

# Chapter 22: Symplectic Manifold

## Symplectic Tensors

**Definition 1.1: Bilinear forms**

Let  $V$  be a vector space, a *bilinear form*  $\omega : V \times V \rightarrow \mathbb{R}$  is a 2-tensor on  $V$ .

**Definition 1.2: Characterization of bilinear forms**

Let  $\omega$  be a bilinear form on  $V$ , it is

- *symmetric* if

$$\omega(x, y) = \omega(y, x)$$

- *skew-symmetric* or *anti-symmetric* if

$$\omega(x, y) = (-1)\omega(y, x)$$

- *alternating* if

$$\omega(x, x) = 0$$

If  $V$  is a vector space over the field  $F$  and  $\text{char}(F) \neq 2$ , then the last two conditions are equivalent. Moreover,

- $V$  is called an *orthogonal geometry* if  $\omega$  is symmetric.
- $V$  is called a *symplectic geometry* if  $\omega$  is alternating.

**Definition 1.3: Metric vector space**

A vector space (not necessarily finite dimensional) is called a *metric vector space* if it is a orthogonal or symplectic geometry.

## Matrices and bilinear forms

**Definition 2.1: Matrix of bilinear form**

If  $B = (b_1, \dots, b_n)$  is an ordered basis for  $V$ , we define the *matrix representation of  $\omega$*  by

$$\mathcal{M}(\omega) = (a_{ij}) = (\omega(b_i, b_j))$$

**Proposition 2.1: Matrix induces a bilinear form**

Let  $A = (a_{ij})$  be a matrix on  $V$  with respect to some basis  $B = (b_n)$  it is clear that  $A$  induces a bilinear form, on  $V$  through  $A(x, y) = [x]_B^T A [y]_B$ , where  $[\cdot]_B$  denotes the canonical isomorphism  $V \cong \mathbb{R}^n$  with respect to the basis  $B$ .

$$[x]_B^T A [y]_B = \begin{bmatrix} x^1 & \dots & x^n \end{bmatrix} A \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}$$

for  $x = x^i b_i$  and  $y = y^j b_j$ .

Moreover,

$$A[x]_B = \begin{bmatrix} A(b_1, x) \\ \vdots \\ A(b_n, x) \end{bmatrix}$$

is a *column* vector  
whose entries are given  
by applying  $x$  on the  
second coordinate

and

$$[x]_B^T A = \begin{bmatrix} A(x, b_1) & \dots & A(x, b_n) \end{bmatrix}$$

is a *row* vector whose  
entries are given by ap-  
plying  $x$  on the first co-  
ordinate

Let  $A_B$  be the matrix representation of  $\omega$  with respect to the  $B$ , if  $C$  is another basis on  $V$ , then how do we compute  $A_C$ ? The answer is simple, recall for any vector  $x \in V$ ,  $x = x_B^i b_i$  and  $x = x_C^j c_j$ , then

$$[x]_B = M_{C,B} [x]_C \quad \text{for some matrix of an automorphism } M_{C,B}$$

$$\omega(x, y) = [x]_B^T A_B [y]_B = ([x]_C^T M_{C,B}^T) A_B (M_{C,B} [y]_C) = [x]_C^T A_C [y]_C, \text{ then}$$

$$M_{C,B}^T A_B M_{C,B} = A_C \tag{1}$$

We can describe this relation between the two matrices  $A_B$  and  $A_C$  by the following

### Definition 2.2: Congruent matrices

Two matrices  $M$  and  $N$  are said to be *congruent*, if there exists an invertible matrix  $P$  for which

$$P^T M P = N$$

Congruence is an equivalence relation on the space of matrices, and the equivalence classes over congruence are called *congruence classes*.

### Proposition 2.2: Characterization of matrices using congruence

Let  $A_1$  and  $A_2$  be matrix representations of two bilinear forms with respect to the basis  $B$ .

$$A_1 = (A_1(b_i, b_j))_{ij} \quad A_2 = (A_2(b_i, b_j))_{ij}$$

They induce the same bilinear form if and only if they are congruent.

### Definition 2.3: Alternate matrices

Let  $M$  be a matrix with  $F$ -coefficients, it is *alternate* if it is skew symmetric and is *hollow*; meaning it

has 0s on the main diagonal. If  $F = \mathbb{R}$  or  $\text{char}(F) \neq 2$ , then alternate matrices are and are precisely the skew-symmetric matrices.

## Orthogonality

For this section,  $(V, \omega)$  will denote a metric vector space, not necessarily finite-dimensional unless we are using matrix representations.

