MATH 263: Section 003, Tutorial 3

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1 Bernoulli Equations (review)

A **Bernoulli equation** is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

When $n \notin \{0, 1\}$, we can let $v = y^{1-n}$, making the ODE linear for v.

Problem 1. Use the substitution $v = y^{1-n}$ to turn the general Bernoulli equation into a linear ODE for v. You do **not** have to solve for v.

Solution: let $v = y^{1-n} \Rightarrow y = v^{\frac{1}{1-n}} \Rightarrow y' = \frac{1}{1-n}v^{\frac{1}{1-n}-1}v' = \frac{1}{1-n}v^{\frac{1}{1-n}-1}v'$. Substituting back in the ODE:

$$\frac{1}{1-n}v^{\frac{1}{1-n}-1}v' + P(x)v^{\frac{1}{1-n}} = Q(x)v^{\frac{1}{1-n}-1}$$

$$\frac{1}{1-n}v' + P(x)\ v = Q(x)$$

$$v' + (1-n)\ P(x)\ v = (1-n)\ Q(x).$$

2 Exact ODEs

An **exact ODE** is of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

$$M(x,y)\ dx + N(x,y)\ dy = 0$$

where

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Then, we can define some $F(x,y)$ such that:

$$d(F(x,y)) = M(x,y)\ dx + N(x,y)\ dy = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$$

Then, find $F(x,y)$, and the relation $F(x,y) = \int 0\ dx = C$ is the solution.

Problem 2.2a. From Boyce and DiPrima, 10th edition (2.6, exercise 14, p.101):
Check whether the differential equation:

$$(9x^2 + y - 1)\ dx + (x - 4y)\ dy = 0$$

is exact. If so, solve the IVP $y(1) = 0$.

Solution: Let $M(x,y) = 9x^2 + y - 1$ and $N(x,y) = x - 4y$. Check for exactness:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = 1.$$

Therefore, the ODE is exact.

$$M(x,y) = \frac{\partial f}{\partial x} = 9x^2 + y - 1$$

Then integrate. With a multivariable function, indefinite integrals must have a function of integration (similar to adding a constant of integration):

$$f = \int 9x^2 + y - 1 \ dx = 3x^3 + xy - x + h(y)$$

$$N(x,y) = \frac{\partial f}{\partial y} = x + h'(y) = x - 4y$$

$$h'(y) = -4y$$

$$h(y) = -2y^2 - C$$

Therefore,

$$f(x,y) = 3x^3 + xy - x - 2y^2 - C = 0$$

$$3x^3 + xy - x - 2y^2 = C$$

IVP, $y(1) = 0$:

$$3 \cdot 1^3 + 1 \cdot 0 - 1 - 2 \cdot 0^2 = C \Rightarrow C = 3 - 1 = 2.$$

$$3x^3 + xy - x - 2y^2 = 2$$

Note: when the ODE is not exact, an integrating factor may make it exact. Let it be $\mu(x,y)$:

$$(\mu(x,y)M(x,y)) + (\mu(x,y)N(x,y))\frac{dy}{dx} = 0$$

To make the ODE exact,

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$$

$$\frac{\partial N}{\partial x}\mu + N\frac{\partial \mu}{\partial x} = \frac{\partial M}{\partial y}\mu + M\frac{\partial \mu}{\partial y}$$

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu + N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = 0$$

Instead of solving a PDE, consider two specific cases:

1. $\mu(x,y) = \mu(x)$:

$$(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\mu = N\frac{d\mu}{dx}$$

$$\mu(x) = \exp[\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} \ dx]$$

2. $\mu(x,y) = \mu(y)$:

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu = M\frac{d\mu}{dy}$$

$$\mu(y) = \exp[\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} \ dy]$$

Try both cases if needed, your integrating factor must be single-variable for it to work. In the general case, both or none may work.

Problem 2.2b. From Boyce and DiPrima, 10th edition (2.6, exercise 28, p.102):
Find the general solution of:

$$y\ dx + (2xy - e^{-2y})\ dy = 0.$$

Solution: Let $M(x, y) = y$ and $N(x, y) = 2xy - e^{-2y}$. Check for exactness:

$$\frac{\partial N}{\partial x} = 2y \neq \frac{\partial M}{\partial y} = 1.$$

Therefore, the equation is not exact without an integrating factor. By observing the two formulas given, one can see that the expression for $\mu(y)$ will be single-variable:

$$\mu(y) = \exp[\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\ dy] = \exp[\int \frac{2y-1}{y}\ dy]$$

$$\mu(y) = \exp[\int (2 - \frac{1}{y})\ dy] = e^{2y} \cdot e^{-ln|y|} \Rightarrow \mu(y) = \frac{1}{y}e^{2y}.$$

Multiplying by the integrating factor:

$$e^{2y}\ dx + (2xe^{2y} - \frac{1}{y})\ dy = 0.$$

Let $M^*(x, y) = e^{2y}$ and $N^*(x, y) = 2xe^{2y} - \frac{1}{y}$. Check for exactness:

$$\frac{\partial N^*}{\partial x} = \frac{\partial M^*}{\partial y} = e^{2y}.$$

$$M^* = \frac{\partial f}{\partial x} = e^{2y}$$

$$f = \int e^{2y}\ dx = xe^{2y} + h(y)$$

$$N^* = \frac{\partial f}{\partial y} = 2xe^{2y} + h'(y) = 2xe^{2y} - \frac{1}{y}$$

$$h'(y) = -\frac{1}{y}$$

$$h(y) = -\ln|y| - C$$

$$f(x, y) = xe^{2y} - \ln|y| - C = 0$$

$$xe^{2y} - \ln|y| = C.$$

3 Change of Variables

As explained in tutorial 2, various substitutions can be used, such as $v = y'(x)$ *or* $v = ax + by$.

