Chapter C: Algebraic Topology

Manifolds Homotopy

Homotopy

This section will follow Munkres Chapters 9 and 13 closely. Possibly other chapters as well.

Definition 1.1: Path

A path is a continuous function from the unit interval $f:[0,1] \to \mathbf{X}$. We say f is a path form x_0 to x_1 if $f(0) = x_0$ and $f(1) = x_1$.

We denote the set of paths from x_0 to x_1 by $Path(x_0, x_1)$. If $f \in Path(x_0, x_1)$, we sometimes denote the reversal of f by $\overline{f} \in Path(x_1, x_0)$, where $\overline{f}(s) \stackrel{\triangle}{=} f(1-s)$.

Definition 1.2: Loop

A loop at $x_0 \in \mathbf{X}$ is a path that begins and ends at x_0 , and $\text{Loop}(x_0) \stackrel{\Delta}{=} \text{Path}(x_0, x_0)$. The constant path (or loop) at x_0 is denoted by $e_{x_0} : [0, 1] \to \mathbf{X}$.

$$e_{x_0}(s) = x_0, \quad \forall s \in [0, 1]$$

Definition 1.3: Homotopy of C(X, Y)

Let f, and g continuous functions from X to Y. f and g are homotopic, denoted by f = g if there exists a continuous function $F \in C(X \times I, Y)$ where

$$F(x,0) = f(x)$$
 and $F(x,1) = g(x)$ (1)

where I = [0, 1].

The function F is called the homotopy between f and g.

If $f \simeq h$, where h is the constant function, we say f is nulhomotopic.

Definition 1.4: Path Homotopy of Path(x_0, x_1)

Two paths f_0 , $f_1 \in \text{Path}(x_0, x_1)$ are said to be *path homotopic*, if there exists a continuous function $F \in C(I \times I, \mathbf{X})$, with

• F is a homotopy between f_0 and f_1 (in the sense of Definition 1.3). For every $s \in [0,1]$,

$$F(s,0) = f_0(s)$$
 and $F(s,1) = f_1(s)$ (2)

• F leaves the endpoints fixed. For every $t \in [0, 1]$, then

$$F(0,t) = x_0$$
 and $F(1,t) = x_1$ (3)

If f_0 and f_1 are path-homotopic, we write $f_0 \simeq_p f_1$.

Manifolds Homotopy

- The function $F \in C(I \times I, \mathbf{X})$ is called the path homotopy between f_0 and f_1 .
- If $f \in \text{Loop}(x_0)$ is path homotopic to the constant path e_{x_0} , then f is nulhomotopic.

• The relation \simeq_p is defined for paths that have the same initial and final points. So it is a relation on $\operatorname{Path}(x_0, x_1)$.

Proposition 1.1: Munkres Lemma 51.1

The relations \simeq and \simeq_p are equivalence relations on $C(\mathbf{X},\mathbf{Y})$ and $\mathrm{Path}(x_0,x_1)$ respectively.

Proof. $(f \simeq f)$: Let $f \in C(\mathbf{X}, \mathbf{Y})$. Define

$$F: \mathbf{X} \times I \to \mathbf{Y}$$
 For every $t \in [0, 1], F(x, t) = f(x)$

F is continuous, since $F = \pi_{\mathbf{X}} \circ (f \times \mathrm{id}_{[0,1]})$, where $f \times \mathrm{id}_{[0,1]}$ is the product of two continuous functions, which is again continuous by Chapter A. Moreover, F(x,0) = f(x) = F(x,1), so F is a homotopy between f and itself.

 $(f \simeq g \implies g \simeq f)$: Let F be the homotopy between f and g. Let G be the 'reversal' in the second coordinate of F, meaning

$$G(x,t) = F(x,1-t)$$
 is continuous, since $G = F \circ (\mathrm{id}_{\mathbf{X}} \times c)$

where $c: I \to I$ that maps $t \mapsto 1 - t$ is continuous, so $id_{\mathbf{X}} \times c$ is continuous; hence G is continuous. Notice for every $x \in \mathbf{X}$,

$$G(x,0) = F(x,1) = g(x)$$
 and $G(x,1) = F(x,0) = f(x)$

therefore G is a homotopy between g and f.

$$H(x,t) = \begin{cases} F(x,2t - \lfloor 2t \rfloor) & \text{for } 0 \le t \le 2^{-1} \\ G(x,2t - \lfloor 2t \rfloor) & \text{for } 2^{-1} \le t \le 1 \end{cases}$$

$$(4)$$

where $|\cdot|$ denotes the floor function.

