Chapter 4: Submersions, Immersions and Embeddings

Let F be a smooth map between two smooth manifolds M and N, with dimensions m and n respectively.

The rank of F at $p \in M$ is the rank of the linear map:

$$dF_p:T_pM\to T_{F(p)}N$$

Example 1.28 (Matrices of Full Rank)

Let $A \in \mathcal{M}(m \times n, \mathbb{R})$ be the set of $m \times n$ matrices with real entries. A has rank m iff there exists some $m \times m$ sub-matrix of A, denoted by S st S is invertible. We wish to show the set of rank-m matrices is invertible. Indeed, let

$$F: \mathcal{M}(m imes n, \mathbb{R}) o \mathbb{R}, \ F(A) = \sum_{S ext{ is a } m imes m ext{ sub-matrix of } A} |\det S|$$

Since $S \mapsto \det S$ is continuous in the entries of S, hence continuous in the entries of A, F is continuous.

So the set
$$\left\{A \in \mathcal{M}(m \times n, \mathbb{R}), \operatorname{rank} A = m\right\} = F^{-1}(\mathbb{R} \setminus \{0\})$$
 is open.

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Before proving the inverse functino theorem, we will need several Lemmas

Proposition 0.1. If A and B are in L(X,Y), then

$$||BA|| \le ||B|| ||A||$$

Proof. Let ||x|| = 1, and

$$||B(Ax)|| \le ||B|| ||Ax|| \le ||B|| ||A|| ||x||$$

this holds for every ||x|| = 1, hence

$$||BA|| \le ||B|| ||A||$$

Proposition 0.2. Let f map a convex open set $U \subseteq \mathbb{R}^n$ into \mathbb{R}^m , if f is differentiable (pointwise) in U, and there exists some M st its derivative its founded (in the operator norm)

$$||Df(x)|| \le M \quad x \in U$$

then, for every pair of elements x_1 , x_2 in U,

$$||f(x_1) - f(x_2)|| \le M||x_1 - x_2||$$

Proof. This proof 'passes the argument' to the scalar-valued version, in short: if x_1 and x_2 are in U. Define

$$c(t) = (1-t)x_1 + tx_2$$

as the convex combination of x_1 and x_2 . The takeaway intuition here is that it suffices to check on the line joining the two points', to obtain an estimate for $||f(x_1) - f(x_2)||$. Indeed, define

$$g(t) = f(c(t))$$
 is a curve $g: \mathbb{R} \to \mathbb{R}^m$

Recall: Theorem 5.19

Proposition 0.3. Let $g:[0,1] \to \mathbb{R}^m$, and g be differentiable on (0,1), then there exists some $x \in (0,1)$ with

$$|f(b) - f(a)| \le (b-a)|f'(x)|$$

Proof. Read from Rudin Theorem 5.19.

Since $Dg(t) = Df(c(t)) \circ Dc(t)$ by the Chain Rule, and Dc(t) = b - a by inspection,

$$\|Dg(t)\| = \|Df(c(t)) \circ Dc(t)\| \le \|Df\| \|Dc\| = \|Df\| (b-a)$$

This holds for every $t \in [0,1]$. Applying Theorem 5.19 gives

$$\underbrace{\|g(1) - g(0)\|}_{\text{curve endpoints}} \le M \|b - a\|$$

Replacing $\|g(1) - g(0)\| = \|f(x_1) - f(x_2)\|$ and $\|Df\| \le M$ we get

$$||f(x_1) - f(x_2)|| \le M||x_1 - x_2||$$

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Rudin Inverse Function Theorem 9.24

Proposition 0.4. Suppose $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$, and Df(a) is invertible for some $a \in \mathbb{R}^n$, and define b = f(a). Then,

- (a) there exist open sets U and V in \mathbb{R}^n such that $a \in U$, $b \in V$, and f is one-to-one on U, and f(U) = V.
- (b) if g is the inverse of f (which exists, by Part a), defined in V by g(f(x)) = x for every $x \in U$ then $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

