Chapter 7

Theorem 7.1

Proposition 1.1. If I is a linear functional on $C_c(X)$, then for every compact $K \subseteq X$, there exists some $C_k \ge 0$ with

$$|I(f)| \le C_K \cdot \|f\|_u$$

Proof. Since supp (f) is compact, by Urysohn's Lemma (Theorem 4.32), there exists a $\phi \in C_c(X, [0, 1])$ such that $\phi = 1$ on K and vanishes outside some compact $\overline{V} \subseteq X$. Then at every x,

$$-\|f\|_u \le f(x) \le +\|f\|_u$$

Implies that

$$(-\|f\|_u)\phi \le f(x) \le (+\|f\|_u)\phi$$

So that $f + ||f||_u \phi \ge 0$ and $+||f||_u - f \ge 0$, and by linearity,

$$(-\|f\|_u)I(\phi) \le I(f) \le (+\|f\|_u)I(\phi)$$

Therefore $|I(f)| \leq I(\phi) ||f||_u$, and taking $C_K = I(\phi)$ will suffice.

Theorem 7.2

Proposition 2.1. The Riesz-Markov-Kakutani Representation Theorem. If (for every) I is a positive linear functional on $C_c(X)$, then there exists a unique Radon measure μ on X, such that

$$I(f)=\int f d\mu$$

for every $f \in C_c(X)$. μ also satisfies, for every open U, and every compact $K \subseteq X$

$$\mu(U) = \sup \{ I(f), f \in \mathcal{C}_c(X), f \prec U \}$$
 (1)

$$\mu(K) = \inf \{ I(f), f \in \mathcal{C}_c(X), f \ge \chi_K \}$$
 (2)

For the sake of completeness, we place the definitions for a Radon measure. Let X be a LCH space, and $\mathbb B$ be its usual σ -algebra, a measure ν is a Radon measure iff

- (i) $\nu(K) < +\infty$ for every compact K.
- (ii) ν is outer-regular on all Borel sets E,

$$\nu(E) = \inf \left\{ \nu(U), \; U \supseteq E, \; U \in \mathcal{T} \right\}$$

Intuition: approximation by open supersets.

(iii) ν is inner-regular on all open sets $U \in \mathcal{T}$,

$$\nu(U) = \sup \left\{ \mu(K), \; K \subseteq U, \; K \text{ compact} \right\}$$

Intuition: approximation by compact subsets

The main proof is extremely long, so we will divide it into several parts. Following Folland's argumentation closely, we will prove (in order)

(a) If μ_1 , μ_2 are Radon measures on X such that for every $f \in C_c(X)$

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

then μ_1 , μ_2 must satisfy (1), and $\mu_1 = \mu_2$ on \mathbb{B} .

(b) If we define, for every open set U, define $\mu: \mathcal{T} \to [0, +\infty]$ such that

$$\mu(U) = \sup \left\{ I(f), \ f \in \mathcal{C}_c(X), \ f \prec U \right\} \tag{3}$$

Then μ is countably subadditive, meaning for every $U \in \mathcal{T}$, $\{U_{j\geq 1}\} \subseteq \mathcal{T}$

$$U = \bigcup U_{j \ge 1} \implies \mu(U) \le \sum \mu(U_{j \ge 1})$$

(c) $\mu(\varnothing) = 0$, $\{\varnothing, X\} \subseteq \mathcal{T}$, so that by Theorem 1.10 μ induces an outer-measure μ^*

$$\mu^*(E) = \inf \left\{ \sum \mu(U_{j \ge 1}), \ U_j \in \mathcal{T}, \ E \subseteq \bigcup U_{j \ge 1} \right\}$$
 (4)

(d) If μ^* is as described above, then if μ is countably subadditive on \mathcal{T} , then

$$\mu^*(E) = \inf \{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$
 (5)

Meaning the two definitions in (4) and (5) are equal.

