Chapter 1: Topological Manifolds

Topological Manifolds

The study of differential geometry begins with tens of pages of definitions.

Definition 1.1: Topological Manifold

Let M be a topological space. M is a topological manifold of dimension m if it is Hausdorff, second-countable, and locally homeomorphic to $\mathbb{R}^n$.

Definition 1.2: Local homeomorphism

M locally homeomorphic to $\mathbb{R}^n$ if every point $x \in M$ an open set U, equipped with a homeomorphism which sends points in U into an open subset of $\mathbb{R}^n$.

$$\phi : U \to \phi(U)$$

The tuple (U, ϕ) is called a coordinate chart.

Definition 1.3: More on coordinate charts

- A coordinate chart (U, ϕ) is centered at $p \in M$ if $p \in U$ and $\phi(p) = 0 \in \mathbb{R}^n$.
- We call U the coordinate domain, and
- we call ϕ the coordinate map.
- If the choice of (U, ϕ) is unambiguous, then the local coordinates of p are simply the coordinates of $\phi(p)$ in $\mathbb{R}^n$, and
- we sometimes also denote $\phi(U)$ by $\hat{U}$ if it is unambiguous to do so.
- If $\hat{U}$ is an open ball/cube, then U is called a coordinate ball/cube.

The central theme of point-set topology (or even metric topology) is that of passing a topological argument to the basis or to a neighbourhood. Manifolds in particular have a nice basis.

Proposition 1.1: Basis of precompact coordinate balls

Every topological manifold has a countable basis of precompact coordinate balls.

Proposition 1.2: Additional facts about topological manifolds

If M is a topological manifold,

- M is locally compact. (Lee, Proposition 1.12)
- M is paracompact, and every open cover has a refinement that is another countably locally finite open cover whose elements are chosen from an arbitrary (but fixed) basis of M. (Lee, Theorem 1.15)
- M is locally-path connected.
- M is connected iff it is path-connected.
- M is metrizable. (Munkres Chapter 6)

Smooth Manifolds

We wish to perform calculus on manifolds.

Definition 2.1: Smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$

Let $F : \mathbb{R}^n \to \mathbb{R}^m$, replacing $\mathbb{R}^n$ and $\mathbb{R}^m$ with open subsets if necessary. F is smooth if its (scalar-valued) component functions has continuous partial derivatives of all orders. The set of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ is sometimes denoted by $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$. If $m = 1$, we sometimes write $C^\infty(\mathbb{R}^n)$, similar to a test function on the Schwartz Space.

Definition 2.2: Transition map from ϕ to ψ

Let (U, ϕ) and (V, ψ) be coordinate charts on M. The composite function (whenever $U \cap V \neq \varnothing$)

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is called the transition map. Notice $\psi \circ \phi^{-1}$ is by definition a homeomorphism.

Definition 2.3: Smoothly compatiable

Two coordinate charts on M, (U, ϕ) and (V, ψ) are called smoothly compatible if either their domains are disjoint, or their transition map is a diffeomorphism on $\mathbb{R}^m$.

Definition 2.4: Smooth atlas

An atlas $\mathcal{A}$ of M is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ whose collection of coordinate domains $\{U_\alpha\}$ for an open cover of M.

It is called a smooth atlas if any two charts in the atlas are pairwise smoothly compatible.

Definition 2.5: Smooth manifold

A smooth atlas $\mathcal{A}$ on M is maximal if it is not contained (properly) in any other smooth atlas as a subset. In other words, if (U', ϕ') is a chart on M that is smoothly compatible with all elements in $\mathcal{A}$, then $(U', \phi') \in \mathcal{A}$ already.

This smooth atlas is often very large, it includes all translations of charts, dilations, and composition with diffeomorphisms in $\mathbb{R}^m$, restrictions onto open subsets, etc. A maximal smooth atlas is sometimes called a complete atlas, or a smooth manifold structure.

A smooth manifold is the tuple $(M, \mathcal{A})$, where $\mathcal{A}$ is some smooth atlas. It can happen if M is originally a topological manifold with a huge number of charts, some of which are smoothly compatible with others, that $\mathcal{A}$ is a strict subset, and both $\mathcal{A}_1$ and $\mathcal{A}_2$ are maximal smooth atlases on M, but $\mathcal{A}_1 \neq \mathcal{A}_2$. We often omit $\mathcal{A}$ and write M if the smooth atlas is understood or not of importance.

Definition 2.6: Smooth coordinate terminologies

Let $(M, \mathcal{A})$ be a smooth manifold.

- Any coordinate chart $(U, \phi) \in \mathcal{A}$ is called a smooth chart, similar to definition 1.3
- We call U the *smooth coordinate domain* or *smooth coordinate neighbourhood* of any $p \in U$, and
- we call ϕ the *smooth coordinate map*.
- The terms *smooth coordinate ball* and *smooth coordinate cube* are used similarly.
- A set $B \subseteq M$ is a *regular coordinate ball* if its image is a smooth coordinate ball centered at the origin; and the closure of this ball in $\mathbb{R}^m$ is a subset of the image of another smooth coordinate ball, centered at the origin.

Definition 2.7: Standard smooth structure on $\mathbb{R}^n$

The maximal smooth atlas containing $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is called the *standard smooth structure on* $\mathbb{R}^n$.

Manifolds with boundary are not as important as regular manifolds for now, but they are worth mentioning.

Definition 2.8: Closed n-dimensional upper half-plane $\mathbb{H}^n \subseteq \mathbb{R}^n$

We define the following symbols for the upper half plane.

- $\mathbb{H}^n = \left\{x \in \mathbb{R}^n,\ x^n \geq 0\right\}$,
- $\operatorname{Int}\mathbb{H}^n = \left\{x \in \mathbb{R}^n,\ x^n > 0\right\}$,
- $\partial\mathbb{H}^n = \left\{x \in \mathbb{R}^n,\ x^n = 0\right\}$

Definition 2.9: Manifolds with boundary

A topological space M is called a manifold with boundary if it is Hausdorff, second-countable, and locally homeomorphic to an open subset of $\mathbb{H}^n$ (endowed with the subspace topology from $\mathbb{R}^n$).

A chart (U, ϕ) is an *interior chart* if its coordinate image is disjoint from the 'boundary' of the upper-half plane. This means $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$. Similarly, (V, ψ) is a *boundary chart* if its range contains a point in $\partial\mathbb{H}^n$; so $\psi(V) \cap \partial\mathbb{H}^n \neq \varnothing$.

Similar to definition 1.3 and definition 2.6, we use the terms *coordinate half-ball*, *coordinate half-cube*, *regular coordinate half-ball*.

Let $p \in M$, it is called an *interior point of M* (not to be confused with the topological interior) if it is in the domain of some interior chart, and p is called a *boundary point of M* if there exists a boundary chart that sends p into $\partial\mathbb{H}^n$. The set of interior points and boundary points of M will be denoted by $\operatorname{Int} M$ and ∂M.

