

# Chapter 1

**Theorem 1.1**

**Proposition 1.1.** *Let  $\mathcal{M}(\mathcal{F})$  be the  $\sigma$ -algebra generated by  $\mathcal{F}$ , if  $\mathcal{E}$  is a subset of  $\mathbb{P}(X)$ , with  $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$ , then  $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$ .*

*Proof.* Notice that because  $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$ ,

$$\mathcal{M}(\mathcal{F}) \in \{\mathcal{M}, \mathcal{E} \subseteq \mathcal{M}, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

Taking the intersection, noting that  $\mathcal{M}(\mathcal{E})$  is the intersection of all  $\sigma$ -algebras containing  $\mathcal{E}$  as a subset, we have

$$\bigcap \{\mathcal{M}(\mathcal{F})\} \supseteq \bigcap \{\mathcal{M}, \mathcal{E} \subseteq \mathcal{M}, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

And

$$\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$$

■

## Theorem 1.2

**Proposition 2.1.** *The Borel  $\sigma$ -algebra of  $\mathbb{R}$ ,  $\mathbb{B}$  is generated by the following*

- The family of open intervals  $\mathcal{E}_\infty = \{(-\infty, b), a < \infty\}$ ,
- The family of closed intervals  $\mathcal{E}_\epsilon = \{[a, b], a < b\}$ ,
- The family of half-open intervals  $\mathcal{E}_\sup = \{(-\infty, b), a < \infty\}$  or  $\mathcal{E}_\Delta = \{[a, b], a < b\}$
- The open rays  $\mathcal{E}_\nabla = \{(-\infty, +\infty), a \in \mathbb{R}\}$  or  $\mathcal{E}_\nearrow = \{(-\infty, a), a \in \mathbb{R}\}$
- The closed rays  $\mathcal{E}_\lrcorner = \{[a, +\infty), a \in \mathbb{R}\}$  or  $\mathcal{E}_\searrow = \{(-\infty, a], a \in \mathbb{R}\}$

*Proof.* By definition,  $\mathbb{B}$  is generated by the family of all open sets in  $\mathbb{R}$ , but every open set is a countable union of open intervals. Therefore

$$\mathcal{T}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E}_\infty) \implies \mathbb{B} \subseteq \mathcal{M}(\mathcal{E}_\infty)$$

Conversely, every open interval is an open set, hence

$$\mathcal{E}_\infty \subseteq \mathcal{T}_\mathbb{R} \subseteq \mathbb{B} \implies \mathcal{M}(\mathcal{E}_\infty) \subseteq \mathbb{B}$$

Every closed interval can also be written as a countable intersection of open intervals, for every  $[a, b]$ , with  $a < b$ , we have

$$[a, b] = \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1}) \quad (1)$$

Indeed, fix any  $x \in [a, b]$  then for every  $n \geq 1$ ,

$$a - n^{-1} < a \leq x \leq b < b + n^{-1}$$

So  $x \in \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1})$ . If  $x$  an element of the left member, then for every  $n \geq 1$ ,

$$a - n^{-1} < x \implies a - x < n^{-1}$$

Similarly for  $x \leq b$ , therefore equation (1) is valid, and  $\mathcal{E}_\epsilon \subseteq \mathbb{B} = \mathcal{M}(\mathcal{E}_\infty)$ . To show the reverse estimate, every open interval can be written as a countable union of closed intervals,

$$(a, b) = \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \quad (2)$$

To show that the above estimate is indeed true, fix any  $x \in (a, b)$ , then

$$\begin{aligned} a < x < b &\iff a < a + n^{-1} \leq x \leq b - n^{-1} < b \\ &\iff x \in \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \end{aligned}$$

So that equation (2) holds. By similar argumentation we have  $\mathcal{E}_\infty \subseteq \mathcal{M}(\mathcal{E}_\epsilon) \implies \mathcal{M}(\mathcal{E}_\epsilon) = \mathcal{M}(\mathcal{E}_\infty)$ .

For  $\mathcal{E}_\exists$ ,  $\mathcal{E}_\Delta$

- $(a, b] = \bigcap_{n \geq 1} (a, b + n^{-1})$ , proves  $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1)$ ,
- $(a, b) = \bigcup_{n \geq 1} (a, b - n^{-1}]$ , proves  $\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3)$ ,
- $[a, b) = \bigcup_{n \geq 1} [a, b - n^{-1}]$ , proves  $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_2)$ ,
- $[a, b] = \bigcap_{n \geq 1} [a, b + n^{-1})$ , proves  $\mathcal{M}(\mathcal{E}_2) \subseteq \mathcal{M}(\mathcal{E}_4)$

So that  $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_4) = \mathbb{B}$ . By taking complements of each element we get  $\mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$  and  $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$ . Notice also that

- $(a, b] = (a, +\infty) \cap (-\infty, b]$ , proves  $\mathcal{E}_\exists \subseteq \mathcal{M}(\mathcal{E}_\nabla)$ , and  $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$ .
- $(a, +\infty) = \bigcup_{n \geq 1} (a, a + n]$ , proves  $\mathcal{E}_\nabla \subseteq \mathcal{M}(\mathcal{E}_\exists)$ , and  $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_3)$ .
- $[a, b) = [a, +\infty) \cap (-\infty, b)$ , proves  $\mathcal{E}_\Delta \subseteq \mathcal{M}(\mathcal{E})$ , and  $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_7)$ ,
- $[a, +\infty) = \bigcup_{n \geq 1} [a, a + n)$ , proves  $\mathcal{E}_\Delta \subseteq \mathcal{M}(\mathcal{E}_\Delta)$ , and  $\mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{M}(\mathcal{E}_4)$ .