- (e) μ^* and μ agree on all open sets, and $\mu^*|_{\mathcal{T}} = \mu$,
- (f) Using again the definition in (4) and (5), we show that every open set $U \in \mathcal{T}_X$ is μ^* -measurable, meaning for every $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

With this, since the set of all outer-measurable (μ^* -measurable) sets, \mathcal{M}^* form a σ -algebra,

$$\mathcal{T} \subset \mathcal{M}^* \implies \mathbb{B} \subset \mathcal{M}^*$$

By Theorem 1.1, and define

$$\mu = \mu^*|_{\mathbb{B}} \tag{6}$$

is a Borel measure. And we note in passing that μ is outer-regular on all $E \in \mathbb{B}$,

$$\mu(E) = \inf \{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$
 (7)

- (g) Using (6) for the definition of μ on \mathbb{B} , we prove that
 - μ is outer-regular on all Borel sets, and
 - μ satisfies Equation (1).
- (h) μ satisfies Equation (2)
- (i) μ is finite on all compact sets.
- (j) μ is inner-regular on all open sets.
- (k) For every $f \in C_c(X, [0, 1])$,

$$I(f) = \int f d\mu \tag{8}$$

(l) For every $f \in C_c(X)$, $I(f) = \int f d\mu \tag{9}$

A small lemma needs to be made before proceeding, that concerns the 'monotonicity' of I on C_cX .

Lemma 2.1 Suppose that $f, g \in C_c(X)$, and $f \geq g \geq 0$ for every X, then $f - g \in C_c(X)$ and $I(f) \geq I(g)$

Proof. Suppose that $x \in X$ where f(x) = 0, then

$$f(x) - g(x) = -g(x) \ge 0 \implies g(x) = 0 \implies f - g = 0$$

Hence

$$\{x, f(x) = 0\} \subseteq \{x, f(x) - g(x) = 0\} \implies \{x, f(x) - g(x) \neq 0\} \subseteq \{x, f(x) \neq 0\}$$

$$\implies \sup (f - g) \subseteq \sup (f)$$

Since supp (f) is compact, and supp (f-g) is a closed subset of supp (f), yields $f-g \in C_c(X)$. And if I is any positive linear functional on $C_c(X)$, then

$$f - g \ge 0 \implies I(f - g) \ge 0$$

 $\implies I(f) > I(g) > 0$

Remark 2.1 If $f \prec U$ and $g \prec U$ for some open subset $U \subseteq X$, then clearly supp $(f - g) \subseteq \text{supp } (f) \subseteq U$, and $1 \geq f \geq f - g \geq 0$ means that $f - g \prec U$ as well.

Part a

Proof. Suppose that μ_1 and μ_2 are Radon measures on X, and for every $f \in C_c(X)$,

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

We first prove (1). Without loss of generality, by monotonicity of L^+ , if $f \prec U$ for some open U, then $0 \leq f \leq ||f||_u \chi_U = \chi_U$ for all x and

$$\int f d\mu_1 \le \int ||f||_u \chi_U d\mu_1 \le \mu_1(U)$$

Therefore $\mu_1(U)$ (resp. $\mu_2(U)$) is an upper-bound for the set

$${I(f), f \in C_c(X), f \prec U}$$

Since μ_1 is inner-regular on $U \in \mathcal{T}$, for every $\varepsilon > 0$ we can find some compact $K \subseteq U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K)$$

By Urysohn's Lemma (Theorem 4.32), there exists some $g \in C_c(X)$ with

- $g \in C_c(X, [0, 1]),$
- g = 1 on $K \subseteq U$,
- g = 0 outside some $\overline{V} \subseteq U$, and
- $g \prec U$.

Hence for every $x \in K$, $g \ge \chi_K$. If $x \notin K$ then $g \ge 0 = \chi_K$; so $g - \chi_K \ge 0$ for every $x \in X$. Since $\chi_K \prec U$, using Lemma 2.1, we get

$$\mu_1(K) \le \int \chi_K d\mu_1 = I(\chi_K) \le I(g)$$

So for every $\varepsilon > 0$, there exists a $g \in C_c(X)$, and $g \prec U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K) \le I(g)$$

Therefore $\mu_1(U) = \sup \{I(f), f \in C_c(X), f \prec U\}$, and the first claim in (a) is proven. To show that μ is indeed unique, since for every open set U, we must have $\mu_1(U) = \mu_2(U)$, and if $E \in \mathbb{B}$ is any Borel set, and by outer-regularity,

$$\mu_1(E) = \inf \{ \mu_1(U), U \supseteq E, U \in \mathcal{T} \} = \inf \{ \mu_2(U), U \supseteq E, U \in \mathcal{T} \} = \mu_2(E)$$

Therefore this measure is unique.

