## Chapter A: Review of Topology

## **Properties of Compact Spaces**

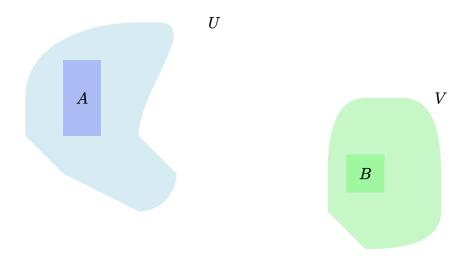


Figure 1: Closed sets A and B within open sets U and V, respectively.

## Proposition 1.1

Let X and Y be topological spaces.

- (a) If  $F \in C(X, Y)$ , and X is compact, then F(X) is compact.
- (b) If **X** is compact and  $F \in C(\mathbf{X}, \mathbb{R})$ , then  $F(\mathbf{X})$  is bounded, and F attains its supremum and infimum on **X**.
- (c) A finite union of compact subspaces of **X** is again compact.
- (d) If **X** is Hausdorff, and A, B are disjoint, compact subspaces of **X**, there exists open U and V, (see fig. 1).
- (e) Every closed subset of a compact space is compact.
- (f) Every compact subset of a Hausdorff space is closed.
- (g) Every compact subset of a metric space is bounded.
- (h) Every finite product of compact spaces is compact.
- (i) Every quotient of a compact space is compact.

Proof of Proposition 1.1 Part A. Let  $f \in C(\mathbf{X}, \mathbf{Y})$  with  $\mathbf{X}$  compact. Fix an open cover of  $f(\mathbf{X})$  in the relative topology,

$$\{U_{\alpha} \cap f(\mathbf{X})\}_{\alpha \in A}$$
 covers  $\mathbf{X}$ ,  $U_{\alpha}$  open in  $\mathbf{Y}$ 

So that  $\bigcup f^{-1}(U_{\alpha}) = \bigcup f^{-1}(U_{\alpha} \cap f(\mathbf{X})) = \mathbf{X}$ . Since  $\{f^{-1}(U_{\alpha})\}_{\alpha \in A}$  is an open cover for  $\mathbf{X}$ , this induces a finite subcollection of indices  $\{\alpha_1, \ldots, \alpha_n\}$  with

$$\bigcup_{j=1}^n f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^n f^{-1}(U_\alpha \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^n f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^n U_{\alpha_j}$$

Proof of Proposition 1.1 Part B. Let X be compact, and  $f \in C(X, \mathbb{R})$ , so that  $f(X) \subseteq \mathbb{R}$  is compact. Compact subsets are closed and bounded in  $\mathbb{R}$ , let  $A = \sup f(X)$  and  $B = \inf f(X)$ . Both A and B are accumulation points of f(X), so A = f(x) and B = f(y) for some x, y in X.

Proof of Proposition 1.1 Part C. Let X be a topological space, and  $K_1, \ldots K_n$  be compact subspaces. Denote  $K = \bigcup_{j=1}^n K_j$ . Let  $\{U_\alpha \cap K\}_{\alpha \in A}$  be an open cover for K, where  $U_\alpha$  is open in X. We can pass the argument to each individual  $K_j$  as follows. Let  $1 \leq j \leq n$ , then  $\{U_\alpha \cap K_j\}_{\alpha \in A}$  is an open cover for  $K_j$ , so there exists as finite subcollection of indices  $I_j \subseteq A$ , (a finite subset of A) whose open sets cover  $K_j$ . Repeat this process for each j and

$$I = \bigcup_{j=1}^{n} I_j$$
 is a finite subset of  $A$ 

with  $K_j \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K)$ . Taking the union over all  $K_j$  reads

$$K = \bigcup_{j=1}^{n} K_j \subseteq \bigcup_{j=1}^{n} \bigcup_{\alpha \in I_j} (U_{\alpha} \cap K) = \bigcup_{\alpha \in I} U_{\alpha} \cap I$$

Proof of Proposition 1.1 Part D. Let X be Hausdorff. We first prove that compact subspaces of X are closed. Indeed, if K is compact in X, fix any  $x \in K^c$ . Let y range through the elements of K, then  $x \neq y$  induces a pair of disjoint open sets  $U_y$  and  $V_y$ , such that

• 
$$x \in U_y$$



Figure 2: In a Hausdorff space, any two distinct points x and y can be separated by disjoint open neighbourhoods U and V.

