

# Folland Reading

me

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# 1 Chapter 4

## 1.1 Theorem 4.1

**WTS.** Suppose that  $A$  is a subset of  $X$ , let  $\text{acc } A$  be the set of accumulation points of  $A$ , then

$$\overline{A} = A \cup \text{acc } (A) \quad (1)$$

and  $A$  is closed if and only if  $\text{acc } (A) \subseteq A$ .

*Proof.* Suppose that  $x \notin \overline{A}$ , then  $x \in (\overline{A})^c = A^{\circ\circ}$ , then  $A^c \in \mathcal{N}_B(x)$ . But this means that  $x \notin \text{acc } (A)$ , since there exists a neighbourhood of  $x$  (in the form of  $A^c$ ), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \emptyset$$

Also,  $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$  which means that

$$x \notin \overline{A} \implies x \notin A$$

Since  $x \notin \overline{A} \implies x \notin A$  and  $x \notin \text{acc } (A)$ ,

$$(\overline{A})^c \subseteq A^c \cap \text{acc } (A)^c = (A \cup \text{acc } (A))^c$$

Now, if  $x \notin \text{acc } (A) \cup A$ , then  $x \notin \text{acc } (A)$ , therefore there exists some  $U \in \mathcal{N}_B(x)$  such that

$$A \cap U \setminus \{x\} = A \cap U = \emptyset$$

Where for the second last equality we used the fact that  $x \notin A \implies A \setminus \{x\} = A$ , and taking complements gives us

$$U \subseteq A^c$$

And since  $U \in \mathcal{N}_B(x)$ , then  $x \in U^o \subseteq A^{co}$  (since  $U^o$  is an open subset of  $A^c$ ). then

$$x \in A^{co} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore  $(A \cup \text{acc}(A))^c \subseteq (\overline{A})^c$ . □

## 1.2 Theorem 4.2

**WTS.** If  $\mathcal{T}_X$  is a topology on  $X$  and  $\mathcal{E} \subseteq \mathcal{T}_X$  then  $\mathcal{E}$  is a base for  $\mathcal{T}_X$  if and only if for every

$$\forall U \in \mathcal{T}_X, U \neq \emptyset, \implies U = \bigcup_{V \in B} V$$

Where  $B$  is a subset of  $\mathcal{E}$ .

*Proof.* Suppose that  $\mathcal{E}$  is a base, then fix any non-empty  $U \in \mathcal{T}_X$ , then for every  $x \in U$ , there exists a neighbourhood base for this  $x$  and a member  $V \in \mathcal{E}$  such that  $x \in V_x \subseteq U$ . Take the union over all  $V_x$  and

$$U \subseteq \bigcup_{x \in U} V_x$$

But each  $V_x \subseteq U$ , so  $U = \bigcup_{x \in U} V_x$ , where  $\{V_x\} \subseteq \mathcal{E}$ .

Conversely, if every non-empty  $U$  is a union of members in  $\mathcal{E}$  then fix any  $x \in X$ , we claim that we have a neighbourhood base in

$$\{V \in \mathcal{E}, x \in V\}$$

The reason is as follows

- $x$  belongs to every  $E \in \{V \in \mathcal{E}, x \in V\}$  and
- For every open  $U$ , if  $x \in U$  then there exists a union of members of  $\mathcal{E}$  such that  $U = \bigcup E_\alpha$ , then  $x \in U \iff \exists E_\alpha \in \{V \in \mathcal{E}, x \in V\}$  and
- Using this particular  $E_\alpha \in \mathcal{E}$  that we just found,  $x \in E_\alpha \subseteq U$ , and we are done.

□

### 1.3 Theorem 4.3

**WTS.** For every  $\mathcal{E} \subseteq \mathbb{P}(X)$ ,  $\mathcal{E}$  is base for a topology on  $X$  if and only if

- (a) each  $x \in X$  is contained in some  $V \in \mathcal{E}$ , and
- (b) if  $U, V \in \mathcal{E}$ , and  $x \in U \cap V$ , then there must exist some  $W \in \mathcal{E}$  with  $x \in W \subseteq U \cap V$ .

*Proof.* Suppose that  $\mathcal{E}$  is a base, then we get a), and b) follows since for every  $U, V \in \mathcal{E} \subseteq \mathcal{T}_X$ , and by closure over finite intersections,  $U \cap V \in \mathcal{T}_X$  implies that there exists some  $W \in \mathcal{E}$  with

$$x \in W \subseteq U \cap V$$

Now, suppose both a) and b) hold, then we claim that this  $\mathcal{E} \subseteq \mathbb{P}(X)$  induces a topology on  $X$

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, \text{ with } x \in V \subseteq U\}$$

Intuitively speaking, this means that  $\mathcal{T}$  is just fine (and not too fine) to satisfy the conditions for  $\mathcal{E} \subseteq \mathcal{T}$  to be a base of  $\mathcal{T}$ .

We first show that  $\mathcal{T}$  is a topology.

- $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$ , the first is trivial and the second is from a)
- Closure under unions: fix  $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ , and  $U = \bigcup U_\alpha$ , and for every  $x \in U$  there exists some  $V_\alpha \in \mathcal{E}$  such that  $x \in V_\alpha \subseteq U_\alpha \subseteq U$ , therefore  $U \in \mathcal{T}$ .
- Closure under finite intersections, fix any  $U_1, U_2$  as elements in  $\mathcal{T}$ , then suppose that they are not disjoint (if they are disjoint then their intersection is the empty set, which is also contained in  $\mathcal{T}$ ). If  $U_1 \cap U_2 \neq \emptyset$ , then for every  $x \in U_1 \cap U_2$  induces two sets  $V_1, V_2 \in \mathcal{E}$  with  $x \in V_1 \subseteq U_1$  and  $x \in V_2 \subseteq U_2$ , taking their intersection and applying b) gives us some  $V \subseteq V_1 \cap V_2$  with  $V \in \mathcal{E}$  therefore  $x \in V \subseteq U_1 \cap U_2$ , and  $\mathcal{T}$  is closed under finite intersections.

Now to show that  $\mathcal{E}$  is a base for  $\mathcal{T}$ ,  $\mathcal{E} \subseteq \mathcal{T}$  is obvious since every  $V \in \mathcal{E}$  satisfies the properties laid out by  $\mathcal{T}$  by simply choosing  $V$  again for any

$x \in V$ . Now fix any member  $U \in \mathcal{T}$ , then for every  $x \in U$ , there exists some  $V \in \mathcal{E}$  with

$$x \in V \subseteq U$$

(This is an immediate consequence of how we defined  $\mathcal{T}$ ). And we can conclude that  $\mathcal{E}$  is a base for this induced topology  $\mathcal{T}$ .  $\square$

## 1.4 Theorem 4.4

**WTS.** If  $\mathcal{E} \subseteq \mathbb{P}(X)$ , the topology  $\mathcal{T}(\mathcal{E})$  generated by  $\mathcal{E}$  consists of  $\emptyset, X$  and all unions of finite intersections of  $\mathcal{E}$ , in symbols

$$\mathcal{T}(\mathcal{E}) = \{\emptyset, X\} \cup \left\{ \bigcup W_\alpha, W_\alpha = \bigcap E_{j \leq n}, E_j \in \mathcal{E} \right\}$$

*Proof.* Denote the set

$$W = \{X\} \cup \left\{ \bigcap V_{j \leq n}, V_j \in \mathcal{E} \right\}$$

We claim this set  $W$  satisfies Theorem 4.3. Since 4.3a) is satisfied with  $X \in W$ . 4.3b) follows since the right member in  $W$  is closed under intersections.

And if we are taking an element from each member,  $E_1 \in \{\emptyset, X\}$  and  $E_2$  is an element in the right member, then it is trivial to verify that their intersection is always contained within  $W$ . Therefore  $W$  induces a topology by Theorem 4.2, and we call this topology  $\mathcal{T}$  — and for the sake of completeness

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, x \in V \subseteq U\}$$

We so claim that if we define  $\overline{W}$  as the union of all members  $w \in W$ , together with the empty set, is equal to the set  $\mathcal{T}$ .

$$\overline{W} = \left\{ \bigcup_{w \in W} w \right\} \cup \{\emptyset\}$$

- We want to show  $\mathcal{T} \subseteq \overline{W}$ , since  $W$  is a base for the topology  $\mathcal{T}$ , every (non-empty)  $U \in \mathcal{T}$  is the union of members in  $W$  (Theorem 4.2), and there exists some  $B \subseteq W$  with

$$U = \bigcup E_{\alpha \in B} \in \overline{W}$$

Now if  $U$  is the empty set then it is trivially contained within  $\overline{W}$ .

- Next, we show that  $\overline{W} \subseteq \mathcal{T}$ , fix any element  $E \in \overline{W}$ , if  $E = \emptyset$  then there is nothing to prove since  $\mathcal{T}$  is a topology. Now for every  $x \in E$ ,

$$x \in E = \bigcup_{w \in W} w \implies x \in w$$

Therefore  $E \in \mathcal{T}$  by definition. This proves that  $\mathcal{T} = \overline{W}$ .



