Chapter 7

Theorem 7.1

Proposition 1.1. *If I is a linear functional on $\mathrm{C}_c(X)$, then for every compact $K \subseteq X$, there exists some $C_k \geq 0$ with*

$$|I(f)| \leq C_K \cdot \|f\|_u$$

Proof. Since $\operatorname{supp}(f)$ is compact, by Urysohn's Lemma (Theorem 4.32), there exists a $\phi \in \mathrm{C}_c(X, [0,1])$ such that $\phi = 1$ on K and vanishes outside some compact $\overline{V} \subseteq X$. Then at every x,

$$-\|f\|_u \leq f(x) \leq +\|f\|_u$$

Implies that

$$(-\|f\|_u)\phi \leq f(x) \leq (+\|f\|_u)\phi$$

So that $f + \|f\|_u \phi \geq 0$ and $+\|f\|_u - f \geq 0$, and by linearity,

$$(-\|f\|_u)I(\phi) \leq I(f) \leq (+\|f\|_u)I(\phi)$$

Therefore $|I(f)| \leq I(\phi)\|f\|_u$, and taking $C_K = I(\phi)$ will suffice. ∎

Theorem 7.2

Proposition 2.1. *The Riesz-Markov-Kakutani Representation Theorem. If (for every) I is a positive linear functional on $\mathrm{C}_c(X)$, then there exists a unique Radon measure μ on X, such that*

$$I(f) = \int f d\mu$$

for every $f \in \mathrm{C}_c(X)$. μ also satisfies, for every open U, and every compact $K \subseteq X$

$$\mu(U) = \sup \{I(f),\ f \in \mathrm{C}_c(X),\ f \prec U\} \tag{1}$$

$$\mu(K) = \inf \{I(f),\ f \in \mathrm{C}_c(X),\ f \geq \chi_K\} \tag{2}$$

For the sake of completeness, we place the definitions for a Radon measure. Let X be a LCH space, and $\mathbb{B}$ be its usual σ-algebra, a measure ν is a Radon measure iff

(i) $\nu(K) < +\infty$ for every compact K.

(ii) ν is outer-regular on all Borel sets E,

$$\nu(E) = \inf \{\nu(U),\ U \supseteq E,\ U \in \mathcal{T}\}$$

Intuition: approximation by open supersets.

(iii) ν is inner-regular on all open sets $U \in \mathcal{T}$,

$$\nu(U) = \sup \{\mu(K),\ K \subseteq U,\ K \text{ compact}\}$$

Intuition: approximation by compact subsets

The main proof is extremely long, so we will divide it into several parts. Following Folland's argumentation closely, we will prove (in order)

(a) If μ_1, μ_2 are Radon measures on X such that for every $f \in \mathrm{C}_c(X)$

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

then μ_1, μ_2 must satisfy (1), and $\mu_1 = \mu_2$ on $\mathbb{B}$.

(b) If we define, for every open set U, define $\mu : \mathcal{T} \to [0, +\infty]$ such that

$$\mu(U) = \sup\{I(f),\ f \in \mathrm{C}_c(X),\ f \prec U\} \tag{3}$$

Then μ is countably subadditive, meaning for every $U \in \mathcal{T}$, $\{U_{j\geq 1}\} \subseteq \mathcal{T}$

$$U = \bigcup U_{j\geq 1} \implies \mu(U) \leq \sum \mu(U_{j\geq 1})$$

(c) $\mu(\varnothing) = 0$, $\{\varnothing, X\} \subseteq \mathcal{T}$, so that by Theorem 1.10 μ induces an outer-measure μ^*

$$\mu^*(E) = \inf\left\{\sum \mu(U_{j\geq 1}),\ U_j \in \mathcal{T},\ E \subseteq \bigcup U_{j\geq 1}\right\} \tag{4}$$

(d) If μ^* is as described above, then if μ is countably subadditive on $\mathcal{T}$, then

$$\mu^*(E) = \inf\{\mu(U),\ U \supseteq E,\ U \in \mathcal{T}\} \tag{5}$$

Meaning the two definitions in (4) and (5) are equal.