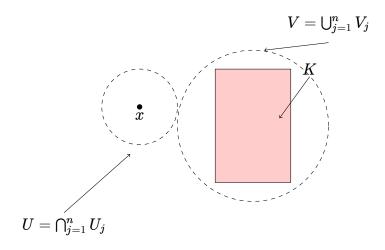


Figure 3: Compact sets are closed in Hausdorff spaces

- $y \in V_y$
- $U_y \cap V_y = \varnothing$
- See fig. 2

Let  $V_y$  range through all possible  $y \in K$ , So that  $\{V_y\}_{y \in K}$  is an open cover. There exists a finite subcollection of 'anchor points' of K,  $y_1, \ldots, y_n$  that corresponds with  $\{V_{y_j}\}_{j=1}^n$ . A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^{n} U_{y_j}$$
 is open

Define  $V = \bigcup_{j=1}^n V_{y_j}$ , so  $V \subseteq K$  and  $U \cap V = \emptyset$  and  $x \in U \subseteq K^c$  (see fig. 3). Therefore K is closed.

Finally, if A and B are disjoint compact sets, then each  $x \in A \subseteq B^c$  induces neighbourhoods  $x \in U_x$ , and  $B \subseteq V_x$  (see fig. 4), let x range through all the elements of A. By compactness of A, this produces a finite subcover, and

$$U = igcup_{j=1}^n U_{x_j} \quad V = igcap_{j=1}^n V_{x_j}$$

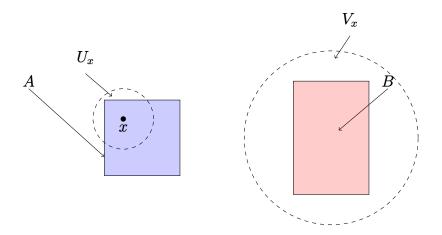


Figure 4: Closed sets A and B, point x in A, and disjoint neighbourhoods U around x and V around B.

are disjoint open sets that contain A and B respectively.

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Proof of Proposition 1.1 Part E. Let  $K \subseteq \mathbf{X}$  be a closed set of a compact space. Let  $\{U_{\alpha} \cap K\}$  be an open cover for K, where each  $U_{\alpha}$  is open in  $\mathbf{X}$ . We can append an extra set  $K^c$  which is open in  $\mathbf{X}$ . The collection

$$W = \{U_{\alpha}\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of  $W_1, \ldots, W_n$  that cover **X** (since **X** is compact by itself). Remove  $K^c$  from this finite subcollection if it exists, and take the intersection with K for each element  $W_i$ , and

$$\{W_1 \cap K, \dots, W_n \cap K\} = \{U_1 \cap K, \dots, U_n \cap K\}$$
 covers  $K$ 

so K is compact.

Proof of Proposition 1.1 Part F. Proven in Part D.

Proof of Proposition 1.1 Part G. let  $K \subseteq X$  be a compact subset of the metric space (X, d). Compact subsets of X are totally bounded, and hence bounded.

Proof of Proposition 1.1 Part H. See Tynchonoff's Theorem in Folland Chapter 4. ■

Proof of Proposition 1.1 Part I. Let X and Y be topological spaces and  $\pi: X \to Y$  be a quotient map. So that Y is endowed with the quotient topology. So that  $\pi$  is a surjective continuous map. and  $\pi(X) = Y$ . Apply Part A, and we see that Y is compact.