# Folland Reading

me

September 24, 2022

## Contents

# 1 Chapter 1

# 2 Chapter 2

# 3 Chapter 3

# 4 Chapter 4

## 4.1 Theorem 4.1

**WTS.** *Suppose that $A$ is a subset of $X$, let $\operatorname{acc} A$ be the set of accumulation points of $A$, then*

$$\overline{A} = A \cup \operatorname{acc}(A) \tag{1}$$

*and $A$ is closed if and only if $\operatorname{acc}(A) \subseteq A$.*

*Proof.* Suppose that $x \notin \overline{A}$, then $x \in (\overline{A})^c = A^{co}$, then $A^c \in \mathcal{N}_B(x)$. But this means that $x \notin \operatorname{acc}(A)$, since there exists a neighbourhood of $x$ (in the form of $A^c$), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \varnothing$$

Also, $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$ which means that

$$x \notin \overline{A} \implies x \notin A$$

Since $x \notin \overline{A} \implies x \notin A$ and $x \notin \operatorname{acc}(A)$,

$$(\overline{A})^c \subseteq A^c \cap \operatorname{acc}(A)^c = (A \cup \operatorname{acc}(A))^c$$

Now, if $x \notin \operatorname{acc}(A) \cup A$, then $x \notin \operatorname{acc}(A)$, therefore there exists some $U \in \mathcal{N}_B(x)$ such that

$$A \cap U \setminus \{x\} = A \cap U = \varnothing$$

Where for the second last equality we used the fact that $x \notin A \implies A \setminus \{x\} = A$, and taking complements gives us

$$U \subseteq A^c$$

And since $U \in \mathcal{N}_B(x)$, then $x \in U^o \subseteq A^{co}$ (since $U^o$ is an open subset of $A^c$). then

$$x \in A^{co} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore $(A \cup \operatorname{acc}(A))^c \subseteq (\overline{A})^c$. ∎

## 4.2 Theorem 4.2

**WTS.**

## 4.3 Theorem 4.3

**WTS.**

*Proof.* $\square$

## 4.4 Theorem 4.4

**WTS.**

## 4.5 Theorem 4.5

**WTS.**

## 4.6 Theorem 4.6

**WTS.**

## 4.7 Theorem 4.7

**WTS.** *$X$ is a $T_1$ space $\iff$ $\{x\}$ is closed for every $x \in X$.*

*Proof.* If $X$ is $T_1$ and $x \in X$, then for every $y \neq x$ there exists some open $U_y$ that contains $y$ but not $x$. Following Folland's argument closely, every $y \neq x$ is is in $\cup U_{y \neq x}$. Hence $\{x\}^c \subseteq \cup U_{y \neq x}$. To show the converse, for every $z \in \cup U_{y \neq x}$ that is open, there exists a $y \neq x$ such that $z \in U_y$. But every $U_y$ does not contain $x$ as an element, so $z \neq x$ implies that $z \notin \{x\}$. And $z \in \{x\}^c$. Hence $\cup U_{y \neq x} = \{x\}^c$.

Now conversely if every $x \in X$ satisfies the fact that $\{x\}^c$ is open, then $\{x\}^c$ is an open set that contains every $y \neq x$. Now fix some $y \neq x$, since $\{y\}$ is also closed, we have $X \cap \{x\}^c$ is an open set that contains $x$ but not y. Also, $\{x\}^c$ is an open set that contains $y$ but not $x$. And therefore $X$ is $T_1$. $\square$

## 4.8 Theorem 4.8

**WTS.**

### 4.9 Theorem 4.9

**WTS.**

## 4.10 Theorem 4.10

**WTS.**

## 4.11 Theorem 4.11

**WTS.**

## 4.12 Theorem 4.12

**WTS.**

### 4.13 Theorem 4.13

**WTS.**

## 4.14 Theorem 4.14

**WTS.** *Suppose that $A$ and $B$ are disjoint closed subsets of the normal space $X$, and let $\Delta = \{k2^{-n} : n \geq 1$ and $0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that*

1. *$A \subseteq U_r \subseteq B^c$ for every $r \in \Delta$ and*
2. *$\overline{U_r} \subseteq U_s$ for $r < s$.*

*Proof.* The goal of this proof is to show that for every $r \in \Delta$, there exists a open $U_r$ that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some $N \geq 1$ that all previous $U_r$ in previous layers have been constructed properly, meaning if $r = k/2^n$, then for every $1 \leq n \leq N - 1$, we have

$$r = \frac{k}{2^n},\ 1 \leq n \leq N - 1,\ 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each $U_r$,

- $A \subseteq U_r \subseteq B^c$ and
- $U_r \in \mathrm{T}_X$

Then for this fixed layer $N \geq 1$, we only have to construct the $U_{k/2^N}$ for every odd $k$, this is because if $k$ is an even number, then $k = 2j$ and $r = 2j/2^N = j/2^{N-1}$ and for this particular $U_r$ is already constructed. So for every odd $k = 2j + 1$, the sets of the form $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every $k - 1 \neq 0$ and $k + 1 \neq 1$. (We will consider these cases later). We claim that for every pair of open sets, $E_1, E_2 \in \mathrm{T}_X$, then there exists some open set $G \in \mathrm{T}_X$ such that if $(E_1, E_2) \in H \subseteq (\mathrm{T}_X \times \mathrm{T}_X)$ where $H$ is defined as the set

$$H = \left\{(E_1, E_2) \subseteq (\mathrm{T}_X \times \mathrm{T}_X) : \overline{E_1} \cap E_2^c = \varnothing\right\}$$

Then there exists some $G = \mathcal{J}(E_1, E_2) \in \mathrm{T}_X$ such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any $(E_1, E_2) \in H$, then this pair induces a pair of disjoint sets $\overline{E_1}$ and $E_2^c$ since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \varnothing$$

And by normality, there exists disjoint open sets $G_1$, $G_2$ such that

- $\overline{E_1} \subseteq G_1 \in \mathrm{T}_X$
- $E_2^c \subseteq G_2 \in \mathrm{T}_X$
- $G_1 \cap G_2 = \varnothing \implies G_1 \subseteq G_2^c \subseteq E$
- Since $G_2^c$ is a closed set that contains $G_1$ as a subset, $\overline{G_1} \subseteq G_2^c \subseteq E$

It is at this point that we will make no further mention of $G_2$ (so we may discard the notion of $G_2$ in our minds). Let us now replace $G$ with $G_1$ then it is an easy task to verify that $G = G_1 = \mathcal{J}(E_1, E_2)$ has the required properties.

