Contents

Chapter 1

Theorem 1.1

Proposition 1.1

Let $\mathcal{M}(\mathcal{F})$ be the σ-algebra generated by $\{$, if $\mathcal{E}$ is a subset of $\mathbb{P}(X)$, with $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.

Proof. Notice that because $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$,

$$\mathcal{M}(\mathcal{F}) \in \{\mathcal{M},\, \mathcal{E} \subseteq \mathcal{M},\, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

Taking the intersection, noting that $\mathcal{M}(\mathcal{E})$ is the intersection of all σ-algebras containing $\mathcal{E}$ as a subset, we have

$$\bigcap\{\mathcal{M}(\mathcal{F})\} \supseteq \bigcap\{\mathcal{M},\, \mathcal{E} \subseteq \mathcal{M},\, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

And

$$\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$$

■

Theorem 1.2

Proposition 2.1

The Borel σ-algebra of $\mathbb{R}$, $\mathbb{B}$ is generated by the following

- The family of open intervals $\mathcal{E}_1 = \{(a, b),\ a < b\}$,
- The family of closed intervals $\mathcal{E}_2 = \{[a, b],\ a < b\}$,
- The family of half-open intervals $\mathcal{E}_3 = \{(a, b],\ a < b\}$ or $\mathcal{E}_4 = \{[a, b),\ a < b\}$
- The open rays $\mathcal{E}_5 = \{(a, +\infty),\ a \in \mathbb{R}\}$ or $\mathcal{E}_6 = \{(-\infty, a),\ a \in \mathbb{R}\}$
- The closed rays $\mathcal{E}_7 = \{[a, +\infty),\ a \in \mathbb{R}\}$ or $\mathcal{E}_8 = \{(-\infty, a],\ a \in \mathbb{R}\}$

Proof. By definition, $\mathbb{B}$ is generated by the family of all open sets in $\mathbb{R}$, but every open set is a countable union of open intervals. Therefore

$$\mathcal{T}_{\mathbb{R}} \subseteq \mathcal{M}(\mathcal{E}_\infty) \implies \mathbb{B} \subseteq \mathcal{M}(\mathcal{E}_\infty)$$

Conversely, every open interval is an open set, hence

$$\mathcal{E}_1 \subseteq \mathcal{T}_{\mathbb{R}} \subseteq \mathbb{B} \implies \mathcal{M}(\mathcal{E}_\infty) \subseteq \mathbb{B}$$

Every closed interval can also be written as a countable intersection of open intervals, for every $[a, b]$, with $a < b$, we have

$$[a, b] = \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1}) \tag{1}$$

Indeed, fix any $x \in [a, b]$ then for every $n \geq 1$,

$$a - n^{-1} < a \leq x \leq b < b + n^{-1}$$

So $x \in \bigcap_{n \geq 1}(a - n^{-1}, b + n^{-1})$. If x an element of the left member, then for every $n \geq 1$,

$$a - n^{-1} < x \implies a - x \leq 0$$

Similarly for $x \leq b$, therefore equation (1) is valid, and $\mathcal{E}_2 \subseteq \mathbb{B} = \mathcal{M}(\mathcal{E}_\infty)$. To show the reverse estimate, every open interval can be written as a countable union of closed intervals,

$$(a, b) = \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \tag{2}$$

To show that the above estimate is indeed true, fix any $x \in (a, b)$, then

$$\begin{aligned} a < x < b &\iff a < a + n^{-1} \leq x \leq b - n^{-1} < b \\ &\iff x \in \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \end{aligned}$$

So that equation (2) holds. By similar argumentation we have $\mathcal{E}_1 \subseteq \mathcal{M}(\mathcal{E}_\in) \implies \mathcal{M}(\mathcal{E}_\in) = \mathcal{M}(\mathcal{E}_\infty)$.

For $\mathcal{E}_3$, $\mathcal{E}_4$

- $(a, b] = \bigcap_{n \geq 1}(a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1)$,
- $(a, b) = \bigcup_{n \geq 1}(a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3)$,
- $[a, b) = \bigcup_{n \geq 1}[a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_2)$,
- $[a, b] = \bigcap_{n \geq 1}[a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_2) \subseteq \mathcal{M}(\mathcal{E}_4)$

So that $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_4) = \mathbb{B}$. By taking complements of each element we get $\mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$ and $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$. Notice also that

- $(a, b] = (a, +\infty) \cap (-\infty, b]$, proves $\mathcal{E}_3 \subseteq \mathcal{M}(\mathcal{E}_5)$, and $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$.
- $(a, +\infty) = \cup_{n \geq 1}(a, a + n]$, proves $\mathcal{E}_5 \subseteq \mathcal{M}(\mathcal{E}_3)$, and $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_3)$.
- $[a, b) = [a, +\infty) \cap (-\infty, b)$, proves $\mathcal{E}_4 \subseteq \mathcal{M}(\mathcal{E}_6)$, and $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_7)$,
- $[a, +\infty) = \bigcup_{n \geq 1}[a, a + n)$, proves $\mathcal{E}_7 \subseteq \mathcal{M}(\mathcal{E}_4)$, and $\mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{M}(\mathcal{E}_4)$.

Finally, $\mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8) = \mathbb{B}$ and $\mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7) = \mathbb{B}$. ■

Theorem 1.3

Proposition 3.1

If A is countable, then $\otimes_{\alpha\in A}\mathcal{M}_\alpha$ is the σ-algebra generated by

$$W \triangleq \left\{\prod_{\alpha\in A} E_\alpha,\ E_\alpha \in \mathcal{M}_\alpha\right\}$$

Proof. We agree to define

$$V \triangleq \left\{\pi_\alpha^{-1}(E_\alpha),\ E_\alpha \in \mathcal{M}_\alpha\right\}$$

By definition, V generates $\otimes_{\alpha\in A}\mathcal{M}_\alpha$. Fix any element in $x = \pi_\alpha^{-1}(E_\alpha) \in V$, then

$$\pi_\alpha(x) \in E_\alpha,\ \pi_{\beta\neq\alpha}(x) \in X_\beta$$

Then $x \in W$ if we choose $x = \prod_{c\in A} E_c$, for $E_c = E_\alpha$ if $c = \alpha$, and $E_c = X_c$ if $c \neq \alpha$. ■

Theorem 1.4

Proposition 4.1

Proof. ■

Theorem 1.5

Proposition 5.1

Proof. ■

Theorem 1.6

Proposition 6.1

Proof. ■

Theorem 1.7

Proposition 7.1

Proof. ∎

Theorem 1.8

Proposition 8.1

Proof. ■

Theorem 1.9

Proposition 9.1

Proof. ∎

Theorem 1.10

Proposition 10.1

Proof. ■

Theorem 1.11

Proposition 11.1: Caratheodory's Theorem

If μ^* is an outer measure on $\mathbf{X}$, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.

Proof. We quote the definition for a set $A \subseteq X$ to be μ^* measurable. For any $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \tag{3}$$

- Show $\mathcal{M}$ is an algebra.
- μ^* is finitely additive on $\mathcal{M}$.
- $\mathcal{M}$ is closed under countable disjoint (this makes $\mathcal{M}$ a sigma algebra, since it is an algebra that is closed under countable disjiont unions)

Lemma 11.1

The family of μ^*-measurable sets is an algebra.

Proof of Lemma 11.1. Clearly $\mathcal{M}$ is closed under complements. To show that it is a σ-algebra, and if $A, B \in \mathcal{M}$, then $\left\{\underbrace{E \cap A}_{1}, \underbrace{E \setminus A}_{2}\right\} \subseteq \mathbb{P}(\mathbf{X})$. And because B is μ^*-measurable,

$$\mu^*(E) = \underbrace{\mu^*(E \cap A \cap B) + \mu^*(E \cap A \setminus B)}_{1} + \underbrace{\mu^*(E \cap B \setminus A) + \mu^*(E \setminus (A \cup B))}_{2}$$

By subadditivity of μ^*, $A \cup B = A \cap B + A \setminus B + B \setminus A$ with $+$ denoting the disjiont union, hence

$$\mu^*(E \cap (A \cap B)) \leq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \setminus B)) + \mu^*(E \cap (B \setminus A))$$

and

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B))$$

■

Lemma 11.2

μ^* is finitely additive on $\mathcal{M}$, the family of μ^*-measurable sets.

Proof of Lemma 11.2. Let A, B be disjoint μ^*-measurable sets. It suffices to show $\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B)$, as the reverse estimate follows from subadditivity. From Lemma 11.1, $A \cup B \in \mathcal{M}$, so

$$\begin{aligned} \mu^*(A \cup B) &= \mu^*(A \cup B \cap A) + \mu^*(A \cup B \setminus A) \\ &= \mu^*(A \cup \varnothing) + \mu^*(A \setminus A \cup B \setminus A) \\ &= \mu^*(A) + \mu^*(B) \end{aligned}$$

■

Corollary 11.1

If $\{A_j\}_{j\geq N} \subseteq \mathcal{M}$ is a finite disjoint family, then

$$\mu^*\left(\bigcup A_{j\leq N}\right) = \sum \mu^*(A_{j\leq N})$$

Lemma 11.3

Let $\{A_j\}_{j\geq 1}$ be a countable disjoint sequence in $\mathcal{M}$, and denote $B_n = \bigcup A_{j\leq n} \in \mathcal{M}$ by Lemma 11.1. For every $E \subseteq \mathbf{X}$,

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j\leq n})$$

Proof of Lemma 11.3. We will proceed by induction. If $n = 1$ then we have equality, suppose the result holds for $n \in \mathbb{N}^+$, and $A_{n+1} \in \mathcal{M}$ so

$$\begin{aligned}
\mu^*(E \cap B_{n+1}) &= \mu^*(E \cap B_{n+1} \cap A_{n+1}) = \mu^*(E \cap B_{n+1} \setminus A_{n+1}) \\
&= \mu^*(E \cap A_{n+1}) + \mu^*(E \cap B_n) \\
&= \sum_{j\leq n+1} \mu^*(E \cap A_j)
\end{aligned}$$

as $A_j \cap A_{n+1} = \varnothing \iff A_j \setminus A_{n+1} = A_j$, and $B_n \cap A_n = A_n \iff A_n \subseteq B_n$. ∎

To show $\mathcal{M}$ is a sigma-algebra, fix any disjoint sequence $\{A_j\}_{j\geq 1} \subseteq \mathcal{M}$, and denote B_n as in lem. 11.3. Define $B = \bigcup A_{j\geq 1} \supseteq B_n$ and for every $n \geq 1$, we have

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \setminus B_n) \\
&= \sum \mu^*(E \cap A_{j\leq n}) + \mu^*(E \setminus B_n) \\
&\geq \sum \mu^*(E \cap A_{j\leq n}) + \mu^*(E \setminus B) && \text{since } B_n \subseteq B \iff B^c \subseteq B_n^c \\
&\geq \sup_n \left[\sum \mu^*(E \cap A_{j\leq n})\right] + \mu^*(E \setminus B)
\end{aligned}$$

Let $J \subseteq \mathbb{N}^+$ be a finite non-empty set. And $\sup J \in \mathbb{N}^+$, $\sup J < +\infty$. By the Archmedean Property we can find a large $N \in \mathbb{N}^+$, with $N > J$ so that

$$\sum_{j\in J} \mu^*(E \cap A_j) \leq \sum_{j\leq N} \mu^*(E \cap A_j)$$

Applying the estimate $\sup_n\left[\sum \mu^*(E \cap A_{j\leq n})\right] + \mu^*(E \setminus B) \leq \mu^*(E)$ reads

$$\left[\sum_{j\in J} \mu^*(E \cap A_j)\right] + \mu^*(E \setminus B) \leq \mu^*(E)$$

Now by Chapter 0, the infinite sum

$$\sum_{j\geq 1}\mu^*(E\cap A_j)=\sup\left\{\sum_{j\in J}\mu^*(E\cap A_j),\ J\subseteq\mathbb{N}^+,\ 0<|J|<+\infty\right\}$$

and $\bigcup A_{j\geq 1}=B$ is μ^*-measurable. Since $\mu^*(\varnothing)=0$, and μ^* is countably additive on $\mathcal{M}$, (perhaps by replacing E with the union over all disjoint sets), μ^* is a measure on $\mathcal{M}$. To show μ^* is a complete measure, fix $A\in\mathcal{M}$ where $\mu^*(A)=0$. Then any $B\subseteq A$ is also null, and for $E\subseteq\mathbf{X}$,

$$\mu^*(E)\geq\underbrace{\mu^*(E\cap B)}_{0}+\mu^*(E\setminus B)\implies B\in\mathcal{M}$$

■

Theorem 1.12

Proposition 12.1

Proof. ■

Theorem 1.13

Proposition 13.1

Proof. ■

Theorem 1.14

Proposition 14.1

Proof. ∎

Theorem 1.15

Proposition 15.1

Proof. If $\{E_j\}_{j\geq 1} \subseteq \mathcal{A}$ such that each $E_j = FDU(I_{ji})$ over finitely many i, and suppose E_j are disjoint, and that $DU(E_j) \in \mathcal{A}$. So that $DU(E_j) = FDU(I_\alpha)$ for some finite collection of half-intervals $\{I_\alpha\}$.

We will first prove the simpler case. Suppose we have already proven:

$$\{E_j\}_{j\geq 1} \subseteq \mathcal{A},\ DU(E_j) = I_\alpha \in \mathcal{A} \implies \mu_0\Big(DU(E_j)\Big) = \sum \mu_0(E_j) = \mu_0(I_\alpha) \quad (4)$$

but each E_j is a FDU of I_{ji}, and for every $j \geq 1$, $E_j \cap I_\alpha \in \mathcal{A}$ (closure under intersections, because the family of FDU of h-intervals is an algebra).

Thus we have a disjoint sequence whose union is one h-interval. In symbols:

$$DU(E_j) = FDU(I_\alpha) \implies I_\alpha = DU(E_j \cap I_\alpha)$$

$\forall j \geq 1,\ E_j \cap I_\alpha \in \mathcal{A} \implies$

$$\begin{aligned}
\mu_0(FDU(I_\alpha)) &= \sum_{\alpha<+\infty} \mu_0(I_\alpha) \\
&= \sum_{\alpha<+\infty} \sum_{j\geq 1} \mu_0(E_j \cap I_\alpha) \\
&= \sum_{j\geq 1} \sum_{\alpha<+\infty} \mu_0(E_j \cap I_\alpha) \\
&= \sum_{j\geq 1} \mu_0(E_j)
\end{aligned}$$

It is permissible to swap the two summations because we are using the supremum definition for a sum of non-negative terms. And we applied finite-additivity (see earlier), to conclude that $\sum_{j\geq 1}\sum_\alpha \mu_0(E_j \cap I_\alpha) = \sum_{j\geq 1} \mu_0(E_j)$.

■

Define

- $\mathcal{H}_1 = \Big\{(a,\ b],\ -\infty \leq a < b < +\infty\Big\}$,
- $\mathcal{H}_2 = \Big\{(a,\ +\infty),\ a \in \mathbb{R} \cup \{-\infty\}\Big\}$,
- $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \{\varnothing\}$. Where $+$ denotes the disjoint union.
- DU: disjoint union, FDU: finite disjoint union.

Steps:

1. Show that $\mathcal{H}$ is an elementary family.
2. Show that if $I_\alpha \in \mathcal{H}_1$, then for every $I_\beta \in \mathcal{H}_1 \cup \mathcal{H}_2$, $I_\alpha \cap I_\beta \in \mathcal{H}_1$. We write this as
$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1,\ I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_1$$
3. Show that if $I_\alpha \in \mathcal{H}_2$, then
$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1,\ I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_2$$
4. Show that $\mu_0((a,\ b]) = \overline{F}(b) - \overline{F}(a)$ is well defined. (modify the proof in Folland to check for $a = -\infty$ with
$$\overline{F} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}, \quad \begin{cases} \overline{F}|_{\mathbb{R}} & = F \\ \overline{F}(+\infty) & = \sup_x F(x), \\ \overline{F}(-\infty) & = \inf_x F(x) \end{cases}$$
5. Show that $\mu_0((a,\ b]) = \overline{F}(b) - \overline{F}(a)$ is well defined for $b < +\infty$. If $E = (a,\ b] \in \mathcal{A}$, then E is an FDU of $\mathcal{H}_1$, and $\mathcal{H}_2$. So we write
$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2) = FDU(\mathcal{H}_1)$$
since E is bounded above, the $\mathcal{H}_2$ part of the FDU must be null. Now fix $E = FDU_{\mathcal{H}_1}(I_j) = FDU_{\mathcal{H}_1}(I_2)$. And follow the proof in Folland to see the 'well-definedness' of μ_0 if $E \in \mathcal{H}_1$.
6. Next, suppose $E \in \mathcal{H}_2$ and
$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2)$$
Clearly $FDU(\mathcal{H}_2) \neq \varnothing$, since E is unbounded above, and $FDU(\mathcal{H}_2)$ consists of exactly one element, so we write
$$E = FDU(\mathcal{H}_1) + (z,\ +\infty)$$
7. Show that $\mu_0((a,\ b]) = \overline{F}(b) - \overline{F}(a)$ is well defined. Hint: use the fact that if $E \in \mathcal{A}$, such that $E = FDU(E, \mathcal{H}_1) + FDU(E, \mathcal{H}_2)$, then $FDU(E, \mathcal{H}_2)$ contains at most one element (after throwing away empty sets), then use this to deduce $E \cap I_\alpha$ has a $FDU(E \cap I_\alpha, \mathcal{H}_2)$ of exactly one $\mathcal{H}_2$ interval, where I_α participates in $FDU(E, \mathcal{H}_2)$, if E is unbounded above. Then take $E \setminus I_\alpha = FDU(E \setminus I_\alpha, \mathcal{H}_1) = FDU(E, \mathcal{H}_1)$.

8. Now show that μ_0 is well-defined on all $E \in \mathcal{A}$.

9. Continue the proof for Folland until you reach the unbounded intervals, then modify the 'right continuity argument' to add an extra $\mathcal{H}_2$ interval. Let $I = \mathcal{H}_1 + \mathcal{H}_2 = I_\alpha + I_\beta$, meaning I can be represented by at most one $\mathcal{H}_1$ and $\mathcal{H}_2$ interval. If $(I_k) \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$, then $\{I_k \cap I_\alpha\} \subseteq \mathcal{H}_1$, and continue the proof as usual.

Theorem 1.16

Proposition 16.1

Proof. ■

Theorem 1.17

Proposition 17.1

Proof. ■

Theorem 1.18

Proposition 18.1

Proof. ■

Exercises

Exercise 1.1

Proposition 1.1

Proof. ■

Exercise 1.2

Proposition 2.1

Proof. ■

Exercise 1.3

Proposition 3.1

Proof. ■

Exercise 1.4

Proposition 4.1

An algebra $\mathcal{A}$ is a σ-algebra $\iff$ it is closed under countable increasing unions.

Proof. $\Longleftarrow$ is trivial. And it suffices to show that $\mathcal{A}$ is closed under countable disjoint unions. Indeed, if $\{E_j\}_{j\geq 1} \subseteq \mathcal{A}$ is a countable disjoint sequence of sets, write

$$F_n = \bigcup E_{j\leq n}$$

Clearly, F_j is increasing, and denote $F = \cup E_{j\geq 1}$, which is a member of $\mathcal{A}$. We claim that

$$\bigcup F_{n\geq 1} = \bigcup E_{j\geq 1}$$

Fix any $x \in \bigcup E_{j\geq 1}$, then x belongs in some $E_j \subseteq F_j$, and $\supseteq$ is proven. Also, if $x \in \bigcup F_{n\geq 1}$, then there exists some F_n for which x is a member of. For this particular F_n, means that $x \in E_j$ where $j \leq n$ and $x \in \bigcup E_{j\geq 1}$. ■

Exercise 1.5

Proposition 5.1

Let $\mathcal{M}(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E} \subseteq X$, and

$$\mathcal{N} = \left\{ \mathcal{M}(\mathcal{F}),\, \mathcal{F} \subseteq \mathcal{E},\, \mathcal{F} \text{ is countable} \right\}$$

Show that $\mathcal{M}(\mathcal{E}) = \mathcal{N}$.

Proof. The outline of the proof is as follows,

1. Prove that $\mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$,
2. Show that $\mathcal{N}$ is a σ-algebra,
3. Show that $\mathcal{N}$ contains $\mathcal{E}$ as a subset, and hence $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{N}$.

First, for any $\mathcal{F} \subseteq \mathcal{E}$, where $\mathcal{F}$ is countable, it follows from Lemma 1.1 that $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}\mathcal{E}$. Taking the union over all of such $\mathcal{F}$, we get $\bigcup \mathcal{M}(\mathcal{F}) = \mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$.

To show that $\mathcal{N}$ is a σ-algebra, fix any $A \in \mathcal{N}$, and A belongs to $\mathcal{M}(\mathcal{F})$, therefore $A^c \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{N}$. To show closure under countable unions, fix a sequence $\{E_j\} \subseteq \mathcal{N}$, then each of these E_j belongs to a corresponding $\mathcal{M}(\mathcal{F}_j)$, for $j \in \{1, 2, \ldots\}$. Now define

$$\overline{\mathcal{F}} = \bigcup \mathcal{F}_{j \geq 1} \subseteq \mathcal{E}$$

and $\overline{\mathcal{F}}$ is obviously countable. Hence for every $j \geq 1$, $\mathcal{M}(\mathcal{F}_j) \subseteq \mathcal{M}(\overline{\mathcal{F}})$ and taking the union yields

$$\bigcup \mathcal{M}(\mathcal{F}_{j \geq 1}) \subseteq \mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$$

It is also clear that our sequence $\{E_j\}$ is contained in $\mathcal{M}(\overline{\mathcal{F}})$, and $E = \bigcup E_j$ belongs to $\mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$ as an element. Therefore $\mathcal{N}$ is a σ-algebra.

Let $\alpha \in A$ index the family of sets in $\mathcal{E}$, (so that $E_\alpha \in \mathcal{E}$) and the singleton set of a set $\{E_\alpha\}$ is a countable subset of $\mathcal{E}$. For every $\alpha \in A$, we have

$$E_\alpha \in \mathcal{M}(\{E_\alpha\}) \subseteq \mathcal{N} \implies \mathcal{E} \subseteq \mathcal{N}$$

And one final application of Lemma 1.1 finishes the proof. ■

Exercise 1.6

Proposition 6.1

Proof. ■

Exercise 1.7

Proposition 7.1

If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$, and $a_1, \ldots, a_n \in [0, +\infty)$, then $\mu = \sum_1^n \mu_j$ is a measure on $(X, \mathcal{M})$.

Proof. If $\{E_j\}$ is a disjoint sequence in $\mathcal{M}$, and denote $E = \bigcup(E_j)$. If for each $k \leq n$, $\mu_k(E) < +\infty$,

$$\mu_k(E) = \sum \mu_k(E_j) \implies a_k\mu_k(E) = \sum a_k\mu_k(E_j)$$

Then,

$$\mu(E) = \sum_{k \leq n} a_k\mu_k(E) = \sum_{k \leq n}\sum_{j \geq 1} a_k\mu_k(E_j) = \sum_{j \geq 1}\sum_{k \leq n} a_k\mu_k(E_j) = \sum_{j \geq 1} \mu(E_j)$$

If there exists some mu_k such that $\mu_k(E) = +\infty$, then

$$\mu(E) = \sum_{k \leq n}\sum_{j \geq 1} a_k\mu_k(E_j)$$

Now if there exists some $\mu_{k'}$ with $\mu_{k'}(E) = +\infty$, then $\mu(E) = \sum_{k \leq n} \mu_k(E) = +\infty$, and

$$\sum_{j \geq 1} \mu(E_j) = \sup_N \sum_{j \leq N}\sum_{k \leq n} a_k\mu_k(E_j) \geq \mu_{k'}(E)$$

Therefore $\mu(E) = \sum_{j \geq 1} \mu(E_j)$, and μ is a measure. ■

Exercise 1.8

Proposition 8.1

If $(X, \mathcal{M}, \mu)$ is a measure space, and $\{E_j\} \subseteq \mathcal{M}$, then $\mu(\liminf E_j) \leq \liminf \mu(E_j)$. Also, $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ provided that $\mu(\bigcup E_{j\geq 1}) < +\infty$

Proof. If $\{E_j\}_{j\geq 1}$ is a sequence in $\mathcal{M}$, and define $F_m = \bigcap_{j\geq m} E_j$

$$\liminf E_j = \bigcup_{m\geq 1} \bigcap_{j\geq m} E_j = \bigcup_{m\geq 1} F_m$$

Also, for every $m \geq 1$, $F_m \subseteq E_m$, and F_m is an increasing sequence, because

$$[m, +\infty) \supseteq [m+1, +\infty) \implies F_m \subseteq F_{m+1}$$

Using continuity above, and writing $F = \bigcup F_{m\geq 1} = \liminf E_j$, we have

$$\begin{aligned} \mu(\liminf E_j) &= \mu(F) \\ &= \liminf \mu(F_m) \\ &\leq \liminf \mu(E_m) \end{aligned}$$

.
The second part of the proof is similar, if $G_m = \bigcup_{j\geq m} E_j$, then

$$\limsup E_j = \bigcap_{m\geq 1} \bigcup_{j\geq m} E_j = \bigcap_{m\geq 1} G_m$$

Similarly, G_m is a decreasing sequence, and since $\mu(\bigcup E_{j\geq 1}) = \mu(G_1)$ is finite, we can use continuity from above in the same manner, and the proof is complete. ∎

Exercise 1.9

Proposition 9.1

Proof. ■

Exercise 1.10

Proposition 10.1

Proof. ■

Exercise 1.11

Proposition 11.1

Proof. ■

Exercise 1.12

Proposition 12.1

Let $(X, \mathcal{M}, \mu)$ be a finite measure space,

- If $E, F \in \mathcal{M}$, and $\mu(E\Delta F) = 0$, then $\mu(E) = \mu(F)$,
- Say that $E \sim F$ if $\mu(E\Delta F) = 0$, then $\sim$ is an equivalence relation on $\mathcal{M}$,
- For every $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Show that ρ defines a metric on the space of $\mathcal{M}/\sim$ equivalence classes.

Proof of Part A. Use the fact that $\mu(F) = \mu(E \cap F) + \mu(F \cap E^c)$, and by monotonicity,

$$\mu(F \cap E^c) \le \mu(E\Delta F) = 0$$

And $\mu(F) = \mu(E \cap F) = \mu(E)$, the last equality follows after a simple modification. ■

Proof of Part B. Suppose that $\mu(E\Delta F) = \mu(F\Delta G) = 0$, then

- $\mu(E \cap F^c) = \mu(F \cap E^c) \le \mu(E\Delta F) = 0$ by monotonicity,
- Similarly, we have $\mu(F \cap G^c)\mu(G \cap F^c) = 0$, and
- By subadditivity,
  - $\mu(E \cap G^c) = \mu(E \cap F^c \cap G^c) + \mu(E \cap F \cap G^c) \le 0$, and $\mu(E \cap G^c) = 0$, and
  - $\mu(G \cap E^c) = 0$
- Therefore $\mu(E\Delta G) = \mu(E \cap G^c) + \mu(G \cap E^c) = 0$

It is clear that the relation is reflexive, since $E\Delta E = \varnothing$, and symmetry is trivial. ■

Proof of Part C. Since $\rho(E, F) = \rho(F, E)$, and $\rho(E, F) \ge 0$ for every $E, F \in \mathcal{M}$, and $\rho(E, F) = 0 \iff E \sim F$. We only have to prove the Triangle Inequality. Notice that

$$\begin{aligned}\mu(E \setminus F) &= \mu(E \cap F^c \cap G) + \mu(E \cap F^c \cap G^c) \\ &\le \mu(F^c \cap G) + \mu(E \cap F^c)\end{aligned}$$

and in the same fashion,

$$\mu(F \setminus E) \le \mu(F \cap G^c) + \mu(E^c \cap F)$$

Combining the two inequalities, and applying additivity finishes the proof. ■

Exercise 1.13

Proposition 13.1

Every σ-finite measure is semi-finite

Proof. Suppose μ is σ-finite then there exists an increasing sequence of sets $E_j \nearrow X$ with $\mu(E_j) < +\infty$. Now for every $W \in \mathcal{M}$, if $\mu(W) = +\infty$ then $\mu(W) = \lim_{j\to\infty} \mu(E_j \cap W) = +\infty$. Since this real-valued limit converges to its supremum $+\infty$, there exists a non-null subset $E_j \cap W$ of positive and finite measure. ∎

Exercise 1.14

Proposition 14.1

If μ is a semi-finite measure, and if $\mu(E) = +\infty$, for every $C > 0$, there exists an $F \subseteq E$ with $0 < \mu(F) < +\infty$.

Proof. Suppose by contradiction that there exists a $C > 0$ so for every $F \subseteq E$, if F is of finite measure, then $0 \leq \mu(F) \leq C$. Let $s = \sup\{\mu(F),\ F \subseteq E,\ 0 < \mu(F) < +\infty\}$, and for any $n^{-1} > 0$, this induces a F_n with measure

$$\mu(F_n) > s - n^{-1}$$

and take $A_n = \bigcup_{j \leq n} F_j$. A simple induction will show that $\mu(A_n) \leq \sum_{j \leq n} \mu(F_j) < +\infty$, therefore $\mu(A_n) \leq s$ for every $n \geq 1$. By continuity from below

$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{j \geq 1} F_j\right) \leq s$$

Next, by monotonicity, denoting the union over A_n by A, for every $n^{-1} > 0$

$$s - n^{-1} \leq \mu(A_n) \leq \mu(A) \leq s \implies \mu(A) = s$$

Now, $E \setminus A$ is a set of infinite measure, and by semi-finiteness. Find a set $B \subseteq E \setminus A$ with strictly positive measure, so that

$$\mu(A \cup B) = \mu(A) + \mu(B) > s$$

And this finishes the proof. ∎

Exercise 1.15

Proposition 15.1

Given a measure μ on $(\mathbf{X}, \mathcal{M})$, and define $\mu_0 = \sup\{\mu(F),\, F \subseteq E,\, \mu(F) < +\infty\}$. Show μ_0 is semi-finite. Then, show that if μ is semi-finite, $\mu = \mu_0$. Lastly, there exists a measure ν on $(\mathbf{X}, \mathcal{M})$, with $\mu = \nu + \mu_0$, where ν only assumes the values 0 or $+\infty$.

Proof. First, a small Lemma. We claim that $\mu_0 = \mu$ on finite sets. Let $E \in \mathcal{M}$, and $\mu(E) < +\infty$, since

$$\mu(E) \in \{\mu(F),\, F \subseteq E,\, \mu(E) < +\infty\} \implies \mu(E) \leq \mu_0(E)$$

Next, for every $W \subseteq E$, $\mu(W) \leq \mu(E)$, so $\mu_0(E) \leq \mu(E)$. This proves the equality.

If E is any measurable subset of $\mathbf{X}$, and suppose also $\mu_0(E) = +\infty$, one can easily find subsets of E, $\{E_n\}_{n\geq 1}$ with

$$n \geq \mu(E_n) < +\infty$$

But E_n is a subset of finite measure, so $0 < \mu(E_n) = \mu_0(E_n) < +\infty$. This proves the semi-finiteness of μ_0.

Next, suppose μ is semi-finite, and fix any measurable set E. If E is if finite measure, then $\mu(E) = \mu_0(E)$, and if $\mu(E) = +\infty$, apply Exercise 14, so there exists a sequence of subsets of finite measure $E_n \subseteq E$ for every $n \geq 1$, with $\mu(E_n) \to \mu(E)$. Therefore $\mu_0(E) = \mu(E)$.

