1 Chapter 4

Notes on Chapter 4

WTS. *Show that every sequentially compact space is countably compact.*

- *Sequential compactness: every sequence has a convergent subsequence.*
- *Countably Compact: every countable open cover has a finite subcover.*

Proof. We will proceed by proving the contrapositive. Suppose $\mathbf{X}$ admits a countable open cover where no finite subcover exists. Denote this open cover by $\{U_n\}_{n\geq 1}$. Since $\{1, 2, \ldots, n\}$ is finite, we can choose a sequence

$$x_n \in \mathbf{X} \setminus \left(\bigcup U_{j\leq n}\right) \neq \varnothing$$

And suppose by contradiction that x_n admits a convergent subsequence in $x_k \to x \in \mathbf{X}$. We claim that $x \in X \setminus (\bigcup U_{j\geq 1})$. Indeed, if

$$x \in \bigcup U_{j\geq 1} \implies x \in U_j$$

for some $j \in \mathbb{N}^+$. So that U_j is an open set that contains x but only finitely many x_k. Therefore $x_k \not\to x$ for every subsequence x_k of x_n, and the proof is complete. $\square$

Theorem 4.1

WTS. *Suppose that A is a subset of X, let $\operatorname{acc} A$ be the set of accumulation points of A, then*

$$\overline{A} = A \cup \operatorname{acc}(A) \tag{1}$$

and A is closed if and only if $\operatorname{acc}(A) \subseteq A$.

Proof. Suppose that $x \notin \overline{A}$, then $x \in (\overline{A})^c = A^{co}$, then $A^c \in \mathcal{N}_B(x)$. But this means that $x \notin \operatorname{acc}(A)$, since there exists a neighbourhood of x (in the form of A^c), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \varnothing$$

Also, $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$ which means that

$$x \notin \overline{A} \implies x \notin A$$

Since $x \notin \overline{A} \implies x \notin A$ and $x \notin \operatorname{acc}(A)$,

$$(\overline{A})^c \subseteq A^c \cap \operatorname{acc}(A)^c = (A \cup \operatorname{acc}(A))^c$$

Now, if $x \notin \operatorname{acc}(A) \cup A$, then $x \notin \operatorname{acc}(A)$, therefore there exists some $U \in \mathcal{N}_B(x)$ such that

$$A \cap U \setminus \{x\} = A \cap U = \varnothing$$

Where for the second last equality we used the fact that $x \notin A \implies A \setminus \{x\} = A$, and taking complements gives us

$$U \subseteq A^c$$

And since $U \in \mathcal{N}_B(x)$, then $x \in U^o \subseteq A^{co}$ (since U^o is an open subset of A^c). then

$$x \in A^{co} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore $(A \cup \operatorname{acc}(A))^c \subseteq (\overline{A})^c$. $\square$

Theorem 4.2

WTS. *If $\mathcal{T}_X$ is a topology on X and $\mathcal{E} \subseteq \mathcal{T}_X$ then $\mathcal{E}$ is a base for $\mathcal{T}_X$ if and only if for every*

$$\forall U \in \mathcal{T}_X,\, U \neq \varnothing,\, \implies U = \bigcup_{V \in B} V$$

Where B is a subset of $\mathcal{E}$.

Proof. Suppose that $\mathcal{E}$ is a base, then fix any non-empty $U \in \mathcal{T}_X$, then for every $x \in U$, there exists a neighbourhood base for this x and a member $V \in \mathcal{E}$ such that $x \in V_x \subseteq U$. Take the union over all V_x and

$$U \subseteq \bigcup_{x \in U} V_x$$

But each $V_x \subseteq U$, so $U = \bigcup_{x \in U} V_x$, where $\{V_x\} \subseteq \mathcal{E}$.

Conversely, if every non-empty U is a union of members in $\mathcal{E}$ then fix any $x \in X$, we claim that we have a neighbourhood base in

$$\{V \in \mathcal{E},\, x \in V\}$$

The reason is as follows

- x belongs to every $E \in \{V \in \mathcal{E},\, x \in V\}$ and
- For every open U, if $x \in U$ then there exists a union of members of $\mathcal{E}$ such that $U = \bigcup E_\alpha$, then $x \in U \iff \exists E_\alpha \in \{V \in \mathcal{E},\, x \in V\}$ and
- Using this particular $E_\alpha \in \mathcal{E}$ that we just found, $x \in E_\alpha \subseteq U$, and we are done.

$\square$

Theorem 4.3

WTS. *For every $\mathcal{E} \subseteq \mathbb{P}(X)$, $\mathcal{E}$ is base for a topology on X if and only if*

(a) each $x \in X$ is contained in some $V \in \mathcal{E}$, and

(b) if $U, V \in \mathcal{E}$, and $x \in U \cap V$, then there must exist some $W \in \mathcal{E}$ with $x \in W \subseteq U \cap V$.

Proof. Suppose that $\mathcal{E}$ is a base, then we get a), and b) follows since for every $U, V \in \mathcal{E} \subseteq \mathcal{T}_X$, and by closure over finite intersections, $U \cap V \in \mathcal{T}_X$ implies that there exists some $W \in \mathcal{E}$ with

$$x \in W \subseteq U \cap V$$

Now, suppose both a) and b) hold, then we claim that this $\mathcal{E} \subseteq \mathbb{P}(X)$ induces a topology on X

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, \text{ with } x \in V \subseteq U\}$$

Intuitively speaking, this means that $\mathcal{T}$ is just fine (and not too fine) to satisfy the conditions for $E \subseteq \mathcal{T}$ to be a base of $\mathcal{T}$.

We first show that $\mathcal{T}$ is a topology.

- $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$, the first is trivial and the second is from a)
- Closure under unions: fix $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$, and $U = \bigcup U_\alpha$, and for every $x \in U$ there exists some $V_\alpha \in \mathcal{E}$ such that $x \in V_\alpha \subseteq U_\alpha \subseteq U$, therefore $U \in \mathcal{T}$.
- Closure under finite intersections, fix any U_1, U_2 as elements in $\mathcal{T}$, then suppose that they are not disjoint (if they are disjoint then their intersection is the empty set, which is also contained in $\mathcal{T}$). If $U_1 \cap U_2 \neq \varnothing$, then for every $x \in U_1 \cap U_2$ induces two sets $V_1, V_2 \in \mathcal{E}$ with $x \in V_1 \subseteq U_2$ and $x \in V_2 \subseteq U_2$, taking their intersection and applying b) gives us some $V \subseteq V_1 \cap V_2$ with $V \in \mathcal{E}$ therefore $x \in V \subseteq U_1 \cap U_2$, and $\mathcal{T}$ is closed under finite intersections.

Now to show that $\mathcal{E}$ is a base for $\mathcal{T}$, $\mathcal{E} \subseteq \mathcal{T}$ is obvious since very $V \in \mathcal{E}$ satisfies the properties laid out by $\mathcal{T}$ by simply choosing V again for any

$x \in V$. Now fix any member $U \in \mathcal{T}$, then for every $x \in U$, there exists some $V \in \mathcal{E}$ with

$$x \in V \subseteq U$$

(This is an immediate consequence of how we defined $\mathcal{T}$). And we can conclude that $\mathcal{E}$ is a base for this induced topology $\mathcal{T}$. $\square$

Theorem 4.4

WTS. *If $\mathcal{E} \subseteq \mathbb{P}(X)$, the topology $\mathcal{T}(\mathcal{E})$ generated by $\mathcal{E}$ consists of $\varnothing, X$ and all unions of finite intersections of $\mathcal{E}$, in symbols*

$$\mathcal{T}(\mathcal{E}) = \{\varnothing, X\} \cup \left\{\bigcup W_\alpha,\ W_\alpha = \bigcap E_{j\leq n},\ E_j \in \mathcal{E}\right\}$$

Proof. Denote the set

$$W = \{X\} \cup \left\{\bigcap V_{j\leq n},\ \ V_j \in \mathcal{E}\right\}$$

We claim this set W satisfies Theorem 4.3. Since 4.3a) is satisfied with $X \in W$. 4.3b) follows since the right member in W is closed under intersections.