(e) μ^* and μ agree on all open sets, and $\mu^*|_{\mathcal{T}} = \mu$,

(f) Using again the definition in (4) and (5), we show that every open set $U \in \mathcal{T}_X$ is μ^*-measurable, meaning for every $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

With this, since the set of all outer-measurable (μ^*-measurable) sets, $\mathcal{M}^*$ form a σ-algebra,

$$\mathcal{T} \subseteq \mathcal{M}^* \implies \mathbb{B} \subseteq \mathcal{M}^*$$

By Theorem 1.1, and define

$$\mu = \mu^*|_{\mathbb{B}} \tag{6}$$

is a Borel measure. And we note in passing that μ is outer-regular on all $E \in \mathbb{B}$,

$$\mu(E) = \inf\{\mu(U),\ U \supseteq E,\ U \in \mathcal{T}\} \tag{7}$$

(g) Using (6) for the definition of μ on $\mathbb{B}$, we prove that

- μ is outer-regular on all Borel sets, and
- μ satisfies Equation (1).

(h) μ satisfies Equation (2)

(i) μ is finite on all compact sets.

(j) μ is inner-regular on all open sets.

(k) For every $f \in \mathrm{C}_c(X, [0, 1])$,

$$I(f) = \int f d\mu \tag{8}$$

(l) For every $f \in \mathrm{C}_c(X)$,

$$I(f) = \int f d\mu \tag{9}$$

A small lemma needs to be made before proceeding, that concerns the 'monotonicity' of I on C_cX.

Lemma 2.1 *Suppose that $f, g \in \mathrm{C}_c(X)$, and $f \geq g \geq 0$ for every X, then $f - g \in \mathrm{C}_c(X)$ and $I(f) \geq I(g)$*

Proof. Suppose that $x \in X$ where $f(x) = 0$, then

$$f(x) - g(x) = -g(x) \geq 0 \implies g(x) = 0 \implies f - g = 0$$

Hence

$$\begin{aligned} \{x,\ f(x) = 0\} \subseteq \{x,\ f(x) - g(x) = 0\} &\implies \{x,\ f(x) - g(x) \neq 0\} \subseteq \{x,\ f(x) \neq 0\} \\ &\implies \operatorname{supp}(f - g) \subseteq \operatorname{supp}(f) \end{aligned}$$

Since $\operatorname{supp}(f)$ is compact, and $\operatorname{supp}(f - g)$ is a closed subset of $\operatorname{supp}(f)$, yields $f - g \in \mathrm{C}_c(X)$. And if I is any positive linear functional on $\mathrm{C}_c(X)$, then

$$\begin{aligned} f - g \geq 0 &\implies I(f - g) \geq 0 \\ &\implies I(f) \geq I(g) \geq 0 \end{aligned}$$

■

Remark 2.1 *If $f \prec U$ and $g \prec U$ for some open subset $U \subseteq X$, then clearly* $\operatorname{supp}(f-g) \subseteq \operatorname{supp}(f) \subseteq U$*, and* $1 \geq f \geq f-g \geq 0$ *means that* $f-g \prec U$ *as well.*

Part a

Proof. Suppose that μ_1 and μ_2 are Radon measures on X, and for every $f \in \mathrm{C}_c(X)$,

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

We first prove (1). Without loss of generality, by monotonicity of L^+, if $f \prec U$ for some open U, then $0 \leq f \leq \|f\|_u \chi_U = \chi_U$ for all x and

$$\int f d\mu_1 \leq \int \|f\|_u \, \chi_U \, d\mu_1 \leq \mu_1(U)$$

Therefore $\mu_1(U)$ (resp. $\mu_2(U)$) is an upper-bound for the set

$$\{I(f),\ f \in \mathrm{C}_c(X),\ f \prec U\}$$

Since μ_1 is inner-regular on $U \in \mathcal{T}$, for every $\varepsilon > 0$ we can find some compact $K \subseteq U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K)$$

By Urysohn's Lemma (Theorem 4.32), there exists some $g \in \mathrm{C}_c(X)$ with

- $g \in \mathrm{C}_c(X, [0,1])$,
- $g = 1$ on $K \subseteq U$,
- $g = 0$ outside some $\overline{V} \subseteq U$, and
- $g \prec U$.