Example 2.1: Sphere as a topological manifold

The n-sphere as a topological manifold. Define

$$S^n = \left\{ x \in \mathbb{R}^{n+1},\ |x| = 1 \right\}$$

We claim that $\{U_i^{\pm}\}_{i=1}^{n+1}$ form an open cover, where

$$U_i^+ = \left\{ x \in S^n, x^i > 0 \right\} \quad U_i^- = \left\{ x \in S^n, x^i < 0 \right\}$$

Each $U_i^{\pm}$ is the inverse image of $\pi_i^{-1}((0, +\infty)) \cap S^n$ or $\pi_i^{-1}((0, -\infty)) \cap S^n$, hence open. For every $x \in S^n$, there exists at least some $1 \le j \le n+1$ that makes the j-th coordinate of x, $x^j \neq 0$. So

$$S^n = \bigcup_i U_i^{\pm}$$

Denote the unit ball $\left\{ x \in \mathbb{R}^n,\ |x| < 1 \right\}$ in $\mathbb{R}^n$ by $\mathbb{B}^n$.

Chapter 2: Smooth Maps

Smooth Maps

Definition 1.1: Smooth functions $C^\infty(M, \mathbb{R}^k)$

Let $F : M \to \mathbb{R}^k$ be a vector-valued function on a smooth manifold M. We say F is a smooth function if for every $p \in M$, there exists a smooth chart $p \in (U, \phi)$ such that the *coordinate representation of F at p, with respect to* (U, ϕ) is a smooth function from $\mathbb{R}^m$ to $\mathbb{R}^k$, denoted by $\hat{F}$ (in the sense of Definition 2.1).

$$\hat{F} = F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^k \in C^\infty(\phi(U), \mathbb{R}^k)$$

if $k = 1$, then we denote the space of *test functions* on M by $C^\infty(M) = C^\infty(M, \mathbb{R})$

Definition 1.2: Smooth maps between manifolds $C^\infty(N, M)$

Let $F : N \to M$ be a map between smooth manifolds N and M (note we switched the order). F is a smooth map if at every $p \in M$, there exists

- a chart in the smooth atlas of N (the domain), $p \in (U, \phi)$,
- another chart in the smooth atlas of M (the range), $F(U) \subseteq (V, \psi)$,
- such that, the *coordinate representation of F at p with respect to* (U, ϕ), *and* (V, ψ) is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}^m$, also denoted by $\hat{F}$.

$$\hat{F} = \psi \circ F \circ \phi^{-1} : \phi(U) \to \psi(V) \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \tag{1}$$

The following propositions summarizes common operations on smooth maps, a few sources of them.

Proposition 1.1: Smooth maps are continuous

If $F : N \to M$ is a smooth map, then F is continuous with respect to the topologies on N and M.

Proof. Let $p \in N$ be fixed, because F is smooth this induces two smooth charts, one in the domain and another in the range; as in definition 1.2. $F(p)$ is a point in M. From eq. (1), $\hat{F}|\phi(U)$ is a smooth hence continuous function. Since $\phi : U \mapsto \phi(U)$ and $\psi : V \mapsto \psi(V)$ are homeomorphisms,

$$F|U = \underbrace{\psi^{-1}}_{\text{continuous}} \circ \underbrace{\hat{F}|\phi(U)}_{\text{smooth}} \circ \underbrace{\phi}_{\text{continuous}} \quad \text{is continuous on} \quad U$$

Let the point p range through all the points in N, so F is continuous at every p, hence on N. ■

Proposition 1.2: Characterizations of Smooth Maps in $C^\infty(N, M)$

Let N and M be smooth manifolds, and $F : N \to M$. F is a smooth map iff

- For every $p \in N$, there exists smooth charts $p \in (U, \phi)$ and $F(p) \in (V, \psi)$ such that $U \cap F^{-1}(V)$ is an open set in N, and the composite map (the coordinate representation)
$$\psi \circ F \circ \phi^{-1}|(U \cap F^{-1}(V)) : \phi(U \cap F^{-1}(V)) \to \psi(V) \quad \text{is smooth}$$
- F is continuous and there exist smooth atlases $\{(U_\alpha, \phi_\alpha)\} \subseteq \mathcal{A}_N$, and $\{(V_\beta, \psi_\beta)\} \subseteq \mathcal{A}_M$ such that the coordinate representation
$$\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi(U_\alpha \cap F^{-1}(V_\beta)) \to \psi(V_\beta) \quad \text{is smooth}$$
whenever it makes sense.
- the restriction of F onto any arbitrary open set U, $F|U : U \mapsto M$ is smooth (in the sense of open submanifold).

Proof. By Proposition 1.1, and the fact that complete atlases are closed under restrictions onto open sets, it is clear that the original definition implies the two. The first definition also clearly implies the original definition, as we can restrict
$$(U, \phi) \mapsto \Big(U \cap F^{-1}(V), \phi|(U \cap F^{-1}(V))\Big)$$
since $U \cap F^{-1}(V)$ is open in the domain manifold.

The second definition implies the original one as well, since the smooth atlases are taken from the maximal atlas, we can pass the argument to any smoothly-compatible chart. Atlases must cover both the domain and the range, and coordinate transitions between smoothly compatible charts are diffeomorphisms. If F is smooth on a subcollection of those charts, meaning
$$\psi_\beta \circ F \circ \phi_\alpha^{-1} \in C^\infty(\phi_\alpha(U_\alpha) \cap F^{-1}(V_\beta), \psi_\beta(V_\beta))$$
it is smooth with respect to every pair of (smooth) charts in the two atlases $\mathcal{A}_N$, $\mathcal{A}_M$, as a composition of smooth maps:
$$\psi \circ \underbrace{\psi^{-1} \circ \psi_\beta}_{\text{smooth}} \circ F \circ \underbrace{\phi_\alpha^{-1} \circ \phi}_{\text{smooth}} \circ \phi^{-1}$$
where we can restrict $\phi_\alpha \mapsto \phi_\alpha|(U_\alpha \cap F^{-1}(V_\beta))$ by continuity of F.
I will prove the third and last equivalence later. ■

Proposition 1.3: Sources of smooth maps

Let N, M, P be smooth manifolds, then

- Every constant map is smooth,
- The identity map $\mathrm{id}_M : M \to M$ is smooth,
- The inclusion map $\iota : W \to M$ is smooth, where W is an open submanifold of M.
- The composition of smooth maps is again a smooth map: if $F \in C^\infty(N, M)$ and $G \in C^\infty(M, P)$, then $(G \circ F) \in C^\infty(N, P)$

Diffeomorphisms

Definition 2.1: Diffeomorphism between Manifolds $\mathcal{D}(N, M)$

Let N and M be smooth manifolds, $F : N \to M$ is a diffeomorphism if it is a smooth bijective map with a smooth inverse. We denote the space of diffeomorphisms from N to M by $\mathcal{D}(N, M)$.

