

# Folland Reading

me

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## 4 Chapter 4

### 4.1 Theorem 4.1

**WTS.** Suppose that  $A$  is a subset of  $X$ , let  $\text{acc } A$  be the set of accumulation points of  $A$ , then

$$\overline{A} = A \cup \text{acc } (A) \quad (1)$$

and  $A$  is closed if and only if  $\text{acc } (A) \subseteq A$ .

*Proof.* Suppose that  $x \notin \overline{A}$ , then  $x \in (\overline{A})^c = A^{\text{co}}$ , then  $A^c \in \mathcal{N}_B(x)$ . But this means that  $x \notin \text{acc } (A)$ , since there exists a neighbourhood of  $x$  (in the form of  $A^c$ ), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \emptyset$$

Also,  $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$  which means that

$$x \notin \overline{A} \implies x \notin A$$

Since  $x \notin \overline{A} \implies x \notin A$  and  $x \notin \text{acc } (A)$ ,

$$(\overline{A})^c \subseteq A^c \cap \text{acc } (A)^c = (A \cup \text{acc } (A))^c$$

Now, if  $x \notin \text{acc } (A) \cup A$ , then  $x \notin \text{acc } (A)$ , therefore there exists some  $U \in \mathcal{N}_B(x)$  such that

$$A \cap U \setminus \{x\} = A \cap U = \emptyset$$

Where for the second last equality we used the fact that  $x \notin A \implies A \setminus \{x\} = A$ , and taking complements gives us

$$U \subseteq A^c$$

And since  $U \in \mathcal{N}_B(x)$ , then  $x \in U^o \subseteq A^{\text{co}}$  (since  $U^o$  is an open subset of  $A^c$ ). then

$$x \in A^{\text{co}} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore  $(A \cup \text{acc } (A))^c \subseteq (\overline{A})^c$ .  $\square$

## 4.2 Theorem 4.2

WTS.

### 4.3 Theorem 4.3

WTS.

*Proof.*

□

**4.4 Theorem 4.4**  
**WTS.**

## 4.5 Theorem 4.5

WTS.

#### 4.6 Theorem 4.6

WTS.



## 4.7 Theorem 4.7

**WTS.**  $X$  is a  $T_1$  space  $\iff \{x\}$  is closed for every  $x \in X$ .

*Proof.* If  $X$  is  $T_1$  and  $x \in X$ , then for every  $y \neq x$  there exists some open  $U_y$  that contains  $y$  but not  $x$ . Following Folland's argument closely, every  $y \neq x$  is in  $\cup U_{y \neq x}$ . Hence  $\{x\}^c \subseteq \cup U_{y \neq x}$ . To show the converse, for every  $z \in \cup U_{y \neq x}$  that is open, there exists a  $y \neq x$  such that  $z \in U_y$ . But every  $U_y$  does not contain  $x$  as an element, so  $z \neq x$  implies that  $z \notin \{x\}$ . And  $z \in \{x\}^c$ . Hence  $\cup U_{y \neq x} = \{x\}^c$ .

Now conversely if every  $x \in X$  satisfies the fact that  $\{x\}^c$  is open, then  $\{x\}^c$  is an open set that contains every  $y \neq x$ . Now fix some  $y \neq x$ , since  $\{y\}$  is also closed, we have  $X \cap \{x\}^c$  is an open set that contains  $x$  but not  $y$ . Also,  $\{x\}^c$  is an open set that contains  $y$  but not  $x$ . And therefore  $X$  is  $T_1$ .  $\square$

#### 4.8 Theorem 4.8

WTS.

#### **4.9 Theorem 4.9**

**WTS.**

#### 4.10 Theorem 4.10

WTS.

#### 4.11 Theorem 4.11

WTS.

#### **4.12 Theorem 4.12**

**WTS.**

### 4.13 Theorem 4.13

WTS.

#### 4.14 Theorem 4.14

**WTS.** Suppose that  $A$  and  $B$  are disjoint closed subsets of the normal space  $X$ , and let  $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$  be the set of dyadic rationals in  $(0, 1)$ . There is a family  $\{U_r : r \in \Delta\}$  of open sets such that

1.  $A \subseteq U_r \subseteq B^c$  for every  $r \in \Delta$  and
2.  $\overline{U_r} \subseteq U_s$  for  $r < s$ .

*Proof.* The goal of this proof is to show that for every  $r \in \Delta$ , there exists a open  $U_r$  that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

For every  $r \in \Delta$ ,  $r = k/2^n$ , where  $k \in \mathbb{N}^+$  sits between 1 and  $2^n - 1$ , this makes  $r \in (0, 1)$ . Each layer of  $r \in \Delta$  is fixed by  $n$ , so for any  $n$  that is held fix, we want to prove that

$$k \in [1, 2^n - 1], r = \frac{k}{2^n} \text{ satisfies:}$$

1.  $U_r$  is open
2.  $A \subseteq U_r \subseteq B^c$
3.  $\overline{U_r} \subseteq U_s$  for every  $r < s$

To get a better feel of the proof, let us suppose that, for a fixed layer  $N$ , that all the previous layers have been constructed, so for every  $n \leq N - 1$ , we have all our  $k/2^n$  where  $1 \leq k \leq 2^n - 1$ . If  $k$  is even such that  $k = 2j$  for some positive integer  $j$ , then

$$\frac{k}{2^N} = \frac{2j}{2^N} = \frac{j}{2^{N-1}}$$

So for every  $k = 2j$ ,  $k/2^N$  is already defined (and it satisfies the above conditions). So we only have to construct for odd number  $k$ . So fix any odd  $k$ , then  $U_r$  is well defined for  $r = (k + 1)/2^N$ , and  $r = (k - 1)/2^N$ . With

