilon > 0$ we can find some compact $K \subseteq U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K)$$

By Urysohn's Lemma (Theorem 4.32), there exists some $g \in C_c(X)$ with

- $g \in C_c(X, [0, 1])$,
- $g = 1$ on $K \subseteq U$,
- $g = 0$ outside some $\bar{V} \subseteq U$, and
- $g \prec U$.

Hence for every $x \in K$, $g \geq \chi_K$. If $x \notin K$ then $g \geq 0 = \chi_K$; so $g - \chi_K \geq 0$ for every $x \in X$. Since $\chi_K \prec U$, using Lemma 2.1, we get

$$\mu_1(K) \leq \int \chi_K d\mu_1 = I(\chi_K) \leq I(g)$$

So for every $\varepsilon > 0$, there exists a $g \in C_c(X)$, and $g \prec U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K) \leq I(g)$$

Therefore $\mu_1(U) = \sup \{I(f), f \in C_c(X), f \prec U\}$, and the first claim in (a) is proven. To show that μ is indeed unique, since for every open set U , we must have $\mu_1(U) = \mu_2(U)$, and if $E \in \mathbb{B}$ is any Borel set, and by outer-regularity,

$$\mu_1(E) = \inf \{\mu_1(U), U \supseteq E, U \in \mathcal{T}\} = \inf \{\mu_2(U), U \supseteq E, U \in \mathcal{T}\} = \mu_2(E)$$

Therefore this measure is unique. ■

Part b

Proof. To show countable subadditivity for μ with equation (7), fix any $U \in \mathcal{T}$ and a sequence $\{U_{j \geq 1}\} \subseteq \mathcal{T}$ with $U = \bigcup U_{j \geq 1}$. It suffices to show that the partial sum of $\sum \mu(U_{j \leq n})$ is greater than $I(f)$ for any $f \in C_c(X)$, $f \prec U$ (hence it is an upper bound).

Fix any f , then denote $K = \text{supp}(f) \subseteq U$, and since $\{U_{j \geq 1}\}$ is an open cover for K , there exists a finite subcollection, $B \subseteq \mathbb{N}^+$ such that

$$K \subseteq \bigcup_{j \in B} U_j$$

Using Theorem 4.41 on this finite cover of K , there exists a partition of unity in $\{g_{j \leq n}\}$ where

- $g_j \in C_c(X, [0, 1])$,
- $g_j \prec U_j \subseteq U$ for every $j \leq n$, and
- $\sum g_j = 1$ on K ,

And notice for every $j \leq n$,

$$\begin{aligned} \{f = 0\} \cup \{g_j = 0\} &\subseteq \{f \cdot g_j = 0\} \implies \{f \cdot g_j \neq 0\} \subseteq \{f \neq 0\} \cap \{g_j \neq 0\} \\ &\implies \text{supp}(f \cdot g_j) \subseteq \text{supp}(f) \cap \text{supp}(g_j) \\ &\implies \text{supp}(f \cdot g_j) \subseteq U_j \subseteq U \end{aligned}$$

Hence $f \cdot g_j \prec U$ and $f \cdot g_j \in C_c(X, [0, 1])$ for every $1 \leq j \leq n$. Moreover, if we take the sum over a finite n , we obtain $f = \sum f \cdot g_{j \leq n}$, this is because for every $x \in X$, so we have

$$\sum_{j \leq n} f(x) \cdot g_j(x) = f(x) \cdot \sum_{j \leq n} g_j(x) = f(x)$$

Then $I(f) = I(\sum f \cdot g_j) = \sum I(f \cdot g_j)$. And by definition of $\mu(U_j)$, since it is the supremum over all $I(h_j)$, where $h_j \in C_c(X, [0, 1])$ and $h_j \prec U_j$

$$I(f \cdot g_j) \leq \mu(U_j), \quad \forall j \leq n$$

Hence

$$I(f) \leq \sum_{j \leq n} \mu(U_j) \leq \sum_{j \geq 1} \mu(U_j)$$

Where for the last estimate we used the fact that μ is non-negative, and since this holds for any f , we can conclude that $\mu(U) \leq \sum_{j \geq 1} \mu(U_j)$. ■

Part c

Proof. By definition of a topology, $\{\emptyset, X\} \subseteq \mathcal{T}$, and $\mu(\emptyset) = \sup\{I(f), f \in C_c(X), f \prec \emptyset\}$, so $\text{supp}(f) = \emptyset$, and $\{x, f(x) \neq 0\} \subseteq \emptyset$, so the set contains one element, namely $I(0) = 0$ by linearity. So $\mu(\emptyset) = 0$. The assumptions for Theorem 1.10 are satisfied and (8) is indeed an outer-measure. ■

Part d

Proof. Denote the right members of (8) and (9) by W_1 and W_2 , we wish to show that $\inf W_1 = \inf W_2$. Clearly $\inf W_1 \leq \inf W_2$, since $W_2 \subseteq W_1$. Now, if μ is countably additive, then for every $\omega \in W_1$ induces a sequence of open sets $\{U_{j \geq 1}\}$ such that $E \subseteq \bigcup U_{j \geq 1}$. Denote the union over $\{U_{j \geq 1}\}$ by U , which is also another open set,

$$\inf W_2 \leq \mu(U) \leq \sum \mu(U_{j \geq 1}) = \omega$$

Since ω is arbitrary, we conclude that $\inf W_2 = \inf W_1$, and this proves (d). ■

Part e

Proof. If U and V are open subsets of X , and if $U \subseteq V$, then

$$\begin{aligned} U \subseteq V &\implies \{f \in C_c(X), f \prec U\} \subseteq \{f \in C_c(X), f \prec V\} \\ &\implies \{I(f), f \in C_c(X), f \prec U\} \subseteq \{I(f), f \in C_c(X), f \prec V\} \end{aligned}$$

Hence $\mu(U) \leq \mu(V)$. Now by equation (9), $\mu^*(U) \leq \mu(U)$. To show the reverse inequality, suppose by contradiction that $\mu^*(U) < \mu(U)$.

