

# Chapter C: Algebraic Topology

## Homotopy

This section will follow Munkres Chapters 9 and 13 closely. Possibly other chapters as well.

### Definition 1.1: Path

A *path* is a continuous function from the unit interval  $f : [0, 1] \rightarrow \mathbf{X}$ . We say  $f$  is a *path from  $x_0$  to  $x_1$*  if  $f(0) = x_0$  and  $f(1) = x_1$ .

We denote the set of paths from  $x_0$  to  $x_1$  by  $\text{Path}(x_0, x_1)$ . If  $f \in \text{Path}(x_0, x_1)$ , we sometimes denote the *reversal* of  $f$  by  $\bar{f} \in \text{Path}(x_1, x_0)$ , where  $\bar{f}(s) \triangleq f(1 - s)$ .

### Definition 1.2: Loop

A *loop* at  $x_0 \in \mathbf{X}$  is a path that begins and ends at  $x_0$ , and  $\text{Loop}(x_0) \triangleq \text{Path}(x_0, x_0)$ . The constant path (or loop) at  $x_0$  is denoted by  $e_{x_0} : [0, 1] \rightarrow \mathbf{X}$ .

$$e_{x_0}(s) = x_0, \quad \forall s \in [0, 1]$$

### Definition 1.3: Homotopy of $C(\mathbf{X}, \mathbf{Y})$

Let  $f$ , and  $g$  continuous functions from  $\mathbf{X}$  to  $\mathbf{Y}$ .  $f$  and  $g$  are homotopic, denoted by  $f \approx g$  if there exists a continuous function  $F \in C(\mathbf{X} \times I, \mathbf{Y})$  where

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x) \quad (1)$$

where  $I = [0, 1]$ .

The function  $F$  is called the *homotopy between  $f$  and  $g$* .

If  $f \simeq h$ , where  $h$  is the constant function, we say  $f$  is *nullhomotopic*.

### Definition 1.4: Path Homotopy of $\text{Path}(x_0, x_1)$

Two paths  $f_0, f_1 \in \text{Path}(x_0, x_1)$  are said to be *path homotopic*, if there exists a continuous function  $F \in C(I \times I, \mathbf{X})$ , with

- $F$  is a *homotopy between  $f_0$  and  $f_1$*  (in the sense of Definition 1.3). For every  $s \in [0, 1]$ ,

$$F(s, 0) = f_0(s) \quad \text{and} \quad F(s, 1) = f_1(s) \quad (2)$$

- $F$  leaves the endpoints fixed. For every  $t \in [0, 1]$ , then

$$F(0, t) = x_0 \quad \text{and} \quad F(1, t) = x_1 \quad (3)$$

If  $f_0$  and  $f_1$  are path-homotopic, we write  $f_0 \simeq_p f_1$ .

- The function  $F \in C(I \times I, \mathbf{X})$  is called the *path homotopy between  $f_0$  and  $f_1$* .
- If  $f \in \text{Loop}(x_0)$  is path homotopic to the constant path  $e_{x_0}$ , then  $f$  is *nullhomotopic*.

- The relation  $\simeq_p$  is defined for paths that have the same initial and final points. So it is a relation on  $\text{Path}(x_0, x_1)$ .

**Proposition 1.1: Munkres Lemma 51.1**

The relations  $\simeq$  and  $\simeq_p$  are equivalence relations on  $C(\mathbf{X}, \mathbf{Y})$  and  $\text{Path}(x_0, x_1)$  respectively.

*Proof.* ( $f \simeq f$ ): Let  $f \in C(\mathbf{X}, \mathbf{Y})$ . Define

$$F : \mathbf{X} \times I \rightarrow \mathbf{Y} \quad \text{For every } t \in [0, 1], F(x, t) = f(x)$$

$F$  is continuous, since  $F = \pi_{\mathbf{X}} \circ (f \times \text{id}_{[0,1]})$ , where  $f \times \text{id}_{[0,1]}$  is the product of two continuous functions, which is again continuous by Chapter A. Moreover,  $F(x, 0) = f(x) = F(x, 1)$ , so  $F$  is a homotopy between  $f$  and itself.