- H is well defined on the overlap $\mathbf{X} \times 2^{-1}$, since F(x,1) = G(x,0) = g(x) at every $x \in \mathbf{X}$.
- If t = 0, then H(x, 1) = F(x, 0) = f(x), and t = 1 gives H(x, 1) = G(x, 1) = h(x).
- Since $H|_{\mathbf{X}\times[0,2^{-1}]}$ and $H|_{\mathbf{X}\times[2^{-1},1]}$ are continuous functions, and they agree on the overlap, H is continuous by the pasting Lemma, and defines a homotopy between f and h.

Now consider paths f, g, h in Path (x_0, x_1) , $(f \simeq_p f)$ is trivial. So is symmetry of \simeq_p , as the reversal in the second coordinate (see above) of the path homotopy between f and g is path homotopy between g and f.

Suppose $f \simeq_p g$, and $g \simeq_p g$. Let F, and G be the path homotopies between f, g and g, h. Write H as in Equation (4), it is a continuous function on $I \times I \to X$, that satisfies

$$H(s,0) = F(s,0) = f(s)$$
 and $H(s,1) = G(s,1) = h(s)$ for every $s \in [0,1]$

If s = 0, it is easy to see from Equation (4) that for every $t \in [0, 1]$,

$$H(0,t) = \begin{cases} F(0,2t - \lfloor 2t \rfloor) = x_0 & \text{for } 0 \le t \le 2^{-1} \\ G(0,2t - \lfloor 2t \rfloor) = x_0 & \text{for } 2^{-1} \le t \le 1 \end{cases} = x_0 \quad \text{and}$$

$$H(1,t) = \begin{cases} F(1,2t - \lfloor 2t \rfloor) = x_1 & \text{for } 0 \le t \le 2^{-1} \\ G(1,2t - \lfloor 2t \rfloor) = x_1 & \text{for } 2^{-1} \le t \le 1 \end{cases} = x_1$$

So the endpoints remain fixed throughout the deformation in t, and H is a path homotopy between f and h. This proves transitivity.

Path and PathClass Products

Definition 2.1: Product of Paths f * g

Let $f \in \text{Path}(x_0, x_1)$ and $g \in \text{Path}(x_1, x_2)$, the product of f and g, denoted by f * g is another path from x_0 to x_2 . For $s \in [0, 1]$,

$$(f * g)(s) \stackrel{\triangle}{=} \begin{cases} f(2s - \lfloor 2s \rfloor) & \text{for } 0 \le s \le 2^{-1} \\ g(2s - \lfloor 2s \rfloor) & \text{for } 2^{-1} \le s \le 1 \end{cases}$$
 (5)

Notice the similarities between Equations (4) and (5),

Proposition 2.1: Properties of the Path Product

Let $f \in \text{Path}(x_0, x_1)$ and $g \in \text{Path}(x_0, x_1)$, let $k \in C(X, Y)$, then

- (i) Invariant under left-multiplication: $f \simeq_p g \implies k \circ f \simeq_p k \circ g$, where $k \circ f$ and $k \circ g$ are elements Paths from $k(x_0)$ to $k(x_1)$, and if F be a path homotopy between f and g, then $k \circ F$ is a path homotopy between $k \circ f$ and $k \circ g$.
- (ii) If we redefine $f \in \text{Path}(x_0, x_1), g \in \text{Path}(x_1, x_2), \text{ and } k \text{ be as above, then}$

$$k \circ (f * q) = (k \circ f) * (k \circ q)$$

Proof.

Proof of Part (i): It is clear that $k \circ f$ and $k \circ g$ are elements of Path $(k(x_0), k(x_1))$, and see Part (ii) for the proof of $k \circ f \simeq_p k \circ g$.