Hence for every $x \in K$, $g \geq \chi_K$. If $x \notin K$ then $g \geq 0 = \chi_K$; so $g - \chi_K \geq 0$ for every $x \in X$. Since $\chi_K \prec U$, using Lemma 2.1, we get

$$\mu_1(K) \leq \int \chi_K \, d\mu_1 = I(\chi_K) \leq I(g)$$

So for every $\varepsilon > 0$, there exists a $g \in \mathrm{C}_c(X)$, and $g \prec U$ where

$$\mu_1(U) - \varepsilon < \mu_1(K) \leq I(g)$$

Therefore $\mu_1(U) = \sup\{I(f),\ f \in \mathrm{C}_c(X),\ f \prec U\}$, and the first claim in (a) is proven. To show that μ is indeed unique, since for every open set U, we must have $\mu_1(U) = \mu_2(U)$, and if $E \in \mathbb{B}$ is any Borel set, and by outer-regularity,

$$\mu_1(E) = \inf\{\mu_1(U), U \supseteq E, U \in \mathcal{T}\} = \inf\{\mu_2(U), U \supseteq E, U \in \mathcal{T}\} = \mu_2(E)$$

Therefore this measure is unique. ■

Part b

Proof. To show countable subadditivity for μ with equation (3), fix any $U \in \mathcal{T}$ and a sequence $\{U_{j\geq 1}\} \subseteq \mathcal{T}$ with $U = \bigcup U_{j\geq 1}$. It suffices to show that the partial sum of $\sum \mu(U_{j\leq n})$ is greater than $I(f)$ for any $f \in \mathrm{C}_c(X)$, $f \prec U$ (hence it is an upper bound).

Fix any f, then denote $K = \operatorname{supp}(f) \subseteq U$, and since $\{U_{j\geq 1}\}$ is an open cover for K, there exists a finite subcollection, $B \subseteq \mathbb{N}^+$ such that

$$K \subseteq \bigcup_{j\in B} U_j$$

Using Theorem 4.41 on this finite cover of K, there exists a partition of unity in $\{g_{j\leq n}\}$ where

- $g_j \in \mathrm{C}_c(X, [0,1])$,
- $g_j \prec U_j \subseteq U$ for every $j \leq n$, and
- $\sum g_j = 1$ on K,

And notice for every $j \leq n$,

$$\begin{aligned}\{f = 0\} \cup \{g_j = 0\} \subseteq \{f \cdot g_j = 0\} &\implies \{f \cdot g_j \neq 0\} \subseteq \{f \neq 0\} \cap \{g_j \neq 0\} \\ &\implies \operatorname{supp}(f \cdot g_j) \subseteq \operatorname{supp}(f) \cap \operatorname{supp}(g_j) \\ &\implies \operatorname{supp}(f \cdot g_j) \subseteq U_j \subseteq U\end{aligned}$$

Hence $f \cdot g_j \prec U$ and $f \cdot g_j \in \mathrm{C}_c(X, [0,1])$ for every $1 \leq j \leq n$. Moreover, if we take the sum over a finite n, we obtain $f = \sum f \cdot g_{j\leq n}$, this is because for every $x \in X$, so we have

$$\sum_{j\leq n} f(x) \cdot g_j x = f(x) \cdot \sum_{j\leq n} g_j(x) = f(x)$$

Then $I(f) = I(\sum f \cdot g_j) = \sum I(f \cdot g_j)$. And by definition of $\mu(U_j)$, since it is the supremum over all $I(h_j)$, where $h_j \in \mathrm{C}_c(X, [0,1])$ and $h_j \prec U_j$

$$I(f \cdot g_j) \leq \mu(U_j), \quad \forall j \leq n$$

Hence

$$I(f) \leq \sum_{j \leq n} \mu(U_j) \leq \sum_{j \geq 1} \mu(U_j)$$

Where for the last estimate we used the fact that μ is non-negative, and since this holds for any f, we can conclude that $\mu(U) \leq \sum_{j \geq 1} \mu(U_j)$. ∎