Now define for every odd $k$, since $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (we note in passing that $\mathcal{J}$ is not a function as the set $G$ may not be unique).

$$U_{k/2^N} = \mathcal{J}\left(U_{(k-1)/2^N}, U_{(k+1)/2^N}\right)$$

Then, if $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$ sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$ and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed' $U_{k/2^N}$ for every $k-1 \neq 0$ and $k+1 \neq 1$. Now let us deal with the remaining pathological cases.

If $k-1$ so happens to be 0, then no $r \in \Delta$ satisfies $r = 0/2^N$, and we substitute

$$\overline{U}_0 = A, \quad \text{or alternatively, } U_0 = A^o$$

Then $U_0 \in \mathrm{T}_X$, $\overline{U}_0 = A \subseteq B^c$. It is at this point that we must mention that $0, 1 \notin \Delta$, so $U_0$ and $U_1$ do not have to obey the rules we have laid out for $U_{r\in\Delta}$.

Now if $k+1$ is equal to $2^N$ (this makes $r = (k+1)/2^N = 1$) we define

$$U_1 = B^c \in \mathrm{T}_X$$

With this, for every $0 \leq m \leq 2^N - 1, U_{m/2^N}$ must staisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (even for when $N = 1$, since $A = \overline{U}_0 \subseteq U_1 = B^c$) and a corresponding $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$ such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when $N = 1$. We would only have to construct for $k = 1$, since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. $\square$

## 4.15 Theorem 4.15

**WTS.** *Urysohn's Lemma. Let $X$ be a normal space, if $A$ and $B$ are disjoint closed subsets of $X$, then there exists a $f \in C(X, [0,1])$ such that $f = 0$ on $A$ and $f = 1$ on $B$.*

*Proof.* Let $r \in \Delta$ be as in Lemma 4.14, and set $U_r$ accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0,1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic $\varepsilon$ argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since $A$ and $B$ are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that $x$ is not a member of any $U_r$, but we set $U_1 = X$. Since none of the $r \in (0,1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The $\varepsilon$ argument follows: suppose for every $\varepsilon > 0$, $(1-\varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannnot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in $X$. So $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of $\Delta$, there exists an $r$, $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r<\alpha} U_r$. So $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r<\alpha} U_r$.

Fix an element $x \in \bigcup_{r<\alpha} U_r$, this induces an $r$ such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r<\alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$.

For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an $r$ (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) — we have $x \notin \overline{U_s}$, but $\left(\overline{U_s}\right)^c$ is open in $X$. It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq \left(\overline{U_s}\right)^c \subseteq \bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$. To show the reverse, fix an element $x$ in the union, then this induces some $x \in \left(\overline{U_s}\right)^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of $W$. This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen $s$ is a lower bound for $W$, and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in $X$, using Theorem 4.9 finishes the proof. $\square$

## 4.16 Theorem 4.16

**WTS.** *The Tietze's Extension Theorem. Let $X$ be a normal space, and for any closed subset $A \subseteq X$, and $f \in C(A, [a, b])$, there exists an $F \in C(X, [a, b])$ which extends $f$.*

*Proof.* We begin with an important lemma that will serve as a 'black box' for the induction.

**Lemma 4.1.** *For every $f \in C(A, [0, 1])$, there exists a $g \in C(X, [0, 1/3])$ such that*

$$0 \leq f - g \leq 2/3 \quad \textit{pointwise on } A \tag{2}$$

*Proof.* Since $f$ is continuous, $B = f^{-1}([0, 1/3])$, and $C = f^{-1}([2/3, 1])$ are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function $g \in C(X, [0, 1])$ such that $g|_B = 0$ and $g|_C = 1$. Relabel $g = g/3$ then $g \in C(X, [0, 1/3])$ (multiplication is continuous).

To show that (2) holds, suppose $x \in B$, then $f(x) \in [0, 1/3]$ and $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$. Now suppose that $x \in C$, then $f(x) \in [2/3, 1]$ and $g(x) = 1/3$ (recall that we relabelled $g$). So we have $0 \leq 1/3 \leq f - g \leq 2/3$. Lastly, for the case where $x \notin (B \cup C)$, then $f(x) \in (1/3, 2/3)$, and $g(x) \in [0, 1/3]$ implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 &\qquad \implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 &\qquad \implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore $0 \leq f(x) - g(x) \leq 2/3$. □

We can assume that $f \in C(A, [0, 1])$, since we can relabel $f = (f - a)/(b - a)$. The main part of this proof consists of constructing a sequence of $\{g_n\} \subseteq C(X, \mathbb{R})$ where $0 \leq g_n \leq (2/3)^n(1/2)$, and $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$ on $A$. Let us begin with the base case with $n = 1$. We can apply Lemma 4.1 to get $g_1 \in C(X, [0, 1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that $\{g_j\}_{j \leq n}$ has been chosen, we will find our $g_{n+1}$ by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by $(2/3)^{-n}$ and we obtain a new function in $C(A, [0, 1])$.

$$0 \le \left( f(x) - \sum_{j \le n} g_j(x) \right) \left( \frac{3}{2} \right)^n \le 1$$

Applying the Lemma 4.1, we get a function $h \in C(X, [0, 1/3])$ such that, for every $x \in A$

$$0 \le \left( f(x) - \sum_{j \le n} g_j(x) \right) \left( \frac{3}{2} \right)^n - h \le 2/3$$

Multiplying across gives

$$0 \le \left( f(x) - \sum_{j \le n} g_j(x) \right) - h \left( \frac{2}{3} \right)^n \le \left( \frac{2}{3} \right)^{n+1}$$

Set $g_{n+1} = h \left( \frac{2}{3} \right)^n$ and $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$. Furthermore, the sum of all $g_j$ pointwise converges uniformly, as

$$\sum_{j \ge 1} \|g_j\|_u \le \sum_{j \ge 1} \left( \frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum $F = \sum g_j$, then this $F \in BC(X)$ (by Theorem 4.9), since every $g_j \in BC(X)$. And

$$\left\| f - \sum_{j \le n} g_j \right\|_u \le \left( \frac{2}{3} \right)^n \longrightarrow 0$$

So $F = f$ on $A$, now if we want to obtain our $F$ on $[a, b]$ we simply relabel $F = F(b - a) + a$. This finishes the proof. $\square$

## 4.17 Theorem 4.17

**WTS.** *If $X$ is a normal space, and $A$ is a closed subspace of $X$, and $f \in C(A)$, then there exists an $F \in C(X)$ such that $F$ extends $f$.*

*Proof.* First we suppose that $f$ is real valued, so $f \in C(X, \mathbb{R})$. And define a $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$, using

$$g = \frac{f}{1 + |f|}$$

Since $g$ satisfies the assumption of Theorem 4.16 (note that we do not require $g$ to be injective), there exists a $G \in C(X, [-1, +1])$ such that $G|_A = g$. Since the set $\{-1, +1\}$ is closed in $\mathbb{R}$, $G^{-1}(\{-1, +1\})$ is closed as well. Since $G^{-1}((-1, +1)) \subseteq A$, this makes $A$ and $B =^{-1} (\{-1, +1\})$ disjoint closed sets in $X$.

