Theorem 8.15

WTS. *If $|\phi(x)| \leq C(1+|x|)^{-n-\varepsilon}$, where $\varepsilon > 0$, and if $f \in L^p$, for $p \in [1, +\infty)$, then*

$$f * \phi_t \to af$$

pointwise for every x in the Lebesgue set of f,

$$\mathcal{L}_f = \left\{ x \in \mathbb{R}^n, \quad \lim_{r\to 0} \frac{1}{m(B(r,x))} \int_{y \in B(r,x)} |f(x) - f(y)| dy = 0 \right\}$$

We also claim that $m(\mathcal{L}_f^c) = 0$, and $x \in \mathcal{L}_f$ at every continuous $f(x)$.

The proof is long, and will be divided into several parts. Let us start with a couple of Lemmas about the Lebesgue Set of f, and several pointwise estimates that will be of use.

Lemma 0.0.1. *If $\phi : \mathbb{R}^n \to \mathbb{C}$, and*

$$|\phi(x)| \leq C(1+|x|)^{n-\varepsilon}, \, \varepsilon > 0 \tag{1}$$

then $\phi \in L^1$. Furthermore, $\phi_t \in L^1$ for every $t > 0$.

Proof. If $x \neq 0$, then

$$|\phi| \leq C \cdot (1+|x|)^{-(n+\varepsilon)} \leq C \cdot |x|^{-(n+\varepsilon)}$$

on some B^c as defined in Theorem 2.52, so $\phi \in L^1(B^c)$. Next,

$$n + \varepsilon > n > n/2 = a$$

and by monotonicity,

$$|\phi| \leq C \cdot (1+|x|)^{-(n+\varepsilon)} \leq C \cdot (1+|x|)^{-(n/2)}$$

so $\phi \in L^1(\mathbb{R}^n)$. Next, if $\phi \in L^1$, then

$$|\phi_t(x)| = t^{-n} |\phi(t^{-1}x)|$$

taking the integral in L^+, and applying Theorem 2.44, with $T : x \mapsto t^{-1}$, and $\det(T) = t^{-n}$, so that

$$\int |\phi_t|(x) dx = |\det(T)| \int |\phi| \circ T(x) dx = \int |\phi|(x) dx < +\infty$$

This completes the Lemma. ∎

Lemma 0.0.2. *If $f : \mathbb{R}^n \to \mathbb{C}$, and if $f \in C(\mathbb{R}^n)$, then $\mathcal{L}_f = \mathbb{R}^n$.*

Proof. Let $x \notin \mathcal{L}_f$, and there exists a sequence $r_k \to 0$ and $\varepsilon_0 > 0$ but

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy \geq \varepsilon_0$$

We claim that for every $k \geq 1$, we can find a $y_k \in B(r_k, x) \setminus \{x\}$ with

$$|f(x) - f(y)| \geq \varepsilon_0$$

Indeed, suppose by contradiction that no such y_k exists, and by monotonicity,

$$\frac{1}{m(B(r_k, x))} \int\limits_{y \in B(r_k, x)} |f(x) - f(y)| dy < \frac{1}{m(B(r_k, x))} \int\limits_{y \in B(r_k, x)} \varepsilon_0 dy = \varepsilon_0$$

So choose y_k as above, and it is clear that $y_k \to x$ as $k \to \infty$, but $f(y_k) \not\to f(x)$. Therefore f is not continuous at x. $\square$

Lemma 0.0.3. *If $x \in \mathcal{L}_f$, then for every $\delta > 0$ there exists a $\eta > 0$, with*

$$r \leq \eta \implies \int_{|y|<r} |f(x - y) - f(x)| dy \leq \delta \cdot r^n$$

Proof. We will start with something trivial.

$$m(B(r)) = r^n m(B(1)) \tag{2}$$

where $B(r) = \{x \in \mathbb{R}^n, |x| < r\}$. By Theorem 2.44,

$$\begin{aligned} m(B(r)) &= \int \chi_B(x/r) dx \\ &= |\det(T)|^{-1} \int \chi_B(x) dx \\ &= r^n m(B(1)) \end{aligned}$$

where $T : x \mapsto x/r$ and $\det(T) = r^{-n}$. Fix $x \in \mathcal{L}_f$, and take $\varepsilon = \delta / m(B(1)) > 0$, and by definition this induces some $\eta > 0$, and for every $r \leq \eta$

$$\frac{1}{m(B(r, x))} \int\limits_{y \in B(r, x)} |f(x) - f(y)| dy \leq \varepsilon$$