Proposition 2.1: Properties of Manifold Diffeomorphisms

Let N, M and P be smooth manifolds, then

- The composition of diffeomorphisms is again a diffeomorphism, that is, if $F \in \mathcal{D}(N, M)$ and $G \in \mathcal{D}(M, P)$, then $(G \circ F) \in \mathcal{D}(N, P)$.
- The open-manifold restriction of a diffeomorphism onto its image is again a diffeomorphism,
- Every diffeomorphism is a homeomorphism and an open map.

Proof. Trivial. ■

Partitions of Unity

See Folland Chapters 4 and 8. Including Urysohn's Lemma, Tietze's Extension Theorem, the usual construction of C_c^∞ bump functions.

Chapter 3: Tangent Spaces

Algebra of Germs on $C^\infty(N)$

The tangent space is a powerful concept that acts almost like the dual in distribution theory.

Definition 1.1: Algebra of Germs at p: $C_p^\infty(N)$

Let N be a smooth manifold and $p \in N$. We define an equivalence relation on the space of test functions on N, $C^\infty(N)$. If $f, g \in C^\infty(N)$, we write $f \sim g$ if $f = g$ for some open neighbourhood about p. We denote this equivalence class by $C_p^\infty(N)$, and it is clear $C^\infty(N)$ is closed under pointwise multiplication by the product rule, and form an algebra; so $C_p^\infty(N)$ is an algebra too.

Tangent spaces of manifolds

Definition 2.1: Vector space of derivations at p: T_pN

Let $\nu : C_p^\infty(N) \to \mathbb{R}$ be a linear functional on the vector space of germs at p. It is called a derivation at p if it satisfies the product rule, if $f, g \in C_p^\infty(N)$,

$$\nu(fg) = g(p)\nu(f) + f(p)\nu(g)$$

then we say

- ν is a tangent vector at p,
- $\nu \in T_pN$,
- ν is an element of the *tangent space at p*.
- ν is a derivation on N at p.

Proposition 2.1: Properties of derivations at p

Let N be a smooth manifold and $p \in N$.

- If $f \in C_p^\infty$ is constant in some neighbourhood of p, then $\nu(f) = 0$ for every $\nu \in T_pN$,
- If $f(p) = g(p) = 0$, then $\nu(fg) = 0$ for tangent vector ν at p.

Tangent spaces of $\mathbb{R}^n$

Proposition 3.1: Basis of $T_p\mathbb{R}^n$

Let $\mathbb{R}^n$ be equipped with the standard smooth structure as in Definition 2.7. The vector space of derivations at $p \in \mathbb{R}^n$ are spanned by the n partial derivatives at p

$$\left.\frac{\partial}{\partial x^j}\right|_p : f \mapsto \left.\frac{\partial}{\partial x^j} f(x)\right|_p, \quad 1 \leq j \leq n,\ f \in C^\infty(\mathbb{R}^n)$$

Moreover, the n vectors form a basis, and $\dim T_p\mathbb{R}^n = n$.

Definition 3.1: Standard Basis of $T_p\mathbb{R}^n$

The standard basis for the tangent space at $p \in \mathbb{R}^n$ is the n partial derivatives at p.

$$T_p\mathbb{R}^n = \left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p \right\} \tag{2}$$

Definition 3.2: Coordinate functions x^j on $\mathbb{R}^n$

Let $x^1, \ldots x^n$ be the standard coordinates on $\mathbb{R}^n$. If $1 \leq j \leq n$, x^j is a function (also represented by the same symbol as the standard coordinate) from $\mathbb{R}^n$ to $\mathbb{R}$, which maps each $p = (p^1, \ldots, p^j, \ldots, p^n)$ to its j-th coordinate $p \mapsto p^j$.

This map is linear, hence smooth, has the matrix representation of $\mathcal{M}\{x^j\} = (\delta_{jk})_{1,k} \in \mathbb{R}^{1\times n}$ where δ_{jk} denotes the discrete mass at j.

Furthermore, the coordinate functions behave like the dual basis for the derivations $\partial/\partial x^j|_p$ at p

$$\left.\frac{\partial}{\partial x^j}\right|_p (x^k) = \delta_{jk}$$

If $\nu \in T_p\mathbb{R}^n$, and has the basis representation

$$\nu = \sum_j \nu^j \left.\frac{\partial}{\partial x^j}\right|_p = \nu^j \left.\frac{\partial}{\partial x^j}\right|_p$$

then

$$\nu = \sum_j \nu(x^j) \left.\frac{\partial}{\partial x^j}\right|_p = \nu(x^j) \left.\frac{\partial}{\partial x^j}\right|_p \tag{3}$$

Differential of a smooth map $F \in C^\infty(N, M)$

Definition 4.1: Differential of a smooth map $dF_p : T_pN \to T_{F(p)}M$

Let $F \in C^\infty(N, M)$, $p \in N$ and $\nu \in T_pN$ be a tangent vector at p. The differential of a smooth map is a linear map that sends tangent vectors in T_pN to tangent vectors in $T_{F(p)}M$. If $f \in C^\infty(M)$ is a test function on M, then $f \circ F$ is a test function on N, and

$$dF_p(\nu)(f) = \nu\Bigg(\underbrace{f \circ F}_{C^\infty(N)}\Bigg)$$

We state the following without proof, as the proof is tedious. It simply involves unboxing the definition of the differential $dF_p : T_pN \to T_pM$, and projecting onto the coordinates.

Proposition 4.1: Properties of the differential

Let N, M and P be smooth manifolds, if $F \in C^\infty(N, M)$, $G \in C^\infty(M, P)$ then

- dF_p is a linear map between T_p and $T_{F(p)}N$,
- $d(G \circ F)_p = dG_{F(p)} \circ dF_p$
- $d(\mathrm{id}_N)_p = \mathrm{id}_{T_pN}$,
- if $F \in \mathcal{D}(N, M)$, then dF_p is a linear isomorphism between T_pN and $T_{F(p)}M$, and
$$(dF_p)^{-1} = d(F^{-1})_{F(p)}$$

Proposition 4.2: Matrix representation of the differential of $F : N \to M$

Let $F \in C^\infty(N, M)$, and $p \in N$ induces two charts $p \in (U, \phi)$ and $F(p) \in (V\, \psi)$. The matrix representation of the differential at p, $dF_p : T_pN \to T_{F(p)}M$ is nothing but the Jacobian matrix of size $m \times n$ of the coordinate representation at p.