Part b

Proof. To show countable subadditivity for μ with equation (3), fix any $U \in \mathcal{T}$ and a sequence $\{U_{j\geq 1}\} \subseteq \mathcal{T}$ with $U = \bigcup U_{j\geq 1}$. It suffices to show that the partial sum of $\sum \mu(U_{j\leq n})$ is greater than I(f) for any $f \in C_c(X)$, $f \prec U$ (hence it is an upper bound).

Fix any f, then denote $K = \text{supp}(f) \subseteq U$, and since $\{U_{j\geq 1}\}$ is an open cover for K, there exists a finite subcollection, $B \subseteq \mathbb{N}^+$ such that

$$K \subseteq \bigcup_{j \in B} U_j$$

Using Theorem 4.41 on this finite cover of K, there exists a partition of unity in $\{g_{j\leq n}\}$ where

- $g_j \in C_c(X, [0, 1]),$
- $g_i \prec U_j \subseteq U$ for every $j \leq n$, and
- $\sum g_j = 1$ on K,

And notice for every $j \leq n$,

$$\{f = 0\} \cup \{g_j = 0\} \subseteq \{f \cdot g_j = 0\} \implies \{f \cdot g_j \neq 0\} \subseteq \{f \neq 0\} \cap \{g_j \neq 0\}$$
$$\implies \operatorname{supp}(f \cdot g_j) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g_j)$$
$$\implies \operatorname{supp}(f \cdot g_i) \subseteq U_i \subseteq U$$

Hence $f \cdot g_j \prec U$ and $f \cdot g_j \in C_c(X, [0, 1])$ for every $1 \leq j \leq n$. Moreover, if we take the sum over a finite n, we obtain $f = \sum f \cdot g_{j \leq n}$, this is because for every $x \in X$, so we have

$$\sum_{j \leq n} f(x) \cdot g_j x = f(x) \cdot \sum_{j \leq n} g_j(x) = f(x)$$

Then $I(f) = I(\sum f \cdot g_j) = \sum I(f \cdot g_j)$. And by definition of $\mu(U_j)$, since it is the supremum over all $I(h_j)$, where $h_j \in C_c(X, [0, 1])$ and $h_j \prec U_j$

$$I(f\cdot g_j) \leq \mu(U_j), \quad \forall j \leq n$$

Hence

$$I(f) \le \sum_{j \le n} \mu(U_j) \le \sum_{j \ge 1} \mu(U_j)$$

Where for the last estimate we used the fact that μ is non-negative, and since this holds for any f, we can conclude that $\mu(U) \leq \sum_{j>1} \mu(U_j)$.

Part c

Proof. By definition of a topology, $\{\emptyset, X\} \subseteq \mathcal{T}$, and $\mu(\emptyset) = \sup\{I(f), f \in C_c(X), f \prec \emptyset\}$, so supp $(f) = \emptyset$, and $\{x, f(x) \neq 0\} \subseteq \emptyset$, so the set contains one element, namely I(0) = 0 by linearity. So $\mu(\emptyset) = 0$. The assumptions for Theorem 1.10 are satisfied and (4) is indeed an outer-measure.

Part d

Proof. Denote the right members of (4) and (5) by W_1 and W_2 , we wish to show that $\inf W_1 = \inf W_2$. Clearly $\inf W_1 \leq \inf W_2$, since $W_2 \subseteq W_1$. Now, if μ is countably additive, then for every $\omega \in W_1$ induces a sequence of open sets $\{U_{j\geq 1}\}$ such that $E \subseteq \bigcup U_{j\geq 1}$. Denote the union over $\{U_{j\geq 1}\}$ by U, which is also another open set,

$$\inf W_2 \le \mu(U) \le \sum \mu(U_{j \ge 1}) = \omega$$

Since ω is arbitrary, we conclude that $\inf W_2 = \inf W_1$, and this proves (d).

Part e

Proof. If U and V are open subsets of X, and if $U \subseteq V$, then

$$U \subseteq V \implies \{f \in C_c(X), \ f \prec U\} \subseteq \{f \in C_c(X), \ f \prec V\}$$
$$\implies \{I(f), \ f \in C_c(X), \ f \prec U\} \subseteq \{I(f), \ f \in C_c(X), \ f \prec V\}$$

Hence $\mu(U) \leq \mu(V)$. Now by equation (5), $\mu^*(U) \leq \mu(U)$. To show the reverse inequality, suppose by contradiction that $\mu^*(U) < \mu(U)$.

