Chapter 7

Proposition 1.1

If I is a linear functional on $C_c(X)$, then for every compact $K \subseteq X$, there exists some $C_k \ge 0$ with

$$|I(f)| \leq C_K \cdot ||f||_u$$

Proof. Since supp (f) is compact, by Urysohn's Lemma (Theorem 4.32), there exists a $\phi \in C_c(X, [0, 1])$ such that $\phi = 1$ on K and vanishes outside some compact $\overline{V} \subseteq X$. Then at every x,

$$-\|f\|_{u} \le f(x) \le +\|f\|_{u}$$

Implies that

$$(-\|f\|_u)\phi \le f(x) \le (+\|f\|_u)\phi$$

So that $f + ||f||_u \phi \ge 0$ and $+||f||_u - f \ge 0$, and by linearity,

$$(-\|f\|_u)I(\phi) \le I(f) \le (+\|f\|_u)I(\phi)$$

Therefore $|I(f)| \leq I(\phi) ||f||_u$, and taking $C_K = I(\phi)$ will suffice.

Proposition 2.1

The Riesz-Markov-Kakutani Representation Theorem. If (for every) I is a positive linear functional on $C_c(X)$, then there exists a unique Radon measure μ on X, such that

$$I(f)=\int f d\mu$$

for every $f \in C_c(X)$. μ also satisfies, for every open U, and every compact $K \subseteq X$

$$\mu(U) = \sup \{ I(f), f \in \mathcal{C}_c(X), f \prec U \}$$
 (1)

$$\mu(K) = \inf\{I(f), f \in \mathcal{C}_c(X), f \ge \chi_K\}$$
(2)

For the sake of completeness, we place the definitions for a Radon measure. Let X be a LCH space, and \mathbb{B} be its usual σ -algebra, a measure ν is a Radon measure iff

- (i) $\nu(K) < +\infty$ for every compact K.
- (ii) ν is outer-regular on all Borel sets E,

$$\nu(E) = \inf \{ \nu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$

Intuition: approximation by open supersets.

(iii) ν is inner-regular on all open sets $U \in \mathcal{T}$,

$$\nu(U) = \sup \{ \mu(K), K \subseteq U, K \text{ compact} \}$$

Intuition: approximation by compact subsets

The main proof is extremely long, so we will divide it into several parts. Following Folland's argumentation closely, we will prove (in order)

(a) If μ_1 , μ_2 are Radon measures on X such that for every $f \in C_c(X)$

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

then μ_1 , μ_2 must satisfy (1), and $\mu_1 = \mu_2$ on \mathbb{B} .

(b) If we define, for every open set U, define $\mu: \mathcal{T} \to [0, +\infty]$ such that

$$\mu(U) = \sup \{ I(f), f \in \mathcal{C}_c(X), f \prec U \}$$
(3)

Then μ is countably subadditive, meaning for every $U \in \mathcal{T}$, $\{U_{j\geq 1}\} \subseteq \mathcal{T}$

$$U = \bigcup U_{j \ge 1} \implies \mu(U) \le \sum \mu(U_{j \ge 1})$$

(c) $\mu(\varnothing) = 0$, $\{\varnothing, X\} \subseteq \mathcal{T}$, so that by Theorem 1.10 μ induces an outer-measure μ^*

$$\mu^*(E) = \inf \left\{ \sum \mu(U_{j\geq 1}), \ U_j \in \mathcal{T}, \ E \subseteq \bigcup U_{j\geq 1} \right\}$$
 (4)

(d) If μ^* is as described above, then if μ is countably subadditive on \mathcal{T} , then

$$\mu^*(E) = \inf \{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$
 (5)

Meaning the two definitions in (4) and (5) are equal.