Part c

Proof. By definition of a topology, $\{\varnothing, X\} \subseteq \mathcal{T}$, and $\mu(\varnothing) = \sup\{I(f),\ f \in \mathrm{C}_c(X),\ f \prec \varnothing\}$, so $\operatorname{supp}(f) = \varnothing$, and $\{x,\ f(x) \neq 0\} \subseteq \varnothing$, so the set contains one element, namely $I(0) = 0$ by linearity. So $\mu(\varnothing) = 0$. The assumptions for Theorem 1.10 are satisfied and (4) is indeed an outer-measure. ∎

Part d

Proof. Denote the right members of (4) and (5) by W_1 and W_2, we wish to show that $\inf W_1 = \inf W_2$. Clearly $\inf W_1 \leq \inf W_2$, since $W_2 \subseteq W_1$. Now, if μ is countably additive, then for every $\omega \in W_1$ induces a sequence of open sets $\{U_{j \geq 1}\}$ such that $E \subseteq \bigcup U_{j \geq 1}$. Denote the union over $\{U_{j \geq 1}\}$ by U, which is also another open set,

$$\inf W_2 \leq \mu(U) \leq \sum \mu(U_{j \geq 1}) = \omega$$

Since ω is arbitrary, we conclude that $\inf W_2 = \inf W_1$, and this proves (d). ∎

Part e

Proof. If U and V are open subsets of X, and if $U \subseteq V$, then

$$\begin{aligned} U \subseteq V &\implies \{f \in \mathrm{C}_c(X),\ f \prec U\} \subseteq \{f \in \mathrm{C}_c(X),\ f \prec V\} \\ &\implies \{I(f),\ f \in \mathrm{C}_c(X),\ f \prec U\} \subseteq \{I(f),\ f \in \mathrm{C}_c(X),\ f \prec V\} \end{aligned}$$

Hence $\mu(U) \leq \mu(V)$. Now by equation (5), $\mu^*(U) \leq \mu(U)$. To show the reverse inequality, suppose by contradiction that $\mu^*(U) < \mu(U)$.

Since $\mu^*(U)$ is an infimum, then for every $\varepsilon > 0$ there exists some $V \supseteq U$ where if we write $\mu^*(U) + \varepsilon = \mu(U)$

$$\mu(V) < \mu^*(U) + \varepsilon = \mu(U) \implies \mu(V) < \mu(U),\, U \subseteq V$$

This contradicts what we have just proven, and therefore $\mu^*(U) = \mu(U)$ for every open set U. ■

Part f

Proof. We wish to show that every open set U is μ^*-measurable. By Theorem 1.10, it suffices to show that for every $E \subseteq X$

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U) \tag{10}$$

because the reverse inequality is given by subadditivity of μ^*, and we can also assume that $\mu^*(E) < +\infty$. Let us assume that E is open, we wish to find some function $h \in \mathrm{C}_c(X)$, $h \prec E$ with

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

The above formula is fussy, but the liberty is taken to show it beforehand to avoid any potential confusion that follows. Since $E \cap U$ is an open subset of X, the definition of $\mu(E \cap U) = \mu^*(E \cap U)$ in (3) tells us that every $\varepsilon > 0$ induces some $f \in \mathrm{C}_c(X)$, $f \prec E \cap U$ where

$$I(f) > \mu(E \cap U) - \varepsilon = \mu^*(E \cap U) - \varepsilon \tag{11}$$

Also, $\mathrm{supp}\,(f)$ is a closed set (compact subsets of Hausdorff spaces are closed), therefore $E \setminus \mathrm{supp}\,(f)$ is an open set. We make a small diversion from the current part of the proof and turn out attention to the fact that

$$\begin{aligned} \mathrm{supp}\,(f) \subseteq U &\implies U^c \subseteq (\mathrm{supp}\,(f))^c \\ &\implies E \setminus U \subseteq E \setminus \mathrm{supp}\,(f) \end{aligned}$$