Since $\mu^*(U)$ is an infimum, then for every $\varepsilon > 0$ there exists some $V \supseteq U$ where if we write $\mu^*(U) + \varepsilon = \mu(U)$

$$\mu(V) < \mu^*(U) + \varepsilon = \mu(U) \implies \mu(V) < \mu(U), \ U \subseteq V$$

This contradicts what we have just proven, and therefore $\mu^*(U) = \mu(U)$ for every open set U.

Part f

Proof. We wish to show that every open set U is μ^* -measurable. By Theorem 1.10, it suffices to show that for every $E \subseteq X$

$$\mu^*(E) \ge \mu^*(E \cap U) + \mu^*(E \setminus U) \tag{10}$$

because the reverse inequality is given by subadditivity of μ^* , and we can also assume that $\mu^*(E) < +\infty$. Let us assume that E is open, we wish to find some function $h \in C_c(X)$, $h \prec E$ with

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

The above formula is fussy, but the liberty is taken to show it beforehand to avoid any potential confusion that follows. Since $E \cap U$ is an open subset of X, the definition of $\mu(E \cap U) = \mu^*(E \cap U)$ in (3) tells us that every $\varepsilon > 0$ induces some $f \in C_c(X)$, $f \prec E \cap U$ where

$$I(f)>\mu(E\cap U)-\varepsilon=\mu^*(E\cap U)-\varepsilon \eqno(11)$$

Also, supp (f) is a closed set (compact subsets of Hausdorff spaces are closed), therefore $E \setminus \text{supp}(f)$ is an open set. We make a small diversion from the current part of the proof and turn out attention to the fact that

$$\operatorname{supp}(f) \subseteq U \implies U^c \subseteq (\operatorname{supp}(f))^c$$
$$\implies E \setminus U \subseteq E \setminus \operatorname{supp}(f)$$

And because the outer-measure μ^* is monotone.

$$\mu^*(U) \le \mu^*(E \setminus \text{supp}(f)) \tag{12}$$

Now, using the definition of $\mu(E \setminus \text{supp}(f))$ (recall that $E \setminus \text{supp}(F)$ is an open set), for every $\varepsilon > 0$, there exists some $g \in C_c(X)$, $g \prec E \setminus \text{supp}(F)$ with

$$I(g) > \mu(E \setminus \text{supp}(f)) - \varepsilon = \mu^*(E \setminus \text{supp}(f)) - \varepsilon \tag{13}$$

It is at this part of the proof where we wish to define h = f + g, but first we must verify

- $f + g \in C_c(X, [0, 1]),$
- $f + g \prec E$

The sum of two non-negative functions is non-negative, and for every $x \in \text{supp}(f), f \leq 1$. Also

$$\operatorname{supp}(g) \subseteq (\operatorname{supp}(f))^c \implies \operatorname{supp}(f) \subseteq (\operatorname{supp}(g))^c$$
$$\implies \operatorname{supp}(f) \subseteq \{g = 0\}$$

The last implication comes from taking complements on both sides of $\{g \neq 0\} \subseteq \text{supp}(g)$. So $x \in \text{supp}(f) \implies f+g \leq 1$. Now if $x \notin \text{supp}(f)$, then $f+g=g \leq 1$. Furthermore, supp (f+g) is a closed subset of compact supp $(f) \cup \text{supp}(g)$. This is because $\{f+g \neq 0\} \subseteq \{f \neq 0\} \cup \{g \neq 0\}$, and the finite union of two compact sets is again again compact.

A moment's thought should yield the fact that the last estimate should be an equality, but it is a needless distraction. Therefore supp (f + g) is compact and $f + g \in C_c(X, [0, 1])$.

