# Chapter 22: Symplectic Manifold

Manifolds Symplectic Tensors

# Symplectic Tensors

## Definition 1.1: Billinear forms

Let V be a vector space, a billinear form  $\omega: V \times V \to \mathbb{R}$  is a 2-tensor on V.

### Definition 1.2: Characterization of billinear forms

Let  $\omega$  be a billinear form on V, it is

• symmetric if

$$\omega(x,y) = \omega(y,x)$$

• skew-symmetric or anti-symmetric if

$$\omega(x,y) = (-1)\omega(y,x)$$

• alternating if

$$\omega(x,x)=0$$

If V is a vector space over the field F and  $\operatorname{char}(F) \neq 2$ , then the last two conditions are equivalent. Moreover,

- V is called an orthogonal geometry if  $\omega$  is symmetric.
- V is called a symplectic geometry if  $\omega$  is alternating.

# Definition 1.3: Metric vector space

A vector space (not necessarily finite dimensional) is called a *metric vector space* if it is a orthogonal or symplectic geometry.

## Matrices and billinear forms

### Definition 2.1: Matrix of billinear form

If  $B = (b_1, \ldots, b_n)$  is an ordered basis for V, we define the matrix representation of  $\omega$  by

$$\mathcal{M}(\omega) = (a_{ij}) = (\omega(b_i,b_j))$$

# Proposition 2.1: Matrix induces a billinear form

Let  $A = (a_{ij})$  be a matrix on V with respect to some basis  $B = (b_{\underline{n}})$  it is clear that A induces a billinear form, on V through  $A(x,y) = [x]_B^T A[y]_B$ , where  $[\cdot]_B$  denotes the canonical isomorphism  $V \cong \mathbb{R}^n$  with respect to the basis B.

$$[x]_B^T A[y]_B = egin{bmatrix} x^1 & \dots & x^n \end{bmatrix} A egin{bmatrix} y^1 \ dots \ y^n \end{bmatrix}$$

for  $x = x^i b_i$  and  $y = y^j b_j$ .

Moreover,

$$A[x]_B = egin{bmatrix} A(b_1,x) \ dots \ A(b_n,x) \end{bmatrix} egin{array}{ll} ext{is a } column ext{ vector} \ ext{whose entries are given} \ ext{by applying } x ext{ on the} \ ext{second coordinate} \end{array}$$

and

$$[x]_B^T A = \begin{bmatrix} A(x,b_1) & \cdots & A(x,b_n) \end{bmatrix}$$
 is a row vector whose entries are given by applying  $x$  on the first coordinate

Let  $A_B$  be the matrix representation of  $\omega$  with respect to the B, if C is another basis on V, then how do we compute  $A_C$ ? The answer is simple, recall for any vector  $x \in V$ ,  $x = x_B^i b_i$  and  $x = x_C^j c_j$ , then

 $[x]_B = M_{C,B}[x]_C$  for some matrix of an automorphism  $M_{C,B}$ 

$$\omega(x,y) = [x]_B^T A_B[y]_B = ([x]_C^T M_{C,B}^T) A_B(M_{C,B}[y]_C) = [x]_C^T A_C[y]_C,$$
 then

$$M_{C,B}^T A_B M_{C,B} = A_C (1)$$

We can describe this relation between the two matrices  $A_B$  and  $A_C$  by the following

## Definition 2.2: Congruent matrices

Two matrices M and N are said to be *congruent*, if there exists an invertible matrix P for which

$$P^TMP = N$$

Congruence is an equivalence relation on the space of matrices, and the equivalence classes over congruence are called *congruence classes*.

## Proposition 2.2: Characterization of matrices using congruence

Let  $A_1$  and  $A_2$  be matrix representations of two billinear forms with respect to the basis B.

$$A_1 = (A_1(b_i, b_j))_{ij}$$
  $A_2 = (A_2(b_i, b_j))_{ij}$ 

They induce the same billinear form if and only if they are congruent.

## Definition 2.3: Alternate matrices

Let M be a matrix with F-coefficients, it is alternate if it is skew symmetric and is hollow; meaning it

Manifolds Orthogonality

has 0s on the main diagonal. If  $F = \mathbb{R}$  or  $\operatorname{char}(F) \neq 2$ , then alternate matrices are and are precisely the skew-symmetric matrices.

# Orthogonality

For this section,  $(V, \omega)$  will denote a metric vector space, not necessarily finite-dimensional unless we are using matrix representations.