For the last part of the proof, let μ be an arbitrary measure. And let $E \in \mathcal{M}$. If $\mu(E) < +\infty$, then $\nu(E) = 0$ would suffice (this proves the first property of the measure). If $\mu(E) = +\infty$, and if $\mu(E)$ is not semi-finite, then set $\nu(E) = +\infty$. So that $\mu_0(E) + \nu(E) = 0 + \infty = \infty = \mu(E)$. The additivity of ν is immediate, since ν can only assume two values. This finishes the proof. ■

Exercise 1.16

Proposition 16.1

Proof. ■

Exercise 1.17

Proposition 17.1

Let $\{A_j\}_{j\geq 1}$ be a countable disjoint sequence in $\mathcal{M}$, and denote $B_n = \bigcup A_{j\leq n} \in \mathcal{M}$. For every $E \subseteq \mathbf{X}$,

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j\leq n})$$

Proof. Proven in Theorem 1.11 as a Lemma. ■

Exercise 1.18

Proposition 18.1

Let $\mathcal{A} \subseteq \mathbb{P}(\mathbf{X})$ be an algebra. $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersection of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$, and μ^* be the induced outer-measure.

(a) For any $E \subseteq \mathbf{X}$, and $\varepsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

(b) If $\mu^*(E) < +\infty$, then E is μ^*-measurable $\iff$ there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.

(c) If μ_0 is σ-finite, the restriction $\mu^*(E) < +\infty$ in (b) is superfluous.

Proof of Part A. Let $E \subseteq \mathbf{X}$ and $\varepsilon > 0$, then by definition of μ^*,

$$\mu^*(E) + \varepsilon \geq \sum \mu_0(A_j) = \sum \mu^*(A_j) \geq A$$

by subadditivity and $A = \bigcup A_j$. ■

Proof of Part B. Suppose E is outer-measurable and of finite outermeasure, then by part A we have a sequence of $A_n \in \mathcal{A}_\sigma$ with

$$\mu^*(E) + n^{-1} \geq \mu^*(A_n) \implies \mu^*(E) = \mu^*(A)$$

if we define $A = \bigcap A_n \supseteq E$. Using the μ^*-measurability of E, we get

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) < +\infty \implies \mu^*(A \setminus E) = 0$$

Conversely, if $\mu^*(A \setminus E) = 0$, for any $V \subseteq \mathbf{X}$, with $\mu^*(V) < +\infty$, we have

$$\begin{aligned}
\mu^*(V) &= \mu^*(V \cap A) + \mu^*(V \setminus A) \\
&\geq \mu^*(V \cap E) + \mu^*(V \setminus A) + \mu^*(V \cap [A \setminus E]) \\
&\geq \mu^*(V \cap E) + \mu^*(V \setminus E)
\end{aligned}$$

■

Proof of Part C. Suppose μ_0 is σ-finite, then $E \in \mathcal{M}^*$ induces a sequence $E_j \nearrow E$, where each E_j is of finite measure. By part b) we obtain $\{A_j\} \subseteq \mathcal{A}_{\sigma\delta}$ with

$$\mu^*(A_j \setminus E_j) = 0$$

Now define $B = \bigcup A_j$, so that $B \in \mathcal{A}_{\sigma\delta}$. Observe $\bigcup(A_j \setminus E_j) = B \setminus E_1 \supseteq B \setminus E$ (verify these). And $\mu^*(B \setminus E) =\leq \sum \mu^*(A_j \setminus E_j) = 0$ by subadditivity. Since $B \supseteq E$, and $B \in \mathcal{A}_{\sigma\delta}$, this proves $\implies$. Conversely, suppose $E \subseteq \mathbf{X}$ and there exists a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$, $\mu^*(B \setminus E) = 0$. Let $\{\mathbf{X}_j\} \nearrow \mathbf{X}$ as a sequence of sets of finite measure. Then,

$$(\mathbf{X}_j \cap B) \setminus (\mathbf{X}_j \cap E) = \mathbf{X}_j \cap (B \setminus E) \subseteq B \setminus E$$

$\mathbf{X}_j \cap B \in \mathcal{A}_{\sigma\delta}$, and $\mathbf{X}_j \cap B \supseteq (\mathbf{X}_j \cap E)$. Each $E_j = \mathbf{X}_j \cap E$ is μ^* measurable by monotonicity, so is their countable union. ■

Exercise 1.19

Proposition 19.1

Let μ^* be an outer measure on $\mathbf{X}$ induced from a finite premeasure μ_0. If $E \subseteq \mathbf{X}$, define the inner measure of E to be $\mu_*(E) = \mu_0(\mathbf{X}) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.

Proof. Suppose $E \subseteq \mathbf{X}$ is μ^*-measurable. Then

$$\mu^*(\mathbf{X}) = \mu^*(\mathbf{X} \cap E) + \mu^*(\mathbf{X} \setminus E) = \mu_0(\mathbf{X})$$

Rearranging gives the result, since all quantities are finite.

If $\mu^*(E) = \mu_*(E)$, then $\mu^*(E^c) = \mu_*(E^c)$, since the definition of μ_* is symmetric. Let $B \in \mathcal{A}_{\sigma\delta}$, with $\mu^*(B) = \mu^*(E)$, $E \subseteq B$. We can always find such a B by taking the intersection over all $B_n \in \mathcal{A}_\sigma$,

$$\mu^*(E) + n^{-1} \geq \sum_j \mu^*(B(j,n)) \geq \mu^*\left(\bigcup_j B(j,n) = B_n\right)$$

Notice $E \subseteq B \iff E^c \supseteq B^c \iff E^c \cap B^c = B^c$.Since B is μ^*-measurable, we have

$$\begin{aligned}
\mu^*(E^c \cap B) + \mu^*(E^c \setminus B) &= \mu^*(E^c) \\
&= \mu^*(\mathbf{X}) - \mu^*(E) \\
\mu^*(B \setminus E) + \mu^*(B^c) &= \mu^*(X) - \mu^*(E) \\
&= \mu^*(B) + \mu^*(B^c) - \mu^*(E) \\
\mu^*(B \setminus E) &= \mu^*(B) - \mu^*(E) \\
&= 0
\end{aligned}$$

■

Exercise 1.20

Proposition 20.1

Proof. ■

Exercise 1.21

Proposition 21.1

Let μ^* be an outermeasure induced from a premeasure, and $\overline{\mu} = \mu^*|_{\mathcal{M}^*}$, where $\mathcal{M}^*$ denotes the family of μ^*-measurable sets. Show that $\overline{\mu}$ is saturated. That is, $\widetilde{\mathcal{M}^*} = \mathcal{M}^*$

Proof. Suppose E is locally measurable (with respect to $\overline{\mu} = \mu^*|_{\mathcal{M}^*}$). Fix $V \subseteq \mathbf{X}$, with $\mu^*(V) < +\infty$. It suffices to show $\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$.

By 18a), find a $V' \in \mathcal{A}_{\sigma\delta}$, with $V \subseteq V'$, and $\mu^*(V') = \mu^*(V) < +\infty$. so that $E \cap V'$ is μ^*-measurable.

$$\mu^*(V) = \mu^*(V \cap E \cap V') + \mu^*(V \setminus (V \cap (V' \cap E))$$

therefore

$$\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$$

■

Exercise 1.22

Proposition 22.1

Proof. To show $\overline{\mu}$ is complete, Fix $U \subseteq F$, where $F \in \mathcal{M}^*$, with $\overline{\mu}(F) = 0$ Let $F' \in \mathcal{A}_{\sigma\delta}$, with $F' \supseteq F$, and

$$\mu^*(F') = \mu^*(F) \geq \mu^*(F' \setminus U)$$

Since $F' \supseteq U$, applying Exercise 18b gives $\overline{\mathcal{M}^*} \subseteq \mathcal{M}^*$. For the other direction,

■

Exercise 1.23

Proposition 23.1

Proof. ∎

Exercise 1.24

Proposition 24.1

If μ is a finite measure on $(X, \mathcal{M})$, and let μ^* be the outer measure. Suppose that $E \subseteq X$ satisfies $\mu^*(E) = \mu^*(X)$ (but $E \notin \mathcal{M}$ necessarily). Show that

(a) For any $A, B \in \mathcal{M}$, and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

(b) Let $\mathcal{M}_E = \{A \cap E,\ A \in \mathcal{M}\}$, and define ν on $\mathcal{M}$ with $\nu(A \cap E) = \mu(A)$. Then $\mathcal{M}_E$ is a σ-algebra, and ν is a measure on $\mathcal{M}_E$.

Proof of Part A.

$$\mu^*(E) = \mu^*(X) \implies \mu^*(X \setminus E) = 0$$

This is a simple consequence of the μ^*-measurability of X, since $X \in \mathcal{M}$, and the μ is a pre-measure on $\mathcal{M}$, b And by monotonicity,

$$\begin{cases} A \cap (X \setminus E) \subseteq (X \setminus E) \\ B \cap (X \setminus E) \subseteq (X \setminus E) \end{cases} \implies \begin{cases} \mu^*(A \cap (X \setminus E)) = 0 \\ \mu^*(B \cap (X \setminus E)) = 0 \end{cases}$$

Write $A \cap X = (A \cap E) \cup (A \cap X \setminus E)$, and by subadditivity of μ^*,

$$\begin{aligned} \mu(A) &= \mu^*(A \cap X) \\ &\leq \mu^*(A \cap E) + \mu^*(X \setminus E) \\ &= \mu^*(B \cap E) \\ &\leq \mu^*(B \cap X) \\ &= \mu(B) \end{aligned}$$

Therefore $\mu(A) \leq \mu(B)$, and $\mu(B) \leq \mu(A)$ is trivial.

■

Proof of Part B. We want to show $\mathcal{M}_E$ is a σ-algebra.

- Closure under complements,

$$\forall A \cap E \in \mathcal{M}_E, A \in \mathcal{M} \implies (E \setminus A^c) \in \mathcal{M}$$

  Therefore $(E \setminus A^c) \cap E \in \mathcal{M}_E$. Note that the question mentions that $\mathcal{M}_E$ is a σ-algebra on E, therefore we take complements relative to E.

- Closure under countable unions. Fix any countable sequence $\{A_j \cap E\} \subseteq \mathcal{M}_E$ where $\{A_j\} \subseteq \mathcal{M}$. It is obvious that $A = \cup A_j \in \mathcal{M}$, therefore $\cup(A_j \cap E) = E \cap A \in \mathcal{M}_E$ as well.

Since $\nu(\varnothing) = \mu(\varnothing \cap E) = 0$, and for countable additivity, fix any disjoint sequence $\{A_j \cap E\}_{j \geq 1} \subseteq \mathcal{M}_E$, where $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$, and let $A = \bigcup A_{j \geq 1}$

$$\begin{aligned}\nu(A \cap E) &= \mu(A) \\ &= \sum \mu(A_{j \geq 1}) \\ &= \sum \nu(A_{j \geq 1} \cap E)\end{aligned}$$

■

Chapter 2

Theorem 2.1

Proposition 1.1

Proof. ■

Chapter 3

Notes on Chapter 3

Proposition 0.2

Prove two things,

1. $\limsup_{r\to R} \phi(r) = \lim_{\varepsilon\to 0} \sup_{0<|r-R|<\varepsilon} \phi(r) = \inf_{\varepsilon>0} \sup_{0<|r-R|<\varepsilon} \phi(r)$,
2. $\lim_{r\to R} \phi(r) = c \iff \limsup_{r\to R} |\phi(r) - c| = 0$

Proof. ■

Proposition 0.3

If $U \subseteq B(1,0) = \{|x| < 1\}$, and $U \in \mathbb{B}$, and if $m(U) > 0$, then the family of sets

$$E_r = \left\{x + ry,\ y \in U\right\}$$

shrinks nicely to $x \in \mathbb{R}^n$.

Proof. Let $r > 0$ be fixed then $\forall z \in E_r \looparrowright z = x + ry$. Hence,

$$\begin{aligned} d(x,z) &= d(x, x+ry) \\ &= |r| d(0,y) < |r| \end{aligned}$$

by translation invariance. ■

Definition 0.1: Signed measure

Let $\mathcal{M}$ be a σ-algebra and $\nu : \mathcal{M} \to [-\infty, +\infty]$ be a set function on $\mathcal{M}$. It is a *signed measure* on $\mathcal{M}$ if

- $\nu(\varnothing) = 0$,
- ν assumes at most one of the values $\pm\infty$,
- If $\{E_j\}_{j\geq 1}$ is a countable, disjoint sequence of sets, the expression

$$\sum_{j\geq 1} \nu(E_j) \quad \text{is unambiguous, and is equal to } \nu\Big(\bigcup E_j\Big)$$

  More precisely,

  - if $|\nu(\bigcup E_j)| < +\infty$, the series $\sum \nu(E_j)$ converges absolutely,
  - if $\nu(\bigcup E_j) = \pm\infty$, the series $\sum \nu(E_j)$ diverges to $\pm\infty$ on every permutation.

Definition 0.2: Positive, negative, null sets

Let ν be a signed measure on $\mathcal{M}$. A measurable set $E \in \mathcal{M}$ is called *positive* (resp. *negative*, *null*) if every measurable subset $F \subseteq E$ satisfies $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0$, $\nu(F)$=0).

Definition 0.3: Mutual singularity

Two signed measures, ν and μ on a common σ-algebra $\mathcal{M}$ are *mutually singular*, denoted by $\nu \perp \mu$ if there exists disjoint, measurable sets E, F whose union is $\mathbf{X}$.

$$\mu \text{ is null on } E, \quad \text{and } \nu \text{ is null on } F$$

Proposition 0.4

Let ν be a signed measure on $(\mathbf{X}, \mathcal{M})$. If $\{E_j\}$ is an increasing sequence in $\mathcal{M}$, $\lim_{n\to+\infty} \nu(E_j) = \nu(\bigcup E_j)$. If $\{E_j\}$ is a decreasing sequence in $\mathcal{M}$, $\lim_{n\to+\infty} \nu(E_j) = \nu(\bigcap E_j)$ provided $\nu(E_1)$ is of finite measure.

Proof. Let ν be a signed measure, and fix any increasing sequence $E_j \nearrow E = \bigcup E_{j\geq 1}$ of sets. This induces a disjoint sequence in $\{F_n\}$. Define $F_1 = E_1$, and if $n \geq 2$,

$$F_n = E_n \setminus \bigcup E_{j\leq n-1}$$

Use σ-additivity of ν, where the sum is 'defined' to be non-ambiguous.

For the second part of the proof, notice if $A \subseteq B$ are measurable sets, if $\nu(A) = \pm\infty$, then $\nu(B) = \pm\infty$, because of the second property of ν. Indeed,

$$\nu(B) = \nu(A) + \nu(B \setminus A) = \pm\infty + c$$

where $c \in \mathbb{R} \cup \{\pm\infty\}$. Therefore $\nu(B) = \nu(A)$. By assumption $\nu(E_1) \in \mathbb{R}$, the contrapositive of the previous argument shows that the intersection $\cap E_j$ is of finite measure as well. We can produce an increasing sequence $G_n = E_1 \setminus E_n$ for $n \in \mathbb{N}^+$. Then

$$\bigcup G_n = \bigcup E_1 \setminus E_n = E_1 \cap \left[\bigcup E_n^c\right] = \left[\bigcap E_j\right]^c$$

We then write

$$E_1 = \left[\bigcup G_n\right] + \left[\bigcap E_n\right]$$

The finiteness of $\nu(E_1)$ on the left hand side implies all the terms in the union converge absolutely. Therefore

$$\begin{aligned}
\nu(E_1) - \nu\Big(\bigcap E_n\Big) &= \lim_{n\to+\infty} \nu(G_n) \\
&= \lim_{n\to+\infty} \nu(E_1) - \nu(E_n) \\
&= \nu(E_1) - \lim_{n\to+\infty} \nu(E_n)
\end{aligned}$$

Cancelling terms finishes the proof. ■

Proposition 0.5

Any measurable subset of a positive set is again positive, and any countable union of positive sets is again positive. Similarly for negative, and null sets.

Proof. Trivial. ■

Proposition 0.6: Hahn Decomposition Theorem

Let ν be a signed measure on the measurable space $(\mathbf{X}, \mathcal{M})$, then there exists positive and negative sets $P, N \in \mathcal{M}$ where $P \cup N = \mathbf{X}$, and $P \cap N = \varnothing$. If P' and N' are another such decomposition,

$$P \Delta P' = N \Delta N' \quad \text{is } \nu\text{-null.}$$

Proof. There are multiple steps to this proof. Suppose ν does not attain $+\infty$. Define

$$m = \sup\left\{\nu(P),\ P \text{ is a positive set}\right\}$$

By assumption $m < +\infty$, let $\{P_j\}$ be a sequence of positive sets with $\nu(P_j) \nearrow m$. We claim the supremum is attained. Indeed, if $P \triangleq \cup P_j$, then P is a positive set as well, by monotonicity $\nu(P) \geq \nu(P_j)$, taking the supremum on both sides reads $\nu(P) = m$.

Wanting to prove $N \triangleq \mathbf{X} \setminus P$ is a ν-negative set,

- Clearly N cannot contain any positive sets $A \subseteq N$ with a non-null measure, since

  $$\nu(A) > 0 \implies \nu(A) + \nu(P) = \nu(A + P) > m$$

  contradicting the supremum.

- Let us examine the properties of subsets of N with *positive measure*. Call this set $A \subseteq N$, where $\nu(A) > 0$.

  The previous bullet point tells us A cannot be a ν-positive set. There exists a $B \subseteq A$ of strictly negative measure,

  $$\nu(A \setminus B) + \nu(B) = \nu(A) \implies \nu(A \setminus B) > \nu(A)$$

  Notice the assumption ν does not attain $+\infty$ allows us to subtract B over.
  Summarizing,

  $$\text{existence of subset of positive measure} \implies \text{subset with even greater positive measure}$$

  We will use the above inductively to construct a measurable subset of N, that is 'small' but has 'large' positive measure at the same time.

- Suppose N is not ν-negative, so it admits a set of positive measure in $A_1 \subseteq N$.

  Let $n_1 = \text{least}\left\{n \in \mathbb{N}^+,\ \exists B \subseteq A_1,\ \nu(B) > \nu(A) + n^{-1}\right\}$, since n_1 is attained, it corresponds to some $A_2 \subseteq A_1$ with $\nu(A_2) > \nu(A_1) + n_1^{-1}$.

Repeating this process inductively, we see

$$\nu(A_k) > \nu(A_{k-1}) + n_k^{-1}$$

Let $A = \bigcap A_k$, this should be a set of large positive measure. A simple induction will show

$$\nu(A_k) > \nu(A_1) + \sum_{j=1}^{k} n_j^{-1} > \sum_{j=1}^{k} n_j^{-1}$$

However, $\nu(A) < +\infty$ by assumption. Upon taking limits and using the estimate above,

$$\sum_{j\geq 1} n_j^{-1} = \lim_{n\to\infty} \nu(A_n) = \nu(A) < +\infty$$

The sum on the left is finite, so its terms must converge to 0. Notice $\nu(A)$ is a subset of N of positive measure, it admits a subset $B \subsetneq A$ with $\nu(B) > \nu(A) + n^{-1}$ for $n \geq 1$.

$n_j^{-1} \to 0$ implies $n_j \to \infty$. So $n < n_j$ for large j. Notice $B \subsetneq A \subsetneq A_j$, and $\nu(B) > \nu(A_j) + n^{-1}$. This contradicts our definition of n_j, stated below for convenience

$$n_j = \text{least}\left\{ n \in \mathbb{N}^+,\ \exists B \subsetneq A_j,\ \nu(B) > \nu(A_j) + n^{-1} \right\}$$

This proves N is ν-negative.

To show this composition is ν-unique, let P' and N' be disjoint, measurable positive and negative sets of $\mathbf{X}$. Then

$$P \setminus P' \subseteq P \quad \text{and} \quad P \setminus P' \setminus \mathbf{X} \setminus P' \subseteq N'$$

So $P \setminus P'$ is at the same time a ν-positive and a ν-negative set, hence it is ν-null by Lemma 3.2.

Finally, the case for when ν attains $+\infty$ can be handled if we consider $-\nu$. P is positive for $-\nu$ iff it is negative for ν, and similarly for N. Relabelling P and N finishes the proof. ■

Theorem 3.4

Proposition 1.1

Proof. ■

Theorem 3.5

Proposition 2.1

Proof. ■

Theorem 3.6

Proposition 3.1

Proof. ■

Theorem 3.7

Proposition 4.1

Proof. ■

Theorem 3.8

Proposition 5.1

Proof. ■

Theorem 3.9

Proposition 6.1

Proof. ■

Theorem 3.10

Proposition 7.1

Proof. ■

Theorem 3.11

Proposition 8.1

Proof. ■

Theorem 3.12

Proposition 9.1

Proof. ■

Theorem 3.13

Proposition 10.1

Proof. ■

Theorem 3.14

Proposition 11.1

Proof. ■

Theorem 3.15

Proposition 12.1

Proof. ■

Theorem 3.16

Proposition 13.1

Proof. ■

Theorem 3.17

Proposition 14.1

Let the maximal function of any measurable $f \in \mathbb{B}_{\mathbb{R}^n}$ be denoted by $Hf(x)$, more precisely,

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

where $A_r|f|$ is the average of $|f|$ on a ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$. In symbols,

$$A_r|f| = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

The maximal theorem makes two claims:

1. $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, and Hf is measurable for every $f \in L^1_{loc}$.
2. There exists a $C > 0$, for every $f \in L^1$
$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1$$
for every $\alpha > 0$.

Proof. Let $\alpha > 0$ and fix $z \in (Hf)^{-1}((\alpha, +\infty))$, so $Hf(z) > \alpha$ and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with $Hf(z) - \alpha > 0$, we get some $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some $r_0 > 0$. Since $A_r|f|$ (a function indexed by $r > 0$) is continuous in $x \in \mathbb{R}^n$, $(A_r|f|)^{-1}((\alpha, +\infty))$ is open, and Hf is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{Hf > \alpha\right\} = \bigcup_{r>0} \{A_r|f| > \alpha\}$$

Let $x \in E_\alpha$, this induces a $r_x > 0$ where $x \in \left\{A_{r_x}|f| > \alpha\right\}$. Rearranging gives

$$\left(\frac{1}{\alpha} \int_{B(r,x)} |f|dz\right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each $x \in E_\alpha \leadsto r_x > 0 \leadsto A_{r_x}|f|$,
- If $U = \bigcup\limits_{x \in E_\alpha} B(r_x, x)$, then $E_\alpha \subseteq U$,
- Choose $c < m(E_\alpha) \leq m(U)$ (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

  $$x_1, \ldots, x_N \in E_\alpha$$

  so that $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$, and $c < 3^n \sum_{j \leq k} m(B_j)$
- Define $B_j = B(r_{x_j}, x_j)$ for all $j \leq k$, and

  $$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

  by finite additivity,

  $$c 3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

  and finally

  $$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of m on $\mathbb{B}$, since

  $$m(E_\alpha) = \sup \left\{ m(K),\ K \in \text{♪}_{\mathbb{R}^n},\ K \subseteq E_\alpha \right\}$$

  for any $K \in \text{♪}_{\mathbb{R}^n}$, $K \subseteq E_\alpha$, we have $m(K) < +\infty$, $m(K) \leq m(E_\alpha)$ and

  $$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

Remark 14.1

We used the properties of a Radon Measure here, without relying on the phrase ‘sending $c \to E_\alpha$‘, which would require us to deal with two cases $m(E_\alpha) < +\infty$ and $m(E_\alpha) = +\infty$.

∎

Theorem 3.18

Proposition 15.1

Proof. ■

Theorem 3.19

Proposition 16.1

Proof. ■

Theorem 3.20

Proposition 17.1

Proof. ■

Theorem 3.21

Proposition 18.1

The Lebesgue Differentiation Theorem. Suppose $f \in L^1_{loc}$, and for every $x \in \mathcal{L}_f$, (so that $x \in \mathbb{R}^n$ a.e). We have

1. $\lim_{r\to 0} \dfrac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)|dy = 0$,
2. $\lim_{r\to 0} \dfrac{1}{m(E_r)} \int_{E_r} f(y)dy = f(x)$,

For every family $\{E_r\}_{r>0}$ that shrinks nicely to $x \in \mathbb{R}^{n\prime}$.

Proof. Since the family $\{E_r\}_{r>0}$ shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r,x)) \implies m(E_r) > \alpha \cdot m(B(r,x))$$

for some $\alpha > 0$, independent on r. Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r,x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)|dy \le \int_{B(r,x)} |f(y) - f(x)|dy$$

Combining the last two results, for every $\varepsilon > 0$, if $0 < r < \varepsilon$, then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)|dy \le m^{-1}B(r,x) \int_{B(r,x)} |f(y) - f(x)|dy$$

Taking the supremum on both sides,

$$\sup_{0<r<\varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)|dy \le \sup_{0<r<\varepsilon} m^{-1}B(r,x) \int_{B(r,x)} |f(y) - f(x)|dy$$

and sending $\varepsilon \to 0$, proves the first claim. The second claim is immediate upon applying the L^1 inequality.

Fix any $\varepsilon > 0$, and

$$\begin{aligned}
\lim_{r\to 0} m^{-1}(E_r) \int_{E_r} f(y)dy = f(x) &\iff \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y)dy - f(x) \right| \\
&\iff \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)]dy \right| \\
&\le \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)|dy \\
&= \lim_{r\to 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)|dy \\
&= 0
\end{aligned}$$

∎

Theorem 3.22

Proposition 19.1

Proof. ∎

Theorem 3.23

Proposition 20.1

Proof. ■

Theorem 3.24

Proposition 21.1

Proof. ■

Theorem 3.25

Proposition 22.1

Proof. ■

Theorem 3.26

Proposition 23.1

Proof. ■

Theorem 3.27

Proposition 24.1

Proof. ■

Theorem 3.28

Proposition 25.1

Proof. ■

Theorem 3.29

Proposition 26.1

Proof. ∎

Theorem 3.30

Proposition 27.1

Proof. ∎

Theorem 3.31

Proposition 28.1

Proof. ■

Theorem 3.32

Proposition 29.1

Proof. ■

Theorem 3.33

Proposition 30.1

Proof. ■

Theorem 3.34

Proposition 31.1

Proof. ∎

Theorem 3.35

Proposition 32.1

Proof. ■

Theorem 3.36

Proposition 33.1

Proof. ■

Chapter 4: Point-set Topology

Topological Spaces

This section will roughly follow Munkres text on Geeneral Topology, in particular we hope to cover Chapters 2, 3, 4 and 9. The rest of the Chapters should be covered proper by the subsequent section.

Definition 1.1: Topology

Let $\mathbf{X}$ be a non-empty set. A topology $\mathcal{T}$ on $\mathbf{X}$, sometimes denoted by $\mathcal{T}_{\mathbf{X}}$ is a family of subsets of $\mathbf{X}$,

- $\{\varnothing, \mathbf{X}\} \subseteq \mathcal{T}$,
- If U_1 and U_2 are elements of $\mathcal{T}$, so is their intersection.
- If $\{U_\alpha\}$ is an arbitrary family of sets in $\mathcal{T}$, their union is also contained in $\mathcal{T}$ as an element.

We call the elements of $\mathcal{T}$ open sets. The complements of elements in $\mathcal{T}$ are closed sets.

Basis of a Topology

Definition 2.1: Basis of a topology

A basis $\mathbb{B}$ is a family of subsets of $\mathbf{X}$, that satisfies:

- Every $x \in \mathbf{X}$ belongs (as an element) in some $V \in \mathbb{B}$.
- If B_1 and B_2 are basis elements, such that their intersection is non-empty. Then every $x \in B_1 \cap B_2$ induces a $B_3 \in \mathbb{B}$ with

$$x \in B_3 \subseteq B_1 \cap B_2$$

This roughly means a basis is 'finitely' fine at every point in x.

If $\mathbb{B}$ is a basis, it 'generates' a topology $\mathcal{T}$ through

$$\mathcal{T} = \left\{ U \subseteq \mathbf{X},\, \forall x \in U,\, x \in B \subseteq U \text{ for some } B \in \mathbb{B} \right\} \tag{5}$$

Notice this is equivalent to $\mathcal{T}$ is the collection of all unions of basis elements in $\mathbb{B}$.

Proposition 2.1

Let $\mathbb{B}$ be a basis as defined in Definition 2.1, then $\mathcal{T}$ as defined in Equation (5) is a valid topology on $\mathbf{X}$. And every member of $\mathcal{T}$ is and is precisely the union of elements in $\mathbb{B}$.

Proof. Every point in $\mathbf{X}$ belongs in some basis element, so $\mathbf{X} \in \mathcal{T}$, so does $\varnothing$. Next, if U_1 and U_2 are in $\mathcal{T}$, then

$$\begin{cases} x \in U_1 \looparrowright x \in B_1 \subseteq U_1 \\ x \in U_2 \looparrowright x \in B_2 \subseteq U_2 \end{cases} \implies x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

for some $B_3 \in \mathbb{B}$, so $\mathcal{T}$ is closed under finite intersections (perhaps after a standard induction argument).

If $\{U_\alpha\} \subseteq \mathcal{T}$, and x belongs in the union of all U_α, then $x \in B_\alpha \subseteq U_\alpha$, which is a subset of the entire union. So the union over U_α is again contained in $\mathcal{T}$, and $\mathcal{T}$ is a topology on $\mathbf{X}$.

It is worth noting that $\mathbb{B} \subseteq \mathcal{T}$. Finally, if $U \in \mathcal{T}$,

$$U = \bigcup_{x \in U} B_x$$

where B_x is the basis element taken to satisfy $x \in B_x \subseteq U$. Every point in U is included in some B_x, and hence is included in the union. For the reverse inclusion, notice the union of subsets of U is again a subset of U.

Now, if $E \subseteq \mathbf{X}$ is the union of basis elements in $\mathbb{B}$, if E is non-empty, then every point $x \in E$ belongs in some B_x. Recycling the previous argument, and we see that E is open in $\mathcal{T}$. If E is empty, we define the 'union' of no sets as the empty set. So $\mathcal{T}$ is precisely the collection of all unions of basis elements $\mathbb{B}$. ■

We are now in a position to compare the relative 'fineness' of topologies.

Definition 2.2: Fineness of topologies

If $\mathcal{T}'$ and $\mathcal{T}$ are both topologies on some non-empty set $\mathbf{X}$. We say $\mathcal{T}'$ is finer than $\mathcal{T}$, or $\mathcal{T}$ is coarser than $\mathcal{T}'$ if

$$\mathcal{T}' \supseteq \mathcal{T}$$

Proposition 2.2

If $\mathbb{B}$ and $\mathbb{B}'$ are bases for $\mathcal{T}'$ and $\mathcal{T}$, the following are equivalent:

- $\mathcal{T}'$ is finer than $\mathcal{T}$,
- If B is an arbitrary basis element in $\mathbb{B}$, then every point $x \in B$ induces a basis element in $\mathbb{B}'$ with
$$x \in \mathbb{B}' \subseteq \mathbb{B}$$

Proof. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Notice $\mathbb{B} \subseteq \mathcal{T}'$ as well. By Equation (5), each $x \in B$ induces a $B' \in \mathbb{B}'$

$$x \in B' \subseteq B$$

Conversely, fix any open set $U \in \mathcal{T}$, and for each $x \in U$,

$$x \in B' \subseteq B \subseteq U$$

Applying Definition 2.1 tells us U is open in $\mathcal{T}'$. ■

The last of the big three 'generating' definitions for topologies will be the sub-basis. It simply means the first condition (but not necessarily) the second, is satisfied in Definition 2.1

Definition 2.3: Sub-basis of a topology

A sub-basis $S \in \mathbb{P}(\mathbf{X})$ is a family of subsets of $\mathbf{X}$ that satisfies one property. Any point x in $\mathbf{X}$ belongs to at least one member of S.