And if we are taking an element from each member, $E_1 \in \{\varnothing, X\}$ and E_2 is an element in the right member, then it is trivial to verify that their intersection is always contained within W. Therefore W induces a topology by Theorem 4.2, and we call this topology $\mathcal{T}$ — and for the sake of completeness

$$\mathcal{T} = \{U \subseteq X,\ \forall x \in U,\ \exists V \in \mathcal{E},\ x \in V \subseteq U\}$$

We so claim that if we define $\overline{W}$ as the union of all members $w \in W$, together with the empty set, is equal to the set $\mathcal{T}$.

$$\overline{W} = \left\{\bigcup_{w\in W} w\right\} \cup \{\varnothing\}$$

- We want to show $\mathcal{T} \subseteq \overline{W}$, since W is a base for the topology $\mathcal{T}$, every (non-empty) $U \in \mathcal{T}$ is the union of members in W (Theorem 4.2), and there exists some $B \subseteq W$ with

  $$U = \bigcup E_{\alpha\in B} \in \overline{W}$$

  Now if U is the empty set then it is trivially contained within $\overline{W}$.

- Next, we show that $\overline{W} \subseteq \mathcal{T}$, fix any element $E \in \overline{W}$, if $E = \varnothing$ then there is nothing to prove since $\mathcal{T}$ is a topology. Now for every $x \in E$,

  $$x \in E = \bigcup_{w\in W} w \implies x \in w$$

  Therefore $E \in \mathcal{T}$ by definition. This proves that $\mathcal{T} = \overline{W}$.

Now that $\overline{W}$ is a topology, that contains $\mathcal{E}$ as a subset, and by definition of $\mathcal{T}(\mathcal{E})$

$$\mathcal{T}(\mathcal{E}) = \bigcap \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Tells us

$$\mathcal{T}(\mathcal{E}) \subseteq \overline{W}, \quad \text{since } \overline{W} \in \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Conversely, fix any member $E \in \overline{W}$, if $E = \varnothing$ then $E \in \mathcal{T}(\mathcal{E})$, if not, then there exists some subset $B \subseteq W$ such that

$$E = \bigcup_{w \in B} w = \bigcup_{w \in B} \bigcap_{j \leq n} V^w_{j \leq n} V_j \in \mathcal{E} \cup \{X\}$$

Since $\mathcal{T}(\mathcal{E})$ is closed under finite intersections and unions, and it contains $\mathcal{E}$ as a subset, $\overline{W} = \mathcal{T}(\mathcal{E})$ and we are done. $\square$

Theorem 4.5

WTS. *Every second countable space is separable. (Countable dense subset).*

Proof. What we wish to prove is that if a space X has a countable base, then it has a countable dense subset. Denote this base of X by $\mathcal{E}$ as usual, then we claim that

$$W = \{x_u,\, U \in \mathcal{E}\}$$

Is a dense subset in X. Note that $(\overline{W})^c = W^{co} \in \mathcal{T}_X$. If $W^{co} = \varnothing$ then we simply take complements and we get $\overline{W} = X$. So suppose that W^{co} is non-empty, then for each $x \in W^{co}$ (by definition of a base), it should induce some $V_x \in \mathcal{E}$ with

$$x \in V_x \subseteq W^{co}$$

But clearly, for every element in $\mathcal{E}$, the second estimate can never be satisfied, since for every $U \in \mathcal{E}$, $x_U \notin W^{co}$ for this particular set W^{co}. Therefore W^{co} must be empty, and this completes the proof. $\square$

Theorem 4.6

WTS. *If X is first countable, then for every $A \subseteq X$, $x \in \overline{A} \iff$ there exists some sequence $\{x_j\}_{j\geq 1} \subseteq A$ such that $x_j \to x$.*

Proof. Suppose that X is first countable, and $A \subseteq X$, and fix any element $x \in \overline{A}$. Since X is first countable, there is a sequence of descending neighbourhoods of $\{U_j\}_{j\geq 1}$ of x such that

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_j \supseteq U_{j+1}$$

If $x \in A$, take $x_n = x$ for all $n \geq 1$. If $x \in \operatorname{acc}(A)$, then take $x_n \in U_n \cap A \setminus \{x\} = U_n \cap A$, which is not empty. Then it remains to show that this sequence converges to x. Fix any neighbourhood $U \in \mathcal{N}_B(x)$ then there exists some N, for every $n \geq N$

$$x \in U^o \implies \exists N \in \mathbb{N}^+, x \in U_N \subseteq U^o$$

Then every $x_n \in A \cap U_N \subseteq A \cap U^o \subseteq U^o$. And this establishes $\implies$.

Now suppose that $x \notin \overline{A}$, so that $x \notin A$ and $x \notin \operatorname{acc}(A)$, then fix any sequence $\{x_j\} \subseteq A$. We wish to show that $x_j \not\to x$.

Since $x \notin \operatorname{acc}(A)$, there exists some $V \in \mathcal{N}_B(X)$ with

$$A \cap V \setminus \{x\} = \varnothing \implies V \subseteq A^c$$

Since $\{x_j\}_{j\geq 1} \subseteq A \implies x_j \notin A^c$ for every $j \geq 1$, then choose V as the neighbourhood around x, and $x_j \not\to x$ for any arbitrary sequence x_j in A. □

Remark. *To truly understand what is going on one should recall that all metric space spaces are first countable.*

Theorem 4.7

WTS. *X is a T_1 space $\iff$ $\{x\}$ is closed for every $x \in X$.*

Proof. If X is T_1 and $x \in X$, then for every $y \neq x$ there exists some open U_y that contains y but not x. Following Folland's argument closely, every $y \neq x$ is is in $\cup U_{y \neq x}$. Hence $\{x\}^c \subseteq \cup U_{y \neq x}$. To show the converse, for every $z \in \cup U_{y \neq x}$ that is open, there exists a $y \neq x$ such that $z \in U_y$. But every U_y does not contain x as an element, so $z \neq x$ implies that $z \notin \{x\}$. And $z \in \{x\}^c$. Hence $\cup U_{y \neq x} = \{x\}^c$.

Now conversely if every $x \in X$ satisfies the fact that $\{x\}^c$ is open, then $\{x\}^c$ is an open set that contains every $y \neq x$. Now fix some $y \neq x$, since $\{y\}$ is also closed, we have $X \cap \{x\}^c$ is an open set that contains x but not y. Also, $\{x\}^c$ is an open set that contains y but not x. And therefore X is T_1. $\square$

Theorem 4.8

WTS. *The map $f : X \to Y$ is continuous if and only if at f is continuous at every $x \in X$.*

Proof. Suppose that f is continuous, then fix any $f(x) \in Y$ and any of its neighbourhood $V \in \mathcal{N}_B(f(x))$,

$$f(x) \in V^o \implies f^{-1}(V^o) \in \mathcal{N}_B(x)$$

But by continuity, $f^{-1}(V^o)$ is an open set that contains x, with

$$f\left(f^{-1}(V^o)\right) \subseteq V^o$$

Therefore f is continuous at x. Now suppose that f is continuous at every $x \in X$, then for every open subset $V \subseteq Y$, and for every point $f(x) \in V = V^o$ means that $V \in \mathcal{N}_B(f(x))$ for all such points $f(x)$. By continuity, for every x in $f^{-1}(V)$, implies that $f^{-1}(V)$ is a neighbourhood of all of its elements, therefore $f^{-1}(V) \subseteq (f^{-1}(V))^o$, and $f^{-1}(V)$ is open. $\square$

Theorem 4.9

WTS. *If $\mathcal{E}_Y$ generates the topology on Y, and f is a mapping from $X \to Y$, then $f : X \to Y$ is continuous if and only if $f^{-1}(V) \in \mathcal{T}_X$ for every $V \in \mathcal{E}_Y$.*

Proof. The inverse image commutes with intersections, complements, and unions. To prove $\Longleftarrow$, use Theorem 4.4, since every $U \in \mathcal{T}_Y$ can be represented the union of finite intersections of elements $\mathcal{E}_Y$, and use the fact that $\mathcal{T}_X$ is closed under arbitrary unions and finite intersections.

