Chapter 1

Theorem 1.1

Proposition 1.1. *Let $\mathcal{M}(\mathcal{F})$ be the σ-algebra generated by $\{$, if $\mathcal{E}$ is a subset of $\mathbb{P}(X)$, with $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$, then $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$.*

Proof. Notice that because $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$,

$$\mathcal{M}(\mathcal{F}) \in \{\mathcal{M},\ \mathcal{E} \subseteq \mathcal{M},\ \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

Taking the intersection, noting that $\mathcal{M}(\mathcal{E})$ is the intersection of all σ-algebras containing $\mathcal{E}$ as a subset, we have

$$\bigcap\{\mathcal{M}(\mathcal{F})\} \supseteq \bigcap\{\mathcal{M},\ \mathcal{E} \subseteq \mathcal{M},\ \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

And

$$\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$$

■

Theorem 1.2

Proposition 2.1. *The Borel σ-algebra of $\mathbb{R}$, $\mathbb{B}$ is generated by the following*

- *The family of open intervals* $\mathcal{E}_\infty = \{(\dashv, \lfloor), \dashv < \lfloor\}$,
- *The family of closed intervals* $\mathcal{E}_\in = \{[\dashv, \lfloor], \dashv < \lfloor\}$,
- *The family of half-open intervals* $\mathcal{E}_\ni = \{(\dashv, \lfloor], \dashv < \lfloor\}$ *or* $\mathcal{E}_\triangle = \{[\dashv, \lfloor), \dashv < \lfloor\}$
- *The open rays* $\mathcal{E}_\bigtriangledown = \{(\dashv, +\infty), \dashv \in \mathbb{R}\}$ *or* $\mathcal{E}_{\not\models} \{(-\infty, \dashv), \dashv \in \mathbb{R}\}$
- *The closed rays* $\mathcal{E}_{\imath} = \{[\dashv, +\infty), \dashv \in \mathbb{R}\}$ *or* $\mathcal{E}_\forall = \{(-\infty, \dashv], \dashv \in \mathbb{R}\}$

Proof. By definition, $\mathbb{B}$ is generated by the family of all open sets in $\mathbb{R}$, but every open set is a countable union of open intervals. Therefore

$$\mathcal{T}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E}_\infty) \implies \mathbb{B} \subseteq \mathcal{M}(\mathcal{E}_\infty)$$

Conversely, every open interval is an open set, hence

$$\mathcal{E}_\infty \subseteq \mathcal{T}_\mathbb{R} \subseteq \mathbb{B} \implies \mathcal{M}(\mathcal{E}_\infty) \subseteq \mathbb{B}$$

Every closed interval can also be written as a countable intersection of open intervals, for every $[a, b]$, with $a < b$, we have

$$[a, b] = \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1}) \tag{1}$$

Indeed, fix any $x \in [a, b]$ then for every $n \geq 1$,

$$a - n^{-1} < a \leq x \leq b < b + n^{-1}$$

So $x \in \bigcap_{n\geq 1}(a - n^{-1}, b + n^{-1})$. If x an element of the left member, then for every $n \geq 1$,

$$a - n^{-1} < x \implies a - x \leq 0$$

Similarly for $x \leq b$, therefore equation (1) is valid, and $\mathcal{E}_\in \subseteq \mathbb{B} = \mathcal{M}(\mathcal{E}_\infty)$. To show the reverse estimate, every open interval can be written as a countable union of closed intervals,

$$(a, b) = \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \tag{2}$$

To show that the above estimate is indeed true, fix any $x \in (a, b)$, then

$$\begin{aligned} a < x < b &\iff a < a + n^{-1} \leq x \leq b - n^{-1} < b \\ &\iff x \in \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \end{aligned}$$

So that equation (2) holds. By similar argumentation we have $\mathcal{E}_\infty \subseteq \mathcal{M}(\mathcal{E}_\in) \implies \mathcal{M}(\mathcal{E}_\in) = \mathcal{M}(\mathcal{E}_\infty)$.

For $\mathcal{E}_\ni$, $\mathcal{E}_\triangle$

- $(a, b] = \bigcap_{n \geq 1} (a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1)$,
- $(a, b) = \bigcup_{n \geq 1} (a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3)$,
- $[a, b) = \bigcup_{n \geq 1} [a, b - n^{-1}]$, proves $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_2)$,
- $[a, b] = \bigcap_{n \geq 1} [a, b + n^{-1})$, proves $\mathcal{M}(\mathcal{E}_2) \subseteq \mathcal{M}(\mathcal{E}_4)$

So that $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_4) = \mathbb{B}$. By taking complements of each element we get $\mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$ and $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$. Notice also that

- $(a, b] = (a, +\infty) \cap (-\infty, b]$, proves $\mathcal{E}_\ni \subseteq \mathcal{M}(\mathcal{E}_\bigtriangledown)$, and $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$.
- $(a, +\infty) = \cup_{n \geq 1} (a, a + n]$, proves $\mathcal{E}_\bigtriangledown \subseteq \mathcal{M}(\mathcal{E}_\ni)$, and $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_3)$.
- $[a, b) = [a, +\infty) \cap (-\infty, b)$, proves $\mathcal{E}_\triangle \subseteq \mathcal{M}(\mathcal{E})$, and $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_7)$,
- $[a, +\infty) = \bigcup_{n \geq 1} [a, a + n)$, proves $\mathcal{E} \subseteq \mathcal{M}(\mathcal{E}_\triangle)$, and $\mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{M}(\mathcal{E}_4)$.