Since $\mu^*(U)$ is an infimum, then for every $\varepsilon > 0$ there exists some $V \supseteq U$ where if we write $\mu^*(U) + \varepsilon = \mu(U)$

$$\mu(V) < \mu^*(U) + \varepsilon = \mu(U) \implies \mu(V) < \mu(U), U \subseteq V$$

This contradicts what we have just proven, and therefore $\mu^*(U) = \mu(U)$ for every open set U . ■

Part f

Proof. We wish to show that every open set U is μ^* -measurable. By Theorem 1.10, it suffices to show that for every $E \subseteq X$

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U) \quad (14)$$

because the reverse inequality is given by subadditivity of μ^* , and we can also assume that $\mu^*(E) < +\infty$. Let us assume that E is open, we wish to find some function $h \in C_c(X)$, $h \prec E$ with

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

The above formula is fussy, but the liberty is taken to show it beforehand to avoid any potential confusion that follows. Since $E \cap U$ is an open subset of X , the definition of $\mu(E \cap U) = \mu^*(E \cap U)$ in (7) tells us that every $\varepsilon > 0$ induces some $f \in C_c(X)$, $f \prec E \cap U$ where

$$I(f) > \mu(E \cap U) - \varepsilon = \mu^*(E \cap U) - \varepsilon \quad (15)$$

Also, $\text{supp}(f)$ is a closed set (compact subsets of Hausdorff spaces are closed), therefore $E \setminus \text{supp}(f)$ is an open set. We make a small diversion from the current part of the proof and turn our attention to the fact that

$$\begin{aligned} \text{supp}(f) \subseteq U &\implies U^c \subseteq (\text{supp}(f))^c \\ &\implies E \setminus U \subseteq E \setminus \text{supp}(f) \end{aligned}$$

And because the outer-measure μ^* is monotone,

$$\mu^*(U) \leq \mu^*(E \setminus \text{supp}(f)) \quad (16)$$

Now, using the definition of $\mu(E \setminus \text{supp}(f))$ (recall that $E \setminus \text{supp}(f)$ is an open set), for every $\varepsilon > 0$, there exists some $g \in C_c(X)$, $g \prec E \setminus \text{supp}(f)$ with

$$I(g) > \mu(E \setminus \text{supp}(f)) - \varepsilon = \mu^*(E \setminus \text{supp}(f)) - \varepsilon \quad (17)$$

It is at this part of the proof where we wish to define $h = f + g$, but first we must verify

- $f + g \in C_c(X, [0, 1])$,
- $f + g \prec E$

The sum of two non-negative functions is non-negative, and for every $x \in \text{supp}(f)$, $f \leq 1$. Also

$$\begin{aligned} \text{supp}(g) \subseteq (\text{supp}(f))^c &\implies \text{supp}(f) \subseteq (\text{supp}(g))^c \\ &\implies \text{supp}(f) \subseteq \{g = 0\} \end{aligned}$$

The last implication comes from taking complements on both sides of $\{g \neq 0\} \subseteq \text{supp}(g)$. So $x \in \text{supp}(f) \implies f + g \leq 1$. Now if $x \notin \text{supp}(f)$, then $f + g = g \leq 1$. Furthermore, $\text{supp}(f + g)$ is a closed subset of compact $\text{supp}(f) \cup \text{supp}(g)$. This is because $\{f + g \neq 0\} \subseteq \{f \neq 0\} \cup \{g \neq 0\}$, and the finite union of two compact sets is again compact.

A moment's thought should yield the fact that the last estimate should be an equality, but it is a needless distraction. Therefore $\text{supp}(f + g)$ is compact and $f + g \in C_c(X, [0, 1])$.

Now both bullet points are satisfied, and we can set $h = f + g$. Adding equation (17) with (15) gives us

$$I(h) = I(f) + I(g) > \mu^*(E \cap U) + \mu^*(E \setminus \text{supp}(f)) - 2\varepsilon$$

Upon applying (16) to the right member of the above estimate, we have

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

But this particular $h \in C_c(X) \cap \{f \prec E\}$, therefore

$$\mu^*(E) \geq I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, equation (14) holds for every open E . Now for any general $E \subseteq X$, fix any $\varepsilon > 0$ and by how we defined $\mu^*(E)$, there exists some open $V \supseteq E$ — recall that $\mu^*(E)$ is the infimum over the set of $\mu(V)$ where V is an open superset of E — hence

$$\mu^*(E) + \varepsilon > \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

By monotonicity (twice) of the outer-measure μ^* , we have

$$\mu^*(E) + \varepsilon > \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Let $\varepsilon \rightarrow 0$, and we get

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Therefore every open $U \subseteq X$ is μ^* -measurable. So $\mu = \mu^*|_{\mathbb{B}}$ is a Borel measure on X . ■

Part g

Proof. To show outer-regularity, fix any $E \in \mathbb{B}$, then by definition,

$$\mu(E) = \mu^*(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \}$$

And for every open U , (5) follows from Equation (7). ■

Part h

Proof. We want to show that for every compact K , Equation (6) holds. To reduce the notational baggage that follows, we agree to define

$$\{I(f), f \in C_c(X), f \prec U\} = \{I(f), f \prec U\}$$

Similarly for $\{I(f), f \geq \chi_K\}$. If $\mu(K) = 0$, then $\mu(K)$ is obviously a lower bound, since $f \geq \chi_K \geq 0$ means that $I(f) \geq 0$, for every $f \geq \chi_K$. So we can suppose $\mu(K) > 0$.

Fix an arbitrary $f \geq \chi_K$, then this particular f induces an open set $U_\alpha = \{f > 1 - \alpha\}$, where $\alpha > 0$. Notice also that

$$K \subseteq \{f \geq 1\} \subseteq \{f > 1 - \alpha\} = U_\alpha$$

Since U_α is an open superset of K , by Equation (11), $\mu(K) \leq \mu(U_\alpha)$, but $\mu(U_\alpha)$ is simply the supremum of $\{I(g), g \prec U_\alpha\}$. If we wish to show that $\mu(K) \leq \mu(U_\alpha) \leq I(f)$, it suffices to show that $I(f)$ is an upper-bound for $\{I(g), g \prec U_\alpha\}$.