A sub-basis can be upgraded to a basis by collecting all of its finite intersections.

Proposition 2.3

Let S be a sub-basis of $\mathbf{X}$, then the collection of all finite intersections of S forms a basis $\mathbb{B}$ of $\mathbf{X}$.

Proof. Every point in $\mathbf{X}$ lies in some element of S, hence in some element of $\mathbb{B}$. The second basis property is immediate, since $\mathbb{B}$ is closed under finite intersections. ■

Product Topology

We will start with products of a finite collection of topological spaces.

Definition 3.1: Finite Product of Topological Spaces

Let $(\mathbf{X}, \mathcal{T}_{\mathbf{X}})$ and $(\mathbf{Y}, \mathcal{T}_{\mathbf{Y}})$ be topological spaces. The product topology (denoted by $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$) on $X \times Y$ is defined as the topology generated by the basis

$$\mathbb{B}_{\mathbf{X}\times\mathbf{Y}} = \left\{ U \times V,\ (U, V) \in \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}} \right\} \tag{6}$$

Since bases are easier to describe than topologies, we have the following statement concerning the basis of the product topology.

Proposition 3.1

If $\mathbb{B}_{\mathbf{X}}$ and $\mathbb{B}_{\mathbf{Y}}$ are bases for $\mathcal{T}_{\mathbf{X}}$ and $\mathcal{T}_{\mathbf{Y}}$, then the product topology (as described in Definition 3.1) is also generated by

$$\mathcal{M} = \left\{ U \times V,\ (U, V) \in \mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \right\} \tag{7}$$

Proof. We will introduce (and use) the technique of 'double inclusion' by proving that the topologies generated are both finer than the other. Let us denote the topology generated by $\mathcal{M}$ in Equation (7) by $\mathcal{T}_{\mathcal{M}}$.

Since $\mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \subseteq \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}$, if $U \times V \in \mathcal{M}$ as in Equation (7), then we can pick the same 'open rectangle' again. We trivially have

$$x \in \underbrace{U \times V}_{\text{member of } \mathcal{T}_{\mathbf{X}}\times\mathcal{T}_{\mathbf{Y}}} \subseteq U \times V$$

and by Proposition 2.2, $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$ is finer than $\mathcal{T}_{\mathcal{M}}$.

Fix any set $U \times V \in \mathbb{B}_{\mathbf{X}\times\mathbf{Y}}$, and if $(p, q) \in U \times V$, each coordinate induces basis elements from $\mathbb{B}_{\mathbf{X}}$ and By, more precisely:

$$\begin{cases} p \in U \implies p \in \text{Basis element of } \mathbb{B}_{\mathbf{X}} \subseteq U \\ q \in V \implies q \in \text{Basis element of } \mathbb{B}_{\mathbf{Y}} \subseteq V \end{cases} \implies (p, q) \in \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{X}}} \times \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{Y}}} \subseteq U \times V$$

by Proposition 2.2, $\mathcal{T}_{\mathcal{M}}$ is finer than $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$ and $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}} = \mathcal{T}_{\mathcal{M}}$. ■

Quotient Topology

Product Topology

The Cartesian Product of an arbitrary family of topological spaces, if equipped with the product topology, preserves a lot of the structure. If $\{X_\alpha\}_{\alpha \in A}$ is a family of topological spaces which are _____, then $\prod X_\alpha$ is _____. Replace _____ with:

1. Hausdorff, (Folland)
2. Regular,
3. Connected, (Munkres chp23, exercise 10)
4. First countable, if A is countable,
5. Second countable, if A is countable,
6. Compact (Tynchonoff's Theorem, Folland)

Connectedness

Definition 6.1: Connectedness

A topological space $\mathbf{X}$ is connected if U and V are disjoint open subsets whose union is $\mathbf{X}$, then at least one of U or V is empty.

See Folland Exercise 4.10 for more properties.

Definition 6.2: Path-connectedness

A topological space $\mathbf{X}$ is path-connected if for any two pair of points $x, y \in \mathbf{X}$. There exists a continuous function $f : [a, b] \to \mathbf{X}$, with $f(a) = x$ and $f(b) = y$.

Definition 6.3: Connected component

The connected components of $\mathbf{X}$ is the family of equivalence classes on $\mathbf{X}$, where $x \sim y$ if there is a connected subspace of $\mathbf{X}$ that contains both of them.

Proposition 6.1

Continuous functions map connected spaces to connected spaces (in the subspace topology).

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces and $f : \mathbf{X} \to \mathbf{Y}$ be continuous. If $f(\mathbf{X})$ is disconnected, then we can find U and V, open and disjoint in $\mathcal{T}_{f(\mathbf{X})}$ such that

$$U \cup V = f(\mathbf{X}) \implies f^{-1}(U) \cup f^{-1}(V) = \mathbf{X}$$

where $f^{-1}(f(\mathbf{X})) = \mathbf{X}$. Both $f^{-1}(U)$ and $f^{-1}(V)$ are open, non-empty, and are pairwise disjoint. So $\mathbf{X}$ is separated. ■

Proposition 6.2

Let $(\mathbf{X}_\alpha, \mathcal{T}_\alpha)$ be a family of connected topological spaces indexed by $\alpha \in A$. Then $\prod_{\alpha \in A} \mathbf{X}_\alpha$ is disconnected in the product topology.

Proof. We will attempt the contrapositive. Suppose $\prod_{\alpha \in A} \mathbf{X}_\alpha$ is disconnected, then ■

Topology in Analysis

Interiors and closures

Definition 7.1: Interior of a set

A^o is defined to be the largest open subset of A,

$$A^o = \bigcup_{\substack{U \text{ open,} \\ U \subseteq A}} U$$

Corollary 7.1

The union of subsets of A is again a subset of A, therefore Corollary 7.1 implies $A^o \subseteq A$ for any $A \subseteq \mathbf{X}$.

Definition 7.2: Closure of a set

and $\overline{A}$ is the smallest closed superset of A,

$$\overline{A} = \bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K$$

Proposition 7.1

The complement of the closure is the interior of the complement, or equivalently: $(\overline{A})^c = A^{co}$

Proof. Taking complements, and the substitution $U = K^c$ reads

$$(\overline{A})^c = \left[\bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K \right]^c$$

$$= \bigcup_{\substack{K \text{ closed,} \\ K^c \subseteq A^c}} K^c$$

$$= \bigcap_{\substack{U \text{ open,} \\ U \subseteq A^c}} U$$

$$= A^{co}$$

■

Remark 7.1

Personally, I remember this as pushing the complement inside and flipping the bar to a c!

Neighbourhoods

The concept of a neighbourhood allows us to characterize the interior of a set 'locally'.

Definition 8.1: Neighbourhood (not necessarily open)

A neighbourhood of $x \in \mathbf{X}$ is a set $U \subseteq \mathbf{X}$ where $x \in U^o$. The set of neighbourhoods for a point $x \in \mathbf{X}$ will sometimes be denoted by $\mathcal{N}(()x)$.

Proposition 8.1: Characterization of the interior

If $W = \left\{ x \in \mathbf{X}, \text{ there exists a neighbourhood } U \text{ of } x, U \subseteq A \right\}$, then $W = A^o$.

Proof. If $x \in A^o$, then A is a neighbourhood of x, and $A \subseteq A$, so $x \in W$. Conversely, if x is a member of W, it has a neighbourhood $U \subseteq A$ (not necessarily open). By monotonicity of the interior,

$$x \in U^o \subseteq A^o$$

and $x \in A^o$. ■

It is easy to see that A is open $\iff A^o = A \iff A$ is a neighbourhood of itself.

- The first equivalence follows from:

  $$E \subseteq \mathbf{X} \implies E^o \subseteq E$$

  and if A is an open set, it is an open subset of itself, by Corollary 7.1 $A \subseteq A^o$. If $A^o = A$, then it suffices to show that A^o is open. Which it is, since it is the arbitrary union of open sets.

- To prove the second equivalence: suppose $A^o = A$, then each $x \in A$ has a neighbourhood contained (as a subset) in A, namely A itself. (This statement is hard to parse, the reader is encouraged to really work through this and be honest).

  $$x \in A^o \subseteq A \implies A \subseteq A^o$$

  so A is a neighbourhood of itself. Conversely, if $A \subseteq A^o$, then $A = A^o$, since the reverse inclusion follows immediately from Corollary 7.1.

Adherent points

Similar to the neighbourhood, the concept of an adherent point of a set allows us to speak of the closure in more concrete terms. The following definition is key in understanding the relationship between the closure, interior, and the boundary.

Definition 9.1: Adherent point of a set

Let $A \subseteq \mathbf{X}$, $x \in \mathbf{X}$ is an adherent point of A if every neighbourhood U of x intersects A. In symbols,

$$U \cap A \neq \varnothing, \quad \forall U \in \mathcal{N}(x)$$

Proposition 9.1: Characterization of the closure

Let $A \subseteq X$, and let W be the set of adherent points of A, then $\overline{A} = W$

Proof. Suppose $x \notin W$, then there exists a neighbourhood U of x where

$$U \cap A = \varnothing \iff U \subseteq A^c$$

this is exactly the definition of the interior of A^c, so $x \in A^{co}$ and recall (from Proposition 7.1) that $(\overline{A})^c = A^{co}$, so $x \notin \overline{A}$. For the reverse inclusion, read the proof backwards, by flipping $\forall \to \exists$ within the set, and we see that

$$W^c = A^{co} = (\overline{A})^c$$

■

Dense and nowhere dense subsets

Definition 10.1: Dense subset

A subset of a topological space $E \subseteq \mathbf{X}$ is dense if $\overline{E} = \mathbf{X}$.

Definition 10.2: Nowhere dense subset

A subset of a topological space $E \subseteq \mathbf{X}$ is nowhere dense if $\overline{E}^o = \varnothing$.
This means E is dense in none of the (non-trivial) open subspaces of $\mathbf{X}$.

Proposition 10.1

E is dense in $\mathbf{X}$ iff for every non-empty, open set $U \subseteq \mathbf{X}$, $U \cap E \neq \varnothing$.

Proof of Proposition 10.1. Suppose E is dense, then $\overline{E} = \mathbf{X}$. Every point of $\mathbf{X}$ is an adherent point of E. Let $U \subseteq \mathbf{X}$ be a non-empty open set. If $x \in U$ then U is a neighbourhood of x, thus U intersects E. Conversely, suppose every non-empty open set U intersects E. Fix any point $x \in \mathbf{X}$, and any neighbourhood U of x. U has a non-empty interior (because it must contain x). But U^o is a non-empty open set, therefore $\varnothing \neq U^o \cap E \subseteq U \cap E$ ∎

Proposition 10.2

Let $f : \mathbf{X} \to \mathbf{X}$ be a homeomorphism. E is nowhere dense iff $f(E)$ is nowhere dense.

Proof. Since f^{-1} is a homeomorphism, suppose $\overline{f^{-1}(E)}^o \neq \varnothing$, there exists a non-empty, open subset $U \subseteq \mathbf{X}$ with

$$\overline{f^{-1}(E)} \cap U = U$$

The direct image yields

$$f\Big((\overline{f^{-1}(E)}) \cap U\Big) = f(U)$$

since f is a bijection (injectivity is necessary here), it commutes with intersections.

$$f(\overline{f^{-1}(E)}) \cap f(U) = f\Big((\overline{f^{-1}(E)}) \cap U\Big) = f(U) \tag{8}$$

and f is continuous, so $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq \mathbf{X}$. For the reverse inclusion, f is a closed map, so $f(\overline{A})$ is a closed superset of $f(A)$ so

$$f(\overline{A}) = \overline{f(A)}$$

Take $A = f^{-1}(E)$, and $f(\overline{f^{-1}(E)}) = \overline{f(f^{-1}(E))} = \overline{E}$. From eq. (8), we see that

$$\overline{E} \cap f(U) = f(U)$$

$f(U)$ is a non-empty open subset of $\mathbf{X}$, since f is an open map, so E is not no-where dense. The reverse implication can be proven by replacing f with f^{-1}. ∎

Urysohn's Lemma Notes

Notes on the construction of the countable 'onion' sequence within a normal space $\mathbf{X}$.

If $\mathbf{X}$ is a normal space, and A and B are disjoint closed subsets, then we can easily find an open U with

$$A \subseteq U \subseteq \overline{U} \subseteq B^c \tag{9}$$

We say that U hides in B^c if the closure of U is contained in B^c. Define $\Delta_n = \left\{k2^{-n},\ 1 < k < 2^n\right\}$, so that $\Delta_n \subseteq (0,1)$ for all $n \geq 1$. Notice

$$\Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

and the even indices for Δ_{n+1} are contained in Δ_n. Suppose Δ_n is well defined, it suffices to choose the odd indices for Δ_{n+1}. If $r = j2^{-(n+1)}$, where j is odd, then r sits in between precisely two elements in $\Delta_n \cup \{0,1\}$. If r sits between an endpoint, then define $\overline{U_0} = A$, and $B^c = U_1$. And denote the closest left and neighbours by s, t respectively. If $s < r < t$, it is clear that $\overline{U_s}$ and U_t^c are disjoint closed sets.

Use the 'normal space' construction to obtain an superset of $\overline{U_s}$ that hides in U_t, denote this open set by U_r, and similar to Equation (9)

$$\overline{U_s} \subseteq U_r \subseteq \overline{U_r} \subseteq U_t$$

Now that the construction of this sequence is complete, we wish to prove Urysohn's Lemma. Let A and B be disjoint closed sets. And define

$$f(x) = \inf\left\{r \in \Delta \cup \{1\},\ x \in U_r\right\}$$

where $U_1 = \mathbf{X}$. So that $0 \leq f(x) \leq 1$ is immediate. If $x \in A$, then x is in all U_r, and by density of $\Delta \subseteq (0,1)$, we have $f(x) = 0$. Conversely, if $x \in B$ then $x \notin U_r$ for all $r \in \Delta$, if E denotes the indices in Δ where $x \in U_s$ when $s \in E$,

$$(-\infty,\ r) \subseteq E^c \iff E \subseteq [r,\ +\infty) \iff \inf(E) \geq r \tag{10}$$

Send $r \to 1$ and $f(x) = 1$. Thus $f|_A = 0$ and $f|_B = 1$.

To show continuity, it suffices to show that the inverse images of the open half $\left\{(x > \alpha),\ (x < \alpha)\right\}_{\alpha \in \mathbb{R}}$ lines are indeed open in $\mathbf{X}$. Let α be fixed. And if $x \in \{f < \alpha\}$, we can 'wiggle' the infimum towards the right (towards α), and using density of Δ within $(0,1)$, there exists a $r \in E$ that satisfies $f(x) < r < \alpha$. This is equivalent to

$$x \in \bigcup_{r<\alpha} U_r$$

If there exists an $r < \alpha$ st x belongs to U_r as an element, then $f(x) \leq r < \alpha$.

If $f(x) > \alpha$, then $(-\infty,\, \alpha) \subseteq E^c$, by Equation (10). Suppose $\alpha < 1$, otherwise $\{f > \alpha\} = \varnothing$. Wiggle $f(x)$ to the left and obtain an $r \in \Delta$, $\alpha < r < f(x)$ with $x \notin U_r$. By density again, take any $s < r$ by a small amount (st $s > \alpha$, $s \in \Delta$), and

$$\overline{U}_s \subseteq U_r \iff U_r^c \subseteq \overline{U}_s$$

so that $x \in \overline{U}_s^c$ for some $s > \alpha$. This is equivalent to

$$x \in \bigcup_{s > \alpha} \overline{U}_s^c$$

Conversely, if $x \notin \overline{U}_s^c$ for some $s > \alpha$, since $\{U_r\}$ (thus $\{\overline{U}_r\}$) is increasing, and $x \notin U_r$ for every $r \leq s$. Hence,

$$(-\infty,\, s] \subseteq E^c \iff E \subseteq (s,\, +\infty) \iff f(x) \geq s > \alpha$$

Compactness

Compactness is one of the most important concepts in topology and analysis.

Definition 12.1: Compact topological space

A topological space $\mathbf{X}$ is compact if every open covering $\{U_\alpha\}$ contains a finite subcover. That is, if $\{U_\alpha\}$ is an arbitrary collection of open sets, then

$$\mathbf{X} = \bigcup U_{\alpha \in A} \implies \bigcup_{j \leq n} U_{\alpha_j}$$

Definition 12.2: Compact set

$E \subseteq \mathbf{X}$ is compact if it is compact in the subspace topology.

Definition 12.3: Precompact set

$E \subseteq \mathbf{X}$ is precompact if its closure is compact (as a subset).

Definition 12.4: Paracompact space

A topological space $\mathbf{X}$ is paracompact if every open covering of $\mathbf{X}$ has a locally finite open refinement that covers $\mathbf{X}$.

Definition 12.5: Locally finite collection of sets

Let $\mathcal{A}$ be a collection of subsets of $\mathbf{X}$. It is called locally finite, if at every point $p \in \mathbf{X}$, we can find a neighbourhood U of p (not necessarily open), that intersects only finitely many members of $\mathcal{A}$. In symbols,

$$U \cap E = \varnothing \quad \text{for all but finitely many} \quad E \in \mathcal{A}$$

We do not require $\mathcal{A}$ to be a cover of $\mathbf{X}$, nor do we require $\mathcal{A}$ to be a collection of open sets.

Definition 12.6: Countably locally finite

A collection $\mathbb{B}$ is countably locally finite if it is the countable union of locally finite families.

$$\mathbb{B} = \overbrace{\bigcup_{\mathbb{N}}}^{\substack{\text{countable}\\ \text{union}}} \mathbb{B}_n, \quad \text{where each } \mathbb{B}_n \text{ is a locally finite collection}$$

Definition 12.7: Refinement

If $\mathcal{A}$ is a collection of sets, $\mathbb{B}$ is a refinement of $\mathcal{A}$ if every element $B \in \mathbb{B}$, induces an element $A \in \mathcal{A}$, such that $B \subseteq A$.

Remark 12.1: Intuition for refinements

If $\mathbb{B}$ is a refinement of $\mathcal{A}$, we can use the 'absolute continuity' muscle. For each element in $\mathbb{B}$ is dominated by some element (through subset inclusion) in $\mathcal{A}$. Recall, if ν and μ are non-negative measures, then $\nu \ll \mu$ if for every measurable set $E \in \mathcal{M}$, $\mu(E) = 0 \implies \nu(E) = 0$.

A refinement of a family of sets is another family of sets, whose elements are dominated by some other element in the un-refined family. *Refining families makes them 'smaller', cover less area.*

Proposition 12.1

Compact Hausdorff spaces are normal, compact subsets of Hausdorff spaces are closed, and closed subsets of compact sets are again compact.

Locally Compact Hausdorff Spaces

Compactness is an intrinsic topological property (in the subspace topology). We see from Proposition 4.25 that compact Hausdorff spaces are normal, which gives a sufficient condition for us to approximate and extend any continuous function; and allows us to extend certain 'local' properties to 'global' properties.

If given a Hausdorff space, not necessarily compact, the natural question is to ask 1) whether a topological space has 'enough' compact subsets to work with, and 2) whether we can embed a given topological space in a larger one to force it to be compact.

Definition 13.1: LCH space

Let $\mathbf{X}$ be a Hausdorff space. We call $\mathbf{X}$ a LCH space if every point $p \in \mathbf{X}$ admits a compact neighbourhood. That is, a compact set K whose interior contains p.

We note in passing that the above definition differs slightly from the usual 'local' definitions.

Definition 13.2: Locally connected

Let $\mathbf{X}$ be a topological space, it is locally connected if for every $x \in \mathbf{X}$, and open neighbourhood U containing x, there exists a connected, open neighbourhood V of x such that $x \in V \subseteq U$.

Definition 13.3: Locally path-connected

Let $\mathbf{X}$ be a topological space, it is locally connected if for every $x \in \mathbf{X}$, and open neighbourhood U containing x, there exists a path-connected, open neighbourhood V of x such that $x \in V \subseteq U$.

Definition 13.4: Local homeomorphism

$\mathbf{X}$ locally homeomorphic to $\mathbb{R}^n$ if every point $x \in \mathbf{X}$ belongs to a coordinate chart (U, ϕ), where U is an open neighbourhood of x and ϕ is a homeomorphism from $U \to \phi(U) \subseteq \mathbb{R}^n$.

Definition 13.5: Local diffeomorphism

Let M be a smooth manifold and $F \in C^\infty(M, N)$. F is a local diffeomorphism if every $p \in M$ in its domain induces a neighbourhood $U \subseteq M$ with $F|U : U \to F(U)$ is a diffeomorphism (in the sense of two open sub-manifolds).

Theorem 4.1

Proposition 1.1

Suppose that A is a subset of X, let $\operatorname{acc} A$ be the set of accumulation points of A, then

$$\overline{A} = A \cup \operatorname{acc}(A) \tag{11}$$

and A is closed if and only if $\operatorname{acc}(A) \subseteq A$.

Proof. Suppose that $x \notin \overline{A}$, then $x \in (\overline{A})^c = A^{co}$, then $A^c \in \mathcal{N}_B(x)$. But this means that $x \notin \operatorname{acc}(A)$, since there exists a neighbourhood of x (in the form of A^c), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \varnothing$$

Also, $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$ which means that

$$x \notin \overline{A} \implies x \notin A$$

Since $x \notin \overline{A} \implies x \notin A$ and $x \notin \operatorname{acc}(A)$,

$$(\overline{A})^c \subseteq A^c \cap \operatorname{acc}(A)^c = (A \cup \operatorname{acc}(A))^c$$

Now, if $x \notin \operatorname{acc}(A) \cup A$, then $x \notin \operatorname{acc}(A)$, therefore there exists some $U \in \mathcal{N}_B(x)$ such that

$$A \cap U \setminus \{x\} = A \cap U = \varnothing$$

Where for the second last equality we used the fact that $x \notin A \implies A \setminus \{x\} = A$, and taking complements gives us

$$U \subseteq A^c$$

And since $U \in \mathcal{N}_B(x)$, then $x \in U^o \subseteq A^{co}$ (since U^o is an open subset of A^c). then

$$x \in A^{co} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore $(A \cup \operatorname{acc}(A))^c \subseteq (\overline{A})^c$. ∎

Theorem 4.2

Proposition 2.1

If $\mathcal{T}_X$ is a topology on X and $\mathcal{E} \subseteq \mathcal{T}_X$ then $\mathcal{E}$ is a base for $\mathcal{T}_X$ if and only if for every

$$\forall U \in \mathcal{T}_X,\, U \neq \varnothing, \implies U = \bigcup_{V \in B} V$$

Where B is a subset of $\mathcal{E}$.

Proof. Suppose that $\mathcal{E}$ is a base, then fix any non-empty $U \in \mathcal{T}_X$, then for every $x \in U$, there exists a neighbourhood base for this x and a member $V \in \mathcal{E}$ such that $x \in V_x \subseteq U$. Take the union over all V_x and

$$U \subseteq \bigcup_{x \in U} V_x$$

But each $V_x \subseteq U$, so $U = \bigcup_{x \in U} V_x$, where $\{V_x\} \subseteq \mathcal{E}$.

Conversely, if every non-empty U is a union of members in $\mathcal{E}$ then fix any $x \in X$, we claim that we have a neighbourhood base in

$$\{V \in \mathcal{E},\, x \in V\}$$

The reason is as follows

- x belongs to every $E \in \{V \in \mathcal{E},\, x \in V\}$ and
- For every open U, if $x \in U$ then there exists a union of members of $\mathcal{E}$ such that $U = \bigcup E_\alpha$, then $x \in U \iff \exists E_\alpha \in \{V \in \mathcal{E},\, x \in V\}$ and
- Using this particular $E_\alpha \in \mathcal{E}$ that we just found, $x \in E_\alpha \subseteq U$, and we are done.

■

Theorem 4.3

Proposition 3.1

For every $\mathcal{E} \subseteq \mathbb{P}(X)$, $\mathcal{E}$ is base for a topology on X if and only if

(a) each $x \in X$ is contained in some $V \in \mathcal{E}$, and

(b) if $U, V \in \mathcal{E}$, and $x \in U \cap V$, then there must exist some $W \in \mathcal{E}$ with $x \in W \subseteq U \cap V$.

Proof. Suppose that $\mathcal{E}$ is a base, then we get a), and b) follows since for every $U, V \in \mathcal{E} \subseteq \mathcal{T}_X$, and by closure over finite intersections, $U \cap V \in \mathcal{T}_X$ implies that there exists some $W \in \mathcal{E}$ with

$$x \in W \subseteq U \cap V$$

Now, suppose both a) and b) hold, then we claim that this $\mathcal{E} \subseteq \mathbb{P}(X)$ induces a topology on X

$$\mathcal{T} = \{U \subseteq X,\ \forall x \in U,\ \exists V \in \mathcal{E},\ \text{with } x \in V \subseteq U\}$$

Intuitively speaking, this means that $\mathcal{T}$ is just fine (and not too fine) to satisfy the conditions for $E \subseteq \mathcal{T}$ to be a base of $\mathcal{T}$.

We first show that $\mathcal{T}$ is a topology.

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$, the first is trivial and the second is from a)
- Closure under unions: fix $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, and $U = \bigcup U_\alpha$, and for every $x \in U$ there exists some $V_\alpha \in \mathcal{E}$ such that $x \in V_\alpha \subseteq U_\alpha \subseteq U$, therefore $U \in \mathcal{T}$.
- Closure under finite intersections, fix any U_1, U_2 as elements in $\mathcal{T}$, then suppose that they are not disjoint (if they are disjoint then their intersection is the empty set, which is also contained in $\mathcal{T}$). If $U_1 \cap U_2 \neq \varnothing$, then for every $x \in U_1 \cap U_2$ induces two sets $V_1, V_2 \in \mathcal{E}$ with $x \in V_1 \subseteq U_2$ and $x \in V_2 \subseteq U_2$, taking their intersection and applying b) gives us some $V \subseteq V_1 \cap V_2$ with $V \in \mathcal{E}$ therefore $x \in V \subseteq U_1 \cap U_2$, and $\mathcal{T}$ is closed under finite intersections.

Now to show that $\mathcal{E}$ is a base for $\mathcal{T}$, $\mathcal{E} \subseteq \mathcal{T}$ is obvious since very $V \in \mathcal{E}$ satisfies the properties laid out by $\mathcal{T}$ by simply choosing V again for any $x \in V$. Now fix any member $U \in \mathcal{T}$, then for every $x \in U$, there exists some $V \in \mathcal{E}$ with

$$x \in V \subseteq U$$

(This is an immediate consequence of how we defined $\mathcal{T}$). And we can conclude that $\mathcal{E}$ is a base for this induced topology $\mathcal{T}$. ■

Theorem 4.4

Proposition 4.1

If $\mathcal{E} \subseteq \mathbb{P}(X)$, the topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ consists of $\varnothing, X$ and all unions of finite intersections of $\mathcal{E}$, in symbols

$$\mathcal{T}(\mathcal{E}) = \{\varnothing, X\} \cup \left\{\bigcup W_\alpha,\ W_\alpha = \bigcap E_{j\leq n},\ E_j \in \mathcal{E}\right\}$$

Proof. Denote the set

$$W = \{X\} \cup \left\{\bigcap V_{j\leq n},\ \ V_j \in \mathcal{E}\right\}$$

We claim this set W satisfies Theorem 4.3. Since 4.3a) is satisfied with $X \in W$. 4.3b) follows since the right member in W is closed under intersections.

And if we are taking an element from each member, $E_1 \in \{\varnothing, X\}$ and E_2 is an element in the right member, then it is trivial to verify that their intersection is always contained within W. Therefore W induces a topology by Theorem 4.2, and we call this topology $\mathcal{T}$ — and for the sake of completeness

$$\mathcal{T} = \{U \subseteq X,\ \forall x \in U,\ \exists V \in \mathcal{E},\ x \in V \subseteq U\}$$

We so claim that if we define $\overline{W}$ as the union of all members $w \in W$, together with the empty set, is equal to the set $\mathcal{T}$.

$$\overline{W} = \left\{\bigcup_{w\in W} w\right\} \cup \{\varnothing\}$$

- We want to show $\mathcal{T} \subseteq \overline{W}$, since W is a base for the topology $\mathcal{T}$, every (non-empty) $U \in \mathcal{T}$ is the union of members in W (Theorem 4.2), and there exists some $B \subseteq W$ with

  $$U = \bigcup E_{\alpha\in B} \in \overline{W}$$

  Now if U is the empty set then it is trivially contained within $\overline{W}$.

- Next, we show that $\overline{W} \subseteq \mathcal{T}$, fix any element $E \in \overline{W}$, if $E = \varnothing$ then there is nothing to prove since $\mathcal{T}$ is a topology. Now for every $x \in E$,

  $$x \in E = \bigcup_{w\in W} w \implies x \in w$$

  Therefore $E \in \mathcal{T}$ by definition. This proves that $\mathcal{T} = \overline{W}$.

Now that $\overline{W}$ is a topology, that contains $\mathcal{E}$ as a subset, and by definition of $\mathcal{T}(\mathcal{E})$

$$\mathcal{T}(\mathcal{E}) = \bigcap\{A,\ \text{is a topology, and } \mathcal{E} \subseteq A\}$$

Tells us

$$\mathcal{T}(\mathcal{E}) \subseteq \overline{W}, \qquad \text{since } \overline{W} \in \{A,\ \text{is a topology, and } \mathcal{E} \subseteq A\}$$

Conversely, fix any member $E \in \overline{W}$, if $E = \varnothing$ then $E \in \mathcal{T}(\mathcal{E})$, if not, then there exists some subset $B \subseteq W$ such that

$$E = \bigcup_{w \in B} w = \bigcup_{w \in B} \bigcap_{j \leq n} V^w_{j \leq n} V_j \in \mathcal{E} \cup \{X\}$$

Since $\mathcal{T}(\mathcal{E})$ is closed under finite intersections and unions, and it contains $\mathcal{E}$ as a subset, $\overline{W} = \mathcal{T}(\mathcal{E})$ and we are done. ■

Theorem 4.5

Proposition 5.1

Every second countable space is separable. (Countable dense subset).

Proof. What we wish to prove is that if a space X has a countable base, then it has a countable dense subset. Denote this base of X by $\mathcal{E}$ as usual, then we claim that

$$W = \{x_u, \, U \in \mathcal{E}\}$$

Is a dense subset in X. Note that $(\overline{W})^c = W^{co} \in \mathcal{T}_X$. If $W^{co} = \varnothing$ then we simply take complements and we get $\overline{W} = X$. So suppose that W^{co} is non-empty, then for each $x \in W^{co}$ (by definition of a base), it should induce some $V_x \in \mathcal{E}$ with

$$x \in V_x \subseteq W^{co}$$

But clearly, for every element in $\mathcal{E}$, the second estimate can never be satisfied, since for every $U \in \mathcal{E}$, $x_U \notin W^{co}$ for this particular set W^{co}. Therefore W^{co} must be empty, and this completes the proof. ■

Theorem 4.6

Proposition 6.1

If X is first countable, then for every $A \subseteq X$, $x \in \overline{A} \iff$ there exists some sequence $\{x_j\}_{j \geq 1} \subseteq A$ such that $x_j \to x$.