Finally, $\mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8) = \mathbb{B}$ and $\mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7) = \mathbb{B}$. ■

Theorem 1.3

Proposition 3.1. *If A is countable, then $\otimes_{\alpha\in A}\mathcal{M}_\alpha$ is the σ-algebra generated by*

$$W := \left\{\prod_{\alpha\in A} E_\alpha,\ E_\alpha \in \mathcal{M}_\alpha\right\}$$

Proof. We agree to define

$$V := \left\{\pi_\alpha^{-1}(E_\alpha),\ E_\alpha \in \mathcal{M}_\alpha\right\}$$

By definition, V generates $\otimes_{\alpha\in A}\mathcal{M}_\alpha$. Fix any element in $x = \pi_\alpha^{-1}(E_\alpha) \in V$, then

$$\pi_\alpha(x) \in E_\alpha,\ \pi_{\beta\neq\alpha}(x) \in X_\beta$$

Then $x \in W$ if we choose $x = \prod_{c\in A} E_c$, for $E_c = E_\alpha$ if $c = \alpha$, and $E_c = X_c$ if $c \neq \alpha$. ■

Theorem 1.4

Proposition 4.1.

Proof. ∎

Theorem 1.5

Proposition 5.1.

Proof. ■

Theorem 1.6

Proposition 6.1.

Proof. ∎

Theorem 1.7

Proposition 7.1.

Proof. ■

Theorem 1.8

Proposition 8.1.

Proof. ■

Theorem 1.9

Proposition 9.1.

Proof. ■

Theorem 1.10

Proposition 10.1.

Proof. ■

Theorem 1.11

Proposition 11.1 (Caratheodory's Theorem). *If μ^* is an outer measure on $\mathbf{X}$, the collection $\mathcal{M}$ of μ^*-measurable sets is a σ-algebra, and the restriction of μ^* to $\mathcal{M}$ is a complete measure.*

Proof. We quote the definition for a set $A \subseteq X$ to be μ^* measurable. For any $E \subseteq X$,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \tag{3}$$

- Show $\mathcal{M}$ is an algebra.
- μ^* is finitely additive on $\mathcal{M}$.
- $\mathcal{M}$ is closed under countable disjoint (this makes $\mathcal{M}$ a sigma algebra, since it is an algebra that is closed under countable disjiont unions)

Lemma 11.1 *The family of μ^*-measurable sets is an algebra.*

Proof of Lemma 11.1. Clearly $\mathcal{M}$ is closed under complements. To show that it is a σ-algebra, and if $A, B \in \mathcal{M}$, then $\left\{\underbrace{E \cap A}_{1}, \underbrace{E \setminus A}_{2}\right\} \subseteq \mathbb{P}(\mathbf{X})$. And because B is μ^*-measurable,

$$\mu^*(E) = \underbrace{\mu^*(E \cap A \cap B) + \mu^*(E \cap A \setminus B)}_{1} + \underbrace{\mu^*(E \cap B \setminus A) + \mu^*(E \setminus (A \cup B))}_{2}$$

By subadditivity of μ^*, $A \cup B = A \cap B + A \setminus B + B \setminus A$ with $+$ denoting the disjiont union, hence

$$\mu^*(E \cap (A \cap B)) \leq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \setminus B)) + \mu^*(E \cap (B \setminus A))$$

and

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B))$$

■

Lemma 11.2 *μ^* is finitely additive on $\mathcal{M}$, the family of μ^*-measurable sets.*

Proof of Lemma 11.2. Let A, B be disjoint μ^*-measurable sets. It suffices to show $\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B)$, as the reverse estimate follows from subadditivity. From Lemma 11.1, $A \cup B \in \mathcal{M}$, so

$$\begin{aligned}\mu^*(A \cup B) &= \mu^*(A \cup B \cap A) + \mu^*(A \cup B \setminus A) \\ &= \mu^*(A \cup \varnothing) + \mu^*(A \setminus A \cup B \setminus A) \\ &= \mu^*(A) + \mu^*(B)\end{aligned}$$

■

Corollary 11.1 *If $\{A_j\}_{j \geq N} \subseteq \mathcal{M}$ is a finite disjoint family, then*

$$\mu^*\left(\bigcup A_{j \leq N}\right) = \sum \mu^*(A_{j \leq N})$$

Lemma 11.3 *Let $\{A_j\}_{j \geq 1}$ be a countable disjoint sequence in $\mathcal{M}$, and denote $B_n = \bigcup A_{j \leq n} \in \mathcal{M}$ by Lemma 11.1. For every $E \subseteq \mathbf{X}$,*

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j \leq n})$$

Proof of Lemma 11.3. We will proceed by induction. If $n = 1$ then we have equality, suppose the result holds for $n \in \mathbb{N}^+$, and $A_{n+1} \in \mathcal{M}$ so

$$\begin{aligned}\mu^*(E \cap B_{n+1}) &= \mu^*(E \cap B_{n+1} \cap A_{n+1}) = \mu^*(E \cap B_{n+1} \setminus A_{n+1}) \\ &= \mu^*(E \cap A_{n+1}) + \mu^*(E \cap B_n) \\ &= \sum_{j \leq n+1} \mu^*(E \cap A_j)\end{aligned}$$

as $A_j \cap A_{n+1} = \varnothing \iff A_j \setminus A_{n+1} = A_j$, and $B_n \cap A_n = A_n \iff A_n \subseteq B_n$. ■

To show $\mathcal{M}$ is a sigma-algebra, fix any disjoint sequence $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$, and denote B_n as in lemma 11.3. Define $B = \bigcup A_{j \geq 1} \supseteq B_n$ and for every $n \geq 1$, we have

$$\mu^*(E) = \mu^*(E \cap B_n) + \mu^*(E \setminus B_n)$$

$$= \sum \mu^*(E \cap A_{j \le n}) + \mu^*(E \setminus B_n)$$

$$\ge \sum \mu^*(E \cap A_{j \le n}) + \mu^*(E \setminus B) \qquad \text{since } B_n \subseteq B \iff B^c \subseteq B_n^c$$

$$\ge \sup_n \left[\sum \mu^*(E \cap A_{j \le n}) \right] + \mu^*(E \setminus B)$$