Finally,  $\mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8) = \mathbb{B}$  and  $\mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7) = \mathbb{B}$ . ■

**Theorem 1.3**

**Proposition 3.1.** *If  $A$  is countable, then  $\otimes_{\alpha \in A} \mathcal{M}_\alpha$  is the  $\sigma$ -algebra generated by*

$$W := \left\{ \prod_{\alpha \in A} E_\alpha, E_\alpha \in \mathcal{M}_\alpha \right\}$$

*Proof.* We agree to define

$$V := \left\{ \pi_\alpha^{-1}(E_\alpha), E_\alpha \in \mathcal{M}_\alpha \right\}$$

By definition,  $V$  generates  $\otimes_{\alpha \in A} \mathcal{M}_\alpha$ . Fix any element in  $x = \pi_\alpha^{-1}(E_\alpha) \in V$ , then

$$\pi_\alpha(x) \in E_\alpha, \pi_{\beta \neq \alpha}(x) \in X_\beta$$

Then  $x \in W$  if we choose  $x = \prod_{c \in A} E_c$ , for  $E_c = E_\alpha$  if  $c = \alpha$ , and  $E_c = X_c$  if  $c \neq \alpha$ . ■

**Theorem 1.4**

**Proposition 4.1.**

*Proof.*



**Theorem 1.5**

**Proposition 5.1.**

*Proof.*



**Theorem 1.6**

**Proposition 6.1.**

*Proof.*





**Theorem 1.7**

**Proposition 7.1.**

*Proof.*



**Theorem 1.8**

**Proposition 8.1.**

*Proof.*



**Theorem 1.9**

**Proposition 9.1.**

*Proof.*



**Theorem 1.10**

**Proposition 10.1.**

*Proof.*



**Theorem 1.11**

**Proposition 11.1** (Caratheodory's Theorem). *If  $\mu^*$  is an outer measure on  $X$ , the collection  $\mathcal{M}$  of  $\mu^*$ -measurable sets is a  $\sigma$ -algebra, and the restriction of  $\mu^*$  to  $\mathcal{M}$  is a complete measure.*

*Proof.* We quote the definition for a set  $A \subseteq X$  to be  $\mu^*$  measurable. For any  $E \subseteq X$ ,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \quad (3)$$

- Show  $\mathcal{M}$  is an algebra.
- $\mu^*$  is finitely additive on  $\mathcal{M}$ .
- $\mathcal{M}$  is closed under countable disjoint (this makes  $\mathcal{M}$  a sigma algebra, since it is an algebra that is closed under countable disjoint unions)

**Lemma 11.1** *The family of  $\mu^*$ -measurable sets is an algebra.*

*Proof of Lemma 11.1.* Clearly  $\mathcal{M}$  is closed under complements. To show that it is a  $\sigma$ -algebra, and if  $A, B \in \mathcal{M}$ , then  $\left\{ \underbrace{E \cap A}_1, \underbrace{E \setminus A}_2 \right\} \subseteq \mathbb{P}(X)$ . And because  $B$  is  $\mu^*$ -measurable,

$$\mu^*(E) = \underbrace{\mu^*(E \cap A \cap B) + \mu^*(E \cap A \setminus B)}_1 + \underbrace{\mu^*(E \cap B \setminus A) + \mu^*(E \setminus (A \cup B))}_2$$

By subadditivity of  $\mu^*$ ,  $A \cup B = A \cap B + A \setminus B + B \setminus A$  with  $+$  denoting the disjoint union, hence

$$\mu^*(E \cap (A \cap B)) \leq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \setminus B)) + \mu^*(E \cap (B \setminus A))$$

and

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B))$$

■

**Lemma 11.2**  *$\mu^*$  is finitely additive on  $\mathcal{M}$ , the family of  $\mu^*$ -measurable sets.*

*Proof of Lemma 11.2.* Let  $A, B$  be disjoint  $\mu^*$ -measurable sets. It suffices to show  $\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B)$ , as the reverse estimate follows from subadditivity. From Lemma 11.1,  $A \cup B \in \mathcal{M}$ , so

$$\begin{aligned}\mu^*(A \cup B) &= \mu^*(A \cup B \cap A) + \mu^*(A \cup B \setminus A) \\ &= \mu^*(A \cup \emptyset) + \mu^*(A \setminus A \cup B \setminus A) \\ &= \mu^*(A) + \mu^*(B)\end{aligned}$$

■

**Corollary 11.1** *If  $\{A_j\}_{j \geq N} \subseteq \mathcal{M}$  is a finite disjoint family, then*

$$\mu^*\left(\bigcup A_{j \leq N}\right) = \sum \mu^*(A_{j \leq N})$$

**Lemma 11.3** *Let  $\{A_j\}_{j \geq 1}$  be a countable disjoint sequence in  $\mathcal{M}$ , and denote  $B_n = \bigcup A_{j \leq n} \in \mathcal{M}$  by Lemma 11.1. For every  $E \subseteq X$ ,*

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j \leq n})$$

*Proof of Lemma 11.3.* We will proceed by induction. If  $n = 1$  then we have equality, suppose the result holds for  $n \in \mathbb{N}^+$ , and  $A_{n+1} \in \mathcal{M}$  so

$$\begin{aligned}\mu^*(E \cap B_{n+1}) &= \mu^*(E \cap B_{n+1} \cap A_{n+1}) = \mu^*(E \cap B_{n+1} \setminus A_{n+1}) \\ &= \mu^*(E \cap A_{n+1}) + \mu^*(E \cap B_n) \\ &= \sum_{j \leq n+1} \mu^*(E \cap A_j)\end{aligned}$$

as  $A_j \cap A_{n+1} = \emptyset \iff A_j \setminus A_{n+1} = A_j$ , and  $B_n \cap A_n = A_n \iff A_n \subseteq B_n$ . ■

To show  $\mathcal{M}$  is a sigma-algebra, fix any disjoint sequence  $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$ , and denote  $B_n$  as in lemma 11.3. Define  $B = \bigcup A_{j \geq 1} \supseteq B_n$  and for every  $n \geq 1$ ,

we have

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \setminus B_n) \\
&= \sum \mu^*(E \cap A_{j \leq n}) + \mu^*(E \setminus B_n) \\
&\geq \sum \mu^*(E \cap A_{j \leq n}) + \mu^*(E \setminus B) \quad \text{since } B_n \subseteq B \iff B^c \subseteq B_n^c \\
&\geq \sup_n \left[ \sum \mu^*(E \cap A_{j \leq n}) \right] + \mu^*(E \setminus B)
\end{aligned}$$