Fix any $I(g) \in \{I(g), g \prec U_\alpha\}$, note that $1 - \alpha \neq 0$ for any α small enough, then

- $f/(1 - \alpha) > 1$ on U_α ,
- $1 \geq g \geq 0$ on U_α , in particular, $f/(1 - \alpha) - g \geq 0$ on U_α ,
- If $x \notin U_\alpha$, then $f/(1 - \alpha) - g = f(1 - \alpha) \geq 0$.
- Therefore $f/(1 - \alpha) - g \geq 0$ for any x , and by Lemma 2.1,

$$I(f/(1 - \alpha)) \geq I(g) \quad \forall g \prec U_\alpha$$

Combining the above estimate with $\mu(K) \leq \mu(U_\alpha)$ gives us

$$\mu(K) \leq \frac{1}{1 - \alpha} I(f)$$

Now write $\varepsilon = \alpha/\mu(K) > 0$ and for every $\varepsilon > 0$ we get

$$\mu(K) - I(f) \leq \alpha\mu(K) = \varepsilon$$

Send $\varepsilon \rightarrow 0$ and $\mu(K) \leq I(f)$ for every $f \geq \chi_K$.

To show that $\mu(K)$ is indeed the infimum for $\{I(f), f \geq \chi_K\}$, notice that for every $\varepsilon > 0$ we can obtain some open superset $U \supseteq K$ (by outer-regularity) where $\mu(U) < \mu(K) + \varepsilon$. By Urysohn's Lemma, there exists some $g \prec U$, $g(x) = 1$ for every $x \in K$.

$$g \in \{I(f), f \prec U\} \cap \{I(f), f \geq \chi_K\}$$

Therefore $I(g) \leq \mu(U) < \mu(K) + \varepsilon$ as desired, and Equation (6) holds. ■

Part i

Proof. $\mu(K) < +\infty$ for every compact K . Indeed, since $I(\chi_K) \in \{I(f), f \geq \chi_K\}$, then by Theorem 7.1, there exists a constant $C_K \geq 0$ that bounds

$$\mu(K) \leq |I(\chi_K)| = I(\chi_K) \leq C_K \cdot \|\chi_K\| = C_K < +\infty$$

■

Part j

Proof. Fix any open set U , then for every $\varepsilon > 0$, there exists some $f \prec U$ with $\mu(U) - \varepsilon < I(f)$. Then denote $K = \text{supp}(f) \subseteq U$. If we take any $I(h) \in \{I(h), h \geq \chi_K\}$, then $h \geq f$ gives us $I(h) \geq I(f)$ by Lemma 2.1. So $I(f)$ is a lower bound of $\{I(h), h \geq \chi_K\}$, therefore

$$\mu(U) - \varepsilon \leq I(f) \leq \mu(K)$$

Since $\text{supp}(f) = K \subseteq U$, this proves inner-regularity of μ on open sets. ■

Part k

Proof. Suppose $f \in C_c(X, [0, 1])$, we first show that Equation (12) holds. We divide the interval $[0, 1]$ into $N \geq 1$ chunks by writing

$$K_j = \{f \geq j/N\}$$

for every $1 \geq j \geq N$. And define $K_0 = \text{supp}(f)$. Each K_j is a closed subset of $\text{supp}(f)$, and therefore compact. More is true,

- $K_{j-1} \supseteq K_j$ for every $1 \leq j \leq N$.
- $x \in K_j$ iff $f(x) \in [\frac{j}{N}, 1]$,
- $x \notin K_j$ iff $f(x) \in [0, \frac{j}{N})$, and
- $x \in (K_{j-1} \setminus K_j)$ iff $f(x) \in [\frac{j-1}{N}, \frac{j}{N})$

Folland constructs a finite sequence of compactly supported functions, $\{f_j\}$, where $1 \leq j \leq N$ such that

- Each $0 \leq f_j \leq 1/N$,
- If $x \in (K_m \setminus K_{m+1})$ iff $f(x) \in [\frac{m}{N}, \frac{m+1}{N})$ means that $f_j = 1$ for all $1 \leq j \leq m$, and
- $f_{m+1} = f - m/N$ on K_m , such that

$$f(x) = \left(\sum f_{j \leq m}(x)\right) + \left(f(x) - \frac{m}{N}\right) = \frac{m}{N} + \left(f(x) - \frac{m}{N}\right)$$

- And for every $m < j \leq N$, $f_j = 0$.
- If $x \notin K_m$ iff $f(x) \in [0, \frac{m}{N})$ then for every $m + 1 \leq j \leq N$, $f_j = 0$.

The illustration for when $N = 5$ below should make things clearer.



It is also trivial to verify that

- For every $x \in K_j$, $f_j = N^{-1}$, and

$$\chi_{K_j} N^{-1} \leq f_j \quad (18)$$

Also, if $x \notin K_j$ then $f_j \geq 0$, therefore $f_j \geq \chi_{K_j} N^{-1}$ at every x .

- If $x \notin K_{j-1}$ then $f_j = 0 \leq \chi_{K_{j-1}} \cdot N^{-1}$. If x is in K_{j-1} then $f_j \leq N^{-1}$ by construction and therefore

$$f_j \leq \chi_{K_{j-1}} N^{-1} \quad (19)$$

for all x .

- $f_j \in C_c(X)$, since $\text{supp}(f_j) \subseteq \text{supp}(f)$.