Let $J \subseteq \mathbb{N}^+$ be a finite non-empty set. And $\sup J \in \mathbb{N}^+$, $\sup J < +\infty$. By the Archmedean Property we can find a large $N \in \mathbb{N}^+$, with $N > J$ so that

$$\sum_{j \in J} \mu^*(E \cap A_j) \le \sum_{j \le N} \mu^*(E \cap A_j)$$

Applying the estimate $\sup_n \left[\sum \mu^*(E \cap A_{j \le n}) \right] + \mu^*(E \setminus B) \le \mu^*(E)$ reads

$$\left[\sum_{j \in J} \mu^*(E \cap A_j) \right] + \mu^*(E \setminus B) \le \mu^*(E)$$

Now by Chapter 0, the infinite sum

$$\sum_{j \ge 1} \mu^*(E \cap A_j) = \sup \left\{ \sum_{j \in J} \mu^*(E \cap A_j),\ J \subseteq \mathbb{N}^+,\ 0 < |J| < +\infty \right\}$$

and $\bigcup A_{j \ge 1} = B$ is μ^*-measurable. Since $\mu^*(\varnothing) = 0$, and μ^* is countably additive on $\mathcal{M}$, (perhaps by replacing E with the union over all disjoint sets), μ^* is a measure on $\mathcal{M}$. To show μ^* is a complete measure, fix $A \in \mathcal{M}$ where $\mu^*(A) = 0$. Then any $B \subseteq A$ is also null, and for $E \subseteq \mathbf{X}$,

$$\mu^*(E) \ge \underbrace{\mu^*(E \cap B)}_{0} + \mu^*(E \setminus B) \implies B \in \mathcal{M}$$

■

Theorem 1.12

Proposition 12.1.

Proof. ■

Theorem 1.13

Proposition 13.1.

Proof. ■

Theorem 1.14

Proposition 14.1.

Proof. ■

Theorem 1.15

Proposition 15.1.

Proof. If $\{E_j\}_{j\geq 1} \subseteq \mathcal{A}$ such that each $E_j = FDU(I_{ji})$ over finitely many i, and suppose E_j are disjoint, and that $DU(E_j) \in \mathcal{A}$. So that $DU(E_j) = FDU(I_\alpha)$ for some finite collection of half-intervals $\{I_\alpha\}$.

We will first prove the simpler case. Suppose we have already proven:

$$\{E_j\}_{j\geq 1} \subseteq \mathcal{A},\ DU(E_j) = I_\alpha \in \mathcal{A} \implies \mu_0\Big(DU(E_j)\Big) = \sum \mu_0(E_j) = \mu_0(I_\alpha) \quad (4)$$

but each E_j is a FDU of I_{ji}, and for every $j \geq 1$, $E_j \cap I_\alpha \in \mathcal{A}$ (closure under intersections, because the family of FDU of h-intervals is an algebra).

Thus we have a disjoint sequence whose union is one h-interval. In symbols:

$$DU(E_j) = FDU(I_\alpha) \implies I_\alpha = DU(E_j \cap I_\alpha)$$

$\forall j \geq 1,\ E_j \cap I_\alpha \in \mathcal{A} \implies$

$$\begin{aligned} \mu_0(FDU(I_\alpha)) &= \sum_{\alpha<+\infty} \mu_0(I_\alpha) \\ &= \sum_{\alpha<+\infty}\sum_{j\geq 1} \mu_0(E_j \cap I_\alpha) \\ &= \sum_{j\geq 1}\sum_{\alpha<+\infty} \mu_0(E_j \cap I_\alpha) \\ &= \sum_{j\geq 1} \mu_0(E_j) \end{aligned}$$

It is permissible to swap the two summations because we are using the supremum definition for a sum of non-negative terms. And we applied finite-additivity (see earlier), to conclude that $\sum_{j\geq 1}\sum_\alpha \mu_0(E_j \cap I_\alpha) = \sum_{j\geq 1} \mu_0(E_j)$. ■

Define

- $\mathcal{H}_1 = \Big\{(a,\ b],\ -\infty \leq a < b < +\infty\Big\}$,
- $\mathcal{H}_2 = \Big\{(a,\ +\infty),\ a \in \mathbb{R} \cup \{-\infty\}\Big\}$,
- $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \{\varnothing\}$. Where $+$ denotes the disjoint union.
- DU: disjoint union, FDU: finite disjoint union.

Steps:

1. Show that $\mathcal{H}$ is an elementary family.
2. Show that if $I_\alpha \in \mathcal{H}_1$, then for every $I_\beta \in \mathcal{H}_1 \cup \mathcal{H}_2$, $I_\alpha \cap I_\beta \in \mathcal{H}_1$. We write this as
$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1,\ I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_1$$
3. Show that if $I_\alpha \in \mathcal{H}_2$, then
$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1,\ I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_2$$
4. Show that $\mu_0((a,\ b]) = \overline{F}(b) - \overline{F}(a)$ is well defined. (modify the proof in Folland to check for $a = -\infty$ with
$$\overline{F} : \overline{\mathbb{R}} \to \overline{\mathbb{R}}, \qquad \begin{cases} \overline{F}|_{\mathbb{R}} & = F \\ \overline{F}(+\infty) & = \sup_x F(x), \\ \overline{F}(-\infty) & = \inf_x F(x) \end{cases}$$
5. Show that $\mu_0((a,\ b]) = \overline{F}(b) - \overline{F}(a)$ is well defined for $b < +\infty$. If $E = (a,\ b] \in \mathcal{A}$, then E is an FDU of $\mathcal{H}_1$, and $\mathcal{H}_2$. So we write
$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2) = FDU(\mathcal{H}_1)$$
since E is bounded above, the $\mathcal{H}_2$ part of the FDU must be null. Now fix $E = FDU_{\mathcal{H}_1}(I_j) = FDU_{\mathcal{H}_1}(I_2)$. And follow the proof in Folland to see the 'well-definedness' of μ_0 if $E \in \mathcal{H}_1$.