$$\mathcal{M}\{dF_p\} = \begin{bmatrix} \frac{\partial \hat{F}^1}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial \hat{F}^1}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^1}{\partial x^n}\Big|_{\phi(p)} \\ \frac{\partial \hat{F}^2}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial \hat{F}^2}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^2}{\partial x^n}\Big|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \hat{F}^m}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial \hat{F}^m}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^m}{\partial x^n}\Big|_{\phi(p)} \end{bmatrix} \tag{4}$$

Alternately, if we write $\hat{p} = \phi(p)$ as the $\mathbb{R}^m$ coordinates at p, then

$$\mathcal{M}\{dF_p\} = \begin{bmatrix} \frac{\partial \hat{F}^1}{\partial x^1}\Big|_{\hat{p}} & \frac{\partial \hat{F}^1}{\partial x^2}\Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^1}{\partial x^n}\Big|_{\hat{p}} \\ \frac{\partial \hat{F}^2}{\partial x^1}\Big|_{\hat{p}} & \frac{\partial \hat{F}^2}{\partial x^2}\Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^2}{\partial x^n}\Big|_{\hat{p}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \hat{F}^m}{\partial x^1}\Big|_{\hat{p}} & \frac{\partial \hat{F}^m}{\partial x^2}\Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^m}{\partial x^n}\Big|_{\hat{p}} \end{bmatrix} \tag{5}$$

Differential of a smooth map $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$

An important application of this is the following. We begin with the $\mathbb{R}^m \to \mathbb{R}^n$ case. We will see that if p and $F(p)$ are represented by another pair of coordinate charts (smoothly compatible with the previous pair), then the rank of dF_p does not change. So the rank of the differential is an invariant of the choice of coordinate chart.

Definition 5.1: Matrix representation of the differential of $F : \mathbb{R}^m \to \mathbb{R}^n$

Let $F \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$, and $p \in \mathbb{R}^m$ induces two charts $p \in (U, \mathrm{id}_{\mathbb{R}^m})$ and $F(p) \in (V\ \mathrm{id}_{\mathbb{R}^n})$, where $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. The matrix representation of the differential at p, $dF_p : T_p\mathbb{R}^m \to T_{F(p)}\mathbb{R}^n$ is nothing but the Jacobian matrix of F at p.

$$\mathcal{M}\{dF_p\} = DF(p) = \begin{bmatrix} \frac{\partial F^1}{\partial x^1}\Big|_{p} & \frac{\partial F^1}{\partial x^2}\Big|_{p} & \cdots & \cdots & \frac{\partial F^1}{\partial x^m}\Big|_{p} \\ \frac{\partial F^2}{\partial x^1}\Big|_{p} & \frac{\partial F^2}{\partial x^2}\Big|_{p} & \cdots & \cdots & \frac{\partial F^2}{\partial x^m}\Big|_{p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial x^1}\Big|_{p} & \frac{\partial F^n}{\partial x^2}\Big|_{p} & \cdots & \cdots & \frac{\partial F^n}{\partial x^m}\Big|_{p} \end{bmatrix} \tag{6}$$

Change of Coordinates Matrix

We will go through the section on the Change of Coordinates, and how different coordinate charts change the representation of a derivation at $p \in M$, where M is some smooth manifold.

Definition 6.1: Standard basis of T_pN

From proposition 2.1, since ϕ^{-1} is a diffeomorphism, $d\Big(\phi^{-1}|_{\phi(p)}\Big) : T_{\phi(p)}\mathbb{R}^n \to T_pN$ is a linear isomorphism. Hence T_pN is a n-dimensional vector space, and the standard basis vectors of $T_p\mathbb{R}^n$ are denoted by

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right\} \tag{7}$$

where each basis vector $\frac{\partial}{\partial x^1}\Big|_p \triangleq d\Big(\phi^{-1}|_{\phi(p)}\Big)$ is the push-forward derivation (through ϕ^{-1}) of the j-th standard basis vector in $T_{\phi(p)}N$.

Proposition 6.1: Differential of $\psi \circ \phi^{-1} : M \to M$

Let M be a smooth manifold, and fix $p \in M$. If $\nu \in T_pM$ is given with respect to the bases

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^m}\bigg|_p \right\} \quad \text{and} \quad \left\{ \frac{\partial}{\partial y^1}\bigg|_p, \dots, \frac{\partial}{\partial y^m}\bigg|_p \right\}$$

Defined by

$$\frac{\partial}{\partial x^j}\bigg|_p \triangleq d\left(\phi^{-1}\Big|_{\phi(p)}\right)\left(\frac{\partial}{\partial x^j}\bigg|_{\phi(p)}\right) \quad \text{and} \quad \frac{\partial}{\partial y^j}\bigg|_p \triangleq d\left(\psi^{-1}\Big|_{\psi(p)}\right)\left(\frac{\partial}{\partial y^j}\bigg|_{\psi(p)}\right)$$

and we write ν in terms of the first basis

$$\nu = \nu^j \, \frac{\partial}{\partial x^j}\bigg|_p = \sum_{j=1}^m \nu^j \, \frac{\partial}{\partial x^j}\bigg|_p$$

and the second basis

$$\nu = \nu^j \, \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p = \sum_{k=1}^m \sum_{j=1}^m \nu^j \, \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p$$

If $f \in C^\infty(M)$, then

$$\nu(f) = \nu^j \, \frac{\partial}{\partial x^j}\bigg|_p f = \nu^j \, \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p f$$

Proof. Recall $\frac{\partial}{\partial x^j}\Big|_p f \triangleq \frac{\partial}{\partial x^j}\Big|_{\phi(p)} f \circ \phi^{-1}$, similarly for $\frac{\partial}{\partial y^j}\Big|_p f$. Deriving f and p and by vector space operations on T_pM, the first basis expansion gives

$$\nu^j \, \frac{\partial}{\partial x^j}\bigg|_p f = \nu^j \, \frac{\partial}{\partial x^j}\bigg|_{\phi(p)} f \circ \phi^{-1} \tag{8}$$

and the second expression reads

$$\nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_p f = \nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_{\psi(p)} f \circ \psi^{-1} \tag{9}$$

Since $f \circ \phi^{-1} \in C^\infty(\mathbb{R}^m, \mathbb{R})$, we see the expressions are indeed equal. By the chain rule, if

$$\psi \circ \phi^{-1}(x^1, \dots x^m) = (y^1, \dots y^m)$$

then

$$D(\psi \circ \phi^{-1})(\phi(p)) = \begin{bmatrix} \left.\frac{\partial y^1}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^1}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^1}{\partial x^m}\right|_{\phi(p)} \\ \left.\frac{\partial y^2}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^2}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^2}{\partial x^m}\right|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial y^m}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^m}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^m}{\partial x^m}\right|_{\phi(p)} \end{bmatrix}$$

It follows from Proposition 3.6d) that the matrix $D(\psi \circ \phi^{-1})|_{\phi(p)}$ is invertible, as $\psi \circ \phi^{-1}$ is a diffeomorphism. ■

Proposition 6.2: Rank of a dF_p is invariant under coordinate change

Let F be a smooth map between M and N, at every $p \in M$, rank dF_p is an invariant over (smoothly compatible) pairs of charts in M and N.