Combining Equations (18) with (19), and by monotonicity in $L^+(X, \mathbb{B}, \mu)$, since $f_j \in L^+$

$$\int \frac{1}{N} \chi_{K_j} d\mu \leq \int f_j d\mu \leq \int \frac{1}{N} \chi_{K_{j-1}} d\mu$$

And for every $1 \leq j \leq N$,

$$\frac{1}{N} \mu(K_j) \leq \int f_j d\mu \leq \frac{1}{N} \mu(K_{j-1}) \quad (20)$$

Furthermore, from Equation (18), since $Nf_j \geq \chi_{K_j}$ then by Equation (6),

$$\mu(K_j) \leq I(Nf_j) \implies \frac{1}{N}\mu(K_j) \leq I(f_j)$$

Now for any arbitrary $I(h) \in \{I(h), h \geq \chi_{K_{j-1}}\}$, since

$$h \geq \chi_{K_{j-1}} \geq Nf_j \implies I(h) \geq I(Nf_j)$$

So $NI(f_j)$ is a lower bound for $\{I(h), h \geq \chi_{K_{j-1}}\}$ and

$$I(f_j) \leq \frac{1}{N}\mu(K_{j-1})$$

Combining the last two results, with $I(f_j)$, we get

$$\frac{1}{N}\mu(K_j) \leq I(f_j) \leq \frac{1}{N}\mu(K_{j-1}) \quad (21)$$

Taking the sum over $1 \leq j \leq N$ for Equations (20) and (21). Define $A = N^{-1} \sum_0^{N-1} \mu(K_j)$, and $B = N^{-1} \sum_1^N \mu(K_j)$

$$B \leq \int f d\mu \leq A$$

And also

$$B \leq I(f) \leq A$$

This is because of finite additivity of both I and the integral, and $f = \sum f_j$ on $K_0 = \text{supp}(f)$. Subtracting the two equations (keeping in mind that $\mu(K_j) < +\infty$ for any compact K_j), we get

$$(-1)(A - B) \leq \left(\int f d\mu - I(f) \right) \leq A - B \implies \left| \int f d\mu - I(f) \right| \leq A - B$$

It is trivial to verify that

$$0 \leq A - B = N^{-1}(\mu(K_0) - \mu(K_N)) \leq N^{-1}\mu(K_0)$$

as $K_N \subseteq K_0$. Let $N \rightarrow \infty$ and

$$\int f d\mu = I(f)$$

Equation (12) holds as desired. ■

Part 1

Proof. Now for any general $f \in C_c(X)$, f must be bounded on the plane since $C_c(X) \subseteq BC(X)$, and $|f| \leq M_0$ for some $M_0 \geq 0$. Since $\text{supp}(f)$ is compact, we know that

$$\int |f| d\mu \leq \int M_0 \chi_{\text{supp}(f)} d\mu \leq M_0 \mu(\text{supp}(f)) < +\infty$$

And $C_c(X) \subseteq L^1(\mu)$. Furthermore,

$$\frac{1}{2}(|\text{Re } f| + |\text{Im } f|) \leq |f| \leq M_0$$

So that $\text{Re } f$ and $\text{Im } f$ are in $C_c(X)$. Without loss of generality, we may assume that f is real. Define $f_1 = \text{Re } f^+/M_0$ and $f_2 = \text{Re } f^-/M_0$ and it immediately follows that $f_1, f_2 \in C_c(X, [0, 1])$.

By linearity of I on $C_c(X)$ and the integral in $L^1(\mu)$,

$$I(f_1 - f_2) = I(f) = \int f d\mu = \int f_1 d\mu - \int f_2 d\mu$$

Then we may apply the above to the real and imaginary parts of a general $f \in C_c(X)$, and this completes the proof. ■

Theorem 7.3

WTS. *See Theorem 7.2*

Proof.

**Theorem 7.4**

WTS. *See Theorem 7.2*

Proof.



Theorem 7.5**WTS.***Proof.*

Theorem 7.6

WTS.

Proof.



Theorem 7.7

WTS.

Proof.



Theorem 7.8

WTS.

Proof.



Theorem 7.9

WTS. *If μ is a Radon measure on X , then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < +\infty$.*

Proof. Theorem 6.7 tells us that the set of L^p simple functions (as Folland calls them), which are

$$\Lambda = \left\{ f, f = \sum_{j \leq n} a_j \chi_{E_j}, a_j \in \mathbb{C}, \mu(E_j) < +\infty \right\}$$

So for every $f \in L^p$, there exists a sequence $\{f_n\} \subseteq \Lambda$ with $f_n \rightarrow f$ pointwise and $f_n \rightarrow f$ in L^p . ■

Theorem 7.10

WTS.

Proof.



Theorem 7.11

WTS.

Proof.



Chapter 8

Theorem 8.1

WTS.

Proof.



Theorem 8.2

WTS.

Proof.



Theorem 8.3

WTS. *If $f \in C^\infty$, then $f \in \mathcal{S}$ if and only if $x^\beta \partial^\alpha f$ is bounded for all multi-indices α, β*

Proof.



Theorem 8.4

WTS.

Proof.



Theorem 8.5

WTS.

Proof.



Theorem 8.6

WTS.

Proof.



Theorem 8.7

WTS.

Proof.



Theorem 8.8

WTS.

Proof.



Theorem 8.9

WTS.

Proof.



Theorem 8.10

WTS.

Proof.



Theorem 8.11

WTS.

Proof.



Theorem 8.12

WTS.

Proof.



Theorem 8.13

WTS.

Proof.



Theorem 8.14

WTS. Suppose $\phi \in L^1$, and $\int \phi(x)dx = a$.

- (a) If $f \in L^p$, $p \in [1, +\infty]$, then $f * \phi_t \rightarrow af$ in the L^p norm as $t \rightarrow 0$.
- (b) If f is bounded and uniformly continuous, then $f * \phi_t \rightarrow af$ uniformly as $t \rightarrow 0$.
- (c) If $f \in L^\infty$ and f is continuous on an open set U , then $f * \phi_t \rightarrow af$ uniformly on compact subsets of U as $t \rightarrow 0$

Proof of Part A. First, the convolution $f * \phi_t$ is in L^p by Young's Inequality (Theorem 8.7). Furthermore,

$$f * \phi_t - af = \int_{y \in \mathbb{R}^n} f(x - y)t^{-n}\phi(t^{-1}y)dy - \int_{y \in \mathbb{R}^n} f(x)\phi(y)dy \quad (22)$$

