# Chapter 7

## Theorem 7.1

**Proposition 1.1.** If I is a linear functional on  $C_c(X)$ , then for every compact  $K \subseteq X$ , there exists some  $C_k \ge 0$  with

$$|I(f)| \le C_K \cdot \|f\|_u$$

*Proof.* Since supp (f) is compact, by Urysohn's Lemma (Theorem 4.32), there exists a  $\phi \in C_c(X, [0, 1])$  such that  $\phi = 1$  on K and vanishes outside some compact  $\overline{V} \subseteq X$ . Then at every x,

$$-\|f\|_u \le f(x) \le +\|f\|_u$$

Implies that

$$(-\|f\|_u)\phi \le f(x) \le (+\|f\|_u)\phi$$

So that  $f + ||f||_u \phi \ge 0$  and  $+||f||_u - f \ge 0$ , and by linearity,

$$(-\|f\|_u)I(\phi) \le I(f) \le (+\|f\|_u)I(\phi)$$

Therefore  $|I(f)| \leq I(\phi) ||f||_u$ , and taking  $C_K = I(\phi)$  will suffice.

### Theorem 7.2

**Proposition 2.1.** The Riesz-Markov-Kakutani Representation Theorem. If (for every) I is a positive linear functional on  $C_c(X)$ , then there exists a unique Radon measure  $\mu$  on X, such that

$$I(f)=\int f d\mu$$

for every  $f \in C_c(X)$ .  $\mu$  also satisfies, for every open U, and every compact  $K \subseteq X$ 

$$\mu(U) = \sup \{ I(f), f \in \mathcal{C}_c(X), f \prec U \}$$
 (1)

$$\mu(K) = \inf \{ I(f), f \in \mathcal{C}_c(X), f \ge \chi_K \}$$
 (2)

For the sake of completeness, we place the definitions for a Radon measure. Let X be a LCH space, and  $\mathbb B$  be its usual  $\sigma$ -algebra, a measure  $\nu$  is a Radon measure iff

- (i)  $\nu(K) < +\infty$  for every compact K.
- (ii)  $\nu$  is outer-regular on all Borel sets E,

$$\nu(E)=\inf\left\{\nu(U),\,U\supseteq E,\,U\in\mathcal{T}\right\}$$

Intuition: approximation by open supersets.

(iii)  $\nu$  is inner-regular on all open sets  $U \in \mathcal{T}$ ,

$$\nu(U) = \sup \left\{ \mu(K), \; K \subseteq U, \; K \text{ compact} \right\}$$

Intuition: approximation by compact subsets

The main proof is extremely long, so we will divide it into several parts. Following Folland's argumentation closely, we will prove (in order)

(a) If  $\mu_1$ ,  $\mu_2$  are Radon measures on X such that for every  $f \in C_c(X)$ 

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

then  $\mu_1$ ,  $\mu_2$  must satisfy (1), and  $\mu_1 = \mu_2$  on  $\mathbb{B}$ .

(b) If we define, for every open set U, define  $\mu: \mathcal{T} \to [0, +\infty]$  such that

$$\mu(U) = \sup \{ I(f), f \in \mathcal{C}_c(X), f \prec U \}$$
 (3)

Then  $\mu$  is countably subadditive, meaning for every  $U \in \mathcal{T}$ ,  $\{U_{j\geq 1}\} \subseteq \mathcal{T}$ 

$$U = \bigcup U_{j \ge 1} \implies \mu(U) \le \sum \mu(U_{j \ge 1})$$

(c)  $\mu(\varnothing)=0,\,\{\varnothing,X\}\subseteq\mathcal{T},\,$  so that by Theorem 1.10  $\mu$  induces an outer-measure  $\mu^*$ 

$$\mu^*(E) = \inf \left\{ \sum \mu(U_{j \ge 1}), \ U_j \in \mathcal{T}, \ E \subseteq \bigcup U_{j \ge 1} \right\}$$
 (4)

(d) If  $\mu^*$  is as described above, then if  $\mu$  is countably subadditive on  $\mathcal{T}$ , then

$$\mu^*(E) = \inf \{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$
 (5)

Meaning the two definitions in (4) and (5) are equal.