Problem 2.3. Find the general solution of:

$$y'' = x^2 + 2xy' + (y')^2$$

Hint: Use the substitution $v(x) = x + y'(x)$.

Solution:

$$y'' = x^2 + 2xy' + (y')^2$$

Using the substitution from above:

$$y''(x) = v'(x) - 1$$
$$y'' = (x + y(x))^2$$
$$v'(x) - 1 = v^2$$
$$\frac{dv}{dx} = v^2 + 1$$
$$\int \frac{dv}{v^2+1}\ dv = \int\ dx$$
$$\arctan v = x + C_1$$
$$v = \tan(x + C_1)$$
$$y(x) = \int v(x)\ dx - \frac{1}{2}x^2 + C_2$$
$$y(x) = -\ln|\cos(x + C_1)| - \frac{1}{2}x^2 + C_2.$$

4 Slope Fields

(From Tutorial 2): A **slope field** is a graphical representation of a family of functions satisfying $y' = f(x, y)$. For some point (x, y), one draws the slope $y' = f(x, y)$ to qualitatively represent solutions. Given a slope field, starting at an initial condition and tracing along the field sketches the particular solution.

Problem 2.4. Draw the slope field of

$$y' = x(y^2 + 2y - 3).$$

Solve for $y' = 0$ for equilibrium points:

$$y' = x(y + 3)(y - 1)$$

From the ODE, it is known that:

- $y'(0) = 0$ for all $x \in \mathbb{R}$.
- The two equilibrium points are $y = -3$ and $y = 1$.

Note: one can solve this ODE, since it is separable.

$$\int \frac{dx}{(y+3)(y-1)} = \int x\ dx$$

The left hand side can be solved using partial fractions:

$$\int (\frac{1}{4(y-1)} - \frac{1}{4(y+3)})\ dy = \int x\ dx$$
$$\ln|y - 1| - \ln|y + 3| = 2x^2 + C$$
$$\frac{y-1}{y+3} = Ke^{2x^2}(K = \pm e^C)$$
$$y = \frac{1 + 3Ke^{2x^2}}{1 - Ke^{2x^2}}.$$

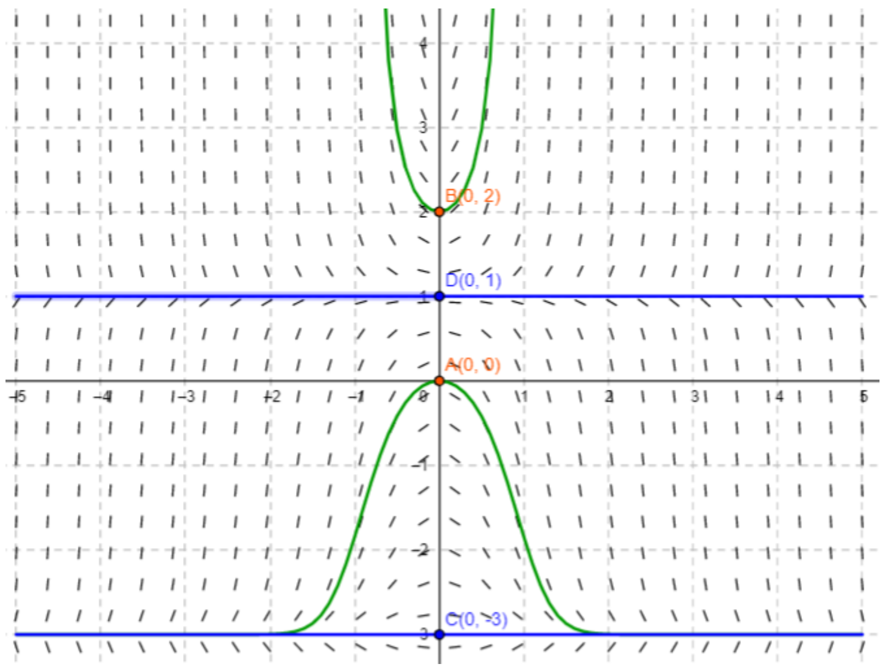

Figure 1: Slope field of $y' = x(y^2 + 2y - 3)$ with equilibrium points in blue and two solutions in green drawn by the initial conditions ($y(0) = 0$ and $y(0) = 2$). $y = -3$ is a "repeller" and $y = 1$ is an "attractor".