( $f \simeq g \implies g \simeq f$ ): Let  $F$  be the homotopy between  $f$  and  $g$ . Let  $G$  be the 'reversal' in the second coordinate of  $F$ , meaning

$$G(x, t) = F(x, 1 - t) \quad \text{is continuous, since } G = F \circ (\text{id}_{\mathbf{X}} \times c)$$

where  $c : I \rightarrow I$  that maps  $t \mapsto 1 - t$  is continuous, so  $\text{id}_{\mathbf{X}} \times c$  is continuous; hence  $G$  is continuous. Notice for every  $x \in \mathbf{X}$ ,

$$G(x, 0) = F(x, 1) = g(x) \quad \text{and} \quad G(x, 1) = F(x, 0) = f(x)$$

therefore  $G$  is a homotopy between  $g$  and  $f$ .

( $f \simeq g, g \simeq h \implies f \simeq h$ ): Let  $F$  be the homotopy between  $f$  and  $g$ , and  $G$  be the homotopy between  $g$  and  $h$ . Define a function  $H : \mathbf{X} \times I \rightarrow \mathbf{Y}$  that morphs  $f$  into  $g$  on  $[0, 2^{-1}]$ , then  $g$  into  $h$  on  $[2^{-1}, 1]$

$$H(x, t) = \begin{cases} F(x, 2t - \lfloor 2t \rfloor) & \text{for } 0 \leq t \leq 2^{-1} \\ G(x, 2t - \lfloor 2t \rfloor) & \text{for } 2^{-1} \leq t \leq 1 \end{cases} \quad (4)$$

where  $\lfloor \cdot \rfloor$  denotes the *floor function*.

- $H$  is well defined on the overlap  $\mathbf{X} \times 2^{-1}$ , since  $F(x, 1) = G(x, 0) = g(x)$  at every  $x \in \mathbf{X}$ .
- If  $t = 0$ , then  $H(x, 1) = F(x, 0) = f(x)$ , and  $t = 1$  gives  $H(x, 1) = G(x, 1) = h(x)$ .
- Since  $H|_{\mathbf{X} \times [0, 2^{-1}]}$  and  $H|_{\mathbf{X} \times [2^{-1}, 1]}$  are continuous functions, and they agree on the overlap,  $H$  is continuous by the pasting Lemma, and defines a homotopy between  $f$  and  $h$ .

Now consider paths  $f, g, h$  in  $\text{Path}(x_0, x_1)$ , ( $f \simeq_p f$ ) is trivial. So is symmetry of  $\simeq_p$ , as the reversal in the second coordinate (see above) of the path homotopy between  $f$  and  $g$  is path homotopy between  $g$  and  $f$ .

Suppose  $f \simeq_p g$ , and  $g \simeq_p h$ . Let  $F$ , and  $G$  be the path homotopies between  $f, g$  and  $g, h$ . Write  $H$  as in Equation (4), it is a continuous function on  $I \times I \rightarrow \mathbf{X}$ , that satisfies

$$H(s, 0) = F(s, 0) = f(s) \quad \text{and} \quad H(s, 1) = G(s, 1) = h(s) \quad \text{for every } s \in [0, 1]$$

If  $s = 0$ , it is easy to see from Equation (4) that for every  $t \in [0, 1]$ ,

$$\begin{aligned} H(0, t) &= \begin{cases} F(0, 2t - \lfloor 2t \rfloor) = x_0 & \text{for } 0 \leq t \leq 2^{-1} \\ G(0, 2t - \lfloor 2t \rfloor) = x_0 & \text{for } 2^{-1} \leq t \leq 1 \end{cases} = x_0 \quad \text{and} \\ H(1, t) &= \begin{cases} F(1, 2t - \lfloor 2t \rfloor) = x_1 & \text{for } 0 \leq t \leq 2^{-1} \\ G(1, 2t - \lfloor 2t \rfloor) = x_1 & \text{for } 2^{-1} \leq t \leq 1 \end{cases} = x_1 \end{aligned}$$