- (e)  $\mu^*$  and  $\mu$  agree on all open sets, and  $\mu^*|_{\mathcal{T}} = \mu$ ,
- (f) Using again the definition in (4) and (5), we show that every open set  $U \in \mathcal{T}_X$  is  $\mu^*$ -measurable, meaning for every  $E \subseteq X$ ,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

With this, since the set of all outer-measurable ( $\mu^*$ -measurable) sets,  $\mathcal{M}^*$  form a  $\sigma$ -algebra,

$$\mathcal{T} \subset \mathcal{M}^* \implies \mathbb{B} \subset \mathcal{M}^*$$

By Theorem 1.1, and define

$$\mu = \mu^*|_{\mathbb{B}} \tag{6}$$

is a Borel measure. And we note in passing that  $\mu$  is outer-regular on all  $E \in \mathbb{B}$ ,

$$\mu(E) = \inf \{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$
 (7)

- (g) Using (6) for the definition of  $\mu$  on  $\mathbb{B}$ , we prove that
  - $\mu$  is outer-regular on all Borel sets, and
  - $\mu$  satisfies Equation (1).
- (h)  $\mu$  satisfies Equation (2)
- (i)  $\mu$  is finite on all compact sets.
- (j)  $\mu$  is inner-regular on all open sets.
- (k) For every  $f \in C_c(X, [0, 1])$ ,

$$I(f) = \int f d\mu \tag{8}$$

(l) For every  $f \in C_c(X)$ ,  $I(f) = \int f d\mu \tag{9}$ 

A small lemma needs to be made before proceeding, that concerns the 'monotonicity' of I on  $C_cX$ .

**Lemma 2.1** Suppose that  $f, g \in C_c(X)$ , and  $f \geq g \geq 0$  for every X, then  $f - g \in C_c(X)$  and  $I(f) \geq I(g)$ 

*Proof.* Suppose that  $x \in X$  where f(x) = 0, then

$$f(x)-g(x)=-g(x)\geq 0 \implies g(x)=0 \implies f-g=0$$

Hence

$$\{x, f(x) = 0\} \subseteq \{x, f(x) - g(x) = 0\} \implies \{x, f(x) - g(x) \neq 0\} \subseteq \{x, f(x) \neq 0\}$$
 
$$\implies \operatorname{supp}(f - g) \subseteq \operatorname{supp}(f)$$

Since supp (f) is compact, and supp (f-g) is a closed subset of supp (f), yields  $f-g \in C_c(X)$ . And if I is any positive linear functional on  $C_c(X)$ , then

$$f - g \ge 0 \implies I(f - g) \ge 0$$
  
 $\implies I(f) > I(g) > 0$ 

**Remark 2.1** If  $f \prec U$  and  $g \prec U$  for some open subset  $U \subseteq X$ , then clearly supp  $(f - g) \subseteq \text{supp } (f) \subseteq U$ , and  $1 \geq f \geq f - g \geq 0$  means that  $f - g \prec U$  as well.

#### Part a

*Proof.* Suppose that  $\mu_1$  and  $\mu_2$  are Radon measures on X, and for every  $f \in C_c(X)$ ,

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

We first prove (1). Without loss of generality, by monotonicity of  $L^+$ , if  $f \prec U$  for some open U, then  $0 \leq f \leq ||f||_u \chi_U = \chi_U$  for all x and

$$\int f d\mu_1 \le \int ||f||_u \chi_U d\mu_1 \le \mu_1(U)$$

Therefore  $\mu_1(U)$  (resp.  $\mu_2(U)$ ) is an upper-bound for the set

$${I(f), f \in C_c(X), f \prec U}$$

Since  $\mu_1$  is inner-regular on  $U \in \mathcal{T}$ , for every  $\varepsilon > 0$  we can find some compact  $K \subseteq U$  where

$$\mu_1(U) - \varepsilon < \mu_1(K)$$

By Urysohn's Lemma (Theorem 4.32), there exists some  $g \in C_c(X)$  with

- $g \in C_c(X, [0, 1]),$
- g = 1 on  $K \subseteq U$ ,
- g = 0 outside some  $\overline{V} \subseteq U$ , and
- $g \prec U$ .