Now that  $\overline{W}$  is a topology, that contains  $\mathcal{E}$  as a subset, and by definition of  $\mathcal{T}(\mathcal{E})$

$$\mathcal{T}(\mathcal{E}) = \bigcap \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Tells us

$$\mathcal{T}(\mathcal{E}) \subseteq \overline{W}, \quad \text{since } \overline{W} \in \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Conversely, fix any member  $E \in \overline{W}$ , if  $E = \emptyset$  then  $E \in \mathcal{T}(\mathcal{E})$ , if not, then there exists some subset  $B \subseteq W$  such that

$$E = \bigcup_{w \in B} w = \bigcup_{w \in B} \bigcap_{j \leq n} V_{j \leq n}^w V_j \in \mathcal{E} \cup \{X\}$$

Since  $\mathcal{T}(\mathcal{E})$  is closed under finite intersections and unions, and it contains  $\mathcal{E}$  as a subset,  $\overline{W} = \mathcal{T}(\mathcal{E})$  and we are done.  $\square$

## 1.5 Theorem 4.5

**WTS.** *Every second countable space is separable. (Countable dense subset).*

*Proof.* What we wish to prove is that if a space  $X$  has a countable base, then it has a countable dense subset. Denote this base of  $X$  by  $\mathcal{E}$  as usual, then we claim that

$$W = \{x_u, U \in \mathcal{E}\}$$

Is a dense subset in  $X$ . Note that  $(\overline{W})^c = W^{co} \in \mathcal{T}_X$ . If  $W^{co} = \emptyset$  then we simply take complements and we get  $\overline{W} = X$ . So suppose that  $W^{co}$  is non-empty, then for each  $x \in W^{co}$  (by definition of a base), it should induce some  $V_x \in \mathcal{E}$  with

$$x \in V_x \subseteq W^{co}$$

But clearly, for every element in  $\mathcal{E}$ , the second estimate can never be satisfied, since for every  $U \in \mathcal{E}$ ,  $x_U \notin W^{co}$  for this particular set  $W^{co}$ . Therefore  $W^{co}$  must be empty, and this completes the proof.  $\square$

## 1.6 Theorem 4.6

**WTS.** *If  $X$  is first countable, then for every  $A \subseteq X$ ,  $x \in \overline{A} \iff$  there exists some sequence  $\{x_j\}_{j \geq 1} \subseteq A$  such that  $x_j \rightarrow x$ .*

*Proof.* Suppose that  $X$  is first countable, and  $A \subseteq X$ , and fix any element  $x \in \overline{A}$ . Since  $X$  is first countable, there is a sequence of descending neighbourhoods of  $\{U_j\}_{j \geq 1}$  of  $x$  such that

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_j \supseteq U_{j+1}$$

If  $x \in A$ , take  $x_n = x$  for all  $n \geq 1$ . If  $x \in \text{acc}(A)$ , then take  $x_n \in U_1 \cap A \setminus \{x\} = U_1 \cap A$ , which is not empty. Then it remains to show that this sequence converges to  $x$ . Fix any neighbourhood  $U \in \mathcal{N}_B(x)$  then there exists some  $N$ , for every  $n \geq N$

$$x \in U^o \implies \exists N \in \mathbb{N}^+, x \in U_N \subseteq U^o$$

Then every  $x_n \in A \cap U_N \subseteq A \cap U^o \subseteq U^o$ . And this establishes  $\implies$ .

Now suppose that  $x \notin \overline{A}$ , so that  $x \notin A$  and  $x \notin \text{acc}(A)$ , then fix any sequence  $\{x_j\} \subseteq A$ . We wish to show that  $x_j \not\rightarrow x$ .

Since  $x \notin \text{acc}(A)$ , there exists some  $V \in \mathcal{N}_B(X)$  with

$$A \cap V \setminus \{x\} = \emptyset \implies V \subseteq A^c$$

Since  $\{x_j\}_{j \geq 1} \subseteq A \implies x_j \notin A^c$  for every  $j \geq 1$ , then choose  $V$  as the neighbourhood around  $x$ , and  $x_j \not\rightarrow x$  for any arbitrary sequence  $x_j$  in  $A$ .  $\square$

## 1.7 Theorem 4.7

**WTS.**  $X$  is a  $T_1$  space  $\iff \{x\}$  is closed for every  $x \in X$ .

*Proof.* If  $X$  is  $T_1$  and  $x \in X$ , then for every  $y \neq x$  there exists some open  $U_y$  that contains  $y$  but not  $x$ . Following Folland's argument closely, every  $y \neq x$  is in  $\cup U_{y \neq x}$ . Hence  $\{x\}^c \subseteq \cup U_{y \neq x}$ . To show the converse, for every  $z \in \cup U_{y \neq x}$  that is open, there exists a  $y \neq x$  such that  $z \in U_y$ . But every  $U_y$  does not contain  $x$  as an element, so  $z \neq x$  implies that  $z \notin \{x\}$ . And  $z \in \{x\}^c$ . Hence  $\cup U_{y \neq x} = \{x\}^c$ .

Now conversely if every  $x \in X$  satisfies the fact that  $\{x\}^c$  is open, then  $\{x\}^c$  is an open set that contains every  $y \neq x$ . Now fix some  $y \neq x$ , since  $\{y\}$  is also closed, we have  $X \cap \{x\}^c$  is an open set that contains  $x$  but not  $y$ . Also,  $\{x\}^c$  is an open set that contains  $y$  but not  $x$ . And therefore  $X$  is  $T_1$ .  $\square$

## 1.8 Theorem 4.8

**WTS.** *The map  $f : X \rightarrow Y$  is continuous if and only if at  $f$  is continuous at every  $x \in X$ .*

*Proof.* Suppose that  $f$  is continuous, then fix any  $f(x) \in Y$  and any of its neighbourhood  $V \in \mathcal{N}_B(f(x))$ ,

$$f(x) \in V^o \implies f^{-1}(V^o) \in \mathcal{N}_B(x)$$

But by continuity,  $f^{-1}(V^o)$  is an open set that contains  $x$ , with

$$f(f^{-1}(V^o)) \subseteq V^o$$

Therefore  $f$  is continuous at  $x$ . Now suppose that  $f$  is continuous at every  $x \in X$ , then for every open subset  $V \subseteq Y$ , and for every point  $f(x) \in V = V^o$  means that  $V \in \mathcal{N}_B(f(x))$  for all such points  $f(x)$ . By continuity, for every  $x$  in  $f^{-1}(V)$ , implies that  $f^{-1}(V)$  is a neighbourhood of all of its elements, therefore  $f^{-1}(V) \subseteq (f^{-1}(V))^o$ , and  $f^{-1}(V)$  is open.  $\square$

## 1.9 Theorem 4.9

**WTS.** *If  $\mathcal{E}_Y$  generates the topology on  $Y$ , and  $f$  is a mapping from  $X \rightarrow Y$ , then  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(V) \in \mathcal{T}_X$  for every  $V \in \mathcal{E}_Y$ .*

*Proof.* The inverse image commutes with intersections, complements, and unions. To prove  $\Leftarrow$ , use Theorem 4.4, since every  $U \in \mathcal{T}_Y$  can be represented the union of finite intersections of elements  $\mathcal{E}_Y$ , and use the fact that  $\mathcal{T}_X$  is closed under arbitrary unions and finite intersections.

To show  $\Rightarrow$ , since  $\mathcal{E}_Y \subseteq \mathcal{T}_Y$ , if  $f^{-1}$  is open for every  $U \in \mathcal{T}_Y$ , then it is open for every  $U \in \mathcal{E}_Y$  as well.  $\square$

### 1.10 Theorem 4.10

**WTS.** *If  $X_\alpha$  is Hausdorff for each  $\alpha \in A$ , then  $X = \prod_{\alpha \in A} X_\alpha$  is Hausdorff.*

*Proof.* If two elements in  $X$ ,  $x \neq y$  then there exists some  $\alpha \in A$  such that  $\pi_\alpha(x) \neq \pi_\alpha(y) \in X_\alpha$ , but this  $X_\alpha$  is Hausdorff, then there exists two open, disjoint sets  $V_x, V_y \subseteq X_\alpha$  such that

- $x \in \pi_\alpha^{-1}(V_x)$ , and  $y \in \pi_\alpha^{-1}(V_y)$
- $\pi_\alpha^{-1}(V_x) \cap \pi_\alpha^{-1}(V_y) = \pi_\alpha^{-1}(V_x \cap V_y) = \emptyset$
- $\pi_\alpha^{-1}(V_x), \pi_\alpha^{-1}(V_y) \in \mathcal{T}_X$

Where for the last bullet point we used the fact that the product topology makes all the projection maps continuous. This proves that  $X$  is Hausdorff.  $\square$

### 1.11 Theorem 4.11

**WTS.** If  $X_\alpha$  and  $Y$  are topological spaces, and  $X = \prod_{\alpha \in A} X_\alpha$ , and  $f : Y \rightarrow X$  is a mapping. Then  $f$  is continuous if and only if  $\pi_\alpha \circ f$  is continuous for each  $\alpha \in A$ .

*Proof.* If  $\pi_\alpha \circ f$  is continuous at each  $\alpha$ , this means that

$$\forall \alpha \in A, \forall E_\alpha \in \mathcal{T}_\alpha, f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{T}_Y$$

But it is exactly sets of the form  $\pi_\alpha^{-1}(E_\alpha)$  which generate the weak topology for  $\mathcal{T}_X$ . Therefore  $f$  is continuous.

Now, suppose that  $f$  is continuous, by definition of the weak topology (as it is generated by the set of inverse projections), for every  $\alpha \in A$ ,  $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{T}_X$  and by continuity of  $f$ , its inverse image is open in  $Y$  as well.  $\square$

**Remark.** The take-away intuition here is that if the range space is generated by some  $\mathcal{E}$ , then a function is continuous if and only if all inverse images of sets in  $\mathcal{E}$  are open in the domain space. Furthermore, if the range space is endowed with the product topology (which is generated by sets of the form  $\pi_\alpha^{-1}(E_\alpha)$ , where  $E_\alpha \in \mathcal{T}_\alpha$ ), then it suffices to check all inverse images of those. And this is equivalent to checking that  $\pi_\alpha(\cdot) \circ f$  is continuous at each  $\alpha$ .



**1.12 Theorem 4.12**

**WTS.**

**1.13 Theorem 4.13**

**WTS.**

### 1.14 Theorem 4.14

**WTS.** Suppose that  $A$  and  $B$  are disjoint closed subsets of the normal space  $X$ , and let  $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$  be the set of dyadic rationals in  $(0, 1)$ . There is a family  $\{U_r : r \in \Delta\}$  of open sets such that

1.  $A \subseteq U_r \subseteq B^c$  for every  $r \in \Delta$ ,
2.  $\overline{U_r} \subseteq U_s$  for  $r < s$ , and
3. For every  $r < s$ ,  $\overline{U_r} \subseteq U_s$

*Proof.* The goal of this proof is to show that for every  $r \in \Delta$ , there exists a open  $U_r$  that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some  $N \geq 1$  that all previous  $U_r$  in previous layers have been constructed properly, meaning if  $r = k/2^n$ , then for every  $1 \leq n \leq N - 1$ , we have

$$r = \frac{k}{2^n}, 1 \leq n \leq N - 1, 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each  $U_r$ ,

- $A \subseteq U_r \subseteq B^c$  and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer  $N \geq 1$ , we only have to construct the  $U_{k/2^N}$  for every odd  $k$ , this is because if  $k$  is an even number, then  $k = 2j$  and  $r = 2j/2^N = j/2^{N-1}$  and for this particular  $U_r$  is already constructed. So for every odd  $k = 2j + 1$ , the sets of the form  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  are already defined, and satisfy

$$A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every  $k - 1 \neq 0$  and  $k + 1 \neq 1$ . (We will consider these cases later). We claim that for every pair of open sets,  $E_1, E_2 \in \mathcal{T}_X$ , then there exists some open set  $G \in \mathcal{T}_X$  such that if  $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$  where  $H$  is defined as the set

$$H = \{(E_1, E_2) \in (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \emptyset\}$$

Then there exists some  $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$  such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any  $(E_1, E_2) \in H$ , then this pair induces a pair of disjoint sets  $\overline{E_1}$  and  $E_2^c$  since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \emptyset$$

And by normality, there exists disjoint open sets  $G_1, G_2$  such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \emptyset \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since  $G_2^c$  is a closed set that contains  $G_1$  as a subset,  $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of  $G_2$  (so we may discard the notion of  $G_2$  in our minds). Let us now replace  $G$  with  $G_1$  then it is an easy task to verify that  $G = G_1 = \mathcal{J}(E_1, E_2)$  has the required properties.