- $\overline{U_0} = A$
- $U_1^c = B$



Then both  $U_0$  and  $U_1$  are open sets. However it should be noted that  $U_0$  and  $U_1$  are not of the form of  $\{U_r : r \in \Delta\}$ . Now for any odd  $k = 2j + 1$ ,

$$\begin{aligned}
\frac{2j+1}{2^N} < 1 &\iff 2j+1 < 2^N \\
&\iff 2j+1 < 2^N \\
&\iff 2j < 2^N - 1 \\
&\iff j < 2^{N-1} - 1/2 \\
&\iff 0 \leq j \leq 2^{N-1}
\end{aligned}$$

Now,  $k - 1 < k + 1$  implies that (since we have assumed that  $U_r$  is well constructed for  $n \leq N - 1$ ,

$$\overline{U}_{\frac{k-1}{2^N}} \subseteq U_{\frac{k+1}{2^N}}$$

Then by normality, since

$$\overline{U}_{\frac{k-1}{2^N}} \cap U_{\frac{k+1}{2^N}}^c = \emptyset$$

The above are two closed sets that are disjoint.

□

## 4.15 Theorem 4.15

**WTS.** *Urysohn's Lemma.* Let  $X$  be a normal space, if  $A$  and  $B$  are disjoint closed subsets of  $X$ , then there exists a  $f \in C(X, [0, 1])$  such that  $f = 0$  on  $A$  and  $f = 1$  on  $B$ .

*Proof.* Let  $r \in \Delta$  be as in Lemma 4.14, and set  $U_r$  accordingly except for  $U_1 = X$ . Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write  $W = \{k : x \in U_k\}$ , Then for every  $x \in A$  we have  $f(x) = 0$ , since by the construction of the 'onion' function in Lemma 4.14, for each  $r \in \Delta \cap (0, 1)$ ,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since  $r > 0$  is arbitrary, and  $0 \in W$ , we can use a classic  $\varepsilon$  argument. If  $f(x) > 0$  then there exists some  $0 < r < f(x)$  by density of the dyadic rationals on the line, if  $f(x) < 0$  then this implies that there exists some  $f(x) < r < 0$  such that  $x \in U_r$ , but no  $r \in \Delta$  can be negative, hence  $f(x) = 0$ .

Now, for every  $x \in B$ , since  $A$  and  $B$  are disjoint, and  $A \subseteq U_r \subseteq B^c$ , then for every  $x \in B$  means that  $x$  is not a member of any  $U_r$ , but we set  $U_1 = X$ . Since none of the  $r \in (0, 1)$  is a member of the set we are taking the infimum, and  $x \in U_1 = X$ . The  $\varepsilon$  argument follows: suppose for every  $\varepsilon > 0$ ,  $(1 - \varepsilon) \notin W$ , and  $1 \in W$ , then  $f(x) = 1$ .

Since  $x \in U_1 = X$ , for every  $x \in X$ ,  $f(x) \leq 1$ , and  $f(x)$  cannot be negative as  $r > 0$  for every  $r \in \Delta$ . So  $0 \leq f(x) \leq 1$ . Now we have to show that this  $f(x)$  is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in  $X$ . So  $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$  and  $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$ .

Suppose that  $f(x) < \alpha$ , so  $\inf W < \alpha$ , and using the density of  $\Delta$ , there exists an  $r$ ,  $f(x) < r < \alpha$  such that  $x \in U_r$  such that  $x \in \bigcup_{r < \alpha} U_r$ . So  $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r < \alpha} U_r$ .

Fix an element  $x \in \bigcup_{r < \alpha} U_r$ , this induces an  $r$  such that  $\inf W \leq r < \alpha$  therefore  $f(x) < \alpha$ , and  $\bigcup_{r < \alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$ .

For the second case, suppose that  $f(x) > \alpha$ , then  $\inf W > \alpha$ , and there exists an  $r$  (by density) such that  $\inf W > r > \alpha$  such that for every  $k \in W$ ,  $k \neq r$ . Therefore  $x \notin U_r$ , but by density again, and using the property of the onion function: for every  $s < r$  we get  $\overline{U_s} \subseteq U_r$ , taking complements (which reverses the estimate) — we have  $x \notin \overline{U_s}$ , but  $(\overline{U_s})^c$  is open in  $X$ . It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq (\overline{U_s})^c \subseteq \bigcup_{s > \alpha} (\overline{U_s})^c$$

So  $f^{-1}((\alpha, +\infty))$  is a subset of  $\bigcup_{s > \alpha} (\overline{U_s})^c$ . To show the reverse, fix an element  $x$  in the union, then this induces some  $x \in (\overline{U_s})^c \subseteq (U_s)^c$ . Then for this  $s > \alpha$ ,  $(-\infty, s)$  contains no elements of  $W$ . This is because for every  $p < s$  implies that  $(U_s)^c \subseteq (U_p)^c$ , so  $p \notin W$ . Our chosen  $s$  is a lower bound for  $W$ , and  $\alpha < s \leq \inf W = f(x)$ .

Since all of the inverse images from the generating set of  $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$  are open in  $X$ , using Theorem 4.9 finishes the proof.  $\square$

## 4.16 Theorem 4.16

**WTS.** *The Tietze's Extension Theorem. Let  $X$  be a normal space, and for any closed subset  $A \subseteq X$ , and  $f \in C(A, [a, b])$ , there exists an  $F \in C(X, [a, b])$  which extends  $f$ .*

*Proof.* We begin with an important lemma that will serve as a 'black box' for the induction.

**Lemma 4.1.** *For every  $f \in C(A, [0, 1])$ , there exists a  $g \in C(X, [0, 1/3])$  such that*

$$0 \leq f - g \leq 2/3 \quad \text{pointwise on } A \quad (2)$$

*Proof.* Since  $f$  is continuous,  $B = f^{-1}([0, 1/3])$ , and  $C = f^{-1}([2/3, 1])$  are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function  $g \in C(X, [0, 1])$  such that  $g|_B = 0$  and  $g|_C = 1$ . Relabel  $g = g/3$  then  $g \in C(X, [0, 1/3])$  (multiplication is continuous).