Hence for every  $x \in K$ ,  $g \ge \chi_K$ . If  $x \notin K$  then  $g \ge 0 = \chi_K$ ; so  $g - \chi_K \ge 0$  for every  $x \in X$ . Since  $\chi_K \prec U$ , using Lemma 2.1, we get

$$\mu_1(K) \le \int \chi_K d\mu_1 = I(\chi_K) \le I(g)$$

So for every  $\varepsilon > 0$ , there exists a  $g \in C_c(X)$ , and  $g \prec U$  where

$$\mu_1(U) - \varepsilon < \mu_1(K) \le I(g)$$

Therefore  $\mu_1(U) = \sup \{I(f), f \in C_c(X), f \prec U\}$ , and the first claim in (a) is proven. To show that  $\mu$  is indeed unique, since for every open set U, we must have  $\mu_1(U) = \mu_2(U)$ , and if  $E \in \mathbb{B}$  is any Borel set, and by outer-regularity,

$$\mu_1(E) = \inf \{ \mu_1(U), U \supseteq E, U \in \mathcal{T} \} = \inf \{ \mu_2(U), U \supseteq E, U \in \mathcal{T} \} = \mu_2(E)$$

Therefore this measure is unique.

#### Part b

*Proof.* To show countable subadditivity for  $\mu$  with equation (3), fix any  $U \in \mathcal{T}$  and a sequence  $\{U_{j\geq 1}\} \subseteq \mathcal{T}$  with  $U = \bigcup U_{j\geq 1}$ . It suffices to show that the partial sum of  $\sum \mu(U_{j\leq n})$  is greater than I(f) for any  $f \in C_c(X)$ ,  $f \prec U$  (hence it is an upper bound).

Fix any f, then denote  $K = \text{supp}(f) \subseteq U$ , and since  $\{U_{j\geq 1}\}$  is an open cover for K, there exists a finite subcollection,  $B \subseteq \mathbb{N}^+$  such that

$$K \subseteq \bigcup_{j \in B} U_j$$

Using Theorem 4.41 on this finite cover of K, there exists a partition of unity in  $\{g_{j\leq n}\}$  where

- $g_j \in C_c(X, [0, 1]),$
- $g_i \prec U_j \subseteq U$  for every  $j \leq n$ , and
- $\sum g_j = 1$  on K,

And notice for every  $j \leq n$ ,

$$\{f = 0\} \cup \{g_j = 0\} \subseteq \{f \cdot g_j = 0\} \implies \{f \cdot g_j \neq 0\} \subseteq \{f \neq 0\} \cap \{g_j \neq 0\}$$
$$\implies \operatorname{supp}(f \cdot g_j) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g_j)$$
$$\implies \operatorname{supp}(f \cdot g_i) \subseteq U_i \subseteq U$$

Hence  $f \cdot g_j \prec U$  and  $f \cdot g_j \in C_c(X, [0, 1])$  for every  $1 \leq j \leq n$ . Moreover, if we take the sum over a finite n, we obtain  $f = \sum f \cdot g_{j \leq n}$ , this is because for every  $x \in X$ , so we have

$$\sum_{j \leq n} f(x) \cdot g_j x = f(x) \cdot \sum_{j \leq n} g_j(x) = f(x)$$

Then  $I(f) = I(\sum f \cdot g_j) = \sum I(f \cdot g_j)$ . And by definition of  $\mu(U_j)$ , since it is the supremum over all  $I(h_j)$ , where  $h_j \in C_c(X, [0, 1])$  and  $h_j \prec U_j$ 

$$I(f \cdot g_i) \le \mu(U_i), \quad \forall j \le n$$

Hence

$$I(f) \le \sum_{j \le n} \mu(U_j) \le \sum_{j \ge 1} \mu(U_j)$$

Where for the last estimate we used the fact that  $\mu$  is non-negative, and since this holds for any f, we can conclude that  $\mu(U) \leq \sum_{j>1} \mu(U_j)$ .

