

# Chapter 3

## Notes on Chapter 3

### Proposition 0.1

Prove two things,

1.  $\limsup_{r \rightarrow R} \phi(r) = \lim_{\varepsilon \rightarrow 0} \sup_{0 < |r-R| < \varepsilon} \phi(r) = \inf_{\varepsilon > 0} \sup_{0 < |r-R| < \varepsilon} \phi(r),$
2.  $\lim_{r \rightarrow R} \phi(r) = c \iff \limsup_{r \rightarrow R} |\phi(r) - c| = 0$

*Proof.*



**Proposition 0.2**

If  $U \subseteq B(1, 0) = \{|x| < 1\}$ , and  $U \in \mathbb{B}$ , and if  $m(U) > 0$ , then the family of sets

$$E_r = \left\{ x + ry, y \in U \right\}$$

shrinks nicely to  $x \in \mathbb{R}^n$ .

*Proof.* Let  $r > 0$  be fixed then  $\forall z \in E_r \ni z = x + ry$ . Hence,

$$\begin{aligned} d(x, z) &= d(x, x + ry) \\ &= |r|d(0, y) < |r| \end{aligned}$$

by translation invariance. ■

**Theorem 3.1****Proposition 1.1**

*Proof.* Let  $\nu$  be a signed measure, and fix any increasing sequence  $E_j \nearrow E = \bigcup E_{j \geq 1}$  of sets. This induces a disjoint sequence in  $\{F_n\}$ . Define  $F_1 = E_1$ , and if  $n \geq 2$ ,

$$F_n = E_n \setminus \bigcup E_{j \leq n-1}$$

and from this, the finite It is clear that  $\bigcup F_{n \geq 1} = E$ , and let us assume  $\nu(E)$  is of finite measure.

By countable additivity, and the absolute convergence of the series  $\sum_{j \leq n} \nu(F_j)$

$$\begin{aligned} \nu\left(\bigcup E_{j \geq 1}\right) &= \sum_{j \geq 1} \nu(F_j) \\ &= \lim_n \sum_{j \leq n} \nu(F_j) \\ &= \lim \nu(E_n) \end{aligned}$$

■

**Theorem 3.2**

**Proposition 2.1**

*Proof.*



**Theorem 3.3****Proposition 3.1**

*Proof.*



**Theorem 3.4**

**Proposition 4.1**

*Proof.*



**Theorem 3.5****Proposition 5.1***Proof.*



**Theorem 3.6****Proposition 6.1***Proof.*

**Theorem 3.7**

**Proposition 7.1**

*Proof.*



**Theorem 3.8**

**Proposition 8.1**

*Proof.*



**Theorem 3.9****Proposition 9.1***Proof.*

**Theorem 3.10****Proposition 10.1***Proof.*

**Theorem 3.11**

**Proposition 11.1**

*Proof.*



**Theorem 3.12**

**Proposition 12.1**

*Proof.*



**Theorem 3.13****Proposition 13.1***Proof.*



**Theorem 3.14****Proposition 14.1***Proof.*

**Theorem 3.15****Proposition 15.1***Proof.*

**Theorem 3.16****Proposition 16.1***Proof.*

**Theorem 3.17****Proposition 17.1**

Let the maximal function of any measurable  $f \in \mathbb{B}_{\mathbb{R}^n}$  be denoted by  $Hf(x)$ , more precisely,

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r, x))} \int_{B(r, x)} f(y) dy$$

where  $A_r|f|$  is the average of  $|f|$  on a ball with radius  $r > 0$  centered at  $x \in \mathbb{R}^n$ . In symbols,

$$A_r|f| = \frac{1}{m(B(r, x))} \int_{B(r, x)} f(y) dy$$

The maximal theorem makes two claims:

1.  $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$ , and  $Hf$  is measurable for every  $f \in L^1_{loc}$ .
2. There exists a  $C > 0$ , for every  $f \in L^1$

$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1$$

for every  $\alpha > 0$ .