To show that (2) holds, suppose  $x \in B$ , then  $f(x) \in [0, 1/3]$  and  $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$ . Now suppose that  $x \in C$ , then  $f(x) \in [2/3, 1]$  and  $g(x) = 1/3$  (recall that we relabelled  $g$ ). So we have  $0 \leq 1/3 \leq f - g \leq 2/3$ . Lastly, for the case where  $x \notin (B \cup C)$ , then  $f(x) \in (1/3, 2/3)$ , and  $g(x) \in [0, 1/3]$  implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 & \implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 & \implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore  $0 \leq f(x) - g(x) \leq 2/3$ . □

We can assume that  $f \in C(A, [0, 1])$ , since we can relabel  $f = (f - a)/(b - a)$ . The main part of this proof consists of constructing a sequence of  $\{g_n\} \subseteq C(X, \mathbb{R})$  where  $0 \leq g_n \leq (2/3)^n(1/2)$ , and  $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$  on  $A$ . Let us begin with the base case with  $n = 1$ . We can apply Lemma 4.1 to get  $g_1 \in C(X, [0, 1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that  $\{g_j\}_{j \leq n}$  has been chosen, we will find our  $g_{n+1}$  by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by  $(2/3)^{-n}$  and we obtain a new function in  $C(A, [0, 1])$ .

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) \left( \frac{3}{2} \right)^n \leq 1$$

Applying the Lemma 4.1, we get a function  $h \in C(X, [0, 1/3])$  such that, for every  $x \in A$

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) \left( \frac{3}{2} \right)^n - h \leq 2/3$$

Multiplying across gives

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) - h \left( \frac{2}{3} \right)^n \leq \left( \frac{2}{3} \right)^{n+1}$$

Set  $g_{n+1} = h \left( \frac{2}{3} \right)^n$  and  $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$ . Furthermore, the sum of all  $g_j$  pointwise converges uniformly, as

$$\sum_{j \geq 1} \|g_j\|_u \leq \sum_{j \geq 1} \left( \frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum  $F = \sum g_j$ , then this  $F \in BC(X)$  (by Theorem 4.9), since every  $g_j \in BC(X)$ . And

$$\left\| f - \sum_{j \leq n} g_j \right\|_u \leq \left( \frac{2}{3} \right)^n \rightarrow 0$$

So  $F = f$  on  $A$ , now if we want to obtain our  $F$  on  $[a, b]$  we simply relabel  $F = F(b - a) + a$ . This finishes the proof.  $\square$

#### 4.17 Theorem 4.17

**WTS.** *If  $X$  is a normal space, and  $A$  is a closed subspace of  $X$ , and  $f \in C(A)$ , then there exists an  $F \in C(X)$  such that  $F$  extends  $f$ .*

*Proof.* First we suppose that  $f$  is real valued, so  $f \in C(X, \mathbb{R})$ . And define a  $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$ , using

$$g = \frac{f}{1 + |f|}$$

Since  $g$  satisfies the assumption of Theorem 4.16 (note that we do not require  $g$  to be injective), there exists a  $G \in C(X, [-1, +1])$  such that  $G|_A = g$ . Since the set  $\{-1, +1\}$  is closed in  $\mathbb{R}$ ,  $G^{-1}(\{-1, +1\})$  is closed as well. Since  $G^{-1}((-1, +1)) \subseteq A$ , this makes  $A$  and  $B = X \setminus G^{-1}((-1, +1))$  disjoint closed sets in  $X$ .

By Urysohn's Lemma, there exists a continuous function  $h \in C(X, [0, 1])$  such that  $h|_B = 0$  and  $h|_A = 1$ , so that the product  $|hG| < 1$  for all  $x \in X$ . We can think of this  $h$  as a continuous indicator function that filters out the parts we do not want, namely  $G^{-1}(\{-1, +1\})$ . Now define  $F$  in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that  $F|_A = g/(1 - |g|) = f$  indeed. Since  $|g| = \frac{|f|}{1+|f|}$ , and  $g(1 + |f|) = f$  implies that  $g/(1 - |g|) = f$ , because  $g \in C(A, (-1, +1))$ . This completes the proof for any  $f \in \mathbb{R}$  if  $f \in C(A)$ , then

1.  $\operatorname{Re}(f) = f_1 \in C(A, \mathbb{R})$
2.  $\operatorname{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in  $C(X, \mathbb{R})$  that extends  $f_1$  and  $f_2$ , and  $F_1 + iF_2 = f$  on  $A$  and  $F_1 + iF_2 \in C(X)$ , and the proof is complete.  $\square$

#### 4.18 Theorem 4.18

**WTS.** *If  $X$  is a topological space, and  $E \subseteq X$  and  $x \in X$ , then  $x \in \text{acc } E \iff$  there exists a net in  $E \setminus \{x\}$  that converges to  $x$ , and  $x \in \overline{E} \iff$  there exists a net in  $E$  that converges to  $x$ .*

*Proof.* Suppose that  $x \in \text{acc } E$ , then for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $E \cap U \setminus \{x\} \neq \emptyset$ , then choose  $\mathcal{N}(x)$  as the set of neighbourhoods directed by reverse inclusion (and this makes  $(\mathcal{N}(x), \lesssim)$  a directed set), and we will define the net as follows.

Map each  $U \in \mathcal{N}(x)$  to some  $x_U \in E \cap U \setminus \{x\}$ , then this net converges to  $x$ . Suppose that we fix a neighbourhood,  $V \in \mathcal{N}(x)$ , then for every  $U \gtrsim V$  we have  $x_U \in U \subseteq V$ . So  $\langle x_U \rangle$  is eventually in  $V$ .

Conversely, if  $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$ , and  $x_\alpha \rightarrow x$ , then every  $U \in \mathcal{N}(x)$  there exists a  $x_\alpha \in E \cap U \setminus \{x\}$  that makes

$$E \cap U \neq \emptyset \quad \forall U \in \mathcal{N}(x)$$

Hence  $x \in \text{acc } E$ .