Now apply Theorem 2.44, with $y \mapsto y/t$, and denote this invertible map by $T \in GL(n, \mathbb{R})$, so that $|\det(T)| = t^{-n}$, then $y = T(y)t$ for every $t > 0$. It follows that

$$\begin{aligned} (f * \phi_t)(x) &= |\det(T)| \cdot \int_{y \in \mathbb{R}^n} f(x - t \cdot Ty)\phi(T(y))dy \\ &= \int_{z \in \mathbb{R}^n} f(x - tz)\phi(z)dz \\ &= \int_{z \in \mathbb{R}^n} \tau_{tz}f(x)\phi(z)dz \end{aligned} \quad (23)$$

Next, $a = \int \phi$ so $af = \int f(x)\phi(z)dz$. Using Equations (22) and (23) we get

$$(f * \phi_t - af)(x) = \int_{z \in \mathbb{R}^n} (\tau_{tz}f - f)\phi(z)dz \quad (24)$$

We wish to apply Minkowski's Inequality for integrals, which states, roughly speaking:

The norm of an integral is less than the integral of the norm.

to Equation (24), and

$$\|f * \phi_t - af\|_p \leq \int_{z \in \mathbb{R}^n} \|(\tau_{tz}f - f)\phi(z)\|_p dz \quad (25)$$

The assumptions for Theorem 6.19 are satisfied by

1. Notice for every $z \in \mathbb{R}^{n'}$,

$$\|(\tau_{tz}f - f)\phi(z)\|_p = \left(\int_{x \in \mathbb{R}^n} |(\tau_{tz}f(x) - f(x))\phi(z)|^p dx \right)^{1/p} \leq |\phi(z)| \left(2\|f\|_p \right) < +\infty$$

Since $\|\phi\|_1 < +\infty$, $|\phi(z)| < +\infty$ almost everywhere.

2. Next, to show $z \mapsto \|\phi(z)(\tau_{tz}f - f)\|_p$ is in $L^1\mathbb{R}^n$, z . Reuse the last estimate in the previous bullet point, and

$$\|\phi(z)(\tau_{tz}f - f)\|_p \leq |\phi(z)| \left(2\|f\|_p \right)$$

Taking the integral in L^+ with respect to z , we get

$$\left\| \left(\|\phi(z)(\tau_{tz}f - f)\|_p \right) \right\|_1 < +\infty$$

so both assumptions are satisfied.

Therefore Equation (25) holds. Next, fix any sequence of $t_n > 0$ with $t_n \rightarrow 0$. The Dominated Convergence Theorem gives, since $|\phi(z)|\|\tau_{t_n z}f - f\|_p$ is dominated by $|\phi(z)| \cdot 2\|f\|_p \in L^1 \cap L^+$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{z \in \mathbb{R}^n} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz &= \int_{z \in \mathbb{R}^n} \lim_{n \rightarrow \infty} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz \\ &= \int_{z \in \mathbb{R}^n} 0 dz \\ &= 0 \end{aligned}$$

The second last equality is from Lemma 8.4, as translation is continuous in the L^p norm for $p \in [1, +\infty)$. So almost every $z \in \mathbb{R}^n$ (since again, $|\phi(z)|$ can be infinite on a null set),

$$\|\tau_{t_n z}f - f\|_p \rightarrow 0 \implies \|\tau_{t_n z}f - f\|_p |\phi(z)| \rightarrow 0$$

as $n \rightarrow +\infty$. It follows that

$$\lim_{n \rightarrow \infty} \|f * \phi_{t_n} - af\|_p = \lim_{n \rightarrow \infty} \left\| \int_{z \in \mathbb{R}^n} [\tau_{t_n z} f(x) - f(x)] \phi(z) dz \right\|_p = 0$$

Since the sequence $t_n \rightarrow 0$ is arbitrary, we conclude that the function $t \mapsto \|f * \phi_t - af\|_p$ has a limit of 0 as $t \rightarrow 0$. \blacksquare

Proof of Part B. Suppose $f \in \text{UBC}(\mathbb{R}^n)$, so that f is uniformly continuous and bounded. We wish to show $f * \phi_t \rightarrow af$ uniformly as $t \rightarrow 0$. In symbols,

$$g : t \mapsto \|f * \phi_t - af\|_u, \quad g \rightarrow 0, \text{ as } t \rightarrow 0$$

The convolution between f and ϕ_t makes sense at every $x \in \mathbb{R}^n$, as

$$\int |\tau_y f(x)| |\phi(y)| dy \leq \|f\|_u \cdot \|\phi\|_1 < +\infty$$

Taking the supremum norm on both sides of Equation (24), we get

$$\begin{aligned} \|f * \phi_t - af\|_u &= \sup_{x \in \mathbb{R}^n} \left| \int_{z \in \mathbb{R}^n} (\tau_{tz} f - f) \cdot \phi(z) dz \right| \\ &\leq \sup_{x \in \mathbb{R}^n} \int_{z \in \mathbb{R}^n} |\tau_{tz} f - f| \cdot |\phi(z)| dz \\ &\leq \int_{z \in \mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |\tau_{tz} f - f| \cdot |\phi(z)| dz \\ &= \int_{z \in \mathbb{R}^n} \|\tau_{tz} f - f\|_u \cdot |\phi(z)| dz \end{aligned} \quad (26)$$

the last equality is a simple consequence of the monotonicity of the integral in L^+ , indeed. For every $x \in \mathbb{R}^n$, the following holds pointwise for almost every z

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f - f\|_u \implies \sup_{x \in \mathbb{R}^n} |\tau_{tz} f - f| \leq \|\tau_{tz} f - f\|_u$$

Apply the Dominated Theorem to the right member of (26), noting that it is dominated by $|\phi(z)| \cdot 2\|f\|_u \in L^1 \cap L^+$ as we have done for Part A of the proof. Since this holds for every sequence $t_n \rightarrow 0$, the proof is complete. \blacksquare

Proof of Part C. Next, suppose that $f \in L^\infty$, and $f \in C(U)$, where U is open in \mathbb{R}^n . We claim that

$$f * \phi_t \rightarrow af$$

within the topology of uniform convergence on compact subsets of U . So that for every $K \in \mathfrak{J}$, $K \subseteq U$

$$\sup_{x \in K} |f * \phi_t - af| \rightarrow 0, \text{ as } t \rightarrow 0$$

First, a small technical Lemma.