#### Part c

*Proof.* By definition of a topology,  $\{\emptyset, X\} \subseteq \mathcal{T}$ , and  $\mu(\emptyset) = \sup\{I(f), f \in C_c(X), f \prec \emptyset\}$ , so supp  $(f) = \emptyset$ , and  $\{x, f(x) \neq 0\} \subseteq \emptyset$ , so the set contains one element, namely I(0) = 0 by linearity. So  $\mu(\emptyset) = 0$ . The assumptions for Theorem 1.10 are satisfied and (4) is indeed an outer-measure.

#### Part d

*Proof.* Denote the right members of (4) and (5) by  $W_1$  and  $W_2$ , we wish to show that  $\inf W_1 = \inf W_2$ . Clearly  $\inf W_1 \leq \inf W_2$ , since  $W_2 \subseteq W_1$ . Now, if  $\mu$  is countably additive, then for every  $\omega \in W_1$  induces a sequence of open sets  $\{U_{j\geq 1}\}$  such that  $E \subseteq \bigcup U_{j\geq 1}$ . Denote the union over  $\{U_{j\geq 1}\}$  by U, which is also another open set,

$$\inf W_2 \le \mu(U) \le \sum \mu(U_{j \ge 1}) = \omega$$

Since  $\omega$  is arbitrary, we conclude that  $\inf W_2 = \inf W_1$ , and this proves (d).

#### Part e

*Proof.* If U and V are open subsets of X, and if  $U \subseteq V$ , then

$$U \subseteq V \implies \{f \in C_c(X), \ f \prec U\} \subseteq \{f \in C_c(X), \ f \prec V\}$$
$$\implies \{I(f), \ f \in C_c(X), \ f \prec U\} \subseteq \{I(f), \ f \in C_c(X), \ f \prec V\}$$

Hence  $\mu(U) \leq \mu(V)$ . Now by equation (5),  $\mu^*(U) \leq \mu(U)$ . To show the reverse inequality, suppose by contradiction that  $\mu^*(U) < \mu(U)$ .

Since  $\mu^*(U)$  is an infimum, then for every  $\varepsilon > 0$  there exists some  $V \supseteq U$  where if we write  $\mu^*(U) + \varepsilon = \mu(U)$ 

$$\mu(V) < \mu^*(U) + \varepsilon = \mu(U) \implies \mu(V) < \mu(U), \ U \subseteq V$$

This contradicts what we have just proven, and therefore  $\mu^*(U) = \mu(U)$  for every open set U.

#### Part f

*Proof.* We wish to show that every open set U is  $\mu^*$ -measurable. By Theorem 1.10, it suffices to show that for every  $E \subseteq X$ 

$$\mu^*(E) \ge \mu^*(E \cap U) + \mu^*(E \setminus U) \tag{10}$$

because the reverse inequality is given by subadditivity of  $\mu^*$ , and we can also assume that  $\mu^*(E) < +\infty$ . Let us assume that E is open, we wish to find some function  $h \in C_c(X)$ ,  $h \prec E$  with

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

The above formula is fussy, but the liberty is taken to show it beforehand to avoid any potential confusion that follows. Since  $E \cap U$  is an open subset of X, the definition of  $\mu(E \cap U) = \mu^*(E \cap U)$  in (3) tells us that every  $\varepsilon > 0$  induces some  $f \in C_c(X)$ ,  $f \prec E \cap U$  where

$$I(f)>\mu(E\cap U)-\varepsilon=\mu^*(E\cap U)-\varepsilon \eqno(11)$$

Also, supp (f) is a closed set (compact subsets of Hausdorff spaces are closed), therefore  $E \setminus \text{supp}(f)$  is an open set. We make a small diversion from the current part of the proof and turn out attention to the fact that

$$\operatorname{supp}(f) \subseteq U \implies U^c \subseteq (\operatorname{supp}(f))^c$$
$$\implies E \setminus U \subseteq E \setminus \operatorname{supp}(f)$$