Proof of Part (ii): Let F be the path homotopy between f and g. The composition $(k \circ F)$ is in $C(\mathbf{X} \times I, \mathbf{Y})$. Equation (2) reads

$$\begin{aligned} &(k\circ F)(s,0)=k(F(s,0))=(k\circ f)(s) \text{ and}\\ &(k\circ F)(s,1)=k(F(s,1))=(k\circ g)(s) \text{ for every } s\in[0,1] \end{aligned}$$

and Equation (3) gives

$$(k \circ F)(0,t) = k(F(0,t)) = k(x_0)$$
 and $(k \circ F)(1,t) = k(F(1,t)) = k(x_1)$ for every $t \in [0,1]$

therefore $k \circ F$ is a path homotopy between the paths $k \circ f$ and $k \circ g$.

Definition 2.2: Path Homotopy class [f]

Let $f \in \text{Path}(x_0, x_1)$, we define the path homotopy class of f as

$$[f] \stackrel{\Delta}{=} \left\{ g \in \operatorname{Path}(x_0, x_1), \ g \simeq_p f
ight\}$$

Definition 2.3: Product of PathClasses [f] * [g]

Let $*: \text{PathClass}(x_0, x_1) \times \text{PathClass}(x_1, x_2) \to \text{PathClass}(x_0, x_2)$ be a binary operation, where

$$[f] * [g] \stackrel{\Delta}{=} [f * g]$$
 is well defined.

for arbitrary $[f] \in \text{PathClass}(x_0, x_1)$ and $[g] \in \text{PathClass}(x_1, x_2)$. This means it is independent of the representative chosen. More formally, if $f \simeq_p f' \in \text{Path}(x_0, x_1)$, and $g \simeq_p g' \in \text{Path}(x_1, x_2)$, then $f * g \simeq_p f' * g'$.

Proposition 2.2: Properties of the PathClass product

Let [f], [g] and [h] be PathClasses from and to the points x_0, x_1, x_2 . Then

- 1. Associativity: ([f] * [g]) * [h] = [f] * ([g] * [h]),
- 2. Left and Right identities: if $[f] \in \text{PathClass}(x_0, x_1)$, e_{x_0} , e_{x_1} denote the constant paths on x_0 and x_1 (the initial and final points of any $f \in [f]$), then

$$[e_{x_0}] * [f] = [f] \quad \text{and} \quad [f] * [e_{x_1}] = [f]$$

3. Left and Right inverses: let $[\overline{f}]$ be the PathClass containing the reversal of f (see Definition 1.1) for the definition, then

$$[\overline{f}]*[f]=[e_{x_1}] \quad ext{and} \quad [f]*[\overline{f}]=[e_{x_0}]$$

4. Generalized Associativity: if $\{[f_j]\}_{j\leq n}$ is a sequence of PathClasses, such that $[f_j] \in \text{PathClass}(x_{j-1}, x_j)$, then

$$\prod [f_j] \stackrel{\triangle}{=} [f_1] * [f_2] * \cdots * [f_n]$$
 is a well-defined object

meaning we can place the brackets wherever we want.

Proof. We will give an outline for the proof of Generalized Associativity, the rest are trivial. Let $\{[f_j]\}$ be defined as above. If $\{a_j\}_{j=0}^n$, and $\{b_j\}_{j=0}^n$ are 'cell partitions' of the unit interval (in the sense of the

Manifolds Fundamental Group

Riemann integral), meaning

$$0 = a_0 < a_1 < \dots < a_n = 1$$
, and $0 = b_0 < b_1 < \dots < b_n = 1$

We agree to define the following

- the lengths of each cell $l_{a_j} \stackrel{\Delta}{=} a_j a_{j-1}$ and $l_{b_j} \stackrel{\Delta}{=} b_j b_{j-1}$, and
- the cells themselves are denoted by $\operatorname{cell}(a_j) = [a_{j-1}, a_j], \operatorname{cell}(b_j) = [b_{j-1}, b_j],$
- $p \in Path(0,1)$, where p is given explicitly by

$$p(s) = \sum_{j=1}^n \chi_{\operatorname{cell}(a_j) \setminus \{a_{j-1}\}} igg(rac{l_{b_j}}{l_{a_j}} (s-a_j) + b_jigg)$$

It is clear p is continuous, and for $j = 1, \ldots, n$,

$$p|_{\text{cell}(a_j)}$$
 is the positive linear map from $\text{cell}(a_j)$ to $\text{cell}(b_j)$

Using the same line of argumentation as in the proof for associativity, we see that any two 'ways' of bracketing the expression has no impact on the path-homotopy class.