To show $\Longrightarrow$, since $\mathcal{E}_Y \subseteq \mathcal{T}_Y$, if f^{-1} is open for every $U \in \mathcal{T}_Y$, then it is open for every $U \in \mathcal{E}_Y$ as well. $\square$

Theorem 4.10

WTS. *If X_α is Hausdorff for each $\alpha \in A$, then $X = \prod_{\alpha \in A} X_\alpha$ is Hausdorff.*

Proof. If two elements in X, $x \neq y$ then there exists some $\alpha \in A$ such that $\pi_\alpha(x) \neq \pi_\alpha(y) \in X_\alpha$, but this X_α is Hausdorff, then there exists two open, disjoint sets $V_x, V_y \subseteq X_\alpha$ such that

- $x \in \pi_\alpha^{-1}(V_x)$, and $y \in \pi_\alpha^{-1}(V_y)$
- $\pi_\alpha^{-1}(V_x) \cap \pi_\alpha^{-1}(V_y) = \pi_\alpha^{-1}(V_x \cap V_y) = \varnothing$
- $\pi_\alpha^{-1}(V_x), \pi_\alpha^{-1}(V_y) \in \mathcal{T}_X$

Where for the last bullet point we used the fact that the product topology makes all the projection maps continuous. This proves that X is Hausdorff. □

Theorem 4.11

WTS. *If X_α and Y are topological spaces, and $X = \prod_{\alpha \in A} X_\alpha$, and $f : Y \to X$ is a mapping. Then f is continuous if and only if $\pi_\alpha \circ f$ is continuous for each $\alpha \in A$.*

Proof. If $\pi_\alpha \circ f$ is continuous at each α, this means that

$$\forall \alpha \in A,\ \forall E_\alpha \in \mathcal{T}_\alpha,\ f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{T}_Y$$

But it is exactly sets of the form $\pi_\alpha^{-1}(E_\alpha)$ which generate the weak topology for $\mathcal{T}_X$. Therefore f is continuous.

Now, suppose that f is continuous, by definition of the weak topology (as it is generated by the set of inverse projections), for every $\alpha \in A$, $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{T}_X$ and by continuity of f, its inverse image is open in Y as well. $\square$

Remark. *The take-away intuition here is that if the range space is generated by some $\mathcal{E}$, then a function is continuous if and only if all inverse images of sets in $\mathcal{E}$ are open in the domain space. Furthermore, if the range space is endowed with the product topology (which is generated by sets of the form $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{T}_\alpha$), then it suffices to check all inverse images of those. And this is equivalent to checking that $\pi_\alpha(\cdot) \circ f$ is continuous at each α.*

Theorem 4.12

WTS. *If X is a topological space, and A is any non-empty set, $\{f_n\} \subseteq X^A$ is a sequence, then $f_n \to f$ with respect to the product topology if and only if $f_n \to f$ pointwise.*

Proof. Suppose that $f_n \to f$ pointwise. Since the product topology $\mathcal{T}_X$ is generated from sets of the form

$$\pi_\alpha^{-1}(E_\alpha),\ E_\alpha \in \mathcal{T}_\alpha$$

And by Theorem 4.4, $\mathcal{T}_X$ consists of $\varnothing, X$ and unions of finite intersections of $\pi_\alpha^{-1}(E_\alpha)$. We claim that for every $f \in X^A$, the following is a valid neighbourhood base for f

$$\left\{ \bigcap_{j \leq n} \pi_{\alpha_j}^{-1}(E_{\alpha_j}),\ E_{\alpha_j} \in \mathcal{T}_{\alpha_j} \cap \mathcal{N}_B(\pi_{\alpha_j}(f)) \right\}$$

A couple things to note

- Each E_{α_j} is open in X_{α_j}, so that its inverse image is also open (in X). Since any neighbourhood base has to be a subset of $\mathcal{T}_X$.
- Only finitely many intersections are involved, so each element in the above set is open in X.
- Each E_{α_j} is a neighbourhood of $\pi_{\alpha_j}(f)$, meaning $f \in E^o_{\alpha_j} = E_{\alpha_j}$.
- Last and perhaps most importantly for intuition, fix any non-empty open set $U \in \mathcal{T}_X$ then by Theorem 4.4 (or my reading of it), U can be written as the union of sets like

  $$\bigcap_{j \leq m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}$$

  Then applying Theorem 4.2, the family of finite intersections of $\pi_\alpha^{-1}(E_\alpha)$ is a base for $\mathcal{T}_X$. Then,

  $$N_{base}(f) = \left\{ V = \bigcap_{j \leq m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}, \quad f \in V \right\}$$

  Has to be a neighbourhood base for any $f \in X$.

Now to show that $f_n \to f$ in the product topology, fix any neighbourhood $U \in \mathcal{N}_B(f)$, then $f \in U^o$, and by definition of a neighbourhood base, there exists some $E \in N_{base}(f)$ such that $f \in E \subseteq U^o$, but this E is just the finite intersection of $\pi_{\alpha_j}^{-1}(E_{\alpha_j})$, then at every α_j

- Let N_j be an integer such that for every $n \geq N_j$, $\pi_{\alpha_j}(f_n) \in E_{\alpha_j}$
- Set $N = \sum_{j \leq m} N_j \geq N_j$ for every $j \leq m$.

Then for every $n \geq N$, $f_n \in E \subseteq U^o \subseteq U$ for any arbitrary neighbourhood U of f. So $f_n \to f$ in the product topology.

Conversely, suppose that $f_n \to f$ in the product topology, then fix any $\alpha \in A$, and for every neighbourhood E_α of $\pi_\alpha(f)$, $\pi_\alpha^{-1}(E_\alpha)$ is a neighbourhood of f. Hence for every $\alpha \in A$, and for every neighbourhood E_α of $\pi_\alpha(f)$, $pi_\alpha(f_n)$ is eventually in E_α. This completes the proof. $\square$

Theorem 4.13

WTS. *If X is a topological space then* $\mathrm{BC}(X)$ *is a closed subspace of* $B(X)$ *in the uniform metric, and $BC(X)$ is complete.*

Proof. Suppose that $\{f_n\} \subseteq \mathrm{BC}(X)$ converges to some f. There are a couple things that we need to show prior to tackling the main proof.

(a) $B(X)$ endowed with the uniform norm of an $f \in B(X)$

$$\|f\|_u = \sup\{|f(x)|,\, x \in X\}$$

Is indeed a normed vector space.

(b) $B(X)$ with its norm (and induced metric), is a complete metric space. So that our $\{f_n\} \to f$ at worst, converges to $f \in B(X)$.

To show that $B(X)$ is a normed vector space, for any $k \in \mathbb{C}$, $f_1, f_2 \in B(X)$, then at every $x \in X$

$$|f_1(x) + kf_2(x)| \leq |f_1(x)| + |k| \cdot |f_2(x)| \leq \|f_1\|_u + |k|\|f_2\|_u$$

And to show absolute homogeneity, note that $\sup|kA| = |k| \cdot \sup A$ for any non-empty bounded above set of reals A. This proves (a).