By translation invariance of m,

$$m(B(r,x)) = m(B(r)) = r^n \cdot m(B(1))$$

and apply the map $y \mapsto x - y$, which is a composition a rotation by $|-1|$ and a translation by $x \in \mathbb{R}^n$. By Theorems 2.44 and 2.42,

$$\int\limits_{|y| \in B(r)} |f(x) - f(x-y)| dy = \int\limits_{y \in B(r,x)} |f(x) - f(y)| dy < \varepsilon m(B(1)) \cdot r^n = \delta r^n$$

where we used the fact that

$$\begin{aligned} d(x-y, x) < r &\iff d(-y, 0) < r \\ &\iff d(y, 0) < r \end{aligned}$$

hence

$$\chi_{B(r,x)}(x-y) = \chi_{B(r,0)}(y)$$

$\square$

Lemma 0.0.4. *Let* $A_j = \left\{ |y| \in [2^{-j}\eta,\, 2^{1-j}\eta) \right\}$, *and if Equation* (1) *holds for* ϕ *then* ϕ_t *satisfies*

$$|\phi_t| \le C \cdot t^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \tag{3}$$

on A_j *for every* $t > 0$, *where* $\alpha = t^{-1}\eta$ *for some* $\eta > 0$.

Moreover, if $A_0 = \left\{ |y| < 2^{-K}\eta \right\}$, *where* $K \ge 0$, *then*

$$|\phi_t(y)| \le C \cdot t^{-n} \tag{4}$$

on A_0

Proof. Notice that

$$t^{-1}y \in [2^{-j} \cdot \eta/t,\, 2^{1-j} \cdot \eta/t) = [2^{-j} \cdot \alpha,\, 2^{1-j} \cdot \alpha)$$

And

$$1 + |t^{-1}y| \ge |t^{-1}y| \ge 2^{-j}\alpha$$

Therefore

$$C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}$$

and applying Equation (1) establishes the first claim.

The second claim follows from Equation (1),

$$|\phi_t(y)| \leq C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}$$

□

Lemma 0.0.5.

Proof. □

Main Proof of Theorem 8.15. The outline of the proof is as follows,

1. $|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)}$ for $\varepsilon > 0$ and
2. $f \in L^p$ for $p \in [1, +\infty)$,
3. for any $x \in \mathcal{L}_f$, we wish to show
$$|f * \phi_t - af|(x) \to 0, \quad \text{as } t \to 0$$
4. To prove this, we fix some $\beta > 0$ and show that
$$|f * \phi_t - af|(x) < \beta$$
since β is arbitrary, the proof will be complete.
5. By Lemma 0.0.3, for every $\delta > 0$ there exists a $\eta > 0$ where $r \leq \eta$ implies
$$\int_{|y|<r} |f(x) - f(x-y)|dy \leq \delta \cdot r^n$$
and using the L^1 inequality,
$$\begin{aligned} |f * \phi_t - af|(x) &= \left| \int [f(x-y) - f(x)] \cdot \phi_t(y) dy \right| \\ &\leq \int |f(x-y) - f(x)| \cdot |\phi_t(y)| dy \\ &= \int\limits_{|y|<\eta} |f(x-y) - f(y)| \cdot |\phi_t(y)| dy + \int\limits_{|y|\geq\eta} |f(x-y) - f(y)| \cdot |\phi_t(y)| dy \\ &= I_1 + I_2 \end{aligned}$$

6. Let $\delta = \beta(2A)^{-1}$, where
$$A = 2^n \cdot C\left[\frac{2^\varepsilon}{2^\varepsilon - 1} + 1\right]$$
we make the claim that this choice of δ will give us $I_1 < \beta/2$

7. After choosing $\delta > 0$, (which induces $\eta > 0$), we will show that $I_2 < \beta/2$ (for a fixed $\eta > 0$) for t sufficiently small, and applying the Triangle Inequality finishes the proof.

Let η be as above, and for $t > 0$ and suppose we can find a $K \in \mathbb{N}^+$ with
$$2^K \leq \eta/t \leq 2^{K+1} \tag{5}$$
and define $\alpha = \eta/t$ for convenience.

Notice for any $K \geq 1$, the interval $[0, 1)$ can be partitioned in the following manner
$$[0,1) = [0, 2^{-K}) \cup \left(\bigcup_{j=1}^{K} [2^{-j}, 2^{1-j})\right)$$
and let us define
$$A_j = \left\{|y| \in [2^{-j}\eta,\, 2^{1-j}\eta)\right\}, \quad A_0 = \left\{|y| \in [0, 2^{-K}\eta)\right\}$$

If no such K exists, then let $A_j = \varnothing$ and set $A_0 = \{|y| \in [0, \eta)\}$. The disjoint union of all $A_{j\geq 0}$ is the open ball $\{|y| \in [0, \eta)\}$. By Lemma 0.0.4 and Lemma 0.0.3 each $j \geq 0$,
$$\begin{aligned}
I_1 &= \sum_{j=0}^{K} \int_{y \in A_j} |f(x-y) - f(y)||\phi_t(y)|dy \\
&\leq Ct^{-n}\delta(2^{-K}\eta)^n + \sum_{j=1}^{K} \int_{y\in A_j} |f(x-y) - f(y)||\phi_t(y)|dy \\
&\leq Ct^{-n}\delta(2^{-K}\eta)^n + \sum_{j=1}^{K} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n
\end{aligned}$$