Now for the second part of the Theorem, suppose that  $x \in \overline{E}$ , if  $x \notin E$  then  $E = E \setminus \{x\}$  and  $x \in \text{acc } E$ , so there exists a net in  $E \setminus \{x\} \subseteq E$  such that  $x_\alpha \rightarrow x$ . If  $x \in E$  then simply choose  $\langle x_\alpha \rangle = x$  for every  $\alpha \in A$ .

Now, suppose that there is a net that converges to  $x$ , and this net  $\langle x_\alpha \rangle \subseteq E$ , if  $x \in E$  then there is nothing to prove, since  $E \subseteq \overline{E}$ , so suppose that  $x \notin E$ , then there exists a net in  $E \setminus \{x\} = E$  such that

$$x_\alpha \rightarrow x \implies x \in \text{acc } E \subseteq \overline{E}$$

□

#### 4.19 Theorem 4.19

**WTS.** Let  $X$  and  $Y$  be topological spaces, then every  $f : X \rightarrow Y$  is continuous at a point  $x \in X \iff$  every net  $\langle x_\alpha \rangle$  that converges to  $x$  implies that  $\langle f(x_\alpha) \rangle$  converges to  $f(x)$ .

*Proof.* If  $f$  is continuous at a point  $x \in X$ , then  $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$ , then for every net  $\langle x_\alpha \rangle$  that converges to this  $x$ , there exists an  $\alpha_0$  such that for every  $\alpha \gtrsim \alpha_0$  implies that  $x_\alpha \in f^{-1}(V)$ . Hence

$$f(x_\alpha) \in f(f^{-1}(V)) \subseteq V$$

And this is equivalent to saying that for every  $V \in \mathcal{N}(f(x))$ ,  $\langle f(x_\alpha) \rangle$  is eventually in  $V$ , and this proves convergence.

Now suppose that  $f$  is not continuous at some  $x$ , then there exists a  $V \in \mathcal{N}(f(x))$  such that  $f^{-1}(V) \notin \mathcal{N}(x)$ , so

$$x \notin (f^{-1}(V))^o \implies x \in (f^{-1}(V))^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our  $x \in \overline{f^{-1}(V^c)}$  induces a net  $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$  that converges to  $x$ . But every element in the net is contained within  $f^{-1}(V^c)$ , and for every  $\alpha \in A$

$$f(x_\alpha) \in f(f^{-1}(V^c)) \subseteq V^c$$

gives  $f(x_\alpha) \notin V$ , but  $V$  is a neighbourhood of  $f(x)$ , hence there exists some  $x_\alpha \rightarrow x$  and  $f(x_\alpha) \not\rightarrow f(x)$ .  $\square$



## 4.20 Theorem 4.20

**WTS.** If  $\langle x_\alpha \rangle$  is a net in  $X$ , and  $x \in X$  is a cluster point of  $\langle x_\alpha \rangle \iff$  there exists a subnet of  $\langle x_\alpha \rangle$  that converges to  $x$ .

*Proof.* Suppose that  $\langle y_\beta \rangle_{\beta \in B}$  is a subnet of  $\langle x_\alpha \rangle$  that converges to  $x$ , then for every neighbourhood  $U \in \mathcal{N}(x)$ , there exists a  $\beta_1$  such that for every  $\beta \gtrsim \beta_1$  we get  $y_\beta = x_{\alpha_\beta} \in U$ .

Furthermore, let us fix a  $\alpha_0 \in A$  to attempt to show that  $\langle x_\alpha \rangle$  is frequently in  $U$ , then by the subnet property of  $\langle y_\beta \rangle$ , there exists some  $\beta_2 \in B$  such that for every  $\beta \gtrsim \beta_2$ ,  $\alpha_\beta \gtrsim \alpha_0$ . (Intuitively this property means that the directed set of  $B$  'grows' as much as the directed set of  $A$ , so we can always find elements that are greater than any fixed  $\alpha_0$ .)

Since  $\langle y_\beta \rangle$  is a net, we there exists some  $\beta \in B$  such that  $\beta \gtrsim \beta_1$  and  $\beta \gtrsim \beta_2$ , we then apply the  $\beta \mapsto \alpha_\beta$  map and we obtain some  $\alpha = \alpha_\beta$  that satisfies:

- $\alpha = \alpha_\beta \gtrsim \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that  $\beta \gtrsim \beta_1$  so that  $y_\beta$  falls into  $U$ .

Conversely, suppose that  $x$  is a cluster point of  $\langle x_\alpha \rangle$ , then by definition

$$\forall U \in \mathcal{N}(x), \forall \alpha_0 \in A, \exists \alpha \gtrsim \alpha_0, x_\alpha \in U$$

Denote the directed neighbourhoods of  $x$  by  $\mathcal{N}(x)$ , and construct our directed set  $B$  for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every  $(U, \gamma) \in B$  we can map it to some  $\alpha_{(U, \gamma)} \in A$ , if we choose some  $\alpha_{(U, \gamma)} \gtrsim \gamma$  and  $\alpha_{(U, \gamma)} \in U$ .

To show that  $B$  is a directed set, we say that  $(U, \gamma) \gtrsim (U', \gamma')$  if and only if  $U \subseteq U'$  and  $\gamma \gtrsim \gamma'$ . And to show that  $\langle y_\beta \rangle = \langle x_{\alpha_{(U, \gamma)}} \rangle$  is indeed a subnet of  $\langle x_\alpha \rangle$ , fix any  $\alpha_0 \in A$ , then simply take any neighbourhood  $U$  of  $x$  (we always

have  $X \in \mathcal{N}(x)$  — and therefore  $(U, \alpha_0) \in B$ .