By Urysohn's Lemma, there exists a continuous function $h \in C(X, [0, 1])$ such that $h|_B = 0$ and $h|_A = 1$, so that the product $|hG| < 1$ for all $x \in X$. We can think of this $h$ as a continuous indicator function that filters out the parts we do not want, namely $G^{-1}\{-1, +1\}$. Now define $F$ in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that $F|_A = g/(1 - |g|) = f$ indeed. Since $|g| = \frac{|f|}{1+|f|}$, and $g(1 + |f|) = f$ implies that $g/(1 - |g|) = f$, because $g \in C(A, (-1, +1))$ This completes the proof for any $f \in \mathbb{R}$ if $f \in C(A)$, then

1. $\mathrm{Re}(f) = f_1 \in C(A, \mathbb{R})$
2. $\mathrm{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in $C(X, \mathbb{R})$ that extends $f_1$ and $f_2$, and $F_1 + iF_2 = f$ on $A$ and $F_1 + iF_2 \in C(X)$, and the proof is complete. $\square$

## 4.18 Theorem 4.18

**WTS.** *If $X$ is a topological space, and $E \subseteq X$ and $x \in X$, then $x \in \operatorname{acc} E \iff$ there exists a net in $E \setminus \{x\}$ that converges to $x$, and $x \in \overline{E} \iff$ there exists a net in $E$ that converges to $x$.*

*Proof.* Suppose that $x \in \operatorname{acc} E$, then for every neighbourhood $U \in \mathcal{N}(x)$, $E \cap U \setminus \{x\} \neq \varnothing$, then choose $\mathcal{N}(x)$ as the set of neighbourhoods directed by reverse inclusion (and this makes $(\mathcal{N}(x), \lesssim)$ a directed set), and we will define the net as follows.

Map each $U \in \mathcal{N}(x)$ to some $x_U \in E \cap U \setminus \{x\}$, then this net converges to $x$. Suppose that we fix a neighbourhood, $V \in \mathcal{N}(x)$, then for every $U \gtrsim V$ we have $x_u \in U \subseteq V$. So $\langle x_U \rangle$ is eventually in $V$.

Conversely, if $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$, and $x_\alpha \to x$, then every $U \in \mathcal{N}(x)$ there exists a $x_\alpha \in E \cap U \setminus \{x\}$ that makes

$$E \cap U \neq \varnothing \quad \forall U \in \mathcal{N}(x)$$

Hence $x \in \operatorname{acc} E$.

Now for the second part of the Theorem, suppose that $x \in \overline{E}$, if $x \notin E$ then $E = E \setminus \{x\}$ and $x \in \operatorname{acc} E$, so there exists a net in $E \setminus \{x\} \subseteq E$ such that $x_\alpha \to x$. If $x \in E$ then simply choose $\langle x_\alpha \rangle = x$ for every $\alpha \in A$.

Now, suppose that there is a net that converges to $x$, and this net $\langle x_\alpha \rangle \subseteq E$, if $x \in E$ then there is nothing to prove, since $E \subseteq \overline{E}$, so suppose that $x \notin E$, then there exists a net in $E \setminus \{x\} = E$ such that

$$x_\alpha \to x \implies x \in \operatorname{acc} E \subseteq \overline{E}$$

$\square$

## 4.19 Theorem 4.19

**WTS.** *Let $X$ and $Y$ be topological spaces, then every $f : X \to Y$ is continuous at a point $x \in X \iff$ every net $\langle x_\alpha \rangle$ that converges to $x$ implies that $\langle f(x_\alpha) \rangle$ converges to $f(x)$.*

*Proof.* If $f$ is continuous at a point $x \in X$, then $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$, then for every net $\langle x_\alpha \rangle$ that converges to this $x$, there there exists an $\alpha_0$ such that for every $\alpha \gtrsim \alpha_0$ implies that $x_\alpha \in f^{-1}(V)$. Hence

$$f(x_\alpha) \in f\left(f^{-1}(V)\right) \subseteq V$$

And this is equivalent to saying that for every $V \in \mathcal{N}(f(x))$, $\langle f(x_\alpha) \rangle$ is eventually in $V$, and this proves convergence.

Now suppose that $f$ is not continuous at some $x$, then there exists a $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$, so

$$x \notin \left(f^{-1}(V)\right)^o \implies x \in \left(f^{-1}(V)\right)^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our $x \in \overline{f^{-1}(V^c)}$ induces a net $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$ that converges to $x$. But every element in the net is contained within $f^{-1}(V^c)$, and for every $\alpha \in A$

$$f(x_\alpha) \in f\left(f^{-1}(V^c)\right) \subseteq V^c$$

gives $f(x_\alpha) \notin V$, but $V$ is a neighbourhood of $f(x)$, hence there exists some $x_\alpha \to x$ and $f(x_\alpha) \not\to f(x)$. $\square$

## 4.20 Theorem 4.20

**WTS.** *If $\langle x_\alpha \rangle$ is a net in $X$, and $x \in X$ is a cluster point of $\langle x_\alpha \rangle$ $\iff$ there exists a subnet of $\langle x_\alpha \rangle$ that converges to $x$.*

*Proof.* Suppose that $\langle y_\beta \rangle_{\beta \in B}$ is a subnet of $\langle x_\alpha \rangle$ that converges to $x$, then for every neighbourhood $U \in \mathcal{N}(x)$ , there exists a $\beta_1$ such that for every $\beta \gtrsim \beta_1$ we get $y_\beta = x_{\alpha_\beta} \in U$.

