MATH 263: Section 003, Midterm Review Document

Mohamed-Amine Azzouz
mohamed-amine.azzouz@mail.mcgill.ca

September 13^{th} 2021

1 Review of the Material from Week 1

1.1 Ordinary and Partial Differential Equations

Ordinary Differential Equations (ODE's) are differential equations involving a single variable function and its derivatives. For example:

$$y''(x) + y(x) = \cos x$$

Partial Differential Equations (PDE's) are differential equations involving a multi-variable function and its partial derivatives. For example:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

1.2 Order of a Differential Equation (DE)

The **order of a DE** corresponds to the highest derivative it contains. For example,

$$y^{(69)}(x) + y(x)^2 = \sin x$$

is a 69^{th} order ODE.

1.3 Verify Whether a Function Solves a DE

Given a solution to verify, one simply needs to compute its derivatives and substitute them in the differential equation.

1.4 Initial and Boundary Value Problems and Conditions

An **initial value problem** (IVP) is a differential equation with initial value conditions. Those conditions are restrictions on the solution's value and derivatives at a point, such as $y(0) = 1,\ y'(1) = 0$.
A **boundary value problem** (BVP) uses boundary value conditions, which are multiple restrictions on the solution's value, such as $y(0) = 1,\ y(1) = -1,\ y(2) = 7$. In general, an n^{th} order ODE will require n initial conditions to produce a unique solution.

1.5 Autonomous ODE's

ALSO IN TUTORIAL 4: **Autonomous ODE's** only contain the dependent variable, they are of the form:

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)})$$

1.6 Linear and Non-Linear ODE's

A **linear ODE** can be written as a linear combination of y and its derivatives as such:

$$\sum_{k=0}^{n} a_k(x)\ y^{(k)} = g(x)$$

An example would be:

$$x^2\ y''(x) + 2x\ y'(x) - y(x) = \cos x$$

Otherwise, the ODE is **non-linear**.
Note: when the right hand side $g(x)$ is 0, the ODE is also **homogeneous**.

1.7 Slope Fields

ALSO IN TUTORIAL 4: A **slope field** is a graphical representation of a family of functions satisfying $y' = f(x, y)$. For some point (x, y), one draws the slope $y' = f(x, y)$ to qualitatively represent the solutions. Given a slope field, starting at an initial condition and tracing along the field sketches the particular solution.

2 Tutorial 2

2.1 Separable ODE's

A **separable ODE** is of the form:

$$\frac{dy}{dx} = f(x)g(y)$$

2.2 Solving First Order Linear ODE's: Integrating Factors

A **first order linear ODE** is of the form:

$$y' + p(x)y = q(x)$$

2.3 Homogeneous First Order ODE's

A **homogeneous ODE** is of the form:

$$\frac{dy}{dx} = F(\frac{y}{x})$$

Note: **not** the same as the definition given in 1.6.
Let $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow y' = xv' + v$. Then substitute and solve for v to find y.

2.4 Bernoulli Equations

A **Bernoulli equation** is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

When $n \notin \{0, 1\}$, we can let $v = y^{1-n}$, making the ODE linear for v.

3 Tutorial 3

3.1 Exact ODEs

An **exact ODE** is of the form:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

$$M(x,y)\ dx + N(x,y)\ dy = 0$$

where

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Then, we can define some $F(x,y)$ such that:

$$d(F(x,y)) = M(x,y)\ dx + N(x,y)\ dy = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$$

Then, find $F(x,y)$, and the relation $F(x,y) = \int 0\ dx = C$ is the solution.

Note: when the ODE is not exact, an integrating factor may make it exact. Let it be $\mu(x,y)$:

$$(\mu(x,y)M(x,y)) + (\mu(x,y)N(x,y))\frac{dy}{dx} = 0$$

To make the ODE exact,

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$$

$$\frac{\partial N}{\partial x}\mu + N\frac{\partial \mu}{\partial x} = \frac{\partial M}{\partial y}\mu + M\frac{\partial \mu}{\partial y}$$

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu + N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = 0.$$

Instead of solving a PDE, consider two specific cases:

1. $\mu(x,y) = \mu(x)$:

$$(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\mu = N\frac{d\mu}{dx}$$

$$\mu(x) = \exp[\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\ dx]$$

2. $\mu(x,y) = \mu(y)$:

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu = M\frac{d\mu}{dy}$$

$$\mu(y) = \exp[\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\ dy.]$$

Try both cases if needed, your integrating factor must be single-variable for it to work. In the general case, both or none may work.

