MATH 263: Section 003, Tutorial 4

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1 Exact ODEs (review)

An **exact ODE** is of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$
$$M(x,y)\ dx + N(x,y)\ dy = 0$$

where

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Then, we can define some $F(x,y)$ such that:

$$d(F(x,y)) = M(x,y)\ dx + N(x,y)\ dy = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$$

Then, find $F(x,y)$, and the relation $F(x,y) = \int 0\ dx = C$ is the solution.
Note: $d(F(x,y)) = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$ is a form of the multivariable chain rule.

Problem 2.2a. From Boyce and DiPrima, 10th edition (2.6, exercise 5, p.101):
Check whether the differential equation:

$$\frac{dy}{dx} = -\frac{ax+by}{bx+cy}$$

is exact. If so, find the general solution.

Note: when the ODE is not exact, an integrating factor may make it exact. Let it be $\mu(x,y)$:

$$(\mu(x,y)M(x,y)) + (\mu(x,y)N(x,y))\frac{dy}{dx} = 0$$

To make the ODE exact,

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$$
$$\frac{\partial N}{\partial x}\mu + N\frac{\partial \mu}{\partial x} = \frac{\partial M}{\partial y}\mu + M\frac{\partial \mu}{\partial y}$$
$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu + N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = 0$$

Instead of solving a PDE, consider two specific cases:
1. $\mu(x,y) = \mu(x)$:

$$(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\mu = N\frac{d\mu}{dx}$$
$$\mu(x) = \exp[\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\ dx]$$

2. $\mu(x,y) = \mu(y)$:

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu = M\frac{d\mu}{dy}$$

$$\mu(y) = \exp[\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\ dy]$$

Try both cases if needed, your integrating factor must be single-variable for it to work. In the general case, both or none may work.

Problem 2.2b. From Boyce and DiPrima, 10th edition (2.6, modified exercise 27, p.102):
Find the solution $x(t)$ of:

$$dt + (\frac{t}{x} - \sin x)\ dx = 0.$$

With IVP $x(0) = -\pi$.

2 Autonomous Equations

(From Tutorial 2): **Autonomous ODE's** only contain the dependent variable, they are of the form:

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)})$$

(From Tutorial 2): A **slope field** is a graphical representation of a family of functions satisfying $y' = f(x,y)$. For some point (x,y), one draws the slope $y' = f(x,y)$ to qualitatively represent solutions. Given a slope field, starting at an initial condition and tracing along the field sketches the particular solution.

Problem 2.4. From Boyce and DiPrima, 10th edition (2.5, exercise 12, p.89):
Consider the following ODE:

$$\frac{d\theta}{dt} = -\theta^2(\theta^2 - 4)$$

Sketch the graph of $f(\theta)$ versus θ, determine and classify the equilibrium points. Then find the general solution.