And because the outer-measure μ^* is monotone,

$$\mu^*(U) \leq \mu^*(E \setminus \mathrm{supp}\,(f)) \tag{12}$$

Now, using the definition of $\mu(E \setminus \mathrm{supp}\,(f))$ (recall that $E \setminus \mathrm{supp}\,(F)$ is an open set), for every $\varepsilon > 0$, there exists some $g \in \mathrm{C}_c(X)$, $g \prec E \setminus \mathrm{supp}\,(F)$ with

$$I(g) > \mu(E \setminus \mathrm{supp}\,(f)) - \varepsilon = \mu^*(E \setminus \mathrm{supp}\,(f)) - \varepsilon \tag{13}$$

It is at this part of the proof where we wish to define $h = f + g$, but first we must verify

- $f + g \in \mathrm{C}_c(X, [0,1])$,
- $f + g \prec E$

The sum of two non-negative functions is non-negative, and for every $x \in \operatorname{supp}(f)$, $f \leq 1$. Also

$$\begin{aligned}\operatorname{supp}(g) \subseteq (\operatorname{supp}(f))^c &\implies \operatorname{supp}(f) \subseteq (\operatorname{supp}(g))^c \\ &\implies \operatorname{supp}(f) \subseteq \{g = 0\}\end{aligned}$$

The last implication comes from taking complements on both sides of $\{g \neq 0\} \subseteq \operatorname{supp}(g)$. So $x \in \operatorname{supp}(f) \implies f + g \leq 1$. Now if $x \notin \operatorname{supp}(f)$, then $f + g = g \leq 1$. Furthermore, $\operatorname{supp}(f + g)$ is a closed subset of compact $\operatorname{supp}(f) \cup \operatorname{supp}(g)$. This is because $\{f + g \neq 0\} \subseteq \{f \neq 0\} \cup \{g \neq 0\}$, and the finite union of two compact sets is again again compact.

A moment's thought should yield the fact that the last estimate should be an equality, but it is a needless distraction. Therefore $\operatorname{supp}(f + g)$ is compact and $f + g \in \mathrm{C}_c(X, [0,1])$.

Now both bullet points are satisfied, and we can set $h = f + g$. Adding equation (13) with (11) gives us

$$I(h) = I(f) + I(g) > \mu^*(E \cap U) + \mu^*(E \setminus \operatorname{supp}(f)) - 2\varepsilon$$

Upon applying (12) to the right member of the above estimate, we have

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

But this particular $h \in \mathrm{C}_c(X) \cap \{f \prec E\}$, therefore

$$\mu^*(E) \geq I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Since $\varepsilon > 0$ is arbitrary, equation (10) holds for every open E. Now for any general $E \subseteq X$, fix any $\varepsilon > 0$ and by how we defined $\mu^*(E)$, there exists some open $V \supseteq E$ —recall that $\mu^*(E)$ is the infimum over the set of $\mu(V)$ where V is an open superset of E — hence

$$\mu^*(E) + \varepsilon > \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

By monotonicity (twice) of the outer-measure μ^*, we have

$$\mu^*(E) + \varepsilon > \mu^*(E \cap U) + \mu^* E \setminus U$$

Let $\varepsilon \to 0$, and we get

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Therefore every open $U \subseteq X$ is μ^*-measurable. So $\mu = \mu^*|_{\mathbb{B}}$ is a Borel measure on X. ■

Part g

Proof. To show outer-regularity, fix any $E \in \mathbb{B}$, then by definition,

$$\mu(E) = \mu^*(E) = \inf \{\mu(U),\, U \supseteq E,\, U \in \mathcal{T}\}$$

And for every open U, (1) follows from Equation (3). ■

Part h

Proof. We want to show that for every compact K, Equation (2) holds. To reduce the notational baggage that follows, we agree to define

$$\{I(f),\, f \in \mathrm{C}_c(X),\, f \prec U\} = \{I(f),\, f \prec U\}$$

Similarily for $\{I(f),\, f \geq \chi_K\}$. If $\mu(K) = 0$, then $\mu(K)$ is obviously a lower bound, since $f \geq \chi_K \geq 0$ means that $I(f) \geq 0$, for every $f \geq \chi_K$. So we can suppose $\mu(K) > 0$.