Furthermore, let us fix a $\alpha_0 \in A$ to attempt to show that $\langle x_\alpha \rangle$ is frequently in $U$, then by the subnet property of $\langle y_\beta \rangle$, there exists some $\beta_2 \in B$ such that for every $\beta \gtrsim \beta_2$, $\alpha_\beta \gtrsim \alpha_0$. (Intuitively this property means that the directed set of $B$ 'grows' as much as the directed set of $A$, so we can always find elements that are greater than any fixed $\alpha_0$.)

Since $\langle y_\beta \rangle$ is a net, we there exists some $\beta \in B$ such that $\beta \gtrsim \beta_1$ and $\beta \gtrsim \beta_2$, we then apply the $\beta \mapsto \alpha_\beta$ map and we obtain some $\alpha = \alpha_\beta$ that satisfies:

- $\alpha = \alpha_\beta \gtrsim \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that $\beta \gtrsim \beta_1$ so that $y_\beta$ falls into $U$.

Conversely, suppose that $x$ is a cluster point of $\langle x_\alpha \rangle$, then by definition

$$\forall U \in \mathcal{N}(x),\ \forall \alpha_0 \in A,\ \exists \alpha \gtrsim \alpha_0,\ x_\alpha \in U$$

Denote the directed neighbourhoods of $x$ by $\mathcal{N}(x)$, and construct our directed set $B$ for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every $(U, \gamma) \in B$ we can map it to some $\alpha_{(U,\gamma)} \in A$, if we choose some $\alpha_{(U,\gamma)} \gtrsim \gamma$ and $\alpha_{(U,\gamma)} \in U$.

To show that $B$ is a directed set, we say that $(U, \gamma) \gtrsim (U', \gamma')$ if and only if $U \subseteq U'$ and $\gamma \gtrsim \gamma'$. And to show that $\langle y_\beta \rangle = \langle x_{\alpha_{(U,\gamma)}} \rangle$ is indeed a subnet of $\langle x_\alpha \rangle$, fix any $\alpha_0 \in A$, then simply take any neighbourhood $U$ of $x$ (we always have $X \in \mathcal{N}(x)$) — and therefore $(U, \alpha_0) \in B$.

Now for every $(U', \alpha_0') \gtrsim (U, \alpha_0)$ implies that $\alpha_0' \gtrsim \alpha_0$, therefore we have

$$\alpha_{(U',\alpha_0')} \gtrsim \alpha_0' \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that $\langle y_\beta \rangle$ indeed converges to $x$, fix any $V \in \mathcal{N}(x)$, then with any $\alpha_0 \in A$, and for every $(V', \alpha_0') \gtrsim (V, \alpha_0) \in B$, we have

$$x_{\alpha_{(V',\alpha_0')}} \in V' \subseteq V$$

So $\langle x_{\alpha_{(U,\gamma)}} \rangle$ converges to $x$. $\square$

## 4.21 Theorem 4.21

**WTS.** *A topological space $X$ is compact $\iff$ every family of closed sets, $\{F_\alpha\}_{\alpha \in A}$ that has the finite intersection property, implies that*

$$\bigcap_{\alpha \in A} F_\alpha \neq \varnothing$$

*Proof.* We first examine the assertion, Theorem 4.21 proposes for any family of closed sets $\{F_\alpha\}_{\alpha \in A}$, and for every finite subset $B \subseteq A$ then,

$$\bigcap_{\alpha \in B} F_\alpha \neq \varnothing \implies \bigcap_{\alpha \in A} F_\alpha \neq \varnothing$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha \in A} F_\alpha = \varnothing \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha \in B} F_\alpha = \varnothing$$

Applying DeMorgan's theorem, and since every $\{F_\alpha\}_{\alpha \in A}$ induces a family of open sets (and vice versa), where $U_\alpha = F_\alpha^c$, so for any familiy of open sets $\{U_\alpha\}_{\alpha \in A}$ we have

$$\bigcup_{\alpha \in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha \in B} U_\alpha = X$$

Which is equivalent to saying that $X$ is compact. $\square$

## 4.22 Theorem 4.22

**WTS.** *A closed subset of a compact space $X$ is compact.*

*Proof.* Suppose $F \subseteq X$ and $F$ is open, then fix an open cover for $F$, so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since $F^c$ is an open set, we can obtain a valid open cover for $X$, then we pick out a finite subcover, for some finite $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left( \bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with $F$ on both sides yields

$$\begin{aligned}
&F = X \cap F \subseteq (F^c \cap F) \cup \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \\
&F = \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\
&F \subseteq \bigcup_{\alpha \in B} U_\alpha
\end{aligned}$$

Therefore every open cover of $F$ has a finite subcover, and $F$ is compact. $\square$

## 4.23 Theorem 4.23

**WTS.** *If $F$ is a compact subset of a Hausdorff space $X$, and $x \notin F$, there are disjoint open sets $U, V$ such that $x \in U$ and $F \subseteq V$.*

*Proof.* Since $x \in F^c$, for every $y \in F$, $x \neq y$ induces two sets $U_y, V_y$ (because $X$ is $T_2$).

- $U_y \cap V_y = \varnothing$
- $x \in U_y$
- $y \in V_y$

But $\{V_y\}_{y \in F}$ is an open cover for the compact set $F$, then there exists a finite subcollection $H \subseteq F$ such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since $H$ is finite, $U = \bigcap_{y \in H} U_y$ is an open set that contains $x$, also define $V = \bigcup_{y \in H} V_y$. If for every $y \in H$, $U_y \cap V_y = \varnothing$, then $U \cap V_y = U \cap V = \varnothing$. This completes the proof. $\square$

**Remark.** *Every metric space $(X, d)$ is first countable, and $T_2$ (it is actually $T_4$, but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element $x \in X$ and we notice that $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$ is a countable neighbourhood base for every $x$. To show that $(X, d)$ is $T_2$, for every pair of elements $x \neq y$, we can take $r = d(x, y)/2$ and there exists disjoint open sets $V_r(x)$ and $V_r(y)$ such that $x \in V_r(x)$ and $y \in V_r(y)$.*

## 4.24 Theorem 4.24

**WTS.** *Every compact subset of a Hausdorff ($T_2$) space is closed.*

*Proof.* If $F$ is compact, then for every $x \in F^c$, by Theorem 4.23, there exists two disjoint open sets such that $x \in U$ and $F \subseteq V$, but

$$U \cap V = \varnothing \implies U \cap F = \varnothing \implies U \subseteq F^c$$