Proof. Let $p \in (U_1, \phi_1) \cap (U_2, \phi_2)$, and $F(p) \in (V_1, \psi_1) \cap (V_2, \psi_2)$. Where all charts are smoothly compatible if it makes sense to talk about it. Both $\phi_2 \circ \phi_1^{-1}$ and $\psi_2 \circ \psi_1^{-1}$ are diffeomorphisms, and the change of basis matrices $D(\phi_2 \circ \phi_1^{-1})\big|_{\phi_1(p)}$ and $D(\psi_2 \circ \psi_1^{-1})\big|_{\psi_1(F(p))}$ are invertible by Proposition 3.6d) again, so the ranks dF_p with respect to any of the two charts are equal.

$$\underbrace{D(\psi_2 \circ \psi_1^{-1})\Big|_{\psi_1(F(p))}}_{\text{invertible}} \Big(\mathcal{M}\{dF_p\}\Big) \underbrace{D(\phi_2 \circ \phi_1^{-1})\Big|_{\phi_1(p)}}_{\text{invertible}}$$

■

Chapter 4: Submersions, Immersions and Embeddings

Matrices

The following is of utmost importance. It states that that rank of a matrix, square or otherwise, is an 'open condition'.

Example 1.1: Lee Example 1.28 (Matrices of Full Rank)

Let $A \in \mathcal{M}(m \times n, \mathbb{R})$ be the set of $m \times n$ matrices with real entries. A has rank m iff there exists some $m \times m$ sub-matrix of A, denoted by S st S is invertible. We wish to show the set of rank-m matrices is invertible. Indeed, let

$$F : \mathcal{M}(m \times n, \mathbb{R}) \to \mathbb{R},\ \Delta_{m\times m}(A) = \sum_{\substack{S \text{ is a } m\times m \\ \text{sub-matrix of } A}} |\det\{S\}|$$

Since $S \mapsto \det\{S\}$ is continuous in the entries of S, hence continuous in the entries of A, $\Delta_{m\times m}$ is continuous.
So the set $\left\{A \in \mathcal{M}(m \times n,\ \mathbb{R}),\ \operatorname{rank} A = m\right\} = F^{-1}(\mathbb{R} \setminus \{0\})$ is open.

Estimates in vector calculus

Before proving the inverse function theorem, we will need several Lemmas

Proposition 2.1: Rudin Theorem 9.7

If A and B are in $L(\mathbf{X}, \mathbf{Y})$, then

$$\|BA\| \leq \|B\|\|A\|$$

Proof. Let $\|x\| = 1$, and

$$\|B(Ax)\| \leq \|B\|\|Ax\| \leq \|B\|\|A\|\|x\|$$

this holds for every $\|x\| = 1$, hence

$$\|BA\| \leq \|B\|\|A\|$$

■

Proposition 2.2: Rudin Theorem 9.19

Let f map a convex open set $U \subseteq \mathbb{R}^n$ into $\mathbb{R}^m$, if f is differentiable (pointwise) in U, and there exists some M st its derivative its bounded (in the operator norm)

$$\|Df(x)\| \leq M \quad x \in U$$

then, for every pair of elements x_1, x_2 in U,

$$\|f(x_1) - f(x_2)\| \leq M\|x_1 - x_2\|$$

Proof. This proof 'passes the argument' to the scalar-valued version, in short: if x_1 and x_2 are in U. Define

$$c(t) = (1-t)x_1 + tx_2$$

as the convex combination of x_1 and x_2. The takeaway intuition here is that it suffices to check on the line joining the two points', to obtain an estimate for $\|f(x_1) - f(x_2)\|$. Indeed, define

$$g(t) = f(c(t)) \text{ is a curve } g : \mathbb{R} \to \mathbb{R}^m$$

Using Theorem 5.19, of which we will state below

Proposition 2.3: Rudin Theorem 5.19

Let $g : [0,1] \to \mathbb{R}^m$, and g be differentiable on $(0,1)$, then there exists some $x \in (0,1)$ with

$$|f(b) - f(a)| \leq (b-a)|f'(x)|$$

Proof. Read from Rudin Theorem 5.19. ∎

Since $Dg(t) = Df(c(t)) \circ Dc(t)$ by the Chain Rule, and $Dc(t) = b - a$ by inspection,

$$\|Dg(t)\| = \|Df(c(t)) \circ Dc(t)\| \leq \|Df\|\|Dc\| = \|Df\|(b-a)$$

This holds for every $t \in [0,1]$. Applying Theorem 5.19 gives

$$\underbrace{\|g(1) - g(0)\|}_{\text{curve endpoints}} \leq M\|b - a\|$$

Replacing $\|g(1) - g(0)\| = \|f(x_1) - f(x_2)\|$ and $\|Df\| \leq M$ we get

$$\|f(x_1) - f(x_2)\| \leq M\|x_1 - x_2\|$$

∎

Inverse Function Theorem (Rudin)

Proposition 3.1: Rudin Theorem 9.24

Suppose $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, and $Df(a)$ is invertible for some $a \in \mathbb{R}^n$, and define $b = f(a)$. Then,

(a) there exist open sets U and V in $\mathbb{R}^n$ such that $a \in U$, $b \in V$, and f is one-to-one on U, and $f(U) = V$.

(b) if g is the inverse of f (which exists, by Part a), defined in V by $g(f(x)) = x$ for every $x \in U$ then $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

Proof of Part A. We define $Df(a) = A \in \mathbb{R}^{n\times n}$, so A is invertible, and $\|A^{-1}\| \neq 0$, where $\|\cdot\|$ denotes the operator norm. Recall all norms on finite-dimensional vector spaces are equivalent, this will be useful later.
Choose $\lambda > 0$ st

$$\lambda = \left\|A^{-1}\right\|^{-1} 2^{-1} \tag{10}$$

By continuity of $Df(x)$ at the point a, let $\lambda > 0$, this induces a $B(\delta, a)$ with $x \in B(\delta, a)$ means

$$\underbrace{\|Df(x) - Df(a)\|}_{\text{operator norm}} < \lambda \tag{11}$$

as $Df : \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n)$ takes a point in $\mathbb{R}^n$ and returns a linear map., with $L(\mathbb{R}^n, \mathbb{R}^n)$ endowed with the usual vector space structure. Fix $y \in \mathbb{R}^n$, and define

$$\phi(x) = \underbrace{x + A^{-1}(y}_{\text{offset}} - f(x))$$

this is now a function solely in x, and $\phi(x) = x \iff f(x) = y$ is clear, but such a fixed point is not necessarily unique. We claim that it is unique in $B(\delta, a)$. We will use the contractive mapping principle.