Let  $J \subseteq \mathbb{N}^+$  be a finite non-empty set. And  $\sup J \in \mathbb{N}^+$ ,  $\sup J < +\infty$ . By the Archimedean Property we can find a large  $N \in \mathbb{N}^+$ , with  $N > J$  so that

$$\sum_{j \in J} \mu^*(E \cap A_j) \leq \sum_{j \leq N} \mu^*(E \cap A_j)$$

Applying the estimate  $\sup_n \left[ \sum \mu^*(E \cap A_{j \leq n}) \right] + \mu^*(E \setminus B) \leq \mu^*(E)$  reads

$$\left[ \sum_{j \in J} \mu^*(E \cap A_j) \right] + \mu^*(E \setminus B) \leq \mu^*(E)$$

Now by Chapter 0, the infinite sum

$$\sum_{j \geq 1} \mu^*(E \cap A_j) = \sup \left\{ \sum_{j \in J} \mu^*(E \cap A_j), J \subseteq \mathbb{N}^+, 0 < |J| < +\infty \right\}$$

and  $\bigcup A_{j \geq 1} = B$  is  $\mu^*$ -measurable. Since  $\mu^*(\emptyset) = 0$ , and  $\mu^*$  is countably additive on  $\mathcal{M}$ , (perhaps by replacing  $E$  with the union over all disjoint sets),  $\mu^*$  is a measure on  $\mathcal{M}$ . To show  $\mu^*$  is a complete measure, fix  $A \in \mathcal{M}$  where  $\mu^*(A) = 0$ . Then any  $B \subseteq A$  is also null, and for  $E \subseteq X$ ,

$$\mu^*(E) \geq \underbrace{\mu^*(E \cap B)}_0 + \mu^*(E \setminus B) \implies B \in \mathcal{M}$$

■

**Theorem 1.12**

**Proposition 12.1.**

*Proof.*





**Theorem 1.13**

**Proposition 13.1.**

*Proof.*



**Theorem 1.14**

**Proposition 14.1.**

*Proof.*



**Theorem 1.15****Proposition 15.1.**

*Proof.* If  $\{E_j\}_{j \geq 1} \subseteq \mathcal{A}$  such that each  $E_j = FDU(I_{ji})$  over finitely many  $i$ , and suppose  $E_j$  are disjoint, and that  $DU(E_j) \in \mathcal{A}$ . So that  $DU(E_j) = FDU(I_\alpha)$  for some finite collection of half-intervals  $\{I_\alpha\}$ .

We will first prove the simpler case. Suppose we have already proven:

$$\{E_j\}_{j \geq 1} \subseteq \mathcal{A}, DU(E_j) = I_\alpha \in \mathcal{A} \implies \mu_0\left(DU(E_j)\right) = \sum \mu_0(E_j) = \mu_0(I_\alpha) \quad (4)$$

but each  $E_j$  is a FDU of  $I_{ji}$ , and for every  $j \geq 1$ ,  $E_j \cap I_\alpha \in \mathcal{A}$  (closure under intersections, because the family of FDU of h-intervals is an algebra).

Thus we have a disjoint sequence whose union is one h-interval. In symbols:

$$DU(E_j) = FDU(I_\alpha) \implies I_\alpha = DU(E_j \cap I_\alpha)$$

$$\forall j \geq 1, E_j \cap I_\alpha \in \mathcal{A} \implies$$

$$\begin{aligned} \mu_0(FDU(I_\alpha)) &= \sum_{\alpha < +\infty} \mu_0(I_\alpha) \\ &= \sum_{\alpha < +\infty} \sum_{j \geq 1} \mu_0(E_j \cap I_\alpha) \\ &= \sum_{j \geq 1} \sum_{\alpha < +\infty} \mu_0(E_j \cap I_\alpha) \\ &= \sum_{j \geq 1} \mu_0(E_j) \end{aligned}$$

It is permissible to swap the two summations because we are using the supremum definition for a sum of non-negative terms. And we applied finite-additivity (see earlier), to conclude that  $\sum_{j \geq 1} \sum_{\alpha} \mu_0(E_j \cap I_\alpha) = \sum_{j \geq 1} \mu_0(E_j)$ . ■

Define

- $\mathcal{H}_1 = \left\{ (a, b], -\infty \leq a < b < +\infty \right\},$
- $\mathcal{H}_2 = \left\{ (a, +\infty), a \in \mathbb{R} \cup \{-\infty\} \right\},$
- $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \{\emptyset\}$ . Where  $+$  denotes the disjoint union.
- $DU$ : disjoint union,  $FDU$ : finite disjoint union.

Steps:

1. Show that  $\mathcal{H}$  is an elementary family.
2. Show that if  $I_\alpha \in \mathcal{H}_1$ , then for every  $I_\beta \in \mathcal{H}_1 \cup \mathcal{H}_2$ ,  $I_\alpha \cap I_\beta \in \mathcal{H}_1$ . We write this as

$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1, I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_1$$

3. Show that if  $I_\alpha \in \mathcal{H}_2$ , then

$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1, I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_2$$

4. Show that  $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$  is well defined. (modify the proof in Folland to check for  $a = -\infty$  with

$$\overline{F} : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}, \quad \begin{cases} \overline{F}|_{\mathbb{R}} &= F \\ \overline{F}(+\infty) &= \sup_x F(x), \\ \overline{F}(-\infty) &= \inf_x F(x) \end{cases}$$

5. Show that  $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$  is well defined for  $b < +\infty$ . If  $E = (a, b] \in \mathcal{A}$ , then  $E$  is an FDU of  $\mathcal{H}_1$ , and  $\mathcal{H}_2$ . So we write

$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2) = FDU(\mathcal{H}_1)$$

since  $E$  is bounded above, the  $\mathcal{H}_2$  part of the FDU must be null. Now fix  $E = FDU_{\mathcal{H}_1}(I_j) = FDU_{\mathcal{H}_1}(I_2)$ . And follow the proof in Folland to see the 'well-definedness' of  $\mu_0$  if  $E \in \mathcal{H}_1$ .

