MATH 263: Section 003, Tutorial 11

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1 Series Solutions Near a Regular Singular Point, Part I and II

Consider the general second order linear ODE:

$$P(x)y"(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$

Where $P(x_0) = 0$, meaning that $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are not analytical at $x = x_0$. $x = x_0$ is then a **singular point**.

Consider the case of **regular singular points**, where $(x - x_0)p(x) = (x - x_0)\frac{Q(x)}{P(x)}$ and $(x - x_0)^2 q(x) = (x-x_0)^2 \frac{R(x)}{P(x)}$ are analytic at $x = x_0$. We can write them as: $(x-x_0)p(x) = (x-x_0)\frac{Q(x)}{P(x)}$ and $(x-x_0)^2 q(x) = (x - x_0)^2 \frac{R(x)}{P(x)}$ are

$$(x - x_0)p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n,$$

and

$$(x - x_0)^2 q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n,$$

Plugging them in the ODE, we get

$$(x - x_0)^2 y" + (x - x_0)[(x - x_0)p(x)]y' + [(x - x_0)^2 q(x)]y = 0$$

$$(x-x_0)^2 y" + (x-x_0)(p_0+p_1(x-x_0)+\cdots+p_n(x-x_0)^n+\dots)y' + (q_0+q_1(x-x_0)+\cdots+q_n(x-x_0)^n+\dots)y = 0.$$

As x approaches x_0, the ODE behaves as an Euler equation as such:

$$(x - x_0)^2 y" + p_0(x - x_0)y' + q_0 y = 0.$$

where $p_0 = \lim_{x\to x_0} \frac{Q(x)}{P(x)}$ and $q_0 = \lim_{x\to x_0} \frac{R(x)}{P(x)}$

The solutions will be of the form of Euler solutions times a power series as such:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n (x - x_0)^n,\ a_0 \neq 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{r+n}$$

Then plug it in the ODE. The Euler characteristic equation will arise along with a recurrence relation for a_n that depends on r.

When these roots are identical or differ by an integer, one can only find one fundamental solution with this method.

Let $x_0 = 0$. Given a double root r_1, the second solution will be of the form:

$$y_2(x) = y_1(x)\ln x + x^{r_1}\sum_{n=1}^{\infty} b_n(r_1)x^n$$

Given roots r_1 and r_2 that differ by an integer (in other words $r_1 - r_2 = N > 0$, where $N \in \mathbb{N}$), the second solution will be of the form:

$$y_2(x) = Cy_1(x)\ln x + x^{r_2}[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n]$$

In each case, we plug this equation back in the ODE to find all constants and coefficients of the second solution. Note: $r_1 > r_2$. Given roots r_1 and r_2 that do not differ by an integer, the general solution will be of the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$y(x) = c_1 x^{r_1}\sum_{n=0}^{\infty} a_n(r_1)x^n + c_2 x^{r_2}\sum_{n=0}^{\infty} a_n(r_2)x^n.$$

Problem 1. From Boyce and DiPrima, 10th edition (5.5, exercise 3, p.286):
Find one fundamental solution of:

$$xy" + y = 0.$$

Solution: Let $x_0 = 0$: $P(x_0) = 0$, making it a singular point.

$$\lim_{x\to x_0}(x - x_0)\frac{Q(x)}{P(x)} = \lim_{x\to 0} x\frac{0}{x} = 0$$

and

$$\lim_{x\to x_0}(x - x_0)\frac{R(x)}{P(x)} = x^2\frac{1}{x} = 0$$

So $x_0 = 0$ is a regular singular point.

Let:

$$y(x) = \sum_{n=0}^{\infty} a_n x^{r+n}$$

$$y'(x) = \sum_{n=0}^{\infty}(r+n)a_n x^{r+n-1}$$

$$y"(x) = \sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2}$$

Note: here since our series is not a Taylor Series, differentiating does not change the index at which the sum starts.