- (e) μ^* and μ agree on all open sets, and $\mu^*|_{\mathcal{T}} = \mu$,
- (f) Using again the definition in (4) and (5), we show that every open set $U \in \mathcal{T}_X$ is μ^* -measurable, meaning for every $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

With this, since the set of all outer-measurable (μ^* -measurable) sets, \mathcal{M}^* form a σ -algebra,

$$\mathcal{T} \subseteq \mathcal{M}^* \implies \mathbb{B} \subseteq \mathcal{M}^*$$

By Theorem 1.1, and define

$$\mu = \mu^*|_{\mathbb{B}} \tag{6}$$

is a Borel measure. And we note in passing that μ is outer-regular on all $E \in \mathbb{B}$,

$$\mu(E) = \inf \{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \}$$
 (7)

- (g) Using (6) for the definition of μ on \mathbb{B} , we prove that
 - μ is outer-regular on all Borel sets, and
 - μ satisfies Equation (1).
- (h) μ satisfies Equation (2)
- (i) μ is finite on all compact sets.
- (j) μ is inner-regular on all open sets.
- (k) For every $f \in C_c(X, [0, 1])$,

$$I(f) = \int f d\mu \tag{8}$$

(l) For every $f \in C_c(X)$,

$$I(f) = \int f d\mu \tag{9}$$

A small lemma needs to be made before proceeding, that concerns the 'monotonicity' of I on C_cX .

Lemma 2.1

Suppose that $f, g \in C_c(X)$, and $f \geq g \geq 0$ for every X, then $f - g \in C_c(X)$ and $I(f) \geq I(g)$

Proof. Suppose that $x \in X$ where f(x) = 0, then

$$f(x) - g(x) = -g(x) \ge 0 \implies g(x) = 0 \implies f - g = 0$$

Hence

$$\{x, f(x) = 0\} \subseteq \{x, f(x) - g(x) = 0\} \implies \{x, f(x) - g(x) \neq 0\} \subseteq \{x, f(x) \neq 0\}$$

$$\implies \operatorname{supp}(f - g) \subseteq \operatorname{supp}(f)$$

Since supp (f) is compact, and supp (f-g) is a closed subset of supp (f), yields $f-g \in C_c(X)$. And if I is any positive linear functional on $C_c(X)$, then

$$f - g \ge 0 \implies I(f - g) \ge 0$$

 $\implies I(f) \ge I(g) \ge 0$

Remark 2.1

If $f \prec U$ and $g \prec U$ for some open subset $U \subseteq X$, then clearly supp $(f - g) \subseteq \text{supp}\,(f) \subseteq U$, and $1 \geq f \geq f - g \geq 0$ means that $f - g \prec U$ as well.

Part a

Proof. Suppose that μ_1 and μ_2 are Radon measures on X, and for every $f \in C_c(X)$,

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

We first prove (1). Without loss of generality, by monotonicity of L^+ , if $f \prec U$ for some open U, then $0 \leq f \leq ||f||_u \chi_U = \chi_U$ for all x and

$$\int f d\mu_1 \le \int \|f\|_u \chi_U d\mu_1 \le \mu_1(U)$$

Therefore $\mu_1(U)$ (resp. $\mu_2(U)$) is an upper-bound for the set

$$\{I(f), f \in C_c(X), f \prec U\}$$

Since μ_1 is inner-regular on $U \in \mathcal{T}$, for every $\varepsilon > 0$ we can find some compact $K \subseteq U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K)$$

By Urysohn's Lemma (Theorem 4.32), there exists some $g \in C_c(X)$ with

- $g \in C_c(X, [0, 1]),$
- g=1 on $K\subseteq U$,
- g = 0 outside some $\overline{V} \subseteq U$, and
- $g \prec U$.

Hence for every $x \in K$, $g \ge \chi_K$. If $x \notin K$ then $g \ge 0 = \chi_K$; so $g - \chi_K \ge 0$ for every $x \in X$. Since $\chi_K \prec U$, using Lemma 2.1, we get

$$\mu_1(K) \le \int \chi_K d\mu_1 = I(\chi_K) \le I(g)$$

So for every $\varepsilon > 0$, there exists a $g \in C_c(X)$, and $g \prec U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K) \le I(g)$$

Therefore $\mu_1(U) = \sup \{I(f), f \in C_c(X), f \prec U\}$, and the first claim in (a) is proven. To show that μ is indeed unique, since for every open set U, we must have $\mu_1(U) = \mu_2(U)$, and if $E \in \mathbb{B}$ is any Borel set, and by outer-regularity,

$$\mu_1(E) = \inf \{ \mu_1(U), U \supseteq E, U \in \mathcal{T} \} = \inf \{ \mu_2(U), U \supseteq E, U \in \mathcal{T} \} = \mu_2(E)$$

Therefore this measure is unique.