Now define for every odd  $k$ , since  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (we note in passing that  $\mathcal{J}$  is not a function as the set  $G$  may not be unique).

$$U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$$

Then, if  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  is 'well constructed' we have

$$A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore  $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$  sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{k/2^N}$  and
- $\overline{U_{k/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed'  $U_{k/2^N}$  for every  $k - 1 \neq 0$  and  $k + 1 \neq 1$ . Now let us deal with the remaining

pathological cases.

If  $k - 1$  so happens to be 0, then no  $r \in \Delta$  satisfies  $r = 0/2^N$ , and we substitute

$$\overline{U}_0 = A, \quad \text{or alternatively, } U_0 = A^o$$

Then  $U_0 \in \mathcal{T}_X$ ,  $\overline{U}_0 = A \subseteq B^c$ . It is at this point that we must mention that  $0, 1 \notin \Delta$ , so  $U_0$  and  $U_1$  do not have to obey the rules we have laid out for  $U_{r \in \Delta}$ .

Now if  $k + 1$  is equal to  $2^N$  (this makes  $r = (k + 1)/2^N = 1$ ) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every  $0 \leq m \leq 2^N - 1$ ,  $U_{m/2^N}$  must staisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (even for when  $N = 1$ , since  $A = \overline{U}_0 \subseteq U_1 = B^c$ ) and a corresponding  $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$  such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when  $N = 1$ . We would only have to construct for  $k = 1$ , since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last.  $\square$

### 1.15 Theorem 4.15

**WTS.** *Urysohn's Lemma.* Let  $X$  be a normal space, if  $A$  and  $B$  are disjoint closed subsets of  $X$ , then there exists a  $f \in C(X, [0, 1])$  such that  $f = 0$  on  $A$  and  $f = 1$  on  $B$ .

*Proof.* Let  $r \in \Delta$  be as in Lemma 4.14, and set  $U_r$  accordingly except for  $U_1 = X$ . Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write  $W = \{k : x \in U_k\}$ , Then for every  $x \in A$  we have  $f(x) = 0$ , since by the construction of the 'onion' function in Lemma 4.14, for each  $r \in \Delta \cap (0, 1)$ ,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since  $r > 0$  is arbitrary, and  $0 \in W$ , we can use a classic  $\varepsilon$  argument. If  $f(x) > 0$  then there exists some  $0 < r < f(x)$  by density of the dyadic rationals on the line, if  $f(x) < 0$  then this implies that there exists some  $f(x) < r < 0$  such that  $x \in U_r$ , but no  $r \in \Delta$  can be negative, hence  $f(x) = 0$ .

Now, for every  $x \in B$ , since  $A$  and  $B$  are disjoint, and  $A \subseteq U_r \subseteq B^c$ , then for every  $x \in B$  means that  $x$  is not a member of any  $U_r$ , but we set  $U_1 = X$ . Since none of the  $r \in (0, 1)$  is a member of the set we are taking the infimum, and  $x \in U_1 = X$ . The  $\varepsilon$  argument follows: suppose for every  $\varepsilon > 0$ ,  $(1 - \varepsilon) \notin W$ , and  $1 \in W$ , then  $f(x) = 1$ .

Since  $x \in U_1 = X$ , for every  $x \in X$ ,  $f(x) \leq 1$ , and  $f(x)$  cannot be negative as  $r > 0$  for every  $r \in \Delta$ . So  $0 \leq f(x) \leq 1$ . Now we have to show that this  $f(x)$  is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in  $X$ . So  $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$  and  $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$ .

Suppose that  $f(x) < \alpha$ , so  $\inf W < \alpha$ , and using the density of  $\Delta$ , there exists an  $r$ ,  $f(x) < r < \alpha$  such that  $x \in U_r$  such that  $x \in \bigcup_{r < \alpha} U_r$ . So  $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r < \alpha} U_r$ .

Fix an element  $x \in \bigcup_{r < \alpha} U_r$ , this induces an  $r$  such that  $\inf W \leq r < \alpha$  therefore  $f(x) < \alpha$ , and  $\bigcup_{r < \alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$ .

For the second case, suppose that  $f(x) > \alpha$ , then  $\inf W > \alpha$ , and there exists an  $r$  (by density) such that  $\inf W > r > \alpha$  such that for every  $k \in W$ ,  $k \neq r$ . Therefore  $x \notin U_r$ , but by density again, and using the property of the onion function: for every  $s < r$  we get  $\overline{U_s} \subseteq U_r$ , taking complements (which reverses the estimate) — we have  $x \notin \overline{U_s}$ , but  $(\overline{U_s})^c$  is open in  $X$ . It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq (\overline{U_s})^c \subseteq \bigcup_{s > \alpha} (\overline{U_s})^c$$

So  $f^{-1}((\alpha, +\infty))$  is a subset of  $\bigcup_{s > \alpha} (\overline{U_s})^c$ . To show the reverse, fix an element  $x$  in the union, then this induces some  $x \in (\overline{U_s})^c \subseteq (U_s)^c$ . Then for this  $s > \alpha$ ,  $(-\infty, s)$  contains no elements of  $W$ . This is because for every  $p < s$  implies that  $(U_s)^c \subseteq (U_p)^c$ , so  $p \notin W$ . Our chosen  $s$  is a lower bound for  $W$ , and  $\alpha < s \leq \inf W = f(x)$ .

Since all of the inverse images from the generating set of  $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$  are open in  $X$ , using Theorem 4.9 finishes the proof.  $\square$

## 1.16 Theorem 4.16

**WTS.** *The Tietze's Extension Theorem. Let  $X$  be a normal space, and for any closed subset  $A \subseteq X$ , and  $f \in C(A, [a, b])$ , there exists an  $F \in C(X, [a, b])$  which extends  $f$ .*

*Proof.* We begin with an important lemma that will serve as a 'black box' for the induction.

**Lemma 1.1.** *For every  $f \in C(A, [0, 1])$ , there exists a  $g \in C(X, [0, 1/3])$  such that*

$$0 \leq f - g \leq 2/3 \quad \text{pointwise on } A \quad (2)$$

*Proof.* Since  $f$  is continuous,  $B = f^{-1}([0, 1/3])$ , and  $C = f^{-1}([2/3, 1])$  are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function  $g \in C(X, [0, 1])$  such that  $g|_B = 0$  and  $g|_C = 1$ . Relabel  $g = g/3$  then  $g \in C(X, [0, 1/3])$  (multiplication is continuous).

To show that (2) holds, suppose  $x \in B$ , then  $f(x) \in [0, 1/3]$  and  $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$ . Now suppose that  $x \in C$ , then  $f(x) \in [2/3, 1]$  and  $g(x) = 1/3$  (recall that we relabelled  $g$ ). So we have  $0 \leq 1/3 \leq f - g \leq 2/3$ . Lastly, for the case where  $x \notin (B \cup C)$ , then  $f(x) \in (1/3, 2/3)$ , and  $g(x) \in [0, 1/3]$  implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 & \implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 & \implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore  $0 \leq f(x) - g(x) \leq 2/3$ . □

We can assume that  $f \in C(A, [0, 1])$ , since we can relabel  $f = (f - a)/(b - a)$ . The main part of this proof consists of constructing a sequence of  $\{g_n\} \subseteq C(X, \mathbb{R})$  where  $0 \leq g_n \leq (2/3)^n(1/2)$ , and  $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$  on  $A$ . Let us begin with the base case with  $n = 1$ . We can apply Lemma 1.1 to get  $g_1 \in C(X, [0, 1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that  $\{g_j\}_{j \leq n}$  has been chosen, we will find our  $g_{n+1}$  by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$



Here is where my proof deviates from that of Folland's, we multiply both sides by  $(2/3)^{-n}$  and we obtain a new function in  $C(A, [0, 1])$ .

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) \left( \frac{3}{2} \right)^n \leq 1$$

Applying the Lemma 1.1, we get a function  $h \in C(X, [0, 1/3])$  such that, for every  $x \in A$

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) \left( \frac{3}{2} \right)^n - h \leq 2/3$$

Multiplying across gives

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) - h \left( \frac{2}{3} \right)^n \leq \left( \frac{2}{3} \right)^{n+1}$$

Set  $g_{n+1} = h \left( \frac{2}{3} \right)^n$  and  $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$ . Furthermore, the sum of all  $g_j$  pointwise converges uniformly, as

$$\sum_{j \geq 1} \|g_j\|_u \leq \sum_{j \geq 1} \left( \frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum  $F = \sum g_j$ , then this  $F \in BC(X)$  (by Theorem 4.9), since every  $g_j \in BC(X)$ . And

$$\left\| f - \sum_{j \leq n} g_j \right\|_u \leq \left( \frac{2}{3} \right)^n \longrightarrow 0$$

So  $F = f$  on  $A$ , now if we want to obtain our  $F$  on  $[a, b]$  we simply relabel  $F = F(b - a) + a$ . This finishes the proof.  $\square$

### 1.17 Theorem 4.17

**WTS.** *If  $X$  is a normal space, and  $A$  is a closed subspace of  $X$ , and  $f \in C(A)$ , then there exists an  $F \in C(X)$  such that  $F$  extends  $f$ .*

*Proof.* First we suppose that  $f$  is real valued, so  $f \in C(X, \mathbb{R})$ . And define a  $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$ , using

$$g = \frac{f}{1 + |f|}$$

Since  $g$  satisfies the assumption of Theorem 4.16 (note that we do not require  $g$  to be injective), there exists a  $G \in C(X, [-1, +1])$  such that  $G|_A = g$ . Since the set  $\{-1, +1\}$  is closed in  $\mathbb{R}$ ,  $G^{-1}(\{-1, +1\})$  is closed as well. Since  $G^{-1}((-1, +1)) \subseteq A$ , this makes  $A$  and  $B =^{-1}(\{-1, +1\})$  disjoint closed sets in  $X$ .