Now both bullet points are satisfied, and we can set h = f + g. Adding equation (13) with (11) gives us

$$I(h) = I(f) + I(g) > \mu^*(E \cap U) + \mu^*(E \setminus \text{supp}(f)) - 2\varepsilon$$

Upon applying (12) to the right member of the above estimate, we have

$$I(h)>\mu^*(E\cap U)+\mu^*(E\setminus U)-2\varepsilon$$

But this particular $h \in C_c(X) \cap \{f \prec E\}$, therefore

$$\mu^*(E) \geq I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, equation (10) holds for every open E. Now for any general $E \subseteq X$, fix any $\varepsilon > 0$ and by how we defined $\mu^*(E)$, there exists some open $V \supseteq E$ —recall that $\mu^*(E)$ is the infimum over the set of $\mu(V)$ where V is an open superset of E—hence

$$\mu^*(E) + \varepsilon > \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

By monotonicity (twice) of the outer-measure μ^* , we have

$$\mu^*(E) + \varepsilon > \mu^*(E \cap U) + \mu^*E \setminus U$$

Let $\varepsilon \to 0$, and we get

$$\mu^*(E) \ge \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Therefore every open $U\subseteq X$ is μ^* -measurable. So $\mu=\mu^*|_{\mathbb{B}}$ is a Borel measure on X.

Part g

Proof. To show outer-regularity, fix any $E \in \mathbb{B}$, then by definition,

$$\mu(E) = \mu^*(E) = \inf \{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$

And for every open U, (1) follows from Equation (3).

Part h

Proof. We want to show that for every compact K, Equation (2) holds. To reduce the notational baggage that follows, we agree to define

$$\{I(f), f \in C_c(X), f \prec U\} = \{I(f), f \prec U\}$$

Similarly for $\{I(f), f \geq \chi_K\}$. If $\mu(K) = 0$, then $\mu(K)$ is obviously a lower bound, since $f \geq \chi_K \geq 0$ means that $I(f) \geq 0$, for every $f \geq \chi_K$. So we can suppose $\mu(K) > 0$.

Fix an arbitrary $f \ge \chi_K$, then this particular f induces an open set $U_{\alpha} = \{f > 1 - \alpha\}$, where $\alpha > 0$. Notice also that

$$K\subseteq \{f\geq 1\}\subseteq \{f>1-\alpha\}=U_\alpha$$

Since U_{α} is an open superset of K, by Equation (7), $\mu(K) \leq \mu(U_{\alpha})$, but $\mu(U_{\alpha})$ is simply the supremum of $\{I(g), g \prec U_{\alpha}\}$. If we wish to show that $\mu(K) \leq \mu(U_{\alpha}) \leq I(f)$, it suffices to show that I(f) is an upper-bound for $\{I(g), g \prec U_{\alpha}\}$.

Fix any $I(g) \in \{I(g), g \prec U_{\alpha}\}$, note that $1 - \alpha \neq 0$ for any α small enough, then

- $f/(1-\alpha) > 1$ on U_{α} ,
- $1 \ge g \ge 0$ on U_{α} , in particular, $f/(1-\alpha) g \ge 0$ on U_{α} ,
- If $x \notin U_{\alpha}$, then $f/(1-\alpha) g = f(1-\alpha) \ge 0$.
- Therefore $f/(1-\alpha)-g\geq 0$ for any x, and by Lemma 2.1,

$$I(f/(1-\alpha)) \ge I(g) \quad \forall g \prec U_{\alpha}$$

Combining the above estimate with $\mu(K) \leq \mu(U_{\alpha})$ gives us

$$\mu(K) \le \frac{1}{1 - \alpha} I(f)$$

Now write $\varepsilon = \alpha/\mu(K) > 0$ and for every $\varepsilon > 0$ we get

$$\mu(K) - I(f) \le \alpha \mu(K) = \varepsilon$$

Send $\varepsilon \to 0$ and $\mu(K) \le I(f)$ for every $f \ge \chi_K$.

To show that $\mu(K)$ is indeed the infimum for $\{I(f), f \geq \chi_K\}$, notice that for every $\varepsilon > 0$ we can obtain some open superset $U \supseteq K$ (by outer-regularity) where $\mu(U) < \mu(K) + \varepsilon$. By Urysohn's Lemma, there exists some $g \prec U$, g(x) = 1 for every $x \in K$.

$$g \in \{I(f), \ f \prec U\} \cap \{I(f), \ f \geq \chi_K\}$$

Therefore $I(g) \le \mu(U) < \mu(K) + \varepsilon$ as desired, and Equation (2) holds.