Fix an arbitrary $f \geq \chi_K$, then this particular f induces an open set $U_\alpha = \{f > 1 - \alpha\}$, where $\alpha > 0$. Notice also that

$$K \subseteq \{f \geq 1\} \subseteq \{f > 1 - \alpha\} = U_\alpha$$

Since U_α is an open superset of K, by Equation (7), $\mu(K) \leq \mu(U_\alpha)$, but $\mu(U_\alpha)$ is simply the supremum of $\{I(g),\, g \prec U_\alpha\}$. If we wish to show that $\mu(K) \leq \mu(U_\alpha) \leq I(f)$, it suffices to show that $I(f)$ is an upper-bound for $\{I(g),\, g \prec U_\alpha\}$.

Fix any $I(g) \in \{I(g),\, g \prec U_\alpha\}$, note that $1 - \alpha \neq 0$ for any α small enough, then

- $f/(1-\alpha) > 1$ on U_α,
- $1 \geq g \geq 0$ on U_α, in particular, $f/(1-\alpha) - g \geq 0$ on U_α,
- If $x \notin U_\alpha$, then $f/(1-\alpha) - g = f(1-\alpha) \geq 0$.
- Therefore $f/(1-\alpha) - g \geq 0$ for any x, and by Lemma 2.1,

$$I(f/(1-\alpha)) \geq I(g) \quad \forall g \prec U_\alpha$$

Combining the above estimate with $\mu(K) \leq \mu(U_\alpha)$ gives us

$$\mu(K) \leq \frac{1}{1-\alpha} I(f)$$

Now write $\varepsilon = \alpha/\mu(K) > 0$ and for every $\varepsilon > 0$ we get

$$\mu(K) - I(f) \leq \alpha\mu(K) = \varepsilon$$

Send $\varepsilon \to 0$ and $\mu(K) \leq I(f)$ for every $f \geq \chi_K$.

To show that $\mu(K)$ is indeed the infimum for $\{I(f),\, f \geq \chi_K\}$, notice that for every $\varepsilon > 0$ we can obtain some open superset $U \supseteq K$ (by outer-regularity) where $\mu(U) < \mu(K) + \varepsilon$. By Urysohn's Lemma, there exists some $g \prec U$, $g(x) = 1$ for every $x \in K$.

$$g \in \{I(f),\, f \prec U\} \cap \{I(f),\, f \geq \chi_K\}$$

Therefore $I(g) \leq \mu(U) < \mu(K) + \varepsilon$ as desired, and Equation (2) holds. ■

Part i

Proof. $\mu(K) < +\infty$ for every compact K. Indeed, since $I(\chi_K) \in \{I(f),\, f \geq \chi_K\}$, then by Theorem 7.1, there exists a constant $C_K \geq 0$ that bounds

$$\mu(K) \leq |I(\chi_K)| = I(\chi_K) \leq C_K \cdot \|\chi_K\| = C_K < +\infty$$

■

Part j

Proof. Fix any open set U, then for every $\varepsilon > 0$, there exists some $f \prec U$ with $\mu(U) - \varepsilon < I(f)$. Then denote $K = \operatorname{supp}(f) \subseteq U$. If we take any $I(h) \in \{I(h),\ h \geq \chi_K\}$, then $h \geq f$ gives us $I(h) \geq I(f)$ by Lemma 2.1. So $I(f)$ is a lower bound of $\{I(h),\ h \geq \chi_K\}$, therefore

$$\mu(U) - \varepsilon \leq I(f) \leq \mu(K)$$

Since $\operatorname{supp}(f) = K \subseteq U$, this proves inner-regularity of μ on open sets. ■

Part k

Proof. Suppose $f \in \mathrm{C}_c(X, [0,1])$, we first show that Equation (8) holds. We divide the interval $[0,1]$ into $N \geq 1$ chunks by writing

$$K_j = \{f \geq j/N\}$$

for every $1 \geq j \geq N$. And define $K_0 = \operatorname{supp}(f)$. Each K_j is a closed subset of $\operatorname{supp}(f)$, and therefore compact. More is true,