By Urysohn's Lemma, there exists a continuous function  $h \in C(X, [0, 1])$  such that  $h|_B = 0$  and  $h|_A = 1$ , so that the product  $|hG| < 1$  for all  $x \in X$ . We can think of this  $h$  as a continuous indicator function that filters out the parts we do not want, namely  $G^{-1}\{-1, +1\}$ . Now define  $F$  in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that  $F|_A = g/(1 - |g|) = f$  indeed. Since  $|g| = \frac{|f|}{1+|f|}$ , and  $g(1 + |f|) = f$  implies that  $g/(1 - |g|) = f$ , because  $g \in C(A, (-1, +1))$ . This completes the proof for any  $f \in \mathbb{R}$  if  $f \in C(A)$ , then

1.  $\text{Re}(f) = f_1 \in C(A, \mathbb{R})$
2.  $\text{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in  $C(X, \mathbb{R})$  that extends  $f_1$  and  $f_2$ , and  $F_1 + iF_2 = f$  on  $A$  and  $F_1 + iF_2 \in C(X)$ , and the proof is complete.  $\square$

### 1.18 Theorem 4.18

**WTS.** *If  $X$  is a topological space, and  $E \subseteq X$  and  $x \in X$ , then  $x \in \text{acc } E \iff$  there exists a net in  $E \setminus \{x\}$  that converges to  $x$ , and  $x \in \overline{E} \iff$  there exists a net in  $E$  that converges to  $x$ .*

*Proof.* Suppose that  $x \in \text{acc } E$ , then for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $E \cap U \setminus \{x\} \neq \emptyset$ , then choose  $\mathcal{N}(x)$  as the set of neighbourhoods directed by reverse inclusion (and this makes  $(\mathcal{N}(x), \lesssim)$  a directed set), and we will define the net as follows.

Map each  $U \in \mathcal{N}(x)$  to some  $x_U \in E \cap U \setminus \{x\}$ , then this net converges to  $x$ . Suppose that we fix a neighbourhood,  $V \in \mathcal{N}(x)$ , then for every  $U \gtrsim V$  we have  $x_U \in U \subseteq V$ . So  $\langle x_U \rangle$  is eventually in  $V$ .

Conversely, if  $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$ , and  $x_\alpha \rightarrow x$ , then every  $U \in \mathcal{N}(x)$  there exists a  $x_\alpha \in E \cap U \setminus \{x\}$  that makes

$$E \cap U \neq \emptyset \quad \forall U \in \mathcal{N}(x)$$

Hence  $x \in \text{acc } E$ .

Now for the second part of the Theorem, suppose that  $x \in \overline{E}$ , if  $x \notin E$  then  $E = E \setminus \{x\}$  and  $x \in \text{acc } E$ , so there exists a net in  $E \setminus \{x\} \subseteq E$  such that  $x_\alpha \rightarrow x$ . If  $x \in E$  then simply choose  $\langle x_\alpha \rangle = x$  for every  $\alpha \in A$ .

Now, suppose that there is a net that converges to  $x$ , and this net  $\langle x_\alpha \rangle \subseteq E$ , if  $x \in E$  then there is nothing to prove, since  $E \subseteq \overline{E}$ , so suppose that  $x \notin E$ , then there exists a net in  $E \setminus \{x\} = E$  such that

$$x_\alpha \rightarrow x \implies x \in \text{acc } E \subseteq \overline{E}$$

□

### 1.19 Theorem 4.19

**WTS.** Let  $X$  and  $Y$  be topological spaces, then every  $f : X \rightarrow Y$  is continuous at a point  $x \in X \iff$  every net  $\langle x_\alpha \rangle$  that converges to  $x$  implies that  $\langle f(x_\alpha) \rangle$  converges to  $f(x)$ .

*Proof.* If  $f$  is continuous at a point  $x \in X$ , then  $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$ , then for every net  $\langle x_\alpha \rangle$  that converges to this  $x$ , there there exists an  $\alpha_0$  such that for every  $\alpha \gtrsim \alpha_0$  implies that  $x_\alpha \in f^{-1}(V)$ . Hence

$$f(x_\alpha) \in f(f^{-1}(V)) \subseteq V$$

And this is equivalent to saying that for every  $V \in \mathcal{N}(f(x))$ ,  $\langle f(x_\alpha) \rangle$  is eventually in  $V$ , and this proves convergence.

Now suppose that  $f$  is not continuous at some  $x$ , then there exists a  $V \in \mathcal{N}(f(x))$  such that  $f^{-1}(V) \notin \mathcal{N}(x)$ , so

$$x \notin (f^{-1}(V))^o \implies x \in (f^{-1}(V))^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our  $x \in \overline{f^{-1}(V^c)}$  induces a net  $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$  that converges to  $x$ . But every element in the net is contained within  $f^{-1}(V^c)$ , and for every  $\alpha \in A$

$$f(x_\alpha) \in f(f^{-1}(V^c)) \subseteq V^c$$

gives  $f(x_\alpha) \notin V$ , but  $V$  is a neighbourhood of  $f(x)$ , hence there exists some  $x_\alpha \rightarrow x$  and  $f(x_\alpha) \not\rightarrow f(x)$ .  $\square$

## 1.20 Theorem 4.20

**WTS.** If  $\langle x_\alpha \rangle$  is a net in  $X$ , and  $x \in X$  is a cluster point of  $\langle x_\alpha \rangle \iff$  there exists a subnet of  $\langle x_\alpha \rangle$  that converges to  $x$ .

*Proof.* Suppose that  $\langle y_\beta \rangle_{\beta \in B}$  is a subnet of  $\langle x_\alpha \rangle$  that converges to  $x$ , then for every neighbourhood  $U \in \mathcal{N}(x)$ , there exists a  $\beta_1$  such that for every  $\beta \succeq \beta_1$  we get  $y_\beta = x_{\alpha_\beta} \in U$ .

Furthermore, let us fix a  $\alpha_0 \in A$  to attempt to show that  $\langle x_\alpha \rangle$  is frequently in  $U$ , then by the subnet property of  $\langle y_\beta \rangle$ , there exists some  $\beta_2 \in B$  such that for every  $\beta \succeq \beta_2$ ,  $\alpha_\beta \succeq \alpha_0$ . (Intuitively this property means that the directed set of  $B$  'grows' as much as the directed set of  $A$ , so we can always find elements that are greater than any fixed  $\alpha_0$ .)

Since  $\langle y_\beta \rangle$  is a net, we there exists some  $\beta \in B$  such that  $\beta \succeq \beta_1$  and  $\beta \succeq \beta_2$ , we then apply the  $\beta \mapsto \alpha_\beta$  map and we obtain some  $\alpha = \alpha_\beta$  that satisfies:

- $\alpha = \alpha_\beta \succeq \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that  $\beta \succeq \beta_1$  so that  $y_\beta$  falls into  $U$ .

Conversely, suppose that  $x$  is a cluster point of  $\langle x_\alpha \rangle$ , then by definition

$$\forall U \in \mathcal{N}(x), \forall \alpha_0 \in A, \exists \alpha \succeq \alpha_0, x_\alpha \in U$$

Denote the directed neighbourhoods of  $x$  by  $\mathcal{N}(x)$ , and construct our directed set  $B$  for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every  $(U, \gamma) \in B$  we can map it to some  $\alpha_{(U, \gamma)} \in A$ , if we choose some  $\alpha_{(U, \gamma)} \succeq \gamma$  and  $\alpha_{(U, \gamma)} \in U$ .

To show that  $B$  is a directed set, we say that  $(U, \gamma) \succeq (U', \gamma')$  if and only if  $U \subseteq U'$  and  $\gamma \succeq \gamma'$ . And to show that  $\langle y_\beta \rangle = \langle x_{\alpha_{(U, \gamma)}} \rangle$  is indeed a subnet of  $\langle x_\alpha \rangle$ , fix any  $\alpha_0 \in A$ , then simply take any neighbourhood  $U$  of  $x$  (we always

have  $X \in \mathcal{N}(x)$  — and therefore  $(U, \alpha_0) \in B$ .

Now for every  $(U', \alpha'_0) \gtrsim (U, \alpha_0)$  implies that  $\alpha'_0 \gtrsim \alpha_0$ , therefore we have

$$\alpha_{(U', \alpha'_0)} \gtrsim \alpha'_0 \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that  $\langle y_\beta \rangle$  indeed converges to  $x$ , fix any  $V \in \mathcal{N}(x)$ , then with any  $\alpha_0 \in A$ , and for every  $(V', \alpha'_0) \gtrsim (V, \alpha_0) \in B$ , we have

$$x_{\alpha_{(V', \alpha'_0)}} \in V' \subseteq V$$

So  $\langle x_{\alpha_{(U, \gamma)}} \rangle$  converges to  $x$ . □

### 1.21 Theorem 4.21

**WTS.** *A topological space  $X$  is compact  $\iff$  every family of closed sets,  $\{F_\alpha\}_{\alpha \in A}$  that has the finite intersection property, implies that*

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

*Proof.* We first examine the assertion, Theorem 4.21 proposes for any family of closed sets  $\{F_\alpha\}_{\alpha \in A}$ , and for every finite subset  $B \subseteq A$  then,

$$\bigcap_{\alpha \in B} F_\alpha \neq \emptyset \implies \bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha \in A} F_\alpha = \emptyset \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha \in B} F_\alpha = \emptyset$$

Applying DeMorgan's theorem, and since every  $\{F_\alpha\}_{\alpha \in A}$  induces a family of open sets (and vice versa), where  $U_\alpha = F_\alpha^c$ , so for any family of open sets  $\{U_\alpha\}_{\alpha \in A}$  we have

$$\bigcup_{\alpha \in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha \in B} U_\alpha = X$$

Which is equivalent to saying that  $X$  is compact. □

### 1.22 Theorem 4.22

**WTS.** *A closed subset of a compact space  $X$  is compact.*

*Proof.* Suppose  $F \subseteq X$  and  $F$  is open, then fix an open cover for  $F$ , so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since  $F^c$  is an open set, we can obtain a valid open cover for  $X$ , then we pick out a finite subcover, for some finite  $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left( \bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with  $F$  on both sides yields

$$\begin{aligned} F &= X \cap F \subseteq (F^c \cap F) \cup \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \\ F &= \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\ F &\subseteq \bigcup_{\alpha \in B} U_\alpha \end{aligned}$$

Therefore every open cover of  $F$  has a finite subcover, and  $F$  is compact.  $\square$



### 1.23 Theorem 4.23

**WTS.** *If  $F$  is a compact subset of a Hausdorff space  $X$ , and  $x \notin F$ , there are disjoint open sets  $U, V$  such that  $x \in U$  and  $F \subseteq V$ .*

*Proof.* Since  $x \in F^c$ , for every  $y \in F$ ,  $x \neq y$  induces two sets  $U_y, V_y$  (because  $X$  is  $T_2$ ).