### Definition 3.1: Orthogonal complements

A vector  $x \in V$  is orthogonal to another vector  $y \in V$ , written  $x \perp y$ , if  $\omega(x, y) = 0$ .

If  $V$  is an orthogonal or symplectic geometry then  $\perp$  is a symmetric relation. If  $E$  is a subset of  $V$ , we denote the *orthogonal complement of  $E$*  by

$$E^\perp \triangleq \{v \in V, v \perp E\}$$

### Definition 3.2: Characterization of metric vector spaces

- A nonzero vector  $x \in V$  is *isotropic*, or *null* if  $\omega(x, x) = 0$
- $V$  is *isotropic* if it contains at least one isotropic vector.
- $V$  is *anisotropic* or *nonisotropic* if for every  $x \in V$ ,  $\omega(x, x) = 0 \implies x = 0$ ,
- $V$  is *totally isotropic* (that is, symplectic if  $\text{char}(F) \neq 2$ ) if  $\omega(x, x) = 0$  for every vector  $x \in V$ .

The first bullet point above is about vectors in  $V$ , while the others are properties of  $V$ .

- A vector  $x \in V$  is called *degenerate* if  $x \perp V$ , that is,

$$\forall y \in V, \omega(x, y) = 0$$

- The *radical* of  $V$ , denoted by  $\text{rad}(V)$  is the set of all degenerate vectors in  $V$ ,

$$\text{rad}(V) \triangleq V^\perp$$

- $V$  is *singular* or *degenerate* if  $\text{rad}(V) \neq \{0\}$ ,
- $V$  is *non-singular* or *non-degenerate* if  $\text{rad}(V) = \{0\}$ ,
- $V$  is *totally singular*, if  $\text{rad}(V) = V$ .

To summarize,

- $V$  is isotropic if there exists a non-zero isotropic vector, meaning  $\omega(x, x) = 0$ , for some  $x \neq 0$ ,
- $V$  is degenerate if there exists a degenerate vector,  $x \perp V$ .

**Proposition 3.1: Matrix invariants under congruence**

Non-singularity, symmetry, and skew-symmetry are invariants under congruence.

*Proof.* ■

**Proposition 3.2: Characterization of non-degeneracy**

$V$  is non-degenerate if and only if every matrix representation  $A$  of  $\omega$  is non-singular.

*Proof.* Suppose  $V$  is non-degenerate, then let  $B = (b_n)$  be a basis for  $V$ , if  $A$  is the matrix representation of  $\omega$  with respect to  $B$ , let  $x$  be a non-zero vector in  $V$ , so  $x \notin \text{rad}(V)$

$$b_i^T A[x]_B = \omega(b_i, x) \neq 0 \implies A[x]_B \neq 0$$

so  $A$  is non-singular. If  $A'$  is another matrix representation with respect to another basis  $C$ , by Equation (1)  $A'$  is non-singular as well.

Conversely, if every matrix representation of  $\omega$  is non-singular, let  $x$  be a non-zero vector in  $V$ , then  $A[x]_B \neq 0$  is a non-zero vector so there exists some basis component  $(A[x]_B)^j$  that is non zero, and

$$[b_j]_B^T A[x]_B = \omega(b_j, x) \neq 0$$

therefore  $V$  is non-degenerate. ■

**Proposition 3.3: Characterisation of bilinear forms from matrix representations**

Let  $\omega$  be a bilinear form on  $V$ , if  $\mathcal{M}(\omega)$  the induced matrix representation relative to any basis. Assume  $V$  is a vector space over  $\mathbb{R}$ , then

- it is symmetric iff  $\mathcal{M}(\omega)$  symmetric as a matrix,
- it is skew-symmetric, iff alternating iff  $\mathcal{M}(\omega)$  is skew-symmetric as a matrix.

**Corollary 3.1: Characterisation of non-singular symplectic form**

Let  $(V, \omega)$  be a finite dimensional vector space over  $\mathbb{R}$ , equipped with a bilinear form  $\omega$ .  $(V, \omega)$  is a non-singular symplectic vector space iff the matrix representation of  $\omega$  with respect to every basis is non-singular and skew-symmetric.

## Riesz Representation Theorems

**Proposition 4.1**

Let  $(V, \omega)$  be a nonsingular metric vector space, the map  $x \mapsto x \lrcorner \omega \in V^*$  defined by

$$x \lrcorner \omega = \omega(x, \cdot), \quad \text{and} \quad (x \lrcorner \omega)(y) = \omega(x, y), \quad \forall y \in V$$

is a linear isomorphism from  $V$  to  $V^*$ .