And because the outer-measure  $\mu^*$  is monotone.

$$\mu^*(U) \le \mu^*(E \setminus \text{supp}(f)) \tag{12}$$

Now, using the definition of  $\mu(E \setminus \text{supp}(f))$  (recall that  $E \setminus \text{supp}(F)$  is an open set), for every  $\varepsilon > 0$ , there exists some  $g \in C_c(X)$ ,  $g \prec E \setminus \text{supp}(F)$  with

$$I(g) > \mu(E \setminus \text{supp}(f)) - \varepsilon = \mu^*(E \setminus \text{supp}(f)) - \varepsilon \tag{13}$$

It is at this part of the proof where we wish to define h = f + g, but first we must verify

- $f + g \in C_c(X, [0, 1]),$
- $f + g \prec E$

The sum of two non-negative functions is non-negative, and for every  $x \in \text{supp}(f)$ ,  $f \leq 1$ . Also

$$\operatorname{supp}(g) \subseteq (\operatorname{supp}(f))^c \implies \operatorname{supp}(f) \subseteq (\operatorname{supp}(g))^c$$
$$\implies \operatorname{supp}(f) \subseteq \{g = 0\}$$

The last implication comes from taking complements on both sides of  $\{g \neq 0\} \subseteq \text{supp}(g)$ . So  $x \in \text{supp}(f) \implies f+g \leq 1$ . Now if  $x \notin \text{supp}(f)$ , then  $f+g=g \leq 1$ . Furthermore, supp (f+g) is a closed subset of compact supp  $(f) \cup \text{supp}(g)$ . This is because  $\{f+g \neq 0\} \subseteq \{f \neq 0\} \cup \{g \neq 0\}$ , and the finite union of two compact sets is again again compact.

A moment's thought should yield the fact that the last estimate should be an equality, but it is a needless distraction. Therefore supp (f + g) is compact and  $f + g \in C_c(X, [0, 1])$ .

Now both bullet points are satisfied, and we can set h = f + g. Adding equation (13) with (11) gives us

$$I(h) = I(f) + I(g) > \mu^*(E \cap U) + \mu^*(E \setminus \text{supp}(f)) - 2\varepsilon$$

Upon applying (12) to the right member of the above estimate, we have

$$I(h)>\mu^*(E\cap U)+\mu^*(E\setminus U)-2\varepsilon$$

But this particular  $h \in C_c(X) \cap \{f \prec E\}$ , therefore

$$\mu^*(E) \geq I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Since  $\varepsilon > 0$  is arbitrary, equation (10) holds for every open E. Now for any general  $E \subseteq X$ , fix any  $\varepsilon > 0$  and by how we defined  $\mu^*(E)$ , there exists some open  $V \supseteq E$ —recall that  $\mu^*(E)$  is the infimum over the set of  $\mu(V)$  where V is an open superset of E—hence

$$\mu^*(E) + \varepsilon > \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

By monotonicity (twice) of the outer-measure  $\mu^*$ , we have

$$\mu^*(E) + \varepsilon > \mu^*(E \cap U) + \mu^*E \setminus U$$

Let  $\varepsilon \to 0$ , and we get

$$\mu^*(E) \ge \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Therefore every open  $U\subseteq X$  is  $\mu^*$ -measurable. So  $\mu=\mu^*|_{\mathbb{B}}$  is a Borel measure on X.

#### Part g

*Proof.* To show outer-regularity, fix any  $E \in \mathbb{B}$ , then by definition,

$$\mu(E) = \mu^*(E) = \inf \{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$

And for every open U, (1) follows from Equation (3).