6. Next, suppose  $E \in \mathcal{H}_2$  and

$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2)$$

Clearly  $FDU(\mathcal{H}_2) \neq \emptyset$ , since  $E$  is unbounded above, and  $FDU(\mathcal{H}_2)$  consists of exactly one element, so we write

$$E = FDU(\mathcal{H}_1) + (z, +\infty)$$

7. Show that  $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$  is well defined. Hint: use the fact that if  $E \in \mathcal{A}$ , such that  $E = FDU(E, \mathcal{H}_1) + FDU(E, \mathcal{H}_2)$ , then  $FDU(E, \mathcal{H}_2)$  contains at most one element (after throwing away empty sets), then use this to deduce  $E \cap I_\alpha$  has a  $FDU(E \cap I_\alpha, \mathcal{H}_2)$  of exactly one  $\mathcal{H}_2$  interval, where  $I_\alpha$  participates in  $FDU(E, \mathcal{H}_2)$ , if  $E$  is unbounded above. Then take  $E \setminus I_\alpha = FDU(E \setminus I_\alpha, \mathcal{H}_1) = FDU(E, \mathcal{H}_1)$ .
8. Now show that  $\mu_0$  is well-defined on all  $E \in \mathcal{A}$ .
9. Continue the proof for Folland until you reach the unbounded intervals, then modify the 'right continuity argument' to add an extra  $\mathcal{H}_2$  interval. Let  $I = \mathcal{H}_1 + \mathcal{H}_2 = I_\alpha + I_\beta$ , meaning  $I$  can be represented by at most one  $\mathcal{H}_1$  and  $\mathcal{H}_2$  interval. If  $(I_k) \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$ , then  $\{I_k \cap I_\alpha\} \subseteq \mathcal{H}_1$ , and continue the proof as usual.

**Theorem 1.16**

**Proposition 16.1.**

*Proof.*



**Theorem 1.17**

**Proposition 17.1.**

*Proof.*



**Theorem 1.18**

**Proposition 18.1.**

*Proof.*





## Exercises

### Exercise 1.1

#### Proposition 1.1.

*Proof.*



**Exercise 1.2****Proposition 2.1.***Proof.*

**Exercise 1.3**

**Proposition 3.1.**

*Proof.*



**Exercise 1.4**

**Proposition 4.1.** *An algebra  $\mathcal{A}$  is a  $\sigma$ -algebra  $\iff$  it is closed under countable increasing unions.*

*Proof.*  $\Leftarrow$  is trivial. And it suffices to show that  $\mathcal{A}$  is closed under countable disjoint unions. Indeed, if  $\{E_j\}_{j \geq 1} \subseteq \mathcal{A}$  is a countable disjoint sequence of sets, write

$$F_n = \bigcup E_{j \leq n}$$

Clearly,  $F_j$  is increasing, and denote  $F = \bigcup E_{j \geq 1}$ , which is a member of  $\mathcal{A}$ . We claim that

$$\bigcup F_{n \geq 1} = \bigcup E_{j \geq 1}$$

Fix any  $x \in \bigcup E_{j \geq 1}$ , then  $x$  belongs in some  $E_j \subseteq F_j$ , and  $\supseteq$  is proven. Also, if  $x \in \bigcup F_{n \geq 1}$ , then there exists some  $F_n$  for which  $x$  is a member of. For this particular  $F_n$ , means that  $x \in E_j$  where  $j \leq n$  and  $x \in \bigcup E_{j \geq 1}$ .  $\blacksquare$

### Exercise 1.5

**Proposition 5.1.** *Let  $\mathcal{M}(\mathcal{E})$  be the  $\sigma$ -algebra generated by  $\mathcal{E} \subseteq X$ , and*

$$\mathcal{N} = \left\{ \mathcal{M}(\mathcal{F}), \mathcal{F} \subseteq \mathcal{E}, \mathcal{F} \text{ is countable} \right\}$$

*Show that  $\mathcal{M}(\mathcal{E}) = \mathcal{N}$ .*

*Proof.* The outline of the proof is as follows,

1. Prove that  $\mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$ ,
2. Show that  $\mathcal{N}$  is a  $\sigma$ -algebra,
3. Show that  $\mathcal{N}$  contains  $\mathcal{E}$  as a subset, and hence  $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{N}$ .

First, for any  $\mathcal{F} \subseteq \mathcal{E}$ , where  $\mathcal{F}$  is countable, it follows from Lemma 1.1 that  $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}(\mathcal{E})$ . Taking the union over all of such  $\mathcal{F}$ , we get  $\bigcup \mathcal{M}(\mathcal{F}) = \mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$ .

To show that  $\mathcal{N}$  is a  $\sigma$ -algebra, fix any  $A \in \mathcal{N}$ , and  $A$  belongs to  $\mathcal{M}(\mathcal{F})$ , therefore  $A^c \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{N}$ . To show closure under countable unions, fix a sequence  $\{E_j\} \subseteq \mathcal{N}$ , then each of these  $E_j$  belongs to a corresponding  $\mathcal{M}(\mathcal{F}_j)$ , for  $j \in \{1, 2, \dots\}$ . Now define

$$\overline{\mathcal{F}} = \bigcup \mathcal{F}_{j \geq 1} \subseteq \mathcal{E}$$

and  $\overline{\mathcal{F}}$  is obviously countable. Hence for every  $j \geq 1$ ,  $\mathcal{M}(\mathcal{F}_j) \subseteq \mathcal{M}(\overline{\mathcal{F}})$  and taking the union yields

$$\bigcup \mathcal{M}(\mathcal{F}_{j \geq 1}) \subseteq \mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$$

It is also clear that our sequence  $\{E_j\}$  is contained in  $\mathcal{M}(\overline{\mathcal{F}})$ , and  $E = \bigcup E_j$  belongs to  $\mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$  as an element. Therefore  $\mathcal{N}$  is a  $\sigma$ -algebra.