But since $x \in F^c$ is arbitrary, and $U$ is an open subset of $F^c$,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that $F^c$ is open and $F$ is closed. $\square$

## 4.25 Theorem 4.25

**WTS.** *Every compact Hausdorff ($T_2$) space is normal ($T_4$).*

*Proof.* Fix $A$, $B$ which are disjoint closed subsets of $X$, by Theorem 4.22, we know that these two sets are compact. Hence for every $y \in B$ there exists two disjoint open sets $U, V_y$ (by Theorem 4.23)

$A \subseteq U_y$ and $y \in V_y$. But the family $\{V_y\}_{y \in B}$ is s a valid open cover for the compact set $B$, hence there exists a finite subcollection $H \subseteq B$ such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \varnothing$$

The second equality holds for every $y \in H$ so that $U_y \cap (\cup V_{y \in H}) = \varnothing$. Define $U = \cap U_{y \in H}$ and $V = \cup V_{y \in H}$, where both of these are disjoint open sets that that contain $A$ and $B$ as subsets, since for each $y \in H$, $A \subseteq U_y$ hence the intersection of all $U_y$ also contains $A$ as a subset. Therefore $X$ is normal. $\square$

## 4.26 Theorem 4.26

**WTS.** *If $X$ is compact, and $f : X \to Y$ is continuous, then $f(X)$ is compact.*

A small lemma.

**Lemma 4.2.** *For every $\{E_j\} \subseteq X, f(\cup E_j) = \cup f(E_j)$.*

The proof is trivial.

*Proof.* If $\{V_{\alpha \in A}\}$ is an open cover for $f(X)$, then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha \in A} V_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(V_\alpha) \subseteq X$$

Since $f$ is continuous, we have an open cover in the form of $\{f^{-1}(V_\alpha)\}$ for $X$, then there exists a finite subset $B \subset A$ such that

$$X \subseteq \bigcup_{\alpha \in B} f^{-1}(V_\alpha)$$

Then we wish to show that for this $B \subseteq A$, $\{V_{\alpha \in B}\}$ is a finite open cover for $f(X)$. Fix any element $y \in f(X)$, then this induces a $x \in X$ such that $y = f(x)$, but because $\{f^{-1}(V_{\alpha \in B})\}$ is an open cover for $X$, there exists some $\alpha \in B$ such that $x \in f^{-1}(V_\alpha)$, hence by definition of the inverse image

$$f(x) \in V_\alpha \implies f(X) \subseteq \bigcup_{\alpha \in B} V_\alpha$$

Therefore $f(X)$ is compact and this completes the proof. $\square$

## 4.27 Theorem 4.27

**WTS.** *If $X$ is compact, then $C(X) = BC(X)$.*

*Proof.* Notice that $BC(X) \subseteq C(X)$, so we only have to show the reverse estimate. Fix any $f \in C(X)$, since $X$ is compact, by Theorem 4.26 we know that $f(X)$ is also compact. Since $\mathbb{C} = \mathbb{R}^2$ is a complete metric space, $f(X)$ is bounded and $f \in BC(X)$. $\square$

## 4.28 Theorem 4.28

**WTS.** *If $X$ is compact, and if $Y$ is Hausdorff, then any continuous bijection $f : X \to Y$ is a homeomorphism.*

*Proof.* If $E \in X$ is closed, then since $X$ is compact, $E$ is compact as well. By continuity of $f$, $f(X)$ is a compact set in $Y$, but compact subsets of $Y$ are closed, so $f$ is continuous.

We used the fact that the inverse of $f^{-1}$ is $f$, since it suffices to check that every inverse image of a closed set is also closed, $f^{-1}$ is continuous. And by definition of a homeomorphism ($f$ has to be bijective and both $f$ and $f^{-1}$ hav eto be continuous), $f$ is a homeomorphism. $\square$

## 4.29 Theorem 4.29

**WTS.** *If $X$ is any topological space, the following are equivalent.*

*(a) $X$ is compact.*

*(b) Every net has a cluster point.*

*(c) Every net in $X$ has a convergent subnet.*

*Proof.* By Theorem 4.20, every net in $X$ has a cluster point $\iff$ there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose $a$) holds, then $X$ is compact, and fix an arbitrary net $\langle x_\alpha \rangle$ in $X$. and define the 'tail' of the net

$$E_\alpha := \{x_\beta,\ \beta \gtrsim \alpha\}$$

We wish to show that the arbitrary intersection of $\bigcap_{\alpha \in A} \overline{E_\alpha} \neq \varnothing$. Where $\overline{E_\alpha}$ is closed, so it suffices to check that every finite $B \subseteq A$, the intersection over $\overline{E_\alpha}$ is non-empty.

Suppose we are given a finite $B \subseteq A$, then fix any two elements $\alpha$ and $\beta \in B$, by the definition of a net there exists a $\gamma \in A$ such that $\gamma \gtrsim \alpha$ and $\gamma \gtrsim \beta$, and

$$\varnothing \neq \subseteq E_\alpha \cap E_\beta \implies \overline{E_\alpha} \cap \overline{E_\beta} \neq \varnothing$$

Therefore for any finite collection of $\{\overline{E_{\alpha \in B}}\}$, then

$$\bigcap_{\alpha \in A} \overline{E_\alpha} \neq \varnothing$$

Now fix an element $x \in \bigcap_{\alpha \in A} \overline{E_\alpha}$. Then for every $\alpha \in A$, $x \in \overline{E_\alpha}$, and for every neighbourhood $U \in \mathcal{N}(x)$, $U \cap E_\alpha \neq \varnothing$. This is because if $x \in E_\alpha$, then $U \cap E_\alpha$ contains at least $\{x\}$, if $x \in \operatorname{acc} E_\alpha$, then by definition of an accumulation point, $U \cap E_\alpha \setminus \{x\} \neq \varnothing$, so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net, $E_\alpha$, if for every $\alpha \in A$, $x \in E_\alpha$ if and only if there exists some $\gamma \gtrsim \alpha$, $x_\gamma \in U \cap E_\alpha$, this is equivalent to saying that $x$ is a cluster point of $\langle x_\alpha \rangle$. So $a) \implies b)$.