To show (b), fix any Cauchy sequence (with respect to the uniform metric), then for every $\varepsilon > 0$, there exists an N so large that for every $n, m \geq N$ we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_u < \varepsilon$$

This shows that $\{f_n(x)\}_{n\geq 1} \subseteq \mathbb{C}$ is a Cauchy, and it makes sense to call its limit $f(x) = \lim f_n(x)$. To show that for this f,

- $f_n \to f$ uniformly, and
- $f \in B(X)$

Fix an $\varepsilon > 0$, and there exists an N so large that for every $m, n \geq N$ implies that

$$\|f_n(x) - f_m(x)\|_u < \varepsilon$$

Since $\lim_{n\to\infty} f_n(x) = f(x)$, this means that

$$\lim_{n\to\infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| \leq \varepsilon$$

In the above we replaced the strict inequality with an inequality since the sequence may converge to its supremum. Since this holds for any $x \in X$, we have

$$\|f_m - f\|_u \leq \varepsilon$$

One can easily replace all the ε with $\varepsilon/2$ to obtain strict inequalities, to finish the proof, simply send $m \to \infty$ (since $f_m \to f$ pointwise everywhere, the uniform norm goes to zero as well). This proves both bullet points.

Now we will prove Theorem 4.13, for any sequence $\{f_n\} \subseteq \mathrm{BC}(X)$, if it does converge to some f uniformly, then we claim that $f \in \mathrm{BC}(X)$. Note that $f \in B(X)$, so it suffices for us to show that f is continuous at every point $x \in X$.

Fix any ball with radius $\varepsilon > 0$ at $f(x) \in \mathbb{C}$, and since

- $\varepsilon/3 > 0$ induces some N such that for every $n \geq N$, at every point $x \in X$
$$|f_n(x) - f(x)| \leq \|f_n - f\|_u < \varepsilon/3$$
- Another $\varepsilon/3$ ball around $f_n(x)$ (using the same point $x \in X$), such that its inverse image is an open set $U \in \mathcal{T}_X$, because $f_n \in \mathrm{BC}(X)$
$$f_n^{-1}(V_{\varepsilon/3}\, f_n(x)) = U \in \mathcal{T}_X$$
- The last $\varepsilon/3$ comes from the fact that $y \in U \subseteq X$ so it satisfies
$$|f_n(y) - f(y)| \leq \|f_n - f\|_i < \varepsilon/3$$

Combining these three,

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f(x) - f_n(x)| + |f_n(x) - f_n(y)| < \varepsilon$$

So there exists some open set $U \in \mathcal{T}_X$ (and hence neighbourhood of every x), for every open ball of radius $\varepsilon > 0$, around every $f(x) \in \mathbb{C}$, such that

$$f(U) \subseteq B \in \mathcal{T}_{\mathbb{C}}$$

Since the open balls are a neighbourhood base at every point in $\mathbb{C}$, and f is continuous at every point $x \in X$, we must conclude that $f \in \mathrm{BC}(X)$. $\square$

Theorem 4.14

WTS. *Suppose that A and B are disjoint closed subsets of the normal space X, and let $\Delta = \{k2^{-n} : n \geq 1$ and $0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that*

1. $A \subseteq U_r \subseteq B^c$ *for every* $r \in \Delta$,
2. $\overline{U_r} \subseteq U_s$ *for* $r < s$, *and*
3. *For every* $r < s$, $\overline{U}_r \subseteq U_s$

Proof. The goal of this proof is to show that for every $r \in \Delta$, there exists a open U_r that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some $N \geq 1$ that all previous U_r in previous layers have been constructed properly, meaning if $r = k/2^n$, then for every $1 \leq n \leq N - 1$, we have

$$r = \frac{k}{2^n},\ 1 \leq n \leq N - 1,\ 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each U_r,

- $A \subseteq U_r \subseteq B^c$ and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer $N \geq 1$, we only have to construct the $U_{k/2^N}$ for every odd k, this is because if k is an even number, then $k = 2j$ and $r = 2j/2^N = j/2^{N-1}$ and for this particular U_r is already constructed. So for every odd $k = 2j + 1$, the sets of the form $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every $k - 1 \neq 0$ and $k + 1 \neq 1$. (We will consider these cases later). We claim that for every pair of open sets, $E_1, E_2 \in \mathcal{T}_X$, then there exists some open set $G \in \mathcal{T}_X$ such that if $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$ where H is defined as the set

$$H = \left\{(E_1, E_2) \subseteq (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \varnothing\right\}$$

Then there exists some $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$ such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any $(E_1, E_2) \in H$, then this pair induces a pair of disjoint sets $\overline{E_1}$ and E_2^c since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \varnothing$$

And by normality, there exists disjoint open sets G_1, G_2 such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \varnothing \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since G_2^c is a closed set that contains G_1 as a subset, $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of G_2 (so we may discard the notion of G_2 in our minds). Let us now replace G with G_1 then it is an easy task to verify that $G = G_1 = \mathcal{J}(E_1, E_2)$ has the required properties.

Now define for every odd k, since $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (we note in passing that $\mathcal{J}$ is not a function as the set G may not be unique).

$$U_{k/2^N} = \mathcal{J}\left(U_{(k-1)/2^N}, U_{(k+1)/2^N}\right)$$

Then, if $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$ sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$ and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed' $U_{k/2^N}$ for every $k-1 \neq 0$ and $k+1 \neq 1$. Now let us deal with the remaining pathological cases.

If $k-1$ so happens to be 0, then no $r \in \Delta$ satisfies $r = 0/2^N$, and we substitute

$$\overline{U}_0 = A, \quad \text{or alternatively, } U_0 = A^o$$

Then $U_0 \in \mathcal{T}_X$, $\overline{U}_0 = A \subseteq B^c$. It is at this point that we must mention that $0, 1 \notin \Delta$, so U_0 and U_1 do not have to obey the rules we have laid out for $U_{r\in\Delta}$.

Now if $k+1$ is equal to 2^N (this makes $r = (k+1)/2^N = 1$) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every $0 \leq m \leq 2^N - 1, U_{m/2^N}$ must staisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (even for when $N = 1$, since $A = \overline{U}_0 \subseteq U_1 = B^c$) and a corresponding $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$ such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when $N = 1$. We would only have to construct for $k = 1$, since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. □

Theorem 4.15

WTS. *Urysohn's Lemma. Let X be a normal space, if A and B are disjoint closed subsets of X, then there exists a $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.*

Proof. Let $r \in \Delta$ be as in Lemma 4.14, and set U_r accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0,1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic ε argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since A and B are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that x is not a member of any U_r, but we set $U_1 = X$. Since none of the $r \in (0,1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The ε argument follows: suppose for every $\varepsilon > 0$, $(1-\varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannnot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in X. So $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of Δ, there exists an r, $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r<\alpha} U_r$. So $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r<\alpha} U_r$.

Fix an element $x \in \bigcup_{r<\alpha} U_r$, this induces an r such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r<\alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$.

For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an r (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) — we have $x \notin \overline{U_s}$, but $\left(\overline{U_s}\right)^c$ is open in X. It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq \left(\overline{U_s}\right)^c \subseteq \bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$. To show the reverse, fix an element x in the union, then this induces some $x \in \left(\overline{U_s}\right)^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of W. This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen s is a lower bound for W, and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in X, using Theorem 4.9 finishes the proof. □

Theorem 4.16

WTS. *The Tietze's Extension Theorem. Let X be a normal space, and for any closed subset $A \subseteq X$, and $f \in C(A, [a, b])$, there exists an $F \in C(X, [a, b])$ which extends f.*

Proof. We begin with an important lemma that will serve as a 'black box' for the induction.