Let  $\alpha \in A$  index the family of sets in  $\mathcal{E}$ , (so that  $E_\alpha \in \mathcal{E}$ ) and the singleton set of a set  $\{E_\alpha\}$  is a countable subset of  $\mathcal{E}$ . For every  $\alpha \in A$ , we have

$$E_\alpha \in \mathcal{M}(\{E_\alpha\}) \subseteq \mathcal{N} \implies \mathcal{E} \subseteq \mathcal{N}$$

And one final application of Lemma 1.1 finishes the proof. ■

**Exercise 1.6****Proposition 6.1.***Proof.*

**Exercise 1.7**

**Proposition 7.1.** *If  $\mu_1, \dots, \mu_n$  are measures on  $(X, \mathcal{M})$ , and  $a_1, \dots, a_n \in [0, +\infty)$ , then  $\mu = \sum_1^n \mu_j$  is a measure on  $(X, \mathcal{M})$ .*

*Proof.* If  $\{E_j\}$  is a disjoint sequence in  $\mathcal{M}$ , and denote  $E = \bigcup (E_j)$ . If for each  $k \leq n$ ,  $\mu_k(E) < +\infty$ ,

$$\mu_k(E) = \sum \mu_k(E_j) \implies a_k \mu_k(E) = \sum a_k \mu_k(E_j)$$

Then,

$$\mu(E) = \sum_{k \leq n} a_k \mu_k(E) = \sum_{k \leq n} \sum_{j \geq 1} a_k \mu_k(E_j) = \sum_{j \geq 1} \sum_{k \leq n} a_k \mu_k(E_j) = \sum_{j \geq 1} \mu(E_j)$$

If there exists some  $\mu_k$  such that  $\mu_k(E) = +\infty$ , then

$$\mu(E) = \sum_{k \leq n} \sum_{j \geq 1} a_k \mu_k(E_j)$$

Now if there exists some  $\mu_{k'}$  with  $\mu_{k'}(E) = +\infty$ , then  $\mu(E) = \sum_{k \leq n} \mu_k(E) = +\infty$ , and

$$\sum_{j \geq 1} \mu(E_j) = \sup_N \sum_{j \leq N} \sum_{k \leq n} a_k \mu_k(E_j) \geq \mu_{k'}(E)$$

Therefore  $\mu(E) = \sum_{j \geq 1} \mu(E_j)$ , and  $\mu$  is a measure. ■

### Exercise 1.8

**Proposition 8.1.** *If  $(X, \mathcal{M}, \mu)$  is a measure space, and  $\{E_j\} \subseteq \mathcal{M}$ , then  $\mu(\liminf E_j) \leq \liminf \mu(E_j)$ . Also,  $\mu(\limsup E_j) \geq \limsup \mu(E_j)$  provided that  $\mu(\bigcup E_{j \geq 1}) < +\infty$*

*Proof.* If  $\{E_j\}_{j \geq 1}$  is a sequence in  $\mathcal{M}$ , and define  $F_m = \bigcap_{j \geq m} E_j$

$$\liminf E_j = \bigcup_{m \geq 1} \bigcap_{j \geq m} E_j = \bigcup_{m \geq 1} F_m$$

Also, for every  $m \geq 1$ ,  $F_m \subseteq E_m$ , and  $F_m$  is an increasing sequence, because

$$[m, +\infty) \supseteq [m+1, +\infty) \implies F_m \subseteq F_{m+1}$$

Using continuity above, and writing  $F = \bigcup F_{m \geq 1} = \liminf E_j$ , we have

$$\begin{aligned} \mu(\liminf E_j) &= \mu(F) \\ &= \liminf \mu(F_m) \\ &\leq \liminf \mu(E_m) \end{aligned}$$

The second part of the proof is similar, if  $G_m = \bigcup_{j \geq m} E_j$ , then

$$\limsup E_j = \bigcap_{m \geq 1} \bigcup_{j \geq m} E_j = \bigcap_{m \geq 1} G_m$$

Similarly,  $G_m$  is a decreasing sequence, and since  $\mu(\bigcup E_{j \geq 1}) = \mu(G_1)$  is finite, we can use continuity from above in the same manner, and the proof is complete. ■



**Exercise 1.9**

**Proposition 9.1.**

*Proof.*



**Exercise 1.10****Proposition 10.1.***Proof.*

**Exercise 1.11**

**Proposition 11.1.**

*Proof.*



## Exercise 1.12

**Proposition 12.1.** *Let  $(X, \mathcal{M}, \mu)$  be a finite measure space,*

- *If  $E, F \in \mathcal{M}$ , and  $\mu(E \Delta F) = 0$ , then  $\mu(E) = \mu(F)$ ,*
- *Say that  $E \sim F$  if  $\mu(E \Delta F) = 0$ , then  $\sim$  is an equivalence relation on  $\mathcal{M}$ ,*
- *For every  $E, F \in \mathcal{M}$ , define  $\rho(E, F) = \mu(E \Delta F)$ . Show that  $\rho$  defines a metric on the space of  $\mathcal{M}/\sim$  equivalence classes.*

*Proof of Part A.* Use the fact that  $\mu(F) = \mu(E \cap F) + \mu(F \cap E^c)$ , and by monotonicity,

$$\mu(F \cap E^c) \leq \mu(E \Delta F) = 0$$

And  $\mu(F) = \mu(E \cap F) = \mu(E)$ , the last equality follows after a simple modification. ■

*Proof of Part B.* Suppose that  $\mu(E \Delta F) = \mu(F \Delta G) = 0$ , then

- $\mu(E \cap F^c) = \mu(F \cap E^c) \leq \mu(E \Delta F) = 0$  by monotonicity,
- Similarly, we have  $\mu(F \cap G^c) = \mu(G \cap F^c) = 0$ , and
- By subadditivity,
  - $\mu(E \cap G^c) = \mu(E \cap F^c \cap G^c) + \mu(E \cap F \cap G^c) \leq 0$ , and  $\mu(E \cap G^c) = 0$ ,  
and
  - $\mu(G \cap E^c) = 0$
- Therefore  $\mu(E \Delta G) = \mu(E \cap G^c) + \mu(G \cap E^c) = 0$