So the endpoints remain fixed throughout the deformation in  $t$ , and  $H$  is a path homotopy between  $f$  and  $h$ . This proves transitivity. ■

## Path and PathClass Products

### Definition 2.1: Product of Paths $f * g$

Let  $f \in \text{Path}(x_0, x_1)$  and  $g \in \text{Path}(x_1, x_2)$ , the product of  $f$  and  $g$ , denoted by  $f * g$  is another path from  $x_0$  to  $x_2$ . For  $s \in [0, 1]$ ,

$$(f * g)(s) \triangleq \begin{cases} f(2s - \lfloor 2s \rfloor) & \text{for } 0 \leq s \leq 2^{-1} \\ g(2s - \lfloor 2s \rfloor) & \text{for } 2^{-1} \leq s \leq 1 \end{cases} \quad (5)$$

Notice the similarities between Equations (4) and (5),

### Proposition 2.1: Properties of the Path Product

Let  $f \in \text{Path}(x_0, x_1)$  and  $g \in \text{Path}(x_1, x_2)$ , let  $k \in C(\mathbf{X}, \mathbf{Y})$ , then

- (i) Invariant under left-multiplication:  $f \simeq_p g \implies k \circ f \simeq_p k \circ g$ , where  $k \circ f$  and  $k \circ g$  are elements Paths from  $k(x_0)$  to  $k(x_1)$ , and if  $F$  be a path homotopy between  $f$  and  $g$ , then  $k \circ F$  is a path homotopy between  $k \circ f$  and  $k \circ g$ .
- (ii) If we redefine  $f \in \text{Path}(x_0, x_1)$ ,  $g \in \text{Path}(x_1, x_2)$ , and  $k$  be as above, then

$$k \circ (f * g) = (k \circ f) * (k \circ g)$$

*Proof.*

Proof of Part (i): It is clear that  $k \circ f$  and  $k \circ g$  are elements of  $\text{Path}(k(x_0), k(x_1))$ , and see Part (ii) for the proof of  $k \circ f \simeq_p k \circ g$ .

Proof of Part (ii): Let  $F$  be the path homotopy between  $f$  and  $g$ . The composition  $(k \circ F)$  is in  $C(\mathbf{X} \times I, \mathbf{Y})$ . Equation (2) reads

$$\begin{aligned} (k \circ F)(s, 0) &= k(F(s, 0)) = (k \circ f)(s) \text{ and} \\ (k \circ F)(s, 1) &= k(F(s, 1)) = (k \circ g)(s) \text{ for every } s \in [0, 1] \end{aligned}$$

and Equation (3) gives

$$\begin{aligned} (k \circ F)(0, t) &= k(F(0, t)) = k(x_0) \text{ and} \\ (k \circ F)(1, t) &= k(F(1, t)) = k(x_1) \text{ for every } t \in [0, 1] \end{aligned}$$

therefore  $k \circ F$  is a path homotopy between the paths  $k \circ f$  and  $k \circ g$ . ■

### Definition 2.2: Path Homotopy class $[f]$

Let  $f \in \text{Path}(x_0, x_1)$ , we define the *path homotopy class* of  $f$  as

$$[f] \triangleq \left\{ g \in \text{Path}(x_0, x_1), g \simeq_p f \right\}$$

### Definition 2.3: Product of PathClasses $[f] * [g]$

Let  $*$  :  $\text{PathClass}(x_0, x_1) \times \text{PathClass}(x_1, x_2) \rightarrow \text{PathClass}(x_0, x_2)$  be a binary operation, where

$$[f] * [g] \triangleq [f * g] \text{ is well defined.}$$

for arbitrary  $[f] \in \text{PathClass}(x_0, x_1)$  and  $[g] \in \text{PathClass}(x_1, x_2)$ . This means it is independent of the representative chosen. More formally, if  $f \simeq_p f' \in \text{Path}(x_0, x_1)$ , and  $g \simeq_p g' \in \text{Path}(x_1, x_2)$ , then  $f * g \simeq_p f' * g'$ .