3.2 Change of Variables

Various substitutions can be used, such as $v = y'(x)\ or\ v = ax + by$.

3.3 Slope Fields

(From Tutorial 2): A **slope field** is a graphical representation of a family of functions satisfying $y' = f(x,y)$. For some point (x,y), one draws the slope $y' = f(x,y)$ to qualitatively represent solutions. Given a slope field, starting at an initial condition and tracing along the field sketches the particular solution.

4 Tutorial 5

4.1 Second Order Linear ODE's

A **second order linear ODE** is of the form:

$$a_0(x)y" + a_1(x)y'(x) + a_2(x)y(x) = g(x)$$

In particular, it is also **homogeneous** when $g(x) = 0$.

4.2 Principle of Superposition

It can be directly shown that when y_1 and y_2 solve the general homogeneous linear ODE:

$$a_0(x)y''(x) + a_1(x)y'(x) + a_2(x)y(x) = 0$$

the linear combination $y_0(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution to the differential equation.

4.3 Homogeneous Equations with Constant Coefficients

Homogeneous Equations with Constant Coefficients are of the form:

$$ay'' + by' + cy = 0$$

this is solved by making the substitution $y = e^{kx}$, leading to **the characteristic equation**

$$ak^2 + bk + c = 0.$$

4.4 Repeated Roots and Reduction of Order

When $ay'' + by' + cy = 0$ and $b^2 = 4ac$, the characteristic equation will have have a unique solution and only the first solution can be found:

$$y_1(x) = e^{\frac{-b}{2a}x}$$

Therefore, a method called **reduction of order** must be used to find the second solution. For any second order homogeneous linear ODE, given one solution $y_1(x) \neq 0$ that solves

$$y'' + p(x)y'(x) + q(x)y(x) = 0$$

let $y(x) = v(x)y_1(x)$, find y''s derivatives and substitute them in the ODE:

$$y'(x) = v'(x)y_1(x) + v(x)y_1'(x)$$
$$y_1v'' + (2y_1' + py_1)v' + (y_1'' + py_1' + qy_1)v = 0$$

$$y_1v'' + (2y_1' + py_1)v' = 0,$$

which reduces to a first order ODE when letting $\gamma(x) = v'(x)$.

For the ODE, $ay'' + by' + cy = 0$ where $b^2 = 4ac$, the solution is then:

$$y(x) = c_1e^{\frac{-b}{2a}x} + c_2xe^{\frac{-b}{2a}x} = (c_1 + c_2x)e^{\frac{-b}{2a}x}$$

4.5 Higher Order Linear ODE's

A **linear ODE of order n** is of the form:

$$\sum_{k=0}^{n} a_k(x)y^{(k)}(x) = g(x)$$

which is **homogeneous** when $g(x) = 0$. For homogeneous ODEs, the principle of superposition still holds.

To solve constant coefficient homogeneous linear ODEs of the form

$$\sum_{k=0}^{n} a_ky^{(k)}(x) = 0,$$

we can still let $y = e^{kx} \Rightarrow y^{(n)} = k^ne^{kx}$, which gives the characteristic polynomial in k:

$$\sum_{k=0}^{n} a_kk^n = 0.$$

Then, the general solution will be the linear combination of all particular solutions. In particular, when all roots are distinct and real:

$$y(x) = \sum_{i=0}^{n} c_i e^{k_i x},$$

finding a particular solution requires n initial values $y(x_0) = y_0,\ y'(x_1) = y_1,\ \dots\ ,\ y^{(n-1)}(x_{n-1}) = y_{n-1}$.