6. Next, suppose $E \in \mathcal{H}_2$ and

$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2)$$

   Clearly $FDU(\mathcal{H}_2) \neq \varnothing$, since E is unbounded above, and $FDU(\mathcal{H}_2)$ consists of exactly one element, so we write

$$E = FDU(\mathcal{H}_1) + (z,\ +\infty)$$

7. Show that $\mu_0((a,\ b]) = \overline{F}(b) - \overline{F}(a)$ is well defined. Hint: use the fact that if $E \in \mathcal{A}$, such that $E = FDU(E, \mathcal{H}_1) + FDU(E, \mathcal{H}_2)$, then $FDU(E, \mathcal{H}_2)$ contains at most one element (after throwing away empty sets), then use this to deduce $E \cap I_\alpha$ has a $FDU(E \cap I_\alpha, \mathcal{H}_2)$ of exactly one $\mathcal{H}_2$ interval, where I_α participates in $FDU(E, \mathcal{H}_2)$, if E is unbounded above. Then take $E \setminus I_\alpha = FDU(E \setminus I_\alpha, \mathcal{H}_1) = FDU(E, \mathcal{H}_1)$.

8. Now show that μ_0 is well-defined on all $E \in \mathcal{A}$.

9. Continue the proof for Folland until you reach the unbounded intervals, then modify the 'right continuity argument' to add an extra $\mathcal{H}_2$ interval. Let $I = \mathcal{H}_1 + \mathcal{H}_2 = I_\alpha + I_\beta$, meaning I can be represented by at most one $\mathcal{H}_1$ and $\mathcal{H}_2$ interval. If $(I_k) \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$, then $\{I_k \cap I_\alpha\} \subseteq \mathcal{H}_1$, and continue the proof as usual.

Theorem 1.16

Proposition 16.1.

Proof. ∎

Theorem 1.17

Proposition 17.1.

Proof. ■

Theorem 1.18

Proposition 18.1.

Proof. ■

Exercises

Exercise 1.1

Proposition 1.1.

Proof. ■

Exercise 1.2

Proposition 2.1.

Proof. ■

Exercise 1.3

Proposition 3.1.

Proof. ■

Exercise 1.4

Proposition 4.1. *An algebra $\mathcal{A}$ is a σ-algebra $\iff$ it is closed under countable increasing unions.*

Proof. $\Longleftarrow$ is trivial. And it suffices to show that $\mathcal{A}$ is closed under countable disjoint unions. Indeed, if $\{E_j\}_{j\geq 1} \subseteq \mathcal{A}$ is a countable disjoint sequence of sets, write

$$F_n = \bigcup E_{j\leq n}$$

Clearly, F_j is increasing, and denote $F = \cup E_{j\geq 1}$, which is a member of $\mathcal{A}$. We claim that

$$\bigcup F_{n\geq 1} = \bigcup E_{j\geq 1}$$

Fix any $x \in \bigcup E_{j\geq 1}$, then x belongs in some $E_j \subseteq F_j$, and $\supseteq$ is proven. Also, if $x \in \bigcup F_{n\geq 1}$, then there exists some F_n for which x is a member of. For this particular F_n, means that $x \in E_j$ where $j \leq n$ and $x \in \bigcup E_{j\geq 1}$. ∎

Exercise 1.5

Proposition 5.1. *Let $\mathcal{M}(\mathcal{E})$ be the σ-algebra generated by $\mathcal{E} \subseteq X$, and*

$$\mathcal{N} = \left\{\mathcal{M}(\mathcal{F}),\, \mathcal{F} \subseteq \mathcal{E},\, \mathcal{F} \text{ is countable}\right\}$$

Show that $\mathcal{M}(\mathcal{E}) = \mathcal{N}$.

Proof. The outline of the proof is as follows,

1. Prove that $\mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$,
2. Show that $\mathcal{N}$ is a σ-algebra,
3. Show that $\mathcal{N}$ contains $\mathcal{E}$ as a subset, and hence $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{N}$.

First, for any $\mathcal{F} \subseteq \mathcal{E}$, where $\mathcal{F}$ is countable, it follows from Lemma 1.1 that $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}\mathcal{E}$. Taking the union over all of such $\mathcal{F}$, we get $\bigcup \mathcal{M}(\mathcal{F}) = \mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$.

To show that $\mathcal{N}$ is a σ-algebra, fix any $A \in \mathcal{N}$, and A belongs to $\mathcal{M}(\mathcal{F})$, therefore $A^c \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{N}$. To show closure under countable unions, fix a sequence $\{E_j\} \subseteq \mathcal{N}$, then each of these E_j belongs to a corresponding $\mathcal{M}(\mathcal{F}_j)$, for $j \in \{1, 2, \ldots\}$. Now define

$$\overline{\mathcal{F}} = \bigcup \mathcal{F}_{j \geq 1} \subseteq \mathcal{E}$$

and $\overline{\mathcal{F}}$ is obviously countable. Hence for every $j \geq 1$, $\mathcal{M}(\mathcal{F}_j) \subseteq \mathcal{M}(\overline{\mathcal{F}})$ and taking the union yields

$$\bigcup \mathcal{M}(\mathcal{F}_{j \geq 1}) \subseteq \mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$$

It is also clear that our sequence $\{E_j\}$ is contained in $\mathcal{M}(\overline{\mathcal{F}})$, and $E = \bigcup E_j$ belongs to $\mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$ as an element. Therefore $\mathcal{N}$ is a σ-algebra.