It is clear that the relation is reflexive, since  $E \Delta E = \emptyset$ , and symmetry is trivial. ■

*Proof of Part C.* Since  $\rho(E, F) = \rho(F, E)$ , and  $\rho(E, F) \geq 0$  for every  $E, F \in \mathcal{M}$ , and  $\rho(E, F) = 0 \iff E \sim F$ . We only have to prove the Triangle Inequality. Notice that

$$\begin{aligned} \mu(E \setminus F) &= \mu(E \cap F^c \cap G) + \mu(E \cap F^c \cap G^c) \\ &\leq \mu(F^c \cap G) + \mu(E \cap F^c) \end{aligned}$$

and in the same fashion,

$$\mu(F \setminus E) \leq \mu(F \cap G^c) + \mu(E^c \cap F)$$

Combining the two inequalities, and applying additivity finishes the proof. ■

**Exercise 1.13**

**Proposition 13.1.** *Every  $\sigma$ -finite measure is semi-finite*

*Proof.* Suppose  $\mu$  is  $\sigma$ -finite then there exists an increasing sequence of sets  $E_j \nearrow X$  with  $\mu(E_j) < +\infty$ . Now for every  $W \in \mathcal{M}$ , if  $\mu(W) = +\infty$  then  $\mu(W) = \lim_{j \rightarrow \infty} \mu(E_j \cap W) = +\infty$ . Since this real-valued limit converges to its supremum  $+\infty$ , there exists a non-null subset  $E_j \cap W$  of positive and finite measure. ■

**Exercise 1.14**

**Proposition 14.1.** *If  $\mu$  is a semi-finite measure, and if  $\mu(E) = +\infty$ , for every  $C > 0$ , there exists an  $F \subseteq E$  with  $0 < \mu(F) < +\infty$ .*

*Proof.* Suppose by contradiction that there exists a  $C > 0$  so for every  $F \subseteq E$ , if  $F$  is of finite measure, then  $0 \leq \mu(F) \leq C$ . Let  $s = \sup\{\mu(F), F \subseteq E, 0 < \mu(F) < +\infty\}$ , and for any  $n^{-1} > 0$ , this induces a  $F_n$  with measure

$$\mu(F_n) > s - n^{-1}$$

and take  $A_n = \bigcup_{j \leq n} F_j$ . A simple induction will show that  $\mu(A_n) \leq \sum_{j \leq n} \mu(F_j) < +\infty$ , therefore  $\mu(A_n) \leq s$  for every  $n \geq 1$ . By continuity from below

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{j \geq 1} F_j\right) \leq s$$

Next, by monotonicity, denoting the union over  $A_n$  by  $A$ , for every  $n^{-1} > 0$

$$s - n^{-1} \leq \mu(A_n) \leq \mu(A) \leq s \implies \mu(A) = s$$

Now,  $E \setminus A$  is a set of infinite measure, and by semi-finiteness. Find a set  $B \subseteq E \setminus A$  with strictly positive measure, so that

$$\mu(A \cup B) = \mu(A) + \mu(B) > s$$

And this finishes the proof. ■

### Exercise 1.15

**Proposition 15.1.** *Given a measure  $\mu$  on  $(X, \mathcal{M})$ , and define  $\mu_0 = \sup\{\mu(F), F \subseteq E, \mu(F) < +\infty\}$ . Show  $\mu_0$  is semi-finite. Then, show that if  $\mu$  is semi-finite,  $\mu = \mu_0$ . Lastly, there exists a measure  $\nu$  on  $(X, \mathcal{M})$ , with  $\mu = \nu + \mu_0$ , where  $\nu$  only assumes the values 0 or  $+\infty$ .*

*Proof.* First, a small Lemma. We claim that  $\mu_0 = \mu$  on finite sets. Let  $E \in \mathcal{M}$ , and  $\mu(E) < +\infty$ , since

$$\mu(E) \in \{\mu(F), F \subseteq E, \mu(F) < +\infty\} \implies \mu(E) \leq \mu_0(E)$$

Next, for every  $W \subseteq E$ ,  $\mu(W) \leq \mu(E)$ , so  $\mu_0(E) \leq \mu(E)$ . This proves the equality.

If  $E$  is any measurable subset of  $X$ , and suppose also  $\mu_0(E) = +\infty$ , one can easily find subsets of  $E$ ,  $\{E_n\}_{n \geq 1}$  with

$$n \geq \mu(E_n) < +\infty$$

But  $E_n$  is a subset of finite measure, so  $0 < \mu(E_n) = \mu_0(E_n) < +\infty$ . This proves the semi-finiteness of  $\mu_0$ .

Next, suppose  $\mu$  is semi-finite, and fix any measurable set  $E$ . If  $E$  is of finite measure, then  $\mu(E) = \mu_0(E)$ , and if  $\mu(E) = +\infty$ , apply Exercise 14, so there exists a sequence of subsets of finite measure  $E_n \subseteq E$  for every  $n \geq 1$ , with  $\mu(E_n) \rightarrow \mu(E)$ . Therefore  $\mu_0(E) = \mu(E)$ .

