MATH 263: Section 003, Tutorial 10

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1 (Review) Series Solutions Near an Ordinary Point, Part I

Consider the general second order linear ODE:

$$P(x)y"(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$

Where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are analytical around $x = x_0$. Such a point where $P(x_0) \neq 0$ is called an **ordinary point**. We can divide both sides by and get: $P(x)$ and get:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

In that case, one can solve it by plugging in the power series

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

and finding the coefficients a_n, usually through a recurrence relation. It usually cannot be solved and one may only give the few first terms of the solution.

Problem 1. From Boyce and DiPrima, 10th edition (5.2, exercise 9, p.263):
Find the general solution of:

$$(1 + x^2)y" - 4xy' + 6y = 0.$$

2 Series Solutions Near an Ordinary Point, Part II

Consider the general second order linear ODE:

$$P(x)y"(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$
$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

where P, Q, and R are polynomials and where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are analytical in the neighborhood of an ordinary point x_0. Assuming the ODE has an analytical solution of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

From the definition of the Taylor series at $x = x_0$, we get:

$$n!\ a_n = y^{(n)}(x_0)$$

Using that, we get

$$y"(x) = -p(x)\ y'(x) - q(x)\ y(x)$$

By taking successive derivatives and evaluating them at $x = x_0$, we can get the coefficients of the Taylor Series corresponding to the ODE's solution.

Problem 2. From Boyce and DiPrima, 10th edition (5.3, exercise 2, p.269):
Find the first terms (up to a_4) of the solution of:

$$y"(x) + \sin x\ y'(x) + \cos x\ y(x) = 0;\ y(0) = 0,\ y'(0) = 1.$$

3 Series Solutions Near a Regular Singular Point, Part I

Consider the general second order linear ODE:

$$P(x)y"(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$

Where $P(x_0) = 0$, meaning that $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are not analytical at $x = x_0$. $x = x_0$ is then a **singular point**.

Consider the case of **regular singular points**, where $(x - x_0)p(x) = (x - x_0)\frac{Q(x)}{P(x)}$ and $(x - x_0)^2 q(x) = (x-x_0)^2\frac{R(x)}{P(x)}$ are analytic at $x = x_0$. We can write them as: $(x-x_0)p(x) = (x-x_0)\frac{Q(x)}{P(x)}$ and $(x-x_0)^2q(x) = (x - x_0)^2\frac{R(x)}{P(x)}$ are

$$(x - x_0)p(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^n,$$

and

$$(x - x_0)^2 q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n,$$

Plugging them in the ODE, we get

$$(x - x_0)^2 y" + (x - x_0)[(x - x_0)p(x)]y' + [(x - x_0)^2 q(x)]y = 0$$

$$(x-x_0)^2y"+(x-x_0)(p_0+p_1(x-x_0)+\cdots+p_n(x-x_0)^n+\dots)y'+(q_0+q_1(x-x_0)+\cdots+q_n(x-x_0)^n+\dots)y = 0.$$

As x approaches x_0, the ODE behaves as an Euler equation as such:

$$(x - x_0)^2 y" + p_0(x - x_0)y' + q_0 y = 0.$$

where $p_0 = \lim_{x\to x_0} \frac{Q(x)}{P(x)}$ and $q_0 = \lim_{x\to x_0} \frac{R(x)}{P(x)}$

The solutions will be of the form of Euler solutions times a power series as such:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n(x - x_0)^n,\ a_0 \neq 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^{r+n}$$

Then plug it in the ODE. The Euler characteristic equation will arise along with a recurrence relation for a_n that depends on r.

Problem 3. From Boyce and DiPrima, 10th edition (5.5, exercise 3, p.286):
Find one fundamental solution of:

$$xy" + y = 0.$$