MATH 263: Section 003, Tutorial 8

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1 Introduction to Nonhomogeneous ODEs: Method of Undetermined Coefficients

To solve a nonhomogeneous linear ODE of the form:

$$y''(x) + p(x)\ y'(x) + q(x)\ y(x) = g(x) \neq 0$$

note that by superposition, if $y_1(x)$ and $y_2(x)$ solve the homogeneous ODE:

$$y''(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

and that some specific $Y(x)$ solves the nonhomogeneous ODE, then:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x)$$

is the general solution to the nonhomogeneous ODE. To find $Y(x)$, we need to make an educated guess on the form of the solution: we want that guessed form of $Y(x)$ and/or its derivatives to be linearly dependent on $g(x)$. One also must make sure that their chosen $Y(x)$ is not $g(x)$, since it already solves the homogeneous equation. Then find $Y(x)$'s coefficients by plugging them in the ODE.

For the particular solution of $ay'' + by' + cy = g(x)$, here's a table directly from Boyce and DiPrima, 10th edition (3.5, table 3.5.1, p.182):

$g(x)$	$Y(x)$
$P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
$P_n(x)e^{\alpha x}$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}$
$P_n(x)e^{\alpha x}\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$x^s[(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}\cos\beta x + (B_0x^n + B_1x^{n-1} + \cdots + B_n)e^{\alpha x}\sin\beta x]$

Note: Here s is the smallest nonnegative integer (s = 0, 1, or 2) that will ensure that no term in $Y(x)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, **s is the number of times 0 is a root of the characteristic equation**, α **is a root of the characteristic equation**, and $\alpha + i\beta$ **is a root of the characteristic equation**, respectively.

Problem 1.1. Solve the IVP:

$$y''(x) + 4y(x) = 2\cos 2x,\ \ y(0) = 1,\ \ y'(0) = 0$$

Solution: First solve the homogeneous ODE:

$$y''(x) + 4y(x) = 0$$

The characteristic equation is:

$$r^2 + 4 = 0$$

The roots being

$$r_{1,2} = \pm 2i.$$

Therefore, the solution to the homogeneous ODE is:

$$y_h(x) = c_1 \cos 2x + c_2 \sin 2x$$

Now, find a particular solution $Y(x)$ to the nonhomogeneous ODE. $Y(x)$ cannot be of the form $A \cos 2x$, since it already solves the homogeneous equation (s = 1). Therefore, let

$$Y(x) = Ax \cos 2x + Bx \sin 2x$$

$$Y'(x) = -2Ax \sin 2x + A \cos 2x + B \sin 2x + 2Bx \cos 2x$$

$$Y"(x) = 4B \cos 2x - 4Ax \cos 2x - 4A \sin 2x - 4Bx \sin 2x$$

Plug it in the ODE:

$$4B \cos 2x - 4Ax \cos 2x - 4A \sin 2x - 4Bx \sin 2x + 4Ax \cos 2x + 4Bx \sin 2x = 2 \cos 2x$$

$$4B \cos 2x - 4A \sin 2x = 2 \cos 2x$$

Now find A and B:

$$\cos 2x : 4B = 2 \Rightarrow B = \frac{1}{2}$$

$$\sin 2x : -4A = 0 \Rightarrow A = 0$$

Therefore,

$$Y(x) = \frac{1}{2} x \sin 2x$$

$$y(x) = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{2} x \sin 2x$$

Solve the IVP $y(0) = 1,\ y'(0) = 0$:

$$y'(x) = -2c_1 \sin 2x + 2c_2 \cos 2x + \frac{1}{2} \sin 2x + x \cos 2x$$

$$y(0) = c_1 = 1$$

$$y'(0) = 2c_2 = 0 \Rightarrow c_2 = 0$$

$$y(x) = \cos 2x + \frac{1}{2} x \sin 2x.$$

Problem 1.2. From Boyce and DiPrima, 10th edition (3.5, exercise 17, p.184):
Solve the IVP:

$$y"(t) - 2y'(t) + y(t) = te^t + 4,\ y(0) = 1,\ y'(0) = 1.$$

Solution: First solve the homogeneous ODE:

$$y"(t) - 2y'(t) + y(t) = 0$$

The characteristic equation is:

$$k^2 - 2k + 1 = (k-1)^2 = 0$$

The double roots being

$$r_{1,2} = 1.$$

From reduction of order, the solution to the homogeneous ODE is:

$$y_h(t) = c_1 e^t + c_2 t e^t$$

Now, find a particular solution $Y(t)$ to the nonhomogeneous ODE. $Y(t)$ cannot contain te^t, since it already solves the homogeneous equation (te^t is found in a double root, so s = 2).
Without forgetting the constant,

$$Y(t) = t^2(Ae^t + Bte^t) + D = At^2e^t + Bt^3e^t + D$$

$$Y'(t) = At^2e^t + 2Ate^t + Bt^3e^t + 3Bt^2e^t$$

$$Y"(t) = 2Ae^t + 4Ate^t + 6Bte^t + At^2e^t + 6Bt^2e^t + Bt^3e^t$$

Now, plug it in the ODE (this looks terrible I know):

$$2Ae^t + 4Ate^t + 6Bte^t + At^2e^t + 6Bt^2e^t + Bt^3e^t - 2At^2e^t - 4Ate^t - 2Bt^3e^t - 6Bt^2e^t + At^2e^t + Bt^3e^t + D = te^t + 4$$

$$2Ae^t + 4Ate^t + 6Bte^t + At^2e^t + 6Bt^2e^t + Bt^3e^t - 2At^2e^t - 4Ate^t - 2Bt^3e^t - 6Bt^2e^t + At^2e^t + Bt^3e^t + D = te^t + 4$$

$$t^3e^t(B - 2B + B) + t^2e^t(A + 6B - 2A - 6B + A) + te^t(4A + 6B - 4A) + 2Ae^t + D = te^t + 4$$

$$6Bte^t + 2Ae^t + D = te^t + 4$$

Now find A, B, and D:

$$e^t : 2A = 0 \Rightarrow A = 0$$

$$te^t : 6B = 1 \Rightarrow B = \frac{1}{6}$$

$$1 : D = 4$$

Note: if your system of equations for coefficients doesn't give you any answer (inconsistent system, all coefficients cancel out, etc), either you made an algebraic mistake or made a wrong choice for $Y(t)$.
The general solution is then:

$$Y(t) = \frac{1}{6}t^3e^t + 4$$

$$y(t) = c_1e^t + c_2te^t + \frac{1}{6}t^3e^t + 4$$

Solve the IVP $y(0) = 1,\ y'(0) = 1$:

$$y'(t) = c_1e^t + c_2e^t + c_2te^t + \frac{1}{2}t^2e^t + \frac{1}{6}t^3e^t$$

$$y(0) = c_1e^0 + 4 = 1 \Rightarrow c_1 = -3$$

$$y'(0) = c_1e^0 + c_2e^0 = 1 \Rightarrow c_1 = 1 - c_2 = 1 + 3 = 4$$

$$y(t) = -3e^t + 4te^t + \frac{1}{6}t^3e^t + 4$$

$$y(t) = (\frac{1}{6}t^3 + 4t - 3)e^t + 4.$$