Let $\alpha \in A$ index the family of sets in $\mathcal{E}$, (so that $E_\alpha \in \mathcal{E}$) and the singleton set of a set $\{E_\alpha\}$ is a countable subset of $\mathcal{E}$. For every $\alpha \in A$, we have

$$E_\alpha \in \mathcal{M}(\{E_\alpha\}) \subseteq \mathcal{N} \implies \mathcal{E} \subseteq \mathcal{N}$$

And one final application of Lemma 1.1 finishes the proof. ∎

Exercise 1.6

Proposition 6.1.

Proof. ■

Exercise 1.7

Proposition 7.1. *If $\mu_1, \ldots, \mu_n$ are measures on $(X, \mathcal{M})$, and $a_1, \ldots, a_n \in [0, +\infty)$, then $\mu = \sum_1^n \mu_j$ is a measure on $(X, \mathcal{M})$.*

Proof. If $\{E_j\}$ is a disjoint sequence in $\mathcal{M}$, and denote $E = \bigcup(E_j)$. If for each $k \leq n$, $\mu_k(E) < +\infty$,

$$\mu_k(E) = \sum \mu_k(E_j) \implies a_k\mu_k(E) = \sum a_k\mu_k(E_j)$$

Then,

$$\mu(E) = \sum_{k\leq n} a_k\mu_k(E) = \sum_{k\leq n}\sum_{j\geq 1} a_k\mu_k(E_j) = \sum_{j\geq 1}\sum_{k\leq n} a_k\mu_k(E_j) = \sum_{j\geq 1} \mu(E_j)$$

If there exists some mu_k such that $\mu_k(E) = +\infty$, then

$$\mu(E) = \sum_{k\leq n}\sum_{j\geq 1} a_k\mu_k(E_j)$$

Now if there exists some $\mu_{k'}$ with $\mu_{k'}(E) = +\infty$, then $\mu(E) = \sum_{k\leq n} \mu_k(E) = +\infty$, and

$$\sum_{j\geq 1} \mu(E_j) = \sup_N \sum_{j\leq N}\sum_{k\leq n} a_k\mu_k(E_j) \geq \mu_{k'}(E)$$

Therefore $\mu(E) = \sum_{j\geq 1} \mu(E_j)$, and μ is a measure. ∎

Exercise 1.8

Proposition 8.1. *If* $(X, \mathcal{M}, \mu)$ *is a measure space, and* $\{E_j\} \subseteq \mathcal{M}$*, then* $\mu(\liminf E_j) \leq \liminf \mu(E_j)$*. Also,* $\mu(\limsup E_j) \geq \limsup \mu(E_j)$ *provided that* $\mu(\bigcup E_{j\geq 1}) < +\infty$

Proof. If $\{E_j\}_{j\geq 1}$ is a sequence in $\mathcal{M}$, and define $F_m = \bigcap_{j\geq m} E_j$

$$\liminf E_j = \bigcup_{m\geq 1} \bigcap_{j\geq m} E_j = \bigcup_{m\geq 1} F_m$$

Also, for every $m \geq 1$, $F_m \subseteq E_m$, and F_m is an increasing sequence, because

$$[m, +\infty) \supseteq [m+1, +\infty) \implies F_m \subseteq F_{m+1}$$

Using continuity above, and writing $F = \bigcup F_{m\geq 1} = \liminf E_j$, we have

$$\begin{aligned} \mu(\liminf E_j) &= \mu(F) \\ &= \liminf \mu(F_m) \\ &\leq \liminf \mu(E_m) \end{aligned}$$

.
The second part of the proof is similar, if $G_m = \bigcup_{j\geq m} E_j$, then

$$\limsup E_j = \bigcap_{m\geq 1} \bigcup_{j\geq m} E_j = \bigcap_{m\geq 1} G_m$$

Similarly, G_m is a decreasing sequence, and since $\mu(\bigcup E_{j\geq 1}) = \mu(G_1)$ is finite, we can use continuity from above in the same manner, and the proof is complete. ■

Exercise 1.9

Proposition 9.1.

Proof. ■

Exercise 1.10

Proposition 10.1.

Proof. ■

Exercise 1.11

Proposition 11.1.

Proof. ∎

Exercise 1.12

Proposition 12.1. *Let $(X, \mathcal{M}, \mu)$ be a finite measure space,*

- *If $E, F \in \mathcal{M}$, and $\mu(E\Delta F) = 0$, then $\mu(E) = \mu(F)$,*
- *Say that $E \sim F$ if $\mu(E\Delta F) = 0$, then $\sim$ is an equivalence relation on $\mathcal{M}$,*
- *For every $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E\Delta F)$. Show that ρ defines a metric on the space of $\mathcal{M}/\sim$ equivalence classes.*

Proof of Part A. Use the fact that $\mu(F) = \mu(E \cap F) + \mu(F \cap E^c)$, and by monotonicity,

$$\mu(F \cap E^c) \leq \mu(E\Delta F) = 0$$

And $\mu(F) = \mu(E \cap F) = \mu(E)$, the last equality follows after a simple modification. ■