Now for every  $(U', \alpha'_0) \gtrsim (U, \alpha_0)$  implies that  $\alpha'_0 \gtrsim \alpha_0$ , therefore we have

$$\alpha_{(U', \alpha'_0)} \gtrsim \alpha'_0 \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that  $\langle y_\beta \rangle$  indeed converges to  $x$ , fix any  $V \in \mathcal{N}(x)$ , then with any  $\alpha_0 \in A$ , and for every  $(V', \alpha'_0) \gtrsim (V, \alpha_0) \in B$ , we have

$$x_{\alpha_{(V', \alpha'_0)}} \in V' \subseteq V$$

So  $\langle x_{\alpha_{(U, \gamma)}} \rangle$  converges to  $x$ . □

## 4.21 Theorem 4.21

**WTS.** *A topological space  $X$  is compact  $\iff$  every family of closed sets,  $\{F_\alpha\}_{\alpha \in A}$  that has the finite intersection property, implies that*

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

*Proof.* We first examine the assertion, Theorem 4.21 proposes for any family of closed sets  $\{F_\alpha\}_{\alpha \in A}$ , and for every finite subset  $B \subseteq A$  then,

$$\bigcap_{\alpha \in B} F_\alpha \neq \emptyset \implies \bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha \in A} F_\alpha = \emptyset \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha \in B} F_\alpha = \emptyset$$

Applying DeMorgan's theorem, and since every  $\{F_\alpha\}_{\alpha \in A}$  induces a family of open sets (and vice versa), where  $U_\alpha = F_\alpha^c$ , so for any family of open sets  $\{U_\alpha\}_{\alpha \in A}$  we have

$$\bigcup_{\alpha \in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha \in B} U_\alpha = X$$

Which is equivalent to saying that  $X$  is compact. □

## 4.22 Theorem 4.22

**WTS.** *A closed subset of a compact space  $X$  is compact.*

*Proof.* Suppose  $F \subseteq X$  and  $F$  is open, then fix an open cover for  $F$ , so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since  $F^c$  is an open set, we can obtain a valid open cover for  $X$ , then we pick out a finite subcover, for some finite  $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left( \bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with  $F$  on both sides yields

$$\begin{aligned} F &= X \cap F \subseteq (F^c \cap F) \cup \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \\ F &= \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\ F &\subseteq \bigcup_{\alpha \in B} U_\alpha \end{aligned}$$

Therefore every open cover of  $F$  has a finite subcover, and  $F$  is compact.  $\square$

### 4.23 Theorem 4.23

**WTS.** *If  $F$  is a compact subset of a Hausdorff space  $X$ , and  $x \notin F$ , there are disjoint open sets  $U, V$  such that  $x \in U$  and  $F \subseteq V$ .*

*Proof.* Since  $x \in F^c$ , for every  $y \in F$ ,  $x \neq y$  induces two sets  $U_y, V_y$  (because  $X$  is  $T_2$ ).

- $U_y \cap V_y = \emptyset$
- $x \in U_y$
- $y \in V_y$

But  $\{V_y\}_{y \in F}$  is an open cover for the compact set  $F$ , then there exists a finite subcollection  $H \subseteq F$  such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since  $H$  is finite,  $U = \bigcap_{y \in H} U_y$  is an open set that contains  $x$ , also define  $V = \bigcup_{y \in H} V_y$ . If for every  $y \in H$ ,  $U_y \cap V_y = \emptyset$ , then  $U \cap V_y = U \cap V = \emptyset$ . This completes the proof.  $\square$

**Remark.** *Every metric space  $(X, d)$  is first countable, and  $T_2$  (it is actually  $T_4$ , but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element  $x \in X$  and we notice that  $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$  is a countable neighbourhood base for every  $x$ . To show that  $(X, d)$  is  $T_2$ , for every pair of elements  $x \neq y$ , we can take  $r = d(x, y)/2$  and there exists disjoint open sets  $V_r(x)$  and  $V_r(y)$  such that  $x \in V_r(x)$  and  $y \in V_r(y)$ .*

#### 4.24 Theorem 4.24

**WTS.** *Every compact subset of a Hausdorff ( $T_2$ ) space is closed.*

*Proof.* If  $F$  is compact, then for every  $x \in F^c$ , by Theorem 4.23, there exists two disjoint open sets such that  $x \in U$  and  $F \subseteq V$ , but

$$U \cap V = \emptyset \implies U \cap F = \emptyset \implies U \subseteq F^c$$

But since  $x \in F^c$  is arbitrary, and  $U$  is an open subset of  $F^c$ ,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that  $F^c$  is open and  $F$  is closed. □

## 4.25 Theorem 4.25

**WTS.** *Every compact Hausdorff ( $T_2$ ) space is normal ( $T_4$ ).*

*Proof.* Fix  $A, B$  which are disjoint closed subsets of  $X$ , by Theorem 4.22, we know that these two sets are compact. Hence for every  $y \in B$  there exists two disjoint open sets  $U, V_y$  (by Theorem 4.23)

$A \subseteq U_y$  and  $y \in V_y$ . But the family  $\{V_y\}_{y \in B}$  is a valid open cover for the compact set  $B$ , hence there exists a finite subcollection  $H \subseteq B$  such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \emptyset$$

The second equality holds for every  $y \in H$  so that  $U_y \cap (\bigcup V_{y \in H}) = \emptyset$ . Define  $U = \bigcap U_{y \in H}$  and  $V = \bigcup V_{y \in H}$ , where both of these are disjoint open sets that contain  $A$  and  $B$  as subsets, since for each  $y \in H$ ,  $A \subseteq U_y$  hence the intersection of all  $U_y$  also contains  $A$  as a subset. Therefore  $X$  is normal.  $\square$

## 4.26 Theorem 4.26

**WTS.** *If  $X$  is compact, and  $f : X \rightarrow Y$  is continuous, then  $f(X)$  is compact.*

A small lemma.

**Lemma 4.2.** *For every  $\{E_j\} \subseteq X$ ,  $f(\cup E_j) = \cup f(E_j)$ .*

The proof is trivial.

