

Chapter 22: Symplectic Manifold

Symplectic Tensors

Definition 1.1: Bilinear forms

Let V be a vector space, a *bilinear form* $\omega : V \times V \rightarrow \mathbb{R}$ is a 2-tensor on V .

Definition 1.2: Characterization of bilinear forms

Let ω be a bilinear form on V , it is

- *symmetric* if

$$\omega(x, y) = \omega(y, x)$$

- *skew-symmetric* or *anti-symmetric* if

$$\omega(x, y) = (-1)\omega(y, x)$$

- *alternating* if

$$\omega(x, x) = 0$$

If V is a vector space over the field F and $\text{char}(F) \neq 2$, then the last two conditions are equivalent. Moreover,

- V is called an *orthogonal geometry* if ω is symmetric.
- V is called a *symplectic geometry* if ω is alternating.

Definition 1.3: Metric vector space

A vector space (not necessarily finite dimensional) is called a *metric vector space* if it is a orthogonal or symplectic geometry.

Matrices and bilinear forms

Definition 1.4: Matrix of bilinear form

If $B = (b_1, \dots, b_n)$ is an ordered basis for V , we define the *matrix representation of ω* by

$$\mathcal{M}(\omega) = (a_{ij}) = (\omega(b_i, b_j))$$

Proposition 1.1: Matrix induces a bilinear form

Let $A = (a_{ij})$ be a matrix on V with respect to some basis $B = (b_n)$ it is clear that A induces a bilinear form, on V through $A(x, y) = [x]_B^T A [y]_B$, where $[\cdot]_B$ denotes the canonical isomorphism $V \cong \mathbb{R}^n$ with respect to the basis B .

$$[x]_B^T A [y]_B = \begin{bmatrix} x^1 & \dots & x^n \end{bmatrix} A \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}$$

for $x = x^i b_i$ and $y = y^j b_j$.

Moreover,

$$A[x]_B = \begin{bmatrix} A(b_1, x) \\ \vdots \\ A(b_n, x) \end{bmatrix}$$

is a *column* vector
whose entries are given
by applying x on the
second coordinate

and

$$[x]_B^T A = \begin{bmatrix} A(x, b_1) & \dots & A(x, b_n) \end{bmatrix}$$

is a *row* vector whose
entries are given by ap-
plying x on the first co-
ordinate

Let A_B be the matrix representation of ω with respect to the B , if C is another basis on V , then how do we compute A_C ? The answer is simple, recall for any vector $x \in V$, $x = x_B^i b_i$ and $x = x_C^j c_j$, then

$$[x]_B = M_{C,B} [x]_C \quad \text{for some matrix of an automorphism } M_{C,B}$$

$$\omega(x, y) = [x]_B^T A_B [y]_B = ([x]_C^T M_{C,B}^T) A_B (M_{C,B} [y]_C) = [x]_C^T A_C [y]_C, \text{ then}$$

$$M_{C,B}^T A_B M_{C,B} = A_C \tag{1}$$

We can describe this relation between the two matrices A_B and A_C by the following

Definition 1.5: Congruent matrices

Two matrices M and N are said to be *congruent*, if there exists an invertible matrix P for which

$$P^T M P = N$$

Congruence is an equivalence relation on the space of matrices, and the equivalence classes over congruence are called *congruence classes*.

Proposition 1.2: Characterization of matrices using congruence

Let A_1 and A_2 be matrix representations of two bilinear forms with respect to the basis B .

$$A_1 = (A_1(b_i, b_j))_{ij} \quad A_2 = (A_2(b_i, b_j))_{ij}$$

They induce the same bilinear form if and only if they are congruent.

Definition 1.6: Alternate matrices

Let M be a matrix with F -coefficients, it is *alternate* if it is skew symmetric and is *hollow*; meaning it

has 0s on the main diagonal. If $F = \mathbb{R}$ or $\text{char}(F) \neq 2$, then alternate matrices are and are precisely the skew-symmetric matrices.

Orthogonality

For this section, (V, ω) will denote a metric vector space, not necessarily finite-dimensional unless we are using matrix representations.