Part b

Proof. To show countable subadditivity for μ with equation (3), fix any $U \in \mathcal{T}$ and a sequence $\{U_{j\geq 1}\}\subseteq \mathcal{T}$ with $U=\bigcup U_{j\geq 1}$. It suffices to show that the partial sum of $\sum \mu(U_{j\leq n})$ is greater than I(f) for any $f\in C_c(X)$, $f\prec U$ (hence it is an upper bound).

Fix any f, then denote $K = \text{supp}(f) \subseteq U$, and since $\{U_{j\geq 1}\}$ is an open cover for K, there exists a finite subcollection, $B \subseteq \mathbb{N}^+$ such that

$$K \subseteq \bigcup_{j \in B} U_j$$

Using Theorem 4.41 on this finite cover of K, there exists a partition of unity in $\{g_{j\leq n}\}$ where

- $g_j \in C_c(X, [0, 1]),$
- $g_j \prec U_j \subseteq U$ for every $j \leq n$, and
- $\sum g_i = 1$ on K,

And notice for every $j \leq n$,

$$\{f = 0\} \cup \{g_j = 0\} \subseteq \{f \cdot g_j = 0\} \implies \{f \cdot g_j \neq 0\} \subseteq \{f \neq 0\} \cap \{g_j \neq 0\}$$
$$\implies \sup (f \cdot g_j) \subseteq \sup (f) \cap \sup (g_j)$$
$$\implies \sup (f \cdot g_j) \subseteq U_j \subseteq U$$

Hence $f \cdot g_j \prec U$ and $f \cdot g_j \in C_c(X, [0, 1])$ for every $1 \leq j \leq n$. Moreover, if we take the sum over a finite n, we obtain $f = \sum f \cdot g_{j \leq n}$, this is because for every $x \in X$, so we have

$$\sum_{j \le n} f(x) \cdot g_j x = f(x) \cdot \sum_{j \le n} g_j(x) = f(x)$$

Then $I(f) = I(\sum f \cdot g_j) = \sum I(f \cdot g_j)$. And by definition of $\mu(U_j)$, since it is the supremum over all $I(h_j)$, where $h_j \in C_c(X, [0, 1])$ and $h_j \prec U_j$

$$I(f \cdot g_j) \le \mu(U_j), \quad \forall j \le n$$

Hence

$$I(f) \le \sum_{j \le n} \mu(U_j) \le \sum_{j \ge 1} \mu(U_j)$$

Where for the last estimate we used the fact that μ is non-negative, and since this holds for any f, we can conclude that $\mu(U) \leq \sum_{j>1} \mu(U_j)$.

Part c

Proof. By definition of a topology, $\{\emptyset, X\} \subseteq \mathcal{T}$, and $\mu(\emptyset) = \sup\{I(f), f \in C_c(X), f \prec \emptyset\}$, so supp $(f) = \emptyset$, and $\{x, f(x) \neq 0\} \subseteq \emptyset$, so the set contains one element, namely I(0) = 0 by linearity. So $\mu(\emptyset) = 0$. The assumptions for Theorem 1.10 are satisfied and (4) is indeed an outer-measure.

Part d

Proof. Denote the right members of (4) and (5) by W_1 and W_2 , we wish to show that $\inf W_1 = \inf W_2$. Clearly $\inf W_1 \leq \inf W_2$, since $W_2 \subseteq W_1$. Now, if μ is countably additive, then for every $\omega \in W_1$ induces a sequence of open sets $\{U_{j\geq 1}\}$ such that $E \subseteq \bigcup U_{j\geq 1}$. Denote the union over $\{U_{j\geq 1}\}$ by U, which is also another open set,

$$\inf W_2 \le \mu(U) \le \sum \mu(U_{j \ge 1}) = \omega$$

Since ω is arbitrary, we conclude that $\inf W_2 = \inf W_1$, and this proves (d).