Lemma 1.1. *For every $f \in C(A, [0, 1])$, there exists a $g \in C(X, [0, 1/3])$ such that*

$$0 \leq f - g \leq 2/3 \quad \textit{pointwise on } A \tag{2}$$

Proof. Since f is continuous, $B = f^{-1}([0, 1/3])$, and $C = f^{-1}([2/3, 1])$ are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function $g \in C(X, [0, 1])$ such that $g|_B = 0$ and $g|_C = 1$. Relabel $g = g/3$ then $g \in C(X, [0, 1/3])$ (multiplication is continuous).

To show that (2) holds, suppose $x \in B$, then $f(x) \in [0, 1/3]$ and $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$. Now suppose that $x \in C$, then $f(x) \in [2/3, 1]$ and $g(x) = 1/3$ (recall that we relabelled g). So we have $0 \leq 1/3 \leq f - g \leq 2/3$. Lastly, for the case where $x \notin (B \cup C)$, then $f(x) \in (1/3, 2/3)$, and $g(x) \in [0, 1/3]$ implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 \qquad & \implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 \qquad & \implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore $0 \leq f(x) - g(x) \leq 2/3$. □

We can assume that $f \in C(A, [0, 1])$, since we can relabel $f = (f - a)/(b - a)$. The main part of this proof consists of constructing a sequence of $\{g_n\} \subseteq C(X, \mathbb{R})$ where $0 \leq g_n \leq (2/3)^n(1/2)$, and $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$ on A. Let us begin with the base case with $n = 1$. We can apply Lemma 1.1 to get $g_1 \in C(X, [0, 1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that $\{g_j\}_{j \leq n}$ has been chosen, we will find our g_{n+1} by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by $(2/3)^{-n}$ and we obtain a new function in $C(A, [0, 1])$.

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) \left(\frac{3}{2} \right)^n \leq 1$$

Applying the Lemma 1.1, we get a function $h \in C(X, [0, 1/3])$ such that, for every $x \in A$

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) \left(\frac{3}{2} \right)^n - h \leq 2/3$$

Multiplying across gives

$$0 \leq \left(f(x) - \sum_{j \leq n} g_j(x) \right) - h \left(\frac{2}{3} \right)^n \leq \left(\frac{2}{3} \right)^{n+1}$$

Set $g_{n+1} = h \left(\frac{2}{3} \right)^n$ and $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$. Furthermore, the sum of all g_j pointwise converges uniformly, as

$$\sum_{j \geq 1} \|g_j\|_u \leq \sum_{j \geq 1} \left(\frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum $F = \sum g_j$, then this $F \in BC(X)$ (by Theorem 4.9), since every $g_j \in BC(X)$. And

$$\left\| f - \sum_{j \leq n} g_j \right\|_u \leq \left(\frac{2}{3} \right)^n \longrightarrow 0$$

So $F = f$ on A, now if we want to obtain our F on $[a, b]$ we simply relabel $F = F(b - a) + a$. This finishes the proof. $\square$

Theorem 4.17

WTS. *If X is a normal space, and A is a closed subspace of X, and $f \in C(A)$, then there exists an $F \in C(X)$ such that F extends f.*

Proof. First we suppose that f is real valued, so $f \in C(X, \mathbb{R})$. And define a $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$, using

$$g = \frac{f}{1 + |f|}$$

Since g satisfies the assumption of Theorem 4.16 (note that we do not require g to be injective), there exists a $G \in C(X, [-1, +1])$ such that $G|_A = g$. Since the set $\{-1, +1\}$ is closed in $\mathbb{R}$, $G^{-1}(\{-1, +1\})$ is closed as well. Since $G^{-1}((-1, +1)) \subseteq A$, this makes A and $B =^{-1} (\{-1, +1\})$ disjoint closed sets in X.

By Urysohn's Lemma, there exists a continuous function $h \in C(X, [0, 1])$ such that $h|_B = 0$ and $h|_A = 1$, so that the product $|hG| < 1$ for all $x \in X$. We can think of this h as a continuous indicator function that filters out the parts we do not want, namely $G^{-1}\{-1, +1\}$. Now define F in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that $F|_A = g/(1 - |g|) = f$ indeed. Since $|g| = \frac{|f|}{1+|f|}$, and $g(1 + |f|) = f$ implies that $g/(1 - |g|) = f$, because $g \in C(A, (-1, +1))$ This completes the proof for any $f \in \mathbb{R}$ if $f \in C(A)$, then

1. $\mathrm{Re}(f) = f_1 \in C(A, \mathbb{R})$
2. $\mathrm{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in $C(X, \mathbb{R})$ that extends f_1 and f_2, and $F_1 + iF_2 = f$ on A and $F_1 + iF_2 \in C(X)$, and the proof is complete. □

Theorem 4.18

WTS. *If X is a topological space, and $E \subseteq X$ and $x \in X$, then $x \in \operatorname{acc} E \iff$ there exists a net in $E \setminus \{x\}$ that converges to x, and $x \in \overline{E} \iff$ there exists a net in E that converges to x.*

Proof. Suppose that $x \in \operatorname{acc} E$, then for every neighbourhood $U \in \mathcal{N}(x)$, $E \cap U \setminus \{x\} \neq \varnothing$, then choose $\mathcal{N}(x)$ as the set of neighbourhoods directed by reverse inclusion (and this makes $(\mathcal{N}(x), \lesssim)$ a directed set), and we will define the net as follows.

Map each $U \in \mathcal{N}(x)$ to some $x_U \in E \cap U \setminus \{x\}$, then this net converges to x. Suppose that we fix a neighbourhood, $V \in \mathcal{N}(x)$, then for every $U \gtrsim V$ we have $x_u \in U \subseteq V$. So $\langle x_U \rangle$ is eventually in V.

Conversely, if $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$, and $x_\alpha \to x$, then every $U \in \mathcal{N}(x)$ there exists a $x_\alpha \in E \cap U \setminus \{x\}$ that makes

$$E \cap U \neq \varnothing \quad \forall U \in \mathcal{N}(x)$$

Hence $x \in \operatorname{acc} E$.

Now for the second part of the Theorem, suppose that $x \in \overline{E}$, if $x \notin E$ then $E = E \setminus \{x\}$ and $x \in \operatorname{acc} E$, so there exists a net in $E \setminus \{x\} \subseteq E$ such that $x_\alpha \to x$. If $x \in E$ then simply choose $\langle x_\alpha \rangle = x$ for every $\alpha \in A$.

Now, suppose that there is a net that converges to x, and this net $\langle x_\alpha \rangle \subseteq E$, if $x \in E$ then there is nothing to prove, since $E \subseteq \overline{E}$, so suppose that $x \notin E$, then there exists a net in $E \setminus \{x\} = E$ such that

$$x_\alpha \to x \implies x \in \operatorname{acc} E \subseteq \overline{E}$$

□

Theorem 4.19

WTS. *Let X and Y be topological spaces, then every $f : X \to Y$ is continuous at a point $x \in X \iff$ every net $\langle x_\alpha \rangle$ that converges to x implies that $\langle f(x_\alpha) \rangle$ converges to $f(x)$.*

Proof. If f is continuous at a point $x \in X$, then $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$, then for every net $\langle x_\alpha \rangle$ that converges to this x, there there exists an α_0 such that for every $\alpha \gtrsim \alpha_0$ implies that $x_\alpha \in f^{-1}(V)$. Hence

$$f(x_\alpha) \in f\left(f^{-1}(V)\right) \subseteq V$$

And this is equivalent to saying that for every $V \in \mathcal{N}(f(x))$, $\langle f(x_\alpha) \rangle$ is eventually in V, and this proves convergence.