Definition 1.7: Orthogonal complements

A vector $x \in V$ is orthogonal to another vector $y \in V$, written $x \perp y$, if $\omega(x, y) = 0$.

If V is an orthogonal or symplectic geometry then \perp is a symmetric relation. If E is a subset of V , we denote the *orthogonal complement of E* by

$$E^\perp \triangleq \{v \in V, v \perp E\}$$

Definition 1.8: Characterization of metric vector spaces

- A nonzero vector $x \in V$ is *isotropic*, or *null* if $\omega(x, x) = 0$
- V is *isotropic* if it contains at least one isotropic vector.
- V is *anisotropic* or *nonisotropic* if for every $x \in V$, $\omega(x, x) = 0 \implies x = 0$,
- V is *totally isotropic* (that is, symplectic if $\text{char}(F) \neq 2$) if $\omega(x, x) = 0$ for every vector $x \in V$.

The first bullet point above is about vectors in V , while the others are properties of V .

- A vector $x \in V$ is called *degenerate* if $x \perp V$, that is,

$$\forall y \in V, \omega(x, y) = 0$$

- The *radical* of V , denoted by $\text{rad}(V)$ is the set of all degenerate vectors in V ,

$$\text{rad}(V) \triangleq V^\perp$$

- V is *singular* or *degenerate* if $\text{rad}(V) \neq \{0\}$,
- V is *non-singular* or *non-degenerate* if $\text{rad}(V) = \{0\}$,
- V is *totally singular*, if $\text{rad}(V) = V$.

To summarize,

- V is isotropic if there exists a non-zero isotropic vector, meaning $\omega(x, x) = 0$, for some $x \neq 0$,
- V is degenerate if there exists a degenerate vector, $x \perp V$.

Proposition 1.3: Matrix invariants under congruence

Non-singularity, symmetry, and skew-symmetry are invariants under congruence.

Proof. ■

Proposition 1.4: Characterization of non-degeneracy

V is non-degenerate if and only if every matrix representation A of ω is non-singular.

Proof. Suppose V is non-degenerate, then let $B = (b_n)$ be a basis for V , if A is the matrix representation of ω with respect to B , let x be a non-zero vector in V , so $x \notin \text{rad}(V)$

$$b_i^T A[x]_B = \omega(b_i, x) \neq 0 \implies A[x]_B \neq 0$$

so A is non-singular. If A' is another matrix representation with respect to another basis C , by Equation (1) A' is non-singular as well.

Conversely, if every matrix representation of ω is non-singular, let x be a non-zero vector in V , then $A[x]_B \neq 0$ is a non-zero vector so there exists some basis component $(A[x]_B)^j$ that is non zero, and

$$[b_j]_B^T A[x]_B = \omega(b_j, x) \neq 0$$

therefore V is non-degenerate. ■

Proposition 1.5: Characterisation of bilinear forms from matrix representations

Let ω be a bilinear form on V , if $\mathcal{M}(\omega)$ the induced matrix representation relative to any basis. Assume V is a vector space over \mathbb{R} , then

- it is symmetric iff $\mathcal{M}(\omega)$ symmetric as a matrix,
- it is skew-symmetric, iff alternating iff $\mathcal{M}(\omega)$ is skew-symmetric as a matrix.

Corollary 1.1: Characterisation of non-singular symplectic form

Let (V, ω) be a finite dimensional vector space over \mathbb{R} , equipped with a bilinear form ω . (V, ω) is a non-singular symplectic vector space iff the matrix representation of ω with respect to every basis is non-singular and skew-symmetric.

Riesz Representation Theorems

Proposition 1.6

Let (V, ω) be a nonsingular metric vector space, the map $x \mapsto x \lrcorner \omega \in V^*$ defined by

$$x \lrcorner \omega = \omega(x, \cdot), \quad \text{and} \quad (x \lrcorner \omega)(y) = \omega(x, y), \quad \forall y \in V$$

is a linear isomorphism from V to V^* .