Proof of Part B. Suppose that $\mu(E\Delta F) = \mu(F\Delta G) = 0$, then

- $\mu(E \cap F^c) = \mu(F \cap E^c) \leq \mu(E\Delta F) = 0$ by monotonicity,
- Similarly, we have $\mu(F \cap G^c)\mu(G \cap F^c) = 0$, and
- By subadditivity,
  - $\mu(E\cap G^c) = \mu(E\cap F^c\cap G^c)+\mu(E\cap F\cap G^c) \leq 0$, and $\mu(E\cap G^c) = 0$, and
  - $\mu(G \cap E^c) = 0$
- Therefore $\mu(E\Delta G) = \mu(E \cap G^c) + \mu(G \cap E^c) = 0$

It is clear that the relation is reflexive, since $E\Delta E = \varnothing$, and symmetry is trivial. ■

Proof of Part C. Since $\rho(E, F) = \rho(F, E)$, and $\rho(E, F) \geq 0$ for every $E, F \in \mathcal{M}$, and $\rho(E, F) = 0 \iff E \sim F$. We only have to prove the Triangle Inequality. Notice that

$$\begin{aligned} \mu(E \setminus F) &= \mu(E \cap F^c \cap G) + \mu(E \cap F^c \cap G^c) \\ &\leq \mu(F^c \cap G) + \mu(E \cap F^c) \end{aligned}$$

and in the same fashion,

$$\mu(F \setminus E) \leq \mu(F \cap G^c) + \mu(E^c \cap F)$$

Combining the two inequalities, and applying additivity finishes the proof. ■

Exercise 1.13

Proposition 13.1. *Every σ-finite measure is semi-finite*

Proof. Suppose μ is σ-finite then there exists an increasing sequence of sets $E_j \nearrow X$ with $\mu(E_j) < +\infty$. Now for every $W \in \mathcal{M}$, if $\mu(W) = +\infty$ then $\mu(W) = \lim_{j\to\infty} \mu(E_j \cap W) = +\infty$. Since this real-valued limit converges to its supremum $+\infty$, there exists a non-null subset $E_j \cap W$ of positive and finite measure. ∎

Exercise 1.14

Proposition 14.1. *If μ is a semi-finite measure, and if $\mu(E) = +\infty$, for every $C > 0$, there exists an $F \subseteq E$ with $0 < \mu(F) < +\infty$.*

Proof. Suppose by contradiction that there exists a $C > 0$ so for every $F \subseteq E$, if F is of finite measure, then $0 \leq \mu(F) \leq C$. Let $s = \sup\{\mu(F), F \subseteq E, 0 < \mu(F) < +\infty\}$, and for any $n^{-1} > 0$, this induces a F_n with measure

$$\mu(F_n) > s - n^{-1}$$

and take $A_n = \bigcup_{j \leq n} F_j$. A simple induction will show that $\mu(A_n) \leq \sum_{j \leq n} \mu(F_j) < +\infty$, therefore $\mu(A_n) \leq s$ for every $n \geq 1$. By continuity from below

$$\lim_{n \to \infty} \mu(A_n) = \mu\left(\bigcup_{j \geq 1} F_j\right) \leq s$$

Next, by monotonicity, denoting the union over A_n by A, for every $n^{-1} > 0$

$$s - n^{-1} \leq \mu(A_n) \leq \mu(A) \leq s \implies \mu(A) = s$$

Now, $E \setminus A$ is a set of infinite measure, and by semi-finiteness. Find a set $B \subseteq E \setminus A$ with strictly positive measure, so that

$$\mu(A \cup B) = \mu(A) + \mu(B) > s$$

And this finishes the proof. ■

Exercise 1.15

Proposition 15.1. *Given a measure μ on $(\mathbf{X}, \mathcal{M})$, and define $\mu_0 = \sup\{\mu(F),\, F \subseteq E,\, \mu(F) < +\infty\}$. Show μ_0 is semi-finite. Then, show that if μ is semi-finite, $\mu = \mu_0$. Lastly, there exists a measure ν on $(\mathbf{X}, \mathcal{M})$, with $\mu = \nu + \mu_0$, where ν only assumes the values 0 or $+\infty$.*

Proof. First, a small Lemma. We claim that $\mu_0 = \mu$ on finite sets. Let $E \in \mathcal{M}$, and $\mu(E) < +\infty$, since

$$\mu(E) \in \{\mu(F),\, F \subseteq E,\, \mu(E) < +\infty\} \implies \mu(E) \leq \mu_0(E)$$

Next, for every $W \subseteq E$, $\mu(W) \leq \mu(E)$, so $\mu_0(E) \leq \mu(E)$. This proves the equality.

If E is any measurable subset of $\mathbf{X}$, and suppose also $\mu_0(E) = +\infty$, one can easily find subsets of E, $\{E_n\}_{n\geq 1}$ with

$$n \geq \mu(E_n) < +\infty$$

But E_n is a subset of finite measure, so $0 < \mu(E_n) = \mu_0(E_n) < +\infty$. This proves the semi-finiteness of μ_0.

Next, suppose μ is semi-finite, and fix any measurable set E. If E is if finite measure, then $\mu(E) = \mu_0(E)$, and if $\mu(E) = +\infty$, apply Exercise 14, so there exists a sequence of subsets of finite measure $E_n \subseteq E$ for every $n \geq 1$, with $\mu(E_n) \to \mu(E)$. Therefore $\mu_0(E) = \mu(E)$.

For the last part of the proof, let μ be an arbitrary measure. And let $E \in \mathcal{M}$. If $\mu(E) < +\infty$, then $\nu(E) = 0$ would suffice (this proves the first property of the measure). If $\mu(E) = +\infty$, and if $\mu(E)$ is not semi-finite, then set $\nu(E) = +\infty$. So that $\mu_0(E) + \nu(E) = 0 + \infty = \infty = \mu(E)$. The additivity of ν is immediate, since ν can only assume two values. This finishes the proof. ■

Exercise 1.16

Proposition 16.1.