#### Part h

*Proof.* We want to show that for every compact K, Equation (2) holds. To reduce the notational baggage that follows, we agree to define

$$\{I(f), f \in C_c(X), f \prec U\} = \{I(f), f \prec U\}$$

Similarly for  $\{I(f), f \geq \chi_K\}$ . If  $\mu(K) = 0$ , then  $\mu(K)$  is obviously a lower bound, since  $f \geq \chi_K \geq 0$  means that  $I(f) \geq 0$ , for every  $f \geq \chi_K$ . So we can suppose  $\mu(K) > 0$ .

Fix an arbitrary  $f \ge \chi_K$ , then this particular f induces an open set  $U_{\alpha} = \{f > 1 - \alpha\}$ , where  $\alpha > 0$ . Notice also that

$$K\subseteq \{f\geq 1\}\subseteq \{f>1-\alpha\}=U_\alpha$$

Since  $U_{\alpha}$  is an open superset of K, by Equation (7),  $\mu(K) \leq \mu(U_{\alpha})$ , but  $\mu(U_{\alpha})$  is simply the supremum of  $\{I(g), g \prec U_{\alpha}\}$ . If we wish to show that  $\mu(K) \leq \mu(U_{\alpha}) \leq I(f)$ , it suffices to show that I(f) is an upper-bound for  $\{I(g), g \prec U_{\alpha}\}$ .

Fix any  $I(g) \in \{I(g), g \prec U_{\alpha}\}$ , note that  $1 - \alpha \neq 0$  for any  $\alpha$  small enough, then

- $f/(1-\alpha) > 1$  on  $U_{\alpha}$ ,
- $1 \ge g \ge 0$  on  $U_{\alpha}$ , in particular,  $f/(1-\alpha) g \ge 0$  on  $U_{\alpha}$ ,
- If  $x \notin U_{\alpha}$ , then  $f/(1-\alpha) g = f(1-\alpha) \ge 0$ .
- Therefore  $f/(1-\alpha)-g\geq 0$  for any x, and by Lemma 2.1,

$$I(f/(1-\alpha)) \ge I(g) \quad \forall g \prec U_{\alpha}$$

Combining the above estimate with  $\mu(K) \leq \mu(U_{\alpha})$  gives us

$$\mu(K) \leq \frac{1}{1-\alpha} I(f)$$

Now write  $\varepsilon = \alpha/\mu(K) > 0$  and for every  $\varepsilon > 0$  we get

$$\mu(K) - I(f) \le \alpha \mu(K) = \varepsilon$$

Send  $\varepsilon \to 0$  and  $\mu(K) \le I(f)$  for every  $f \ge \chi_K$ .

To show that  $\mu(K)$  is indeed the infimum for  $\{I(f), f \geq \chi_K\}$ , notice that for every  $\varepsilon > 0$  we can obtain some open superset  $U \supseteq K$  (by outer-regularity) where  $\mu(U) < \mu(K) + \varepsilon$ . By Urysohn's Lemma, there exists some  $g \prec U$ , g(x) = 1 for every  $x \in K$ .

$$g \in \{I(f), \ f \prec U\} \cap \{I(f), \ f \geq \chi_K\}$$

Therefore  $I(g) \le \mu(U) < \mu(K) + \varepsilon$  as desired, and Equation (2) holds.

#### Part i

*Proof.*  $\mu(K) < +\infty$  for every compact K. Indeed, since  $I(\chi_K) \in \{I(f), f \ge \chi_K\}$ , then by Theorem 7.1, there exists a constant  $C_K \ge 0$  that bounds

$$\mu(K) \leq |I(\chi_K)| = I(\chi_K) \leq C_K \cdot \|\chi_K\| = C_K < +\infty$$

#### Part j

*Proof.* Fix any open set U, then for every  $\varepsilon > 0$ , there exists some  $f \prec U$  with  $\mu(U) - \varepsilon < I(f)$ . Then denote  $K = \text{supp}(f) \subseteq U$ . If we take any  $I(h) \in \{I(h), h \geq \chi_K\}$ , then  $h \geq f$  gives us  $I(h) \geq I(f)$  by Lemma 2.1. So I(f) is a lower bound of  $\{I(h), h \geq \chi_K\}$ , therefore

$$\mu(U) - \varepsilon \le I(f) \le \mu(K)$$

Since supp  $(f) = K \subseteq U$ , this proves inner-regularity of  $\mu$  on open sets.