Part e

Proof. If U and V are open subsets of X, and if $U \subseteq V$, then

$$U \subseteq V \implies \{f \in C_c(X), \ f \prec U\} \subseteq \{f \in C_c(X), \ f \prec V\}$$
$$\implies \{I(f), \ f \in C_c(X), \ f \prec U\} \subseteq \{I(f), \ f \in C_c(X), \ f \prec V\}$$

Hence $\mu(U) \leq \mu(V)$. Now by equation (5), $\mu^*(U) \leq \mu(U)$. To show the reverse inequality, suppose by contradiction that $\mu^*(U) < \mu(U)$.

Since $\mu^*(U)$ is an infimum, then for every $\varepsilon > 0$ there exists some $V \supseteq U$ where if we write $\mu^*(U) + \varepsilon = \mu(U)$

$$\mu(V) < \mu^*(U) + \varepsilon = \mu(U) \implies \mu(V) < \mu(U), \ U \subseteq V$$

This contradicts what we have just proven, and therefore $\mu^*(U) = \mu(U)$ for every open set U.

Part f

Proof. We wish to show that every open set U is μ^* -measurable. By Theorem 1.10, it suffices to show that for every $E \subseteq X$

$$\mu^*(E) \ge \mu^*(E \cap U) + \mu^*(E \setminus U) \tag{10}$$

because the reverse inequality is given by subadditivity of μ^* , and we can also assume that $\mu^*(E) < +\infty$. Let us assume that E is open, we wish to find some function $h \in C_c(X)$, $h \prec E$ with

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

The above formula is fussy, but the liberty is taken to show it beforehand to avoid any potential confusion that follows. Since $E \cap U$ is an open subset of X, the definition of $\mu(E \cap U) = \mu^*(E \cap U)$ in (3) tells us that every $\varepsilon > 0$ induces some $f \in C_c(X)$, $f \prec E \cap U$ where

$$I(f) > \mu(E \cap U) - \varepsilon = \mu^*(E \cap U) - \varepsilon \tag{11}$$

Also, supp (f) is a closed set (compact subsets of Hausdorff spaces are closed), therefore $E \setminus \text{supp}(f)$ is an open set. We make a small diversion from the current part of the proof and turn out attention to the fact that

$$\operatorname{supp}(f) \subseteq U \implies U^c \subseteq (\operatorname{supp}(f))^c$$
$$\implies E \setminus U \subseteq E \setminus \operatorname{supp}(f)$$

And because the outer-measure μ^* is monotone,

$$\mu^*(U) \le \mu^*(E \setminus \text{supp}(f)) \tag{12}$$

Now, using the definition of $\mu(E \setminus \text{supp}(f))$ (recall that $E \setminus \text{supp}(F)$ is an open set), for every $\varepsilon > 0$, there exists some $g \in C_c(X)$, $g \prec E \setminus \text{supp}(F)$ with

$$I(g) > \mu(E \setminus \text{supp}(f)) - \varepsilon = \mu^*(E \setminus \text{supp}(f)) - \varepsilon \tag{13}$$

It is at this part of the proof where we wish to define h = f + g, but first we must verify

- $f + g \in C_c(X, [0, 1]),$
- $f+g \prec E$

The sum of two non-negative functions is non-negative, and for every $x \in \text{supp}(f)$, $f \leq 1$. Also

$$\operatorname{supp}(g) \subseteq (\operatorname{supp}(f))^c \implies \operatorname{supp}(f) \subseteq (\operatorname{supp}(g))^c$$
$$\implies \operatorname{supp}(f) \subseteq \{g = 0\}$$

The last implication comes from taking complements on both sides of $\{g \neq 0\} \subseteq \text{supp}(g)$. So $x \in \text{supp}(f) \implies f + g \leq 1$. Now if $x \notin \text{supp}(f)$, then $f + g = g \leq 1$. Furthermore, supp (f + g) is a closed subset of compact supp $(f) \cup \text{supp}(g)$. This is because $\{f + g \neq 0\} \subseteq \{f \neq 0\} \cup \{g \neq 0\}$, and the finite union of two compact sets is again again compact.

A moment's thought should yield the fact that the last estimate should be an equality, but it is a needless distraction. Therefore supp (f + g) is compact and $f + g \in C_c(X, [0, 1])$.