## Definition 3.1: Orthogonal complements

A vector  $x \in V$  is orthogonal to another vector  $y \in V$ , written  $x \perp y$ , if  $\omega(x,y) = 0$ .

If V is an orthogonal or symplectic geometry then  $\bot$  is a symmetric relation. If E is a subset of V, we denote the *orthogonal complement of* E by

$$E^{\perp} \stackrel{\Delta}{=} \left\{ v \in V, \ v \perp E 
ight\}$$

# Definition 3.2: Characterization of metric vector spaces

- A nonzero vector  $x \in V$  is *isotropic*, or null if  $\omega(x,x) = 0$
- ullet V is isotropic if it contains at least one isotropic vector.
- V is anisotropic or nonisotropic if for every  $x \in V$ ,  $\omega(x,x) = 0 \implies x = 0$ ,
- V is totally isotropic (that is, symplectic if  $\operatorname{char}(F) \neq 2$ ) if  $\omega(x,x) = 0$  for every vector  $x \in V$ . The first bullet point above is about vectors in V, while the others are properties of V.
  - A vector  $x \in V$  is called degenerate if  $x \perp V$ , that is,

$$\forall y \in V, \ \omega(x,y) = 0$$

• The radical of V, denoted by rad(V) is the set of all degenerate vectors in V,

$$\mathrm{rad}(V) \stackrel{\Delta}{=} V^{\perp}$$

- V is singular or degenerate if  $rad(V) \neq \{0\}$ ,
- V is non-singular or non-degenerate if  $rad(V) = \{0\},\$
- V is totally singular, if rad(V) = V.

To summarize,

- V is isotropic if there exists a non-zero isotropic vector, meaning  $\omega(x,x)=0$ , for some  $x\neq 0$ ,
- V is degenerate if there exists a degenerate vector,  $x \perp V$ .

# Proposition 3.1: Matrix invariants under congruence

Non-singularity, symmetry, and skew-symmetry are invariants under congruence.

Proof.

# Proposition 3.2: Characterization of non-degeneracy

V is non-degenerate if and only if every matrix representation A of  $\omega$  is non-singular.

*Proof.* Suppose V is non-degenerate, then let  $B = (b_{\underline{n}})$  be a basis for V, if A is the matrix representation of  $\omega$  with respect to B, let x be a non-zero vector in V, so  $x \notin \operatorname{rad}(V)$ 

$$b_i^T A[x]_B = \omega(b_i, x) \neq 0 \implies A[x]_B \neq 0$$

so A is non-singular. If A' is another matrix representation with respect to another basis C, by Equation (1) A' is non-singular as well.

Conversely, if every matrix representation of  $\omega$  is non-singular, let x be a non-zero vector in V, then  $A[x]_B \neq 0$  is a non-zero vector so there exists some basis component  $(A[x]_B)^j$  that is non zero, and

$$[b_j]_B^T A[x]_B = \omega(b_j, x) \neq 0$$

therefore V is non-degenerate.

### Proposition 3.3: Characterisation of billinear forms from matrix representations

Let  $\omega$  be a billinear form on V, if  $\mathcal{M}(\omega)$  the induced matrix representation relative to any basis. Assume V is a vector space over  $\mathbb{R}$ , then

- it is symmetric iff  $\mathcal{M}(\omega)$  symmetric as a matrix,
- it is skew-symmetric, iff alternating iff  $\mathcal{M}(\omega)$  is skew-symmetric as a matrix.

## Corollary 3.1: Characterisation of non-singular symplectic form

Let  $(V, \omega)$  be a finite dimensional vector space over  $\mathbb{R}$ , equipped with a billinear form  $\omega$ .  $(V, \omega)$  is a non-singular symplectic vector space iff the matrix representation of  $\omega$  with respect to every basis is non-singular and skew-symmetric.

### Riesz Representation Theorems

# Proposition 4.1

Let  $(V,\omega)$  be a nonsingular metric vector space, the map  $x\mapsto x\lrcorner\omega\in V^*$  defined by

$$x\lrcorner\omega=\omega(x,\cdot),\quad ext{and}\quad (x\lrcorner\omega)(y)=\omega(x,y),\quad \forall y\in V$$

Manifolds Isometries

is a linear isomorphism from V to  $V^*$ .