Differentiating $\phi(x)$ reads

$$D\phi(x) = \underbrace{I}_{I = A^{-1}A} - A^{-1}Df(x) = A^{-1}(A - Df(x))$$

Proposition 2.1 tells us the norm of a product is bounded above by the product of the norms. Using eqs. (10) and (11), if $x \in U$ we have

$$\|D\phi(x)\| = \left\|A^{-1}(A - Df(x))\right\| \leq \left\|A^{-1}\right\| \|A - Df(x)\| \leq 2^{-1}$$

The total derivative of ϕ is uniformly bounded in U, applying Proposition 2.2 tells us that ϕ is a contractive mapping

$$\|D\phi(x)\| \leq 2^{-1} \implies \|\phi(x_1) - \phi(x_2)\| \leq 2^{-1}\|x_1 - x_2\|$$

for x_1, x_2 in U.

To show $f|U$ is indeed a bijection, fix $y \in f(U)$ so $y = f(x)$ for some $x \in U$, and there can only be one fixed point stemming from $\phi|U$, with $\phi(z) = z + A^{-1}(y - f(z))$ being the 'fixed point detector'. Write $(f|U)^{-1}(y) = \lim\{(\phi|U)(x_n)\}_n$ and every point in $f(U)$ has a unique inverse.

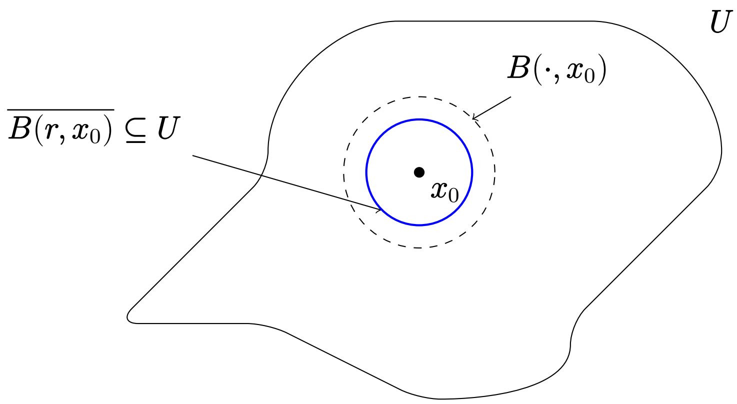

Figure 1: Every point x_0 in an open set U admits an open ball that hides in U

For the last part of the proof, we wish to show $V = f(U)$ is open. Let $y_0 \in V$ and we can 'hone into' the inverse of y_0 using the same construction as earlier. So $f(x_0) = y_0$ for some unique $x_0 \in U$.

If x_0 is in U, it induces an open ball (see fig. 1) st

$$x_0 \in B(r, x_0) \subseteq \overline{B(r, x_0)} \subseteq U, \quad r > 0$$

We claim the open ball $B(\lambda r, y_0) \subseteq V$. Indeed, suppose $y \in \mathbb{R}^n$ with

$$d(y, y_0) < \lambda r$$

If ϕ is the 'fixed-point detector' with respect to y (the point we are trying to prove that is in $f(U)$), in fact: we will prove $y \in f(\overline{B(r, x_0)}) \subseteq f(U)$.

$$\underbrace{\phi(x_0) - x_0}_{\text{removing the offset from } \phi(x_0)} = A^{-1}(y - f(x_0)) = A^{-1}(y - y_0)$$

using the operator norm on $A^{-1}(y - y_0)$ reads

$$\|\phi(x_0) - x_0\| = \left\|A^{-1}(y - y_0)\right\| \leq \left\|A^{-1}\right\| \|y - y_0\| \leq \left\|A^{-1}\right\| \lambda r = r2^{-1}$$

We will drag y into the image of the closed ball as follows: suppose x is another point that lies in the closed ball, ϕ is contractive on $\overline{B} \subseteq U$ regardless of the point y that induces ϕ. But $\overline{B}$ is closed, hence it is complete. So the Cauchy sequence (from the contractive mapping theorem) produces exactly one point in $\overline{B}$. It remains to show that if we start our sequence at some point $x \in \overline{B}$, then $\phi(x) \in \overline{B}$ as well, and a simple induction will produce our contractive sequence.

To this, fix $x \in \overline{B}$, and

$$\begin{aligned} |\phi(x) - x_0| &\leq |\phi(x) - \phi(x_0)| + |\phi(x_0) - x_0| \\ &\leq \overbrace{2^{-1}|x - x_0|}^{\text{contraction on } \overline{B} \subseteq U} + \overbrace{r2^{-1}}^{\text{earlier}} \\ &= r \end{aligned}$$

therefore ϕ contracts to a fixed point $x^* \in \overline{B}$, and $f(x^*) = y$. So $y \in f(\overline{B}) \subseteq f(U)$ as desired. ■

Proof of Part B. The proof is quite long, and we will only focus on the important bits. Rudin uses the technique of approximating smooth functions using first-order terms. He writes

$$\begin{cases} f(x) &= y \\ f(x+h) &= y + k \end{cases} \implies k = f(x+h) - f(x)$$

Furthermore, if $x \in U$, then the derivative $Df(x)$ is invertible, this is from Theorem 9.8, obtains an estimate on the open ball in $GL(n, \mathbb{R})$. Roughly speaking, this open ball 'drags' other matrices into $GL(n, \mathbb{R})$. If A is invertible, and B is a conformable matrix with A, then

$$\underbrace{\|B - A\|}_{\substack{\text{distance} \\ \text{between} \\ A,B}} \left\|A^{-1}\right\| < 1 \implies B \in GL(n, \mathbb{R})$$

If $x \in B(\delta, a)$, then Equation (11) reads

$$\|Df(x) - A\| < \lambda \implies \|Df(x) - A\| \left\|A^{-1}\right\| < 2^{-1} < 1$$

so $Df(x)$ is invertible with inverse T.