Proof of Part A. We define $Df(a) = A \in \mathbb{R}^{n \times n}$, so A is invertible, and $||A^{-1}|| \neq 0$, where $||\cdot||$ denotes the operator norm. Recall all norms on finite-dimensional vector spaces are equivalent, this will be useful later. Choose $\lambda > 0$ st

$$\lambda = \|A^{-1}\|^{-1}2^{-1} \tag{1}$$

By continuity of Df(x) at the point a, let $\lambda > 0$, this induces a $B(\delta, a)$ with $x \in B(\delta, a)$ means

$$\underbrace{\|Df(x) - Df(a)\|}_{\text{operator norm}} < \lambda \tag{2}$$

as $Df: \mathbb{R}^n \to L(\mathbb{R}^n, \mathbb{R}^n)$ takes a point in \mathbb{R}^n and returns a linear map., with $L(\mathbb{R}^n, \mathbb{R}^n)$ endowed with the usual vector space structure. Fix $y \in \mathbb{R}^n$, and define

$$\phi(x) = \underbrace{x + A^{-1}(y - f(x))}_{\text{offset}}$$

this is now a function solely in x, and $\phi(x) = x \iff f(x) = y$ is clear, but such a fixed point is not necessarily unique. We claim that it is unique in $B(\delta, a)$. We will use the contractive mapping principle.

Differentiating $\phi(x)$ reads

$$D\phi(x) = \underbrace{I}_{I=A^{-1}A} - A^{-1}Df(x) = A^{-1}(A - Df(x))$$

Proposition 0.1 tells us the norm of a product is bounded above by the product of the norms. Using eqs. (1) and (2), if $x \in U$ we have

$$\|D\phi(x)\| = \|A^{-1}(A - Df(x))\| \le \|A^{-1}\| \|A - Df(x)\| \le 2^{-1}$$

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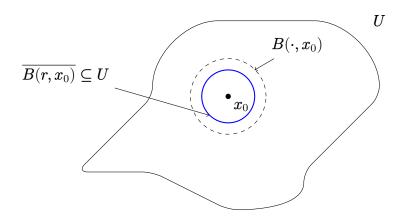


Figure 1: Every point x_0 in an open set U admits an open ball that hides in U

The total derivative of ϕ is uniformly bounded in U, applying Proposition 0.2 tells us that ϕ is a contractive mapping

$$||D\phi(x)|| \le 2^{-1} \implies ||\phi(x_1) - \phi(x_2)|| \le 2^{-1}||x_1 - x_2||$$

for x_1 , x_2 in U.

To show f|U is indeed a bijection, fix $y \in f(U)$ so y = f(x) for some $x \in U$, and there can only be one fixed point stemming from $\phi|U$, with $\phi(z) = z + A^{-1}(y - f(z))$ being the 'fixed point detector'. Write $(f|U)^{-1}(y) = \lim\{(\phi|U)(x_n)\}_n$ and every point in f(U) has a unique inverse.

For the last part of the proof, we wish to show V = f(U) is open. Let $y_0 \in V$ and we can 'hone into' the inverse of y_0 using the same construction as earlier. So $f(x_0) = y_0$ for some unique $x_0 \in U$.

If x_0 is in U, it induces an open ball (see fig. 1) st

$$x_0 \in B(r, x_0) \subseteq \overline{B(r, x_0)} \subseteq U, \quad r > 0$$

We claim the open ball $B(\lambda r, y_0) \subseteq V$. Indeed, suppose $y \in \mathbb{R}^n$ with

$$d(y, y_0) < \lambda r$$

If ϕ is the 'fixed-point detector' with respect to y (the point we are trying to prove that is in f(U)), in fact: we will prove $y \in f(\overline{B(r,x_0)}) \subseteq f(U)$.