- $U_y \cap V_y = \emptyset$
- $x \in U_y$
- $y \in V_y$

But  $\{V_y\}_{y \in F}$  is an open cover for the compact set  $F$ , then there exists a finite subcollection  $H \subseteq F$  such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since  $H$  is finite,  $U = \bigcap_{y \in H} U_y$  is an open set that contains  $x$ , also define  $V = \bigcup_{y \in H} V_y$ . If for every  $y \in H$ ,  $U_y \cap V_y = \emptyset$ , then  $U \cap V_y = U \cap V = \emptyset$ . This completes the proof.  $\square$

**Remark.** *Every metric space  $(X, d)$  is first countable, and  $T_2$  (it is actually  $T_4$ , but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element  $x \in X$  and we notice that  $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$  is a countable neighbourhood base for every  $x$ . To show that  $(X, d)$  is  $T_2$ , for every pair of elements  $x \neq y$ , we can take  $r = d(x, y)/2$  and there exists disjoint open sets  $V_r(x)$  and  $V_r(y)$  such that  $x \in V_r(x)$  and  $y \in V_r(y)$ .*

## 1.24 Theorem 4.24

**WTS.** *Every compact subset of a Hausdorff ( $T_2$ ) space is closed.*

*Proof.* If  $F$  is compact, then for every  $x \in F^c$ , by Theorem 4.23, there exists two disjoint open sets such that  $x \in U$  and  $F \subseteq V$ , but

$$U \cap V = \emptyset \implies U \cap F = \emptyset \implies U \subseteq F^c$$

But since  $x \in F^c$  is arbitrary, and  $U$  is an open subset of  $F^c$ ,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that  $F^c$  is open and  $F$  is closed. □

### 1.25 Theorem 4.25

**WTS.** *Every compact Hausdorff ( $T_2$ ) space is normal ( $T_4$ ).*

*Proof.* Fix  $A, B$  which are disjoint closed subsets of  $X$ , by Theorem 4.22, we know that these two sets are compact. Hence for every  $y \in B$  there exists two disjoint open sets  $U, V_y$  (by Theorem 4.23)

$A \subseteq U_y$  and  $y \in V_y$ . But the family  $\{V_y\}_{y \in B}$  is a valid open cover for the compact set  $B$ , hence there exists a finite subcollection  $H \subseteq B$  such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \emptyset$$

The second equality holds for every  $y \in H$  so that  $U_y \cap (\cup V_{y \in H}) = \emptyset$ . Define  $U = \cap U_{y \in H}$  and  $V = \cup V_{y \in H}$ , where both of these are disjoint open sets that contain  $A$  and  $B$  as subsets, since for each  $y \in H$ ,  $A \subseteq U_y$  hence the intersection of all  $U_y$  also contains  $A$  as a subset. Therefore  $X$  is normal.  $\square$

## 1.26 Theorem 4.26

**WTS.** *If  $X$  is compact, and  $f : X \rightarrow Y$  is continuous, then  $f(X)$  is compact.*

A small lemma.

**Lemma 1.2.** *For every  $\{E_j\} \subseteq X$ ,  $f(\cup E_j) = \cup f(E_j)$ .*

The proof is trivial.

*Proof.* If  $\{V_{\alpha \in A}\}$  is an open cover for  $f(X)$ , then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha \in A} V_{\alpha}\right) = \bigcup_{\alpha \in A} f^{-1}(V_{\alpha}) \subseteq X$$

Since  $f$  is continuous, we have an open cover in the form of  $\{f^{-1}(V_{\alpha})\}$  for  $X$ , then there exists a finite subset  $B \subset A$  such that

$$X \subseteq \bigcup_{\alpha \in B} f^{-1}(V_{\alpha})$$

Then we wish to show that for this  $B \subseteq A$ ,  $\{V_{\alpha \in B}\}$  is a finite open cover for  $f(X)$ . Fix any element  $y \in f(X)$ , then this induces a  $x \in X$  such that  $y = f(x)$ , but because  $\{f^{-1}(V_{\alpha \in B})\}$  is an open cover for  $X$ , there exists some  $\alpha \in B$  such that  $x \in f^{-1}(V_{\alpha})$ , hence by definition of the inverse image

$$f(x) \in V_{\alpha} \implies f(X) \subseteq \bigcup_{\alpha \in B} V_{\alpha}$$

Therefore  $f(X)$  is compact and this completes the proof.  $\square$

### 1.27 Theorem 4.27

**WTS.** *If  $X$  is compact, then  $C(X) = BC(X)$ .*

*Proof.* Notice that  $BC(X) \subseteq C(X)$ , so we only have to show the reverse estimate. Fix any  $f \in C(X)$ , since  $X$  is compact, by Theorem 4.26 we know that  $f(X)$  is also compact. Since  $\mathbb{C} = \mathbb{R}^2$  is a complete metric space,  $f(X)$  is bounded and  $f \in BC(X)$ .  $\square$

## 1.28 Theorem 4.28

**WTS.** *If  $X$  is compact, and if  $Y$  is Hausdorff, then any continuous bijection  $f : X \rightarrow Y$  is a homeomorphism.*

*Proof.* If  $E \subset X$  is closed, then since  $X$  is compact,  $E$  is compact as well. By continuity of  $f$ ,  $f(E)$  is a compact set in  $Y$ , but compact subsets of  $Y$  are closed, so  $f$  is continuous.

We used the fact that the inverse of  $f^{-1}$  is  $f$ , since it suffices to check that every inverse image of a closed set is also closed,  $f^{-1}$  is continuous. And by definition of a homeomorphism ( $f$  has to be bijective and both  $f$  and  $f^{-1}$  have to be continuous),  $f$  is a homeomorphism.  $\square$

## 1.29 Theorem 4.29

**WTS.** *If  $X$  is any topological space, the following are equivalent.*

- (a)  *$X$  is compact.*
- (b) *Every net has a cluster point.*
- (c) *Every net in  $X$  has a convergent subnet.*

*Proof.* By Theorem 4.20, every net in  $X$  has a cluster point  $\iff$  there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose a) holds, then  $X$  is compact, and fix an arbitrary net  $\langle x_\alpha \rangle$  in  $X$ . and define the 'tail' of the net

$$E_\alpha := \{x_\beta, \beta \succeq \alpha\}$$

We wish to show that the arbitrary intersection of  $\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \emptyset$ . Where  $\overline{E}_\alpha$  is closed, so it suffices to check that every finite  $B \subseteq A$ , the intersection over  $\overline{E}_\alpha$  is non-empty.

Suppose we are given a finite  $B \subseteq A$ , then fix any two elements  $\alpha$  and  $\beta \in B$ , by the definition of a net there exists a  $\gamma \in A$  such that  $\gamma \succeq \alpha$  and  $\gamma \succeq \beta$ , and

$$\emptyset \neq E_\alpha \cap E_\beta \implies \overline{E}_\alpha \cap \overline{E}_\beta \neq \emptyset$$

Therefore for any finite collection of  $\{\overline{E}_{\alpha \in B}\}$ , then

$$\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \emptyset$$

Now fix an element  $x \in \bigcap_{\alpha \in A} \overline{E}_\alpha$ . Then for every  $\alpha \in A$ ,  $x \in \overline{E}_\alpha$ , and for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $U \cap E_\alpha \neq \emptyset$ . This is because if  $x \in E_\alpha$ , then  $U \cap E_\alpha$  contains at least  $\{x\}$ , if  $x \in \text{acc } E_\alpha$ , then by definition of an accumulation point,  $U \cap E_\alpha \setminus \{x\} \neq \emptyset$ , so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net,  $E_\alpha$ , if for every  $\alpha \in A$ ,  $x \in E_\alpha$  if and only if there exists some  $\gamma \succeq \alpha$ ,  $x_\gamma \in U \cap E_\alpha$ ,

this is equivalent to saying that  $x$  is a cluster point of  $\langle x_\alpha \rangle$ . So  $a) \implies b)$ .

Now let us suppose that  $X$  is not compact, then there exists an open cover  $\{U_{\alpha \in A}\}$  of  $X$  that has no finite subcover. Let  $\mathbb{B}$  be the collection of all finite subsets of  $A$ , directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every  $B \in \mathbb{B}$ , find some  $x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c$ . So we have a net in  $X$ . Now we will show that no  $x \in X$  can be a cluster point of this net. Suppose not, then take a neighbourhood  $U_\beta$  with  $\beta \in A$  such that  $U_\beta$  belongs to the open cover we first discussed. Then for any  $B \in \mathbb{B}$  such that  $B \gtrsim \{\beta\}$  (meaning that  $\{\beta\} \subseteq B$ , where  $B$  is a finite set), then

$$x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c \implies x_B \notin \left(\bigcup_{\alpha \in \{\beta\}} U_\alpha\right) \implies x_B \in U_\beta^c$$

Hence no point in  $X$  can be a cluster point for this net, and the proof is complete.  $\square$



### 1.30 Theorem 4.30

**WTS.** *If  $X$  is a LCH space, and for every  $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$ , there exists a compact  $N \subseteq U$  where  $N \in \mathcal{N}_B(x)$ .*

*Proof.* For every  $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$ , we can find an  $E$  open subset of  $U$  that has a compact closure, since every  $x \in X$  induces some compact  $F \in \mathcal{N}_B(x)$ , therefore

$$E := U \cap F^\circ \implies \overline{E} \subseteq F$$

Since closed subsets of compact sets are compact (by Theorem 4.22),  $\overline{E}$  is compact. More is true, since  $E$  is open,

$$x \in U \cap F^\circ \implies x \in E^\circ \implies E \in \mathcal{N}_B(x)$$

Now it suffices to show that there exists some compact  $N \subseteq E \subseteq U$  such that  $N \in \mathcal{N}_B(x)$ . Since  $\overline{E}$  is compact, the closed subset  $\partial E = \overline{E} \cap \overline{E}^c$  of  $\overline{E}$  is also compact.