Lemma 14.1 *If $\phi \in L^1(\mathbb{R}^n)$, then for every $\varepsilon > 0$, there exists $E \in \mathfrak{J}$, with*

$$\int_{E^c} |\phi| = \|\phi \chi_{E^c}\|_1 < +\varepsilon$$

Proof. Assume that $\phi \geq 0$, if not, replace ϕ by $|\phi|$. Since $C_c(\mathbb{R}^n)$ is dense in L^1 for every $\varepsilon 2^{-1} > 0$ there exists some $\psi \in C_c(\mathbb{R}^n)$ with $\|\psi - \phi\|_1 < \varepsilon^{-1}$, and denote $E = \text{supp}(\psi) \in \mathfrak{J}$, then

$$\| |\psi| - |\phi| \|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

So we can assume $\psi \geq 0$ as well, perhaps by relabelling ψ by $|\psi|$. Then,

$$\|\psi - \chi_E \phi\|_1 = \|\chi_E(\psi - \phi)\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

by monotonicity in L^+ . The Triangle Inequality in L^1 gives

$$\|\chi_{E^c} \phi\|_1 = \|\phi - \chi_E \phi\|_1 = \|\phi(1 - \chi_E)\|_1 \leq \|\phi - \psi\|_1 + \|\psi - \chi_E \phi\|_1 < \varepsilon$$

■

Back to the main proof of Part C, fix any $\varepsilon > 0$, then by Lemma 14.1, ϕ induces some $E \in \mathfrak{J}$ with $\|\chi_{E^c} \phi\|_1 < +\varepsilon$. By Lemma 8.4, $\chi_K f \in C_c(\mathbb{R}^n) \subseteq \text{UBC}(\mathbb{R}^n)$. Uniform continuity of $\chi_K f$ gives us the continuity of translations. Now for the same $\varepsilon > 0$, there exists $r > 0$, for every $w \in \mathbb{R}^n$,

$$|w| < r \implies \|\tau_w \chi_K f - \chi_K f\|_u < +\varepsilon \quad (27)$$

Since $E \in \mathfrak{J}$, it is bounded, and let t be a small positive number such that for every $z \in E$,

$$|tz| < t \cdot (1 + \sup_{z \in E} |z|) < r$$

There exists such a t , namely $t = r2^{-1}(1 + \sup_{z \in E} |z|)^{-1}$. And for this $t > 0$, it follows that for every $z \in E$,

$$\sup_{x \in K} |\tau_{tz}f - f| < +\varepsilon$$

Since this holds for every $z \in E$, we write

$$\sup_{x \in K, z \in E} |\tau_{tz}f - f| < +\varepsilon$$

And

$$|\phi(z)| \left[\sup_{x \in K, z \in E} |\tau_{tz}f - f| \right] < |\phi(z)|\varepsilon$$

Monotonicity in $L^+(E, z)$ reads, for every $x \in K$,

$$\int_{z \in E} |\phi(z)(\tau_{tz}f - f)| dz \leq \int_{z \in E} |\phi|\varepsilon dz = \varepsilon \|\chi_E \phi\|_1 \leq \varepsilon \|\phi\|_1$$

Since this holds for every $x \in \mathbb{R}^n$,

$$\sup_{x \in K} \left\{ \int_{z \in E} |\phi(z)| \cdot |\tau_{tz}f - f| dz \right\} \leq \varepsilon \|\phi\|_1 \quad (28)$$

Next, notice for every t, z , we have

$$|\tau_{tz}f - f| \leq \|\tau_{tz}f\|_u + \|f\|_u \leq 2 \cdot \|f\|_u$$

And the following holds $z \in E^c$ a.e,

$$|\phi(z)| \cdot |\tau_{tz}f - f| \leq |\phi(z)| \cdot 2\|f\|_u$$

Taking the integral, and applying the condition we imposed on E from Lemma (14.1), so that

$$\int_{z \in E^c} |\phi(z)| \cdot |\tau_{tz}f - f| dz \leq 2\|f\|_u \int_{z \in E^c} |\phi(z)| dz \leq 2\|f\|_u \varepsilon$$

Taking the supremum of the above estimate, so

$$\sup_{x \in K} \left\{ \int_{z \in E^c} |\phi(z)(\tau_{tz}f - f)| dz \right\} \leq 2\|f\|_u \varepsilon \quad (29)$$

Combining Equations (28) and (29). Applying the additivity of the supremum (of $x \in K$), since both members are finite,

$$\sup_{x \in K} \left\{ \int_E |\phi(z)|(\tau_{tz}f - f)dz + \int_{E^c} |\phi(z)|(\tau_{tz}f - f)dz \right\} < \varepsilon(2\|f\|_u + \|\phi\|_1)$$

The left member above is equal to $\sup_{x \in K} |f * \phi_t - af|$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of Part C. ■

Theorem 8.15

WTS. If $|\phi(x)| \leq C(1 + |x|)^{-n-\varepsilon}$, where $\varepsilon > 0$, and if $f \in L^p$, for $p \in [1, +\infty)$, then

$$f * \phi_t \rightarrow af$$

pointwise for every x in the Lebesgue set of f ,

$$\mathcal{L}_f = \left\{ x \in \mathbb{R}^n, \lim_{r \rightarrow 0} \frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy = 0 \right\}$$

We also claim that $m(\mathcal{L}_f^c) = 0$, and $x \in \mathcal{L}_f$ at every continuous $f(x)$.

The proof is long, and will be divided into several parts. Let us start with a couple of Lemmas about the Lebesgue Set of f , and several pointwise estimates that will be of use.

Lemma 15.1 If $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$, and

$$|\phi(x)| \leq C(1 + |x|)^{n-\varepsilon}, \varepsilon > 0 \quad (30)$$

then $\phi \in L^1$. Furthermore, $\phi_t \in L^1$ for every $t > 0$.