For the last part of the proof, let  $\mu$  be an arbitrary measure. And let  $E \in \mathcal{M}$ . If  $\mu(E) < +\infty$ , then  $\nu(E) = 0$  would suffice (this proves the first property of the measure). If  $\mu(E) = +\infty$ , and if  $\mu(E)$  is not semi-finite, then set  $\nu(E) = +\infty$ . So that  $\mu_0(E) + \nu(E) = 0 + \infty = \infty = \mu(E)$ . The additivity of  $\nu$  is immediate, since  $\nu$  can only assume two values. This finishes the proof. ■



**Exercise 1.16****Proposition 16.1.***Proof.*

## Exercise 1.17

**Proposition 17.1.** *Let  $\{A_j\}_{j \geq 1}$  be a countable disjoint sequence in  $\mathcal{M}$ , and denote  $B_n = \bigcup A_{j \leq n} \in \mathcal{M}$ . For every  $E \subseteq X$ ,*

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j \leq n})$$

*Proof.* Proven in Theorem 1.11 as a Lemma. ■

### Exercise 1.18

**Proposition 18.1.** Let  $\mathcal{A} \subseteq \mathbb{P}(\mathbf{X})$  be an algebra.  $\mathcal{A}_\sigma$  the collection of countable unions of sets in  $\mathcal{A}$ , and  $\mathcal{A}_{\sigma\delta}$  the collection of countable intersection of sets in  $\mathcal{A}_\sigma$ . Let  $\mu_0$  be a premeasure on  $\mathcal{A}$ , and  $\mu^*$  be the induced outer-measure.

- (a) For any  $E \subseteq \mathbf{X}$ , and  $\varepsilon > 0$ , there exists  $A \in \mathcal{A}_\sigma$  with  $E \subseteq A$  and  $\mu^*(A) \leq \mu^*(E) + \varepsilon$ .
- (b) If  $\mu^*(E) < +\infty$ , then  $E$  is  $\mu^*$ -measurable  $\iff$  there exists  $B \in \mathcal{A}_{\sigma\delta}$  with  $E \subseteq B$  and  $\mu^*(B \setminus E) = 0$ .
- (c) If  $\mu_0$  is  $\sigma$ -finite, the restriction  $\mu^*(E) < +\infty$  in (b) is superfluous.

*Proof of Part A.* Let  $E \subseteq \mathbf{X}$  and  $\varepsilon > 0$ , then by definition of  $\mu^*$ ,

$$\mu^*(E) + \varepsilon \geq \sum \mu_0(A_j) = \sum \mu^*(A_j) \geq \mu^*(A)$$

by subadditivity and  $A = \bigcup A_j$ . ■

*Proof of Part B.* Suppose  $E$  is outer-measurable and of finite outermeasure, then by part A we have a sequence of  $A_n \in \mathcal{A}_\sigma$  with

$$\mu^*(E) + n^{-1} \geq \mu^*(A_n) \implies \mu^*(E) = \mu^*(A)$$

if we define  $A = \bigcap A_n \supseteq E$ . Using the  $\mu^*$ -measurability of  $E$ , we get

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) < +\infty \implies \mu^*(A \setminus E) = 0$$

Conversely, if  $\mu^*(A \setminus E) = 0$ , for any  $V \subseteq \mathbf{X}$ , with  $\mu^*(V) < +\infty$ , we have

$$\begin{aligned} \mu^*(V) &= \mu^*(V \cap A) + \mu^*(V \setminus A) \\ &\geq \mu^*(V \cap E) + \mu^*(V \setminus A) + \mu^*(V \cap [A \setminus E]) \\ &\geq \mu^*(V \cap E) + \mu^*(V \setminus E) \end{aligned}$$

■

*Proof of Part C.* Suppose  $\mu_0$  is  $\sigma$ -finite, then  $E \in \mathcal{M}^*$  induces a sequence  $E_j \nearrow E$ , where each  $E_j$  is of finite measure. By part b) we obtain  $\{A_j\} \subseteq \mathcal{A}_{\sigma\delta}$  with

$$\mu^*(A_j \setminus E_j) = 0$$

Now define  $B = \bigcup A_j$ , so that  $B \in \mathcal{A}_{\sigma\delta}$ . Observe  $\bigcup (A_j \setminus E_j) = B \setminus E_1 \supseteq B \setminus E$  (verify these). And  $\mu^*(B \setminus E) \leq \sum \mu^*(A_j \setminus E_j) = 0$  by subadditivity. Since  $B \supseteq E$ , and  $B \in \mathcal{A}_{\sigma\delta}$ , this proves  $\implies$ .

Conversely, suppose  $E \subseteq X$  and there exists a  $B \in \mathcal{A}_{\sigma\delta}$  with  $E \subseteq B$ ,  $\mu^*(B \setminus E) = 0$ . Let  $\{X_j\} \nearrow X$  as a sequence of sets of finite measure. Then,

$$(X_j \cap B) \setminus (X_j \cap E) = X_j \cap (B \setminus E) \subseteq B \setminus E$$

$X_j \cap B \in \mathcal{A}_{\sigma\delta}$ , and  $X_j \cap B \supseteq (X_j \cap E)$ . Each  $E_j = X_j \cap E$  is  $\mu^*$  measurable by monotonicity, so is their countable union.  $\blacksquare$

### Exercise 1.19

**Proposition 19.1.** *Let  $\mu^*$  be an outer measure on  $\mathbf{X}$  induced from a finite premeasure  $\mu_0$ . If  $E \subseteq \mathbf{X}$ , define the inner measure of  $E$  to be  $\mu_*(E) = \mu_0(\mathbf{X}) - \mu^*(E^c)$ . Then  $E$  is  $\mu^*$ -measurable iff  $\mu^*(E) = \mu_*(E)$ .*

*Proof.* Suppose  $E \subseteq \mathbf{X}$  is  $\mu^*$ -measurable. Then

$$\mu^*(\mathbf{X}) = \mu^*(\mathbf{X} \cap E) + \mu^*(\mathbf{X} \setminus E) = \mu_0(\mathbf{X})$$

Rearranging gives the result, since all quantities are finite.