*Proof.* Let  $\alpha > 0$  and fix  $z \in (Hf)^{-1}((\alpha, +\infty))$ , so  $Hf(z) > \alpha$  and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with  $Hf(z) - \alpha > 0$ , we get some  $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let  $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$ , it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some  $r_0 > 0$ . Since  $A_r|f|$  (a function indexed by  $r > 0$ ) is continuous in  $x \in \mathbb{R}^n$ ,  $(A_r|f|)^{-1}((\alpha, +\infty))$  is open, and  $Hf$  is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{ Hf > \alpha \right\} = \bigcup_{r>0} \{A_r|f| > \alpha\}$$

Let  $x \in E_\alpha$ , this induces a  $r_x > 0$  where  $x \in \left\{ A_{r_x} |f| > \alpha \right\}$ . Rearranging gives

$$\left( \frac{1}{\alpha} \int_{B(r,x)} |f| dz \right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each  $x \in E_\alpha \mapsto r_x > 0 \mapsto A_{r_x} |f|$ ,
- If  $U = \bigcup_{x \in E_\alpha} B(r_x, x)$ , then  $E_\alpha \subseteq U$ ,
- Choose  $c < m(E_\alpha) \leq m(U)$  (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

$$x_1, \dots, x_N \in E_\alpha$$

so that  $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$ , and  $c < 3^n \sum_{j \leq k} m(B_j)$

- Define  $B_j = B(r_{x_j}, x_j)$  for all  $j \leq k$ , and

$$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

by finite additivity,

$$c 3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

and finally

$$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of  $m$  on  $\mathbb{B}$ , since

$$m(E_\alpha) = \sup \left\{ m(K), K \in \mathcal{J}_{\mathbb{R}^n}, K \subseteq E_\alpha \right\}$$

for any  $K \in \mathcal{J}_{\mathbb{R}^n}$ ,  $K \subseteq E_\alpha$ , we have  $m(K) < +\infty$ ,  $m(K) \leq m(E_\alpha)$  and

$$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

**Remark 17.1**

We used the properties of a Radon Measure here, without relying on the phrase ‘sending  $c \rightarrow E_\alpha$ ’, which would require us to deal with two cases  $m(E_\alpha) < +\infty$  and  $m(E_\alpha) = +\infty$ .

■

**Theorem 3.18****Proposition 18.1***Proof.*

**Theorem 3.19****Proposition 19.1***Proof.*



**Theorem 3.20****Proposition 20.1***Proof.*

## Theorem 3.21

**Proposition 21.1**

The Lebesgue Differentiation Theorem. Suppose  $f \in L^1_{loc}$ , and for every  $x \in \mathcal{L}_f$ , (so that  $x \in \mathbb{R}^n$  a.e). We have

1.  $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0,$
2.  $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x),$

For every family  $\{E_r\}_{r>0}$  that shrinks nicely to  $x \in \mathbb{R}^{n'}$ .

*Proof.* Since the family  $\{E_r\}_{r>0}$  shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r, x)) \implies m(E_r) > \alpha \cdot m(B(r, x))$$

for some  $\alpha > 0$ , independent on  $r$ . Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r, x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r, x)} |f(y) - f(x)| dy$$

Combining the last two results, for every  $\varepsilon > 0$ , if  $0 < r < \varepsilon$ , then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0 < r < \varepsilon} m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

and sending  $\varepsilon \rightarrow 0$ , proves the first claim. The second claim is immediate upon applying the  $L^1$  inequality.

Fix any  $\varepsilon > 0$ , and

$$\begin{aligned} \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} f(y) dy = f(x) &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y) dy - f(x) \right| \\ &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)] dy \right| \\ &\leq \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\ &= \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\ &= 0 \end{aligned}$$

■

**Theorem 3.22**

**Proposition 22.1**

*Proof.*



**Theorem 3.23****Proposition 23.1***Proof.*

**Theorem 3.24****Proposition 24.1***Proof.*

**Theorem 3.25****Proposition 25.1***Proof.*

**Theorem 3.26****Proposition 26.1***Proof.*

**Theorem 3.27****Proposition 27.1***Proof.*



**Theorem 3.28****Proposition 28.1***Proof.*

**Theorem 3.29**

**Proposition 29.1**

*Proof.*



**Theorem 3.30****Proposition 30.1***Proof.*

**Theorem 3.31****Proposition 31.1***Proof.*

**Theorem 3.32****Proposition 32.1***Proof.*

**Theorem 3.33****Proposition 33.1**

*Proof.*



**Theorem 3.34****Proposition 34.1***Proof.*

**Theorem 3.35****Proposition 35.1**

*Proof.*





**Theorem 3.36****Proposition 36.1***Proof.*