### Proposition 2.2: Properties of the PathClass product

Let  $[f]$ ,  $[g]$  and  $[h]$  be PathClasses from and to the points  $x_0, x_1, x_2$ . Then

1. Associativity:  $([f] * [g]) * [h] = [f] * ([g] * [h])$ ,
2. Left and Right identities: if  $[f] \in \text{PathClass}(x_0, x_1)$ ,  $e_{x_0}, e_{x_1}$  denote the constant paths on  $x_0$  and  $x_1$  (the initial and final points of any  $f \in [f]$ ), then

$$[e_{x_0}] * [f] = [f] \quad \text{and} \quad [f] * [e_{x_1}] = [f]$$

3. Left and Right inverses: let  $[\bar{f}]$  be the PathClass containing the reversal of  $f$  (see Definition 1.1) for the definition, then

$$[\bar{f}] * [f] = [e_{x_1}] \quad \text{and} \quad [f] * [\bar{f}] = [e_{x_0}]$$

4. Generalized Associativity: if  $\{[f_j]\}_{j \leq n}$  is a sequence of PathClasses, such that  $[f_j] \in \text{PathClass}(x_{j-1}, x_j)$ , then

$$\prod [f_j] \triangleq [f_1] * [f_2] * \cdots * [f_n] \text{ is a well-defined object}$$

meaning we can place the brackets wherever we want.

*Proof.* We will give an outline for the proof of Generalized Associativity, the rest are trivial. Let  $\{[f_j]\}$  be defined as above. If  $\{a_j\}_{j=0}^n$ , and  $\{b_j\}_{j=0}^n$  are 'cell partitions' of the unit interval (in the sense of the Riemann integral), meaning

$$0 = a_0 < a_1 < \cdots < a_n = 1, \quad \text{and} \quad 0 = b_0 < b_1 < \cdots < b_n = 1$$

We agree to define the following

- the lengths of each cell  $l_{a_j} \triangleq a_j - a_{j-1}$  and  $l_{b_j} \triangleq b_j - b_{j-1}$ , and
- the cells themselves are denoted by  $\text{cell}(a_j) = [a_{j-1}, a_j]$ ,  $\text{cell}(b_j) = [b_{j-1}, b_j]$ ,
- $p \in \text{Path}(0, 1)$ , where  $p$  is given explicitly by

$$p(s) = \sum_{j=1}^n \chi_{\text{cell}(a_j) \setminus \{a_{j-1}\}} \left( \frac{l_{b_j}}{l_{a_j}} (s - a_j) + b_j \right)$$

It is clear  $p$  is continuous, and for  $j = 1, \dots, n$ ,

$$p|_{\text{cell}(a_j)} \text{ is the positive linear map from } \text{cell}(a_j) \text{ to } \text{cell}(b_j)$$

Using the same line of argumentation as in the proof for associativity, we see that any two 'ways' of bracketing the expression has no impact on the path-homotopy class.  $\blacksquare$

## Fundamental Group

### Definition 3.1: Fundamental group $\pi_1(\mathbf{X}, x_0)$

Let  $x_0 \in \mathbf{X}$ , the *fundamental group of  $\mathbf{X}$  relative to (base point)  $x_0$*  is denoted by  $\pi_1(\mathbf{X}, x_0) = \text{PathClass}(x_0, x_0)$ .

**Definition 3.2: Isomorphism induced by  $\text{Path}(x_0, x_1)$**

Suppose  $\alpha \in \text{Path}(x_0, x_1)$ , we define a map  $\hat{\alpha} : \pi_1(\mathbf{X}, x_0) \rightarrow \pi_1(\mathbf{X}, x_1)$ , with

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

where  $\bar{\alpha}$  is the reversal of  $\alpha$ . We call  $\hat{\alpha}$  the *isomorphism induced by  $\alpha$*  (Munkres Theorem 52.1).