Now both bullet points are satisfied, and we can set h = f + g. Adding equation (13) with (11) gives us

$$I(h) = I(f) + I(g) > \mu^*(E \cap U) + \mu^*(E \setminus \text{supp}(f)) - 2\varepsilon$$

Upon applying (12) to the right member of the above estimate, we have

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

But this particular $h \in C_c(X) \cap \{f \prec E\}$, therefore

$$\mu^*(E) \ge I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, equation (10) holds for every open E. Now for any general $E \subseteq X$, fix any $\varepsilon > 0$ and by how we defined $\mu^*(E)$, there exists some open $V \supseteq E$ —recall that $\mu^*(E)$ is the infimum over the set of $\mu(V)$ where V is an open superset of E — hence

$$\mu^*(E) + \varepsilon > \mu(V) \ge \mu^*(V \cap U) + \mu^*(V \setminus U)$$

By monotonicity (twice) of the outer-measure μ^* , we have

$$\mu^*(E) + \varepsilon > \mu^*(E \cap U) + \mu^*E \setminus U$$

Let $\varepsilon \to 0$, and we get

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Therefore every open $U \subseteq X$ is μ^* -measurable. So $\mu = \mu^*|_{\mathbb{B}}$ is a Borel measure on X.

Part g

Proof. To show outer-regularity, fix any $E \in \mathbb{B}$, then by definition,

$$\mu(E) = \mu^*(E) = \inf \left\{ \mu(U), \ U \supseteq E, \ U \in \mathcal{T} \right\}$$

And for every open U, (1) follows from Equation (3).

Part h

Proof. We want to show that for every compact K, Equation (2) holds. To reduce the notational baggage that follows, we agree to define

$${I(f), f \in C_c(X), f \prec U} = {I(f), f \prec U}$$

Similarly for $\{I(f), f \geq \chi_K\}$. If $\mu(K) = 0$, then $\mu(K)$ is obviously a lower bound, since $f \geq \chi_K \geq 0$ means that $I(f) \geq 0$, for every $f \geq \chi_K$. So we can suppose $\mu(K) > 0$.

Fix an arbitrary $f \ge \chi_K$, then this particular f induces an open set $U_{\alpha} = \{f > 1 - \alpha\}$, where $\alpha > 0$. Notice also that

$$K \subseteq \{f \ge 1\} \subseteq \{f > 1 - \alpha\} = U_{\alpha}$$

Since U_{α} is an open superset of K, by Equation (7), $\mu(K) \leq \mu(U_{\alpha})$, but $\mu(U_{\alpha})$ is simply the supremum of $\{I(g), g \prec U_{\alpha}\}$. If we wish to show that $\mu(K) \leq \mu(U_{\alpha}) \leq I(f)$, it suffices to show that I(f) is an upper-bound for $\{I(g), g \prec U_{\alpha}\}$.

Fix any $I(g) \in \{I(g), g \prec U_{\alpha}\}$, note that $1 - \alpha \neq 0$ for any α small enough, then

- $f/(1-\alpha) > 1$ on U_{α} ,
- $1 \ge g \ge 0$ on U_{α} , in particular, $f/(1-\alpha) g \ge 0$ on U_{α} ,
- If $x \notin U_{\alpha}$, then $f/(1-\alpha) g = f(1-\alpha) \ge 0$.
- Therefore $f/(1-\alpha)-g\geq 0$ for any x, and by Lemma 2.1,

$$I(f/(1-\alpha)) \ge I(g) \quad \forall g \prec U_{\alpha}$$

Combining the above estimate with $\mu(K) \leq \mu(U_{\alpha})$ gives us

$$\mu(K) \leq \frac{1}{1-\alpha} I(f)$$

Now write $\varepsilon = \alpha/\mu(K) > 0$ and for every $\varepsilon > 0$ we get

$$\mu(K) - I(f) \leq \alpha \mu(K) = \varepsilon$$

Send $\varepsilon \to 0$ and $\mu(K) \le I(f)$ for every $f \ge \chi_K$.

To show that $\mu(K)$ is indeed the infimum for $\{I(f), f \geq \chi_K\}$, notice that for every $\varepsilon > 0$ we can obtain some open superset $U \supseteq K$ (by outer-regularity) where $\mu(U) < \mu(K) + \varepsilon$. By Urysohn's Lemma, there exists some $g \prec U$, g(x) = 1 for every $x \in K$.

$$g \in \{I(f), \ f \prec U\} \cap \{I(f), \ f \geq \chi_K\}$$

Therefore $I(g) \leq \mu(U) < \mu(K) + \varepsilon$ as desired, and Equation (2) holds.

