MATH 263: Section 003, Tutorial 12

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1 The Laplace Transform

The **Laplace Transform** is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt$$

where the integral is defined.

Note that the **Laplace Transform** is an **integral Transform**, meaning that it is linear (due to the properties of the integral). Therefore, for all $a, b \in \mathbb{R}$:

$$\mathcal{L}\{a\ f(t) + b\ g(t)\} = a\ \mathcal{L}\{f(t)\} + b\ \mathcal{L}\{g(t)\}$$

Problem 1a. From Boyce and DiPrima, 10th edition (6.1, exercise 9, p.315):
Find the Laplace Transform of:

$$f(t) = e^{at}\cosh bt$$

Solution: (Review of hyperbolic trigonometric functions) Recall that

$$\cosh bt = \frac{1}{2}(e^{bt} + e^{-bt})$$

$$\sinh bt = \frac{1}{2}(e^{bt} - e^{-bt})$$

$$F(s) = \int_0^\infty e^{-st}\ e^{at}\ \cosh bt\ dt$$

$$F(s) = \int_0^\infty \frac{1}{2}\ e^{-st}\ e^{at}\ (e^{bt} + e^{-bt})\ dt$$

$$F(s) = \int_0^\infty \frac{1}{2}\ (e^{-(s-a-b)t} + e^{-(s-a+b)t})\ dt$$

$$F(s) = \frac{1}{2}\int_0^\infty e^{-(s-a-b)t}\ dt + \frac{1}{2}\int_0^\infty e^{-(s-a+b)t}\ dt$$

$$F(s) = \frac{-1}{2(s-a-b)}\ e^{-(s-a-b)t}\Big|_{t=0}^{\infty} - \frac{1}{2(s-a+b)}\ e^{-(s-a+b)t}\Big|_{t=0}^{\infty}$$

$$F(s) = \frac{1}{2(s-a-b)} + \frac{1}{2(s-a+b)}$$

$$F(s) = \frac{s-a}{(s-a)^2 - b^2}$$

Problem 1b. From Boyce and DiPrima, 10th edition (6.1, exercise 22, p.315):
Find the Laplace Transform of:

$$f(t) = \begin{cases} t & 0 \leq t < 1 \\ 0 & t \geq 1 \end{cases}$$

Solution:

$$F(s) = \int_0^\infty e^{-st}\ f(t)\ dt$$

We can split the piece-wise function into intervals to find its Laplace Transform:

$$F(s) = \int_0^1 te^{-st}\ dt + \int_1^\infty 0\ e^{-st}\ dt$$

$$F(s) = \int_0^1 te^{-st}\ dt$$

Integrating by parts, we get:

$$F(s) = e^{-st}(-\frac{t}{s} - \frac{1}{s^2})\Big|_{t=0}^{1}$$

$$F(s) = e^{-s}(-\frac{1}{s} - \frac{1}{s^2}) - e^{-0}(-\frac{0}{s} - \frac{1}{s^2})$$

$$F(s) = \frac{1}{s^2} - \frac{e^{-s}(s+1)}{s^2}$$

$$F(s) = \frac{1 - e^{-s}(s+1)}{s^2}$$

2 Solution of Initial Value Problems using The Laplace Transform

Using integration by parts, we can find the Laplace Transform of a derivative:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

note that this Laplace Transform is also in terms of the function's initial conditions. Now, from induction, we can find the Laplace Transform of the n'th derivative of a function:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

This Laplace Transform has $n-1$ initial conditions. Note that this way, we can solve first order constant coefficient ODE of the form:

$$ay"(t) + by'(t) + cy(t) = g(t)$$

By taking the Laplace Transform on both sides, using the formula to take the Laplace Transform of a derivative, solve for the solution $Y(s)$, and apply the Inverse Laplace Transform ($\mathcal{L}^{-1}\{F(s)\}$) to find the solution to the IVP. To find inverse Laplace Transforms in this course, one should decompose $Y(s)$ into simpler terms (usually using partial fractions) and use a Laplace Transform table to find the solution $y(t)$.

Problem 2a. From Boyce and DiPrima, 10th edition (6.2, exercise 4, p.324):
Find the inverse Laplace Transform of:

$$F(s) = \frac{3s}{s^2 - s - 6}.$$

Solution:

$$F(s) = \frac{3s}{s^2 - s - 6} = \frac{3s}{(s+2)(s-3)}$$

$$F(s) = \frac{A}{s+2} + \frac{B}{s-3}$$

$$F(s) = \frac{As - 3A + Bs + 2B}{(s+2)(s-3)} = \frac{3s}{(s+2)(s-3)}$$

Now, solve for A and B:

$$s : A + B = 3$$

$$1 : -3A + 2B = 0$$

$$\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \frac{1}{5} \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 6/5 \\ 9/5 \end{pmatrix}$$

Therefore,

$$F(s) = \frac{3s}{s^2 - s - 6} = \frac{6/5}{s+2} + \frac{9/5}{s-3}$$

Now use linearity and the inverse Laplace Transform using linearity and the Laplace Transform table:

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{6}{5}e^{-2t} + \frac{9}{5}e^{3t}$$

Problem 2b. From Boyce and DiPrima, 10th edition (6.2, exercise 14, p.325):
Solve this IVP using the Laplace Transform:

$$y\text{''} - 4y' + 4y = 0;\ y(0) = 1,\ y'(0) = 1$$

Solution: First, take the Laplace Transform on both sides:

$$\mathcal{L}\{y\text{''}(t) - 4y'(t) + 4y(t)\} = \mathcal{L}\{0\}$$

$$\mathcal{L}\{y\text{''}(t)\} - 4\mathcal{L}\{y'(t)\} + 4\mathcal{L}\{y(t)\} = 0$$

Let $Y(s) = \mathcal{L}\{y(t)\}$:

$$(s^2\mathcal{L}\{y(t)\} - sy(0) - y'(0)) - 4(\mathcal{L}\{y'(t)\} = s\mathcal{L}\{y(t)\} - y(0)) + 4\mathcal{L}\{y(t)\} = 0$$

$$(s^2Y(s) - sy(0) - y'(0)) - 4(sY(s) - y(0)) + 4Y(s) = 0$$

$$(s^2 - 4s + 4)Y(s) - (s-4)y(0) - y'(0) = 0$$

$$(s^2 - 4s + 4)Y(s) = (s-4)y(0) + y'(0)$$

Since $y(0) = 1,\ y'(0) = 1$:

$$(s-2)^2Y(s) = (s-4) + 1 = s - 3$$

$$Y(s) = \frac{s-3}{(s-2)^2}$$

Use partial fractions to split up $Y(s)$, which is of the form:

$$Y(s) = \frac{A}{s-2} + \frac{B}{(s-2)^2}$$

$$Y(s) = \frac{As - 2A + B}{(s-2)^2} = \frac{s-3}{(s-2)^2}$$

Now, solve for A and B:

$$s : A = 1$$

$$1 : -2A + B = -3 \Rightarrow B = 2 - 3 = -1$$

Therefore,

$$Y(s) = \frac{s-3}{(s-2)^2} = \frac{1}{s-2} - \frac{1}{(s-2)^2}$$

Then, find the inverse Laplace Transform using linearity and the Laplace Transform table:

$$y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\{\frac{1}{s-2} - \frac{1}{(s-2)^2}\}$$

$$y(t) = \mathcal{L}^{-1}\{\frac{1}{s-2}\} - \mathcal{L}^{-1}\{\frac{1}{(s-2)^2}\}$$

$$y(t) = e^{2t} - te^{2t}$$