## Isometries

## Definition 5.1: Isometry between MVS

Let  $(V, \omega)$  and  $(W, \eta)$  be metric vector spaces. An isometry  $\tau \in L(V, W)$  is a linear isomorphism that preserves the billinear form.

$$\omega(u,v)=\eta(\tau u,\tau v)$$

## Definition 5.2: Orthogonal, symplectic groups

Let V be a nonsingular metric vector space. If V is an orthogonal (resp. symplectic) geometry, the set of all isometries on V is called the *orthogonal (resp. symplectic) group on* V. It is a group under composition, and is denoted by  $\mathcal{O}(V)$  (resp.  $\mathrm{Sp}(V)$ ).

# Hyperbolic spaces, nonsingular completions

**Canonical Forms** 

Symplectic Manifolds

Darboux's Theorem

## Proposition 9.1: Lie Derivatives of Tensor Fields (along time-varying vector fields)

Let M be a smooth manifold. Suppose  $V: J \times M \to TM$  is a smooth time-varying vector field on M. Denote the time-varying flow of V by  $\psi: \mathcal{E} \to M$ . Let  $A \in \mathcal{T}^k(M)$  be a smooth time-invariant covariant k-tensor field on M. For every  $(t_1, t_0, p) \in \mathcal{E}$ ,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=t_1} (\psi_{t,t_0}^* A)_p = (\psi_{t_1,t_0}^* (\mathcal{L}V_{t_1} A))_p$$
(2)

# Chapter Hofer book

Manifolds Darboux's Theorem

# Definition 0.1: Symplectic vector space

Let V be a finite dimensional vector space over  $\mathbb{R}$ . It is a *symplectic vector space* if it admits a non-singular, antisymmetric billinear form  $\omega: V \times V \to \mathbb{R}$ .

$$\omega(u,v) = -\omega(v,u)$$

for  $u,v\in V.$  By the previous section on Riesz Representation, the linear map

$$\hat{\omega}: V \to V^*, \quad v \mapsto \omega(v, \cdot)$$

is a linear isomorphism of V onto its dual vector  $V^*$ .

We define the standard symplectic vector space  $(\mathbb{R}^{2n}, \omega_0)$ , where  $n \in \mathbb{N}^+$ , where

$$\omega_0(u,v) = < Ju, v> \quad J \stackrel{\Delta}{=} egin{bmatrix} 0 & I_n \ -I_n & 0 \end{bmatrix}$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product on  $\mathbb{R}^{2n}$ .

$$\omega_0(u, v) = \langle Ju, v \rangle = \langle u, J^T v \rangle = u^T J^T v$$
 (3)

 $J^T = -J$  by Corollary 3.1.

We will mainly deal with non-singular symplectic forms because of Riesz isomorphism.

## Definition 0.2: Symplectic linear map

Let  $(V, \omega)$  be a symplectic vector space. A linear map  $F \in \text{Hom}(V)$  is *symplectic* if it preserves symplectic form  $\omega$ . For every  $u \in V$ ,

$$< u, v> = < Au, Av> \stackrel{\triangle}{=} A^*\omega(u, v)$$

where  $A^*: \mathcal{T}^*(V) \to \mathcal{T}^*(V)$  denotes the tensor pullback by precomposing any tensor S by A

$$orall S \in \mathcal{T}^k(V), \quad A^*S(v_{\underline{k}}) \stackrel{\Delta}{=} S(Av_{\underline{k}})$$

The set of linear symplectic maps on a 2n-dimensional vector space form a group under composition. It is a Lie Group denoted by Sp(n).

## Proposition 0.2: Symplectic Maps are Area-preserving

Let  $(\mathbb{R}^{2n}, \omega_0)$  denote the standard symplectic space. If  $\varphi \in \operatorname{Sp}(n)$ , then  $\det \varphi = 1$ .

*Proof.* See page 4.

Manifolds Darboux's Theorem

$$(\Lambda_{s-|\alpha|}\partial^{\alpha}f)^{\hat{}} = (1+|\zeta|^2)^{s/2-|\alpha|/2} \cdot (\partial^{\alpha}f)^{\hat{}}$$

$$\tag{4}$$

$$= (1 + |\zeta|^2)^{s/2 - |\alpha|/2} \cdot (2\pi i \zeta)^{\alpha} \cdot \hat{f}$$

$$\tag{5}$$

$$= (2\pi i)^{|\alpha|} (1 + |\zeta|^2)^{(s-|\alpha|)/2} \cdot |\zeta|^{|\alpha|} \cdot \hat{f}$$
 (6)

$$\leq |\alpha|(1+|\zeta|^2)^{s/2}\hat{f} \tag{7}$$