Proof. Suppose that X is first countable, and $A \subseteq X$, and fix any element $x \in \overline{A}$. Since X is first countable, there is a sequence of descending neighbourhoods of $\{U_j\}_{j \geq 1}$ of x such that

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_j \supseteq U_{j+1}$$

If $x \in A$, take $x_n = x$ for all $n \geq 1$. If $x \in \operatorname{acc}(A)$, then take $x_n \in U_n \cap A \setminus \{x\} = U_n \cap A$, which is not empty. Then it remains to show that this sequence converges to x. Fix any neighbourhood $U \in \mathcal{N}_B(x)$ then there exists some N, for every $n \geq N$

$$x \in U^o \implies \exists N \in \mathbb{N}^+, \, x \in U_N \subseteq U^o$$

Then every $x_n \in A \cap U_N \subseteq A \cap U^o \subseteq U^o$. And this establishes $\implies$.

Now suppose that $x \notin \overline{A}$, so that $x \notin A$ and $x \notin \operatorname{acc}(A)$, then fix any sequence $\{x_j\} \subseteq A$. We wish to show that $x_j \not\to x$.

Since $x \notin \operatorname{acc}(A)$, there exists some $V \in \mathcal{N}_B(X)$ with

$$A \cap V \setminus \{x\} = \varnothing \implies V \subseteq A^c$$

Since $\{x_j\}_{j \geq 1} \subseteq A \implies x_j \notin A^c$ for every $j \geq 1$, then choose V as the neighbourhood around x, and $x_j \not\to x$ for any arbitrary sequence x_j in A. ∎

Remark 6.1

To truly understand what is going on one should recall that all metric space spaces are first countable.

Theorem 4.7

Proposition 7.1

X is a T_1 space $\iff$ $\{x\}$ is closed for every $x \in X$.

Proof. If X is T_1 and $x \in X$, then for every $y \neq x$ there exists some open U_y that contains y but not x. Following Folland's argument closely, every $y \neq x$ is is in $\cup U_{y\neq x}$. Hence $\{x\}^c \subseteq \cup U_{y\neq x}$. To show the converse, for every $z \in \cup U_{y\neq x}$ that is open, there exists a $y \neq x$ such that $z \in U_y$. But every U_y does not contain x as an element, so $z \neq x$ implies that $z \notin \{x\}$. And $z \in \{x\}^c$. Hence $\cup U_{y\neq x} = \{x\}^c$.

Now conversely if every $x \in X$ satisfies the fact that $\{x\}^c$ is open, then $\{x\}^c$ is an open set that contains every $y \neq x$. Now fix some $y \neq x$, since $\{y\}$ is also closed, we have $X \cap \{x\}^c$ is an open set that contains x but not y. Also, $\{x\}^c$ is an open set that contains y but not x. And therefore X is T_1. ■

Theorem 4.8

Proposition 8.1

The map $f : X \to Y$ is continuous if and only if at f is continuous at every $x \in X$.

Proof. Suppose that f is continuous, then fix any $f(x) \in Y$ and any of its neighbourhood $V \in \mathcal{N}_B(f(x))$,

$$f(x) \in V^o \implies f^{-1}(V^o) \in \mathcal{N}_B(x)$$

But by continuity, $f^{-1}(V^o)$ is an open set that contains x, with

$$f\left(f^{-1}(V^o)\right) \subseteq V^o$$

Therefore f is continuous at x. Now suppose that f is continuous at every $x \in X$, then for every open subset $V \subseteq Y$, and for every point $f(x) \in V = V^o$ means that $V \in \mathcal{N}_B(f(x))$ for all such points $f(x)$. By continuity, for every x in $f^{-1}(V)$, implies that $f^{-1}(V)$ is a neighbourhood of all of its elements, therefore $f^{-1}(V) \subseteq (f^{-1}(V))^o$, and $f^{-1}(V)$ is open. ■

Theorem 4.9

Proposition 9.1

If $\mathcal{E}_Y$ generates the topology on Y, and f is a mapping from $X \to Y$, then $f : X \to Y$ is continuous if and only if $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{E}_Y$.

Proof. The inverse image commutes with intersections, complements, and unions. To prove $\Longleftarrow$, use Theorem 4.4, since every $U \in \mathcal{T}_Y$ can be represented the union of finite intersections of elements $\mathcal{E}_Y$, and use the fact that $\mathcal{T}_X$ is closed under arbitrary unions and finite intersections.

To show $\Longrightarrow$, since $\mathcal{E}_Y \subseteq \mathcal{T}_Y$, if f^{-1} is open for every $U \in \mathcal{T}_Y$, then it is open for every $U \in \mathcal{E}_Y$ as well. ■

Theorem 4.10

Proposition 10.1

If X_α is Hausdorff for each $\alpha \in A$, then $X = \prod_{\alpha \in A} X_\alpha$ is Hausdorff.

Proof. If two elements in X, $x \neq y$ then there exists some $\alpha \in A$ such that $\pi_\alpha(x) \neq \pi_\alpha(y) \in X_\alpha$, but this X_α is Hausdorff, then there exists two open, disjoint sets $V_x, V_y \subseteq X_\alpha$ such that

- $x \in \pi_\alpha^{-1}(V_x)$, and $y \in \pi_\alpha^{-1}(V_y)$
- $\pi_\alpha^{-1}(V_x) \cap \pi_\alpha^{-1}(V_y) = \pi_\alpha^{-1}(V_x \cap V_y) = \varnothing$
- $\pi_\alpha^{-1}(V_x), \pi_\alpha^{-1}(V_y) \in \mathcal{T}_X$

Where for the last bullet point we used the fact that the product topology makes all the projection maps continuous. This proves that X is Hausdorff. ■

Theorem 4.11

Proposition 11.1

If X_α and Y are topological spaces, and $X = \prod_{\alpha \in A} X_\alpha$, and $f : Y \to X$ is a mapping. Then f is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in A$.

Proof. If $\pi_\alpha \circ f$ is continuous at each α, this means that

$$\forall \alpha \in A,\ \forall E_\alpha \in \mathcal{T}_\alpha,\ f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{T}_Y$$

But it is exactly sets of the form $\pi_\alpha^{-1}(E_\alpha)$ which generate the weak topology for $\mathcal{T}_X$. Therefore f is continuous.

Now, suppose that f is continuous, by definition of the weak topology (as it is generated by the set of inverse projections), for every $\alpha \in A$, $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{T}_X$ and by continuity of f, its inverse image is open in Y as well. ■

Remark 11.1

The take-away intuition here is that if the range space is generated by some $\mathcal{E}$, then a function is continuous if and only if all inverse images of sets in $\mathcal{E}$ are open in the domain space. Furthermore, if the range space is endowed with the product topology (which is generated by sets of the form $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{T}_\alpha$), then it suffices to check all inverse images of those. And this is equivalent to checking that $\pi_\alpha(\cdot) \circ f$ is continuous at each α.

Theorem 4.12

Proposition 12.1

If X is a topological space, and A is any non-empty set, $\{f_n\} \subseteq X^A$ is a sequence, then $f_n \to f$ with respect to the product topology if and only if $f_n \to f$ pointwise.

Proof. Suppose that $f_n \to f$ pointwise. Since the product topology $\mathcal{T}_X$ is generated from sets of the form

$$\pi_\alpha^{-1}(E_\alpha),\ E_\alpha \in \mathcal{T}_\alpha$$

And by Theorem 4.4, $\mathcal{T}_X$ consists of $\varnothing, X$ and unions of finite intersections of $\pi_\alpha^{-1}(E_\alpha)$. We claim that for every $f \in X^A$, the following is a valid neighbourhood base for f

$$\left\{\bigcap_{j\le n} \pi_{\alpha_j}^{-1}(E_{\alpha_j}),\ E_{\alpha_j} \in \mathcal{T}_{\alpha_j} \cap \mathcal{N}_B(\pi_{\alpha_j}(f))\right\}$$

A couple things to note

- Each E_{α_j} is open in X_{α_j}, so that its inverse image is also open (in X). Since any neighbourhood base has to be a subset of $\mathcal{T}_X$.
- Only finitely many intersections are involved, so each element in the above set is open in X.
- Each E_{α_j} is a neighbourhood of $\pi_{\alpha_j}(f)$, meaning $f \in E_{\alpha_j}^o = E_{\alpha_j}$.
- Last and perhaps most importantly for intuition, fix any non-empty open set $U \in \mathcal{T}_X$ then by Theorem 4.4 (or my reading of it), U can be written as the union of sets like

  $$\bigcap_{j\le m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}$$

  Then applying Theorem 4.2, the family of finite intersections of $\pi_\alpha^{-1}(E_\alpha)$ is a base for $\mathcal{T}_X$. Then,

  $$N_{base}(f) = \left\{V = \bigcap_{j\le m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}, \quad f \in V\right\}$$

  Has to be a neighbourhood base for any $f \in X$.

Now to show that $f_n \to f$ in the product topology, fix any neighbourhood $U \in \mathcal{N}_B(f)$, then $f \in U^o$, and by definition of a neighbourhood base, there exists some $E \in N_{base}(f)$ such that $f \in E \subseteq U^o$, but this E is just the finite intersection of $\pi_{\alpha_j}^{-1}(E_{\alpha_j})$, then at every α_j

- Let N_j be an integer such that for every $n \ge N_j$, $\pi_{\alpha_j}(f_n) \in E_{\alpha_j}$
- Set $N = \sum_{j\le m} N_j \ge N_j$ for every $j \le m$.

Then for every $n \geq N$, $f_n \in E \subseteq U^o \subseteq U$ for any arbitrary neighbourhood U of f. So $f_n \to f$ in the product topology.

Conversely, suppose that $f_n \to f$ in the product topology, then fix any $\alpha \in A$, and for every neighbourhood E_α of $\pi_\alpha(f)$, $\pi_\alpha^{-1}(E_\alpha)$ is a neighbourhood of f. Hence for every $\alpha \in A$, and for every neighbourhood E_α of $\pi_\alpha(f)$, $pi_\alpha(f_n)$ is eventually in E_α. This completes the proof. ■

Theorem 4.13

Proposition 13.1

If X is a topological space then $\mathrm{BC}(X)$ is a closed subspace of $B(X)$ in the uniform metric, and $BC(X)$ is complete.

Proof. We will prove four things, the last two are just book-keeping. Parts (b, d) imply Part (c), as the closure of any set under a complete metric space is again complete.

(a) $B(X)$ endowed with the uniform norm of an $f \in B(X)$

$$\|f\|_u = \sup\{|f(x)|,\ x \in X\}$$

Is indeed a normed vector space.

(b) $B(X)$ with its norm (and induced metric), is a complete metric space. So that our $\{f_n\} \to f$ at worst, converges to $f \in B(X)$.

(c) If $\{f_n\}_{n\geq 1} \subseteq \mathrm{BC}(X)$ is a uniformly Cauchy sequence, and $f_n \to f$, then $f \in \mathrm{BC}(X)$.

(d) If f is an adherent point of $\mathrm{BC}(X)$, then $f \in \mathrm{BC}(X)$.

To show that $B(X)$ is a normed vector space, for any $k \in \mathbb{C}$, $f_1, f_2 \in B(X)$, then at every $x \in X$

$$|f_1(x) + kf_2(x)| \leq |f_1(x)| + |k| \cdot |f_2(x)| \leq \|f_1\|_u + |k|\|f_2\|_u$$

And to show absolute homogeneity, note that $\sup|kA| = |k| \cdot \sup A$ for any non-empty bounded above set of reals A. This proves (a).

To show (b), fix any Cauchy sequence in $B(X)$ (with respect to the uniform metric), then for every $\varepsilon > 0$, there exists an N so large that for every $n, m \geq N$ we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_u < \varepsilon$$

This shows that $\{f_n(x)\}_{n\geq 1} \subseteq \mathbb{C}$ is Cauchy, and it makes sense to call its limit $f(x) = \lim f_n(x)$. To show that for this f,

- $f_n \to f$ uniformly, and
- $f \in B(X)$

Fix an $\varepsilon > 0$, and there exists an N so large that for every $m, n \geq N$ implies that

$$\|f_n(x) - f_m(x)\|_u < \varepsilon$$

Since $\lim_{n\to\infty} f_n(x) = f(x)$, this means that

$$\lim_{n\to\infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| \leq \varepsilon$$

The above holds for any x, hence

$$\|f_m - f\|_u \le \varepsilon \implies \|f\|_u \le \|f_m - f\|_u + \|f_m\|_u < +\infty$$

This proves both bullet points.

Proof of (c): Now we will prove Theorem 4.13, for any sequence $\{f_n\} \subseteq \mathrm{BC}(X)$, if it does converge to some f uniformly, then we claim that $f \in \mathrm{BC}(X)$. Note that $f \in B(X)$, so it suffices to show continuity at every $x_0 \in X$.

Fix any ball with radius $\varepsilon > 0$ at $f(x_0) \in \mathbb{C}$, and since

- $\varepsilon/3 > 0$ induces some N such that for every $n \ge N$, at every point $x \in X$
$$|f_n(x_0) - f(x_0)| \le \|f_n - f\|_u < \varepsilon/3$$
- Another $\varepsilon/3$ gives us an open ball around $f_n(x_0)$ in $\mathbb{C}$ (using the same point $x_0 \in X$). Continuity of f_n gives us
$$f_n^{-1}(\ B(\varepsilon/3,\ f_n(x_0))\ ) = U \in \mathcal{T}_X$$
- If x is a point in U,
$$|f_n(x) - f(x)| \le \|f_n - f\|_i < \varepsilon/3$$
this gives us the last $\varepsilon/3$.

Combining these three,

$$|f(x) - f(x_0)| \le \underbrace{|f(x) - f_n(x)| + |f(x_0) - f_n(x_0)|}_{\text{uniform convergence}} + \underbrace{|f_n(x_0) - f_n(x)|}_{\text{continuity of } f_n} < \varepsilon$$

So there exists some open set $U \in \mathcal{T}_X$ (and hence neighbourhood of every x), for every open ball of radius $\varepsilon > 0$, around every $f(x) \in \mathbb{C}$, such that

$$f(U) \subseteq B \in \mathcal{T}_{\mathbb{C}}$$

Since the open balls are a neighbourhood base at every point in $\mathbb{C}$, and f is continuous at every point $x \in X$, we must conclude that $f \in \mathrm{BC}(X)$.

Part (d): Let $f \in \overline{\mathrm{BC}(X)}$. Notice $\mathrm{BC}(X)$ is a metric space, hence first countable. There exists a sequence $\{f_n\} \subseteq \mathrm{BC}(X)$ that converges to f. Convergent sequeneces in any metric space is Cauchy, apply Part (c) finishes the proof. ∎

Theorem 4.14

Proposition 14.1

Suppose that A and B are disjoint closed subsets of the normal space X, and let $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$ be the set of dyadic rationals in $(0,1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that

1. $A \subseteq U_r \subseteq B^c$ for every $r \in \Delta$,
2. $\overline{U_r} \subseteq U_s$ for $r < s$, and
3. For every $r < s$, $\overline{U}_r \subseteq U_s$

Proof. The goal of this proof is to show that for every $r \in \Delta$, there exists a open U_r that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some $N \geq 1$ that all previous U_r in previous layers have been constructed properly, meaning if $r = k/2^n$, then for every $1 \leq n \leq N-1$, we have

$$r = \frac{k}{2^n},\ 1 \leq n \leq N-1,\ 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each U_r,

- $A \subseteq U_r \subseteq B^c$ and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer $N \geq 1$, we only have to construct the $U_{k/2^N}$ for every odd k, this is because if k is an even number, then $k = 2j$ and $r = 2j/2^N = j/2^{N-1}$ and for this particular U_r is already constructed. So for every odd $k = 2j+1$, the sets of the form $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every $k-1 \neq 0$ and $k+1 \neq 1$. (We will consider these cases later). We claim that for every pair of open sets, $E_1, E_2 \in \mathcal{T}_X$, then there exists some open set $G \in \mathcal{T}_X$ such that if $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$ where H is defined as the set

$$H = \left\{(E_1, E_2) \subseteq (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \varnothing\right\}$$

Then there exists some $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$ such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any $(E_1, E_2) \in H$, then this pair induces a pair of disjoint sets $\overline{E_1}$ and E_2^c since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \varnothing$$

And by normality, there exists disjoint open sets G_1, G_2 such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \varnothing \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since G_2^c is a closed set that contains G_1 as a subset, $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of G_2 (so we may discard the notion of G_2 in our minds). Let us now replace G with G_1 then it is an easy task to verify that $G = G_1 = \mathcal{J}(E_1, E_2)$ has the required properties.

Now define for every odd k, since $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (we note in passing that $\mathcal{J}$ is not a function as the set G may not be unique).

$$U_{k/2^N} = \mathcal{J}\left(U_{(k-1)/2^N}, U_{(k+1)/2^N}\right)$$

Then, if $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$ sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$ and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed' $U_{k/2^N}$ for every $k-1 \neq 0$ and $k+1 \neq 1$. Now let us deal with the remaining pathological cases.

If $k-1$ so happens to be 0, then no $r \in \Delta$ satisfies $r = 0/2^N$, and we substitute

$$\overline{U}_0 = A, \qquad \text{or alternatively, } U_0 = A^o$$

Then $U_0 \in \mathcal{T}_X$, $\overline{U}_0 = A \subseteq B^c$. It is at this point that we must mention that $0, 1 \notin \Delta$, so U_0 and U_1 do not have to obey the rules we have laid out for $U_{r\in\Delta}$.

Now if $k+1$ is equal to 2^N (this makes $r = (k+1)/2^N = 1$) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every $0 \leq m \leq 2^N - 1, U_{m/2^N}$ must staisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (even for when $N = 1$, since $A = \overline{U}_0 \subseteq U_1 = B^c$) and a corresponding $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$ such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when $N = 1$. We would only have to construct for $k = 1$, since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■

Theorem 4.15

Proposition 15.1

Urysohn's Lemma. Let X be a normal space, if A and B are disjoint closed subsets of X, then there exists a $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Proof. Let $r \in \Delta$ be as in Lemma 4.14, and set U_r accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0,1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic ε argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since A and B are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that x is not a member of any U_r, but we set $U_1 = X$. Since none of the $r \in (0,1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The ε argument follows: suppose for every $\varepsilon > 0$, $(1-\varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannnot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in X. So $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of Δ, there exists an r, $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r<\alpha} U_r$. So $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r<\alpha} U_r$.

Fix an element $x \in \bigcup_{r<\alpha} U_r$, this induces an r such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r<\alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$.
For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an r (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) — we have $x \notin \overline{U_s}$, but $\left(\overline{U_s}\right)^c$ is open in X. It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq \left(\overline{U_s}\right)^c \subseteq \bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$. To show the reverse, fix an element x in the union, then this induces some $x \in \left(\overline{U_s}\right)^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of W. This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen s is a lower bound for W, and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in X, using Theorem 4.9 finishes the proof. ■

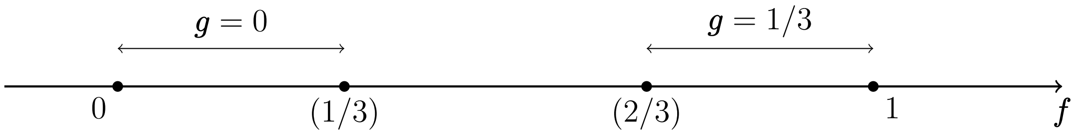


Figure 1: Lemma 16.1 for Theorem 4.16: Separate the range of $f \in C(A,\ [0,1])$ into three parts. Subtract an additional g that reduces the error even further.

Theorem 4.16

Proposition 16.1

The Tietze's Extension Theorem. Let X be a normal space, and for any closed subset $A \subseteq X$, and $f \in C(A,\ [a,b])$, there exists an $F \in C(X, [a,b])$ which extends f.

Proof. We begin with an important lemma that will serve as a 'black box' for the induction.

Lemma 16.1

For every $f \in C(A,\ [0,1])$, there exists a $g \in C(X, [0,1/3])$ such that

$$0 \leq f - g \leq 2/3 \quad \text{pointwise on } A \tag{12}$$

Proof. Since f is continuous, $B = f^{-1}([0,1/3])$, and $C = f^{-1}([2/3,1])$ are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function $g \in C(X,[0,1])$ such that $g|_B = 0$ and $g|_C = 1$. Rescale g by a factor of $1/3$, and $g \in C(X,[0,1/3])$.

To show that Equation (12) holds, suppose $x \in B$, then $f(x) \in [0,1/3]$ and $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$. Now suppose that $x \in C$, then $f(x) \in [2/3,1]$ and $g(x) = 1/3$ (recall that we relabelled g). So we have $0 \leq 1/3 \leq f - g \leq 2/3$. Lastly, for the case where $x \notin (B \cup C)$, then $f(x) \in (1/3,2/3)$, and $g(x) \in [0,1/3]$ implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 &\qquad \implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 &\qquad \implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore $0 \leq f(x) - g(x) \leq 2/3$. See Figure 1.

■

We can assume that $f \in C(A,\ [0,1])$, since we can relabel $f = (f-a)/(b-a)$. The main part of this proof consists of constructing a sequence of $\{g_n\} \subseteq C(X,\mathbb{R})$ where $0 \leq g_n \leq (2/3)^n(1/2)$, and $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$ on A. Let us begin with the base case with $n = 1$. We can apply Lemma 16.1 to get $g_1 \in C(X,\ [0,1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that $\{g_j\}_{j\le n}$ has been chosen, we will find our g_{n+1} by noting that

$$0 \le f(x) - \sum_{j\le n} g_j(x) \le (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by $(2/3)^{-n}$ and we obtain a new function in $C(A,\ [0,1])$.

$$0 \le \left(f(x) - \sum_{j\le n} g_j(x)\right)\left(\frac{3}{2}\right)^n \le 1$$

Applying Lemma 16.1, we get a function $h \in C(X,[0,1/3])$ that reduces the error between f and the partial sums of $g_{j\le n-1}$. For every $x \in A$

$$0 \le \left(f(x) - \sum_{j\le n} g_j(x)\right)\left(\frac{3}{2}\right)^n - h \le 2/3$$

Multiplying across gives

$$0 \le \left(f(x) - \sum_{j\le n} g_j(x)\right) - h\left(\frac{2}{3}\right)^n \le \left(\frac{2}{3}\right)^{n+1}$$

Set $g_{n+1} = h\left(\frac{2}{3}\right)^n$ and $g_{n+1} \in C(X,[0,2^n/3^{n+1}])$. Furthermore, the sum of all g_j pointwise converges uniformly, as

$$\sum_{j\ge 1} \|g_j\|_u \le \sum_{j\ge 1}\left(\frac{2}{3}\right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum $F = \sum g_j$, then this $F \in BC(X)$ (by Theorem 4.13 and 5.1). And

$$\left\| f - \sum_{j\le n} g_j \right\|_u \le \left(\frac{2}{3}\right)^n \longrightarrow 0$$

So $F = f$ on A, now if we want to obtain our F on $[a,b]$ we simply relabel $F = F(b-a)+a$. This finishes the proof. ∎

Theorem 4.17

Proposition 17.1

If X is a normal space, and A is a closed subspace of X, and $f \in C(A)$, then there exists an $F \in C(X)$ such that F extends f.

Proof. First we suppose that f is real valued, so $f \in C(X, \mathbb{R})$. And define a $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$, using

$$g = \frac{f}{1 + |f|}$$

Since g satisfies the assumption of Theorem 4.16 (note that we do not require g to be injective), there exists a $G \in C(X, [-1, +1])$ such that $G|_A = g$. Since the set $\{-1, +1\}$ is closed in $\mathbb{R}$, $G^{-1}(\{-1, +1\})$ is closed as well. Since $G^{-1}((-1, +1)) \subseteq A$, this makes A and $B =^{-1} (\{-1, +1\})$ disjoint closed sets in X.

By Urysohn's Lemma, there exists a continuous function $h \in C(X, [0, 1])$ such that $h|_B = 0$ and $h|_A = 1$, so that the product $|hG| < 1$ for all $x \in X$. We can think of this h as a continuous indicator function that filters out the parts we do not want, namely $G^{-1}\{-1, +1\}$. Now define F in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that $F|_A = g/(1 - |g|) = f$ indeed. Since $|g| = \frac{|f|}{1+|f|}$, and $g(1 + |f|) = f$ implies that $g/(1 - |g|) = f$, because $g \in C(A, (-1, +1))$ This completes the proof for any $f \in \mathbb{R}$ if $f \in C(A)$, then

1. $\mathrm{Re}(f) = f_1 \in C(A, \mathbb{R})$
2. $\mathrm{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in $C(X, \mathbb{R})$ that extends f_1 and f_2, and $F_1 + iF_2 = f$ on A and $F_1 + iF_2 \in C(X)$, and the proof is complete. ■

Theorem 4.18

Proposition 18.1

If X is a topological space, and $E \subseteq X$ and $x \in X$, then $x \in \operatorname{acc} E \iff$ there exists a net in $E \setminus \{x\}$ that converges to x, and $x \in \overline{E} \iff$ there exists a net in E that converges to x.

Proof. Suppose that $x \in \operatorname{acc} E$, then for every neighbourhood $U \in \mathcal{N}(x)$, $E \cap U \setminus \{x\} \neq \varnothing$, then choose $\mathcal{N}(x)$ as the set of neighbourhoods directed by reverse inclusion (and this makes $(\mathcal{N}(x), \lesssim)$ a directed set), and we will define the net as follows.

Map each $U \in \mathcal{N}(x)$ to some $x_U \in E \cap U \setminus \{x\}$, then this net converges to x. Suppose that we fix a neighbourhood, $V \in \mathcal{N}(x)$, then for every $U \gtrsim V$ we have $x_u \in U \subseteq V$. So $\langle x_U \rangle$ is eventually in V.

Conversely, if $\left\langle x_\alpha \right\rangle \subseteq E \setminus \{x\}$, and $x_\alpha \to x$, then every $U \in \mathcal{N}(x)$ there exists a $x_\alpha \in E \cap U \setminus \{x\}$ that makes

$$E \cap U \neq \varnothing \quad \forall U \in \mathcal{N}(x)$$

Hence $x \in \operatorname{acc} E$.

Now for the second part of the Theorem, suppose that $x \in \overline{E}$, if $x \notin E$ then $E = E \setminus \{x\}$ and $x \in \operatorname{acc} E$, so there exists a net in $E \setminus \{x\} \subseteq E$ such that $x_\alpha \to x$. If $x \in E$ then simply choose $\left\langle x_\alpha \right\rangle = x$ for every $\alpha \in A$.

Now, suppose that there is a net that converges to x, and this net $\left\langle x_\alpha \right\rangle \subseteq E$, if $x \in E$ then there is nothing to prove, since $E \subseteq \overline{E}$, so suppose that $x \notin E$, then there exists a net in $E \setminus \{x\} = E$ such that

$$x_\alpha \to x \implies x \in \operatorname{acc} E \subseteq \overline{E}$$

■

Theorem 4.19

Proposition 19.1

Let X and Y be topological spaces, then every $f : X \to Y$ is continuous at a point $x \in X \iff$ every net $\left\langle x_\alpha \right\rangle$ that converges to x implies that $\left\langle f(x_\alpha) \right\rangle$ converges to $f(x)$.

Proof. If f is continuous at a point $x \in X$, then $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$, then for every net $\left\langle x_\alpha \right\rangle$ that converges to this x, there there exists an α_0 such that for every $\alpha \gtrsim \alpha_0$ implies that $x_\alpha \in f^{-1}(V)$. Hence

$$f(x_\alpha) \in f\left(f^{-1}(V)\right) \subseteq V$$

And this is equivalent to saying that for every $V \in \mathcal{N}(f(x))$, $\left\langle f(x_\alpha) \right\rangle$ is eventually in V, and this proves convergence.

Now suppose that f is not continuous at some x, then there exists a $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$, so

$$x \notin \left(f^{-1}(V)\right)^{o} \implies x \in \left(f^{-1}(V)\right)^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our $x \in \overline{f^{-1}(V^c)}$ induces a net $\left\langle x_\alpha \right\rangle \subseteq f^{-1}(V^c)$ that converges to x. But every element in the net is contained within $f^{-1}(V^c)$, and for every $\alpha \in A$

$$f(x_\alpha) \in f\left(f^{-1}(V^c)\right) \subseteq V^c$$

gives $f(x_\alpha) \notin V$, but V is a neighbourhood of $f(x)$, hence there exists some $x_\alpha \to x$ and $f(x_\alpha) \not\to f(x)$. ■

Theorem 4.20

Proposition 20.1

If $\left\langle x_\alpha \right\rangle$ is a net in X, and $x \in X$ is a cluster point of $\left\langle x_\alpha \right\rangle \iff$ there exists a subnet of $\left\langle x_\alpha \right\rangle$ that converges to x.

Proof. Suppose that $\left\langle y_\beta \right\rangle_{\beta \in B}$ is a subnet of $\left\langle x_\alpha \right\rangle$ that converges to x, then for every neighbourhood $U \in \mathcal{N}(x)$, there exists a β_1 such that for every $\beta \gtrsim \beta_1$ we get $y_\beta = x_{\alpha_\beta} \in U$.

Furthermore, let us fix a $\alpha_0 \in A$ to attempt to show that $\left\langle x_\alpha \right\rangle$ is frequently in U, then by the subnet property of $\left\langle y_\beta \right\rangle$, there exists some $\beta_2 \in B$ such that for every $\beta \gtrsim \beta_2$, $\alpha_\beta \gtrsim \alpha_0$. (Intuitively this property means that the directed set of B 'grows' as much as the directed set of A, so we can always find elements that are greater than any fixed α_0.)

Since $\left\langle y_\beta \right\rangle$ is a net, we there exists some $\beta \in B$ such that $\beta \gtrsim \beta_1$ and $\beta \gtrsim \beta_2$, we then apply the $\beta \mapsto \alpha_\beta$ map and we obtain some $\alpha = \alpha_\beta$ that satisfies:

- $\alpha = \alpha_\beta \gtrsim \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that $\beta \gtrsim \beta_1$ so that y_β falls into U.

Conversely, suppose that x is a cluster point of $\left\langle x_\alpha \right\rangle$, then by definition

$$\forall U \in \mathcal{N}(x),\ \forall \alpha_0 \in A,\ \exists \alpha \gtrsim \alpha_0,\ x_\alpha \in U$$

Denote the directed neighbourhoods of x by $\mathcal{N}(x)$, and construct our directed set B for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every $(U, \gamma) \in B$ we can map it to some $\alpha_{(U,\gamma)} \in A$, if we choose some $\alpha_{(U,\gamma)} \gtrsim \gamma$ and $\alpha_{(U,\gamma)} \in U$.

To show that B is a directed set, we say that $(U, \gamma) \gtrsim (U', \gamma')$ if and only if $U \subseteq U'$ and $\gamma \gtrsim \gamma'$. And to show that $\left\langle y_\beta \right\rangle = \left\langle x_{\alpha_{(U,\gamma)}} \right\rangle$ is indeed a subnet of $\left\langle x_\alpha \right\rangle$, fix any $\alpha_0 \in A$, then simply take any neighbourhood U of x (we always have $X \in \mathcal{N}(x)$) — and therefore $(U, \alpha_0) \in B$.