Now let us suppose that $X$ is not compact, then there exists an open cover $\{U_{\alpha \in A}\}$ of $X$ that has no finite subcover. Let $\mathbb{B}$ be the collection of all finite subsets of $A$, directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every $B \in \mathbb{B}$, find some $x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c$. So we have a net in $X$. Now we will show that no $x \in X$ can be a cluster point of this net. Suppose not, then take a neighbourhood $U_\beta$ with $\beta \in A$ such that $U_\beta$ belongs to the open cover we first discussed. Then for any $B \in \mathbb{B}$ such that $B \gtrsim \{\beta\}$ (meaning that $\{\beta\} \subseteq B$, where $B$ is a finite set), then

$$x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c \implies x_B \notin \left(\bigcup_{\alpha \in \{\beta\}} U_\beta\right) \implies x_B \in U_\beta^c$$

Hence no point in $X$ can be a cluster point for this net, and the proof is complete. □

## 4.30 Theorem 4.30

**WTS.** *If $X$ is a LCH space, and for every $U \in \mathcal{N}_B(x) \cap \mathrm{T}_X$, there exists a compact $N \subseteq U$ where $N \in \mathcal{N}_B(x)$.*

*Proof.* For every $U \in \mathcal{N}_B(x) \cap \mathrm{T}_x$, we can find an $E$ open subset of $U$ that has a compact closure, since every $x \in X$ induces some compact $F \in \mathcal{N}_B(x)$, therefore

$$E \coloneqq U \cap F^o \implies \overline{E} \subseteq F$$

Since closed subsets of compact sets are compact (by Theorem 4.22), $\overline{E}$ is compact. More is true, since $E$ is open,

$$x \in U \cap F^o \implies x \in E^o \implies E \in \mathcal{N}_B(x)$$

Now it suffices to show that there exists some compact $N \subseteq E \subseteq U$ such that $N \in \mathcal{N}_B(x)$. Since $\overline{E}$ is compact, the closed subset $\partial E = \overline{E} \cap \overline{E^c}$ of $\overline{E}$ is also compact.

Since $\partial E \cap E^o = \varnothing$, $x \in E^o = E$ means that $x \notin \partial E$. Applying Theorem 4.23 to the compact set $\partial E$ and $x \notin \partial E$ gives us two disjoint open sets $V'$ and $W'$. We list their properties

1. $V', W' \in \mathrm{T}_X$
2. $x \in V'$
3. $\partial E \subseteq W'$
4. $V' \cap W' = \varnothing$

The two disjoint pairs induce another pair of open sets relative to $\overline{E}$, recall the definition of the topology relative to $\overline{E}$,

$$\mathrm{T}_{\overline{E}} = \left\{ A \cap \overline{E} : A \in \mathrm{T}_X \right\}$$

We now agree to define

- $V = V' \cap \overline{E}$
- $W = W' \cap \overline{E}$

Then evidently $V, W \in \mathrm{T}_{\overline{E}}$ and

1. $x \in V' \cap \overline{E} \implies x \in V$

2. $\partial E \subseteq \overline{E} \implies \partial E \subseteq W$

3. $V' \cap W' = \varnothing \implies V \cap W = \varnothing$

Furthermore,

$$\partial E \subseteq W \implies W^c \subseteq (\partial E)^c = E^o \cup E^{co}$$

Taking the intersection over $\overline{E}$ gives us

$$\overline{E} \setminus W \subseteq \overline{E} \cap (E^o \cup E^{co})$$

Note that $E^{co} = (\overline{E})^c$, since $(E^c)^{oc} = \overline{(E^{cc})} = \overline{E}$ therefore $\overline{E} \cap E^{oc} = \varnothing$, hence

$$\overline{E} \setminus W \subseteq \overline{E} \cap E^o = E^o$$

Using the fact from 3, $V \subseteq W^c$ and $V \subseteq \overline{E}$ and $V \subseteq W^c$ implies that $V \subseteq \overline{E} \setminus W$. Compiling everything, we have

$$V \subseteq \overline{E} \setminus W \subseteq E$$

Note that the set $\overline{E} \setminus W$ is closed in $\mathrm{T}_X$ (and hence closed in $\overline{E}$) by closure over intersections, $\overline{V}$ is therefore a closed subset of $\overline{E} \setminus W$, and $\overline{V}$ is compact. Also

$$\overline{V} \subseteq \overline{E} \setminus W \subseteq E$$

To check that $\overline{V} \in \mathcal{N}_B(x)$, note that

$$x \in V^o \subseteq (\overline{V})^o \implies \overline{V} \in \mathcal{N}_B(x)$$

The subset relation $V^o \subseteq \overline{V}^o$ comes from the fact that $V^o$ is an open subset of $\overline{V}$, and hence is contained in $(\overline{V})^o$ as a subset. Now let us define $N = \overline{V}$, and $N$ satisfies the assertions in the Theorem, since

- $N \in \mathcal{N}_B(x)$

- $N$ is compact

- $N \subseteq E \subseteq U$

And this completes the proof. □

**Remark.** *Intuitively speaking, this means that if $X$ is any LCH space, then for every open neighbourhood $U \in \mathcal{N}_B(x)$, there exists a compact $E \in \mathcal{N}_B(x)$ such that $x \in E \subseteq U^o$. This property is indeed a very strong one as it allows us to have effectively 'infinite' descending compact neighbourhoods of $x$.*

## 4.31 Theorem 4.31

**WTS.** *$X$ is a LCH space, and $K \subseteq U \subseteq X$ where $K$ is compact, and $U$ is open, then there exists some precompact, open $V$ with*

$$K \subseteq V \subseteq \overline{V} \subseteq U$$

*Proof.* For every $x \in K$, we can apply Proposition 4.30, since $x \in K \subseteq U$, this induces some compact $F_x \subseteq U$ where $F_x \in \mathcal{N}_B(x)$. Then we can obtain an open cover of $U$ in the form of $\{F_x^o\}_{x \in K}$. By compactness of $K$, there exists a finite $B \subseteq K$ such that

$$K \subseteq \bigcup_{x \in B} F_x^o$$

Let $V = \bigcup_{x \in B} F_x^o$, then clearly $V$ is open, and $K \subseteq V$. Since each $F_x$ is closed (compact sets are closed in any Hausdorff Space), we have

$$V \subseteq \bigcup_{x \in B} F_x \implies \overline{V} \subseteq \bigcup_{x \in B} F_x$$

Since $\bigcup_{x \in B} F_x$ is a finite union of compact sets, we claim that it is also compact. Consider two compact sets $E_1$ and $E_2$, then if $\{U_\alpha\}_{\alpha \in A}$ is any open cover of $E_1 \cup E_2$, it must be an open cover for $E_1$ and $E_2$ as well, because

$$E_1, E_2 \subseteq E_1 \cup E_2 \bigcup_{\alpha \in A} U_\alpha$$

Since $E_1$ and $E_2$ are both compact sets, they each induce two finite subsets of $B_1$, $B_2$ of $A$ whose union $B = B_1 \cup B_2$ is also compact. Therefore

$$E_1 \cup E_2 \subseteq \bigcup_{\alpha \in B} U_\alpha$$

Then a simple proof by induction will show that if $\{E_{j \leq n}\}$ is a family of compact sets, then $E = \bigcup E_{j \leq n}$ is also compact.