Proof of 15.1. If $x \neq 0$, then

$$|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)} \leq C \cdot |x|^{-(n+\varepsilon)}$$

on some B^c as defined in Theorem 2.52, so $\phi \in L^1(B^c)$. Next,

$$n + \varepsilon > n > n/2 = a$$

and by monotonicity,

$$|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)} \leq C \cdot (1 + |x|)^{-(n/2)}$$

so $\phi \in L^1(\mathbb{R}^n)$. Next, if $\phi \in L^1$, then

$$|\phi_t(x)| = t^{-n} |\phi(t^{-1}x)|$$

taking the integral in L^+ , and applying Theorem 2.44, with $T : x \mapsto t^{-1}x$, and $\det(T) = t^{-n}$, so that

$$\int |\phi_t|(x) dx = |\det(T)| \int |\phi| \circ T(x) dx = \int |\phi|(x) dx < +\infty$$

This completes the Lemma. ■

Lemma 15.2 *If $f : \mathbb{R}^n \rightarrow \mathbb{C}$, and if $f \in C(\mathbb{R}^n)$, then $\mathcal{L}_f = \mathbb{R}^n$.*

Proof of 15.2. Let $x \notin \mathcal{L}_f$, and there exists a sequence $r_k \rightarrow 0$ and $\varepsilon_0 > 0$ but

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy \geq \varepsilon_0$$

We claim that for every $k \geq 1$, we can find a $y_k \in B(r_k, x) \setminus \{x\}$ with

$$|f(x) - f(y_k)| \geq \varepsilon_0$$

Indeed, suppose by contradiction that no such y_k exists, and by monotonicity,

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy < \frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} \varepsilon_0 dy = \varepsilon_0$$

So choose y_k as above, and it is clear that $y_k \rightarrow x$ as $k \rightarrow \infty$, but $f(y_k) \not\rightarrow f(x)$. Therefore f is not continuous at x . \blacksquare

Lemma 15.3 *If $x \in \mathcal{L}_f$, then for every $\delta > 0$ there exists a $\eta > 0$, with*

$$r \leq \eta \implies \int_{|y| < r} |f(x - y) - f(x)| dy \leq \delta \cdot r^n$$

Proof of 15.3. We will start with something trivial.

$$m(B(r)) = r^n m(B(1)) \tag{31}$$

where $B(r) = \{x \in \mathbb{R}^n, |x| < r\}$. By Theorem 2.44,

$$\begin{aligned} m(B(r)) &= \int \chi_B(x/r) dx \\ &= |\det(T)|^{-1} \int \chi_B(x) dx \\ &= r^n m(B(1)) \end{aligned}$$

where $T : x \mapsto x/r$ and $\det(T) = r^{-n}$. Fix $x \in \mathcal{L}_f$, and take $\varepsilon = \delta/m(B(1)) > 0$, and by definition this induces some $\eta > 0$, and for every $r \leq \eta$

$$\frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy \leq \varepsilon$$

By translation invariance of m ,

$$m(B(r, x)) = m(B(r)) = r^n \cdot m(B(1))$$

and apply the map $y \mapsto x - y$, which is a composition a rotation by $| - 1 |$ and a translation by $x \in \mathbb{R}^n$. By Theorems 2.44 and 2.42,

$$\int_{|y| \in B(r)} |f(x) - f(x - y)| dy = \int_{y \in B(r, x)} |f(x) - f(y)| dy < \varepsilon m(B(1)) \cdot r^n = \delta r^n$$

where we used the fact that

$$\begin{aligned} d(x - y, x) < r &\iff d(-y, 0) < r \\ &\iff d(y, 0) < r \end{aligned}$$

hence

$$\chi_{B(r, x)}(x - y) = \chi_{B(r, 0)}(y)$$

■

Lemma 15.4 *Let $A_j = \left\{ |y| \in [2^{-j}\eta, 2^{1-j}\eta] \right\}$, and if Equation (30) holds for ϕ then ϕ_t satisfies*

$$|\phi_t| \leq C \cdot t^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \quad (32)$$

on A_j for every $t > 0$, where $\alpha = t^{-1}\eta$ for some $\eta > 0$.

Moreover, if $A_0 = \left\{ |y| < 2^{-K}\eta \right\}$, where $K \geq 0$, then

$$|\phi_t(y)| \leq C \cdot t^{-n} \quad (33)$$

on A_0

Proof of 15.4. Notice that

$$t^{-1}y \in [2^{-j} \cdot \eta/t, 2^{1-j} \cdot \eta/t) = [2^{-j} \cdot \alpha, 2^{1-j} \cdot \alpha)$$

And

$$1 + |t^{-1}y| \geq |t^{-1}y| \geq 2^{-j}\alpha$$

Therefore

$$C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}$$

and applying Equation (30) establishes the first claim.

The second claim follows from Equation (30),

$$|\phi_t(y)| \leq C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}$$

■

Lemma 15.5

Proof.

■

Main Proof of Theorem 8.15. The outline of the proof is as follows,

1. $|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)}$ for $\varepsilon > 0$ and
2. $f \in L^p$ for $p \in [1, +\infty)$,
3. for any $x \in \mathcal{L}_f$, we wish to show

$$|f * \phi_t - af|(x) \rightarrow 0, \quad \text{as } t \rightarrow 0$$

4. To prove this, we fix some $\beta > 0$ and show that

$$|f * \phi_t - af|(x) < \beta$$

since β is arbitrary, the proof will be complete.

5. By Lemma 15.3, for every $\delta > 0$ there exists a $\eta > 0$ where $r \leq \eta$ implies

$$\int_{|y| < r} |f(x) - f(x - y)| dy \leq \delta \cdot r^n$$

and using the L^1 inequality,

$$\begin{aligned} |f * \phi_t - af|(x) &= \left| \int [f(x - y) - f(x)] \cdot \phi_t(y) dy \right| \\ &\leq \int |f(x - y) - f(x)| \cdot |\phi_t(y)| dy \\ &= \int_{|y| < \eta} |f(x - y) - f(y)| \cdot |\phi_t(y)| dy + \int_{|y| \geq \eta} |f(x - y) - f(y)| \cdot |\phi_t(y)| dy \\ &= I_1 + I_2 \end{aligned}$$

6. Let $\delta = \beta(2A)^{-1}$, where

$$A = 2^n \cdot C \left[\frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$$

we make the claim that this choice of δ will give us $I_1 < \beta/2$

7. After choosing $\delta > 0$, (which induces $\eta > 0$), we will show that $I_2 < \beta/2$ (for a fixed $\eta > 0$) for t sufficiently small, and applying the Triangle Inequality finishes the proof.

