Notation

We will use the following notation to simplify computations with multilinear maps. Let E and F be sets, and $v_1, \dots, v_k \in E$. $f : E \to F$.

- Listing individual elements: $v_{\underline{k}}$ means $v_1, \dots, v_k$ as separate elements.
- Creating a k-list: $(v_{\underline{k}}) = (v_1, \dots, v_k) \in \prod E_{j \le k}$ if $v_i \in E_i$ for $i = \underline{k}$.
- Double indices: $(v_{\underline{n_k}}) = (v_{n_{\underline{k}}}) = (v_{n_1}, \dots, v_{n_k})$, and

$$(v_{\underline{n_k}}) \neq (v_{n_{(}1,\dots,k)})$$

- Closest bracket convention:

$$(v_{(n_{\underline{k}})}) = (v_{(n_1,\dots,n_k)}) \quad \text{and} \quad (v_{n_{(\underline{k})}}) = (v_{n_{(1,\dots,k)}})$$

- Underlining 0 means it is iterated 0 times:

$$(v_{\underline{0}}, a, b, c) = (a, b, c)$$

- Skipping an index:

$$(v_{\underline{i-1}}, v_{i+\underline{k-i}}) = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$$

  for $i = \underline{k}$.

- Applying f to a particular index:

$$(v_{\underline{i-1}}, f(v_i), v_{i+\underline{k-i}}) = (v_1, \dots, v_{i-1}, f(v_i), v_{i+1}, \dots, v_k)$$

  Of course, if $i = 1$, then the above expression reads $(f(v_1), v_2 \dots, v_k)$ by the $\underline{0}$ interpretation.

- In any list using this 'underline' notation, we can find the size of a list by summing over all the underlined terms, and the number of terms with no underline.
- If $\wedge : E \times E \to F$ is any associative binary operation,

$$\bigwedge(v_{\underline{k}}) = v_1 \wedge \cdots \wedge v_k$$

Remark 1.1: Preview of exterior calculus

We can write the formula for the determinant of a $\mathbb{R}^{k\times k}$ matrix in this notation. Suppose $a_i \in \mathbb{R}$, and $b_i \in \mathbb{R}^{k-1}$ for $i = \underline{k}$.

$$M = \begin{bmatrix} a_1 & \cdots & a_k \\ | & & | \\ b_1 & \cdots & b_k \\ | & & | \end{bmatrix}$$

The determinant of M, can then be written as

$$\det(M) = \sum_{i=\underline{k}} (-1)^{i-1} a_i \det\Big(b_{\underline{i-1}}, b_{i+\underline{k-i}}\Big)$$

k-linear maps

Definition 2.1: k-linear maps

Let $E_{\underline{k}}$, F be Banach spaces. A map $\varphi : \prod E_{\underline{k}}$ is k-linear if for every $i = \underline{k}$, $v_i \in E_i$,

$$\varphi(\cdot^{\underline{i-1}}, v_i, \cdot^{\underline{k-i}}) : \prod(E_{\underline{i-1}}, E_{i+\underline{k-i}}) \to F \quad \text{is } (k-1)\text{-linear}$$

The following theorem should give confidence to the notation we have adopted to use.

Proposition 2.1

Let $E_{\underline{k}}$ and F be Banach spaces, a k-linear map $\varphi : \prod E_{\underline{k}} \to F$ is continuous iff there exists a $C > 0$, such that for every $x_i \in E_i$, $i = \underline{k}$

$$\Big|\varphi(x_{\underline{k}})\Big| \leq C \prod \Big|x_{\underline{k}}\Big|$$

Proof. Suppose φ is continuous, then it is continuous at the origin. Picking $\varepsilon = 1$ induces a $\delta > 0$ such that for $\Big|(x_{\underline{k}})\Big| \leq \delta$, $\Big|\varphi(x_{\underline{k}})\Big| \leq 1$. The usual trick of normalizing an arbitrary vector $(x_{\underline{k}}) \in \prod E_{\underline{k}}$ does the job:

$$\Big|\varphi(x_k \cdot \Big|x_{\underline{k}}\Big|^{-1} \cdot \delta)\Big| \leq 1 \implies \Big|\varphi(x_{\underline{k}})\Big| \leq \delta^{-k} \prod \Big|x_{\underline{k}}\Big|$$