Since  $\partial E \cap E^\circ = \emptyset$ ,  $x \in E^\circ = E$  means that  $x \notin \partial E$ . Applying Theorem 4.23 to the compact set  $\partial E$  and  $x \notin \partial E$  gives us two disjoint open sets  $V'$  and  $W'$ . We list their properties

1.  $V', W' \in \mathcal{T}_X$
2.  $x \in V'$
3.  $\partial E \subseteq W'$
4.  $V' \cap W' = \emptyset$

The two disjoint pairs induce another pair of open sets relative to  $\overline{E}$ , recall the definition of the topology relative to  $\overline{E}$ ,

$$\mathcal{T}_{\overline{E}} = \{A \cap \overline{E} : A \in \mathcal{T}_X\}$$

We now agree to define

- $V = V' \cap \overline{E}$
- $W = W' \cap \overline{E}$

Then evidently  $V, W \in \mathcal{T}_{\overline{E}}$  and

1.  $x \in V' \cap \overline{E} \implies x \in V$
2.  $\partial E \subseteq \overline{E} \implies \partial E \subseteq W$
3.  $V' \cap W' = \emptyset \implies V \cap W = \emptyset$

Furthermore,

$$\partial E \subseteq W \implies W^c \subseteq (\partial E)^c = E^o \cup E^{co}$$

Taking the intersection over  $\overline{E}$  gives us

$$\overline{E} \setminus W \subseteq \overline{E} \cap (E^o \cup E^{co})$$

Note that  $E^{co} = (\overline{E})^c$ , since  $(E^c)^{oc} = \overline{(E^{cc})} = \overline{E}$  therefore  $\overline{E} \cap E^{oc} = \emptyset$ , hence

$$\overline{E} \setminus W \subseteq \overline{E} \cap E^o = E^o$$

Using the fact from 3,  $V \subseteq W^c$  and  $V \subseteq \overline{E}$  and  $V \subseteq W^c$  implies that  $V \subseteq \overline{E} \setminus W$ . Compiling everything, we have

$$V \subseteq \overline{E} \setminus W \subseteq E$$

Note that the set  $\overline{E} \setminus W$  is closed in  $\mathcal{T}_X$  (and hence closed in  $\overline{E}$ ) by closure over intersections,  $\overline{V}$  is therefore a closed subset of  $\overline{E} \setminus W$ , and  $\overline{V}$  is compact. Also

$$\overline{V} \subseteq \overline{E} \setminus W \subseteq E$$

To check that  $\overline{V} \in \mathcal{N}_B(x)$ , note that

$$x \in V^o \subseteq (\overline{V})^o \implies \overline{V} \in \mathcal{N}_B(x)$$

The subset relation  $V^o \subseteq \overline{V}^o$  comes from the fact that  $V^o$  is an open subset of  $\overline{V}$ , and hence is contained in  $(\overline{V})^o$  as a subset. Now let us define  $N = \overline{V}$ , and  $N$  satisfies the assertions in the Theorem, since

- $N \in \mathcal{N}_B(x)$
- $N$  is compact
- $N \subseteq E \subseteq U$

And this completes the proof.  $\square$

**Remark.** *Intuitively speaking, this means that if  $X$  is any LCH space, then for every open neighbourhood  $U \in \mathcal{N}_B(x)$ , there exists a compact  $E \in \mathcal{N}_B(x)$  such that  $x \in E \subseteq U^o$ . This property is indeed a very strong one as it allows us to have effectively 'infinite' descending compact neighbourhoods of  $x$ .*

### 1.31 Theorem 4.31

**WTS.**  *$X$  is a LCH space, and  $K \subseteq U \subseteq X$  where  $K$  is compact, and  $U$  is open, then there exists some precompact, open  $V$  with*

$$K \subseteq V \subseteq \overline{V} \subseteq U$$

*Proof.* For every  $x \in K$ , we can apply Proposition 4.30, since  $x \in K \subseteq U$ , this induces some compact  $F_x \subseteq U$  where  $F_x \in \mathcal{N}_B(x)$ . Then we can obtain an open cover of  $U$  in the form of  $\{F_x^o\}_{x \in K}$ . By compactness of  $K$ , there exists a finite  $B \subseteq K$  such that

$$K \subseteq \bigcup_{x \in B} F_x^o$$

Let  $V = \bigcup_{x \in B} F_x^o$ , then clearly  $V$  is open, and  $K \subseteq V$ . Since each  $F_x$  is closed (compact sets are closed in any Hausdorff Space), we have

$$V \subseteq \bigcup_{x \in B} F_x \implies \overline{V} \subseteq \bigcup_{x \in B} F_x$$

Since  $\bigcup_{x \in B} F_x$  is a finite union of compact sets, we claim that it is also compact. Consider two compact sets  $E_1$  and  $E_2$ , then if  $\{U_\alpha\}_{\alpha \in A}$  is any open cover of  $E_1 \cup E_2$ , it must be an open cover for  $E_1$  and  $E_2$  as well, because

$$E_1, E_2 \subseteq E_1 \cup E_2 \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since  $E_1$  and  $E_2$  are both compact sets, they each induce two finite subsets of  $B_1, B_2$  of  $A$  whose union  $B = B_1 \cup B_2$  is also compact. Therefore

$$E_1 \cup E_2 \subseteq \bigcup_{\alpha \in B} U_\alpha$$

Then a simple proof by induction will show that if  $\{E_{j \leq n}\}$  is a family of compact sets, then  $E = \bigcup E_{j \leq n}$  is also compact.

Returning to the main part of the proof,  $\bigcup_{x \in B} F_x$  is a compact set, therefore  $\overline{V}$  is also compact. Moreover

$$\forall x \in K, F_x \subseteq U \implies \overline{V} \subseteq \bigcup_{x \in B} F_x \subseteq U$$

Combining, we have

- $K \subseteq V \subseteq \overline{V}$ ,
- $V$  is open and  $\overline{V}$  is compact, and
- $\overline{V} \subseteq U$

This completes the proof.

□

### 1.32 Theorem 4.32

**WTS.** *Urysohn's Lemma, Locally Compact Version.* For any LCH space  $X$ , and if  $K \subseteq U \subseteq X$  where  $K$  is compact and  $U$  is open, then there exists some  $f \in C(X, [0, 1])$  with

- $f = 1$  on  $K$
- $f = 0$  outside some compact  $\bar{V} \subseteq U$

*Proof.* Let  $V$  be as in Theorem 4.31, for our fixed  $K \subseteq U \subseteq X$ , there exists a pre-compact, open  $V$  that satisfies

$$K \subseteq V \subseteq \bar{V} \subseteq X$$

It follows that this  $(\bar{V}, \mathcal{T}_{\bar{V}})$  is a normal space by Theorem 4.25 (compact Hausdorff spaces are normal), and by Urysohn's Lemma (Theorem 4.15) on normal spaces, since we can easily find two disjoint closed subsets of  $\bar{V}$  in the form of

- $K \subseteq V^\circ = V \subseteq \bar{V}$  (compact sets in Hausdorff spaces are closed)
- $\partial V = \bar{V} \cap \bar{V}^c$  (closed sets in compact spaces are compact)
- $K \subseteq V^\circ$  implies that  $K \cap \partial V = K \cap (\bar{V} \setminus V^\circ) = \emptyset$

Then there exists a continuous  $f|_{\bar{V}} \in C(\bar{V}, [0, 1])$  that evaluates to

- $f|_{\bar{V}} = 1$  on closed  $K$
- $f|_{\bar{V}} = 0$  on closed  $\partial V$

Now let us extend  $f|_{\bar{V}}$  to  $f$  by defining

$$f|_{(\bar{V})^c} = 0$$

We will show that this extension of  $f$  is indeed continuous. Indeed, for every closed set  $E \subseteq [0, 1]$  that does not contain 0, we have:

$$0 \notin E \implies \{0\} \cap E = \emptyset \implies f^{-1}(\{0\}) \cap f^{-1}(E) = \emptyset$$

But  $(\bar{V})^c \subseteq f^{-1}(\{0\})$  therefore

$$(\bar{V})^c \cap f^{-1}(\{0\}) \cap f^{-1}(E) = (\bar{V})^c \cap f^{-1}(E) = \emptyset \implies f^{-1}(E) \subseteq \bar{V}$$

We can write

$$f^{-1}(E) = f|_{\overline{V}}^{-1}(E)$$

But we know that  $f|_{\overline{V}}$  is continuous, so  $f|_{\overline{V}}^{-1}(E)$  must be closed (with respect to  $\overline{V}$ ), and therefore is closed wrt  $X$ , since  $\overline{V}$  is closed wrt  $X$ .

For the case where  $0 \in E$ , note that

$$f^{-1}(E) = (f^{-1}(E) \cap \overline{V}) \cup (f^{-1}(E) \cap (\overline{V})^c) = (f|_{\overline{V}})^{-1}(E) \cup (f|_{\overline{V}^c})^{-1}(E)$$

The above equalities are messy in print. They are but a simple consequence of disjoint decomposition of the pre-images, since

$$\overline{V} \cap f^{-1}(E) = \{x \in \overline{V} : f(x) \in E\} = f|_{\overline{V}}^{-1}(E)$$

Back to our main discussion, recall that for every  $x \in \partial V$

$$f(x) = 0 \in f^{-1}(\{0\}) \subseteq f^{-1}|_{\overline{V}}(E)$$

Therefore  $\partial V \subseteq f^{-1}|_{\overline{V}}(E)$ , and  $(\overline{V})^c = f^{-1}|_{(\overline{V})^c}(E)$  gives us (since  $V^c$  is closed),

$$\begin{aligned} f^{-1}(E) &= f^{-1}|_{\overline{V}}(E) \cup \partial V \cup (\overline{V})^c \\ &= f^{-1}|_{\overline{V}}(E) \cup \overline{(V^c)} \cup (\overline{V})^c \\ &= f^{-1}|_{\overline{V}}(E) \cup (V^c \cup V^{co}) \\ &= f^{-1}|_{\overline{V}}(E) \cup V^c \end{aligned}$$

Since  $f^{-1}|_{\overline{V}}(E)$  and  $V^c$  are closed subsets of  $X$ , then  $f^{-1}(E)$  is also closed, and  $f \in C(X, [0, 1])$ .  $\square$

### 1.33 Theorem 4.33

**WTS.** *Every LCH space is completely regular (or  $T_{3.5}$ ).*

*Proof.* Recall that a space  $X$  is completely regular if it is  $T_1$  and every closed subset  $A$  and every  $x \notin A$  there exists some

$$f \in C(X, [0, 1]), \quad f(x) = 1, \quad f|_A = 0$$

Fix a closed set  $A \subseteq X$ , then for every  $x \in A^c$ , there exists a compact  $E_x \in \mathcal{N}_B(x)$  with  $E_x \subseteq A^c$  (by Theorem 4.30).