Returning to the main part of the proof, $\bigcup_{x \in B} F_x$ is a compact set, therefore $\overline{V}$ is also compact. Moreover

$$\forall x \in K,\ F_x \subseteq U \implies \overline{V} \subseteq \bigcup_{x \in B} F_x \subseteq U$$

Combining, we have

- $K \subseteq V \subseteq \overline{V}$,
- $V$ is open and $\overline{V}$ is compact, and
- $\overline{V} \subseteq U$

This completes the proof. $\square$

# 5 Chapter 5

# 6 Chapter 6

## 6.1 Theorem 6.15

**WTS.**

First suppose that $(X, \mathcal{M}, \mu)$ is finite measure space. If $\mu(X) < +\infty$, then for every $E \in \mathcal{M}$, by monotonicity $E \subseteq X$ yields $\mu(E) \leq \mu(X) < +\infty$. Next, for any $p < +\infty$, $\|\chi_E\|_p^p < +\infty$ and $\|\chi_E\|_{+\infty} \leq 1 < +\infty$. So all indicator functions are in $L^p$.

It follows that every simple function is also in $L^p$, since it is a finite linear combination of indicators. We now define $\nu(E) = \phi(\chi_E)$, we wish to show that $\nu : \mathcal{M} \longrightarrow \mathbb{C}$ is a complex measure which is absolutely continuous with respect to $\mu$.

To show $\sigma$-additivity, fix any disjoint sequence $\{E_j\}_{j\geq 1} \subseteq \mathcal{M}$. Where we also note that $\mu(E) = \mu(\cup E_j) < +\infty$. Now suppose that $p < +\infty$, then the following converges in the $p$-norm

$$\chi_E = \sum_{j\geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left(\bigcup E_{j\leq n}\right) = \left(\bigcup E_{j\geq 1}\right) \setminus \left(\bigcup E_{j\leq n}\right) = \bigcup E_{j\geq n+1}$$

and define $F_{n+1}$ as the rightmost member above. Then $\{F_{n\geq 1}\}$ is a decreasing sequence of sets. All sets are of finite measure, hence $\mu(E) - \mu(\cup E_{j\leq n}) = \mu(F_{n+1}) \to 0$.

Now, for any fixed $n \geq 1$,

$$\left|\chi_E - \sum \chi_{E_{j\leq n}}\right| = \left|\sum \chi_{E_{j\geq n+1}}\right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the $p$th power does not change pointwise, and

$$\left|\sum \chi_{E_{j\geq n+1}}\right|^p = \left|\sum \chi_{E_{j\geq n+1}}\right| = \sum \chi_{E_{j\geq n+1}}$$

Convergence in $p$-norm is given by

$$\left\|\chi_E - \sum \chi_{E_{j\leq n}}\right\| = \left\|\sum \chi_{E_{j\geq n+1}}\right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our $\phi \in L^{p*}$

$$\begin{aligned}
\nu(E) &= \phi(\chi_E) \\
&= \phi\left(\lim_{n\to\infty} \sum \chi_{E_{j\leq n}}\right) \\
&= \lim_{n\to\infty} \phi\left(\sum \chi_{E_{j\leq n}}\right) \\
&= \lim_{n\to\infty} \sum \phi\left(\chi_{E_{j\leq n}}\right) \\
&= \lim_{n\to\infty} \sum \nu(E_{j\leq n})
\end{aligned}$$

To show absolute convergence, recall that for any $\phi(\chi_{E_j}) \in \mathbb{C}$, define $\beta_j = \overline{\operatorname{sgn}(\|\phi(\chi_{E_j})\|)}$ then multiplication yields

$$\|\phi(\chi_{E_j})\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the $p$-norm.

$$\left\|\sum_{j\geq 1} \beta_j \chi_{E_j} - \sum_{j\leq n} \beta_j \chi_{E_j}\right\|_p = \left\|\sum_{j\geq n+1} \beta_j \chi_{E_j}\right\|_p$$

And because $\left|\sum_{j\geq n+1} \beta_j \chi_{E_j}\right|$ is pointwise equal to $\left|\sum_{j\geq n+1} \chi_{E_j}\right|$, since $|\beta_j| = 1$ for every $j \geq 1$. We can reuse the same continuity and linearity argument. We also note that $\sum_{j\geq 1} \beta_j \chi_{E_j} \in L^p$ since its $p$-norm is equal to $\mu(E)^{1/p}$.

$$
\begin{aligned}
\sum_{j\geq 1}|\nu(E_j)| &= \sup_{n\geq 1}\sum_{j\leq n}\|\nu(E_{j\leq n})\| \\
&= \lim_{n\to\infty}\sum_{j\leq n}\|\phi(\chi_{E_j})\| \\
&= \lim_{n\to\infty}\sum_{j\leq n}\beta_j\phi(\chi_{E_j}) \\
&= \lim_{n\to\infty}\phi\left(\sum_{j\leq n}\beta_j\chi_{E_j}\right) \\
&= \phi\left(\lim_{n\to\infty}\sum_{j\leq n}\beta_j\chi_{E_j}\right) \\
&\leq \|\phi\|\left\|\sum_{j\geq 1}\beta_j\chi_{E_j}\right\|_p \\
&< +\infty
\end{aligned}
$$

Assuming the above estimate holds, then we only need $\nu(E) = \phi(\chi_E) = \mu(E) = 0$ ($\nu$ is now a measure and $\nu \ll \mu$), As the indicator of a null set is equal to the zero element in $L^p$. Then by Radon-Nikodym we can have some $g \in L^1(\mu)$ such that

$$d\nu = gd\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function $g$. For every $\chi_E$ measurable, $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$, by monotonicity of the integral in $L^+$. So any simple function, $\alpha = \sum a_j \cdot \chi_{E_j}$ means that $\alpha g$ is in $L^1(\mu)$, and

$$\phi(\alpha) = \int \alpha g d\mu$$

If $\|\alpha\|_p = 1$, then

$$\left|\int \alpha g\right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$$M_q(g) = \sup\left\{\left|\int \alpha \cdot g\right|, \|\alpha\|_p = 1, \text{ and } \alpha \text{ is simple and vanishes outside a set of finite measure.}\right\}$$