*Proof.* If  $\{V_{\alpha \in A}\}$  is an open cover for  $f(X)$ , then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha \in A} V_{\alpha}\right) = \bigcup_{\alpha \in A} f^{-1}(V_{\alpha}) \subseteq X$$

Since  $f$  is continuous, we have an open cover in the form of  $\{f^{-1}(V_{\alpha})\}$  for  $X$ , then there exists a finite subset  $B \subset A$  such that

$$X \subseteq \bigcup_{\alpha \in B} f^{-1}(V_{\alpha})$$

Then we wish to show that for this  $B \subseteq A$ ,  $\{V_{\alpha \in B}\}$  is a finite open cover for  $f(X)$ . Fix any element  $y \in f(X)$ , then this induces a  $x \in X$  such that  $y = f(x)$ , but because  $\{f^{-1}(V_{\alpha \in B})\}$  is an open cover for  $X$ , there exists some  $\alpha \in B$  such that  $x \in f^{-1}(V_{\alpha})$ , hence by definition of the inverse image

$$f(x) \in V_{\alpha} \implies f(X) \subseteq \bigcup_{\alpha \in B} V_{\alpha}$$

Therefore  $f(X)$  is compact and this completes the proof. □



#### 4.27 Theorem 4.27

**WTS.** *If  $X$  is compact, then  $C(X) = BC(X)$ .*

*Proof.* Notice that  $BC(X) \subseteq C(X)$ , so we only have to show the reverse estimate. Fix any  $f \in C(X)$ , since  $X$  is compact, by Theorem 4.26 we know that  $f(X)$  is also compact. Since  $\mathbb{C} = \mathbb{R}^2$  is a complete metric space,  $f(X)$  is bounded and  $f \in BC(X)$ .  $\square$

## 4.28 Theorem 4.28

**WTS.** *If  $X$  is compact, and if  $Y$  is Hausdorff, then any continuous bijection  $f : X \rightarrow Y$  is a homeomorphism.*

*Proof.* If  $E \subset X$  is closed, then since  $X$  is compact,  $E$  is compact as well. By continuity of  $f$ ,  $f(E)$  is a compact set in  $Y$ , but compact subsets of  $Y$  are closed, so  $f$  is continuous.

We used the fact that the inverse of  $f^{-1}$  is  $f$ , since it suffices to check that every inverse image of a closed set is also closed,  $f^{-1}$  is continuous. And by definition of a homeomorphism ( $f$  has to be bijective and both  $f$  and  $f^{-1}$  have to be continuous),  $f$  is a homeomorphism.  $\square$

## 4.29 Theorem 4.29

**WTS.** *If  $X$  is any topological space, the following are equivalent.*

- (a)  *$X$  is compact.*
- (b) *Every net has a cluster point.*
- (c) *Every net in  $X$  has a convergent subnet.*

*Proof.* By Theorem 4.20, every net in  $X$  has a cluster point  $\iff$  there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose a) holds, then  $X$  is compact, and fix an arbitrary net  $\langle x_\alpha \rangle$  in  $X$ . and define the 'tail' of the net

$$E_\alpha := \{x_\beta, \beta \succeq \alpha\}$$

We wish to show that the arbitrary intersection of  $\bigcap_{\alpha \in A} \overline{E_\alpha} \neq \emptyset$ . Where  $\overline{E_\alpha}$  is closed, so it suffices to check that every finite  $B \subseteq A$ , the intersection over  $\overline{E_\alpha}$  is non-empty.

Suppose we are given a finite  $B \subseteq A$ , then fix any two elements  $\alpha$  and  $\beta \in B$ , by the definition of a net there exists a  $\gamma \in A$  such that  $\gamma \succeq \alpha$  and  $\gamma \succeq \beta$ , and

$$\emptyset \neq E_\alpha \cap E_\beta \implies \overline{E_\alpha} \cap \overline{E_\beta} \neq \emptyset$$

Therefore for any finite collection of  $\{\overline{E_{\alpha \in B}}\}$ , then

$$\bigcap_{\alpha \in A} \overline{E_\alpha} \neq \emptyset$$

Now fix an element  $x \in \bigcap_{\alpha \in A} \overline{E_\alpha}$ . Then for every  $\alpha \in A$ ,  $x \in \overline{E_\alpha}$ , and for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $U \cap E_\alpha \neq \emptyset$ . This is because if  $x \in E_\alpha$ , then  $U \cap E_\alpha$  contains at least  $\{x\}$ , if  $x \in \text{acc } E_\alpha$ , then by definition of an accumulation point,  $U \cap E_\alpha \setminus \{x\} \neq \emptyset$ , so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net,  $E_\alpha$ , if for every  $\alpha \in A$ ,  $x \in E_\alpha$  if and only if there exists some  $\gamma \succeq \alpha$ ,  $x_\gamma \in U \cap E_\alpha$ ,

this is equivalent to saying that  $x$  is a cluster point of  $\langle x_\alpha \rangle$ . So  $a) \implies b)$ .

Now let us suppose that  $X$  is not compact, then there exists an open cover  $\{U_{\alpha \in A}\}$  of  $X$  that has no finite subcover. Let  $\mathbb{B}$  be the collection of all finite subsets of  $A$ , directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every  $B \in \mathbb{B}$ , find some  $x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c$ . So we have a net in  $X$ . Now we will show that no  $x \in X$  can be a cluster point of this net. Suppose not, then take a neighbourhood  $U_\beta$  with  $\beta \in A$  such that  $U_\beta$  belongs to the open cover we first discussed. Then for any  $B \in \mathbb{B}$  such that  $B \gtrsim \{\beta\}$  (meaning that  $\{\beta\} \subseteq B$ , where  $B$  is a finite set), then

$$x_B \in \left(\bigcup_{\alpha \in B} U_\alpha\right)^c \implies x_B \notin \left(\bigcup_{\alpha \in \{\beta\}} U_\alpha\right) \implies x_B \in U_\beta^c$$

Hence no point in  $X$  can be a cluster point for this net, and the proof is complete.  $\square$

## 5 Chapter 5

## 6 Chapter 6

### 6.1 Theorem 6.15

WTS.