Let η be as above, and for $t > 0$ and suppose we can find a $K \in \mathbb{N}^+$ with

$$2^K \leq \eta/t \leq 2^{K+1} \quad (34)$$

and define $\alpha = \eta/t$ for convenience.

Notice for any $K \geq 1$, the interval $[0, 1)$ can be partitioned in the following manner

$$[0, 1) = [0, 2^{-K}) \cup \left(\bigcup_{j=1}^K [2^{-j}, 2^{1-j}) \right)$$

and let us define

$$A_j = \left\{ |y| \in [2^{-j}\eta, 2^{1-j}\eta) \right\}, \quad A_0 = \left\{ |y| \in [0, 2^{-K}\eta) \right\}$$

If no such K exists, then let $A_j = \emptyset$ and set $A_0 = \{|y| \in [0, \eta)\}$. The disjoint union of all $A_{j \geq 0}$ is the open ball $\{|y| \in [0, \eta)\}$. By Lemma 15.4 and Lemma 15.3 each $j \geq 0$,

$$\begin{aligned} I_1 &= \sum_{j=0}^K \int_{y \in A_j} |f(x-y) - f(y)| |\phi_t(y)| dy \\ &\leq Ct^{-n} \delta (2^{-K}\eta)^n + \sum_{j=1}^K \int_{y \in A_j} |f(x-y) - f(y)| |\phi_t(y)| dy \\ &\leq Ct^{-n} \delta (2^{-K}\eta)^n + \sum_{j=1}^K Ct^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \delta (2^{1-j}\eta)^n \end{aligned}$$

The left member reads,

$$\begin{aligned} Ct^{-n}\delta(2^{-K}\eta)^n &\leq C\delta\alpha^n 2^{-Kn} \\ &\leq C\delta 2^{n(K+1)} 2^{-Kn} \\ &= C\delta 2^n \end{aligned}$$

and termwise for the right,

$$\begin{aligned} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n &= C\delta \cdot t^\varepsilon \cdot 2^{j\varepsilon+n}\eta^{-\varepsilon} \\ &= (C\delta 2^n \alpha^{-\varepsilon}) \cdot 2^{j\varepsilon} \end{aligned}$$

Summing over the geometric series,

$$\begin{aligned} \sum_{j=1}^K 2^{j\varepsilon} &= 2^\varepsilon \sum_{j=0}^{K-1} 2^{j\varepsilon} \\ &= \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} \end{aligned}$$

using the estimate for α in Equation (34)

$$\alpha \in [2^K, 2^K + 1) \implies \alpha^{-\varepsilon} \in [2^{-\varepsilon(K+1)}, 2^{-\varepsilon K})$$

and combining the last few equations, the right member becomes

$$\begin{aligned} (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} &\leq (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)}}{2^\varepsilon - 1} \\ &\leq (C\delta 2^n) \cdot \frac{2^\varepsilon}{2^\varepsilon - 1} \end{aligned}$$

Finally, $I_1 \leq (C\delta 2^n) \left[\frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$, and by Step 6, $I_1 \leq \beta/2$.

Obtaining an estimate for I_2 is another laborious enterprise. Let us define $W = \{|y| \geq \eta\}$, and

- By Holder's Inequality,

$$I_2 \leq \|f\|_p \|\chi_W \cdot \phi_t\|_q + |f(x)| \|\chi_W \cdot \phi_t\|_1$$

where q is the conjugate exponent to p . Since $p \in [1, +\infty)$, it suffices to show $\|\chi_W \cdot \phi_t\|_q \rightarrow 0$ as $t \rightarrow 0$ for $q \in [1, +\infty]$.

- Suppose $q = +\infty$,

$$y \in W \iff |y| \geq \eta \iff |t^{-1}y| \geq \alpha$$

$$\text{then } \|\chi_W \cdot \phi_t\|_\infty \leq Ct^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq Ct^\varepsilon \eta^{-(n+\varepsilon)}$$

- Now suppose $q \in [1, +\infty)$, by polar integration and Theorems 2.51, 2.52 (brace yourselves):

$$\begin{aligned} \|\chi_W \cdot \phi_t\|_q^q &= t^{-nq} \cdot \int_{y \in W} C^q \cdot |t^{-1}y|^{-q \cdot (n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \int_{|y| \geq \eta} |y|^{-q \cdot (n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \sigma(S^{n-1}) \int_{r \geq \eta} r^{n-1} \cdot r^{-q \cdot (n+\varepsilon)} dr \\ &= \frac{C^q t^{\varepsilon q}}{n - q \cdot (n + \varepsilon)} r^{n - q \cdot (n + \varepsilon)} \Big|_\eta^\infty \\ &= \frac{C^q t^{\varepsilon q}}{q \cdot (n + \varepsilon) - n} \eta^{n - q \cdot (n + \varepsilon)} \\ \|\chi_W \cdot \phi_t\|_q &= \left[\frac{C}{(q \cdot (n + \varepsilon) - n)^{1/q}} \left(\eta^{n - q \cdot (n + \varepsilon)} \right)^{1/q} \right] t^\varepsilon \\ &= C_3(q) t^\varepsilon \end{aligned}$$

- Find a t sufficiently small so that

$$t^\varepsilon < \min \left\{ \beta(4C_3(1)|f(x)|)^{-1}, \beta(4C_3(q)\|f\|_p)^{-1}, \beta(4C \cdot \eta^{-(n+\varepsilon)})^{-1} \right\}$$

- Therefore $I_2 < \beta/2$, and the proof is complete upon sending $\beta \rightarrow 0$.

■

Theorem 8.16

WTS. *See Theorem 8.15*

Proof.



Theorem 8.17

WTS.

Proof.



Theorem 8.18**WTS.***Proof.*

Theorem 8.19

WTS.

Proof.



Theorem 8.20

WTS.

Proof.