And we estimate the deviation $|k|^{-1} \leq \lambda |h|^{-1}$ by using the contraction inequality with y as the basepoint for ϕ. Skipping a few lines ahead (to the confusing part), we see that

$$|h| \leq |h - A^{-1}k| + |A^{-1}k| \leq 2^{-1}|h| + |A^{-1}k|$$

subtracting over, and multiplying across gives a upper bound on $|k|^{-1}$

$$2^{-1}|h| \leq |A^{-1}k| \implies 2^{-1}|h| \leq \left\|A^{-1}\right\| |k| \implies |k|^{-1} \leq \underbrace{\frac{2}{\|A^{-1}\|}}_{\lambda} |h|^{-1}$$

Notice $2\lambda \|A^{-1}\| = 1$, so $2/\|A^{-1}\| = \lambda$. Finally, we 'factor out' $-T$ on the line just before the difference quotient.

$$\begin{aligned}\overbrace{g(y+k)-g(y)-Tk}^{\substack{\text{numerator in}\\ \text{difference quotient}}} &= h - Tk\\ &= -T\Big(\underbrace{f(x+h)-f(x)}_{=k} - \underbrace{Df(x)h}_{=T^{-1}h}\Big)\end{aligned}$$

We see that $T = Dg(y)$, indeed:

$$\begin{aligned}\frac{|g(y+k)-g(y)-Tk|}{|k|} &\le \frac{\|T\|}{\lambda}\frac{|f(x+h)-f(x)-Df(x)h|}{|h|}\\ &\lesssim \frac{|f(x+h)-f(x)-Df(x)h|}{|h|}\\ &= \underbrace{o(h) = o(k)}_{|h|\lesssim|k|} \to 0\end{aligned}$$

Finally, $Df|U : U \to GL(n,\mathbb{R})$ is a continuous mapping. By Theorem 9.8, $(Df|U)^{-1} : U \to GL(n,\mathbb{R})$ is continuous as well. Therefore $g \in C^1(U,U)$, and $f|U$ is a C^1-diffeomorphism. ■

Remark 3.1

The inverse function theorem is extremely powerful. If a f is a C^1 map from and into $\mathbb{R}^n$, and the total differential of f is full rank (hence invertible, as it is square) at some point $a \in \mathbb{R}^n$, the theorem states three things:

- For points x within a small enough neighbourhood a, the total differential $Df(x)$ is invertible,
- On this same neighbourhood (denoted by U), $f(U)$ is a bijection,
- the inverse of f is a C^1 map. This makes $f|U$ a C^1-diffeomorphism

Inverse Function Theorem on Manifolds

Let F be a smooth map between two smooth manifolds M and N, with dimensions m and n respectively.

Definition 4.1: Rank of a map

The rank of F at $p \in M$ is the rank of the linear map:

$$dF_p : T_pM \to T_{F(p)}N$$

Definition 4.2: Constant rank maps

A smooth map $F \in C^\infty(M, N)$ has constant rank if its differential $dF_p : T_pM \to T_{F(p)}N$ has the same rank at every point $p \in M$.

There are three types of constant rank maps that are of interest.

Definition 4.3: Smooth submersion

F is a smooth submersion if dF_p is a surjection onto $T_{F(p)}N$ at p-everywhere. That is, $\operatorname{rank} dF_p = \dim T_{F(p)}N = \dim N$

Definition 4.4: Smooth immersion

F is a smooth immersion if dF_p is an injection onto $T_{F(p)}N$ at p-everywhere. That is, $\operatorname{rank} dF_p = \dim T_pM = \dim M$

Definition 4.5: Smooth embedding

F is a smooth embedding if it is a smooth immersion, and it is a homeomorphism onto its range $F(M) \subseteq N$.

Definition 4.6: Local diffeomorphism

F is a local diffeomorphism if every $p \in M$ in its domain induces a neighbourhood $U \subseteq M$ with $F|U : U \to F(U)$ is a diffeomorphism (in the sense of two open sub-manifolds).

Proposition 4.1: Rank as an open condition

Suppose $F : M \to N$ is a smooth map, and $p \in M$. If dF_p is a surjection (resp. injection), pointwise at p, there exists a neighbourhood U of p where $F|U$ is a smooth submersion (resp. immersion)

Proof. Trivial. See Example 1.1. ■

Proposition 4.2: Inverse Function Theorem on Manifolds

Let M and N be smooth manifolds, and $F : M \to N$ be a smooth map. Suppose the differential of F is invertible at some point $p \in M$, then there exists connected neighbourhoods U_0 of p, and V_0 of $F(p)$ such that $F|U_0 : U_0 \to V_0$ is a diffeomorphism.

Proof. Trivial. See the regular inverse function theorem Proposition 3.1 on Euclidean space, and pass the argument back to the manifolds using coordinate charts. ■

Proposition 4.3: Rank Theorem for Manifolds

Let $F : M \to N$ be a smooth map with constant rank r, then at every $p \in M$, there exists smooth charts $p \in (U, \phi)$ and $F(U) \subseteq (V, \psi)$, where the coordinate representation of F takes the form

$$\hat{F}(x) = \begin{bmatrix} \mathrm{id}_{r\times r} & 0_{r\times m-r} \\ 0_{n-r\times r} & 0_{n-r\times m-r} \end{bmatrix} x, \qquad \text{or equivalently} \tag{12}$$

$$\hat{F}(x^1, \ldots, x^r, x^{r+1}, \ldots, x^m) = (x^1, \ldots, x^r) \tag{13}$$

Proof. Tedious. However, some techinques are worth remembering:

- Passing the argument to the Euclidean case as usual,
- We are free to shrink the sizes of open cubes and balls, and exploit local compactness,
- Suppose we are given a matrix of size $m \times n$, which has rank r, then we can attach a sub-matrix to make it square and invertible, then rehearse the usual arguments with the Inverse Function Theorems Propositions 3.1 and 4.2 to obtain a neighbourhood small enough that preserves the rank of the square matrix. Then pass the argument back to the smaller sub-matrix.

The last bullet point is worth elaborating, suppose we are given a rectangular matrix, where A is square and invertible. Take $z = (x, y)^T$ with dimensions that make the formulas below make sense.

$$Mz = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \implies \begin{bmatrix} M \\ I \end{bmatrix} (z, y)^T = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} \quad \text{is square and invertible}$$

by Proposition 4.1, we see that there exists a neighbourhood about the square matrix $\begin{bmatrix} M \\ I \end{bmatrix}$ such that it remains invertible, hence a neighbourhood about M that makes A invertible (as a sub-matrix), so the rank of M is preserved. ■

Corollary 4.1: Rank Theorem for Manifolds - Special Cases

Let $F : M \to N$ be a smooth map with constant rank. If F is a smooth immersion, then Equation (12) takes the form:

$$\hat{F}(x) = \begin{bmatrix} \mathrm{id}_{m\times m} \\ 0_{n-m\times m} \end{bmatrix} x, \qquad \text{or equivalently} \tag{14}$$

$$\hat{F}(x^1, \ldots, x^m) = (x^1, \ldots, x^m, 0 \ldots, 0) \tag{15}$$

If F is a smooth submersion,

$$\hat{F}(x) = \begin{bmatrix} \mathrm{id}_{n\times n} & 0_{n\times m-n} \end{bmatrix} x, \qquad \text{or equivalently} \tag{16}$$

$$\hat{F}(x^1, \ldots, x^n, x^{n+1}, \ldots, x^m) = (x^1, \ldots, x^n) \tag{17}$$

More on immersions and embeddings

Proposition 5.1: Characterization of smooth immersions

F is a smooth immersion iff every point $p \in M$ has a neighbourhood $U \subseteq M$ where $F|U : U \to N$ is a smooth embedding.