Part i

Proof. $\mu(K) < +\infty$ for every compact K. Indeed, since $I(\chi_K) \in \{I(f), f \ge \chi_K\}$, then by Theorem 7.1, there exists a constant $C_K \ge 0$ that bounds

$$\mu(K) \leq |I(\chi_K)| = I(\chi_K) \leq C_K \cdot \|\chi_K\| = C_K < +\infty$$

Part j

Proof. Fix any open set U, then for every $\varepsilon > 0$, there exists some $f \prec U$ with $\mu(U) - \varepsilon < I(f)$. Then denote $K = \text{supp}(f) \subseteq U$. If we take any $I(h) \in \{I(h), h \geq \chi_K\}$, then $h \geq f$ gives us $I(h) \geq I(f)$ by Lemma 2.1. So I(f) is a lower bound of $\{I(h), h \geq \chi_K\}$, therefore

$$\mu(U) - \varepsilon \le I(f) \le \mu(K)$$

Since supp $(f) = K \subseteq U$, this proves inner-regularity of μ on open sets.

Part k

Proof. Suppose $f \in C_c(X, [0, 1])$, we first show that Equation (8) holds. We divide the interval [0, 1] into $N \ge 1$ chunks by writing

$$K_j = \{f \ge j/N\}$$

for every $1 \ge j \ge N$. And define $K_0 = \text{supp}(f)$. Each K_j is a closed subset of supp (f), and therefore compact. More is true,

- $K_{j-1} \supseteq K_j$ for every $1 \le j \le N$.
- $x \in K_j$ iff $f(x) \in \left[\frac{j}{N}, 1\right]$,
- $x \notin K_j$ iff $f(x) \in \left[0, \frac{j}{N}\right)$, and
- $x \in (K_{j-1} \setminus K_j)$ iff $f(x) \in \left[\frac{j-1}{N}, \frac{j}{N}\right]$

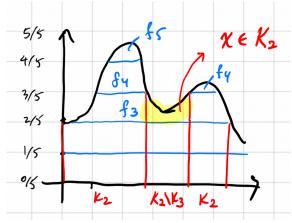
Folland constructs a finite sequence of compactly supported functions, $\{f_j\}$, where $1 \leq j \leq N$ such that

- Each $0 \le f_i \le 1/N$,
- If $x \in (K_m \setminus K_{m+1})$ iff $f(x) \in \left[\frac{m}{N}, \frac{m+1}{N}\right)$ means that $f_j = 1$ for all $1 \le j \le m$, and
- $f_{m+1} = f m/N$ on K_m , such that

$$f(x) = \left(\sum f_{j \leq m}(x)
ight) + \left(f(x) - rac{m}{N}
ight) = rac{m}{N} + \left(f(x) - rac{m}{N}
ight)$$

- And for every $m < j \le N, f_j = 0.$
- If $x \notin K_m$ iff $f(x) \in [0, \frac{m}{N}]$ then for every $m+1 \le j \le N$, $f_j = 0$.

The illustration for when N=5 below should make things clearer.



It is also trivial to verify that

• For every $x \in K_j$, $f_j = N^{-1}$, and

$$\chi_{K_i} N^{-1} \le f_j \tag{14}$$

Also, if $x \notin K_j$ then $f_j \ge 0$, therefore $f_j \ge \chi_{K_j} N^{-1}$ at every x.

• If $x \notin K_{j-1}$ then $f_j = 0 \le \chi_{K_{j-1}} \cdot N^{-1}$. If x is in K_{j-1} then $f_j \le N^{-1}$ by construction and therefore

$$f_j \le \chi_{K_{j-1}} N^{-1} \tag{15}$$

for all x.

• $f_j \in C_c(X)$, since supp $(f_j) \subseteq \text{supp } (f)$.