5 Tutorial 6

5.1 Complex Numbers, Euler and DeMoivre's Formulas

The ODE $ay" + by' + cy = 0$ gives the characteristic equation $ak^2 + bk + c = 0$, which does not have solutions in $\mathbb{R}$ when $b^2 < 4ac$. Therefore, we can define a number, called i (the imaginary unit), such that $i^2 = -1$ ($i = \sqrt{-1}$). This lets us work with a new kind of numbers, called the **complex numbers**, denoted as $\mathbb{C}$.

$$\mathbb{C} = \{z : z = a + bi, a \in \mathbb{R}, b \in \mathbb{R}\}$$

Where a is the **real part** of z, and b is the **imaginary part** of z.

$$Re(z) = a,\ Im(z) = b$$

Note: In Electrical Engineering, j is used instead as a complex unit. This is to not confuse the imaginary unit i with the current variable i. Complex numbers can be added, subtracted, multiplied, and divided the same way as if i were an algebraic variable, while keeping in mind that $i^2 = -1$. Examples:

$$(3 + 2i) + (2 - 4i) = (3 + 2) + (2 - 4)i = 5 - 2i$$
$$(3 + 2i) - (2 - 4i) = (3 - 2) + (2 + 4)i = 1 + 6i$$
$$(3 + 2i)(2 - 4i) = 6 - 12i + 4i - 8^2 = 6 - 12i + 4i - 8(-1) = 14 - 8i$$
$$\frac{3 + 2i}{2 - 4i} = \frac{-2 + 16i}{2^2 - (4i)^2} = \frac{-2 + 16i}{2^2 + 4^2} = \frac{-2 + 16i}{20} = \frac{1}{10}(-1 + 8i).$$

Note that the division process consists of multiplying the numerator and denominator by the complex conjugate of the denominator. In general, the complex conjugate of $z = a + bi$, often denoted as $z^* = a - bi$. Complex numbers can also be written in a polar form, $z = a + bi = r[\cos(\theta) + i\sin(\theta)]$, where $r = \sqrt{a^2 + b^2}$ is the norm, and $\theta = \arctan(\frac{b}{a})$ is the argument. Those two representations can be illustrated using the **complex plane**.

Defining exponentiation for complex numbers as $e^z = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we can let:

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{even\ n} \frac{i^n x^n}{n!} + \sum_{odd\ n} \frac{i^n x^n}{n!}$$

For even numbers, let $n = 2k \Rightarrow i^n = i^{2k} = (-1)^k$. For odd numbers, $n = 2k + 1 \Rightarrow i^n = i^{2k+1} = i \cdot i^{2k} = i(-1)^k$.

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$
$$e^{ix} = \cos x + i \sin x,$$

which is **Euler's Formula**. Therefore, the polar form can be written as

$$z = a + bi = r[\cos(\theta) + i\sin(\theta)] = re^{i\theta}.$$

Similarly, **DeMoivre's Formula** is:

$$(e^{ix})^n = e^{inx} = \cos(nx) + i\sin(nx).$$

This formula can be used to multiply and find powers of complex numbers:

$$z^n = re^{in\theta} = r^n[\cos(n\theta) + i\sin(n\theta)].$$

5.2 The Wronskian and Abel's Theorem

Given two solutions of a second order linear ODE, $y_1(x),\ y_2(x)$, they are independent if their **Wronskian** is not 0, which is given by:

$$W = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x)$$

Given a differential equation of the form:

$$y" + p(x)y' + q(x)y = 0$$

Abel's Theorem states that:

$$W = c\ \exp[-\int p(x)\ dx]$$

where the constant can be found with a initial condition on the Wronskian.

5.3 Euler's Equation

Euler's Equations are of the form:

$$ax^2y" + bxy' + cy = 0$$

this is solved by making the substitution $y = x^r$, $x > 0$. The characteristic polynomial becomes:

$$ar^2 + (b-a)r + c = 0.$$

For complex roots, the solution would be of the form:

$$y(x) = c_1x^{r_1} + c_2x^{r_2},\ x > 0$$

In the case where $x < 0$, one can make the substitution $t = -x > 0$ and $y(x) = u(t)$, which would give the same ODE. Therefore, for all $x \neq 0$, the solution is:

$$y(x) = c_1|x|^{r_1} + c_2|x|^{r_2}$$

For complex roots $r_{1,2} = \lambda \pm i\mu$, the solution would be of the form:

$$y(x) = c_1x^{\lambda+i\mu} + c_2x^{\lambda-i\mu},\ x > 0$$

Knowing that $x^{i\mu} = e^{i\mu \ln x} = \cos(\mu \ln x) + i\sin(\mu \ln x)$, the final real solution would be:

$$y(x) = x^\lambda[k_1\cos(\mu \ln|x|) + k_2\sin(\mu \ln|x|)]$$

Given a double root r_1, reduction of order gives us a solution of:

$$y(x) = (c_1 + c_2 \ln|x|)|x|^{r_1}$$

6 Tutorial 7

6.1 Higher Order Homogeneous Equations and the Wronskian

Given n solutions to an n^{th} order linear ODE, showing their independence can also be shown by their **Wronskian**, which must be nonzero for linear independence. In general, it is of the form:

$$W(y_1,\ y_2,\ y_3,\ \dots\ y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1" & y_2" & y_3" & \cdots & y_n" \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Note: **Abel's Theorem** also can show the Wronskian for higher order ODEs. For an n^{th} order linear ODE of the form:

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0$$

Given n fundamental solutions $y_1,\ y_2,\ y_3,\ \ldots\ y_n$, **Abel's Theorem** states that:

$$W(y_1,\ y_2,\ y_3,\ \ldots\ y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_n \\ y_1' & y_2' & y_3' & \cdots & y_n' \\ y_1" & y_2" & y_3" & \cdots & y_n" \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \cdots & y_n^{(n)} \end{vmatrix} = C\ \exp[-\int p_{n-1}(x)\ dx]$$

6.2 Existence and Uniqueness Theorem

Given the IVP:

$$y" + p(x)y' + q(x)y = g(x),\ y(x_0) = y_0,\ y'(x_0) = y_0$$

if p, q, and g are continuous on the open interval I that contains the point x_0, then the solution to the IVP is **unique, differentiable, and exists** on the interval I.

6.3 Reduction of Order

Recall in **Tutorial 5** that given the ODE:

$$y" + p(x)y'(x) + q(x)y(x) = 0$$

and one solution $y_1(x)$, we can find **a general solution of the form** $y(x) = v(x)y_1(x)$. Then, find y"s derivatives and substitute them in the ODE to find v and y:

$$y'(x) = v'(x)y_1(x) + v(x)y_1'(x)$$
$$y_1v" + (2y_1' + py_1)v' + (y_1" + py_1' + qy_1)v = 0$$
$$y_1v" + (2y_1' + py_1)v' = 0$$

6.4 Euler's Equations: Change of Variables

More general Euler Equations of the form:

$$a(x - x_0)^2y" + b(x - x_0)y' + cy = 0$$

can solved by simply letting $t = x - x_0,\ dt = dx$.
Then, the solution is solved the same way as done in **Tutorial 6** to find in general:

$$y = y(t) = y(x - x_0),\ x \neq x_0.$$

7 Tutorial 8

7.1 Introduction to Nonhomogeneous ODEs: Method of Undetermined Coefficients

To solve a nonhomogeneous linear ODE of the form:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = g(x) \neq 0$$

note that by superposition, if $y_1(x)$ and $y_2(x)$ solve the homogeneous ODE:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

and that some specific $Y(x)$ solves the nonhomogeneous ODE, then:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x)$$

is the general solution to the nonhomogeneous ODE. To find $Y(x)$, we need to make an educated guess on the form of the solution: we want that guessed form of $Y(x)$ and/or its derivatives to be linearly dependent on $g(x)$. One also must make sure that their chosen $Y(x)$ is not $g(x)$, since it already solves the homogeneous equation. Then find $Y(x)$'s coefficients by plugging them in the ODE.

For the particular solution of $ay" + by' + cy = g(x)$, here's a table directly from Boyce and DiPrima, 10th edition (3.5, table 3.5.1, p.182):

$g(x)$	$Y(x)$
$P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
$P_n(x)e^{\alpha x}$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}$
$P_n(x)e^{\alpha x} \begin{cases} \cos\beta x \\ \sin\beta x \end{cases}$	$x^s[(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}\cos\beta x + (B_0x^n + B_1x^{n-1} + \cdots + B_n)e^{\alpha x}\sin\beta x]$

Note: Here s is the smallest nonnegative integer (s = 0, 1, or 2) that will ensure that no term in $Y(x)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, **s is the number of times 0 is a root of the characteristic equation**, α **is a root of the characteristic equation**, and $\alpha + i\beta$ **is a root of the characteristic equation**, respectively.

Note: if your system of equations for coefficients doesn't give you any answer (inconsistent system, all coefficients cancel out, etc), either you made an algebraic mistake or made a wrong choice for $Y(x)$.