Proof. ■

Exercise 1.17

Proposition 17.1. *Let* $\{A_j\}_{j\geq 1}$ *be a countable disjoint sequence in* $\mathcal{M}$, *and denote* $B_n = \bigcup A_{j\leq n} \in \mathcal{M}$. *For every* $E \subseteq \mathbf{X}$,

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j\leq n})$$

Proof. Proven in Theorem 1.11 as a Lemma. ■

Exercise 1.18

Proposition 18.1. *Let $\mathcal{A} \subseteq \mathbb{P}(\mathbf{X})$ be an algebra. $\mathcal{A}_\sigma$ the collection of countable unions of sets in $\mathcal{A}$, and $\mathcal{A}_{\sigma\delta}$ the collection of countable intersection of sets in $\mathcal{A}_\sigma$. Let μ_0 be a premeasure on $\mathcal{A}$, and μ^* be the induced outer-measure.*

(a) For any $E \subseteq \mathbf{X}$, and $\varepsilon > 0$, there exists $A \in \mathcal{A}_\sigma$ with $E \subseteq A$ and $\mu^*(A) \leq \mu^*(E) + \varepsilon$.

(b) If $\mu^*(E) < +\infty$, then E is μ^*-measurable $\iff$ there exists $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$ and $\mu^*(B \setminus E) = 0$.

(c) If μ_0 is σ-finite, the restriction $\mu^*(E) < +\infty$ in (b) is superfluous.

Proof of Part A. Let $E \subseteq \mathbf{X}$ and $\varepsilon > 0$, then by definition of μ^*,

$$\mu^*(E) + \varepsilon \geq \sum \mu_0(A_j) = \sum \mu^*(A_j) \geq A$$

by subadditivity and $A = \bigcup A_j$. ∎

Proof of Part B. Suppose E is outer-measurable and of finite outermeasure, then by part A we have a sequence of $A_n \in \mathcal{A}_\sigma$ with

$$\mu^*(E) + n^{-1} \geq \mu^*(A_n) \implies \mu^*(E) = \mu^*(A)$$

if we define $A = \bigcap A_n \supseteq E$. Using the μ^*-measurability of E, we get

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) < +\infty \implies \mu^*(A \setminus E) = 0$$

Conversely, if $\mu^*(A \setminus E) = 0$, for any $V \subseteq \mathbf{X}$, with $\mu^*(V) < +\infty$, we have

$$\begin{aligned}\mu^*(V) &= \mu^*(V \cap A) + \mu^*(V \setminus A) \\ &\geq \mu^*(V \cap E) + \mu^*(V \setminus A) + \mu^*(V \cap [A \setminus E]) \\ &\geq \mu^*(V \cap E) + \mu^*(V \setminus E)\end{aligned}$$

∎

Proof of Part C. Suppose μ_0 is σ-finite, then $E \in \mathcal{M}^*$ induces a sequence $E_j \nearrow E$, where each E_j is of finite measure. By part b) we obtain $\{A_j\} \subseteq \mathcal{A}_{\sigma\delta}$ with

$$\mu^*(A_j \setminus E_j) = 0$$

Now define $B = \bigcup A_j$, so that $B \in \mathcal{A}_{\sigma\delta}$. Observe $\bigcup(A_j \setminus E_j) = B \setminus E_1 \supseteq B \setminus E$ (verify these). And $\mu^*(B \setminus E) = \leq \sum \mu^*(A_j \setminus E_j) = 0$ by subadditivity. Since $B \supseteq E$, and $B \in \mathcal{A}_{\sigma\delta}$, this proves $\implies$.
Conversely, suppose $E \subseteq \mathbf{X}$ and there exists a $B \in \mathcal{A}_{\sigma\delta}$ with $E \subseteq B$, $\mu^*(B \setminus E) = 0$. Let $\{\mathbf{X}_j\} \nearrow \mathbf{X}$ as a sequence of sets of finite measure. Then,

$$(\mathbf{X}_j \cap B) \setminus (\mathbf{X}_j \cap E) = \mathbf{X}_j \cap (B \setminus E) \subseteq B \setminus E$$

$\mathbf{X}_j \cap B \in \mathcal{A}_{\sigma\delta}$, and $\mathbf{X}_j \cap B \supseteq (\mathbf{X}_j \cap E)$. Each $E_j = \mathbf{X}_j \cap E$ is μ^* measurable by monotonicity, so is their countable union. ■

Exercise 1.19

Proposition 19.1. *Let μ^* be an outer measure on $\mathbf{X}$ induced from a finite premeasure μ_0. If $E \subseteq \mathbf{X}$, define the inner measure of E to be $\mu_*(E) = \mu_0(\mathbf{X}) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$.*

Proof. Suppose $E \subseteq \mathbf{X}$ is μ^*-measurable. Then

$$\mu^*(\mathbf{X}) = \mu^*(\mathbf{X} \cap E) + \mu^*(\mathbf{X} \setminus E) = \mu_0(\mathbf{X})$$

Rearranging gives the result, since all quantities are finite.