Isometries

Definition 1.9: Isometry between MVS

Let (V, ω) and (W, η) be metric vector spaces. An *isometry* $\tau \in L(V, W)$ is a linear isomorphism that preserves the bilinear form.

$$\omega(u, v) = \eta(\tau u, \tau v)$$

Definition 1.10: Orthogonal, symplectic groups

Let V be a nonsingular metric vector space. If V is an orthogonal (resp. symplectic) geometry, the set of all isometries on V is called the *orthogonal* (resp. *symplectic*) *group on V* . It is a group under composition, and is denoted by $\mathcal{O}(V)$ (resp. $\text{Sp}(V)$).

Hyperbolic spaces, nonsingular completions

Canonical Forms

Symplectic Manifolds

Darboux's Theorem

Proposition 1.7: Lie Derivatives of Tensor Fields (along time-varying vector fields)

Let M be a smooth manifold. Suppose $V : J \times M \rightarrow TM$ is a smooth time-varying vector field on M . Denote the time-varying flow of V by $\psi : \mathcal{E} \rightarrow M$. Let $A \in \mathcal{T}^k(M)$ be a smooth time-invariant covariant k -tensor field on M . For every $(t_1, t_0, p) \in \mathcal{E}$,

$$\left. \frac{d}{dt} \right|_{t=t_1} (\psi_{t,t_0}^* A)_p = (\psi_{t_1,t_0}^* (\mathcal{L}V_{t_1} A))_p \quad (2)$$

Chapter Hofer book

Definition 2.1: Symplectic vector space

Let V be a finite dimensional vector space over \mathbb{R} . It is a *symplectic vector space* if it admits a non-singular, antisymmetric bilinear form $\omega : V \times V \rightarrow \mathbb{R}$.

$$\omega(u, v) = -\omega(v, u)$$

for $u, v \in V$. By the previous section on Riesz Representation, the linear map

$$\hat{\omega} : V \rightarrow V^*, \quad v \mapsto \omega(v, \cdot)$$

is a linear isomorphism of V onto its dual vector V^* .

We define the *standard symplectic vector space* $(\mathbb{R}^{2n}, \omega_0)$, where $n \in \mathbb{N}^+$, where

$$\omega_0(u, v) = \langle Ju, v \rangle \quad J \triangleq \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^{2n} .

$$\omega_0(u, v) = \langle Ju, v \rangle = \langle u, J^T v \rangle = u^T J^T v \quad (3)$$

$J^T = -J$ by Corollary 1.1.

We will mainly deal with non-singular symplectic forms because of Riesz isomorphism.

Definition 2.2: Symplectic linear map

Let (V, ω) be a symplectic vector space. A linear map $F \in \text{Hom}(V)$ is *symplectic* if it preserves symplectic form ω . For every $u \in V$,

$$\langle u, v \rangle = \langle Au, Av \rangle \triangleq A^* \omega(u, v)$$

where $A^* : \mathcal{T}^*(V) \rightarrow \mathcal{T}^*(V)$ denotes the tensor pullback by precomposing any tensor S by A

$$\forall S \in \mathcal{T}^k(V), \quad A^* S(v_{\underline{k}}) \triangleq S(Av_{\underline{k}})$$

The set of linear symplectic maps on a $2n$ -dimensional vector space form a group under composition. It is a Lie Group denoted by $\text{Sp}(n)$.

Proposition 2.1: Symplectic Maps are Area-preserving

Let $(\mathbb{R}^{2n}, \omega_0)$ denote the standard symplectic space. If $\varphi \in \text{Sp}(n)$, then $\det \varphi = 1$.

Proof. See page 4. ■

$$(\Lambda_{s-|\alpha|} \partial^\alpha f)^\wedge = (1 + |\zeta|^2)^{s/2-|\alpha|/2} \cdot (\partial^\alpha f)^\wedge \quad (4)$$

$$= (1 + |\zeta|^2)^{s/2-|\alpha|/2} \cdot (2\pi i \zeta)^\alpha \cdot \hat{f} \quad (5)$$

$$= (2\pi i)^{|\alpha|} (1 + |\zeta|^2)^{(s-|\alpha|)/2} \cdot |\zeta|^{|\alpha|} \cdot \hat{f} \quad (6)$$

$$\leq |\alpha| (1 + |\zeta|^2)^{s/2} \hat{f} \quad (7)$$