Note that  $E_x \subseteq A^c$  where  $E_x$  is compact and  $A^c$  is closed, then an application of Theorem 4.31 tell us that there exists an  $f \in C(X, [0, 1])$  such that for every  $x \in E_x$ ,  $f(x) = 1$  and for points  $y \notin A^c$  (which means that  $y \in A$ ),  $f(y) = 0$ . Therefore  $X$  is completely regular.  $\square$

**1.34 Theorem 4.34**

**WTS.**

*Proof.*





### 1.35 Theorem 4.35

**WTS.** *If  $X$  is a LCH space, we claim that*

$$\overline{C_c(X)} = C_0(X)$$

*Proof.* We begin by proving several things that are mentioned before this Theorem, namely

$$C_c(X) \subseteq C_0(X) \subseteq BC(X)$$

Fix an  $f \in C_c(X)$ , and for every  $\varepsilon > 0$ ,

$$x \in |f|^{-1}([\varepsilon, +\infty)) \implies |f(x)| \geq \varepsilon > 0$$

Therefore  $|f|^{-1}([\varepsilon, +\infty))$  is a closed subset of  $\text{supp}(f)$ , since  $(-\infty, \varepsilon)$  is open in  $\mathbb{R}$ , then  $[\varepsilon, +\infty)$  is a closed set. And by continuity of  $|\cdot| \circ f$  (a composition of two continuous functions),  $|f|^{-1}([\varepsilon, +\infty))$  is closed. Using the fact that closed subsets of compact  $\text{supp}(f)$  are also compact, we get  $f \in C_0(X)$ .

Next, we show that  $C_0(X) \subseteq BC(X)$ . Fix any element  $f$  of  $C_0(X)$  with an arbitrary  $\varepsilon > 0$ , then  $E_\varepsilon = \{x \in X : |f(x)| \geq \varepsilon\}$  is compact. The continuity of  $f$  guarantees that the direct image of a compact set is another compact set (Theorem 4.26)

$$|f|(E_\varepsilon) \text{ is a compact subset of } \mathbb{R}$$

And therefore for every  $x \in E_\varepsilon \implies |f(x)| \in |f|(E_\varepsilon)$ , then by Heine-Borel, there exists some  $M \geq 0$  such that  $|f(x)| \leq M$ . If  $x \notin E_\varepsilon$ , then by definition of  $E_\varepsilon$ , implies that  $|f(x)| < \varepsilon$ . Then  $|f(x)| \leq M + \varepsilon$  for every  $x \in X$ . Hence  $f \in BC(X)$ .

Here I wish to offer an alternate proof for  $C_0(X) \subseteq BC(X)$ , we begin by constructing an open cover for  $\text{supp}(f)$  such that

$$\{U_n\}_{n>0} = \{x \in X : |f(x)| < n\}$$

Then there exists a finite subcollection of  $\{U_n\}_{n \in B}$  where  $B$  is a finite set, then define  $M = 1 + \sum_{n \in B} n$  and for every  $x \in \text{supp}(f)$  we have  $|f(x)| < n$  and since  $n > 0$  this holds for every  $x \in X$  too. Therefore  $f \in BC(X)$ .

For the main proof of Theorem 4.35, since  $\text{BC}(X)$  is endowed with the uniform metric, it is also first countable, and therefore by Theorem 4.6, it suffices to show that every sequence  $\{f_n\}_{n \geq 1} \subseteq C_c(X)$  converges in  $C_0(X)$ . And every element  $f \in C_0(X)$  has a convergence sequence in  $C_c(X)$ .

Fix a convergent sequence  $\{f_n\}_{n \geq 1} \subseteq C_c(X)$  that converges uniformly to some  $f \in \text{BC}(X)$  (since  $\text{BC}(X)$  is a closed subset of  $C(X)$  with respect to the uniform norm), then for every  $\varepsilon > 0$ , there exists some  $n \geq 1$  with

$$\|f_n - f\|_u < \varepsilon$$

We aim to show that  $(\text{supp}(f_n))^c \subseteq |f|^{-1}((-\infty, \varepsilon))$ , so fix any  $x \notin \text{supp}(f_n)$ , then

$$|f(x) - f_n(x)| = |f(x)| \leq \|f - f_n\|_u < \varepsilon$$

This establishes the estimate, and taking complements

$$|f|^{-1}([\varepsilon, +\infty)) \subseteq \text{supp}(f_n)$$

Therefore for any arbitrary  $\varepsilon > 0$ ,  $\{x \in X, |f(x)| \geq \varepsilon\}$  is compact, and  $\overline{C_c(X)} \subseteq C_0(X)$ . Conversely, fix any  $f \in C_0(X)$ , and for every  $n \geq 1$ , define

$$K_n = \{x \in X, |f(x)| \geq 1/n\}$$

Using Urysohn's Lemma for our LCH space  $X$ , there exists some  $g_n$  that has a compact support, and  $g_n(x) = 1$  for every  $x \in K_n$ . We then write  $f_n = g_n \cdot f \in C_c(X)$ . We wish to show that  $f_n \rightarrow f$  uniformly. Notice that for any fixed  $n \geq 1$ , if  $x \in K_n$  then

$$f_n(x) = f(x) \implies |f_n - f|(x) = 0$$

If  $x \notin K_n$ ,  $|f(x)| < 1/n$  (recall what  $K_n$  does), and  $f_n = g_n \cdot f \in [0, 1]$  by definition of  $g_n$  from Theorem 4.32, hence

$$|f_n(x) - f(x)| = |f(x)| \cdot |1 - g_n| \leq |f(x)| < 1/n$$

Taking the supremum over  $x \in X$ , we have

$$\|f_n - f\|_u < 1/n \rightarrow 0$$

As we send  $n$  to  $+\infty$ , and  $f_n \rightarrow f$  uniformly. This completes the proof.  $\square$

**1.36 Theorem 4.36**

**WTS.**

*Proof.*



### 1.37 Theorem 4.37

**WTS.** *If  $X$  is an LCH space and  $E \subseteq X$ .  $E$  is closed if and only if  $E \cap K$  is closed for every compact  $K \subseteq X$ .*

*Proof.* Suppose that  $E$  is closed, then  $E \cap K$  is closed, since compact subsets of Hausdorff spaces are closed, and  $E \cap K \subseteq K$  tells us that  $E \cap K$  is indeed compact.

Now suppose that  $E$  is not closed, by Theorem 4.1,  $E \neq \overline{E}$ , so pick some  $x \in (\overline{E} \setminus E) = \text{acc}(E) \cap E^c$ , since  $X$  is locally compact, let  $K_x$  be a compact neighbourhood of  $x$ , then for every neighbourhood  $U \in \mathcal{N}_B(x)$ , we have

$$x \in U^o, x \in K_x^o, \implies x \in (U^o \cap K_x^o) \subseteq (U \cap K_x)^o$$

Since  $(U^o \cap K_x^o)$  is an open subset of  $(U \cap K_x)$ , then  $(U \cap K_x) \in \mathcal{N}_B(x)$ , and recall that  $x \in \text{acc}(E)$ , therefore

$$(U \cap K_x) \cap E \setminus \{x\} = U \cap (K_x \cap E) \neq \emptyset$$

But  $x \notin E \implies x \notin E \cap K_x$ . So  $x$  is an accumulation point of  $E \cap K_x$  that is not in  $E \cap K_x$ . Therefore there exists some  $E \cap K_x$  (with  $K_x$  compact) that is not closed.  $\square$

**1.38 Theorem 4.38**

**WTS.**

*Proof.*

□

**1.39 Theorem 4.39**

**WTS.**

*Proof.*



**1.40 Theorem 4.40**

**WTS.**

*Proof.*



**1.41 Theorem 4.41**

**WTS.**

*Proof.*





## 2 Chapter 6

### 2.1 Theorem 6.1

**WTS.** *For every  $a, b \geq 0$ , and  $0 < \lambda < 1$ , then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b$$

## 2.2 Theorem 6.2

WTS.

*Proof.*



## 2.3 Theorem 6.3

WTS.

*Proof.*



## 2.4 Theorem 6.4

**WTS.**

*Proof.*

□

## 2.5 Theorem 6.5

**WTS.**

*Proof.*



## 2.6 Theorem 6.6

WTS.

## 2.7 Theorem 6.7

WTS.

## 2.8 Theorem 6.8

WTS.



## 2.9 Theorem 6.9

**WTS.**

*Proof.*

□

## 2.10 Theorem 6.10

WTS.

*Proof.*



## 2.11 Theorem 6.11

WTS.

*Proof.*



## 2.12 Theorem 6.12

WTS.

*Proof.*



### **2.13 Theorem 6.13**

**WTS.**

## 2.14 Theorem 6.14

WTS.

## 2.15 Theorem 6.15

WTS.

First suppose that  $(X, \mathcal{M}, \mu)$  is finite measure space. If  $\mu(X) < +\infty$ , then for every  $E \in \mathcal{M}$ , by monotonicity  $E \subseteq X$  yields  $\mu(E) \leq \mu(X) < +\infty$ . Next, for any  $p < +\infty$ ,  $\|\chi_E\|_p^p < +\infty$  and  $\|\chi_E\|_{+\infty} \leq 1 < +\infty$ . So all indicator functions are in  $L^p$ .

It follows that every simple function is also in  $L^p$ , since it is a finite linear combination of indicators. We now define  $\nu(E) = \phi(\chi_E)$ , we wish to show that  $\nu : \mathcal{M} \rightarrow \mathbb{C}$  is a complex measure which is absolutely continuous with respect to  $\mu$ .

To show  $\sigma$ -additivity, fix any disjoint sequence  $\{E_j\}_{j \geq 1} \subseteq \mathcal{M}$ . Where we also note that  $\mu(E) = \mu(\cup E_j) < +\infty$ . Now suppose that  $p < +\infty$ , then the following converges in the  $p$ -norm

$$\chi_E = \sum_{j \geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left( \bigcup_{j \leq n} E_j \right) = \left( \bigcup_{j \geq 1} E_j \right) \setminus \left( \bigcup_{j \leq n} E_j \right) = \bigcup_{j \geq n+1} E_j$$

and define  $F_{n+1}$  as the rightmost member above. Then  $\{F_{n \geq 1}\}$  is a decreasing sequence of sets. All sets are of finite measure, hence  $\mu(E) - \mu(\cup E_{j \leq n}) = \mu(F_{n+1}) \rightarrow 0$ .