Now suppose that f is not continuous at some x, then there exists a $V \in \mathcal{N}(f(x))$ such that $f^{-1}(V) \notin \mathcal{N}(x)$, so

$$x \notin \left(f^{-1}(V)\right)^o \implies x \in \left(f^{-1}(V)\right)^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our $x \in \overline{f^{-1}(V^c)}$ induces a net $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$ that converges to x. But every element in the net is contained within $f^{-1}(V^c)$, and for every $\alpha \in A$

$$f(x_\alpha) \in f\left(f^{-1}(V^c)\right) \subseteq V^c$$

gives $f(x_\alpha) \notin V$, but V is a neighbourhood of $f(x)$, hence there exists some $x_\alpha \to x$ and $f(x_\alpha) \not\to f(x)$. □

Theorem 4.20

WTS. *If $\langle x_\alpha \rangle$ is a net in X, and $x \in X$ is a cluster point of $\langle x_\alpha \rangle \iff$ there exists a subnet of $\langle x_\alpha \rangle$ that converges to x.*

Proof. Suppose that $\langle y_\beta \rangle_{\beta \in B}$ is a subnet of $\langle x_\alpha \rangle$ that converges to x, then for every neighbourhood $U \in \mathcal{N}(x)$, there exists a β_1 such that for every $\beta \gtrsim \beta_1$ we get $y_\beta = x_{\alpha_\beta} \in U$.

Furthermore, let us fix a $\alpha_0 \in A$ to attempt to show that $\langle x_\alpha \rangle$ is frequently in U, then by the subnet property of $\langle y_\beta \rangle$, there exists some $\beta_2 \in B$ such that for every $\beta \gtrsim \beta_2$, $\alpha_\beta \gtrsim \alpha_0$. (Intuitively this property means that the directed set of B 'grows' as much as the directed set of A, so we can always find elements that are greater than any fixed α_0.)

Since $\langle y_\beta \rangle$ is a net, we there exists some $\beta \in B$ such that $\beta \gtrsim \beta_1$ and $\beta \gtrsim \beta_2$, we then apply the $\beta \mapsto \alpha_\beta$ map and we obtain some $\alpha = \alpha_\beta$ that satisfies:

- $\alpha = \alpha_\beta \gtrsim \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that $\beta \gtrsim \beta_1$ so that y_β falls into U.

Conversely, suppose that x is a cluster point of $\langle x_\alpha \rangle$, then by definition

$$\forall U \in \mathcal{N}(x),\ \forall \alpha_0 \in A,\ \exists \alpha \gtrsim \alpha_0,\ x_\alpha \in U$$

Denote the directed neighbourhoods of x by $\mathcal{N}(x)$, and construct our directed set B for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every $(U, \gamma) \in B$ we can map it to some $\alpha_{(U,\gamma)} \in A$, if we choose some $\alpha_{(U,\gamma)} \gtrsim \gamma$ and $\alpha_{(U,\gamma)} \in U$.

To show that B is a directed set, we say that $(U, \gamma) \gtrsim (U', \gamma')$ if and only if $U \subseteq U'$ and $\gamma \gtrsim \gamma'$. And to show that $\langle y_\beta \rangle = \langle x_{\alpha_{(U,\gamma)}} \rangle$ is indeed a subnet of $\langle x_\alpha \rangle$, fix any $\alpha_0 \in A$, then simply take any neighbourhood U of x (we always have $X \in \mathcal{N}(x))$ — and therefore $(U, \alpha_0) \in B$.

Now for every $(U', \alpha_0') \gtrsim (U, \alpha_0)$ implies that $\alpha_0' \gtrsim \alpha_0$, therefore we have

$$\alpha_{(U', \alpha_0')} \gtrsim \alpha_0' \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that $\langle y_\beta \rangle$ indeed converges to x, fix any $V \in \mathcal{N}(x)$, then with any $\alpha_0 \in A$, and for every $(V', \alpha_0') \gtrsim (V, \alpha_0) \in B$, we have

$$x_{\alpha_{(V', \alpha_0')}} \in V' \subseteq V$$

So $\langle x_{\alpha_{(U,\gamma)}} \rangle$ converges to x. $\square$

Theorem 4.21

WTS. *A topological space X is compact $\iff$ every family of closed sets, $\{F_\alpha\}_{\alpha\in A}$ that has the finite intersection property, implies that*

$$\bigcap_{\alpha\in A} F_\alpha \neq \varnothing$$

Proof. We first examine the assertion, Theorem 4.21 proposes for any family of closed sets $\{F_\alpha\}_{\alpha\in A}$, and for every finite subset $B \subseteq A$ then,

$$\bigcap_{\alpha\in B} F_\alpha \neq \varnothing \implies \bigcap_{\alpha\in A} F_\alpha \neq \varnothing$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha\in A} F_\alpha = \varnothing \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha\in B} F_\alpha = \varnothing$$

Applying DeMorgan's theorem, and since every $\{F_\alpha\}_{\alpha\in A}$ induces a family of open sets (and vice versa), where $U_\alpha = F_\alpha^c$, so for any familiy of open sets $\{U_\alpha\}_{\alpha\in A}$ we have

$$\bigcup_{\alpha\in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha\in B} U_\alpha = X$$

Which is equivalent to saying that X is compact. $\square$

Theorem 4.22

WTS. *A closed subset of a compact space X is compact.*

Proof. Suppose $F \subseteq X$ and F is open, then fix an open cover for F, so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since F^c is an open set, we can obtain a valid open cover for X, then we pick out a finite subcover, for some finite $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left(\bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with F on both sides yields

$$\begin{aligned}
&F = X \cap F \subseteq (F^c \cap F) \cup \left(F \cap \left(\bigcup_{\alpha \in B} U_\alpha \right) \right) \\
&F = \left(F \cap \left(\bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\
&F \subseteq \bigcup_{\alpha \in B} U_\alpha
\end{aligned}$$

Therefore every open cover of F has a finite subcover, and F is compact. □

Theorem 4.23

WTS. *If F is a compact subset of a Hausdorff space X, and $x \notin F$, there are disjoint open sets U, V such that $x \in U$ and $F \subseteq V$.*

Proof. Since $x \in F^c$, for every $y \in F$, $x \neq y$ induces two sets U_y, V_y (because X is T_2).

- $U_y \cap V_y = \varnothing$
- $x \in U_y$
- $y \in V_y$

But $\{V_y\}_{y \in F}$ is an open cover for the compact set F, then there exists a finite subcollection $H \subseteq F$ such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since H is finite, $U = \bigcap_{y \in H} U_y$ is an open set that contains x, also define $V = \bigcup_{y \in H} V_y$. If for every $y \in H$, $U_y \cap V_y = \varnothing$, then $U \cap V_y = U \cap V = \varnothing$. This completes the proof. □

Remark. *Every metric space (X, d) is first countable, and T_2 (it is actually T_4, but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element $x \in X$ and we notice that $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$ is a countable neighbourhood base for every x. To show that (X, d) is T_2, for every pair of elements $x \neq y$, we can take $r = d(x, y)/2$ and there exists disjoint open sets $V_r(x)$ and $V_r(y)$ such that $x \in V_r(x)$ and $y \in V_r(y)$.*

Theorem 4.24

WTS. *Every compact subset of a Hausdorff (T_2) space is closed.*

Proof. If F is compact, then for every $x \in F^c$, by Theorem 4.23, there exists two disjoint open sets such that $x \in U$ and $F \subseteq V$, but

$$U \cap V = \varnothing \implies U \cap F = \varnothing \implies U \subseteq F^c$$

But since $x \in F^c$ is arbitrary, and U is an open subset of F^c,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that F^c is open and F is closed. $\square$

Theorem 4.25

WTS. *Every compact Hausdorff (T_2) space is normal (T_4).*

Proof. Fix A, B which are disjoint closed subsets of X, by Theorem 4.22, we know that these two sets are compact. Hence for every $y \in B$ there exists two disjoint open sets U, V_y (by Theorem 4.23)

$A \subseteq U_y$ and $y \in V_y$. But the family $\{V_y\}_{y \in B}$ is s a valid open cover for the compact set B, hence there exists a finite subcollection $H \subseteq B$ such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \varnothing$$

The second equality holds for every $y \in H$ so that $U_y \cap (\cup V_{y \in H}) = \varnothing$. Define $U = \cap U_{y \in H}$ and $V = \cup V_{y \in H}$, where both of these are disjoint open sets that that contain A and B as subsets, since for each $y \in H$, $A \subseteq U_y$ hence the intersection of all U_y also contains A as a subset. Therefore X is normal. $\square$

Theorem 4.26

WTS. *If X is compact, and $f : X \to Y$ is continuous, then $f(X)$ is compact.*

A small lemma.