First suppose that  $(X, \mathcal{M}, \mu)$  is finite measure space. If  $\mu(X) < +\infty$ , then for every  $E \in \mathcal{M}$ , by monotonicity  $E \subseteq X$  yields  $\mu(E) \leq \mu(X) < +\infty$ . Next, for any  $p < +\infty$ ,  $\|\chi_E\|_p^p < +\infty$  and  $\|\chi_E\|_{+\infty} \leq 1 < +\infty$ . So all indicator functions are in  $L^p$ .

It follows that every simple function is also in  $L^p$ , since it is a finite linear combination of indicators. We now define  $\nu(E) = \phi(\chi_E)$ , we wish to show that  $\nu : \mathcal{M} \rightarrow \mathbb{C}$  is a complex measure which is absolutely continuous with respect to  $\mu$ .

To show  $\sigma$ -additivity, fix any disjoint sequence  $\{E_j\}_{j \geq 1} \subseteq \mathcal{M}$ . Where we also note that  $\mu(E) = \mu(\cup E_j) < +\infty$ . Now suppose that  $p < +\infty$ , then the following converges in the  $p$ -norm

$$\chi_E = \sum_{j \geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left( \bigcup E_{j \leq n} \right) = \left( \bigcup E_{j \geq 1} \right) \setminus \left( \bigcup E_{j \leq n} \right) = \bigcup E_{j \geq n+1}$$

and define  $F_{n+1}$  as the rightmost member above. Then  $\{F_{n \geq 1}\}$  is a decreasing sequence of sets. All sets are of finite measure, hence  $\mu(E) - \mu(\cup E_{j \leq n}) = \mu(F_{n+1}) \rightarrow 0$ .

Now, for any fixed  $n \geq 1$ ,

$$\left| \chi_E - \sum \chi_{E_{j \leq n}} \right| = \left| \sum \chi_{E_{j \geq n+1}} \right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the  $p$ th power does not change pointwise, and

$$\left| \sum \chi_{E_{j \geq n+1}} \right|^p = \left| \sum \chi_{E_{j \geq n+1}} \right| = \sum \chi_{E_{j \geq n+1}}$$

Convergence in  $p$ -norm is given by

$$\left\| \chi_E - \sum \chi_{E_{j \leq n}} \right\| = \left\| \sum \chi_{E_{j \geq n+1}} \right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our  $\phi \in L^{p*}$

$$\begin{aligned} \nu(E) &= \phi(\chi_E) \\ &= \phi\left(\lim_{n \rightarrow \infty} \sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \phi\left(\sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \sum \phi(\chi_{E_{j \leq n}}) \\ &= \lim_{n \rightarrow \infty} \sum \nu(E_{j \leq n}) \end{aligned}$$

To show absolute convergence, recall that for any  $\phi(\chi_{E_j}) \in \mathbb{C}$ , define  $\beta_j = \overline{\text{sgn}(\|\phi(\chi_{E_j})\|)}$  then multiplication yields

$$\|\phi(\chi_{E_j})\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the  $p$ -norm.

$$\left\| \sum_{j \geq 1} \beta_j \chi_{E_j} - \sum_{j \leq n} \beta_j \chi_{E_j} \right\|_p = \left\| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right\|_p$$

And because  $\left| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right|$  is pointwise equal to  $\left| \sum_{j \geq n+1} \chi_{E_j} \right|$ , since  $|\beta_j| = 1$  for every  $j \geq 1$ . We can reuse the same continuity and linearity argument. We also note that  $\sum_{j \geq 1} \beta_j \chi_{E_j} \in L^p$  since its  $p$ -norm is equal to  $\mu(E)^{1/p}$ .

$$\begin{aligned}
\sum_{j \geq 1} |\nu(E_j)| &= \sup_{n \geq 1} \sum_{j \leq n} \|\nu(E_{j \leq n})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \|\phi(\chi_{E_j})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \phi(\chi_{E_j}) \\
&= \lim_{n \rightarrow \infty} \phi \left( \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&= \phi \left( \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&\leq \|\phi\| \left\| \sum_{j \geq 1} \beta_j \chi_{E_j} \right\|_p \\
&< +\infty
\end{aligned}$$

Assuming the above estimate holds, then we only need  $\nu(E) = \phi(\chi_E) = \mu(E) = 0$  ( $\nu$  is now a measure and  $\nu \ll \mu$ ), As the indicator of a null set is equal to the zero element in  $L^p$ . Then by Radon-Nikodym we can have some  $g \in L^1(\mu)$  such that

$$d\nu = g d\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function  $g$ . For every  $\chi_E$  measurable,  $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$ , by monotonicity of the integral in  $L^+$ . So any simple function,  $\alpha = \sum a_j \cdot \chi_{E_j}$  means that  $\alpha g$  is in  $L^1(\mu)$ , and

$$\phi(\alpha) = \int \alpha g d\mu$$

If  $\|\alpha\|_p = 1$ , then

$$\left| \int \alpha g \right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$



Then

$$M_q(g) = \sup \left\{ \left| \int \alpha \cdot g \right|, \|\alpha\|_p = 1, \text{ and } \alpha \text{ is simple and vanishes outside a set of finite measure.} \right\}$$

Since  $S_g = \{x \in X, g(x) \neq 0\}$  is  $\sigma$ -finite, an application of Theorem 6.14 tells us that  $g \in L^q$ , and  $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$ . Now that we know  $g$  is in  $L^q$  we can use the density of  $\alpha$  in  $L^p$  to show, for every single  $f \in L^p$

$$\phi(f) = \int f g d\mu$$

Conjure a sequence of ' $\alpha$ 's, and call them  $\{f_n\} \rightarrow f$  p.w.a.e, then each  $f_n \cdot g \in L^1$ . An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n g d\mu = \int f g d\mu = \phi(f)$$

This completes the proof for when  $\mu$  is finite.