Fundamental Group

Definition 3.1: Fundamental group $\pi_1(\mathbf{X}, x_0)$

Let $x_0 \in \mathbf{X}$, the fundamental group of \mathbf{X} relative to (base point) x_0 is denoted by $\pi_1(\mathbf{X}, x_0) = \operatorname{PathClass}(x_0, x_0)$.

Definition 3.2: Isomorphism induced by $Path(x_0, x_1)$

Suppose $\alpha \in \text{Path}(x_0, x_1)$, we define a map $\hat{\alpha} : \pi_1(\mathbf{X}, x_0) \to \pi_1(\mathbf{X}, x_1)$, with

$$\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha]$$

where $\overline{\alpha}$ is the reversal of α . We call $\hat{\alpha}$ the isomorphism induced by α (Munkres Theorem 52.1).

Isomorphism proof. Let [f] and [g] be elements of $\pi_1(\mathbf{X}, x_0)$, then

$$\begin{split} \widehat{\alpha}([f]*[g]) &= ([\overline{\alpha}]*[f]*[\alpha])*([\overline{\alpha}]*[g]*[\alpha]) \\ &= [\overline{\alpha}]*([f]*[g])*[\alpha] \\ &= \widehat{\alpha}([f])*\widehat{\alpha}([g]) \end{split}$$

and $\hat{\alpha}$ is a homomorphism. We claim inverse of $\hat{\alpha}$ is $\hat{\overline{\alpha}}$. Fix $[f] \in \pi_1(\mathbf{X}, x_0), [g] \in \pi_1(\mathbf{X}, x_1)$, then

$$(\widehat{\overline{\alpha}}\circ\widehat{\alpha})([f])=[\alpha]*([\overline{\alpha}]*[f]*[\alpha])*[\overline{\alpha}]=[f]$$

so $\hat{\alpha}$ is the left-inverse for $\hat{\alpha}$. A similar argument shows it is the right inverse as well with $(\hat{\alpha} \circ \hat{\alpha})([g]) = [g]$. Therefore $\pi_1(\mathbf{X}, x_0)$ is group isomorphic to $\pi_1(\mathbf{X}, x_1)$.

Homomorphisms

Definition 4.1: Homomorphism induced by a continuous map

Let $h \in C(\mathbf{X}, \mathbf{Y})$, and $y_0 = h(x_0)$, it induces a map between loops at x_0 and y_0 .

$$h_*: \operatorname{Loop}(x_0) \to \operatorname{Loop}(y_0), f \mapsto h \circ f$$

It is a also a group homomorphism between fundamental groups. We use the same symbol for the two maps, relying on context to distinguish between the two.

$$h_*: \pi_1(\mathbf{X}, x_0) \to \pi_1(\mathbf{Y}, y_0), [f] \mapsto [h \circ f]$$

is well defined because of Proposition 2.2, it is a homomorphism (again by Proposition 2.2) because h 'distributes' over *

$$h\circ (f*g)=(h\circ f)*(h\circ g)$$

Remark 4.1: Functorial properties of the h_*

If $x_0 \in \mathbf{X}$, the tuple (x_0, \mathbf{X}) is an object in the category of pointed topological spaces, and the map h_* is a covariant functor from the category of pointed topological spaces to the category of groups.

Follows from Munkres Theorem 52.4, if the expressions below make sense,

$$(g \circ f)_* = g_* \circ f_*$$
 and $h_* \circ (g \circ f)_* = (h \circ g)_* \circ f_*$

And the identity map $i: \mathbf{X} \to \mathbf{X}$ gets 'sent' to the identity homomorphism in $\operatorname{Hom}(\pi_1(\mathbf{X}, x_0), \pi_1(\mathbf{X}, x_0))$. And if h is a homeomorphism between \mathbf{X} and \mathbf{Y} , then h_* is an isomorphism at every point.

Simply connected space

Definition 5.1: Simply connected space

A topological space **X** is *simply connected* if it is path-connected, and $\pi_1(\mathbf{X}, x_0) = \{[e_{x_0}]\}$ for some $x_0 \in \mathbf{X}$. Notice this implies every fundamental group of **X** is trivial.

Proposition 5.1: Properties of simply connected spaces

If **X** is a simply connected space, then PathClass (x_0, x_1) consists of one element. That is to say, if f and g are Paths from x_0 to x_1 , then $f \simeq_p g$.

Covering maps

Definition 6.1: Covering maps and spaces