#### Part k

*Proof.* Suppose  $f \in C_c(X, [0, 1])$ , we first show that Equation (8) holds. We divide the interval [0, 1] into  $N \ge 1$  chunks by writing

$$K_j = \{f \ge j/N\}$$

for every  $1 \ge j \ge N$ . And define  $K_0 = \text{supp}(f)$ . Each  $K_j$  is a closed subset of supp (f), and therefore compact. More is true,

- $K_{j-1} \supseteq K_j$  for every  $1 \le j \le N$ .
- $x \in K_j$  iff  $f(x) \in \left[\frac{j}{N}, 1\right]$ ,
- $x \notin K_j$  iff  $f(x) \in \left[0, \frac{j}{N}\right)$ , and
- $x \in (K_{j-1} \setminus K_j)$  iff  $f(x) \in \left[\frac{j-1}{N}, \frac{j}{N}\right)$

Folland constructs a finite sequence of compactly supported functions,  $\{f_j\}$ , where  $1 \leq j \leq N$  such that

- Each  $0 \le f_i \le 1/N$ ,
- If  $x \in (K_m \setminus K_{m+1})$  iff  $f(x) \in \left[\frac{m}{N}, \frac{m+1}{N}\right)$  means that  $f_j = 1$  for all  $1 \le j \le m$ , and
- $f_{m+1} = f m/N$  on  $K_m$ , such that

$$f(x) = \left(\sum f_{j \leq m}(x)
ight) + \left(f(x) - rac{m}{N}
ight) = rac{m}{N} + \left(f(x) - rac{m}{N}
ight)$$

- And for every  $m < j \le N, f_j = 0.$
- If  $x \notin K_m$  iff  $f(x) \in [0, \frac{m}{N}]$  then for every  $m+1 \le j \le N$ ,  $f_j = 0$ .

The illustration for when N=5 below should make things clearer.



It is also trivial to verify that

• For every  $x \in K_j$ ,  $f_j = N^{-1}$ , and

$$\chi_{K_i} N^{-1} \le f_j \tag{14}$$

Also, if  $x \notin K_j$  then  $f_j \ge 0$ , therefore  $f_j \ge \chi_{K_j} N^{-1}$  at every x.

• If  $x \notin K_{j-1}$  then  $f_j = 0 \le \chi_{K_{j-1}} \cdot N^{-1}$ . If x is in  $K_{j-1}$  then  $f_j \le N^{-1}$  by construction and therefore

$$f_j \le \chi_{K_{j-1}} N^{-1} \tag{15}$$

for all x.

•  $f_j \in C_c(X)$ , since supp  $(f_j) \subseteq \text{supp } (f)$ .

Combining Equations (14) with (15), and by monotonicity in  $L^+(X, \mathbb{B}, \mu)$ , since  $f_j \in L^+$ 

$$\int \frac{1}{N} \chi_{K_j} d\mu \le \int f_j d\mu \le \int \frac{1}{N} \chi_{K_{j-1}} d\mu$$

And for every  $1 \le j \le N$ ,

$$\frac{1}{N}\mu(K_j) \le \int f_j d\mu \le \frac{1}{N}\mu(K_{j-1}) \tag{16}$$

Furthermore, from Equation (14), since  $Nf_j \ge \chi_{K_j}$  then by Equation (2),

