MATH 263: Section 003, Tutorial 2

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1 Review of the Material from Week 1

1.1 Ordinary and Partial Differential Equations

Ordinary Differential Equations (ODE's) are differential equations involving a single variable function and its derivatives. For example:

$$y''(x) + y(x) = \cos x$$

Partial Differential Equations (PDE's) are differential equations involving a multi-variable function and its partial derivatives. For example:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

1.2 Order of a Differential Equation (DE)

The **order of a DE** corresponds to the highest derivative it contains. For example,

$$y^{(69)}(x) + y(x)^2 = \sin x$$

is a 69^{th} order ODE.

1.3 Verify Whether a Function Solves a DE

Given a solution to verify, one simply needs to compute its derivatives and substitute them in the differential equation.

1.4 Initial and Boundary Value Problems and Conditions

An **initial value problem** (IVP) is a differential equation with initial value conditions. Those conditions are restrictions on the solution's value and derivatives at a point, such as $y(0) = 1,\ y'(1) = 0$.
A **boundary value problem** (BVP) uses boundary value conditions, which are multiple restrictions on the solution's value, such as $y(0) = 1,\ y(1) = -1,\ y(2) = 7$. In general, an n^{th} order ODE will require n initial conditions to produce a unique solution.

1.5 Autonomous ODE's

Autonomous ODE's only contain the dependent variable, they are of the form:

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)})$$

1.6 Linear and Non-Linear ODE's

A **linear ODE** can be written as a linear combination of y and its derivatives as such:

$$\sum_{k=0}^{n} a_k(x)\ y^{(k)} = g(x)$$

An example would be:

$$x^2\ y''(x) + 2x\ y'(x) - y(x) = \cos x$$

Otherwise, the ODE is **non-linear**.
Note: when the right hand side $g(x)$ is 0, the ODE is also **homogeneous**.

1.7 Slope Fields

A **slope field** is a graphical representation of a family of functions satisfying $y' = f(x,y)$. For some point (x,y), one draws the slope $y' = f(x,y)$ to qualitatively represent the solutions. Given a slope field, starting at an initial condition and tracing along the field sketches the particular solution.

2 Tutorial 2

2.1 Separable ODE's

A **separable ODE** is of the form:

$$\frac{dy}{dx} = f(x)g(y)$$

Problem 2.1. Solve the IVP:

$$\frac{dy}{dx} = \frac{x}{y}\sqrt{1+x^2}$$

for $y(0) = -\sqrt{\frac{5}{3}}$.

Solution: Bring all the x's and dx's on one side, and all the y's and dy's on the other side:

$$y\ dy = x\sqrt{1+x^2}\ dx$$

Then, integrate both sides:

$$\int y\ dy = \int x\sqrt{1+x^2}\ dx$$

Using integration by parts, we obtain:

$$\frac{1}{2}y^2 + C_1 = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + C_2$$

Note: don't forget your constants of integration!

$$y^2 = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + 2(C_2 - C_1)$$

Let $C_0 = 2(C_2 - C_1)$:

$$y^2 = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C_0$$

$$y = \pm\sqrt{\frac{2}{3}(1+x^2)^{\frac{3}{2}} + C_0}$$

$y(0) = -\sqrt{\frac{5}{3}}$. Since $y(x) \leqslant 0)$, take the negative root:

$$y(0) = -\sqrt{\frac{5}{3}} = -\sqrt{\frac{2}{3} + C_0}$$

$$\frac{5}{3} = \frac{2}{3} + C_0$$
$$C_0 = 1.$$

Therefore,

$$y(x) = -\sqrt{1 + \frac{2}{3}(1+x^2)^{\frac{3}{2}}}.$$

2.2 Solving First Order Linear ODE's: Integrating Factors

A **first order linear ODE** is of the form:

$$y' + p(x)y = q(x)$$

Problem 2.2a. Determine the general solution of:

$$xy' + 2y = e^{-x}$$

Then, determine the solution's long term behaviour.

Solution: First divide both sides by x:

$$y' + \frac{2}{x}\ y = \frac{1}{x}\ e^{-x}$$

Then, find an integrating factor μ to simplify the left side:

$$\mu y' + (\frac{2}{x}\mu)\ y = \mu\frac{1}{x}\ e^{-x}$$

We want $\frac{2}{x}\mu = \mu' = \frac{d\mu}{dx}$ since $\frac{d}{dx}(\mu y) = \mu y + \mu' y$:

$$\frac{2}{x}dx = \frac{1}{\mu}d\mu$$

$$\mu = e^{\int \frac{2}{x}\ dx} = e^{2\ln|x|}$$
$$\mu = x^2$$

Now, undo the product rule from the left side:

$$\frac{d}{dx}(x^2 y) = x^2\ \frac{1}{x}\ e^{-x} = x\ e^{-x}$$

$$x^2 y = \int x\ e^{-x}\ dx$$

$$x^2 y = -x\ e^{-x} - e^{-x} + C$$
$$y(x) = -e^{-x}\ \frac{x+1}{x^2} + \frac{C}{x^2}$$

To find the long term behaviour, find $\lim_{x\to\infty} y(x)$:

$$\lim_{x\to\infty} -e^{-x}\ \frac{x+1}{x^2} + \frac{C}{x^2}$$

$$= \lim_{x\to\infty} -e^{-x}\ (\frac{1}{x} + \frac{1}{x^2}) + \frac{C}{x^2} = 0$$

Note: since $p(x) = \frac{2}{x}$, which is not defined at $x = 0$, $\lim_{x\to 0} y(x)$ does not exist (Existence and Uniqueness Theorem).