*Isomorphism proof.* Let  $[f]$  and  $[g]$  be elements of  $\pi_1(\mathbf{X}, x_0)$ , then

$$\begin{aligned} \hat{\alpha}([f] * [g]) &= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \\ &= [\bar{\alpha}] * ([f] * [g]) * [\alpha] \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g]) \end{aligned}$$

and  $\hat{\alpha}$  is a homomorphism. We claim inverse of  $\hat{\alpha}$  is  $\hat{\bar{\alpha}}$ . Fix  $[f] \in \pi_1(\mathbf{X}, x_0)$ ,  $[g] \in \pi_1(\mathbf{X}, x_1)$ , then

$$(\hat{\bar{\alpha}} \circ \hat{\alpha})([f]) = [\alpha] * ([\bar{\alpha}] * [f] * [\alpha]) * [\bar{\alpha}] = [f]$$

so  $\hat{\bar{\alpha}}$  is the left-inverse for  $\hat{\alpha}$ . A similar argument shows it is the right inverse as well with  $(\hat{\alpha} \circ \hat{\bar{\alpha}})([g]) = [g]$ . Therefore  $\pi_1(\mathbf{X}, x_0)$  is group isomorphic to  $\pi_1(\mathbf{X}, x_1)$ . ■

## Homomorphisms

**Definition 4.1: Homomorphism induced by a continuous map**

Let  $h \in C(\mathbf{X}, \mathbf{Y})$ , and  $y_0 = h(x_0)$ , it induces a map between loops at  $x_0$  and  $y_0$ .

$$h_* : \text{Loop}(x_0) \rightarrow \text{Loop}(y_0), f \mapsto h \circ f$$

It is also a group homomorphism between fundamental groups. We use the same symbol for the two maps, relying on context to distinguish between the two.

$$h_* : \pi_1(\mathbf{X}, x_0) \rightarrow \pi_1(\mathbf{Y}, y_0), [f] \mapsto [h \circ f]$$

is well defined because of Proposition 2.2, it is a homomorphism (again by Proposition 2.2) because  $h$  'distributes' over  $*$

$$h \circ (f * g) = (h \circ f) * (h \circ g)$$

**Remark 4.1: Functorial properties of the  $h_*$**

If  $x_0 \in \mathbf{X}$ , the tuple  $(x_0, \mathbf{X})$  is an object in the category of *pointed topological spaces*, and the map  $h_*$  is a *covariant functor* from the category of pointed topological spaces to the category of groups.

Follows from Munkres Theorem 52.4, if the expressions below make sense,

$$(g \circ f)_* = g_* \circ f_* \quad \text{and} \quad h_* \circ (g \circ f)_* = (h \circ g)_* \circ f_*$$

And the identity map  $i : \mathbf{X} \rightarrow \mathbf{X}$  gets 'sent' to the identity homomorphism in  $\text{Hom}(\pi_1(\mathbf{X}, x_0), \pi_1(\mathbf{X}, x_0))$ . And if  $h$  is a homeomorphism between  $\mathbf{X}$  and  $\mathbf{Y}$ , then  $h_*$  is an isomorphism at every point.

## Simply connected space

**Definition 5.1: Simply connected space**

A topological space  $\mathbf{X}$  is *simply connected* if it is path-connected, and  $\pi_1(\mathbf{X}, x_0) = \{[e_{x_0}]\}$  for some  $x_0 \in \mathbf{X}$ . Notice this implies every fundamental group of  $\mathbf{X}$  is trivial.

**Proposition 5.1: Properties of simply connected spaces**

If  $\mathbf{X}$  is a simply connected space, then  $\text{PathClass}(x_0, x_1)$  consists of one element. That is to say, if  $f$  and  $g$  are Paths from  $x_0$  to  $x_1$ , then  $f \simeq_p g$ .

**Covering maps****Definition 6.1: Covering maps and spaces**