$$\mu(K_j) \leq I(Nf_j) \implies \frac{1}{N} \mu(K_j) \leq I(f_j)$$

Now for any arbitrary  $I(h) \in \{I(h), h \ge \chi_{K_{j-1}}\}$ , since

$$h \ge \chi_{K_{i-1}} \ge Nf_i \implies I(h) \ge I(Nf_i)$$

So  $NI(f_j)$  is a lower bound for  $\{I(h), \ h \geq \chi_{K_{j-1}}\}$  and

$$I(f_j) \le \frac{1}{N} \mu(K_{j-1})$$

Combining the last two results, with  $I(f_i)$ , we get

$$\frac{1}{N}\mu(K_j) \le I(f_j) \le \frac{1}{N}\mu(K_{j-1}) \tag{17}$$

Taking the sum over  $1 \le j \le N$  for Equations (16) and (17). Define  $A = N^{-1} \sum_{i=0}^{N-1} \mu(K_j)$ , and  $B = N^{-1} \sum_{i=0}^{N} \mu(K_j)$ 

$$B \le \int f d\mu \le A$$

And also

$$B \leq I(f) \leq A$$

This is because of finite additivity of both I and the integral, and  $f = \sum f_j$  on  $K_0 = \text{supp}(f)$ . Subtracting the two equations (keeping in mind that  $\mu(K_j) < +\infty$  for any compact  $K_j$ ), we get

$$(-1)(A-B) \leq \left( \int f d\mu - I(f) \right) \leq A - B \implies \left| \int f d\mu - I(f) \right| \leq A - B$$

It is trivial to verify that

$$0 < A - B = N^{-1}(\mu(K_0) - \mu(K_N)) < N^{-1}\mu(K_0)$$

as  $K_N \subseteq K_0$ . Let  $N \to \infty$  and

$$\int f d\mu = I(f)$$

Equation (8) holds as desired.

#### Part 1

*Proof.* Now for any general  $f \in C_c(X)$ , f must be bounded on the plane since  $C_c(X) \subseteq BC(X)$ , and  $|f| \leq M_0$  for some  $M_0 \geq 0$ . Since supp (f) is compact, we know that

$$\int |f| d\mu \le \int M_0 \chi_{\operatorname{supp}(f)} d\mu \le M_0 \mu(\operatorname{supp}(f)) < +\infty$$

And  $C_c(X) \subseteq L^1(\mu)$ . Furthermore,

$$\frac{1}{2}(|\operatorname{Re} f| + |\operatorname{Im} f|) \le |f| \le M_0$$

So that Re f and Im f are in  $C_c(X)$ . Without loss of generality, we may assume that f is real. Define  $f_1 = \operatorname{Re} f^+/M_0$  and  $f_2 = \operatorname{Re} f^-/M_0$  and it immediately follows that  $f_1, f_2 \in C_c(X, [0, 1])$ .

By linearity of I on  $C_c(X)$  and the integral in  $L^1(\mu)$ ,

$$I(f_1-f_2)=I(f)=\int f d\mu =\int f_1 d\mu -\int f_2 d\mu$$

Then we may apply the above to the real and imaginary parts of a general  $f \in C_c(X)$ , and this completes the proof.

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## Folland Reading

Theorem 7.3

## Theorem 7.3

Proposition 3.1. See Theorem 7.2

Proof.

## Theorem 7.4

Proposition 4.1. See Theorem 7.2

Theorem 7.5

Proposition 5.1.

Theorem 7.6

Proposition 6.1.

Theorem 7.7

Proposition 7.1.

Theorem 7.8

Proposition 8.1.

Proof.

## Theorem 7.9

**Proposition 9.1.** If  $\mu$  is a Radon measure on X, then  $C_c(X)$  is dense in  $L^p(\mu)$  for  $1 \leq p < +\infty$ .

*Proof.* Theorem 6.7 tells us that the set of  $L^p$  simple functions (as Folland calls them), which are

$$\Lambda = \left\{f,\, f = \sum_{j \leq n} a_j \chi_{E_j}, \; a_j \in \mathbb{C}, \, \mu(E_j) < +\infty 
ight\}$$

So for every  $f \in L^p$ , there exists a sequence  $\{f_n\} \subseteq \Lambda$  with  $f_n \to f$  pointwise and  $f_n \to f$  in  $L^p$ .

Theorem 7.10

Proposition 10.1.

Theorem 7.11

Proposition 11.1.