Now, for any fixed  $n \geq 1$ ,

$$\left| \chi_E - \sum \chi_{E_{j \leq n}} \right| = \left| \sum \chi_{E_{j \geq n+1}} \right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the  $p$ th power does not change pointwise, and

$$\left| \sum \chi_{E_{j \geq n+1}} \right|^p = \left| \sum \chi_{E_{j \geq n+1}} \right| = \sum \chi_{E_{j \geq n+1}}$$

Convergence in  $p$ -norm is given by

$$\left\| \chi_E - \sum \chi_{E_{j \leq n}} \right\| = \left\| \sum \chi_{E_{j \geq n+1}} \right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our  $\phi \in L^{p*}$

$$\begin{aligned} \nu(E) &= \phi(\chi_E) \\ &= \phi\left(\lim_{n \rightarrow \infty} \sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \phi\left(\sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \sum \phi(\chi_{E_{j \leq n}}) \\ &= \lim_{n \rightarrow \infty} \sum \nu(E_{j \leq n}) \end{aligned}$$

To show absolute convergence, recall that for any  $\phi(\chi_{E_j}) \in \mathbb{C}$ , define  $\beta_j = \overline{\text{sgn}(\|\phi(\chi_{E_j})\|)}$  then multiplication yields

$$\|\phi(\chi_{E_j})\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the  $p$ -norm.

$$\left\| \sum_{j \geq 1} \beta_j \chi_{E_j} - \sum_{j \leq n} \beta_j \chi_{E_j} \right\|_p = \left\| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right\|_p$$

And because  $\left| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right|$  is pointwise equal to  $\left| \sum_{j \geq n+1} \chi_{E_j} \right|$ , since  $|\beta_j| = 1$  for every  $j \geq 1$ . We can reuse the same continuity and linearity argument. We also note that  $\sum_{j \geq 1} \beta_j \chi_{E_j} \in L^p$  since its  $p$ -norm is equal to  $\mu(E)^{1/p}$ .



$$\begin{aligned}
\sum_{j \geq 1} |\nu(E_j)| &= \sup_{n \geq 1} \sum_{j \leq n} \|\nu(E_{j \leq n})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \|\phi(\chi_{E_j})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \phi(\chi_{E_j}) \\
&= \lim_{n \rightarrow \infty} \phi \left( \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&= \phi \left( \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&\leq \|\phi\| \left\| \sum_{j \geq 1} \beta_j \chi_{E_j} \right\|_p \\
&< +\infty
\end{aligned}$$

Assuming the above estimate holds, then we only need  $\nu(E) = \phi(\chi_E) = \mu(E) = 0$  ( $\nu$  is now a measure and  $\nu \ll \mu$ ), As the indicator of a null set is equal to the zero element in  $L^p$ . Then by Radon-Nikodym we can have some  $g \in L^1(\mu)$  such that

$$d\nu = g d\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function  $g$ . For every  $\chi_E$  measurable,  $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$ , by monotonicity of the integral in  $L^+$ . So any simple function,  $\alpha = \sum a_j \cdot \chi_{E_j}$  means that  $\alpha g$  is in  $L^1(\mu)$ , and

$$\phi(\alpha) = \int \alpha g d\mu$$

If  $\|\alpha\|_p = 1$ , then

$$\left| \int \alpha g \right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$$M_q(g) = \sup \left\{ \left| \int \alpha \cdot g \right|, \|\alpha\|_p = 1, \text{ and } \alpha \text{ is simple and vanishes outside a set of finite measure.} \right\}$$

Since  $S_g = \{x \in X, g(x) \neq 0\}$  is  $\sigma$ -finite, an application of Theorem 6.14 tells us that  $g \in L^q$ , and  $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$ . Now that we know  $g$  is in  $L^q$  we can use the density of  $\alpha$  in  $L^p$  to show, for every single  $f \in L^p$

$$\phi(f) = \int f g d\mu$$

Conjure a sequence of ' $\alpha$ 's, and call them  $\{f_n\} \rightarrow f$  p.w.a.e, then each  $f_n \cdot g \in L^1$ . An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n g d\mu = \int f g d\mu = \phi(f)$$

This completes the proof for when  $\mu$  is finite.

Let us upgrade our  $\mu$  into a  $\sigma$ -finite measure. Then there exists an increasing sequence  $\{E_n\} \nearrow X$  such that each  $E_n$  is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in  $P_n$  vanishes outside a set of finite measure and is also in  $L^p$ . And  $Q_n$  is defined in a similar manner. Now, fix our  $\phi \in L^{p*}$ , and for each  $f \in P_n$ , there exists a corresponding  $g_n \in Q_n$ . Then  $p \in [1, +\infty)$  tells us that  $q \in (1, +\infty]$ , and the assumptions for Theorem 6.13 all hold. Therefore for each  $g_n \in Q_n$ , there is a corresponding bounded linear operator  $\phi_{g_n} \in (P_n)^*$  such that

$$\phi(f) = \phi|_{P_n}(f) = \int f g_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of  $g_n$  towards some  $g \in L^q$ . We claim that this limit makes sense. As for any  $n < m$ , such that  $E_n \subseteq E_m$  then  $g_n = g_m$  on  $E_n$  pointwise. The proof is simple since each the restriction of our  $\phi \in L^{p*}$  onto  $E_n$  and  $E_m$  spawns two functions  $g_n$  and  $g_m \in L^1$ . To verify, take any subset  $Z \subseteq E_n$  then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So  $g_n = g_m$  pointwise a.e on  $E_n$ . Now we define  $g$  measurable such that  $g|_{E_n} = g_n$  for every  $n$ . And

$$\begin{aligned} |g_n| &= \chi_{E_n} \cdot |g_m| \implies \\ |g_n| &\leq |g_{n+1}| \implies \\ \|g_n\|_q &\leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty \end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with  $(P_n \subseteq P_{n+1})$ . If  $q = +\infty$  then  $g \in L^\infty$  is trivial, but for any  $q < +\infty$ . We wish to show that  $g \in L^q$ . Since  $|g_n| \leq |g|$  pointwise for every  $n$ , and for each  $x \in X$ , there exists a  $N$ , where  $n \geq N$  implies  $|g(x)| = |g_n(x)|$ , so  $|g(x)|$  is an upperbound that is actually attained by the sequence  $|g_n(x)|$ . So,  $|g(x)| = \sup_{n \geq 1} \{|g_n(x)|\}$ .

Using the Monotone Convergence Theorem on  $|g_n|$ ,

$$\begin{aligned} \int \lim_{n \rightarrow \infty} |g_n|^q d\mu &= \int \sup_{n \geq 1} |g_n|^q d\mu \\ &= \int |g|^q d\mu \\ &= \lim \int |g_n|^q d\mu \end{aligned}$$

Which yields  $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$ . It follows that  $g \in L^q$ .

Finally, we will show that  $\phi(f) = \int fg$  for every  $f \in L^p$ . Redefine  $f_n = f \cdot \chi_{E_n} \in P_n$  for every  $n \geq 1$ . We claim that  $f_n \rightarrow f$  in the  $p$ -norm.

$$\begin{aligned} |f_n - f| &\leq |f_n| + |f| \\ &\leq |f| + |f| \\ &\leq 2|f| \end{aligned}$$

And  $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$ . Now it is permissible to apply the Dominated Theorem, and we will do so.

$$\begin{aligned}
\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\
\lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\
&= 0
\end{aligned}$$

And we have  $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}
\phi(f) &= \lim \phi|_{P_n}(f_n) \\
&= \lim \int f_n \cdot g_n \\
&= \lim \int f \cdot g \cdot \chi_{E_n} \\
&= \int \lim (f g \cdot \chi_{E_n}) \\
&= \int f g
\end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of  $f g \chi_{E_n} \rightarrow f g$  pointwise and Holder's Inequality. This completes the proof for when  $\mu$  is of  $\sigma$ -finite measure, and  $p \in [1, +\infty)$ .

Suppose now  $\mu$  is arbitrary, and  $p \in (1, +\infty)$ , then  $q < +\infty$ . Now let us agree to define, for every  $\sigma$ -finite  $E \subseteq X$

$$P_E = \{L^p, |f| = |f| \cdot \chi_E\}$$

Where  $Q_E$  does not hold any surprises. Then for each  $E$  we have a  $\phi|_E$  which induces a  $g_E$  that vanishes outside  $E$ . We are ready for the final part of the proof.

First, if  $E \subseteq F$  and both  $E$  and  $F$  are  $\sigma$ -finite, then  $\|g_E\|_q \leq \|g_F\|_q$ . This is a simple consequence of monotonicity in  $L^+$  if we take  $|g_E|^q \leq |g_F|^q$ .

Second, we define

$$W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}$$

Let  $M$  be the supremum of  $W$ , then there exists a sequence of  $\sigma$ -finite sets,  $\{E_n\}$  where  $\|g_{E_n}\|_q \rightarrow M \leq \|\phi\|_{p*}$ . Take a set  $F = \cup E_{n \geq 1}$ , which is also  $\sigma$ -finite, so that  $\|g_F\|_q = M$ . Now assume there exists another  $\sigma$ -finite superset of  $F$ , let us call it  $A$ . Then

$$\int |g_F|^q + \int |g_{A \setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have  $g_{A \setminus F} = 0$  pointwise a.e. For any  $f \in L^p$ , the spots where  $f$  does not vanish is  $\sigma$ -finite. This comes from  $\int |f|^p < +\infty$ . So it suffices to integrate over this  $\sigma$ -finite set. But we already know, even if this set  $A$  contains  $F$  as a subset,  $\int f g_F = \int f g_A$ .

We now define  $g = g_F$ , and the proof is complete. As for every  $\phi \in L^{p*}$ , there exists a  $g \in L^q$  such that the evaluation of any  $f \in L^p$  is given by integrating  $f$  with  $g$ . ■

## 2.16 Theorem 6.16

WTS.

*Proof.*



**2.17 Theorem 6.17**

**WTS.**

*Proof.*



## 2.18 Theorem 6.18

**WTS.** *For every pair of  $\sigma$ -finite measure spaces,*



**2.19 Theorem 6.19**

**WTS.**

**2.20 Theorem 6.20**

**WTS.**

*Proof.*



## 2.21 Theorem 6.21

WTS.

*Proof.*



## 2.22 Theorem 6.22

WTS.

**2.23 Theorem 6.23**

**WTS.**

**2.24 Theorem 6.24**

**WTS.**

*Proof.*



**2.25 Theorem 6.25**

**WTS.**

*Proof.*



**2.26 Theorem 6.26**

**WTS.**

*Proof.*





**2.27 Theorem 6.27**

**WTS.**