Problem 2.2b. Solve the IVP:

$$\cos x\ y' + \sin x\ y = \tan x$$

for $y(x_0) = 1,\ 0 \le x_0 \le \frac{\pi}{2}$. For which value(s) of x_0 does the IVP have no solution?

Solution: divide both sides by $\cos x$:

$$y' + \tan x\ y = \tan x \sec x$$

The integrating factor μ is

$$\mu = e^{\int \tan x\ dx} = e^{\ln|\sec x|} = |\sec x| = \sec x$$

Note: $|\sec x| = \sec x$ since $\sec x = \frac{1}{\cos x} \ge 0$ for $0 \le x_0 < \frac{\pi}{2}$. Multiplying by the integrating factor we get:

$$\frac{d}{dx}(y \sec x) = \tan x \sec^2 x$$

$$y \sec x = \int \tan x \sec^2 x\ dx$$

Making the substitution $u = \tan x$, we get:

$$y \sec x = \frac{1}{2}\tan^2 x + C_0 = \frac{1}{2}\sec^2 x + C_0 - \frac{1}{2} = \frac{1}{2}\sec^2 x + C_1$$

$$y(x) = \frac{1}{2}\sec x + C_1 \cos x.$$

IVP: $y(x_0) = 1$. Note that for $x_0 = \frac{\pi}{2}$, the IVP does not have a solution. Applying the Existence and Uniqueness Theorem, this is because $p(x) = \tan x$ and $q(x) = \tan x \sec x$, which are not defined at $x_0 = \frac{\pi}{2}$. A more appropriate IVP with a unique solution would be $y(x_0 = 0) = 1$:

$$y(0) = 1 = \frac{1}{2}\sec 0 + C_1 \cos 0.$$

$$1 = \frac{1}{2} \cdot 1 + C_1 \cdot 1.$$

$$C_1 = \frac{1}{2}$$

Therefore, the solution to the second IVP is:

$$y(x) = \frac{1}{2}\sec x + \frac{1}{2}\cos x.$$

2.3 Homogeneous First Order ODE's

A **homogeneous ODE** is of the form:

$$\frac{dy}{dx} = F(\frac{y}{x})$$

Note: **not** the same as the definition given in 1.6.
Let $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow y' = xv' + v$. Then substitute and solve for v to find y.

Problem 2.3. Determine the general solution of:

$$xy' = y + x\ e^{\frac{y}{x}}$$

Solution:

$$y' = \frac{y}{x} + e^{\frac{y}{x}}$$

Using the substitution from above:

$$xv' + v = v + e^v$$
$$xv' = x\frac{dv}{dx} = e^v$$
$$\int e^{-v}\ dv = \int \frac{1}{x}\ dx$$
$$-e^{-v} = C + \ln|x|$$
$$-v = \ln(C_0 - \ln|x|),\ (C_0 = -C)$$
$$v = -\ln(C_0 - \ln|x|)$$
$$y = vx = -x\ln(C_0 - \ln|x|)$$

Note: Other types of substitution to solve ODE's exist, such as $v = y'(x)$ *or* $v = ax + by$.

2.4 Bernoulli Equations

A **Bernoulli equation** is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

When $n \notin \{0, 1\}$, we can let $v = y^{1-n}$, making the ODE linear for v.

Problem 2.4. Solve the IVP:

$$y' + \frac{y}{x} = xy^3$$

for $x > 0$ and $y(1) = \frac{1}{2}$.

Solution: $n = 3$, so let $v = y^{1-n} = y^{-2} \Rightarrow y = v^{\frac{-1}{2}} \Rightarrow y' = \frac{-1}{2}v^{\frac{-3}{2}}v'$. Substituting back in the ODE:

$$\frac{-1}{2}v^{\frac{-3}{2}}v' + \frac{1}{x}\ v^{\frac{-1}{2}} = x\ v^{\frac{-3}{2}}$$

Multiply everything by $v^{\frac{3}{2}}$, which makes the ODE linear for v:

$$\frac{-1}{2}v' + \frac{1}{x}\ v = x$$
$$v' - \frac{2}{x}\ v = -2x$$
$$\mu = e^{\int \frac{-2}{x}\ dx} = x^{-2}$$
$$\frac{d}{dx}(x^{-2}v) = -2xx^{-2} = \frac{-2}{x}$$
$$(x^{-2}v) = \int \frac{-2}{x}\ dx = -2\ln x + C$$
$$v = x^2(C - 2\ln x)$$
$$y = v^{\frac{-1}{2}} = \frac{1}{\sqrt{v}} = \frac{1}{x\sqrt{(C - 2\ln x)}}$$

Now, the constant C is:

$$y(1) = \frac{1}{2} = \frac{1}{1\sqrt{(C - 2\ln 2)}}$$
$$2 = \sqrt{C} \Rightarrow C = 4.$$

Therefore,

$$y(x) = \frac{1}{x\sqrt{(4 - 2\ln x)}}.$$