## Isometries

### Definition 5.1: Isometry between MVS

Let  $(V, \omega)$  and  $(W, \eta)$  be metric vector spaces. An *isometry*  $\tau \in L(V, W)$  is a linear isomorphism that preserves the bilinear form.

$$\omega(u, v) = \eta(\tau u, \tau v)$$

### Definition 5.2: Orthogonal, symplectic groups

Let  $V$  be a nonsingular metric vector space. If  $V$  is an orthogonal (resp. symplectic) geometry, the set of all isometries on  $V$  is called the *orthogonal* (resp. *symplectic*) *group on  $V$* . It is a group under composition, and is denoted by  $\mathcal{O}(V)$  (resp.  $\text{Sp}(V)$ ).

## Hyperbolic spaces, nonsingular completions

### Canonical Forms

### Symplectic Manifolds

### Darboux's Theorem

#### Proposition 9.1: Lie Derivatives of Tensor Fields (along time-varying vector fields)

Let  $M$  be a smooth manifold. Suppose  $V : J \times M \rightarrow TM$  is a smooth time-varying vector field on  $M$ . Denote the time-varying flow of  $V$  by  $\psi : \mathcal{E} \rightarrow M$ . Let  $A \in \mathcal{T}^k(M)$  be a smooth time-invariant covariant  $k$ -tensor field on  $M$ . For every  $(t_1, t_0, p) \in \mathcal{E}$ ,

$$\left. \frac{d}{dt} \right|_{t=t_1} (\psi_{t,t_0}^* A)_p = (\psi_{t_1,t_0}^* (\mathcal{L}V_{t_1} A))_p \quad (2)$$

# Chapter Hofer book

**Definition 0.1: Symplectic vector space**

Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$ . It is a *symplectic vector space* if it admits a non-singular, antisymmetric bilinear form  $\omega : V \times V \rightarrow \mathbb{R}$ .

$$\omega(u, v) = -\omega(v, u)$$

for  $u, v \in V$ . By the previous section on Riesz Representation, the linear map

$$\hat{\omega} : V \rightarrow V^*, \quad v \mapsto \omega(v, \cdot)$$

is a linear isomorphism of  $V$  onto its dual vector  $V^*$ .

We define the *standard symplectic vector space*  $(\mathbb{R}^{2n}, \omega_0)$ , where  $n \in \mathbb{N}^+$ , where

$$\omega_0(u, v) = \langle Ju, v \rangle \quad J \triangleq \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product on  $\mathbb{R}^{2n}$ .

$$\omega_0(u, v) = \langle Ju, v \rangle = \langle u, J^T v \rangle = u^T J^T v \quad (3)$$

$J^T = -J$  by Corollary 3.1.

We will mainly deal with non-singular symplectic forms because of Riesz isomorphism.

**Definition 0.2: Symplectic linear map**

Let  $(V, \omega)$  be a symplectic vector space. A linear map  $F \in \text{Hom}(V)$  is *symplectic* if it preserves symplectic form  $\omega$ . For every  $u \in V$ ,

$$\langle u, v \rangle = \langle Au, Av \rangle \triangleq A^* \omega(u, v)$$

where  $A^* : \mathcal{T}^*(V) \rightarrow \mathcal{T}^*(V)$  denotes the tensor pullback by precomposing any tensor  $S$  by  $A$

$$\forall S \in \mathcal{T}^k(V), \quad A^* S(v_{\underline{k}}) \triangleq S(Av_{\underline{k}})$$

The set of linear symplectic maps on a  $2n$ -dimensional vector space form a group under composition. It is a Lie Group denoted by  $\text{Sp}(n)$ .

**Proposition 0.2: Symplectic Maps are Area-preserving**

Let  $(\mathbb{R}^{2n}, \omega_0)$  denote the standard symplectic space. If  $\varphi \in \text{Sp}(n)$ , then  $\det \varphi = 1$ .

*Proof.* See page 4. ■



$$(\Lambda_{s-|\alpha|} \partial^\alpha f)^\wedge = (1 + |\zeta|^2)^{s/2-|\alpha|/2} \cdot (\partial^\alpha f)^\wedge \quad (4)$$

$$= (1 + |\zeta|^2)^{s/2-|\alpha|/2} \cdot (2\pi i \zeta)^\alpha \cdot \hat{f} \quad (5)$$

$$= (2\pi i)^{|\alpha|} (1 + |\zeta|^2)^{(s-|\alpha|)/2} \cdot |\zeta|^{|\alpha|} \cdot \hat{f} \quad (6)$$

$$\leq |\alpha| (1 + |\zeta|^2)^{s/2} \hat{f} \quad (7)$$