Theorem 3.21

WTS. *The Lebesgue Differentiation Theorem. Suppose $f \in L^1_{loc}$, and for every $x \in \mathcal{L}_f$, (so that $x \in \mathbb{R}^n$ a.e). We have*

1. $\lim_{r \to 0} \dfrac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0,$
2. $\lim_{r \to 0} \dfrac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x),$

For every family $\{E_r\}_{r>0}$ that shrinks nicely to $x \in \mathbb{R}^{n\prime}$.

Proof. Since the family $\{E_r\}_{r>0}$ shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r,x)) \implies m(E_r) > \alpha \cdot m(B(r,x))$$

for some $\alpha > 0$, independent on r. Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r,x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r,x)} |f(y) - f(x)| dy$$

Combining the last two results, for every $\varepsilon > 0$, if $0 < r < \varepsilon$, then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1} B(r,x) \int_{B(r,x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0<r<\varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0<r<\varepsilon} m^{-1} B(r,x) \int_{B(r,x)} |f(y) - f(x)| dy$$

and sending $\varepsilon \to 0$, proves the first claim. The second claim is immediate upon applying the L^1 inequality.

Fix any $\varepsilon > 0$, and

$$\begin{aligned}
\lim_{r\to 0} m^{-1}(E_r)\int_{E_r} f(y)dy = f(x) &\iff \lim_{\varepsilon\to 0}\sup_{0<r<\varepsilon}\left|m^{-1}(E_r)\int_{E_r} f(y)dy - f(x)\right| \\
&\iff \lim_{\varepsilon\to 0}\sup_{0<r<\varepsilon}\left|m^{-1}(E_r)\int_{E_r} [f(y) - f(x)]dy\right| \\
&\leq \lim_{\varepsilon\to 0}\sup_{0<r<\varepsilon} m^{-1}(E_r)\int_{E_r} |f(y) - f(x)|dy \\
&= \lim_{r\to 0} m^{-1}(E_r)\int_{E_r} |f(y) - f(x)|dy \\
&= 0
\end{aligned}$$

□