- $K_{j-1} \supseteq K_j$ for every $1 \leq j \leq N$.
- $x \in K_j$ iff $f(x) \in \left[\frac{j}{N},\ 1\right]$,
- $x \notin K_j$ iff $f(x) \in \left[0,\ \frac{j}{N}\right)$, and
- $x \in (K_{j-1} \setminus K_j)$ iff $f(x) \in \left[\frac{j-1}{N},\ \frac{j}{N}\right)$

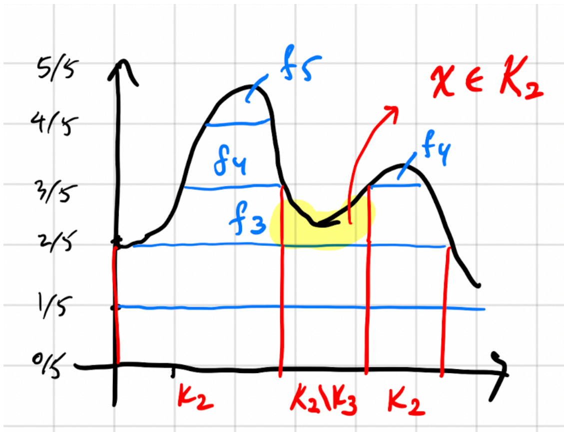
Folland constructs a finite sequence of compactly supported functions, $\{f_j\}$, where $1 \leq j \leq N$ such that

- Each $0 \leq f_j \leq 1/N$,
- If $x \in (K_m \setminus K_{m+1})$ iff $f(x) \in \left[\frac{m}{N},\ \frac{m+1}{N}\right)$ means that $f_j = 1$ for all $1 \leq j \leq m$, and
- $f_{m+1} = f - m/N$ on K_m, such that

$$f(x) = \left(\sum f_{j \leq m}(x)\right) + \left(f(x) - \frac{m}{N}\right) = \frac{m}{N} + \left(f(x) - \frac{m}{N}\right)$$

- And for every $m < j \leq N$, $f_j = 0$.
- If $x \notin K_m$ iff $f(x) \in \left[0, \frac{m}{N}\right)$ then for every $m+1 \leq j \leq N$, $f_j = 0$.

The illustration for when $N = 5$ below should make things clearer.



It is also trivial to verify that

- For every $x \in K_j$, $f_j = N^{-1}$, and
$$\chi_{K_j} N^{-1} \leq f_j \tag{14}$$
Also, if $x \notin K_j$ then $f_j \geq 0$, therefore $f_j \geq \chi_{K_j} N^{-1}$ at every x.
- If $x \notin K_{j-1}$ then $f_j = 0 \leq \chi_{K_{j-1}} \cdot N^{-1}$. If x is in K_{j-1} then $f_j \leq N^{-1}$ by construction and therefore
$$f_j \leq \chi_{K_{j-1}} N^{-1} \tag{15}$$
for all x.
- $f_j \in \mathrm{C}_c(X)$, since $\operatorname{supp}(f_j) \subseteq \operatorname{supp}(f)$.

Combining Equations (14) with (15), and by monotonicity in $L^+(X, \mathbb{B}, \mu)$, since $f_j \in L^+$
$$\int \frac{1}{N} \chi_{K_j} d\mu \leq \int f_j d\mu \leq \int \frac{1}{N} \chi_{K_{j-1}} d\mu$$
And for every $1 \leq j \leq N$,
$$\frac{1}{N} \mu(K_j) \leq \int f_j d\mu \leq \frac{1}{N} \mu(K_{j-1}) \tag{16}$$