Since $S_g = \{x \in X, g(x) \neq 0\}$ is $\sigma$-finite, an application of Theorem 6.14 tells us that $g \in L^q$, and $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$. Now that we know $g$ is in $L^q$ we can use the density of $\alpha$ in $L^p$ to show, for every single $f \in L^p$

$$\phi(f) = \int fgd\mu$$

Conjure a sequence of '$\alpha$'s, and call them $\{f_n\} \to f$ p.w.a.e, then each $f_n \cdot g \in L^1$. An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n\, gd\mu = \int fgd\mu = \phi(f)$$

This completes the proof for when $\mu$ is finite.

Let us upgrade our $\mu$ into a $\sigma$-finite measure. Then there exists an increasing sequence $\{E_n\} \nearrow X$ such that each $E_n$ is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in $P_n$ vanishes outside a set of finite measure and is also in $L^p$. And $Q_n$ is defined in a similar manner. Now, fix our $\phi \in L^{p*}$, and for each $f \in P_n$, there exists a corresponding $g_n \in Q_n$. Then $p \in [1, +\infty)$ tells us that $q \in (1, +\infty]$, and the assumptions for Theorem 6.13 all hold. Therefore for each $g_n \in Q_n$, there is a corresponding bounded linear operator $\phi_{g_n} \in (P_n)^*$ such that

$$\phi(f) = \phi|_{P_n}(f) = \int fg_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of $g_n$ towards some $g \in L^q$. We claim that this limit makes sense. As for any $n < m$, such that $E_n \subseteq E_m$ then $g_n = g_m$ on $E_n$ pointwise. The proof is simple since each the restriction of our $\phi \in L^{p*}$ onto $E_n$ and $E_m$ spawns two functions $g_n$ and $g_m \in L^1$. To verify, take any subset $Z \subseteq E_n$ then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So $g_n = g_m$ pointwise a.e on $E_n$. Now we define $g$ measurable such that $g|_{E_n} = g_n$ for every $n$. And

$$\begin{aligned}
|g_n| = \chi_{E_n} \cdot |g_m| &\implies \\
|g_n| \leq |g_{n+1}| &\implies \\
&\|g_n\|_q \leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty
\end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with ($P_n \subseteq P_{n+1}$). If $q = +\infty$ then $g \in L^\infty$ is trivial, but for any $q < +\infty$. We wish to show that $g \in L^q$. Since $|g_n| \leq |g|$ pointwise for every $n$, and for each $x \in X$, there exists a $N$, where $n \geq N$ implies $|g(x)| = |g_n(x)|$, so $|g(x)|$ is an upperbound that is actually attained by the sequence $|g_n(x)|$. So, $|g(x)| = \sup_{n\geq 1}\{|g_n(x)|\}$.

Using the Monotone Convergence Theorem on $|g_n|$,

$$\begin{aligned}
\int \lim_{n\to\infty} |g_n|^q d\mu &= \int \sup_{n\geq 1} |g_n|^q d\mu \\
&= \int |g|^q d\mu \\
&= \lim \int |g_n|^q d\mu
\end{aligned}$$

Which yields $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$. It follows that $g \in L^q$.

Finally, we will show that $\phi(f) = \int fg$ for every $f \in L^p$. Redefine $f_n = f \cdot \chi_{E_n} \in P_n$ for every $n \geq 1$. We claim that $f_n \to f$ in the $p$-norm.

$$\begin{aligned}
|f_n - f| &\leq |f_n| + |f| \\
&\leq |f| + |f| \\
&\leq 2|f|
\end{aligned}$$

And $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$. Now it is permissiable to apply the Dominated Theorem, and we will do so.

$$\begin{aligned}
\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\
\lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\
&= 0
\end{aligned}$$

And we have $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}
\phi(f) &= \lim \phi|_{P_n}(f_n) \\
&= \lim \int f_n \cdot g_n \\
&= \lim \int f \cdot g \cdot \chi_{E_n} \\
&= \int \lim (fg \cdot \chi_{E_n}) \\
&= \int fg
\end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of $fg\chi_{E_n} \to fg$ pointwise and Holder's Inequality. This completes the proof for when $\mu$ is of $\sigma$-finite measure, and $p \in [1, +\infty)$.

Suppose now $\mu$ is arbitrary, and $p \in (1, +\infty)$, then $q < +\infty$. Now let us agree to define, for every $\sigma$-finite $E \subseteq X$

$$P_E = \{L^p, |f| = |f| \cdot \chi_E\}$$

Where $Q_E$ does not hold any surprises. Then for each $E$ we have a $\phi|_E$ which induces a $g_E$ that vanishes outside $E$. We are ready for the final part of the proof.

First, if $E \subseteq F$ and both $E$ and $F$ are $\sigma$-finite, then $\|g_E\|_q \leq \|g_F\|_q$. This is a simple consequence of monotonicity in $L^+$ if we take $|g_E|^q \leq |g_F|^q$.

Second, we define

$$W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}$$

Let $M$ be the supremum of $W$, then there exists a sequence of $\sigma$-finite sets, $\{E_n\}$ where $\|g_{E_n}\|_q \to M \leq \|\phi\|_{p*}$. Take a set $F = \cup E_{n\geq 1}$, which is also $\sigma$-finite, so that $\|g_F\|_q = M$. Now assume there exists another $\sigma$-finite superset of $F$, let us call it $A$. Then

$$\int |g_F|^q + \int |g_{A\setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have $g_{A\setminus F} = 0$ pointwise a.e. For any $f \in L^p$, the spots where $f$ does not vanish is $\sigma$-finite. This comes from $\int |f|^p < +\infty$. So it suffices to integrate over this $\sigma$-finite set. But we already know, even if this set $A$ contains $F$ as a subset, $\int f g_F = \int f g_A$.

We now define $g = g_F$, and the proof is complete. As for every $\phi \in L^{p*}$, there exists a $g \in L^q$ such that the evaluation of any $f \in L^p$ is given by integrating $f$ with $g$. ■