If  $\mu^*(E) = \mu_*(E)$ , then  $\mu^*(E^c) = \mu_*(E^c)$ , since the definition of  $\mu_*$  is symmetric. Let  $B \in \mathcal{A}_{\sigma\delta}$ , with  $\mu^*(B) = \mu^*(E)$ ,  $E \subseteq B$ . We can always find such a  $B$  by taking the intersection over all  $B_n \in \mathcal{A}_\sigma$ ,

$$\mu^*(E) + n^{-1} \geq \sum_j \mu^*(B(j, n)) \geq \mu^*\left(\bigcup_j B(j, n) = B_n\right)$$

Notice  $E \subseteq B \iff E^c \supseteq B^c \iff E^c \cap B^c = B^c$ . Since  $B$  is  $\mu^*$ -measurable, we have

$$\begin{aligned} \mu^*(E^c \cap B) + \mu^*(E^c \setminus B) &= \mu^*(E^c) \\ &= \mu^*(\mathbf{X}) - \mu^*(E) \\ \mu^*(B \setminus E) + \mu^*(B^c) &= \mu^*(\mathbf{X}) - \mu^*(E) \\ &= \mu^*(B) + \mu^*(B^c) - \mu^*(E) \\ \mu^*(B \setminus E) &= \mu^*(B) - \mu^*(E) \\ &= 0 \end{aligned}$$

■

**Exercise 1.20****Proposition 20.1.***Proof.*

### Exercise 1.21

**Proposition 21.1.** *Let  $\mu^*$  be an outermeasure induced from a premeasure, and  $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$ , where  $\mathcal{M}^*$  denotes the family of  $\mu^*$ -measurable sets. Show that  $\bar{\mu}$  is saturated. That is,  $\widetilde{\mathcal{M}^*} = \mathcal{M}^*$*

*Proof.* Suppose  $E$  is locally measurable (with respect to  $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$ ). Fix  $V \subseteq \mathbf{X}$ , with  $\mu^*(V) < +\infty$ . It suffices to show  $\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$ .

By 18a), find a  $V' \in \mathcal{A}_{\sigma\delta}$ , with  $V \subseteq V'$ , and  $\mu^*(V') = \mu^*(V) < +\infty$ . so that  $E \cap V'$  is  $\mu^*$ -measurable.

$$\mu^*(V) = \mu^*(V \cap E \cap V') + \mu^*(V \setminus (V \cap (V' \cap E)))$$

therefore

$$\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$$

■

## Exercise 1.22

### Proposition 22.1.

*Proof.* To show  $\bar{\mu}$  is complete, Fix  $U \subseteq F$ , where  $F \in \mathcal{M}^*$ , with  $\bar{\mu}(F) = 0$ . Let  $F' \in \mathcal{A}_{\sigma\delta}$ , with  $F' \supseteq F$ , and

$$\mu^*(F') = \mu^*(F) \geq \mu^*(F' \setminus U)$$

Since  $F' \supseteq U$ , applying Exercise 18b gives  $\overline{\mathcal{M}^*} \subseteq \mathcal{M}^*$ . For the other direction,

■



**Exercise 1.23****Proposition 23.1.***Proof.*

### Exercise 1.24

**Proposition 24.1.** *If  $\mu$  is a finite measure on  $(X, \mathcal{M})$ , and let  $\mu^*$  be the outer measure. Suppose that  $E \subseteq X$  satisfies  $\mu^*(E) = \mu^*(X)$  (but  $E \notin \mathcal{M}$  necessarily). Show that*

- (a) *For any  $A, B \in \mathcal{M}$ , and  $A \cap E = B \cap E$ , then  $\mu(A) = \mu(B)$ .*
- (b) *Let  $\mathcal{M}_E = \{A \cap E, A \in \mathcal{M}\}$ , and define  $\nu$  on  $\mathcal{M}$  with  $\nu(A \cap E) = \mu(A)$ . Then  $\mathcal{M}_E$  is a  $\sigma$ -algebra, and  $\nu$  is a measure on  $\mathcal{M}_E$ .*

*Proof of Part A.*

$$\mu^*(E) = \mu^*(X) \implies \mu^*(X \setminus E) = 0$$

This is a simple consequence of the  $\mu^*$ -measurability of  $X$ , since  $X \in \mathcal{M}$ , and the  $\mu$  is a pre-measure on  $\mathcal{M}$ , and by monotonicity,

$$\begin{cases} A \cap (X \setminus E) \subseteq (X \setminus E) \\ B \cap (X \setminus E) \subseteq (X \setminus E) \end{cases} \implies \begin{cases} \mu^*(A \cap (X \setminus E)) = 0 \\ \mu^*(B \cap (X \setminus E)) = 0 \end{cases}$$

Write  $A \cap X = (A \cap E) \cup (A \cap X \setminus E)$ , and by subadditivity of  $\mu^*$ ,

$$\begin{aligned} \mu(A) &= \mu^*(A \cap X) \\ &\leq \mu^*(A \cap E) + \mu^*(X \setminus E) \\ &= \mu^*(B \cap E) \\ &\leq \mu^*(B \cap X) \\ &= \mu(B) \end{aligned}$$

Therefore  $\mu(A) \leq \mu(B)$ , and  $\mu(B) \leq \mu(A)$  is trivial. ■

*Proof of Part B.* We want to show  $\mathcal{M}_E$  is a  $\sigma$ -algebra.

- Closure under complements,

$$\forall A \cap E \in \mathcal{M}_E, A \in \mathcal{M} \implies (E \setminus A^c) \in \mathcal{M}_E$$

Therefore  $(E \setminus A^c) \cap E \in \mathcal{M}_E$ . Note that the question mentions that  $\mathcal{M}_E$  is a  $\sigma$ -algebra on  $E$ , therefore we take complements relative to  $E$ .

- Closure under countable unions. Fix any countable sequence  $\{A_j \cap E\} \subseteq \mathcal{M}_E$  where  $\{A_j\} \subseteq \mathcal{M}$ . It is obvious that  $A = \cup A_j \in \mathcal{M}$ , therefore  $\cup(A_j \cap E) = E \cap A \in \mathcal{M}_E$  as well.

Since  $\nu(\emptyset) = \mu(\emptyset \cap E) = 0$ , and for countable additivity, fix any disjoint sequence  $\{A_j \cap E\}_{j \geq 1} \subseteq \mathcal{M}_E$ , where  $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$ , and let  $A = \cup A_{j \geq 1}$

$$\begin{aligned} \nu(A \cap E) &= \mu(A) \\ &= \sum \mu(A_{j \geq 1}) \\ &= \sum \nu(A_{j \geq 1} \cap E) \end{aligned}$$

■