Conversely, fix a sequence (indexed by n, in k elements in the product space $\prod E_{\underline{k}}$), so

$$(x_n^{\underline{k}}) \to (x^{\underline{k}}) \quad \text{as } n \to +\infty \tag{1}$$

To proceed any further, we need to prove an important equation that decomposes a difference in φ.

$$\varphi(b^{\underline{k}}) - \varphi(a^{\underline{k}}) = \sum_{i=\underline{k}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}}) \tag{2}$$

where $(b^{\underline{k}})$ and $(a^{\underline{k}})$ are elements in $\prod E_{\underline{k}}$, and $\Delta_i = b^i - a^i$ for $i = \underline{k}$. The proof is in the following note, which is in more detail than usual - to help the reader ease into the new notation.

Note 2.1

We proceed by induction, and eq. (2) follows by setting $m = k$ in

$$\varphi(a^{\underline{k}}) = \varphi(b^{\underline{m}}, a^{m+\underline{k-m}}) - \sum_{i=\underline{m}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}}) \tag{3}$$

Base case: set $m = 1$, by definition of k-linearity (definition 2.1) of φ. Since $a^1 = b^1 - \Delta_1$,

$$\varphi(a^{\underline{k}}) = \varphi(b^1 - \Delta_1, a^{1+\underline{k-1}}) = \varphi(b^1, a^{1+\underline{k-1}}) - \varphi(\Delta_1, a^{1+\underline{k-1}})$$

Induction hypothesis: suppose eq. (3) holds for a fixed m. Since $a^{m+1} = b^{m+1} - \Delta_{m+1}$,

$$\begin{aligned}\varphi(a^{\underline{k}}) &= \varphi(b^{\underline{m}}, a^{m+\underline{k-m}}) - \sum_{i=\underline{m}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}}) \\ &= \varphi(b^{\underline{m}}, a^{m+1}, a^{(m+1)+\underline{k-(m+1)}}) - \sum_{i=\underline{m}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}}) \\ &= \varphi(b^{\underline{m+1}}, a^{(m+1)+\underline{k-(m+1)}}) - \varphi(b^{\underline{m+1}}, \Delta_{m+1}, a^{(m+1)+\underline{k-(m+1)}}) - \sum_{i=\underline{m}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}})\end{aligned}$$

and this proves eq. (2)

We substitute $a^i = x^i$, and $b^i = x_n^i$ for $i = \underline{k}$, and eq. (2) becomes eq. (4)

$$\varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) = \sum_{i=\underline{k}} \varphi(x_n^{\underline{i-1}}, x_n^i - x^i, x^{i+\underline{k-i}}) \tag{4}$$

Then the triangle inequality reads

$$\begin{aligned}\left|\varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}})\right| &\leq \sum_{i=\underline{k}} \left|\varphi(x_n^{\underline{i-1}}, x_n^i - x^i, x^{i+\underline{k-i}})\right| \\ &\leq \sum_{i=\underline{k}} |\varphi| \cdot \Pi\left(x_n{}^{\underline{i-1}}, \Delta_i, x^{i+\underline{k-i}}\right) \\ &\leq \sum_{i=\underline{k}} |\varphi| \cdot \left|x_n^i - x^i\right| \, \Pi \, \left(x_n^{\underline{i-1}}, x^{i+\underline{k-i}}\right) \\ &\lesssim_n |\varphi| \sup_{i=\underline{k}} |x_n^i - x^i| \to 0\end{aligned}$$

where we identify the product $\Pi\left(v^{\underline{k}}\right)$ with the product of their norms $\Pi\left(\left|v^{\underline{k}}\right|\right)$. ■

Remark 2.1

The k-linear variant of **??** holds. We will use but not prove this fact.

We will discuss (covariant) *tensors* on a single Banach space E, which are simply k-linear maps from the product space of E. That is, $E_1 = \cdots = E_k = E$, and we write $L(E_{\underline{k}}; F) = L(E^k, F) = L^k(E, F)$.