Furthermore, from Equation (14), since $Nf_j \geq \chi_{K_j}$ then by Equation (2),

$$\mu(K_j) \leq I(Nf_j) \implies \frac{1}{N}\mu(K_j) \leq I(f_j)$$

Now for any arbitrary $I(h) \in \{I(h),\ h \geq \chi_{K_{j-1}}\}$, since

$$h \geq \chi_{K_{j-1}} \geq Nf_j \implies I(h) \geq I(Nf_j)$$

So $NI(f_j)$ is a lower bound for $\{I(h),\ h \geq \chi_{K_{j-1}}\}$ and

$$I(f_j) \leq \frac{1}{N}\mu(K_{j-1})$$

Combining the last two results, with $I(f_j)$, we get

$$\frac{1}{N}\mu(K_j) \leq I(f_j) \leq \frac{1}{N}\mu(K_{j-1}) \tag{17}$$

Taking the sum over $1 \leq j \leq N$ for Equations (16) and (17). Define $A = N^{-1}\sum_0^{N-1}\mu(K_j)$, and $B = N^{-1}\sum_1^N \mu(K_j)$

$$B \leq \int f d\mu \leq A$$

And also

$$B \leq I(f) \leq A$$

This is because of finite additivity of both I and the integral, and $f = \sum f_j$ on $K_0 = \operatorname{supp}(f)$. Subtracting the two equations (keeping in mind that $\mu(K_j) < +\infty$ for any compact K_j), we get

$$(-1)(A-B) \leq \left(\int f d\mu - I(f)\right) \leq A - B \implies \left|\int f d\mu - I(f)\right| \leq A - B$$

It is trivial to verify that

$$0 \leq A - B = N^{-1}(\mu(K_0) - \mu(K_N)) \leq N^{-1}\mu(K_0)$$

as $K_N \subseteq K_0$. Let $N \to \infty$ and

$$\int f d\mu = I(f)$$

Equation (8) holds as desired. ■

Part 1

Proof. Now for any general $f \in \mathrm{C}_c(X)$, f must be bounded on the plane since $\mathrm{C}_c(X) \subseteq \mathrm{BC}(X)$, and $|f| \leq M_0$ for some $M_0 \geq 0$. Since $\operatorname{supp}(f)$ is compact, we know that

$$\int |f| d\mu \leq \int M_0 \chi_{\operatorname{supp}(f)} d\mu \leq M_0 \mu(\operatorname{supp}(f)) < +\infty$$

And $\mathrm{C}_c(X) \subseteq L^1(\mu)$. Furthermore,

$$\frac{1}{2}(|\operatorname{Re} f| + |\operatorname{Im} f|) \leq |f| \leq M_0$$

So that $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $\mathrm{C}_c(X)$. Without loss of generality, we may assume that f is real. Define $f_1 = \operatorname{Re} f^+ / M_0$ and $f_2 = \operatorname{Re} f^- / M_0$ and it immediately follows that $f_1, f_2 \in \mathrm{C}_c(X, [0, 1])$.

By linearity of I on $\mathrm{C}_c(X)$ and the integral in $L^1(\mu)$,

$$I(f_1 - f_2) = I(f) = \int f d\mu = \int f_1 d\mu - \int f_2 d\mu$$

Then we may apply the above to the real and imaginary parts of a general $f \in \mathrm{C}_c(X)$, and this completes the proof. ■

Theorem 7.3

Proposition 3.1. *See Theorem 7.2*

Proof. ■

Theorem 7.4

Proposition 4.1. *See Theorem 7.2*

Proof. ■

Theorem 7.5

Proposition 5.1.

Proof. ■

Theorem 7.6

Proposition 6.1.

Proof. ■

Theorem 7.7

Proposition 7.1.

Proof. ■

Theorem 7.8

Proposition 8.1.

Proof. ∎

Theorem 7.9

Proposition 9.1. *If μ is a Radon measure on X, then $\mathrm{C}_c(X)$ is dense in $L^p(\mu)$ for $1 \leq p < +\infty$.*

Proof. Theorem 6.7 tells us that the set of L^p simple functions (as Folland calls them), which are

$$\Lambda = \left\{ f,\ f = \sum_{j \leq n} a_j \chi_{E_j},\ a_j \in \mathbb{C},\ \mu(E_j) < +\infty \right\}$$

So for every $f \in L^p$, there exists a sequence $\{f_n\} \subseteq \Lambda$ with $f_n \to f$ pointwise and $f_n \to f$ in L^p. ∎

Theorem 7.10

Proposition 10.1.

Proof. ■

Theorem 7.11

Proposition 11.1.

Proof. ■