Lemma 1.2. *For every $\{E_j\} \subseteq X, f(\cup E_j) = \cup f(E_j)$.*

The proof is trivial.

Proof. If $\{V_{\alpha\in A}\}$ is an open cover for $f(X)$, then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha\in A} V_\alpha\right) = \bigcup_{\alpha\in A} f^{-1}(V_\alpha) \subseteq X$$

Since f is continuous, we have an open cover in the form of $\{f^{-1}(V_\alpha)\}$ for X, then there exists a finite subset $B \subset A$ such that

$$X \subseteq \bigcup_{\alpha\in B} f^{-1}(V_\alpha)$$

Then we wish to show that for this $B \subseteq A$, $\{V_{\alpha\in B}\}$ is a finite open cover for $f(X)$. Fix any element $y \in f(X)$, then this induces a $x \in X$ such that $y = f(x)$, but because $\{f^{-1}(V_{\alpha\in B})\}$ is an open cover for X, there exists some $\alpha \in B$ such that $x \in f^{-1}(V_\alpha)$, hence by definition of the inverse image

$$f(x) \in V_\alpha \implies f(X) \subseteq \bigcup_{\alpha\in B} V_\alpha$$

Therefore $f(X)$ is compact and this completes the proof. $\square$

Theorem 4.27

WTS. *If X is compact, then $C(X) = BC(X)$.*

Proof. Notice that $BC(X) \subseteq C(X)$, so we only have to show the reverse estimate. Fix any $f \in C(X)$, since X is compact, by Theorem 4.26 we know that $f(X)$ is also compact. Since $\mathbb{C} = \mathbb{R}^2$ is a complete metric space, $f(X)$ is bounded and $f \in BC(X)$. $\square$

Theorem 4.28

WTS. *If X is compact, and if Y is Hausdorff, then any continuous bijection $f : X \to Y$ is a homeomorphism.*

Proof. If $E \in X$ is closed, then since X is compact, E is compact as well. By continuity of f, $f(X)$ is a compact set in Y, but compact subsets of Y are closed, so f is continuous.

We used the fact that the inverse of f^{-1} is f, since it suffices to check that every inverse image of a closed set is also closed, f^{-1} is continuous. And by definition of a homeomorphism (f has to be bijective and both f and f^{-1} hav eto be continuous), f is a homeomorphism. □

Theorem 4.29

WTS. *If X is any topological space, the following are equivalent.*

(a) X is compact.

(b) Every net has a cluster point.

(c) Every net in X has a convergent subnet.

Proof. By Theorem 4.20, every net in X has a cluster point $\iff$ there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose a) holds, then X is compact, and fix an arbitrary net $\langle x_\alpha \rangle$ in X. and define the 'tail' of the net

$$E_\alpha \coloneqq \{x_\beta,\ \beta \gtrsim \alpha\}$$

We wish to show that the arbitrary intersection of $\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \varnothing$. Where $\overline{E}_\alpha$ is closed, so it suffices to check that every finite $B \subseteq A$, the intersection over $\overline{E}_\alpha$ is non-empty.

Suppose we are given a finite $B \subseteq A$, then fix any two elements α and $\beta \in B$, by the definition of a net there exists a $\gamma \in A$ such that $\gamma \gtrsim \alpha$ and $\gamma \gtrsim \beta$, and

$$\varnothing \neq \subseteq E_\alpha \cap E_\beta \implies \overline{E}_\alpha \cap \overline{E}_\beta \neq \varnothing$$

Therefore for any finite collection of $\{\overline{E}_{\alpha \in B}\}$, then

$$\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \varnothing$$

Now fix an element $x \in \bigcap_{\alpha \in A} \overline{E}_\alpha$. Then for every $\alpha \in A$, $x \in \overline{E}_\alpha$, and for every neighbourhood $U \in \mathcal{N}(x)$, $U \cap E_\alpha \neq \varnothing$. This is because if $x \in E_\alpha$, then $U \cap E_\alpha$ contains at least $\{x\}$, if $x \in \operatorname{acc} E_\alpha$, then by definition of an accumulation point, $U \cap E_\alpha \setminus \{x\} \neq \varnothing$, so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net, E_α, if for every $\alpha \in A$, $x \in E_\alpha$ if and only if there exists some $\gamma \gtrsim \alpha$, $x_\gamma \in U \cap E_\alpha$, this is equivalent to saying that x is a cluster point of $\langle x_\alpha \rangle$. So $a) \implies b)$.

Now let us suppose that X is not compact, then there exists an open cover $\{U_{\alpha \in A}\}$ of X that has no finite subcover. Let $\mathbb{B}$ be the collection of all finite subsets of A, directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every $B \in \mathbb{B}$, find some $x_B \in (\bigcup_{\alpha \in B} U_\alpha)^c$. So we have a net in X. Now we will show that no $x \in X$ can be a cluster point of this net. Suppose not, then take a neighbourhood U_β with $\beta \in A$ such that U_β belongs to the open cover we first discussed. Then for any $B \in \mathbb{B}$ such that $B \gtrsim \{\beta\}$ (meaning that $\{\beta\} \subseteq B$, where B is a finite set), then

$$x_B \in \left(\bigcup_{\alpha \in B} U_\alpha \right)^c \implies x_B \notin \left(\bigcup_{\alpha \in \{\beta\}} U_\beta \right) \implies x_B \in U_\beta^c$$

Hence no point in X can be a cluster point for this net, and the proof is complete. $\square$

Theorem 4.30

WTS. *If X is a LCH space, and for every $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$, there exists a compact $N \subseteq U$ where $N \in \mathcal{N}_B(x)$.*

Proof. For every $U \in \mathcal{N}_B(x) \cap \mathcal{T}_x$, we can find an E open subset of U that has a compact closure, since every $x \in X$ induces some compact $F \in \mathcal{N}_B(x)$, therefore

$$E \coloneqq U \cap F^o \implies \overline{E} \subseteq F$$

Since closed subsets of compact sets are compact (by Theorem 4.22), $\overline{E}$ is compact. More is true, since E is open,

$$x \in U \cap F^o \implies x \in E^o \implies E \in \mathcal{N}_B(x)$$

Now it suffices to show that there exists some compact $N \subseteq E \subseteq U$ such that $N \in \mathcal{N}_B(x)$. Since $\overline{E}$ is compact, the closed subset $\partial E = \overline{E} \cap \overline{E^c}$ of $\overline{E}$ is also compact.