Now for every $(U', \alpha_0') \gtrsim (U, \alpha_0)$ implies that $\alpha_0' \gtrsim \alpha_0$, therefore we have

$$\alpha_{(U',\alpha_0')} \gtrsim \alpha_0' \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that $\left\langle y_\beta \right\rangle$ indeed converges to x, fix any $V \in \mathcal{N}(x)$, then with any $\alpha_0 \in A$, and for every $(V', \alpha_0') \gtrsim (V, \alpha_0) \in B$, we have

$$x_{\alpha_{(V',\alpha_0')}} \in V' \subseteq V$$

So $\left\langle x_{\alpha_{(U,\gamma)}} \right\rangle$ converges to x. $\blacksquare$

Theorem 4.21

Proposition 21.1

A topological space X is compact $\iff$ every family of closed sets, $\{F_\alpha\}_{\alpha\in A}$ that has the finite intersection property, implies that

$$\bigcap_{\alpha\in A} F_\alpha \neq \varnothing$$

Proof. We first examine the assertion, Theorem 4.21 proposes for any family of closed sets $\{F_\alpha\}_{\alpha\in A}$, and for every finite subset $B \subseteq A$ then,

$$\bigcap_{\alpha\in B} F_\alpha \neq \varnothing \implies \bigcap_{\alpha\in A} F_\alpha \neq \varnothing$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha\in A} F_\alpha = \varnothing \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha\in B} F_\alpha = \varnothing$$

Applying DeMorgan's theorem, and since every $\{F_\alpha\}_{\alpha\in A}$ induces a family of open sets (and vice versa), where $U_\alpha = F_\alpha^c$, so for any familiy of open sets $\{U_\alpha\}_{\alpha\in A}$ we have

$$\bigcup_{\alpha\in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha\in B} U_\alpha = X$$

Which is equivalent to saying that X is compact. ■

Theorem 4.22

Proposition 22.1

A closed subset of a compact space X is compact.

Proof. Suppose $F \subseteq X$ and F is open, then fix an open cover for F, so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since F^c is an open set, we can obtain a valid open cover for X, then we pick out a finite subcover, for some finite $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left(\bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with F on both sides yields

$$\begin{aligned}
&F = X \cap F \subseteq (F^c \cap F) \cup \left(F \cap \left(\bigcup_{\alpha \in B} U_\alpha \right) \right) \\
&F = \left(F \cap \left(\bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\
&F \subseteq \bigcup_{\alpha \in B} U_\alpha
\end{aligned}$$

Therefore every open cover of F has a finite subcover, and F is compact. ■

Theorem 4.23

Proposition 23.1

If F is a compact subset of a Hausdorff space X, and $x \notin F$, there are disjoint open sets U, V such that $x \in U$ and $F \subseteq V$.

Proof. Since $x \in F^c$, for every $y \in F$, $x \neq y$ induces two sets U_y, V_y (because X is T_2).

- $U_y \cap V_y = \varnothing$
- $x \in U_y$
- $y \in V_y$

But $\{V_y\}_{y \in F}$ is an open cover for the compact set F, then there exists a finite subcollection $H \subseteq F$ such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since H is finite, $U = \bigcap_{y \in H} U_y$ is an open set that contains x, also define $V = \bigcup_{y \in H} V_y$. If for every $y \in H$, $U_y \cap V_y = \varnothing$, then $U \cap V_y = U \cap V = \varnothing$. This completes the proof. ■

Remark 23.1

Every metric space (X, d) is first countable, and T_2 (it is actually T_4, but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element $x \in X$ and we notice that $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$ is a countable neighbourhood base for every x. To show that (X, d) is T_2, for every pair of elements $x \neq y$, we can take $r = d(x, y)/2$ and there exists disjoint open sets $V_r(x)$ and $V_r(y)$ such that $x \in V_r(x)$ and $y \in V_r(y)$.

Theorem 4.24

Proposition 24.1

Every compact subset of a Hausdorff (T_2) space is closed.

Proof. If F is compact, then for every $x \in F^c$, by Theorem 4.23, there exists two disjoint open sets such that $x \in U$ and $F \subseteq V$, but

$$U \cap V = \varnothing \implies U \cap F = \varnothing \implies U \subseteq F^c$$

But since $x \in F^c$ is arbitrary, and U is an open subset of F^c,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that F^c is open and F is closed. ■

Theorem 4.25

Proposition 25.1

Every compact Hausdorff (T_2) space is normal (T_4).

Proof. Fix A, B which are disjoint closed subsets of X, by Theorem 4.22, we know that these two sets are compact. Hence for every $y \in B$ there exists two disjoint open sets U, V_y (by Theorem 4.23)

$A \subseteq U_y$ and $y \in V_y$. But the family $\{V_y\}_{y \in B}$ is s a valid open cover for the compact set B, hence there exists a finite subcollection $H \subseteq B$ such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \varnothing$$

The second equality holds for every $y \in H$ so that $U_y \cap (\cup V_{y \in H}) = \varnothing$. Define $U = \cap U_{y \in H}$ and $V = \cup V_{y \in H}$, where both of these are disjoint open sets that that contain A and B as subsets, since for each $y \in H$, $A \subseteq U_y$ hence the intersection of all U_y also contains A as a subset. Therefore X is normal. ■

Theorem 4.26

Proposition 26.1

If X is compact, and $f : X \to Y$ is continuous, then $f(X)$ is compact.

A small lemma.

Lemma 26.1

For every $\{E_j\} \subseteq X, f(\cup E_j) = \cup f(E_j)$.

The proof is trivial.

Proof. If $\{V_{\alpha \in A}\}$ is an open cover for $f(X)$, then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha) \subseteq X$$

Since f is continuous, we have an open cover in the form of $\{f^{-1}(V_\alpha)\}$ for X, then there exists a finite subset $B \subset A$ such that

$$X \subseteq \bigcup_{\alpha \in B} f^{-1}(V_\alpha)$$

Then we wish to show that for this $B \subseteq A$, $\{V_{\alpha \in B}\}$ is a finite open cover for $f(X)$. Fix any element $y \in f(X)$, then this induces a $x \in X$ such that $y = f(x)$, but because $\{f^{-1}(V_{\alpha \in B})\}$ is an open cover for X, there exists some $\alpha \in B$ such that $x \in f^{-1}(V_\alpha)$, hence by definition of the inverse image

$$f(x) \in V_\alpha \implies f(X) \subseteq \bigcup_{\alpha \in B} V_\alpha$$

Therefore $f(X)$ is compact and this completes the proof. ■

Theorem 4.27

Proposition 27.1

If X is compact, then $C(X) = BC(X)$.

Proof. Notice that $BC(X) \subseteq C(X)$, so we only have to show the reverse estimate. Fix any $f \in C(X)$, since X is compact, by Theorem 4.26 we know that $f(X)$ is also compact. Since $\mathbb{C} = \mathbb{R}^2$ is a complete metric space, $f(X)$ is bounded and $f \in BC(X)$. ∎

Theorem 4.28

Proposition 28.1

If X is compact, and if Y is Hausdorff, then any continuous bijection $f : X \to Y$ is a homeomorphism.

Proof. If $E \in X$ is closed, then since X is compact, E is compact as well. By continuity of f, $f(X)$ is a compact set in Y, but compact subsets of Y are closed, so f is continuous.

We used the fact that the inverse of f^{-1} is f, since it suffices to check that every inverse image of a closed set is also closed, f^{-1} is continuous. And by definition of a homeomorphism (f has to be bijective and both f and f^{-1} hav eto be continuous), f is a homeomorphism. ■

Theorem 4.29

Proposition 29.1

If X is any topological space, the following are equivalent.

(a) X is compact.

(b) Every net has a cluster point.

(c) Every net in X has a convergent subnet.

Proof. By Theorem 4.20, every net in X has a cluster point $\iff$ there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose a) holds, then X is compact, and fix an arbitrary net $\left\langle x_\alpha \right\rangle$ in X. and define the 'tail' of the net

$$E_\alpha \triangleq \{x_\beta,\ \beta \gtrsim \alpha\}$$

We wish to show that the arbitrary intersection of $\bigcap_{\alpha\in A} \overline{E}_\alpha \neq \varnothing$. Where $\overline{E}_\alpha$ is closed, so it suffices to check that every finite $B \subseteq A$, the intersection over $\overline{E}_\alpha$ is non-empty.

Suppose we are given a finite $B \subseteq A$, then fix any two elements α and $\beta \in B$, by the definition of a net there exists a $\gamma \in A$ such that $\gamma \gtrsim \alpha$ and $\gamma \gtrsim \beta$, and

$$\varnothing \neq \subseteq E_\alpha \cap E_\beta \implies \overline{E}_\alpha \cap \overline{E}_\beta \neq \varnothing$$

Therefore for any finite collection of $\{\overline{E}_{\alpha\in B}\}$, then

$$\bigcap_{\alpha\in A} \overline{E}_\alpha \neq \varnothing$$

Now fix an element $x \in \bigcap_{\alpha\in A} \overline{E}_\alpha$. Then for every $\alpha \in A$, $x \in \overline{E}_\alpha$, and for every neighbourhood $U \in \mathcal{N}(x)$, $U \cap E_\alpha \neq \varnothing$. This is because if $x \in E_\alpha$, then $U \cap E_\alpha$ contains at least $\{x\}$, if $x \in \operatorname{acc} E_\alpha$, then by definition of an accumulation point, $U \cap E_\alpha \setminus \{x\} \neq \varnothing$, so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net, E_α, if for every $\alpha \in A$, $x \in E_\alpha$ if and only if there exists some $\gamma \gtrsim \alpha$, $x_\gamma \in U \cap E_\alpha$, this is equivalent to saying that x is a cluster point of $\left\langle x_\alpha \right\rangle$. So $a) \implies b)$.

Now let us suppose that X is not compact, then there exists an open cover $\{U_{\alpha\in A}\}$ of X that has no finite subcover. Let $\mathbb{B}$ be the collection of all finite subsets of A, directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every $B \in \mathbb{B}$, find some $x_B \in \left(\bigcup_{\alpha\in B} U_\alpha\right)^c$. So we have a net in X. Now we will show that no $x \in X$ can be a cluster point of this net. Suppose not, then take a neighbourhood U_β with

$\beta \in A$ such that U_β belongs to the open cover we first discussed. Then for any $B \in \mathbb{B}$ such that $B \gtrsim \{\beta\}$ (meaning that $\{\beta\} \subseteq B$, where B is a finite set), then

$$x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c \implies x_B \notin \left(\bigcup_{\alpha \in \{\beta\}} U_\beta\right) \implies x_B \in U_\beta^c$$

Hence no point in X can be a cluster point for this net, and the proof is complete. ■

Theorem 4.30

Proposition 30.1

If X is a LCH space, and for every $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$, there exists a compact $N \subseteq U$ where $N \in \mathcal{N}_B(x)$.

Proof. For every $U \in \mathcal{N}_B(x) \cap \mathcal{T}_x$, we can find an E open subset of U that has a compact closure, since every $x \in X$ induces some compact $F \in \mathcal{N}_B(x)$, therefore

$$E \triangleq U \cap F^o \implies \overline{E} \subseteq F$$

Since closed subsets of compact sets are compact (by Theorem 4.22), $\overline{E}$ is compact. More is true, since E is open,

$$x \in U \cap F^o \implies x \in E^o \implies E \in \mathcal{N}_B(x)$$

Now it suffices to show that there exists some compact $N \subseteq E \subseteq U$ such that $N \in \mathcal{N}_B(x)$. Since $\overline{E}$ is compact, the closed subset $\partial E = \overline{E} \cap \overline{E^c}$ of $\overline{E}$ is also compact.

Since $\partial E \cap E^o = \varnothing$, $x \in E^o = E$ means that $x \notin \partial E$. Applying Theorem 4.23 to the compact set ∂E and $x \notin \partial E$ gives us two disjoint open sets V' and W'. We list their properties

1. $V', W' \in \mathcal{T}_X$
2. $x \in V'$
3. $\partial E \subseteq W'$
4. $V' \cap W' = \varnothing$

The two disjoint pairs induce another pair of open sets relative to $\overline{E}$, recall the definition of the topology relative to $\overline{E}$,

$$\mathcal{T}_{\overline{E}} = \left\{ A \cap \overline{E} : A \in \mathcal{T}_X \right\}$$

We now agree to define

- $V = V' \cap \overline{E}$
- $W = W' \cap \overline{E}$

Then evidently $V, W \in \mathcal{T}_{\overline{E}}$ and

1. $x \in V' \cap \overline{E} \implies x \in V$
2. $\partial E \subseteq \overline{E} \implies \partial E \subseteq W$
3. $V' \cap W' = \varnothing \implies V \cap W = \varnothing$

Furthermore,

$$\partial E \subseteq W \implies W^c \subseteq (\partial E)^c = E^o \cup E^{co}$$

Taking the intersection over $\overline{E}$ gives us

$$\overline{E} \setminus W \subseteq \overline{E} \cap (E^o \cup E^{co})$$

Note that $E^{co} = (\overline{E})^c$, since $(E^c)^{oc} = \overline{(E^{cc})} = \overline{E}$ therefore $\overline{E} \cap E^{oc} = \varnothing$, hence

$$\overline{E} \setminus W \subseteq \overline{E} \cap E^o = E^o$$

Using the fact from 3, $V \subseteq W^c$ and $V \subseteq \overline{E}$ and $V \subseteq W^c$ implies that $V \subseteq \overline{E} \setminus W$. Compiling everything, we have

$$V \subseteq \overline{E} \setminus W \subseteq E$$

Note that the set $\overline{E} \setminus W$ is closed in $\mathcal{T}_X$ (and hence closed in $\overline{E}$) by closure over intersections, $\overline{V}$ is therefore a closed subset of $\overline{E} \setminus W$, and $\overline{V}$ is compact. Also

$$\overline{V} \subseteq \overline{E} \setminus W \subseteq E$$

To check that $\overline{V} \in \mathcal{N}_B(x)$, note that

$$x \in V^o \subseteq (\overline{V})^o \implies \overline{V} \in \mathcal{N}_B(x)$$

The subset relation $V^o \subseteq \overline{V}^o$ comes from the fact that V^o is an open subset of $\overline{V}$, and hence is contained in $(\overline{V})^o$ as a subset. Now let us define $N = \overline{V}$, and N satisfies the assertions in the Theorem, since

- $N \in \mathcal{N}_B(x)$
- N is compact
- $N \subseteq E \subseteq U$

And this completes the proof. ■

Remark 30.1

Intuitively speaking, this means that if X is any LCH space, then for every open neighbourhood $U \in \mathcal{N}_B(x)$, there exists a compact $E \in \mathcal{N}_B(x)$ such that $x \in E \subseteq U^o$. This property is indeed a very strong one as it allows us to have effectively 'infinite' descending compact neighbourhoods of x.

Theorem 4.31

Proposition 31.1

X is a LCH space, and $K \subseteq U \subseteq X$ where K is compact, and U is open, then there exists some precompact, open V with

$$K \subseteq V \subseteq \overline{V} \subseteq U$$

Proof. For every $x \in K$, we can apply Proposition 4.30, since $x \in K \subseteq U$, this induces some compact $F_x \subseteq U$ where $F_x \in \mathcal{N}_B(x)$. Then we can obtain an open cover of U in the form of $\{F_x^o\}_{x \in K}$. By compactness of K, there exists a finite $B \subseteq K$ such that

$$K \subseteq \bigcup_{x \in B} F_x^o$$

Let $V = \bigcup_{x \in B} F_x^o$, then clearly V is open, and $K \subseteq V$. Since each F_x is closed (compact sets are closed in any Hausdorff Space), we have

$$V \subseteq \bigcup_{x \in B} F_x \implies \overline{V} \subseteq \bigcup_{x \in B} F_x$$

Since $\bigcup_{x \in B} F_x$ is a finite union of compact sets, we claim that it is also compact. Consider two compact sets E_1 and E_2, then if $\{U_\alpha\}_{\alpha \in A}$ is any open cover of $E_1 \cup E_2$, it must be an open cover for E_1 and E_2 as well, because

$$E_1, E_2 \subseteq E_1 \cup E_2 \bigcup_{\alpha \in A} U_\alpha$$

Since E_1 and E_2 are both compact sets, they each induce two finite subsets of B_1, B_2 of A whose union $B = B_1 \cup B_2$ is also compact. Therefore

$$E_1 \cup E_2 \subseteq \bigcup_{\alpha \in B} U_\alpha$$

Then a simple proof by induction will show that if $\{E_{j \leq n}\}$ is a family of compact sets, then $E = \bigcup E_{j \leq n}$ is also compact.

Returning to the main part of the proof, $\bigcup_{x \in B} F_x$ is a compact set, therefore $\overline{V}$ is also compact. Moreover

$$\forall x \in K,\ F_x \subseteq U \implies \overline{V} \subseteq \bigcup_{x \in B} F_x \subseteq U$$

Combining, we have

- $K \subseteq V \subseteq \overline{V}$,
- V is open and $\overline{V}$ is compact, and
- $\overline{V} \subseteq U$

This completes the proof. ∎

Theorem 4.32

Proposition 32.1

Urysohn's Lemma, Locally Compact Version. For any LCH space X, and if $K \subseteq U \subseteq X$ where K is compact and U is open, then there exists some $f \in C(X, [0,1])$ with

- $f = 1$ on K
- $f = 0$ outside some compact $\overline{V} \subseteq U$

Proof. Let V be as in Theorem 4.31, for our fixed $K \subseteq U \subseteq X$, there exists a pre-compact, open V that satisfies

$$K \subseteq V \subseteq \overline{V} \subseteq X$$

It follows that this $(\overline{V}, \mathcal{T}_{\overline{V}})$ is a normal space by Theorem 4.25 (compact Hausdorff spaces are normal), and by Urysohn's Lemma (Theorem 4.15) on normal spaces, since we can easily find two disjoint closed subsets of $\overline{V}$ in the form of

- $K \subseteq V^o = V \subseteq \overline{V}$ (compact sets in Hausdorff spaces are closed)
- $\partial V = \overline{V} \cap \overline{V^c}$ (closed sets in compact spaces are compact)
- $K \subseteq V^o$ implies that $K \cap \partial V = K \cap (\overline{V} \setminus V^o) = \varnothing$

Then there exists a continuous $f|_{\overline{V}} \in C(\overline{V}, [0,1])$ that evaluates to

- $f|_{\overline{V}} = 1$ on closed K
- $f|_{\overline{V}} = 0$ on closed ∂V

Now let us extend $f|_{\overline{V}}$ to f by defining

$$f|_{(\overline{V})^c} = 0$$

We will show that this extension of f is indeed continuous. Indeed, for every closed set $E \subseteq [0,1]$ that does not contain 0, we have:

$$0 \notin E \implies \{0\} \cap E = \varnothing \implies f^{-1}(\{0\}) \cap f^{-1}(E) = \varnothing$$

But $(\overline{V})^c \subseteq f^{-1}(\{0\})$ therefore

$$(\overline{V})^c \cap f^{-1}(\{0\}) \cap f^{-1}(E) = (\overline{V})^c \cap f^{-1}(E) = \varnothing \implies f^{-1}(E) \subseteq \overline{V}$$

We can write

$$f^{-1}(E) = f|_{\overline{V}}^{-1}(E)$$

But we know that $f|_{\overline{V}}$ is continuous, so $f|_{\overline{V}}^{-1}(E)$ must be closed (with respect to $\overline{V}$), and therefore is closed wrt X, since $\overline{V}$ is closed wrt X.

For the case where $0 \in E$, note that

$$f^{-1}(E) = \left(f^{-1}(E) \cap \overline{V}\right) \cup \left(f^{-1}(E) \cap (\overline{V})^c\right) = \left(f|_{\overline{V}}\right)^{-1}(E) \cup \left(f|_{\overline{V}^c}\right)^{-1}(E)$$

The above equalities are messy in print. They are but a simple consequence of disjoint decomposition of the pre-images, since

$$\overline{V} \cap f^{-1}(E) = \{x \in \overline{V} : f(x) \in E\} = f|_{\overline{V}}^{-1}(E)$$

Back to our main discussion, recall that for every $x \in \partial V$

$$f(x) = 0 \in f^{-1}(\{0\}) \subseteq f^{-1}|_{\overline{V}}(E)$$

Therefore $\partial V \subseteq f^{-1}|_{\overline{V}}(E)$, and $(\overline{V})^c = f^{-1}|_{(\overline{V})^c}(E)$ gives us (since V^c is closed),

$$\begin{aligned} f^{-1}(E) &= f^{-1}|_{\overline{V}}(E) \cup \partial V \cup (\overline{V})^c \\ &= f^{-1}|_{\overline{V}}(E) \cup \overline{(V^c)} \cup (\overline{V})^c \\ &= f^{-1}|_{\overline{V}}(E) \cup (V^c \cup V^{co}) \\ &= f^{-1}|_{\overline{V}}(E) \cup V^c \end{aligned}$$

Since $f^{-1}|_{\overline{V}}(E)$ and V^c are closed subsets of X, then $f^{-1}(E)$ is also closed, and $f \in C(X, [0,1])$. ■

Theorem 4.33

Proposition 33.1

Every LCH space is completely regular (or $T_{3.5}$).

Proof. Recall that a space X is completely regular if it is T_1 and every closed subset A and every $x \notin A$ there exists some

$$f \in C(X, [0,1]), \quad f(x) = 1, \ f|_A = 0$$

Fix a closed set $A \subseteq X$, then for every $x \in A^c$, there exists a compact $E_x \in \mathcal{N}_B(x)$ with $E_x \subseteq A^c$ (by Theorem 4.30).

Note that $E_x \subseteq A^c$ where E_x is compact and A^c is closed, then an application of Theorem 4.31 tell us that there exists an $f \in C(X, [0,1])$ such that for every $x \in E_x$, $f(x) = 1$ and for points $y \notin A^c$ (which means that $y \in A$), $f(y) = 0$. Therefore X is completely regular. ■

Theorem 4.34

Proposition 34.1

Proof. ■

Theorem 4.35

Proposition 35.1

If X is a LCH space, we claim that

$$\overline{C_c(X)} = C_0(X)$$

Proof. We begin by proving several things that are mentioned before this Theorem, namely

$$C_c(X) \subseteq C_0(X) \subseteq BC(X)$$

Fix an $f \in C_c(X)$, and for every $\varepsilon > 0$,

$$x \in |f|^{-1}([\varepsilon, +\infty)) \implies |f(x)| \geq \varepsilon > 0$$

Therefore $|f|^{-1}([\varepsilon, +\infty))$ is a closed subset of $\operatorname{supp}(f)$, since $(-\infty, \varepsilon)$ is open in $\mathbb{R}$, then $[\varepsilon, +\infty)$ is a closed set. And by continuity of $|\cdot| \circ f$ (a composition of two continuous functions), $|f|^{-1}([\varepsilon, +\infty))$ is closed. Using the fact that closed subsets of compact $\operatorname{supp}(f)$ are also compact, we get $f \in C_0(X)$.

Next, we show that $C_0(X) \subseteq BC(X)$. Fix any element f of $C_0(X)$ with an arbitrary $\varepsilon > 0$, then $E_\varepsilon = \{x \in X : |f(x)| \geq \varepsilon\}$ is compact. The continuity of f guarantees that the direct image of a compact set is another compact set (Theorem 4.26)

$$|f|(E_\varepsilon) \text{ is a compact subset of } \mathbb{R}$$

And therefore for every $x \in E_\varepsilon \implies |f(x)| \in f(E\varepsilon)$, then by Heine-Borel, there exists some $M \geq 0$ such that $|f(x)| \leq M$. If $x \notin E_\varepsilon$, then by definition of E_ε, implies that $|f(x)| < \varepsilon$. Then $|f(x)| \leq M + \varepsilon$ for every $x \in X$. Hence $f \in BC(X)$.

Here I wish to offer an alternate proof for $C_0(X) \subseteq BC(X)$, we begin by constructing an open cover for $\operatorname{supp}(f)$ such that

$$\{U_n\}_{n>0} = \{x \in X | f(x)| < n\}$$

Then there exists a finite subcollection of $\{U_n\}_{n \in B}$ where B is a finite set, then define $M = 1 + \sum_{n \in B} n$ and for every $x \in \operatorname{supp}(f)$ we have $|f(x)| < n$ and since $n > 0$ this holds for every $x \in X$ too. Therefore $f \in BC(X)$.

For the main proof of Theorem 4.35, since $BC(X)$ is endowed with the uniform metric, it is also first countable, and therefore by Theorem 4.6, it suffices to show that every sequence $\{f_n\}_{n \geq 1} \subseteq C_c(X)$ converges in $C_0(X)$. And every element $f \in C_0(X)$ has a convergence sequence in $C_c(X)$.

Fix a convergent sequence $\{f_n\}_{n \geq 1} \subseteq C_c(X)$ that converges uniformly to some $f \in BC(X)$ (since $BC(X)$ is a closed subset of $C(X)$ with respect to the uniform norm), then for every $\varepsilon > 0$, there exists some $n \geq 1$ with

$$\|f_n - f\|_u < \varepsilon$$

We aim to show that $(\operatorname{supp}(f_n))^c \subseteq |f|^{-1}((-\infty, \varepsilon))$, so fix any $x \notin \operatorname{supp}(f_n)$, then

$$|f(x) - f_n(x)| = |f(x)| \leq \|f - f_n\|_u < \varepsilon$$

This establishes the estimate, and taking complements

$$|f|^{-1}([\varepsilon, +\infty)) \subseteq \operatorname{supp}(f_n)$$

Therefore for any arbitrary $\varepsilon > 0, \{x \in X,\ |f(x)| \geq \varepsilon\}$ is compact, and $\overline{\mathrm{C}_c(X)} \subseteq \mathrm{C}_0(X)$. Conversely, fix any $f \in \mathrm{C}_0(X)$, and for every $n \geq 1$, define

$$K_n = \{x \in X,\ |f(x)| \geq 1/n\}$$

Using Urysohn's Lemma for our LCH space X, there exists some g_n that has a compact support, and $g_n(x) = 1$ for every $x \in K_n$. We then write $f_n = g_n \cdot f \in \mathrm{C}_c(X)$. We wish to show that $f_n \to f$ uniformly. Notice that for any fixed $n \geq 1$, if $x \in K_n$ then

$$f_n(x) = f(x) \implies |f_n - f|(x) = 0$$

If $x \notin K_n$, $|f(x)| < 1/n$ (recall what K_n does), and $f_n = g_n \cdot f \in [0, 1]$ by definition of g_n from Theorem 4.32, hence

$$|f_n(x) - f(x)| = |f(x)| \cdot |1 - g_n| \leq |f(x)| < 1/n$$

Taking the supremum over $x \in X$, we have

$$\|f_n - f\|_u < 1/n \to 0$$

As we send n to $+\infty$, and $f_n \to f$ uniformly. This completes the proof. ■

Theorem 4.36

Proposition 36.1

Proof. ■

Theorem 4.37

Proposition 37.1

If X is an LCH space and $E \subseteq X$. E is closed if and only if $E \cap K$ is closed for every compact $K \subseteq X$.

Proof. Suppose that E is closed, then $E \cap K$ is closed, since compact subsets of Hausdorff spaces are closed, and $E \cap K \subseteq K$ tells us that $E \cap K$ is indeed compact.

Now suppose that E is not closed, by Theorem 4.1, $E \neq \overline{E}$, so pick some $x \in (\overline{E} \setminus E) = \operatorname{acc}(E) \cap E^c$, since X is locally compact, let K_x be a compact neighbourhood of x, then for every neighbourhood $U \in \mathcal{N}_B(x)$, we have

$$x \in U^o,\, x \in K_x^o, \implies x \in (U^o \cap K_x^o) \subseteq (U \cap K_x)^o$$

Since $(U^o \cap K_x^o)$ is an open subset of $(U \cap K_x)$, then $(U \cap K_x) \in \mathcal{N}_B(x)$, and recall that $x \in \operatorname{acc}(E)$, therefore

$$(U \cap K_x) \cap E \setminus \{x\} = U \cap (K_x \cap E) \neq \varnothing$$

But $x \notin E \implies x \notin E \cap K_x$. So x is an accumulation point of $E \cap K_x$ that is not in $E \cap K_x$. Therefore there exists some $E \cap K_x$ (with K_x compact) that is not closed. ■

Theorem 4.38

Proposition 38.1

If $\mathbf{X}$ is an LCH space, $C(\mathbf{X})$ is a closed subspace of $\mathbb{C}^{\mathbf{X}}$ in the topology of uniform convergence on compact sets. (We will sometimes refer to this as the topology of compact convergence.)

Proof. Let $E \subseteq \mathbf{X}$ be closed and endowed with the subspace topology.

$$\mathcal{T}_E = \left\{ U \cap E,\, U \in \mathcal{T}_{\mathbf{X}} \right\}$$

Then $A \cap E$ is closed relative to E iff it it is closed relative to $\mathbf{X}$. The proof for for this can be found in the Notes.

Let f be an adherent point of $C(\mathbf{X})$ endowed with the topology of compact convergence. If W is closed in $\mathbb{C}$, let K range through compact sets of $\mathbf{X}$. $f|K$ is in the closure of $C(K, \mathbb{C})$, therefore continuous by Proposition 4.13, as $C_c(K) \subseteq BC(K)$. So $f|K$ is continuous, and $(f|K)^{-1}(W)$ is closed Rel. K. Notice

$$(f|K)^{-1}(W) = f^{-1}(W) \cap K \tag{13}$$

since we can write $(f|K)(x) = (f \circ \iota_K)(x)$. Where $\iota_K : K \to \mathbf{X}$ is the inclusion map, which is an embedding. Equation (13) follows immediately. Therefore $f^{-1}(W) \cap K = (f|K)^{-1}(W)$ is closed Rel. K, so it is closed Rel. $\mathbf{X}$. This holds for every compact K, so $f^{-1}(W)$ is closed for any closed $W \subseteq \mathbb{C}$, and f is continuous. ∎

Theorem 4.39

Proposition 39.1

Proof. ■

Theorem 4.40

Proposition 40.1

Proof. ∎

Theorem 4.41

Proposition 41.1

Proof. ■

Exercises

Exercise 4.1

Proposition 1.1

If card $\mathbf{X} \geq 2$, there is a topology on $\mathbf{X}$ that is T_0 but not T_1.

Proof. Let $\mathcal{T}_{\mathbf{X}} = \{\varnothing\} \cup \{\{x\} \cup B,\ B \subseteq \mathbf{X}\}$, where $x \in \mathbf{X}$ is any point in $\mathbf{X}$. Suppose U_1, and U_2 are open sets in $\mathcal{T}_{\mathbf{X}}$, if either is empty then their intersection must be contained in $\mathcal{T}_{\mathbf{X}}$. Otherwise $U_1 = \{x\} \cup B_1$, and $U_2 = \{x\} \cup B_2$, where B_1 and B_2 are subsets of $\mathbf{X}$.

$$U_1 \cap U_2 = \{x\}(B_1 \cap B_2) \in \mathcal{T}_{\mathbf{X}}$$

Notice also $\{\varnothing,\ \mathbf{X}\} \subseteq \mathcal{T}_{\mathbf{X}}$. Fix an arbitrary family of open sets $\{U_\alpha\}_{\alpha \in A}$, in similar fashion we have $\bigcup U_\alpha = \{x\} \cup \left(\bigcup B_{\alpha \in A}\right)$ so their union is contained in $\mathcal{T}_{\mathbf{X}}$ as well.

This topology is T_0. Fix $y \neq z$ in $\mathbf{X}$, if either y or z is x, then choosing $\{x\}$ does the job. So assume $x \neq y \neq z \neq x$, and $\{y\} \cup \{x\}$ is an open set that does not contain z. This topology cannot be T_1, as x sticks onto every open set, so there are no open sets which separate x from the other points in $\mathbf{X}$. ∎

Exercise 4.2

Proposition 2.1

If $\mathbf{X}$ is an infinite set, the cofinite topology on $\mathbf{X}$ is T_1 but not T_2, and is first countable iff $\mathbf{X}$ is countable.

Proof. We will first verify that the cofinite topology $\mathcal{T}_{\mathbf{X}}$ is a topology.

$$\mathcal{T}_{\mathbf{X}} = \left\{U, \quad U^c \text{ is finite.}\right\} \cup \{\varnothing\}$$

So that $\{\varnothing,\ \mathbf{X}\} \subseteq \mathcal{T}_{\mathbf{X}}$. Let U_1 and U_2 be a pair of open sets, assuming if neither of them are empty, then U_2^c and U_1^c are finite sets, so that $U_1^c \cup U_2^c$ is finite as well. Use DeMorgan to see that $U_1 \cap U_2 \in \mathcal{T}_{\mathbf{X}}$.