Combining Equations (14) with (15), and by monotonicity in $L^+(X, \mathbb{B}, \mu)$, since $f_j \in L^+$

$$\int \frac{1}{N} \chi_{K_j} d\mu \le \int f_j d\mu \le \int \frac{1}{N} \chi_{K_{j-1}} d\mu$$

And for every $1 \le j \le N$,

$$\frac{1}{N}\mu(K_j) \le \int f_j d\mu \le \frac{1}{N}\mu(K_{j-1}) \tag{16}$$

Furthermore, from Equation (14), since $Nf_j \ge \chi_{K_j}$ then by Equation (2),

$$\mu(K_j) \leq I(Nf_j) \implies \frac{1}{N} \mu(K_j) \leq I(f_j)$$

Now for any arbitrary $I(h) \in \{I(h), h \ge \chi_{K_{j-1}}\}$, since

$$h \ge \chi_{K_{i-1}} \ge Nf_i \implies I(h) \ge I(Nf_i)$$

So $NI(f_j)$ is a lower bound for $\{I(h), \ h \geq \chi_{K_{j-1}}\}$ and

$$I(f_j) \le \frac{1}{N} \mu(K_{j-1})$$

Combining the last two results, with $I(f_i)$, we get

$$\frac{1}{N}\mu(K_j) \le I(f_j) \le \frac{1}{N}\mu(K_{j-1}) \tag{17}$$

Taking the sum over $1 \le j \le N$ for Equations (16) and (17). Define $A = N^{-1} \sum_{i=0}^{N-1} \mu(K_j)$, and $B = N^{-1} \sum_{i=0}^{N} \mu(K_j)$

$$B \le \int f d\mu \le A$$

And also

$$B \leq I(f) \leq A$$

This is because of finite additivity of both I and the integral, and $f = \sum f_j$ on $K_0 = \text{supp}(f)$. Subtracting the two equations (keeping in mind that $\mu(K_j) < +\infty$ for any compact K_j), we get

$$(-1)(A-B) \leq \left(\int f d\mu - I(f) \right) \leq A - B \implies \left| \int f d\mu - I(f) \right| \leq A - B$$

It is trivial to verify that

$$0 < A - B = N^{-1}(\mu(K_0) - \mu(K_N)) < N^{-1}\mu(K_0)$$

as $K_N \subseteq K_0$. Let $N \to \infty$ and

$$\int f d\mu = I(f)$$

Equation (8) holds as desired.

Part 1

Proof. Now for any general $f \in C_c(X)$, f must be bounded on the plane since $C_c(X) \subseteq BC(X)$, and $|f| \leq M_0$ for some $M_0 \geq 0$. Since supp (f) is compact, we know that

$$\int |f| d\mu \le \int M_0 \chi_{\operatorname{supp}(f)} d\mu \le M_0 \mu(\operatorname{supp}(f)) < +\infty$$

And $C_c(X) \subseteq L^1(\mu)$. Furthermore,

$$\frac{1}{2}(|\operatorname{Re} f| + |\operatorname{Im} f|) \le |f| \le M_0$$

So that Re f and Im f are in $C_c(X)$. Without loss of generality, we may assume that f is real. Define $f_1 = \operatorname{Re} f^+/M_0$ and $f_2 = \operatorname{Re} f^-/M_0$ and it immediately follows that $f_1, f_2 \in C_c(X, [0, 1])$.

By linearity of I on $C_c(X)$ and the integral in $L^1(\mu)$,

$$I(f_1-f_2)=I(f)=\int f d\mu =\int f_1 d\mu -\int f_2 d\mu$$

Then we may apply the above to the real and imaginary parts of a general $f \in C_c(X)$, and this completes the proof.

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Folland Reading

Theorem 7.3

Theorem 7.3

Proposition 3.1. See Theorem 7.2

Proof.

Theorem 7.4

Proposition 4.1. See Theorem 7.2

Theorem 7.5

Proposition 5.1.

Theorem 7.6

Proposition 6.1.

Theorem 7.7

Proposition 7.1.

Theorem 7.8

Proposition 8.1.

Proof.

Theorem 7.9

Proposition 9.1. If μ is a Radon measure on X, then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < +\infty$.

Proof. Theorem 6.7 tells us that the set of L^p simple functions (as Folland calls them), which are

$$\Lambda = \left\{f,\, f = \sum_{j \leq n} a_j \chi_{E_j}, \; a_j \in \mathbb{C}, \, \mu(E_j) < +\infty
ight\}$$

So for every $f \in L^p$, there exists a sequence $\{f_n\} \subseteq \Lambda$ with $f_n \to f$ pointwise and $f_n \to f$ in L^p .

Theorem 7.10

Proposition 10.1.

Theorem 7.11

Proposition 11.1.