$$\underbrace{\phi(x_0)-x_0}_{ ext{removing the offset from }\phi(x_0)}=A^{-1}(y-f(x_0))=A^{-1}(y-y_0)$$

using the operator norm on $A^{-1}(y-y_0)$ reads

$$\|\phi(x_0) - x_0\| = \|A^{-1}(y - y_0)\| \le \|A^{-1}\| \|y - y_0\| \le \|A^{-1}\| \lambda r = r2^{-1}$$

We will drag y into the image of the closed ball as follows: suppose x is another point that lies in the closed ball, ϕ is contractive on $\overline{B} \subseteq U$ regardless of the point y that induces ϕ . But \overline{B} is closed, hence it is complete. So the Cauchy sequence (from the contractive mapping theorem) produces exactly one point in \overline{B} . It remains to show that if we start our sequence at some point $x \in \overline{B}$, then $\phi(x) \in \overline{B}$ as well, and a simple induction will produce our contractive sequence.

To this, fix $x \in \overline{B}$, and

$$|\phi(x) - x_0| \le |\phi(x) - \phi(x_0)| + |\phi(x_0) - x_0|$$

$$\le \frac{\text{contraction on } \overline{B} \subseteq U}{2^{-1}|x - x_0|} + r2^{-1}$$

$$= r$$

therefore ϕ contracts to a fixed point $x^* \in \overline{B}$, and $f(x^*) = y$. So $y \in f(\overline{B}) \subseteq f(U)$ as desired.

Proof of Part b. The proof is quite long, and we will only focus on the important bits. Rudin uses the technique of approximating smooth functions using first-order terms. He writes

$$egin{cases} f(x) &= y \ f(x+h) &= y+k \end{cases} \implies k = f(x+h) - f(x)$$

Furthermore, if $x \in U$, then the derivative Df(x) is invertible, this is from Theorem 9.8, obtains an estimate on the open ball in $GL(n,\mathbb{R})$. Roughly

speaking, this open ball 'drags' other matrices into $GL(n, \mathbb{R})$. If A is invertible, and B is a conformable matrix with A, then

If $x \in B(\delta, a)$, then Equation (2) reads

$$||Df(x) - A|| < \lambda \implies ||Df(x) - A|| ||A^{-1}|| < 2^{-1} < 1$$

so Df(x) is invertible with inverse T.

And we estimate the deviation $|k|^{-1} \leq \lambda |h|^{-1}$ by using the contraction inequality with y as the basepoint for ϕ . Skipping a few lines ahead (to the confusing part), we see that

$$|h| \le |h - A^{-1}k| + |A^{-1}k| \le 2^{-1}|h| + |A^{-1}k|$$

subtracting over, and multiplying across gives a upper bound on $|k|^{-1}$

$$2^{-1}|h| \le |A^{-1}k| \implies 2^{-1}|h| \le ||A^{-1}|||k| \implies |k|^{-1} \le \frac{2}{||A^{-1}||}|h|^{-1}$$

Notice $2\lambda ||A^{-1}|| = 1$, so $2/||A^{-1}|| = \lambda$. Finally, we 'factor out' -T on the line just before the difference quotient.

numerator in difference quotient
$$g(y+k)-g(y)-Tk=h-Tk$$

$$=-T\bigg(\underbrace{f(x+h)-f(x)}_{=k}-\underbrace{Df(x)h}_{=T^{-1}h}\bigg)$$

We see that T = Dg(y), indeed:

$$\begin{split} \frac{|g(y+k)-g(y)-Tk|}{|k|} &\leq \frac{\|T\|}{\lambda} \frac{|f(x+h)-f(x)-Df(x)h|}{|h|} \\ &\lesssim \frac{|f(x+h)-f(x)-Df(x)h|}{|h|} \\ &= \underbrace{o(h)=o(k)}_{|h|\lesssim |k|} \to 0 \end{split}$$

Finally, $Df|U:U\to GL(n,\mathbb{R})$ is a continuous mapping. By Theorem 9.8, $(Df|U)^{-1}:U\to GL(n,\mathbb{R})$ is continuous as well. Therefore $g\in C^1(U,U)$, and f|U is a C^1 -diffeomorphism.