If $\{U_\alpha\}_{\alpha\in A}$ is an arbitrary collection of open sets, then

$$\bigcap_{\alpha\in A} U_\alpha^c \subseteq U_\beta^c$$

where $\beta \in A$ is arbitrary, so U_β^c is finite. And the union $\bigcup U_\alpha$ is contained in $\mathcal{T}_{\mathbf{X}}$.

To show that $\mathcal{T}_{\mathbf{X}}$ is T_1, every singleton set is closed. To show that $\mathcal{T}_{\mathbf{X}}$ is not T_2, fix $x \neq y$. If B_x and B_y are open sets that contain x and y respectively. If B_x and B_y disjoint, then

$$B_x \subseteq B_y^c$$

Which means B_x is an open, finite subset. But the only open and finite subset of $\mathbf{X}$ is the empty set. This contradicts $x \in B_x$.

If $\mathbf{X}$ is countable, we will find a neighbourhood base $\mathcal{N}_B(x)$ for any $x \in \mathbf{X}$ as follows:

- We can index $\mathbf{X}$ using $\mathbb{N}^+ \cup \{0\}$, so without loss of generality, let $x_0 = x$, and
- Define $U_1 = \{x_1\}^c$, and $U_n = \bigcap_{j=1}^n \{x_j\}^c$ are open sets that contain x. Equivalently,

$$U_n = \left\{x_j,\ j \geq n+1\right\} \cup \{x_0\}$$

- If V is an open set that contains x_0, then V^c is finite, let $M \in \mathbb{N}^+$ be the largest index of $x_j \notin V$ (the negation of this is that if $j \geq M+1$, then $x_j \in V$) then $U_{M+1} \subseteq V$ as needed, and $\mathbf{X}$ is first countable.

Conversely, if $\mathbf{X}$ is first countable, we can find a descending sequence of neighbourhoods which form a neighbourhood base, $\{U_j\}_{j\geq 1} \subseteq \mathcal{T}_{\mathbf{X}}$. And each U_j^c is finite, so $\bigcup U_j^c$ is countable. Assume for contradiction that $\mathbf{X}$ is uncountable, then

$$\bigcup U_j^c = \left(\bigcap U_j\right)^c$$

is countable, hence the intersection $\bigcap U_j$ must be uncountable (hence infinite). Pick $y \neq x$, where y belongs in the intersection of all neighbourhoods U_j. This contradicts the fact that $\{U_j\}$ is a neighbourhood base, as x is an element in the open set $\{y\}^c$ therefore there must be a U_k

$$x \in U_k \subseteq \{y\}^c$$

But $y \in U_k$ for each U_k and the proof is complete. ■

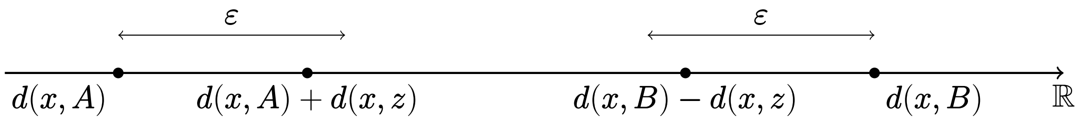


Figure 2: Exercise 4.3: Finding a ε small enough that fits within $d(x, A) < d(x, B)$

Exercise 4.3

Proposition 3.1

Every metric space is normal. (If A, B are disjoint closed sets in the metric space, consider the set of points x where $d(x, A) < d(x, B)$ or $d(x, A) > d(x, B)$).

Proof. First, we show that if A is closed, then $d(x, A) = 0 \iff x \in A$. If $x \in A$, then $d(x, A) \leq d(x, x) = 0$. if $x \notin A$, then there exists a ball of radius $\varepsilon > 0$ where $B(\varepsilon, x) \cap A = \varnothing$. Hence, ε is a lower bound for the set $\{d(x, y), y \in A\}$, taking the infimum over this set we see that $d(x, A) \geq \varepsilon > 0$.

Fix some $x \in \Phi_A$ where $\Phi_A = \left\{ y \in \mathbf{X}, d(y, A) < d(y, B) \right\}$. We wish to find an open ball about x that is contained in Φ_A. The Triangle Inequality works for this definition of distance as well, as

$$f(a) \leq g(a), \forall a \in A \implies \inf_{a \in A} f(a) \leq \inf_{a \in A} g(a) \tag{14}$$

If $a \in A$, then $d(z, A) \leq d(x, z) + d(x, A)$, using Equation (14) yields

$$\begin{cases} d(z, A) \leq d(x, A) + d(x, z) \\ d(x, B) - d(x, z) \leq d(z, B) \end{cases}$$

where $z \in B(\varepsilon, x)$ so $d(x, z) \precsim \varepsilon$. The second estimate above is found by 'flipping an upper bound to become a lower bound'. We can choose $d(x, z)$ sufficiently small that

$$d(x, A) + d(x, z) < d(x, B) - d(x, z)$$

in order to 'pipe' the two inequalities, so

$$2d(x, z) < d(x, B) - d(x, A) \tag{15}$$

Take $\varepsilon = [d(x, B) - d(x, A)]3^{-1}$, then $z \in B(\varepsilon, x)$ implies $d(x, z) < \varepsilon$, and Equation (15) holds. See Figure 2 for details.

■

Exercise 4.4

Proposition 4.1

Proof. ■

Exercise 4.5

Proposition 5.1

Every separable metric space is second countable.

Proof. We wish to show that if $\mathbf{X}$ is a metric space, then

$$\text{second countable} \iff \text{separable}$$

Suppose $\mathbf{X}$ is separable, where A is a countable dense subset in $\mathbf{X}$, and $x \in \mathbf{X}$. Let U be an open set that contains x, so $B(\varepsilon,\ x) \subseteq U$ for some $\varepsilon > 0$. $B(\varepsilon/2,\ x)$ is a non-empty open set, therefore contains some $y \in A$ (this follows from the definition of density). If we choose $r \in \mathbb{Q}$ wisely,

$$d(x, y) < r < \varepsilon/2$$

So that $x \in B(r,\ y)$, and if $z \in B(r,\ y)$, then

$$d(x, z) \leq d(x, y) + d(z, y) < r + r < \varepsilon$$

So $x \in B(r,\ y) \subseteq U$. But $\{B(r, y),\ r \in \mathbb{Q}^+,\ y \in A\}$ is countable. Therefore $\mathbf{X}$ is second countable.

Conversely, Proposition 4.5 gives us the $\Longleftarrow$ direction. But we will repeat anyway, if $\mathbf{X}$ is second-countable with $\mathcal{E}$ as a countable base, then take

$$W = \left\{ x_\alpha \in U, \quad U \in \mathcal{E} \right\}$$

by picking a point from each set, we claim W is dense in $\mathbf{X}$, so $\overline{W} = \mathbf{X}$. If not, then $\overline{W}^c \neq \varnothing$, and

$$\overline{W}^c = (W^c)^o \neq \varnothing$$

Pick a point $x \in W^{co}$, which is an open set containing x. But the way we chose W does not allow for any open set $U \in \mathcal{E}$ with $x \in \mathcal{E} \subseteq W^{co}$, since

> By picking one point from each of the base sets, grouping these points and call it W, and flipping to the complement. Each $U \in \mathcal{E}$ admits a point that escapes W^{co}. Therefore we can ensure no $U \in \mathcal{E}$ can be a subset of W^{co}.

■

Exercise 4.6

Proposition 6.1

Proof. ■

Exercise 4.7

Proposition 7.1

If $\mathbf{X}$ is a topological space, a point $x \in \mathbf{X}$ is called a cluster point of the sequence $\{x_j\}$ if for every neighbourhood $U \in \mathcal{N}(x)$, $x_j \in U$ for infinitely many j. If $\mathbf{X}$ is first countable, x is a cluster point of $\{x_j\}$ iff some subsequence of $\{x_j\}$ converges to x.

Proof. Suppose $\{x_n\}$ has a cluster point in $z \in \mathbf{X}$. Fix a descending sequence of neighbourhoods $U_k \subseteq \mathcal{N}(z)$, where

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k$$

Define $n_k = \text{least}\left\{j \in \mathbb{N}^+,\ j > n_{k-1},\ x_j \in U_k\right\}$ with $n_0 = 0$, so that for every $m \geq k$, $x_{n_m} \in U_k$ eventually. And $\{x_{n_j}\}_{j\geq 1}$ is a subsequence which converges to z. This proves ($\Longrightarrow$).

Conversely (this part does not require that $\mathbf{X}$ be first countable), if $\{x_{n_k}\}_{k\geq 1}$ is a subsequence that converges to $z \in \mathbf{X}$. Every neighbourhood of z must intersect all but infinitely many x_{n_k}, therefore z is a cluster point of $\{x_n\}$. ■

Exercise 4.8

Proposition 8.1

If $\mathbf{X}$ is an infinite set with the cofinite topology and $\{x_j\}$ is a sequence of distinct points in $\mathbf{X}$, then $x_j \to x$ for every $x \in \mathbf{X}$.

Proof. The intuition here is that the cofinite topology does not distinguish between points, so it acts as a type of jelly that hides the points.

Let $x \in \mathbf{X}$ be arbitrary, if $U \in \mathcal{N}(x)$ then $U^o \in \mathcal{N}(x)$, so that $\{y_j\}_{j \leq k}$ are the k points that are required to extend U^o to $\mathbf{X}$. (All but finitely many points are in any open set of $\mathbf{X}$).

There exists a large $N \in \mathbb{N}^+$ so that for every $n \geq N$,

$$x_j \notin \{y_j\}_{j \leq k} \implies x_j \in U^o$$

eventually. And $x_j \to x$. ■

Exercise 4.9

Proposition 9.1

Proof. ■

Exercise 4.10

Proposition 10.1

A topological space $\mathbf{X}$ is called disconnected if there exists non-empty, disjoint open sets U, V and $U \cup V = \mathbf{X}$; otherwise $\mathbf{X}$ is connected. When we speak of connected or disconnected subsets of $\mathbf{X}$, we refer to the relative topology on them

(a) $\mathbf{X}$ is connected iff $\varnothing$ and $\mathbf{X}$ are the only two clopen sets.

(b) If $\{E_\alpha\}_{\alpha \in A}$ is a collection of connected subsets of $\mathbf{X}$, and $bigcapE_{\alpha \in A}$ is non-empty, then $\bigcup E_{\alpha \in A}$ is connected.

(c) If $A \subseteq \mathbf{X}$ is connected, then $\overline{A}$ is connected,

(d) Every point in $x \in \mathbf{X}$ contained in a unique maximal connected subset of $\mathbf{X}$, and this subset is closed. It is called the connected component of x.

Proof. The proof is rather long, so we will split it in several parts. A topological space is disconnected iff it can be written as a disjoint union of two non-empty open sets. Often it is easier to show that a space is disconnected rather than connected.

Part A: Suppose $\mathbf{X}$ is disconnected, this induces a pair of non-empty open sets, A, and B whose union is $\mathbf{X}$, and

$$A \cap B = \varnothing \iff A \subseteq B^c$$

their union is $\mathbf{X}$, hence

$$A \cup B = \mathbf{X} \iff B^c \subseteq A$$

combining the last two estimates, we see that $B = A^c$, so both A and $A^c = B$ are closed. This proves ($\Longleftarrow$).
Now suppose $\{A, A^c\} \neq \{\varnothing, \mathbf{X}\}$ are both clopen. Clearly A is disjoint from its complement, and their union is $\mathbf{X}$.

Part B: We will attempt the contrapositive. Suppose $E = \bigcup E_{\alpha \in A}$ is disconnected. This induces D and D^c which are clopen in the relative topology of E, (by Part A). More precisely,

$$\bigcup E_\alpha = \underbrace{\bigcup (E_\alpha \cap D)}_{\neq \varnothing} + \underbrace{\bigcup (E_\alpha \setminus D)}_{\neq \varnothing} \tag{16}$$

The intersection $\bigcap E_{\alpha \in A}$ is non-trivial, hence

$$\bigcap E_\alpha = \underbrace{\bigcap (E_\alpha \cap D)}_{\neq \varnothing} + \bigcap (E_\alpha \setminus D) \neq \varnothing \tag{17}$$

so at least one of the members on the right are non-empty. Assume without loss of generality that $\bigcap (E_\alpha \cap D)$ is not empty. This tells us $E_\alpha \cap D = \neq \varnothing$ for each $\alpha \in A$. But by Equation (16), if we concentrate on the right member,

$$\bigcup(E_\alpha \setminus D) \neq \varnothing \implies \exists \beta \in A,\ E_\alpha \setminus D \neq \varnothing$$

And for this particular $\beta \in A$, we see that both D and D^c are non-trivially open in E_β, and the proof is complete. A poetic way to summarize the proof would be:

> If the whole is disconnected, and there exists common ground over which the family of sets covers, and because the common ground (intersection) is non-trivial, either D or D^c is non-trivially open in all E_α. The intersection gives us "$\forall$", while the union gives us "$\exists$" for a non-trivially open D or D^c.

There is an alternate way of proving Part B, without using the clopen definition of connectedness. Let C and D be non-empty, disjoint, open sets in $\bigcup E_\alpha$ whose union is $\bigcup E_\alpha$.

$$\bigcap E_\alpha = \bigcap[E_\alpha \cap C] + \bigcap[E_\alpha \cap D] \neq \varnothing$$

Pick $p \in \bigcap E_\alpha$, without loss of generality, assume $p \in \bigcap[E_\alpha \cap C]$, then for every α we have

$$p \in E_\alpha \cap C \implies E_\alpha \cap C \neq \varnothing$$

Since E_α is connected, $E_\alpha \cap D = \varnothing$ for each α. Taking the union over all $E_\alpha \cap D$, we see that

$$\bigcup[E_\alpha \cap D] = \varnothing$$

which contradicts the assumption $D \neq \varnothing$.
Part C: Suppose $\overline{A}$ is disconnected, this induces a non-trivial clopen set D relative to $\overline{A}$.

- Since $\overline{A} \cap D \neq \varnothing$, choose any $y \in \overline{A} \cap D \subseteq \overline{A}$, since D is a neighbourhood of y, and y is an adherent point of A. It is immediate that $A \cap D$ is non-empty.
- Similarly for $A \setminus D \neq \varnothing$,

therefore $\{D, D^c\}$ is non-trivially clopen in A, and A is disconnected.

Part D: The idea here is to use Part B. Let x be fixed, and $\{E_\alpha\}_{\alpha \in A}$ be the family of all connected sets containing x, since their intersection is non-trivial, their union, E is connected. The closure of their union is then the maximal connected component containing x. Indeed, if G is a connected set containing x, then $G \subseteq \bigcup E_\alpha = E$, so $G \subseteq \overline{E}$. ■

Exercise 4.11

Proposition 11.1

If $E_1, \dots E_n$ are subsets of a topological space, the closure of $\bigcup_1^n E_j$ is $\bigcup_1^n \overline{E_j}$

Proof. The finite union of closed sets is again closed, so

$$\forall j \leq n,\ E_j \subseteq \overline{E_j} \implies \overline{\bigcup_1^n E_j} \subseteq \bigcup_1^n \overline{E_j}$$

For the reverse estimate, $E_j \subseteq \bigcup_1^n E_j \subseteq \overline{\bigcup_1^n E_j}$ is a closed set that contains each E_j, therefore

$$\forall j \leq n,\ \overline{E_j} \subseteq \overline{\bigcup_1^n E_j} \implies \bigcup_1^n \overline{E_j} \subseteq \overline{\bigcup_1^n E_j}$$

■

Corollary 11.1

The interior operator distributes over intersections. If A and B are subsets of $\mathbf{X}$, then

$$\overline{(A^c \cup B^c)} = (\overline{A^c} \cup \overline{B^c})$$
$$\left(\overline{(A^c \cup B^c)}\right)^c = A^o \cap B^o$$
$$\left(A^c \cup B^c\right)^{co} = A^o \cap B^o$$
$$(A \cap B)^o = A^o \cap B^o$$

Exercise 4.12

Proposition 12.1

Let $\mathbf{X}$ be a set. A Kuratowski closure operator on $\mathbf{X}$ is a map $A \mapsto A^*$ from $\mathbb{P}(\mathbf{X})$ to itself satisfying

(i) $\varnothing^* = \varnothing$ (does nothing to the empty set),

(ii) $A \subseteq A^*$ (monotonicity),

(iii) $(A^*)^* = A^*$ (idempotence)

(iv) $(A \cup B)^* = A^* \cup B^*$ (distributes over finite unions)

Prove

(a) If $\mathbf{X}$ is a topological space, the map $A \mapsto \overline{A}$ is a Kuratowski closure operator. (Use Exercise 11.)

(b) Conversely, given a Kuratowski closure operator, let $\mathcal{F} = \{A \subseteq \mathbf{X},\ A = A^*\}$ and $\mathcal{T} = \{U \subseteq \mathbf{X},\ U^c \in \mathcal{F}\}$, then $\mathcal{T}$ is a topology on $\mathbf{X}$, and for any set $A \subseteq \mathbf{X}$, A^* will be its closure with respect to $\mathcal{T}$.

Proof. Part A: The empty set is closed, so $\overline{\varnothing} = \varnothing$, and $\overline{A}$ is the smallest closed superset of A, so $A \subseteq \overline{A}$ for every $A \subseteq \mathbf{X}$. $A \subseteq \mathbf{X}$ is closed iff $\overline{A} = A$, so idempotence holds. Distributivity follows from Exercise 11 directly.

Part B: We first show that $\mathcal{T}$ is indeed a topology. Fix U_1 and U_2 in $\mathcal{T}$, so that $U_1^c \cup U_2^c = (U_1 \cap U_2)^c$. The map $A \mapsto A^*$ distributes over finite unions, hence

$$(U_1^c \cup U_2^c)^* = (U_1^c)^* \cup (U_2^c)^* = U_1^c \cup U_2^c$$

Therefore $U_1 \cap U_2 \in \mathcal{T}$. Now suppose $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, then

$$\left(\bigcup U_\alpha \right)^c = \bigcap U_\alpha^c$$

by monotonicity (Property ii): $\bigcap U_\alpha^c \subseteq \left(\bigcap U_\alpha^c \right)^*$. To prove the reverse inclusion, notice if α is held fixed,

$$\bigcap U_\alpha^c \subseteq U_\alpha^c \implies \left(\bigcap U_\alpha^c \right)^* \subseteq U_\alpha^{c*}$$

this follows from 'monotonicity' of the closure operator: if A is a subset of B, then we can write

$$B = A + (B \setminus A) \implies A^* \subseteq A^* + (B \setminus A)^* = B^*$$

Take the intersection over all $\alpha \in A$ on the right member,

$$\left(\bigcap U_\alpha^c\right)^* \subseteq \bigcap U_\alpha^{c*} = \bigcap U_\alpha^c$$

Hence $\left(\bigcap U_\alpha^c\right)^* = \bigcap U_\alpha^c$. The empty set and $\mathbf{X}$ are elements of $\mathcal{F}$. Since $\mathbf{X} \subseteq \mathbf{X}^* \subseteq \mathbf{X}$, and $\{\varnothing, \mathbf{X}\} \subseteq \mathcal{T}$. So $\mathcal{T}$ is a topology.

Finally, A^* is a closed superset of A and suppose K is another closed superset,

$$A \subseteq K \implies A^* \subseteq K^*$$

So A^* is the smallest closed superset of A and this proves the last claim. ■

Exercise 4.13

Proposition 13.1

If $\mathbf{X}$ is a topological space, U is open in $\mathbf{X}$ and A is dense in $\mathbf{X}$, then $\overline{U} = \overline{U \cap A}$.

Proof. The takeaway here is that if A is dense in $\mathbf{X}$, every point $z \in U$ can be approximated by points in $U \cap A$. And and important technique of 'demoting' the neighbourhood to become the interior of the neighbourhood can yield some nice properties. Since the interior of a neighbourhood is again a neighbourhood. This allows intersection with open sets to inherit the 'neighbourhood-ness' of the set.

Let $z \in \overline{U}$, and fix a neighbourhood $V \in \mathcal{N}(z)$, so that the interior of V is also a neighbourhood. By the alternate definition of $\overline{U}$ in terms of adherent points (see Proposition 9.1) of $\overline{U}$, $V^o \cap U \neq \varnothing$. This is a non-empty open set, therefore it must intersect A non-trivially.

$$x \in (V^o \cap U) \cap A = V^o \cap (U \cap A)$$

and $z \in \overline{U \cap A}$.

■

Remark 13.1

We simply used the fact

$$\overline{E} = \left\{ x \in \mathbf{X},\ \forall V \in \mathcal{N}(x),\ V \cap E \neq \varnothing \right\}$$

and the following equivalent characterization of density

$$E \text{ is dense in } \mathbf{X} \iff \text{For every non-empty open set } U,\ U \cap E \neq \varnothing$$

Exercise 4.14

Proposition 14.1

If $\mathbf{X}$ and $\mathbf{Y}$ are topological spaces, $f : \mathbf{X} \to \mathbf{Y}$ is continuous iff $f(\overline{A}) \subseteq \overline{f(A)}$ for every $A \subseteq \mathbf{X}$ iff $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for all $B \subseteq Y$.

Proof. First Equivalence: If f is continuous, fix any $A \subseteq \mathbf{X}$, and $z \in \overline{A}$, by Proposition 9.1 (I will spare you the flipping by including):

$$\overline{A} = \{x \in \mathbf{X},\ \forall U \in \mathcal{N}(X),\ U \cap A \neq \varnothing\}$$

Let $U \in \mathcal{N}(f(z))$, so that $f^{-1}(U^o)$ is an open set containing z and $f^{-1}(U^o) \in \mathcal{N}(z)$, so

$$f^{-1}(U^o) \cap A \neq \varnothing \implies U^o \cap f(A) \subseteq U \cap f(A)$$

so $f(\overline{A}) \subseteq \overline{f(A)}$. Conversely, suppose $f(\overline{A}) \subseteq \overline{f(A)}$ holds for every $A \subseteq \mathbf{X}$. The following is a sequence of symbolic manipulations that I found but have zero intuitive understanding about. First take the inverse image

$$\overline{A} \subseteq f^{-1}\left(f(\overline{A})\right) \subseteq f^{-1}\left(\overline{f(A)}\right)$$

Next, let F be a closed set in $\mathbf{Y}$, and make the substitution $A = f^{-1}(F)$, hence

$$\overline{f^{-1}(F)} \subseteq f^{-1}\left(\overline{f(f^{-1}(F))}\right) \subseteq f^{-1}(\overline{F}) = f^{-1}(F)$$

for the second inclusion we used the monotonicity of the closure, and since $\overline{f^{-1}(F)} = f^{-1}(F)$, we are done.

Second Equivalence: Suppose $f \in C(\mathbf{X}, \mathbf{Y})$, then $\overline{B} \subseteq \mathbf{Y}$ is a closed set, so $f^{-1}(\overline{B})$ is closed in $\mathbf{X}$. By monotonicity of the inverse image,

$$f^{-1}(B) \subseteq f^{-1}(\overline{B}) \implies \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$$

Conversely, if $\overline{f^{-1}(B)} \subseteq f^{-1}(B)$ for any $B \subseteq \mathbf{Y}$, take any closed $B \subseteq \mathbf{Y}$, and

$$\overline{f^{-1}(B)} \subseteq f^{-1}(B) \subseteq \overline{f^{-1}(B)}$$

so $f^{-1}(B)$ is closed, and f is in $C(\mathbf{X}, \mathbf{Y})$. ∎

Exercise 4.16

Proposition 15.1

Proof.

(a) Let $x \in \{f \neq g\}$, then there exists disjoint open subsets of $\mathbf{Y}$, $f(x) \in U$ and $g(x) \in V$, $U \cap V = \varnothing$, but $f^{-1}(U) \cap g^{-1}(V)$ is an open set in $\mathbf{X}$ that contains x. Therefore $\{f \neq g\}$ is open in $\mathbf{X}$.

(b) Suppose $\{f = g\} = E$ is dense in $\mathbf{X}$. Let $x \in E$, induces two disjoint open sets exactly like in part a. This is an open set that contains x, and $y \in f^{-1}(U) \cap g^{-1}(V) \cap E$. Since $y \in E$, it follows that $f(y) = g(y)$, and

$$\begin{cases} y \in f^{-1}(U) \implies & f(y) \in U \\ y \in g^{-1}(V) \implies & g(y) \in V \end{cases}$$

■

Exercise 4.17

Chapter 5

Theorem 5.1

Proposition 1.1

Proof. ■

Theorem 5.2

Proposition 2.1

Proof. ∎

Theorem 5.3

Proposition 3.1

Proof. ■

Theorem 5.4

Proposition 4.1

Proof. ■

Theorem 5.5

Proposition 5.1

Proof. ■

Theorem 5.6

Proposition 6.1

Proof. ■

Theorem 5.7

Proposition 7.1

Proof. ■

Theorem 5.8

Proposition 8.1

Proof. ■

Chapter 6

Theorem 6.1

Proposition 1.1

For every $a, b \geq 0$, and $0 < \lambda < 1$, then

$$a^{\lambda} b^{1-\lambda} \leq \lambda a + (1-\lambda) b$$

Theorem 6.2

Proposition 2.1

Proof. ■

Theorem 6.3

Proposition 3.1

Proof. ■

Theorem 6.4

Proposition 4.1

Proof. ∎

Theorem 6.5

Proposition 5.1

Proof. ■

Theorem 6.6

Proposition 6.1

Theorem 6.7

Proposition 7.1

Theorem 6.8

Proposition 8.1

Theorem 6.9

Proposition 9.1

Proof. ∎

Theorem 6.10

Proposition 10.1

Proof. ■

Theorem 6.11

Proposition 11.1

Proof. ■

Theorem 6.12

Proposition 12.1

Proof. ∎

Theorem 6.13

Proposition 13.1

Theorem 6.14

Proposition 14.1

Theorem 6.15

Proposition 15.1

Proof. First suppose that $(X, \mathcal{M}, \mu)$ is finite measure space. If $\mu(X) < +\infty$, then for every $E \in \mathcal{M}$, by monotonicity $E \subseteq X$ yields $\mu(E) \leq \mu(X) < +\infty$. Next, for any $p < +\infty$, $\|\chi_E\|_p^p < +\infty$ and $\|\chi_E\|_{+\infty} \leq 1 < +\infty$. So all indicator functions are in L^p.

It follows that every simple function is also in L^p, since it is a finite linear combination of indicators. We now define $\nu(E) = \phi(\chi_E)$, we wish to show that $\nu : \mathcal{M} \longrightarrow \mathbb{C}$ is a complex measure which is absolutely continuous with respect to μ.

To show σ-additivity, fix any disjoint sequence $\{E_j\}_{j\geq 1} \subseteq \mathcal{M}$. Where we also note that $\mu(E) = \mu(\cup E_j) < +\infty$. Now suppose that $p < +\infty$, then the following converges in the p-norm

$$\chi_E = \sum_{j\geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left(\bigcup E_{j\leq n}\right) = \left(\bigcup E_{j\geq 1}\right) \setminus \left(\bigcup E_{j\leq n}\right) = \bigcup E_{j\geq n+1}$$

and define F_{n+1} as the rightmost member above. Then $\{F_{n\geq 1}\}$ is a decreasing sequence of sets. All sets are of finite measure, hence $\mu(E) - \mu(\cup E_{j\leq n}) = \mu(F_{n+1}) \to 0$.

Now, for any fixed $n \geq 1$,

$$\left|\chi_E - \sum \chi_{E_{j\leq n}}\right| = \left|\sum \chi_{E_{j\geq n+1}}\right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the pth power does not change pointwise, and

$$\left|\sum \chi_{E_{j\geq n+1}}\right|^p = \left|\sum \chi_{E_{j\geq n+1}}\right| = \sum \chi_{E_{j\geq n+1}}$$

Convergence in p-norm is given by

$$\left\|\chi_E - \sum \chi_{E_{j\leq n}}\right\| = \left\|\sum \chi_{E_{j\geq n+1}}\right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our $\phi \in L^{p*}$

$$\begin{aligned}
\nu(E) &= \phi(\chi_E) \\
&= \phi\left(\lim_{n\to\infty} \sum \chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \phi\left(\sum \chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \sum \phi\left(\chi_{E_{j\le n}}\right) \\
&= \lim_{n\to\infty} \sum \nu(E_{j\le n})
\end{aligned}$$

To show absolute convergence, recall that for any $\phi(\chi_{E_j}) \in \mathbb{C}$, define $\beta_j = \overline{\operatorname{sgn}(\left\|\phi(\chi_{E_j})\right\|)}$ then multiplication yields

$$\left\|\phi(\chi_{E_j})\right\| = \beta_j\phi(\chi_{E_j}) = \phi(\beta_j\chi_{E_j})$$

Then, the following series converges in the p-norm.

$$\left\|\sum_{j\ge1} \beta_j\chi_{E_j} - \sum_{j\le n} \beta_j\chi_{E_j}\right\|_p = \left\|\sum_{j\ge n+1} \beta_j\chi_{E_j}\right\|_p$$

And because $\left|\sum_{j\ge n+1} \beta_j\chi_{E_j}\right|$ is pointwise equal to $\left|\sum_{j\ge n+1} \chi_{E_j}\right|$, since $|\beta_j| = 1$ for every $j \ge 1$. We can reuse the same continuity and linearity argument. We also note that $\sum_{j\ge1} \beta_j\chi_{E_j} \in L^p$ since its p-norm is equal to $\mu(E)^{1/p}$.

$$\begin{aligned}
\sum_{j\ge1} |\nu(E_j)| &= \sup_{n\ge1} \sum_{j\le n} \|\nu(E_{j\le n})\| \\
&= \lim_{n\to\infty} \sum_{j\le n} \|\phi(\chi_{E_j})\| \\
&= \lim_{n\to\infty} \sum_{j\le n} \beta_j\phi(\chi_{E_j}) \\
&= \lim_{n\to\infty} \phi\left(\sum_{j\le n} \beta_j\chi_{E_j}\right) \\
&= \phi\left(\lim_{n\to\infty} \sum_{j\le n} \beta_j\chi_{E_j}\right) \\
&\le \|\phi\| \left\|\sum_{j\ge1} \beta_j\chi_{E_j}\right\|_p \\
&< +\infty
\end{aligned}$$

Assuming the above estimate holds, then we only need $\nu(E) = \phi(\chi_E) = \mu(E) = 0$ (ν is now a measure and $\nu \ll \mu$), As the indicator of a null set is equal to the zero element in L^p. Then by Radon-Nikodym we can have some $g \in L^1(\mu)$ such that

$$d\nu = gd\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function g. For every χ_E measurable, $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$, by monotonicity of the integral in L^+. So any simple function, $\alpha = \sum a_j \cdot \chi_{E_j}$ means that αg is in $L^1(\mu)$, and

$$\phi(\alpha) = \int \alpha g d\mu$$

If $\|\alpha\|_p = 1$, then

$$\left|\int \alpha g\right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$$M_q(g) = \sup\left\{\left|\int \alpha \cdot g\right|, \ \|\alpha\|_p = 1, \quad \begin{array}{l}\text{and } \alpha \text{ is simple, and vanishes out-}\\ \text{side a set of finite measure.}\end{array}\right\} < \infty$$

Since $S_g = \{x \in X, g(x) \neq 0\}$ is σ-finite, an application of Theorem 6.14 tells us that $g \in L^q$, and $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$. Now that we know g is in L^q we can use the density of α in L^p to show, for every single $f \in L^p$

$$\phi(f) = \int fgd\mu$$

Conjure a sequence of 'α's, and call them $\{f_n\} \to f$ p.w.a.e, then each $f_n \cdot g \in L^1$. An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n \, gd\mu = \int fgd\mu = \phi(f)$$

This completes the proof for when μ is finite.