The left member reads,

$$\begin{aligned} Ct^{-n}\delta(2^{-K}\eta)^n &\leq C\delta\alpha^n 2^{-Kn} \\ &\leq C\delta 2^{n(K+1)}2^{-Kn} \\ &= C\delta 2^n \end{aligned}$$

and termwise for the right,

$$\begin{aligned} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n &= C\delta \cdot t^{\varepsilon} \cdot 2^{j\varepsilon+n}\eta^{-\varepsilon} \\ &= (C\delta 2^n\alpha^{-\varepsilon}) \cdot 2^{j\varepsilon} \end{aligned}$$

Summing over the geometric series,

$$\begin{aligned} \sum_{j=1}^{K} 2^{j\varepsilon} &= 2^{\varepsilon}\sum_{j=0}^{K-1} 2^{j\varepsilon} \\ &= \frac{2^{\varepsilon(K+1)} - 2^{\varepsilon}}{2^{\varepsilon} - 1} \end{aligned}$$

using the estimate for α in Equation (5)

$$\alpha \in [2^K, 2^K + 1) \implies \alpha^{-\varepsilon} \in [2^{-\varepsilon(K+1)}, 2^{-\varepsilon K})$$

and combining the last few equations, the right member becomes

$$\begin{aligned} (C\delta 2^n) \cdot \alpha^{-\varepsilon}\frac{2^{\varepsilon(K+1)} - 2^{\varepsilon}}{2^{\varepsilon} - 1} &\leq (C\delta 2^n) \cdot \alpha^{-\varepsilon}\frac{2^{\varepsilon(K+1)}}{2^{\varepsilon} - 1} \\ &\leq (C\delta 2^n) \cdot \frac{2^{\varepsilon}}{2^{\varepsilon} - 1} \end{aligned}$$

Finally, $I_1 \leq (C\delta 2^n)\left[\dfrac{2^{\varepsilon}}{2^{\varepsilon} - 1} + 1\right]$, and by Step 6, $I_1 \leq \beta/2$.

Obtaining an estimate for I_2 is another laborious entreprise. Let us define $W = \{|y| \geq \eta\}$, and

- By Holder's Inequality,

  $$I_2 \leq \|f\|_p \|\chi_W \cdot \phi_t\|_q + |f(x)| \|\chi_W \cdot \phi_t\|_1$$

  where q is the conjugate exponent to p. Since $p \in [1, +\infty)$, it suffices to show $\|\chi_W \cdot \phi_t\|_q \to 0$ as $t \to 0$ for $q \in [1, +\infty]$.

- Suppose $q = +\infty$,

$$y \in W \iff |y| \geq \eta \iff |t^{-1}y| \geq \alpha$$

then $\|\chi_W \cdot \phi_t\|_\infty \leq Ct^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq Ct^\varepsilon \eta^{-(n+\varepsilon)}$

- Now suppose $q \in [1, +\infty)$, by polar integration and Theorems 2.51, 2.52 (brace yourselves):

$$\begin{aligned}
\|\chi_W \cdot \phi_t\|_q^q &= t^{-nq} \cdot \int_{y \in W} C^q \cdot |t^{-1}y|^{-q\cdot(n+\varepsilon)} dy \\
&= C^q \cdot t^{\varepsilon q} \int_{|y| \geq \eta} |y|^{-q\cdot(n+\varepsilon)} dy \\
&= C^q \cdot t^{\varepsilon q} \sigma(S^{n-1}) \int_{r \geq \eta} r^{n-1} \cdot r^{-q\cdot(n+\varepsilon)} dr \\
&= \frac{C^q t^{\varepsilon q}}{n - q \cdot (n+\varepsilon)} r^{n-q\cdot(n+\varepsilon))} \Big]_\eta^\infty \\
&= \frac{C^q t^{\varepsilon q}}{q \cdot (n+\varepsilon) - n} \eta^{n-q\cdot(n+\varepsilon)} \\
\|\chi_W \cdot \phi_t\|_q &= \left[\frac{C}{(q \cdot (n+\varepsilon) - n)^{1/q}} \left(\eta^{n-q\cdot(n+\varepsilon)} \right)^{1/q} \right] t^\varepsilon \\
&= C_3(q) t^\varepsilon
\end{aligned}$$

- Find a t sufficiently small so that

$$t^\varepsilon < \min\left\{ \beta(4C_3(1)|f(x)|)^{-1},\ \beta(4C_3(q)\|f\|_p)^{-1},\ \beta(4C \cdot \eta^{-(n+\varepsilon)})^{-1} \right\}$$

- Therefore $I_2 < \beta/2$, and the proof is complete upon sending $\beta \to 0$.

$\square$