Part i

Proof. $\mu(K) < +\infty$ for every compact K. Indeed, since $I(\chi_K) \in \{I(f), f \geq \chi_K\}$, then by Theorem 7.1, there exists a constant $C_K \geq 0$ that bounds

$$\mu(K) \leq |I(\chi_K)| = I(\chi_K) \leq C_K \cdot \|\chi_K\| = C_K < +\infty$$

Part j

Proof. Fix any open set U, then for every $\varepsilon > 0$, there exists some $f \prec U$ with $\mu(U) - \varepsilon < I(f)$. Then denote $K = \text{supp}(f) \subseteq U$. If we take any $I(h) \in \{I(h), h \geq \chi_K\}$, then $h \geq f$ gives us $I(h) \geq I(f)$ by Lemma 2.1. So I(f) is a lower bound of $\{I(h), h \geq \chi_K\}$, therefore

$$\mu(U) - \varepsilon \le I(f) \le \mu(K)$$

Since supp $(f) = K \subseteq U$, this proves inner-regularity of μ on open sets.

Part k

Proof. Suppose $f \in C_c(X, [0, 1])$, we first show that Equation (8) holds. We divide the interval [0, 1] into $N \ge 1$ chunks by writing

$$K_j = \{f \ge j/N\}$$

for every $1 \geq j \geq N$. And define $K_0 = \text{supp}(f)$. Each K_j is a closed subset of supp (f), and therefore compact. More is true,

- $K_{j-1} \supseteq K_j$ for every $1 \le j \le N$.
- $x \in K_j$ iff $f(x) \in \left[\frac{j}{N}, 1\right]$,
- $x \notin K_j$ iff $f(x) \in \left[0, \frac{j}{N}\right)$, and
- $x \in (K_{j-1} \setminus K_j)$ iff $f(x) \in \left[\frac{j-1}{N}, \frac{j}{N}\right)$

Folland constructs a finite sequence of compactly supported functions, $\{f_j\}$, where $1 \leq j \leq N$ such that

- Each $0 \le f_j \le 1/N$,
- If $x \in (K_m \setminus K_{m+1})$ iff $f(x) \in \left[\frac{m}{N}, \frac{m+1}{N}\right)$ means that $f_j = 1$ for all $1 \le j \le m$, and
- $f_{m+1} = f m/N$ on K_m , such that

$$f(x) = \left(\sum f_{j \leq m}(x)
ight) + \left(f(x) - rac{m}{N}
ight) = rac{m}{N} + \left(f(x) - rac{m}{N}
ight)$$

- And for every $m < j \le N$, $f_j = 0$.
- If $x \notin K_m$ iff $f(x) \in \left[0, \frac{m}{N}\right)$ then for every $m+1 \leq j \leq N$, $f_j = 0$.

5/5
4/5
3/5
2/5
1/5

The illustration for when N=5 below should make things clearer.

It is also trivial to verify that

0/5

• For every $x \in K_j$, $f_j = N^{-1}$, and

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$$\chi_{K_j} N^{-1} \le f_j \tag{14}$$

Also, if $x \notin K_j$ then $f_j \ge 0$, therefore $f_j \ge \chi_{K_j} N^{-1}$ at every x.

• If $x \notin K_{j-1}$ then $f_j = 0 \le \chi_{K_{j-1}} \cdot N^{-1}$. If x is in K_{j-1} then $f_j \le N^{-1}$ by construction and therefore

$$f_j \le \chi_{K_{j-1}} N^{-1} \tag{15}$$

for all x.

• $f_j \in C_c(X)$, since supp $(f_j) \subseteq \text{supp}(f)$.