Let us upgrade our μ into a σ-finite measure. Then there exists an increasing sequence $\{E_n\} \nearrow X$ such that each E_n is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in P_n vanishes outside a set of finite measure and is also in L^p. And Q_n is defined in a similar manner. Now, fix our $\phi \in L^{p*}$, and for each $f \in P_n$, there exists a corresponding $g_n \in Q_n$. Then $p \in [1, +\infty)$ tells us that $q \in (1, +\infty]$, and the assumptions for Theorem 6.13 all hold. Therefore for each $g_n \in Q_n$, there is a corresponding bounded linear operator $\phi_{g_n} \in (P_n)^*$ such that

$$\phi(f) = \phi|_{P_n}(f) = \int fg_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of g_n towards some $g \in L^q$. We claim that this limit makes sense. As for any $n < m$, such that $E_n \subseteq E_m$ then $g_n = g_m$ on E_n pointwise. The proof is simple since each the restriction of our $\phi \in L^{p*}$ onto E_n and E_m spawns two functions g_n and $g_m \in L^1$. To verify, take any subset $Z \subseteq E_n$ then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So $g_n = g_m$ pointwise a.e on E_n. Now we define g measurable such that $g|_{E_n} = g_n$ for every n. And

$$\begin{aligned} |g_n| = \chi_{E_n} \cdot |g_m| &\implies \\ |g_n| \leq |g_{n+1}| &\implies \\ &\|g_n\|_q \leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty \end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with $(P_n \subseteq P_{n+1})$. If $q = +\infty$ then $g \in L^\infty$ is trivial, but for any $q < +\infty$. We wish to show that $g \in L^q$. Since $|g_n| \leq |g|$ pointwise for every n, and for each $x \in X$, there exists a N, where $n \geq N$ implies $|g(x)| = |g_n(x)|$, so $|g(x)|$ is an upperbound that is actually attained by the sequence $|g_n(x)|$. So, $|g(x)| = \sup_{n\geq 1}\{|g_n(x)|\}$.

Using the Monotone Convergence Theorem on $|g_n|$,

$$\begin{aligned} \int \lim_{n\to\infty} |g_n|^q d\mu &= \int \sup_{n\geq 1} |g_n|^q d\mu \\ &= \int |g|^q d\mu \\ &= \lim \int |g_n|^q d\mu \end{aligned}$$

Which yields $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$. It follows that $g \in L^q$.

Finally, we will show that $\phi(f) = \int fg$ for every $f \in L^p$. Redefine $f_n = f \cdot \chi_{E_n} \in P_n$ for every $n \geq 1$. We claim that $f_n \to f$ in the p-norm.

$$\begin{aligned} |f_n - f| &\leq |f_n| + |f| \\ &\leq |f| + |f| \\ &\leq 2|f| \end{aligned}$$

And $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$. Now it is permissible to apply the Dominated Theorem, and we will do so.

$$\begin{aligned} \lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\ \lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\ &= 0 \end{aligned}$$

And we have $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}\phi(f) &= \lim \phi|_{P_n}(f_n) \\ &= \lim \int f_n \cdot g_n \\ &= \lim \int f \cdot g \cdot \chi_{E_n} \\ &= \int \lim (fg \cdot \chi_{E_n}) \\ &= \int fg\end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of $fg\chi_{E_n} \to fg$ pointwise and Holder's Inequality. This completes the proof for when μ is of σ-finite measure, and $p \in [1, +\infty)$.

Suppose now μ is arbitrary, and $p \in (1, +\infty)$, then $q < +\infty$. Now let us agree to define, for every σ-finite $E \subseteq X$

$$P_E = \{L^p, |f| = |f| \cdot \chi_E\}$$

Where Q_E does not hold any surprises. Then for each E we have a $\phi|_E$ which induces a g_E that vanishes outside E. We are ready for the final part of the proof.

First, if $E \subseteq F$ and both E and F are σ-finite, then $\|g_E\|_q \leq \|g_F\|_q$. This is a simple consequence of monotonicity in L^+ if we take $|g_E|^q \leq |g_F|^q$.

Second, we define

$$W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}$$

Let M be the supremum of W, then there exists a sequence of σ-finite sets, $\{E_n\}$ where $\|g_{E_n}\|_q \to M \leq \|\phi\|_{p*}$. Take a set $F = \cup E_{n\geq 1}$, which is also σ-finite, so that $\|g_F\|_q = M$. Now assume there exists another σ-finite superset of F, let us call it A. Then

$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have $g_{A\setminus F} = 0$ pointwise a.e. For any $f \in L^p$, the spots where f does not vanish is σ-finite. This comes from $\int |f|^p < +\infty$. So it suffices to integrate over this σ-finite set. But we already know, even if this set A contains F as a subset, $\int fg_F = \int fg_A$.

We now define $g = g_F$, and the proof is complete. As for every $\phi \in L^{p*}$, there exists a $g \in L^q$ such that the evaluation of any $f \in L^p$ is given by integrating f with g. ■

Chapter 7

Theorem 7.1

Proposition 1.1

If I is a linear functional on $\mathrm{C}_c(X)$, then for every compact $K \subseteq X$, there exists some $C_k \geq 0$ with

$$|I(f)| \leq C_K \cdot \|f\|_u$$

Proof. Since $\operatorname{supp}(f)$ is compact, by Urysohn's Lemma (Theorem 4.32), there exists a $\phi \in \mathrm{C}_c(X, [0,1])$ such that $\phi = 1$ on K and vanishes outside some compact $\overline{V} \subseteq X$. Then at every x,

$$-\|f\|_u \leq f(x) \leq +\|f\|_u$$

Implies that

$$(-\|f\|_u)\phi \leq f(x) \leq (+\|f\|_u)\phi$$

So that $f + \|f\|_u\phi \geq 0$ and $+\|f\|_u - f \geq 0$, and by linearity,

$$(-\|f\|_u)I(\phi) \leq I(f) \leq (+\|f\|_u)I(\phi)$$

Therefore $|I(f)| \leq I(\phi)\|f\|_u$, and taking $C_K = I(\phi)$ will suffice. ■

Theorem 7.2

Proposition 2.1

The Riesz-Markov-Kakutani Representation Theorem. If (for every) I is a positive linear functional on $C_c(X)$, then there exists a unique Radon measure μ on X, such that

$$I(f) = \int f d\mu$$

for every $f \in C_c(X)$. μ also satisfies, for every open U, and every compact $K \subseteq X$

$$\mu(U) = \sup\{I(f),\ f \in C_c(X),\ f \prec U\} \tag{18}$$

$$\mu(K) = \inf\{I(f),\ f \in C_c(X),\ f \geq \chi_K\} \tag{19}$$

For the sake of completeness, we place the definitions for a Radon measure. Let X be a LCH space, and $\mathbb{B}$ be its usual σ-algebra, a measure ν is a Radon measure iff

(i) $\nu(K) < +\infty$ for every compact K.

(ii) ν is outer-regular on all Borel sets E,

$$\nu(E) = \inf\{\nu(U),\ U \supseteq E,\ U \in \mathcal{T}\}$$

Intuition: approximation by open supersets.

(iii) ν is inner-regular on all open sets $U \in \mathcal{T}$,

$$\nu(U) = \sup\{\mu(K),\ K \subseteq U,\ K \text{ compact}\}$$

Intuition: approximation by compact subsets

The main proof is extremely long, so we will divide it into several parts. Following Folland's argumentation closely, we will prove (in order)

(a) If μ_1, μ_2 are Radon measures on X such that for every $f \in C_c(X)$

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

then μ_1, μ_2 must satisfy (18), and $\mu_1 = \mu_2$ on $\mathbb{B}$.

(b) If we define, for every open set U, define $\mu : \mathcal{T} \to [0, +\infty]$ such that

$$\mu(U) = \sup\{I(f),\ f \in C_c(X),\ f \prec U\} \tag{20}$$

Then μ is countably subadditive, meaning for every $U \in \mathcal{T}$, $\{U_{j\geq 1}\} \subseteq \mathcal{T}$

$$U = \bigcup U_{j\geq 1} \implies \mu(U) \leq \sum \mu(U_{j\geq 1})$$

(c) $\mu(\varnothing) = 0$, $\{\varnothing, X\} \subseteq \mathcal{T}$, so that by Theorem 1.10 μ induces an outer-measure μ^*

$$\mu^*(E) = \inf\left\{\sum \mu(U_{j\geq 1}),\, U_j \in \mathcal{T},\, E \subseteq \bigcup U_{j\geq 1}\right\} \tag{21}$$

(d) If μ^* is as described above, then if μ is countably subadditive on $\mathcal{T}$, then

$$\mu^*(E) = \inf\{\mu(U),\, U \supseteq E,\, U \in \mathcal{T}\} \tag{22}$$

Meaning the two definitions in (21) and (22) are equal.

(e) μ^* and μ agree on all open sets, and $\mu^*|_{\mathcal{T}} = \mu$,

(f) Using again the definition in (21) and (22), we show that every open set $U \in \mathcal{T}_X$ is μ^*-measurable, meaning for every $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

With this, since the set of all outer-measurable (μ^*-measurable) sets, $\mathcal{M}^*$ form a σ-algebra,

$$\mathcal{T} \subseteq \mathcal{M}^* \implies \mathbb{B} \subseteq \mathcal{M}^*$$

By Theorem 1.1, and define

$$\mu = \mu^*|_{\mathbb{B}} \tag{23}$$

is a Borel measure. And we note in passing that μ is outer-regular on all $E \in \mathbb{B}$,

$$\mu(E) = \inf\{\mu(U),\, U \supseteq E,\, U \in \mathcal{T}\} \tag{24}$$

(g) Using (23) for the definition of μ on $\mathbb{B}$, we prove that

- μ is outer-regular on all Borel sets, and
- μ satisfies Equation (18).

(h) μ satisfies Equation (19)

(i) μ is finite on all compact sets.

(j) μ is inner-regular on all open sets.

(k) For every $f \in \mathrm{C}_c(X, [0,1])$,

$$I(f) = \int f d\mu \tag{25}$$

(l) For every $f \in \mathrm{C}_c(X)$,

$$I(f) = \int f d\mu \tag{26}$$

A small lemma needs to be made before proceeding, that concerns the 'monotonicity' of I on $C_c X$.

Lemma 2.1

Suppose that $f,\, g \in \mathrm{C}_c(X)$, and $f \geq g \geq 0$ for every X, then $f - g \in \mathrm{C}_c(X)$ and $I(f) \geq I(g)$

Proof. Suppose that $x \in X$ where $f(x) = 0$, then

$$f(x) - g(x) = -g(x) \geq 0 \implies g(x) = 0 \implies f - g = 0$$

Hence

$$\begin{aligned}\{x,\ f(x) = 0\} \subseteq \{x,\ f(x) - g(x) = 0\} &\implies \{x,\ f(x) - g(x) \neq 0\} \subseteq \{x,\ f(x) \neq 0\} \\ &\implies \operatorname{supp}(f - g) \subseteq \operatorname{supp}(f)\end{aligned}$$

Since $\operatorname{supp}(f)$ is compact, and $\operatorname{supp}(f - g)$ is a closed subset of $\operatorname{supp}(f)$, yields $f - g \in \mathrm{C}_c(X)$. And if I is any positive linear functional on $\mathrm{C}_c(X)$, then

$$\begin{aligned}f - g \geq 0 &\implies I(f - g) \geq 0 \\ &\implies I(f) \geq I(g) \geq 0\end{aligned}$$

■

Remark 2.1

If $f \prec U$ and $g \prec U$ for some open subset $U \subseteq X$, then clearly $\operatorname{supp}(f - g) \subseteq \operatorname{supp}(f) \subseteq U$, and $1 \geq f \geq f - g \geq 0$ means that $f - g \prec U$ as well.

Part a

Proof. Suppose that μ_1 and μ_2 are Radon measures on X, and for every $f \in \mathrm{C}_c(X)$,

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

We first prove (18). Without loss of generality, by monotonicity of L^+, if $f \prec U$ for some open U, then $0 \leq f \leq \|f\|_u \chi_U = \chi_U$ for all x and

$$\int f d\mu_1 \leq \int \|f\|_u \chi_U \, d\mu_1 \leq \mu_1(U)$$

Therefore $\mu_1(U)$ (resp. $\mu_2(U)$) is an upper-bound for the set

$$\{I(f),\ f \in \mathrm{C}_c(X),\ f \prec U\}$$

Since μ_1 is inner-regular on $U \in \mathcal{T}$, for every $\varepsilon > 0$ we can find some compact $K \subseteq U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K)$$

By Urysohn's Lemma (Theorem 4.32), there exists some $g \in \mathrm{C}_c(X)$ with

- $g \in \mathrm{C}_c(X, [0,1])$,
- $g = 1$ on $K \subseteq U$,
- $g = 0$ outside some $\overline{V} \subseteq U$, and
- $g \prec U$.

Hence for every $x \in K$, $g \geq \chi_K$. If $x \notin K$ then $g \geq 0 = \chi_K$; so $g - \chi_K \geq 0$ for every $x \in X$. Since $\chi_K \prec U$, using Lemma 2.1, we get

$$\mu_1(K) \leq \int \chi_K \, d\mu_1 = I(\chi_K) \leq I(g)$$

So for every $\varepsilon > 0$, there exists a $g \in \mathrm{C}_c(X)$, and $g \prec U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K) \leq I(g)$$

Therefore $\mu_1(U) = \sup\{I(f),\ f \in \mathrm{C}_c(X),\ f \prec U\}$, and the first claim in (a) is proven. To show that μ is indeed unique, since for every open set U, we must have $\mu_1(U) = \mu_2(U)$, and if $E \in \mathbb{B}$ is any Borel set, and by outer-regularity,

$$\mu_1(E) = \inf\{\mu_1(U), U \supseteq E, U \in \mathcal{T}\} = \inf\{\mu_2(U), U \supseteq E, U \in \mathcal{T}\} = \mu_2(E)$$

Therefore this measure is unique. ■

Part b

Proof. To show countable subadditivity for μ with equation (20), fix any $U \in \mathcal{T}$ and a sequence $\{U_{j\geq 1}\} \subseteq \mathcal{T}$ with $U = \bigcup U_{j\geq 1}$. It suffices to show that the partial sum of $\sum \mu(U_{j\leq n})$ is greater than $I(f)$ for any $f \in \mathrm{C}_c(X)$, $f \prec U$ (hence it is an upper bound).

Fix any f, then denote $K = \operatorname{supp}(f) \subseteq U$, and since $\{U_{j\geq 1}\}$ is an open cover for K, there exists a finite subcollection, $B \subseteq \mathbb{N}^+$ such that

$$K \subseteq \bigcup_{j \in B} U_j$$

Using Theorem 4.41 on this finite cover of K, there exists a partition of unity in $\{g_{j\leq n}\}$ where

- $g_j \in \mathrm{C}_c(X, [0,1])$,
- $g_j \prec U_j \subseteq U$ for every $j \leq n$, and
- $\sum g_j = 1$ on K,

And notice for every $j \leq n$,

$$\begin{aligned}
\{f = 0\} \cup \{g_j = 0\} \subseteq \{f \cdot g_j = 0\} &\implies \{f \cdot g_j \neq 0\} \subseteq \{f \neq 0\} \cap \{g_j \neq 0\} \\
&\implies \operatorname{supp}(f \cdot g_j) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g_j) \\
&\implies \operatorname{supp}(f \cdot g_j) \subseteq U_j \subseteq U
\end{aligned}$$

Hence $f \cdot g_j \prec U$ and $f \cdot g_j \in \mathrm{C}_c(X, [0,1])$ for every $1 \leq j \leq n$. Moreover, if we take the sum over a finite n, we obtain $f = \sum f \cdot g_{j \leq n}$, this is because for every $x \in X$, so we have

$$\sum_{j \leq n} f(x) \cdot g_j x = f(x) \cdot \sum_{j \leq n} g_j(x) = f(x)$$

Then $I(f) = I(\sum f \cdot g_j) = \sum I(f \cdot g_j)$. And by definition of $\mu(U_j)$, since it is the supremum over all $I(h_j)$, where $h_j \in \mathrm{C}_c(X, [0,1])$ and $h_j \prec U_j$

$$I(f \cdot g_j) \leq \mu(U_j), \quad \forall j \leq n$$

Hence

$$I(f) \leq \sum_{j \leq n} \mu(U_j) \leq \sum_{j \geq 1} \mu(U_j)$$

Where for the last estimate we used the fact that μ is non-negative, and since this holds for any f, we can conclude that $\mu(U) \leq \sum_{j \geq 1} \mu(U_j)$. ■

Part c

Proof. By definition of a topology, $\{\varnothing, X\} \subseteq \mathcal{T}$, and $\mu(\varnothing) = \sup\{I(f),\ f \in \mathrm{C}_c(X),\ f \prec \varnothing\}$, so $\operatorname{supp}(f) = \varnothing$, and $\{x,\ f(x) \neq 0\} \subseteq \varnothing$, so the set contains one element, namely $I(0) = 0$ by linearity. So $\mu(\varnothing) = 0$. The assumptions for Theorem 1.10 are satisfied and (21) is indeed an outer-measure. ■

Part d

Proof. Denote the right members of (21) and (22) by W_1 and W_2, we wish to show that $\inf W_1 = \inf W_2$. Clearly $\inf W_1 \leq \inf W_2$, since $W_2 \subseteq W_1$. Now, if μ is countably additive, then for every $\omega \in W_1$ induces a sequence of open sets $\{U_{j \geq 1}\}$ such that $E \subseteq \bigcup U_{j \geq 1}$. Denote the union over $\{U_{j \geq 1}\}$ by U, which is also another open set,

$$\inf W_2 \leq \mu(U) \leq \sum \mu(U_{j \geq 1}) = \omega$$

Since ω is arbitrary, we conclude that $\inf W_2 = \inf W_1$, and this proves (d). ■

Part e

Proof. If U and V are open subsets of X, and if $U \subseteq V$, then

$$\begin{aligned} U \subseteq V &\implies \{f \in \mathrm{C}_c(X),\ f \prec U\} \subseteq \{f \in \mathrm{C}_c(X),\ f \prec V\} \\ &\implies \{I(f),\ f \in \mathrm{C}_c(X),\ f \prec U\} \subseteq \{I(f),\ f \in \mathrm{C}_c(X),\ f \prec V\} \end{aligned}$$

Hence $\mu(U) \leq \mu(V)$. Now by equation (22), $\mu^*(U) \leq \mu(U)$. To show the reverse inequality, suppose by contradiction that $\mu^*(U) < \mu(U)$.

Since $\mu^*(U)$ is an infimum, then for every $\varepsilon > 0$ there exists some $V \supseteq U$ where if we write $\mu^*(U) + \varepsilon = \mu(U)$

$$\mu(V) < \mu^*(U) + \varepsilon = \mu(U) \implies \mu(V) < \mu(U),\ U \subseteq V$$

This contradicts what we have just proven, and therefore $\mu^*(U) = \mu(U)$ for every open set U. ■

Part f

Proof. We wish to show that every open set U is μ^*-measurable. By Theorem 1.10, it suffices to show that for every $E \subseteq X$

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U) \tag{27}$$

because the reverse inequality is given by subadditivity of μ^*, and we can also assume that $\mu^*(E) < +\infty$. Let us assume that E is open, we wish to find some function $h \in \mathrm{C}_c(X)$, $h \prec E$ with

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

The above formula is fussy, but the liberty is taken to show it beforehand to avoid any potential confusion that follows. Since $E \cap U$ is an open subset of X, the definition of $\mu(E \cap U) = \mu^*(E \cap U)$ in (20) tells us that every $\varepsilon > 0$ induces some $f \in \mathrm{C}_c(X)$, $f \prec E \cap U$ where

$$I(f) > \mu(E \cap U) - \varepsilon = \mu^*(E \cap U) - \varepsilon \tag{28}$$

Also, $\mathrm{supp}\,(f)$ is a closed set (compact subsets of Hausdorff spaces are closed), therefore $E \setminus \mathrm{supp}\,(f)$ is an open set. We make a small diversion from the current part of the proof and turn out attention to the fact that

$$\begin{aligned} \mathrm{supp}\,(f) \subseteq U &\implies U^c \subseteq (\mathrm{supp}\,(f))^c \\ &\implies E \setminus U \subseteq E \setminus \mathrm{supp}\,(f) \end{aligned}$$

And because the outer-measure μ^* is monotone,

$$\mu^*(U) \leq \mu^*(E \setminus \mathrm{supp}\,(f)) \tag{29}$$

Now, using the definition of $\mu(E \setminus \mathrm{supp}\,(f))$ (recall that $E \setminus \mathrm{supp}\,(F)$ is an open set), for every $\varepsilon > 0$, there exists some $g \in \mathrm{C}_c(X)$, $g \prec E \setminus \mathrm{supp}\,(F)$ with

$$I(g) > \mu(E \setminus \mathrm{supp}\,(f)) - \varepsilon = \mu^*(E \setminus \mathrm{supp}\,(f)) - \varepsilon \tag{30}$$

It is at this part of the proof where we wish to define $h = f + g$, but first we must verify

- $f + g \in \mathrm{C}_c(X, [0, 1])$,
- $f + g \prec E$

The sum of two non-negative functions is non-negative, and for every $x \in \mathrm{supp}\,(f)$, $f \leq 1$. Also

$$\begin{aligned} \mathrm{supp}\,(g) \subseteq (\mathrm{supp}\,(f))^c &\implies \mathrm{supp}\,(f) \subseteq (\mathrm{supp}\,(g))^c \\ &\implies \mathrm{supp}\,(f) \subseteq \{g = 0\} \end{aligned}$$

The last implication comes from taking complements on both sides of $\{g \neq 0\} \subseteq \mathrm{supp}\,(g)$. So $x \in \mathrm{supp}\,(f) \implies f + g \leq 1$. Now if $x \notin \mathrm{supp}\,(f)$, then $f + g = g \leq 1$. Furthermore, $\mathrm{supp}\,(f + g)$ is a closed subset of compact $\mathrm{supp}\,(f) \cup \mathrm{supp}\,(g)$. This is because $\{f + g \neq 0\} \subseteq \{f \neq 0\} \cup \{g \neq 0\}$, and the finite union of two compact sets is again again compact.

A moment's thought should yield the fact that the last estimate should be an equality, but it is a needless distraction. Therefore $\operatorname{supp}(f+g)$ is compact and $f+g \in \mathrm{C}_c(X,[0,1])$.

Now both bullet points are satisfied, and we can set $h = f+g$. Adding equation (30) with (28) gives us

$$I(h) = I(f) + I(g) > \mu^*(E \cap U) + \mu^*(E \setminus \operatorname{supp}(f)) - 2\varepsilon$$

Upon applying (29) to the right member of the above estimate, we have

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

But this particular $h \in \mathrm{C}_c(X) \cap \{f \prec E\}$, therefore

$$\mu^*(E) \geq I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, equation (27) holds for every open E. Now for any general $E \subseteq X$, fix any $\varepsilon > 0$ and by how we defined $\mu^*(E)$, there exists some open $V \supseteq E$ —recall that $\mu^*(E)$ is the infimum over the set of $\mu(V)$ where V is an open superset of E — hence

$$\mu^*(E) + \varepsilon > \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

By monotonicity (twice) of the outer-measure μ^*, we have

$$\mu^*(E) + \varepsilon > \mu^*(E \cap U) + \mu^* E \setminus U$$

Let $\varepsilon \to 0$, and we get

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Therefore every open $U \subseteq X$ is μ^*-measurable. So $\mu = \mu^*|_{\mathbb{B}}$ is a Borel measure on X. ■

Part g

Proof. To show outer-regularity, fix any $E \in \mathbb{B}$, then by definition,

$$\mu(E) = \mu^*(E) = \inf\{\mu(U),\, U \supseteq E,\, U \in \mathcal{T}\}$$

And for every open U, (18) follows from Equation (20). ■

Part h

Proof. We want to show that for every compact K, Equation (19) holds. To reduce the notational baggage that follows, we agree to define

$$\{I(f),\, f \in \mathrm{C}_c(X),\, f \prec U\} = \{I(f),\, f \prec U\}$$

Similarily for $\{I(f),\, f \geq \chi_K\}$. If $\mu(K) = 0$, then $\mu(K)$ is obviously a lower bound, since $f \geq \chi_K \geq 0$ means that $I(f) \geq 0$, for every $f \geq \chi_K$. So we can suppose $\mu(K) > 0$.

Fix an arbitrary $f \geq \chi_K$, then this particular f induces an open set $U_\alpha = \{f > 1 - \alpha\}$, where $\alpha > 0$. Notice also that

$$K \subseteq \{f \geq 1\} \subseteq \{f > 1 - \alpha\} = U_\alpha$$

Since U_α is an open superset of K, by Equation (24), $\mu(K) \leq \mu(U_\alpha)$, but $\mu(U_\alpha)$ is simply the supremum of $\{I(g),\, g \prec U_\alpha\}$. If we wish to show that $\mu(K) \leq \mu(U_\alpha) \leq I(f)$, it suffices to show that $I(f)$ is an upper-bound for $\{I(g),\, g \prec U_\alpha\}$.

Fix any $I(g) \in \{I(g),\, g \prec U_\alpha\}$, note that $1 - \alpha \neq 0$ for any α small enough, then

- $f/(1-\alpha) > 1$ on U_α,
- $1 \geq g \geq 0$ on U_α, in particular, $f/(1-\alpha) - g \geq 0$ on U_α,
- If $x \notin U_\alpha$, then $f/(1-\alpha) - g = f(1-\alpha) \geq 0$.
- Therefore $f/(1-\alpha) - g \geq 0$ for any x, and by Lemma 2.1,

$$I(f/(1-\alpha)) \geq I(g) \quad \forall g \prec U_\alpha$$

Combining the above estimate with $\mu(K) \leq \mu(U_\alpha)$ gives us

$$\mu(K) \leq \frac{1}{1-\alpha} I(f)$$

Now write $\varepsilon = \alpha/\mu(K) > 0$ and for every $\varepsilon > 0$ we get

$$\mu(K) - I(f) \leq \alpha\mu(K) = \varepsilon$$

Send $\varepsilon \to 0$ and $\mu(K) \leq I(f)$ for every $f \geq \chi_K$.

To show that $\mu(K)$ is indeed the infimum for $\{I(f),\, f \geq \chi_K\}$, notice that for every $\varepsilon > 0$ we can obtain some open superset $U \supseteq K$ (by outer-regularity) where $\mu(U) < \mu(K) + \varepsilon$. By Urysohn's Lemma, there exists some $g \prec U$, $g(x) = 1$ for every $x \in K$.

$$g \in \{I(f),\, f \prec U\} \cap \{I(f),\, f \geq \chi_K\}$$

Therefore $I(g) \leq \mu(U) < \mu(K) + \varepsilon$ as desired, and Equation (19) holds. ■

Part i

Proof. $\mu(K) < +\infty$ for every compact K. Indeed, since $I(\chi_K) \in \{I(f),\, f \geq \chi_K\}$, then by Theorem 7.1, there exists a constant $C_K \geq 0$ that bounds

$$\mu(K) \leq |I(\chi_K)| = I(\chi_K) \leq C_K \cdot \|\chi_K\| = C_K < +\infty$$

■

Part j

Proof. Fix any open set U, then for every $\varepsilon > 0$, there exists some $f \prec U$ with $\mu(U) - \varepsilon < I(f)$. Then denote $K = \operatorname{supp}(f) \subseteq U$. If we take any $I(h) \in \{I(h),\ h \geq \chi_K\}$, then $h \geq f$ gives us $I(h) \geq I(f)$ by Lemma 2.1. So $I(f)$ is a lower bound of $\{I(h),\ h \geq \chi_K\}$, therefore

$$\mu(U) - \varepsilon \leq I(f) \leq \mu(K)$$

Since $\operatorname{supp}(f) = K \subseteq U$, this proves inner-regularity of μ on open sets. ■

Part k

Proof. Suppose $f \in \mathrm{C}_c(X, [0,1])$, we first show that Equation (25) holds. We divide the interval $[0,1]$ into $N \geq 1$ chunks by writing

$$K_j = \{f \geq j/N\}$$

for every $1 \geq j \geq N$. And define $K_0 = \operatorname{supp}(f)$. Each K_j is a closed subset of $\operatorname{supp}(f)$, and therefore compact. More is true,

- $K_{j-1} \supseteq K_j$ for every $1 \leq j \leq N$.
- $x \in K_j$ iff $f(x) \in \left[\frac{j}{N},\ 1\right]$,
- $x \notin K_j$ iff $f(x) \in \left[0,\ \frac{j}{N}\right)$, and
- $x \in (K_{j-1} \setminus K_j)$ iff $f(x) \in \left[\frac{j-1}{N},\ \frac{j}{N}\right)$

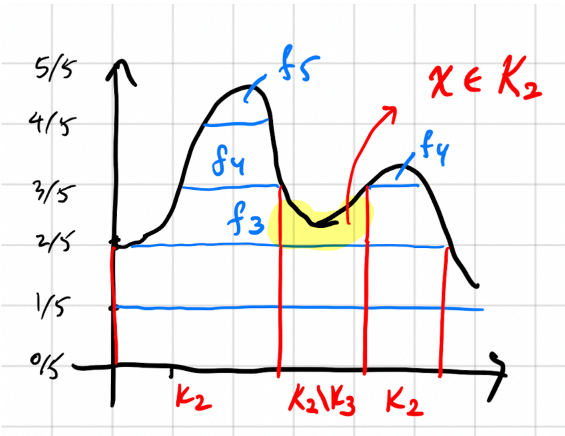
Folland constructs a finite sequence of compactly supported functions, $\{f_j\}$, where $1 \leq j \leq N$ such that

- Each $0 \leq f_j \leq 1/N$,
- If $x \in (K_m \setminus K_{m+1})$ iff $f(x) \in \left[\frac{m}{N},\ \frac{m+1}{N}\right)$ means that $f_j = 1$ for all $1 \leq j \leq m$, and
- $f_{m+1} = f - m/N$ on K_m, such that

$$f(x) = \left(\sum f_{j \leq m}(x)\right) + \left(f(x) - \frac{m}{N}\right) = \frac{m}{N} + \left(f(x) - \frac{m}{N}\right)$$

- And for every $m < j \leq N$, $f_j = 0$.
- If $x \notin K_m$ iff $f(x) \in \left[0,\ \frac{m}{N}\right)$ then for every $m+1 \leq j \leq N$, $f_j = 0$.

The illustration for when $N = 5$ below should make things clearer.



It is also trivial to verify that

- For every $x \in K_j$, $f_j = N^{-1}$, and
$$\chi_{K_j} N^{-1} \le f_j \tag{31}$$
Also, if $x \notin K_j$ then $f_j \ge 0$, therefore $f_j \ge \chi_{K_j} N^{-1}$ at every x.
- If $x \notin K_{j-1}$ then $f_j = 0 \le \chi_{K_{j-1}} \cdot N^{-1}$. If x is in K_{j-1} then $f_j \le N^{-1}$ by construction and therefore
$$f_j \le \chi_{K_{j-1}} N^{-1} \tag{32}$$
for all x.
- $f_j \in \mathrm{C}_c(X)$, since $\mathrm{supp}\,(f_j) \subseteq \mathrm{supp}\,(f)$.