Let us upgrade our  $\mu$  into a  $\sigma$ -finite measure. Then there exists an increasing sequence  $\{E_n\} \nearrow X$  such that each  $E_n$  is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in  $P_n$  vanishes outside a set of finite measure and is also in  $L^p$ . And  $Q_n$  is defined in a similar manner. Now, fix our  $\phi \in L^{p*}$ , and for each  $f \in P_n$ , there exists a corresponding  $g_n \in Q_n$ . Then  $p \in [1, +\infty)$  tells us that  $q \in (1, +\infty]$ , and the assumptions for Theorem 6.13 all hold. Therefore for each  $g_n \in Q_n$ , there is a corresponding bounded linear operator  $\phi_{g_n} \in (P_n)^*$  such that

$$\phi(f) = \phi|_{P_n}(f) = \int f g_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of  $g_n$  towards some  $g \in L^q$ . We claim that this limit makes sense. As for any  $n < m$ , such that  $E_n \subseteq E_m$  then  $g_n = g_m$  on  $E_n$  pointwise. The proof is simple since each the restriction of our  $\phi \in L^{p*}$  onto  $E_n$  and  $E_m$  spawns two functions  $g_n$  and  $g_m \in L^1$ . To verify, take any subset  $Z \subseteq E_n$  then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So  $g_n = g_m$  pointwise a.e on  $E_n$ . Now we define  $g$  measurable such that  $g|_{E_n} = g_n$  for every  $n$ . And

$$\begin{aligned} |g_n| &= \chi_{E_n} \cdot |g_m| \implies \\ |g_n| &\leq |g_{n+1}| \implies \\ \|g_n\|_q &\leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty \end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with  $(P_n \subseteq P_{n+1})$ . If  $q = +\infty$  then  $g \in L^\infty$  is trivial, but for any  $q < +\infty$ . We wish to show that  $g \in L^q$ . Since  $|g_n| \leq |g|$  pointwise for every  $n$ , and for each  $x \in X$ , there exists a  $N$ , where  $n \geq N$  implies  $|g(x)| = |g_n(x)|$ , so  $|g(x)|$  is an upperbound that is actually attained by the sequence  $|g_n(x)|$ . So,  $|g(x)| = \sup_{n \geq 1} \{|g_n(x)|\}$ .

Using the Monotone Convergence Theorem on  $|g_n|$ ,

$$\begin{aligned} \int \lim_{n \rightarrow \infty} |g_n|^q d\mu &= \int \sup_{n \geq 1} |g_n|^q d\mu \\ &= \int |g|^q d\mu \\ &= \lim \int |g_n|^q d\mu \end{aligned}$$

Which yields  $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$ . It follows that  $g \in L^q$ .

Finally, we will show that  $\phi(f) = \int fg$  for every  $f \in L^p$ . Redefine  $f_n = f \cdot \chi_{E_n} \in P_n$  for every  $n \geq 1$ . We claim that  $f_n \rightarrow f$  in the  $p$ -norm.

$$\begin{aligned} |f_n - f| &\leq |f_n| + |f| \\ &\leq |f| + |f| \\ &\leq 2|f| \end{aligned}$$

And  $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$ . Now it is permissible to apply the Dominated Theorem, and we will do so.

$$\begin{aligned}
\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\
\lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\
&= 0
\end{aligned}$$

And we have  $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}
\phi(f) &= \lim \phi|_{P_n}(f_n) \\
&= \lim \int f_n \cdot g_n \\
&= \lim \int f \cdot g \cdot \chi_{E_n} \\
&= \int \lim (f g \cdot \chi_{E_n}) \\
&= \int f g
\end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of  $f g \chi_{E_n} \rightarrow f g$  pointwise and Holder's Inequality. This completes the proof for when  $\mu$  is of  $\sigma$ -finite measure, and  $p \in [1, +\infty)$ .

Suppose now  $\mu$  is arbitrary, and  $p \in (1, +\infty)$ , then  $q < +\infty$ . Now let us agree to define, for every  $\sigma$ -finite  $E \subseteq X$

$$P_E = \{L^p, |f| = |f| \cdot \chi_E\}$$

Where  $Q_E$  does not hold any surprises. Then for each  $E$  we have a  $\phi|_E$  which induces a  $g_E$  that vanishes outside  $E$ . We are ready for the final part of the proof.

First, if  $E \subseteq F$  and both  $E$  and  $F$  are  $\sigma$ -finite, then  $\|g_E\|_q \leq \|g_F\|_q$ . This is a simple consequence of monotonicity in  $L^+$  if we take  $|g_E|^q \leq |g_F|^q$ .

Second, we define

$$W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}$$

Let  $M$  be the supremum of  $W$ , then there exists a sequence of  $\sigma$ -finite sets,  $\{E_n\}$  where  $\|g_{E_n}\|_q \rightarrow M \leq \|\phi\|_{p*}$ . Take a set  $F = \cup E_{n \geq 1}$ , which is also  $\sigma$ -finite, so that  $\|g_F\|_q = M$ . Now assume there exists another  $\sigma$ -finite superset of  $F$ , let us call it  $A$ . Then

$$\int |g_F|^q + \int |g_{A \setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have  $g_{A \setminus F} = 0$  pointwise a.e. For any  $f \in L^p$ , the spots where  $f$  does not vanish is  $\sigma$ -finite. This comes from  $\int |f|^p < +\infty$ . So it suffices to integrate over this  $\sigma$ -finite set. But we already know, even if this set  $A$  contains  $F$  as a subset,  $\int f g_F = \int f g_A$ .

We now define  $g = g_F$ , and the proof is complete. As for every  $\phi \in L^{p*}$ , there exists a  $g \in L^q$  such that the evaluation of any  $f \in L^p$  is given by integrating  $f$  with  $g$ . ■