Plug in the ODE:

$$x\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-2} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

$$\sum_{n=0}^{\infty}(r+n)(r+n-1)a_n x^{r+n-1} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

$$\sum_{n=-1}^{\infty}(r+n+1)(r+n)a_{n+1}x^{r+n} + \sum_{n=0}^{\infty} a_n x^{r+n} = 0.$$

Take out the $n = -1$ term and combine the sums:

$$r(r-1)a_0x^{r-1} + \sum_{n=0}^{\infty}[(r+n+1)(r+n)a_{n+1} + a_n]x^{r+n} = 0.$$

Therefore, as before, we have:

$$r(r-1) = 0 \Rightarrow r_1 = 1, \; r_2 = 0$$

and

$$(r+n+1)(r+n)a_{n+1} + a_n = 0.$$

$$a_{n+1} = -\frac{a_n}{(r+n+1)(r+n)}$$

Note that when the Euler characteristic equation roots are the same or differ by an integer, like here, we can only find one fundamental solution. Else, the two roots correspond to two fundamental solutions with their respective recurrence relations for a_n. This issue will be discussed later, for now let's work with $r_1 = 1$, to find one of the solutions and plug it in the recurrence relation:

$$a_{n+1}(1) = -\frac{a_n(1)}{(n+1)(n+2)}$$

$$a_1 = -\frac{a_0}{1 \cdot 2}$$

$$a_2 = -\frac{a_1}{2 \cdot 3} = \frac{a_0}{1 \cdot 2 \cdot 2 \cdot 3} = \frac{a_0}{2! \cdot 3!}$$

$$a_3 = -\frac{a_2}{3 \cdot 4} = -\frac{a_0}{1 \cdot 2 \cdot 3 \cdot 1 \cdot 2 \cdot 3 \cdot 4} = -\frac{a_0}{3! \cdot 4!}$$

From looking at the pattern, we get

$$a_n(1) = (-1)^n \frac{a_0(1)}{n!(n+1)!}$$

and letting $a_0 = 1$ we get **one** of the fundamental solutions, which is:

$$y_1(x) = \sum_{n=0}^{\infty} a_n(1)x^{r+n} = x^r \sum_{n=0}^{\infty} a_n(1)x^n$$

$$y_1(x) = x \; (1 - \frac{1}{1 \cdot 2}x + \frac{1}{2! \cdot 3!}x^2 + \dots) = x[\sum_{n=0}^{\infty}(-1)^n \frac{a_0}{n!(n+1)!}x^n]$$

In this case, $r_1 - r_2 = 1 - 0 = 1 \in \mathbb{N}$, and the second solution will be of the form:

$$y_2(x) = Cy_1(x)\ln x + x^{r_2}[1 + \sum_{n=1}^{\infty} c_n(r_2)x^n]$$

$$y_2(x) = Cy_1(x)\ln x + [1 + \sum_{n=1}^{\infty} c_n(0)x^n]$$

Take this solution's derivatives:

$$y_2'(x) = Cy_1'(x)\ln x + C\frac{y_1(x)}{x} + \sum_{n=1}^{\infty} nc_n(0)x^{n-1}$$

$$y_2''(x) = Cy_1''(x)\ln x + 2C\frac{y_1'(x)}{x} - C\frac{y_1(x)}{x^2} + \sum_{n=2}^{\infty} n(n-1)c_n(0)x^{n-2},$$

$$xy_2''(x) = Cxy_1''(x)\ln x + 2Cy_1'(x) - C\frac{y_1(x)}{x} + \sum_{n=2}^{\infty} n(n-1)c_n(0)x^{n-1},$$

where

$$y_1(x) = \sum_{n=0}^{\infty} a_n(1)x^{r_1+n} = x\sum_{n=0}^{\infty} a_n(1)x^n = x[1 + \sum_{n=1}^{\infty} a_n(1)x^n]$$

$$y_1'(x) = 1 + \sum_{n=1}^{\infty} a_n(1)x^n + x\sum_{n=1}^{\infty} na_n(1)x^{n-1}$$

$$y_1'(x) = 1 + \sum_{n=1}^{\infty}(1+n)a_n(1)x^n$$

Plugging those values into the ODE, we get:

$$xy_2" + y_2 = 0$$

$$Cxy_1''(x)\ln x + 2Cy_1'(x) - C\frac{y_1(x)}{x} + \sum_{n=2}^{\infty} n(n-1)c_n(0)x^{n-1} + Cy_1(x)\ln x + [1 + \sum_{n=1}^{\infty} c_n(0)x^n] = 0$$

$$C(xy_1''(x) + y_1(x))\ln x + 2Cy_1'(x) - C\frac{y_1(x)}{x} + \sum_{n=2}^{\infty} n(n-1)c_n(0)x^{n-1} + [1 + \sum_{n=1}^{\infty} c_n(0)x^n] = 0$$

$xy_1''(x) + y_1(x) = 0$, so cancel it out and plug in $y_1(x)$:

$$2C[1 + \sum_{n=1}^{\infty}(1+n)a_n(1)x^n] - C\frac{1}{x}(x[1 + \sum_{n=1}^{\infty} a_n(1)x^n]) + \sum_{n=2}^{\infty} n(n-1)c_n(0)x^{n-1} + [1 + \sum_{n=1}^{\infty} c_n(0)x^n] = 0$$

$$2C[1 + \sum_{n=1}^{\infty}(1+n)a_n(1)x^n] - C([1 + \sum_{n=1}^{\infty} a_n(1)x^n]) + \sum_{n=1}^{\infty} n(n+1)c_{n+1}(0)x^n + [1 + \sum_{n=1}^{\infty} c_n(0)x^n] = 0$$

$$1 + C + \sum_{n=1}^{\infty}[C(1+2n)a_n(1) + n(n+1)c_{n+1}(0) + c_n(0)]x^n = 0$$

All terms are independent, so they must all be 0. Therefore,

$$1 + C = 0 \Rightarrow C = -1$$

$$C(1+2n)a_n(1) + n(n+1)c_{n+1}(0) + c_n(0) = 0$$
$$-(1+2n)a_n(1) + n(n+1)c_{n+1}(0) + c_n(0) = 0$$
$$n(n+1)c_{n+1}(0) = (1+2n)a_n(1) - c_n(0)$$
$$c_{n+1}(0) = \frac{(1+2n)a_n(1) - c_n(0)}{n(n+1)},$$

where

$$a_n(1) = (-1)^n \frac{1}{n!(n+1)!}$$

Since there's no restrictions on $c_1(0)$, we can just let it be 0: $n = 1$:

$$c_2(0) = \frac{(1+2)a_1(1) - c_1(0)}{1(2)}$$

$$a_1(1) = (-1)^1 \frac{1}{1!(1+1)!} = \frac{-1}{2}$$

$$c_2(0) = \frac{3 \cdot \frac{-1}{2}}{1(2)} = \frac{-3}{4}$$

$n = 2:$

$$c_3(0) = \frac{(1+4)a_1(1) - c_2(0)}{2(3)}$$

$$a_2(1) = (-1)^2 \frac{1}{2!(1+2)!} = \frac{1}{12}$$

$$c_3(0) = \frac{5 \cdot \frac{1}{12} + \frac{3}{4}}{6} = \frac{7}{36}$$

$n = 3:$

$$c_4(0) = \frac{(1+6)a_1(1) - c_3(0)}{3(4)}$$

$$a_3(1) = (-1)^3 \frac{1}{3!(1+3)!} = \frac{-1}{144}$$

$$c_4(0) = \frac{7 \cdot \frac{-1}{144} - \frac{7}{36}}{12} = \frac{-35}{1728}.$$

Therefore,

$$y_2(x) = -y_1(x) \ln x + 1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \dots$$

$$y(x) = c_1 y_1(x) - c_2[y_1(x) \ln x + 1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \dots]$$

$$y(x) = c_1(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \dots) - c_2[(1 - \frac{1}{2}x + \frac{1}{12}x^2 - \frac{1}{144}x^3 + \frac{1}{2880}x^4 - \dots) \ln x + 1 - \frac{3}{4}x^2 + \frac{7}{36}x^3 - \frac{35}{1728}x^4 + \dots$$