Combining Equations (31) with (32), and by monotonicity in $L^+(X, \mathbb{B}, \mu)$, since $f_j \in L^+$

$$\int \frac{1}{N} \chi_{K_j} d\mu \le \int f_j d\mu \le \int \frac{1}{N} \chi_{K_{j-1}} d\mu$$

And for every $1 \le j \le N$,

$$\frac{1}{N} \mu(K_j) \le \int f_j d\mu \le \frac{1}{N} \mu(K_{j-1}) \tag{33}$$

Furthermore, from Equation (31), since $N f_j \ge \chi_{K_j}$ then by Equation (19),

$$\mu(K_j) \le I(N f_j) \implies \frac{1}{N} \mu(K_j) \le I(f_j)$$

Now for any arbitrary $I(h) \in \{I(h),\ h \ge \chi_{K_{j-1}}\}$, since

$$h \ge \chi_{K_{j-1}} \ge N f_j \implies I(h) \ge I(N f_j)$$

So $N I(f_j)$ is a lower bound for $\{I(h),\ h \ge \chi_{K_{j-1}}\}$ and

$$I(f_j) \le \frac{1}{N} \mu(K_{j-1})$$

Combining the last two results, with $I(f_j)$, we get

$$\frac{1}{N}\mu(K_j) \leq I(f_j) \leq \frac{1}{N}\mu(K_{j-1}) \tag{34}$$

Taking the sum over $1 \leq j \leq N$ for Equations (33) and (34). Define $A = N^{-1}\sum_0^{N-1}\mu(K_j)$, and $B = N^{-1}\sum_1^N \mu(K_j)$

$$B \leq \int f d\mu \leq A$$

And also

$$B \leq I(f) \leq A$$

This is because of finite additivity of both I and the integral, and $f = \sum f_j$ on $K_0 = \operatorname{supp}(f)$. Subtracting the two equations (keeping in mind that $\mu(K_j) < +\infty$ for any compact K_j), we get

$$(-1)(A-B) \leq \left(\int f d\mu - I(f)\right) \leq A - B \implies \left|\int f d\mu - I(f)\right| \leq A - B$$

It is trivial to verify that

$$0 \leq A - B = N^{-1}(\mu(K_0) - \mu(K_N)) \leq N^{-1}\mu(K_0)$$

as $K_N \subseteq K_0$. Let $N \to \infty$ and

$$\int f d\mu = I(f)$$

Equation (25) holds as desired. ■

Part 1

Proof. Now for any general $f \in \mathrm{C}_c(X)$, f must be bounded on the plane since $\mathrm{C}_c(X) \subseteq \mathrm{BC}(X)$, and $|f| \leq M_0$ for some $M_0 \geq 0$. Since $\operatorname{supp}(f)$ is compact, we know that

$$\int |f| d\mu \leq \int M_0 \chi_{\operatorname{supp}(f)} d\mu \leq M_0 \mu(\operatorname{supp}(f)) < +\infty$$

And $\mathrm{C}_c(X) \subseteq L^1(\mu)$. Furthermore,

$$\frac{1}{2}(|\operatorname{Re} f| + |\operatorname{Im} f|) \leq |f| \leq M_0$$

So that $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $\mathrm{C}_c(X)$. Without loss of generality, we may assume that f is real. Define $f_1 = \operatorname{Re} f^+/M_0$ and $f_2 = \operatorname{Re} f^-/M_0$ and it immediately follows that $f_1, f_2 \in \mathrm{C}_c(X, [0,1])$.

By linearity of I on $\mathrm{C}_c(X)$ and the integral in $L^1(\mu)$,

$$I(f_1 - f_2) = I(f) = \int f d\mu = \int f_1 d\mu - \int f_2 d\mu$$

Then we may apply the above to the real and imaginary parts of a general $f \in \mathrm{C}_c(X)$, and this completes the proof. ■

Theorem 7.3

Proposition 3.1

See Theorem 7.2

Proof. ■

Theorem 7.4

Proposition 4.1

See Theorem 7.2

Proof. ■

Theorem 7.5

Proposition 5.1

Proof. ■

Theorem 7.6

Proposition 6.1

Proof. ■

Theorem 7.7

Proposition 7.1

Proof. ■

Theorem 7.8

Proposition 8.1

Proof. ■

Theorem 7.9

Proposition 9.1

If μ is a Radon measure on X, then $\mathrm{C}_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < +\infty$.

Proof. Theorem 6.7 tells us that the set of L^p simple functions (as Folland calls them), which are

$$\Lambda = \left\{ f,\, f = \sum_{j \leq n} a_j \chi_{E_j},\, a_j \in \mathbb{C},\, \mu(E_j) < +\infty \right\}$$

So for every $f \in L^p$, there exists a sequence $\{f_n\} \subseteq \Lambda$ with $f_n \to f$ pointwise and $f_n \to f$ in L^p. ■

Theorem 7.10

Proposition 10.1

Proof. ■

Theorem 7.11

Proposition 11.1

Proof. ■

Chapter 8

Theorem 8.1

Proposition 1.1

Proof. ∎

Theorem 8.2

Proposition 2.1

Proof. ■

Theorem 8.3

Proposition 3.1

If $f \in C^\infty$, then $f \in \mathcal{S}$ if and only if $x^\beta \partial^\alpha f$ is bounded for all multi-indices α, β

Proof. ∎

Theorem 8.4

Proposition 4.1

Proof. ■

Theorem 8.5

Proposition 5.1

Proof. ■

Theorem 8.6

Proposition 6.1

Proof. ■

Theorem 8.7

Proposition 7.1

Proof. ■

Theorem 8.8

Proposition 8.1

Proof. ■

Theorem 8.9

Proposition 9.1

Proof. ∎

Theorem 8.10

Proposition 10.1

Proof. ■

Theorem 8.11

Proposition 11.1

Proof. ■

Theorem 8.12

Proposition 12.1

Proof. ■

Theorem 8.13

Proposition 13.1

Proof. ■

Theorem 8.14

Proposition 14.1

Suppose $\phi \in L^1$, and $\int \phi(x)dx = a$.

(a) If $f \in L^p$, $p \in [1, +\infty]$, then $f * \phi_t \to af$ in the L^p norm as $t \to 0$.

(b) If f is bounded and uniformly continuous, then $f * \phi_t \to af$ uniformly as $t \to 0$.

(c) If $f \in L^\infty$ and f is continuous on an open set U, then $f * \phi_t \to af$ uniformly on compact subsets of U as $t \to 0$

Proof of Part A. First, the convolution $f * \phi_t$ is in L^p by Young's Inequality (Theorem 8.7). Furthermore,

$$f * \phi_t - af = \int\limits_{y \in \mathbb{R}^n} f(x-y)t^{-n}\phi(t^{-1}y)dy - \int\limits_{y \in \mathbb{R}^n} f(x)\phi(y)dy \tag{35}$$

Now apply Theorem 2.44, with $y \mapsto y/t$, and denote this invertible map by $T \in GL(n, \mathbb{R})$, so that $|\det(T)| = t^{-n}$, then $y = T(y)t$ for every $t > 0$. It follows that

$$\begin{aligned}(f * \phi_t)(x) &= |\det(T)| \cdot \int\limits_{y \in \mathbb{R}^n} f(x - t \cdot Ty)\phi(T(y))dy \\ &= \int\limits_{z \in \mathbb{R}^n} f(x - tz)\phi(z)dz \\ &= \int\limits_{z \in \mathbb{R}^n} \tau_{tz} f(x)\phi(z)dz \end{aligned} \tag{36}$$

Next, $a = \int \phi$ so $af = \int f(x)\phi(z)dz$. Using Equations (35) and (36) we get

$$(f * \phi_t - af)(x) = \int\limits_{z \in \mathbb{R}^n} (\tau_{tz} f - f)\phi(z)dz \tag{37}$$

We wish to apply Minkowski's Inequality for integrals, which states, roughly speaking:

> The norm of an integral is less than the integral of the norm.

to Equation (37), and

$$\|f * \phi_t - af\|_p \leq \int_{z \in \mathbb{R}^n} \|(\tau_{tz} f - f)\phi(z)\|_p dz \tag{38}$$

The assumptions for Theorem 6.19 are satisfied by

1. Notice for every $z \in \mathbb{R}^{n'}$,

$$\|(\tau_{tz} f - f)\phi(z)\|_p = \left(\int\limits_{x \in \mathbb{R}^n} |(\tau_{tz} f(x) - f(x))\phi(z)|^p dx\right)^{1/p} \leq |\phi(z)|\left(2\|f\|_p\right) < +\infty$$

   Since $\|\phi\|_1 < +\infty$, $|\phi(z)| < +\infty$ almost everywhere.

2. Next, to show $z \mapsto \|\phi(z)(\tau_{tz}f - f)\|_p$ is in $L^1\mathbb{R}^n, z$. Reuse the last estimate in the previous bullet point, and
$$\|\phi(z)(\tau_{tz}f - f)\|_p \le |\phi(z)|\left(2\|f\|_p\right)$$
Taking the integral in L^+ with respect to z, we get
$$\left\|\left(\|\phi(z)(\tau_{tz}f - f)\|_p\right)\right\|_1 < +\infty$$
so both assumptions are satisfied.

Therefore Equation (38) holds. Next, fix any sequence of $t_n > 0$ with $t_n \to 0$. The Dominated Convergence Theorem gives, since $|\phi(z)|\|\tau_{t_n z}f - f\|_p$ is dominated by $|\phi(z)| \cdot 2\|f\|_p \in L^1 \cap L^+$

$$\begin{aligned}
\lim_{n\to\infty} \int_{z\in\mathbb{R}^n} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz &= \int_{z\in\mathbb{R}^n} \lim_{n\to\infty} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz \\
&= \int_{z\in\mathbb{R}^n} 0 dz \\
&= 0
\end{aligned}$$

The second last equality is from Lemma 8.4, as translation is continuous in the L^p norm for $p \in [1, +\infty)$. So almost every $z \in \mathbb{R}^n$ (since again, $|\phi(z)|$ can be infinite on a null set),
$$\|\tau_{t_n z}f - f\|_p \to 0 \implies \|\tau_{t_n z}f - f\|_p |\phi(z)| \to 0$$
as $n \to +\infty$. It follows that
$$\lim_{n\to\infty} \|f * \phi_{t_n} - af\|_p = \lim_{n\to\infty} \left\| \int_{z\in\mathbb{R}^n} [\tau_{t_n z}f(x) - f(x)]\phi(z) dz \right\|_p = 0$$
Since the sequence $t_n \to 0$ is arbitrary, we conclude that the function $t \mapsto \|f * \phi_t - af\|_p$ has a limit of 0 as $t \to 0$. ■

Proof of Part B. Suppose $f \in \mathrm{UBC}(\mathbb{R}^n)$, so that f is uniformly continuous and bounded. We wish to show $f * \phi_t \to af$ uniformly as $t \to 0$. In symbols,
$$g : t \mapsto \|f * \phi_t - af\|_u,\ g \to 0, \text{ as } t \to 0$$
The convolution between f and ϕ_t makes sense at every $x \in \mathbb{R}^n$, as
$$\int |\tau_y f(x)||\phi(y)| dy \le \|f\|_u \cdot \|\phi\|_1 < +\infty$$

Taking the supremum norm on both sides of Equation (37), we get

$$\begin{aligned}
\|f * \phi_t - af\|_u &= \sup_{x\in\mathbb{R}^n}\left|\int_{z\in\mathbb{R}^n} (\tau_{tz}f - f)\cdot\phi(z)dz\right| \\
&\leq \sup_{x\in\mathbb{R}^n}\int_{z\in\mathbb{R}^n} |\tau_{tz}f - f|\cdot|\phi(z)|dz \\
&\leq \int_{z\in\mathbb{R}^n} \sup_{x\in\mathbb{R}^n}|\tau_{tz}f - f|\cdot|\phi(z)|dz \\
&= \int_{z\in\mathbb{R}^n} \|\tau_{tz}f - f\|_u\cdot|\phi(z)|dz
\end{aligned} \tag{39}$$

the last equality is a simple consequence of the monotonicity of the integral in L^+, indeed. For every $x \in \mathbb{R}^n$, the following holds pointwise for almost every z

$$|\tau_{tz}f - f| \leq \|\tau_{tz}f - f\|_u \implies \sup_{x\in\mathbb{R}^n}|\tau_{tz}f - f| \leq \|\tau_{tz}f - f\|_u$$

Apply the Dominated Theorem to the right member of (39), noting that it is dominated by $|\phi(z)|\cdot 2\|f\|_u \in L^1\cap L^+$ as we have done for Part A of the proof. Since this holds for every sequence $t_n \to 0$, the proof is complete. ■

Proof of Part C. Next, suppose that $f \in L^\infty$, and $f \in C(U)$, where U is open in $\mathbb{R}^n$. We claim that

$$f * \phi_t \to af$$

within the topology of uniform convergence on compact subsets of U. So that for every $K \in ♪$, $K \subseteq U$

$$\sup_{x\in K}\left|f * \phi_t - af\right| \to 0, \text{ as } t \to 0$$

First, a small technical Lemma.

Lemma 14.1

If $\phi \in L^1(\mathbb{R}^n)$, then for every $\varepsilon > 0$, there exists $E \in ♪$, with

$$\int_{E^c}|\phi| = \|\phi\chi_{E^c}\|_1 < +\varepsilon$$

Proof. Assume that $\phi \geq 0$, if not, replace ϕ by $|\phi|$. Since $\mathrm{C}_c(\mathbb{R}^n)$ is dense in L^1 for every $\varepsilon 2^{-1} > 0$ there exists some $\psi \in \mathrm{C}_c(\mathbb{R}^n)$ with $\|\psi - \phi\|_1 < \varepsilon^{-1}$, and denote $E = \mathrm{supp}\,(\psi) \in ♪$, then

$$\||\psi| - |\phi|\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

So we can assume $\psi \geq 0$ as well, perhaps by relabelling ψ by $|\psi|$. Then,

$$\|\psi - \chi_E\phi\|_1 = \|\chi_E(\psi - \phi)\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

by monotonicity in L^+. The Triangle Inequality in L^1 gives

$$\|\chi_{E^c}\phi\|_1 = \|\phi - \chi_E\phi\|_1 = \|\phi(1 - \chi_E)\|_1 \leq \|\phi - \psi\|_1 + \|\psi - \chi_E\phi\|_1 < \varepsilon$$

■

Back to the main proof of Part C, fix any $\varepsilon > 0$, then by Lemma 14.1, ϕ induces some $E \in ♪$ with $\|\chi_{E^c}\phi\|_1 < +\varepsilon$. By Lemma 8.4, $\chi_K f \in C_c(\mathbb{R}^n) \subseteq \mathrm{UBC}(\mathbb{R}^n)$. Uniform continuity of $\chi_K f$ gives us the continuity of translations. Now for the same $\varepsilon > 0$, there exists $r > 0$, for every $w \in \mathbb{R}^n$,

$$|w| < r \implies \|\tau_w \chi_K f - \chi_K f\|_u < +\varepsilon \tag{40}$$

Since $E \in ♪$, it is bounded, and let t be a small positive number such that for every $z \in E$,

$$|tz| < t \cdot (1 + \sup_{z \in E} |z|) < r$$

There exists such a a t, namely $t = r2^{-1}(1 + \sup_{z \in E} |z|)^{-1}$. And for this $t > 0$, it follows that for every $z \in E$,

$$\sup_{x \in K} |\tau_{tz} f - f| < +\varepsilon$$

Since this holds for every $z \in E$, we write

$$\sup_{x \in K, z \in E} |\tau_{tz} f - f| < +\varepsilon$$

And

$$|\phi(z)| \left[\sup_{x \in K, z \in E} |\tau_{tz} f - f| \right] < |\phi(z)|\varepsilon$$

Monotonicity in $L^+(E, z)$ reads, for every $x \in K$,

$$\int\limits_{z \in E} |\phi(z)(\tau_{tz} f - f)| dz \leq \int\limits_{z \in E} |\phi| \varepsilon dz = \varepsilon \|\chi_E \phi\|_1 \leq \varepsilon \|\phi\|_1$$

Since this holds for every $x \in \mathbb{R}^n$,

$$\sup_{x \in K} \left\{ \int_{z \in E} |\phi(z)| \cdot |\tau_{tz} f - f| dz \right\} \leq \varepsilon \|\phi\|_1 \tag{41}$$

Next, notice for every t, z, we have

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f\|_u + \|f\|_u \leq 2 \cdot \|f\|_u$$

And the following holds $z \in E^c$ a.e,

$$|\phi(z)| \cdot |\tau_{tz} f - f| \leq |\phi(z)| \cdot 2\|f\|_u$$

Taking the integral, and applying the condition we imposed on E from Lemma (14.1), so that

$$\int_{z\in E^c} |\phi(z)| \cdot |\tau_{tz}f - f| dz \leq 2\|f\|_u \int_{z\in E^c} |\phi(z)| dz \leq 2\|f\|_u \varepsilon$$

Taking the supremum of the above estimate, so

$$\sup_{x\in K}\left\{\int_{z\in E^c} |\phi(z)(\tau_{tz}f - f)| dz\right\} \leq 2\|f\|_u \varepsilon \tag{42}$$

Combining Equations (41) and (42). Applying the additivity of the supremum (of $x \in K$), since both members are finite,

$$\sup_{x\in K}\left\{\int_E |\phi(z)|(\tau_{tz}f - f)dz + \int_{E^c} |\phi(z)|(\tau_{tz}f - f)dz\right\} < \varepsilon(2\|f\|_u + \|\phi\|_1)$$

The left member above is equal to $\sup_{x\in K} |f * \phi_t - af|$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of Part C. ■

Theorem 8.15

Proposition 15.1

If $|\phi(x)| \leq C(1+|x|)^{-n-\varepsilon}$, where $\varepsilon > 0$, and if $f \in L^p$, for $p \in [1, +\infty)$, then

$$f * \phi_t \to af$$

pointwise for every x in the Lebesgue set of f,

$$\mathcal{L}_f = \left\{ x \in \mathbb{R}^n, \quad \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{y \in B(r,x)} |f(x) - f(y)| dy = 0 \right\}$$

We also claim that $m(\mathcal{L}_f^c) = 0$, and $x \in \mathcal{L}_f$ at every continuous $f(x)$.

The proof is long, and will be divided into several parts. Let us start with a couple of Lemmas about the Lebesgue Set of f, and several pointwise estimates that will be of use.

Lemma 15.1

If $\phi : \mathbb{R}^n \to \mathbb{C}$, and

$$|\phi(x)| \leq C(1+|x|)^{n-\varepsilon},\ \varepsilon > 0 \tag{43}$$

then $\phi \in L^1$. Furthermore, $\phi_t \in L^1$ for every $t > 0$.

Proof of 15.1. If $x \neq 0$, then

$$|\phi| \leq C \cdot (1+|x|)^{-(n+\varepsilon)} \leq C \cdot |x|^{-(n+\varepsilon)}$$

on some B^c as defined in Theorem 2.52, so $\phi \in L^1(B^c)$. Next,

$$n + \varepsilon > n > n/2 = a$$

and by monotonicity,

$$|\phi| \leq C \cdot (1+|x|)^{-(n+\varepsilon)} \leq C \cdot (1+|x|)^{-(n/2)}$$

so $\phi \in L^1(\mathbb{R}^n)$. Next, if $\phi \in L^1$, then

$$|\phi_t(x)| = t^{-n} |\phi(t^{-1}x)|$$

taking the integral in L^+, and applying Theorem 2.44, with $T : x \mapsto t^{-1}$, and $\det(T) = t^{-n}$, so that

$$\int |\phi_t|(x)dx = |\det(T)| \int |\phi| \circ T(x) dx = \int |\phi|(x) dx < +\infty$$

This completes the Lemma. ■

Lemma 15.2

If $f : \mathbb{R}^n \to \mathbb{C}$, and if $f \in C(\mathbb{R}^n)$, then $\mathcal{L}_f = \mathbb{R}^n$.

Proof of 15.2. Let $x \notin \mathcal{L}_f$, and there exists a sequence $r_k \to 0$ and $\varepsilon_0 > 0$ but

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy \geq \varepsilon_0$$

We claim that for every $k \geq 1$, we can find a $y_k \in B(r_k, x) \setminus \{x\}$ with

$$|f(x) - f(y)| \geq \varepsilon_0$$

Indeed, suppose by contradiction that no such y_k exists, and by monotonicity,

$$\frac{1}{m(B(r_k, x))} \int\limits_{y \in B(r_k, x)} |f(x) - f(y)| dy < \frac{1}{m(B(r_k, x))} \int\limits_{y \in B(r_k, x)} \varepsilon_0 dy = \varepsilon_0$$

So choose y_k as above, and it is clear that $y_k \to x$ as $k \to \infty$, but $f(y_k) \not\to f(x)$. Therefore f is not continuous at x. ■

Lemma 15.3

If $x \in \mathcal{L}_f$, then for every $\delta > 0$ there exists a $\eta > 0$, with

$$r \leq \eta \implies \int_{|y| < r} |f(x - y) - f(x)| dy \leq \delta \cdot r^n$$

Proof of 15.3. We will start with something trivial.

$$m(B(r)) = r^n m(B(1)) \tag{44}$$

where $B(r) = \{x \in \mathbb{R}^n, \ |x| < r\}$. By Theorem 2.44,

$$\begin{aligned} m(B(r)) &= \int \chi_B(x/r) dx \\ &= |\det(T)|^{-1} \int \chi_B(x) dx \\ &= r^n m(B(1)) \end{aligned}$$

where $T : x \mapsto x/r$ and $\det(T) = r^{-n}$. Fix $x \in \mathcal{L}_f$, and take $\varepsilon = \delta / m(B(1)) > 0$, and by definition this induces some $\eta > 0$, and for every $r \leq \eta$

$$\frac{1}{m(B(r, x))} \int\limits_{y \in B(r, x)} |f(x) - f(y)| dy \leq \varepsilon$$

By translation invariance of m,

$$m(B(r, x)) = m(B(r)) = r^n \cdot m(B(1))$$

and apply the map $y \mapsto x - y$, which is a composition a rotation by $|-1|$ and a translation by $x \in \mathbb{R}^n$. By Theorems 2.44 and 2.42,

$$\int_{|y| \in B(r)} |f(x) - f(x-y)| dy = \int_{y \in B(r,x)} |f(x) - f(y)| dy < \varepsilon m(B(1)) \cdot r^n = \delta r^n$$

where we used the fact that

$$\begin{aligned} d(x-y, x) < r &\iff d(-y, 0) < r \\ &\iff d(y, 0) < r \end{aligned}$$

hence

$$\chi_{B(r,x)}(x-y) = \chi_{B(r,0)}(y)$$

■

Lemma 15.4

Let $A_j = \left\{ |y| \in [2^{-j}\eta,\, 2^{1-j}\eta) \right\}$, and if Equation (43) holds for ϕ then ϕ_t satisfies

$$|\phi_t| \leq C \cdot t^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \tag{45}$$

on A_j for every $t > 0$, where $\alpha = t^{-1}\eta$ for some $\eta > 0$.

Moreover, if $A_0 = \left\{ |y| < 2^{-K}\eta \right\}$, where $K \geq 0$, then

$$|\phi_t(y)| \leq C \cdot t^{-n} \tag{46}$$

on A_0

Proof of 15.4. Notice that

$$t^{-1}y \in [2^{-j} \cdot \eta/t,\, 2^{1-j} \cdot \eta/t) = [2^{-j} \cdot \alpha,\, 2^{1-j} \cdot \alpha)$$

And

$$1 + |t^{-1}y| \geq |t^{-1}y| \geq 2^{-j}\alpha$$

Therefore

$$C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}$$

and applying Equation (43) establishes the first claim.

The second claim follows from Equation (43),

$$|\phi_t(y)| \leq C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}$$

■

Main Proof of Theorem 8.15. The outline of the proof is as follows,

1. $|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)}$ for $\varepsilon > 0$ and
2. $f \in L^p$ for $p \in [1, +\infty)$,
3. for any $x \in \mathcal{L}_f$, we wish to show
$$|f * \phi_t - af|(x) \to 0, \quad \text{as } t \to 0$$
4. To prove this, we fix some $\beta > 0$ and show that
$$|f * \phi_t - af|(x) < \beta$$
since β is arbitrary, the proof will be complete.
5. By Lemma 15.3, for every $\delta > 0$ there exists a $\eta > 0$ where $r \leq \eta$ implies
$$\int_{|y|<r} |f(x) - f(x-y)| dy \leq \delta \cdot r^n$$
and using the L^1 inequality,
$$\begin{aligned} |f * \phi_t - af|(x) &= \left| \int [f(x-y) - f(x)] \cdot \phi_t(y) dy \right| \\ &\leq \int |f(x-y) - f(x)| \cdot |\phi_t(y)| dy \\ &= \int\limits_{|y|<\eta} |f(x-y) - f(y)| \cdot |\phi_t(y)| dy + \int\limits_{|y|\geq\eta} |f(x-y) - f(y)| \cdot |\phi_t(y)| dy \\ &= I_1 + I_2 \end{aligned}$$
6. Let $\delta = \beta(2A)^{-1}$, where
$$A = 2^n \cdot C \left[\frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$$
we make the claim that this choice of δ will give us $I_1 < \beta/2$
7. After choosing $\delta > 0$, (which induces $\eta > 0$), we will show that $I_2 < \beta/2$ (for a fixed $\eta > 0$) for t sufficiently small, and applying the Triangle Inequality finishes the proof.

Let η be as above, and for $t > 0$ and suppose we can find a $K \in \mathbb{N}^+$ with
$$2^K \leq \eta/t \leq 2^{K+1} \tag{47}$$
and define $\alpha = \eta/t$ for convenience.

Notice for any $K \geq 1$, the interval $[0, 1)$ can be partitioned in the following manner
$$[0, 1) = [0, 2^{-K}) \cup \left(\bigcup_{j=1}^{K} [2^{-j}, 2^{1-j}) \right)$$

and let us define

$$A_j = \left\{ |y| \in [2^{-j}\eta,\, 2^{1-j}\eta) \right\}, \quad A_0 = \left\{ |y| \in [0, 2^{-K}\eta) \right\}$$

If no such K exists, then let $A_j = \varnothing$ and set $A_0 = \{|y| \in [0, \eta)\}$. The disjoint union of all $A_{j\geq 0}$ is the open ball $\{|y| \in [0, \eta)\}$. By Lemma 15.4 and Lemma 15.3 each $j \geq 0$,

$$\begin{aligned} I_1 &= \sum_{j=0}^{K} \int_{y \in A_j} |f(x-y) - f(y)||\phi_t(y)|dy \\ &\leq Ct^{-n}\delta(2^{-K}\eta)^n + \sum_{j=1}^{K} \int_{y \in A_j} |f(x-y) - f(y)||\phi_t(y)|dy \\ &\leq Ct^{-n}\delta(2^{-K}\eta)^n + \sum_{j=1}^{K} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n \end{aligned}$$

The left member reads,

$$\begin{aligned} Ct^{-n}\delta(2^{-K}\eta)^n &\leq C\delta\alpha^n 2^{-Kn} \\ &\leq C\delta 2^{n(K+1)}2^{-Kn} \\ &= C\delta 2^n \end{aligned}$$

and termwise for the right,

$$\begin{aligned} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n &= C\delta \cdot t^{\varepsilon} \cdot 2^{j\varepsilon+n}\eta^{-\varepsilon} \\ &= (C\delta 2^n\alpha^{-\varepsilon}) \cdot 2^{j\varepsilon} \end{aligned}$$

Summing over the geometric series,

$$\begin{aligned} \sum_{j=1}^{K} 2^{j\varepsilon} &= 2^{\varepsilon}\sum_{j=0}^{K-1} 2^{j\varepsilon} \\ &= \frac{2^{\varepsilon(K+1)} - 2^{\varepsilon}}{2^{\varepsilon} - 1} \end{aligned}$$

using the estimate for α in Equation (47)

$$\alpha \in [2^K, 2^K + 1) \implies \alpha^{-\varepsilon} \in [2^{-\varepsilon(K+1)}, 2^{-\varepsilon K})$$

and combining the last few equations, the right member becomes

$$\begin{aligned} (C\delta 2^n) \cdot \alpha^{-\varepsilon}\frac{2^{\varepsilon(K+1)} - 2^{\varepsilon}}{2^{\varepsilon} - 1} &\leq (C\delta 2^n) \cdot \alpha^{-\varepsilon}\frac{2^{\varepsilon(K+1)}}{2^{\varepsilon} - 1} \\ &\leq (C\delta 2^n) \cdot \frac{2^{\varepsilon}}{2^{\varepsilon} - 1} \end{aligned}$$

Finally, $I_1 \leq (C\delta 2^n)\left[\frac{2^\varepsilon}{2^\varepsilon - 1} + 1\right]$, and by Step 6, $I_1 \leq \beta/2$.

Obtaining an estimate for I_2 is another laborious entreprise. Let us define $W = \{|y| \geq \eta\}$, and

- By Holder's Inequality,

$$I_2 \leq \|f\|_p \|\chi_W \cdot \phi_t\|_q + |f(x)| \|\chi_W \cdot \phi_t\|_1$$

  where q is the conjugate exponent to p. Since $p \in [1, +\infty)$, it suffices to show $\|\chi_W \cdot \phi_t\|_q \to 0$ as $t \to 0$ for $q \in [1, +\infty]$.

- Suppose $q = +\infty$,

$$y \in W \iff |y| \geq \eta \iff |t^{-1}y| \geq \alpha$$

  then $\|\chi_W \cdot \phi_t\|_\infty \leq Ct^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq Ct^\varepsilon \eta^{-(n+\varepsilon)}$

- Now suppose $q \in [1, +\infty)$, by polar integration and Theorems 2.51, 2.52 (brace yourselves):

$$\begin{aligned}
\|\chi_W \cdot \phi_t\|_q^q &= t^{-nq} \cdot \int_{y \in W} C^q \cdot |t^{-1}y|^{-q\cdot(n+\varepsilon)} dy \\
&= C^q \cdot t^{\varepsilon q} \int_{|y| \geq \eta} |y|^{-q\cdot(n+\varepsilon)} dy \\
&= C^q \cdot t^{\varepsilon q} \sigma(S^{n-1}) \int_{r \geq \eta} r^{n-1} \cdot r^{-q\cdot(n+\varepsilon)} dr \\
&= \left.\frac{C^q t^{\varepsilon q}}{n - q\cdot(n+\varepsilon)} r^{n-q\cdot(n+\varepsilon))}\right]_\eta^\infty \\
&= \frac{C^q t^{\varepsilon q}}{q\cdot(n+\varepsilon) - n} \eta^{n-q\cdot(n+\varepsilon)} \\
\|\chi_W \cdot \phi_t\|_q &= \left[\frac{C}{(q\cdot(n+\varepsilon) - n)^{1/q}} \left(\eta^{n-q\cdot(n+\varepsilon)}\right)^{1/q}\right] t^\varepsilon \\
&= C_3(q) t^\varepsilon
\end{aligned}$$

- Find a t sufficiently small so that

$$t^\varepsilon < \min\left\{\beta(4C_3(1)|f(x)|)^{-1},\ \beta(4C_3(q)\|f\|_p)^{-1},\ \beta(4C \cdot \eta^{-(n+\varepsilon)})^{-1}\right\}$$

- Therefore $I_2 < \beta/2$, and the proof is complete upon sending $\beta \to 0$.

■

Theorem 8.16

Proposition 16.1

See Theorem 8.15

Proof. ∎

Theorem 8.17

Proposition 17.1

Proof. ■

Theorem 8.18

Proposition 18.1

Proof. ■

Theorem 8.19

Proposition 19.1

Proof. ■

Theorem 8.20

Proposition 20.1

Proof. ∎