MATH 263: Section 003, Tutorial 9

Mohamed-Amine Azzouz
mohamed-amine.azzouz@mail.mcgill.ca

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1 Nonhomogeneous ODEs: Variation of Parameters

To solve a nonhomogeneous linear ODE of the form:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = g(x) \neq 0$$

the idea is to guess a solution of the form:

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Plugging it in the ODE, we can derive formulas for $u_1(x)$ and $u_2(x)$:

$$u_1(x) = \int \frac{-y_2(x)g(x)}{W(y_1, y_2)(x)} dx + c_1$$
$$u_2(x) = \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)} dx + c_2$$

This method is more systematic and works for methods that the method of undetermined coefficients cannot solve, but it requires computing integrals, which may be tedious or impossible to solve. Note: a similar, although more complex derivation can be used with higher order linear ODEs. Again, using this method for higher order ODEs can become rather convoluted.

Problem 1.1. Problem from Assignment 1 (question 5) solved another way, find the general solution of:

$$x"(t) + 3x'(t) = 2te^{-3t}$$

Problem 1.2. Find the general solution of:

$$y"(x) + a^2 y(x) = g(x),\ a \neq 0$$

2 Review of Power Series

A **Power Series** centered at $x = x_0$ is of the form:

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

which is well defined on a **radius of convergence** on which the series converges. The two main ways to find the radius of convergence of a Power Series:

The **Ratio Test**:
If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty} |\frac{a_{n+1}(x - x_0)^{n+1}}{a_n (x - x_0)^n}| = |x - x_0| \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |x - x_0| L,$$

then the power series converges absolutely at that value of x if $|x-x_0|L < 1$ and diverges if $|x-x_0|L > 1$. If $|x-x_0|L = 1$, the test is inconclusive.

The **Root Test**: If, for a fixed value of x,

$$\lim_{n\to\infty} \sqrt[n]{|a_n(x-x_0)^n|} = |x-x_0| \lim_{n\to\infty} \sqrt[n]{|a_n|} = |x-x_0|L,$$

then the power series converges absolutely at that value of x if $|x-x_0|L < 1$ and diverges if $|x-x_0|L > 1$. If $|x-x_0|L = 1$, the test is inconclusive.

In particular, a **Taylor Series** centered at $x = x_0$ is of the form:

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

Problem 2. From Boyce and DiPrima, 10th edition (5.1, exercise 2, p.253):
Find the radius of convergence of the Power Series:

$$\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$$

3 Series Solutions Near an Ordinary Point, Part I

Consider the general second order linear ODE:

$$P(x)y"(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$

Where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are analytical around $x = x_0$. Such a point where $P(x_0) \neq 0$ is called an **ordinary point**. We can divide both sides by and get: $P(x)$ and get:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

In that case, one can solve it by plugging in the power series

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n,$$

and finding the coefficients a_n, usually through a recurrence relation. It usually cannot be solved and one may only give the few first terms of the solution.

Problem 3. From Boyce and DiPrima, 10th edition (5.2, exercise 2, p.263):
Find the general solution of:

$$y" - xy' - y = 0.$$