Combining Equations (14) with (15), and by monotonicity in $L^+(X, \mathbb{B}, \mu)$, since $f_j \in L^+$

$$\int \frac{1}{N} \chi_{K_j} d\mu \le \int f_j d\mu \le \int \frac{1}{N} \chi_{K_{j-1}} d\mu$$

And for every $1 \le j \le N$,

$$\frac{1}{N}\mu(K_j) \le \int f_j d\mu \le \frac{1}{N}\mu(K_{j-1}) \tag{16}$$

Furthermore, from Equation (14), since $Nf_j \ge \chi_{K_j}$ then by Equation (2),

$$\mu(K_j) \leq I(Nf_j) \implies \frac{1}{N} \mu(K_j) \leq I(f_j)$$

Now for any arbitrary $I(h) \in \{I(h), h \ge \chi_{K_{j-1}}\}$, since

$$h \ge \chi_{K_{j-1}} \ge Nf_j \implies I(h) \ge I(Nf_j)$$

So $NI(f_j)$ is a lower bound for $\{I(h), h \geq \chi_{K_{j-1}}\}$ and

$$I(f_j) \leq \frac{1}{N} \mu(K_{j-1})$$

Combining the last two results, with $I(f_i)$, we get

$$\frac{1}{N}\mu(K_j) \le I(f_j) \le \frac{1}{N}\mu(K_{j-1}) \tag{17}$$

Taking the sum over $1 \le j \le N$ for Equations (16) and (17). Define $A = N^{-1} \sum_{j=0}^{N-1} \mu(K_j)$, and $B = N^{-1} \sum_{j=1}^{N} \mu(K_j)$

$$B \le \int f d\mu \le A$$

And also

$$B \leq I(f) \leq A$$

This is because of finite additivity of both I and the integral, and $f = \sum f_j$ on $K_0 = \text{supp}(f)$. Subtracting the two equations (keeping in mind that $\mu(K_j) < +\infty$ for any compact K_j), we get

$$(-1)(A-B) \leq \left(\int f d\mu - I(f) \right) \leq A - B \implies \left| \int f d\mu - I(f) \right| \leq A - B$$

It is trivial to verify that

$$0 \le A - B = N^{-1}(\mu(K_0) - \mu(K_N)) \le N^{-1}\mu(K_0)$$

as $K_N \subseteq K_0$. Let $N \to \infty$ and

$$\int f d\mu = I(f)$$

Equation (8) holds as desired.

Part 1

Proof. Now for any general $f \in C_c(X)$, f must be bounded on the plane since $C_c(X) \subseteq BC(X)$, and $|f| \leq M_0$ for some $M_0 \geq 0$. Since supp (f) is compact, we know that

$$\int |f| d\mu \le \int M_0 \chi_{\text{supp}(f)} d\mu \le M_0 \mu(\text{supp}(f)) < +\infty$$

And $C_c(X) \subseteq L^1(\mu)$. Furthermore,

$$\frac{1}{2}(|\operatorname{Re} f| + |\operatorname{Im} f|) \le |f| \le M_0$$

So that Re f and Im f are in $C_c(X)$. Without loss of generality, we may assume that f is real. Define $f_1 = \text{Re } f^+/M_0$ and $f_2 = \text{Re } f^-/M_0$ and it immediately follows that $f_1, f_2 \in C_c(X, [0, 1])$.

By linearity of I on $C_c(X)$ and the integral in $L^1(\mu)$,

$$I(f_1-f_2)=I(f)=\int f d\mu=\int f_1 d\mu-\int f_2 d\mu$$

Then we may apply the above to the real and imaginary parts of a general $f \in C_c(X)$, and this completes the proof.

Proposition 3.1

See Theorem 7.2

Proof.

Theorem 7.4

Proposition 4.1

See Theorem 7.2

Proposition 5.1

Proposition 6.1

Proposition 7.1

Proposition 8.1

Proposition 9.1

If μ is a Radon measure on X, then $C_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < +\infty$.

Proof. Theorem 6.7 tells us that the set of L^p simple functions (as Folland calls them), which are

$$\Lambda = \left\{ f, \, f = \sum_{j \leq n} a_j \chi_{E_j}, \; a_j \in \mathbb{C}, \, \mu(E_j) < +\infty
ight\}$$

So for every $f \in L^p$, there exists a sequence $\{f_n\} \subseteq \Lambda$ with $f_n \to f$ pointwise and $f_n \to f$ in L^p .

Proposition 10.1

Proposition 11.1