If $\mu^*(E) = \mu_*(E)$, then $\mu^*(E^c) = \mu_*(E^c)$, since the definition of μ_* is symmetric. Let $B \in \mathcal{A}_{\sigma\delta}$, with $\mu^*(B) = \mu^*(E)$, $E \subseteq B$. We can always find such a B by taking the intersection over all $B_n \in \mathcal{A}_\sigma$,

$$\mu^*(E) + n^{-1} \geq \sum_j \mu^*(B(j,n)) \geq \mu^*\left(\bigcup_j B(j,n) = B_n\right)$$

Notice $E \subseteq B \iff E^c \supseteq B^c \iff E^c \cap B^c = B^c$.Since B is μ^*-measurable, we have

$$\begin{aligned}
\mu^*(E^c \cap B) + \mu^*(E^c \setminus B) &= \mu^*(E^c) \\
&= \mu^*(\mathbf{X}) - \mu^*(E) \\
\mu^*(B \setminus E) + \mu^*(B^c) &= \mu^*(X) - \mu^*(E) \\
&= \mu^*(B) + \mu^*(B^c) - \mu^*(E) \\
\mu^*(B \setminus E) &= \mu^*(B) - \mu^*(E) \\
&= 0
\end{aligned}$$

■

Exercise 1.20

Proposition 20.1.

Proof. ■

Exercise 1.21

Proposition 21.1. *Let μ^* be an outermeasure induced from a premeasure, and $\overline{\mu} = \mu^*|_{\mathcal{M}^*}$, where $\mathcal{M}^*$ denotes the family of μ^*-measurable sets. Show that $\overline{\mu}$ is saturated. That is, $\widetilde{\mathcal{M}^*} = \mathcal{M}^*$*

Proof. Suppose E is locally measurable (with respect to $\overline{\mu} = \mu^*|_{\mathcal{M}^*}$). Fix $V \subseteq \mathbf{X}$, with $\mu^*(V) < +\infty$. It suffices to show $\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$.

By 18a), find a $V' \in \mathcal{A}_{\sigma\delta}$, with $V \subseteq V'$, and $\mu^*(V') = \mu^*(V) < +\infty$. so that $E \cap V'$ is μ^*-measurable.

$$\mu^*(V) = \mu^*(V \cap E \cap V') + \mu^*(V \setminus (V \cap (V' \cap E))$$

therefore

$$\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$$

∎

Exercise 1.22

Proposition 22.1.

Proof. To show $\overline{\mu}$ is complete, Fix $U \subseteq F$, where $F \in \mathcal{M}^*$, with $\overline{\mu}(F) = 0$ Let $F' \in \mathcal{A}_{\sigma\delta}$, with $F' \supseteq F$, and

$$\mu^*(F') = \mu^*(F) \geq \mu^*(F' \setminus U)$$

Since $F' \supseteq U$, applying Exercise 18b gives $\overline{\mathcal{M}^*} \subseteq \mathcal{M}^*$. For the other direction,

∎

Exercise 1.23

Proposition 23.1.

Proof. ■

Exercise 1.24

Proposition 24.1. *If μ is a finite measure on $(X, \mathcal{M})$, and let μ^* be the outer measure. Suppose that $E \subseteq X$ satisfies $\mu^*(E) = \mu^*(X)$ (but $E \notin \mathcal{M}$ necessarily). Show that*

(a) For any $A, B \in \mathcal{M}$, and $A \cap E = B \cap E$, then $\mu(A) = \mu(B)$.

(b) Let $\mathcal{M}_E = \{A \cap E,\ A \in \mathcal{M}\}$, and define ν on $\mathcal{M}$ with $\nu(A \cap E) = \mu(A)$. Then $\mathcal{M}_E$ is a σ-algebra, and ν is a measure on $\mathcal{M}_E$.

Proof of Part A.

$$\mu^*(E) = \mu^*(X) \implies \mu^*(X \setminus E) = 0$$

This is a simple consequence of the μ^*-measurability of X, since $X \in \mathcal{M}$, and the μ is a pre-measure on $\mathcal{M}$, b And by monotonicity,

$$\begin{cases} A \cap (X \setminus E) \subseteq (X \setminus E) \\ B \cap (X \setminus E) \subseteq (X \setminus E) \end{cases} \implies \begin{cases} \mu^*(A \cap (X \setminus E)) = 0 \\ \mu^*(B \cap (X \setminus E)) = 0 \end{cases}$$

Write $A \cap X = (A \cap E) \cup (A \cap X \setminus E)$, and by subadditivity of μ^*,

$$\begin{aligned} \mu(A) &= \mu^*(A \cap X) \\ &\leq \mu^*(A \cap E) + \mu^*(X \setminus E) \\ &= \mu^*(B \cap E) \\ &\leq \mu^*(B \cap X) \\ &= \mu(B) \end{aligned}$$

Therefore $\mu(A) \leq \mu(B)$, and $\mu(B) \leq \mu(A)$ is trivial.

■

Proof of Part B. We want to show $\mathcal{M}_E$ is a σ-algebra.

- Closure under complements,

$$\forall A \cap E \in \mathcal{M}_E, A \in \mathcal{M} \implies (E \setminus A^c) \in \mathcal{M}$$

  Therefore $(E \setminus A^c) \cap E \in \mathcal{M}_E$. Note that the question mentions that $\mathcal{M}_E$ is a σ-algebra on E, therefore we take complements relative to E.

- Closure under countable unions. Fix any countable sequence $\{A_j \cap E\} \subseteq \mathcal{M}_E$ where $\{A_j\} \subseteq \mathcal{M}$. It is obvious that $A = \cup A_j \in \mathcal{M}$, therefore $\cup(A_j \cap E) = E \cap A \in \mathcal{M}_E$ as well.

Since $\nu(\varnothing) = \mu(\varnothing \cap E) = 0$, and for countable additivity, fix any disjoint sequence $\{A_j \cap E\}_{j\geq 1} \subseteq \mathcal{M}_E$, where $\{A_j\}_{j\geq 1} \subseteq \mathcal{M}$, and let $A = \bigcup A_{j\geq 1}$

$$\begin{aligned}
\nu(A \cap E) &= \mu(A) \\
&= \sum \mu(A_{j\geq 1}) \\
&= \sum \nu(A_{j\geq 1} \cap E)
\end{aligned}$$

■