Proof. We will prove it for when M and N are smooth manifolds, see Lee for the full proof with boundary. It involves extending the argument by composing F with an inclusion map. From Lemma 3.11 (Lee), if $a \in \partial\mathbb{H}^n$, then the differential of the inclusion map $\iota : \mathbb{H}^n \to \mathbb{R}^n$ is a linear isomorphism between tangent spaces.

$$d\iota_a : T_a\mathbb{H}^n \to T_a\mathbb{R}^n, \quad \underbrace{T_a\mathbb{H}^n \cong T_a\mathbb{R}^n}_{\text{isomorphic}}$$

If for every $p \in M$, there exists a neighbourhood U of p with $F|U : U \to N$ a smooth embedding, then $dF|U_p$ has rank m, so dF_p has rank m, and the differential is injective pointwise everywhere. Conversely, if dF_p is a smooth immersion, the Rank Theorem (Proposition 4.3) tells us there exists connected neighbourhoods of p and $F(p)$, where F has coordinate representation in Equation (14) with respect to an appropriate choice of coordinate charts centered at p, so $\hat{F}(\hat{p}) = 0 \in \mathbb{R}^n$. Let $\hat{p} \in \hat{U}$ and $\hat{F}(\hat{p}) \in \hat{V}$, the proof then devolves into a linear-map problem. $\hat{F}$ given by the expression in Equation (14) is clearly injective. Therefore it is bijective onto its range, its inverse is nothing but the map that removes the extra zeroes at the end. Therefore $F|U$ is a smooth embedding. ∎

Definition 5.1: Section of $\pi : M \to N$

If $\pi : M \to N$ is a continuous map, a *section of* π is a continuous right inverse for π, i.e $\sigma : N \to M$, $\sigma \in C(N, M)$, $\pi \circ \sigma = \mathrm{id}_N$.

A *local section for* π is a continuous function σ from an open set $U \subseteq V$ into M with $\pi \circ \sigma = \mathrm{id}_U$.

Proposition 5.2: Characterization of smooth submersion

Let $\pi : M \to N$ be smooth, then π is a smooth submersion iff every point of M is in the image of a smooth local section of π.

Proof. Suppose π is a smooth submersion, and fix $p \in M$, by the Rank Theorem Proposition 4.3, and Equation (16), π has the coordinate representation

$$\hat{\pi}(x^1, \ldots, x^n, x^{n+1}, x^m) = (x^1, \ldots, x^n)$$

between two open sets $U \subseteq M$ and $V \subseteq N$, (it really does not matter). Now, define

$$\sigma : V \to M, (x^1, \ldots, x^n) \mapsto \underbrace{(x^1, \ldots, x^n, 0, \ldots, 0)}_{\mathbb{R}^m} \in U$$

the charts by assumption are centered, and $\pi \circ \sigma$ is clearly smooth (check coordinate-wise), so σ reaches p. Conversely, recall if the composition of maps $(g \circ f)$ is a surjection, then g is a surjection. Now, fix $p \in M$, this induces an open set V containing $\pi(p)$, and a smooth local section $\sigma_V : V \to M$. By Proposition 4.1, *the differential of a composition is equal to the composition of the differentials*

$$\mathrm{id}_{T_qN} = d(\mathrm{id}_U) = d(\pi)_{\sigma(q)} \circ d(\sigma)_q$$

so $d(\pi)_{\sigma q} = d(\pi)_p$ is a surjection and the proof is complete. ■

Regular values and level sets

If $F : \mathbf{X} \to \mathbf{Y}$, and $c \in \mathbf{Y}$, we call the $F^{-1}(\{y\})$ a *level set at* c, and c the *level value*. We often write $F^{-1}(y)$ in place of $F^{-1}(\{y\})$. If $\mathbf{Y} = \mathbb{R}^k$, then $F^{-1}(0)$ is the *zero set of* F.

Definition 6.1: Critical point of $F \in C^\infty(N, M)$

$p \in M$ is a *critical point* of F if the differential $dF_p : T_pN \to T_{F(p)}M$ fails to be surjective at p, otherwise p is a *regular point* of F.

Definition 6.2: Critical value of $f \in C^\infty(N, \mathbb{R})$

Let f be a test function on N, $c \in \mathbb{R}$ is a critical value if there exists a $p \in f^{-1}(c)$ where df_p is not surjective.

Otherwise c is called a regular value. Caveat: if c is not in the image of f, we call c a regular value as well. It is clear c is a regular value iff $c \notin f(M)$ or for every

$p \in f^{-1}(c)$, df_p is a surjection.

Notice $df_p : T_pN \to T_{f(p)}\mathbb{R}$ is not surjective $\iff$ all partials of the coordinate representation of f vanish. Since the matrix representation of df_p has the form

$$\mathcal{M}\{df_p\} = \begin{bmatrix} \frac{\partial f}{\partial x^1}\Big|_p & \cdots & \frac{\partial f}{\partial x^m}\Big|_p \end{bmatrix}$$

$\operatorname{rank} \mathcal{M}\{df_p\} \neq 1 \iff \frac{\partial f}{\partial x^j}\Big|_p = 0$ for $1 \leq j \leq m$.

Definition 6.3: Regular level set of $f \in C^\infty(N, \mathbb{R})$

If c is a regular value of f, then $f^{-1}(c)$ is called a regular level set.

Define $g = f - c$, the partials at $p \in N$ for both f and g agree, so the matrix representations of df_p and dg_p are identical (whereas their ranges might not be),

$$\begin{aligned}
\mathcal{M}\{dg_p\} &= \begin{bmatrix} \frac{\partial g}{\partial x^1}\Big|_p & \cdots & \frac{\partial g}{\partial x^m}\Big|_p \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f - c}{\partial x^1}\Big|_p & \cdots & \frac{\partial f - c}{\partial x^m}\Big|_p \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f}{\partial x^1}\Big|_p & \cdots & \frac{\partial f}{\partial x^m}\Big|_p \end{bmatrix} \\
&= \mathcal{M}\{df_p\}
\end{aligned}$$

Commentary

Proposition 4.1 roughly states that, if the differential of F at some point p is injective or surjective, then there exists a neighbourhood U about p such that $dF|U(p)$ is an injection or surjection. The continuity of the map $dF|U(p) \mapsto \Delta_{m\times m}(dF|U(p))$, induces a neighbourhood in the vector space of matrices about the differential $dF|U(p)$. This vector space is endowed with any of the equivalent norms on $\mathcal{M}(m \times n, \mathbb{R})$, which is equivalent to the entrywise 2-norm. Since all partials of the form $\frac{\partial \hat{F}^k}{\partial x^j}\Big|_{\hat{p}}$ are continuous, we take the intersection over all $n \times m$ partials such that $dF|U(p)$ is an injection or surjection. Finally, send this neighbourhood about $\hat{p}$ through to p by using the continuity of ϕ.