Since $\partial E \cap E^o = \varnothing$, $x \in E^o = E$ means that $x \notin \partial E$. Applying Theorem 4.23 to the compact set ∂E and $x \notin \partial E$ gives us two disjoint open sets V' and W'. We list their properties

1. $V', W' \in \mathcal{T}_X$
2. $x \in V'$
3. $\partial E \subseteq W'$
4. $V' \cap W' = \varnothing$

The two disjoint pairs induce another pair of open sets relative to $\overline{E}$, recall the definition of the topology relative to $\overline{E}$,

$$\mathcal{T}_{\overline{E}} = \left\{ A \cap \overline{E} : A \in \mathcal{T}_X \right\}$$

We now agree to define

- $V = V' \cap \overline{E}$
- $W = W' \cap \overline{E}$

Then evidently $V, W \in \mathcal{T}_{\overline{E}}$ and

1. $x \in V' \cap \overline{E} \implies x \in V$
2. $\partial E \subseteq \overline{E} \implies \partial E \subseteq W$
3. $V' \cap W' = \varnothing \implies V \cap W = \varnothing$

Furthermore,

$$\partial E \subseteq W \implies W^c \subseteq (\partial E)^c = E^o \cup E^{co}$$

Taking the intersection over $\overline{E}$ gives us

$$\overline{E} \setminus W \subseteq \overline{E} \cap (E^o \cup E^{co})$$

Note that $E^{co} = (\overline{E})^c$, since $(E^c)^{oc} = \overline{(E^{cc})} = \overline{E}$ therefore $\overline{E} \cap E^{oc} = \varnothing$, hence

$$\overline{E} \setminus W \subseteq \overline{E} \cap E^o = E^o$$

Using the fact from 3, $V \subseteq W^c$ and $V \subseteq \overline{E}$ and $V \subseteq W^c$ implies that $V \subseteq \overline{E} \setminus W$. Compiling everything, we have

$$V \subseteq \overline{E} \setminus W \subseteq E$$

Note that the set $\overline{E} \setminus W$ is closed in $\mathcal{T}_X$ (and hence closed in $\overline{E}$) by closure over intersections, $\overline{V}$ is therefore a closed subset of $\overline{E} \setminus W$, and $\overline{V}$ is compact. Also

$$\overline{V} \subseteq \overline{E} \setminus W \subseteq E$$

To check that $\overline{V} \in \mathcal{N}_B(x)$, note that

$$x \in V^o \subseteq (\overline{V})^o \implies \overline{V} \in \mathcal{N}_B(x)$$

The subset relation $V^o \subseteq \overline{V}^o$ comes from the fact that V^o is an open subset of $\overline{V}$, and hence is contained in $(\overline{V})^o$ as a subset. Now let us define $N = \overline{V}$, and N satisfies the assertions in the Theorem, since

- $N \in \mathcal{N}_B(x)$
- N is compact
- $N \subseteq E \subseteq U$

And this completes the proof. $\square$

Remark. *Intuitively speaking, this means that if X is any LCH space, then for every open neighbourhood $U \in \mathcal{N}_B(x)$, there exists a compact $E \in \mathcal{N}_B(x)$ such that $x \in E \subseteq U^o$. This property is indeed a very strong one as it allows us to have effectively 'infinite' descending compact neighbourhoods of x.*

Theorem 4.31

WTS. *X is a LCH space, and $K \subseteq U \subseteq X$ where K is compact, and U is open, then there exists some precompact, open V with*

$$K \subseteq V \subseteq \overline{V} \subseteq U$$

Proof. For every $x \in K$, we can apply Proposition 4.30, since $x \in K \subseteq U$, this induces some compact $F_x \subseteq U$ where $F_x \in \mathcal{N}_B(x)$. Then we can obtain an open cover of U in the form of $\{F_x^o\}_{x \in K}$. By compactness of K, there exists a finite $B \subseteq K$ such that

$$K \subseteq \bigcup_{x \in B} F_x^o$$

Let $V = \bigcup_{x \in B} F_x^o$, then clearly V is open, and $K \subseteq V$. Since each F_x is closed (compact sets are closed in any Hausdorff Space), we have

$$V \subseteq \bigcup_{x \in B} F_x \implies \overline{V} \subseteq \bigcup_{x \in B} F_x$$

Since $\bigcup_{x \in B} F_x$ is a finite union of compact sets, we claim that it is also compact. Consider two compact sets E_1 and E_2, then if $\{U_\alpha\}_{\alpha \in A}$ is any open cover of $E_1 \cup E_2$, it must be an open cover for E_1 and E_2 as well, because

$$E_1, E_2 \subseteq E_1 \cup E_2 \bigcup_{\alpha \in A} U_\alpha$$

Since E_1 and E_2 are both compact sets, they each induce two finite subsets of B_1, B_2 of A whose union $B = B_1 \cup B_2$ is also compact. Therefore

$$E_1 \cup E_2 \subseteq \bigcup_{\alpha \in B} U_\alpha$$

Then a simple proof by induction will show that if $\{E_{j \leq n}\}$ is a family of compact sets, then $E = \bigcup E_{j \leq n}$ is also compact.

Returning to the main part of the proof, $\bigcup_{x \in B} F_x$ is a compact set, therefore $\overline{V}$ is also compact. Moreover

$$\forall x \in K,\ F_x \subseteq U \implies \overline{V} \subseteq \bigcup_{x \in B} F_x \subseteq U$$

Combining, we have

- $K \subseteq V \subseteq \overline{V}$,
- V is open and $\overline{V}$ is compact, and
- $\overline{V} \subseteq U$

This completes the proof. $\square$

Theorem 4.32

WTS. *Urysohn's Lemma, Locally Compact Version. For any LCH space X, and if $K \subseteq U \subseteq X$ where K is compact and U is open, then there exists some $f \in C(X, [0,1])$ with*

- $f = 1$ *on* K
- $f = 0$ *outside some compact* $\overline{V} \subseteq U$

Proof. Let V be as in Theorem 4.31, for our fixed $K \subseteq U \subseteq X$, there exists a pre-compact, open V that satisfies

$$K \subseteq V \subseteq \overline{V} \subseteq X$$

It follows that this $(\overline{V}, \mathcal{T}_{\overline{V}})$ is a normal space by Theorem 4.25 (compact Hausdorff spaces are normal), and by Urysohn's Lemma (Theorem 4.15) on normal spaces, since we can easily find two disjoint closed subsets of $\overline{V}$ in the form of

- $K \subseteq V^o = V \subseteq \overline{V}$ (compact sets in Hausdorff spaces are closed)
- $\partial V = \overline{V} \cap \overline{V^c}$ (closed sets in compact spaces are compact)
- $K \subseteq V^o$ implies that $K \cap \partial V = K \cap (\overline{V} \setminus V^o) = \varnothing$

Then there exists a continuous $f|_{\overline{V}} \in C(\overline{V}, [0,1])$ that evaluates to

- $f|_{\overline{V}} = 1$ on closed K
- $f|_{\overline{V}} = 0$ on closed ∂V

Now let us extend $f|_{\overline{V}}$ to f by defining

$$f|_{(\overline{V})^c} = 0$$

We will show that this extension of f is indeed continuous. Indeed, for every closed set $E \subseteq [0,1]$ that does not contain 0, we have:

$$0 \notin E \implies \{0\} \cap E = \varnothing \implies f^{-1}(\{0\}) \cap f^{-1}(E) = \varnothing$$

But $(\overline{V})^c \subseteq f^{-1}(\{0\})$ therefore

$$(\overline{V})^c \cap f^{-1}(\{0\}) \cap f^{-1}(E) = (\overline{V})^c \cap f^{-1}(E) = \varnothing \implies f^{-1}(E) \subseteq \overline{V}$$

We can write

$$f^{-1}(E) = f|_{\overline{V}}^{-1}(E)$$

But we know that $f|_{\overline{V}}$ is continuous, so $f|_{\overline{V}}^{-1}(E)$ must be closed (with respect to $\overline{V}$), and therefore is closed wrt X, since $\overline{V}$ is closed wrt X.

For the case where $0 \in E$, note that

$$f^{-1}(E) = \left(f^{-1}(E) \cap \overline{V}\right) \cup \left(f^{-1}(E) \cap (\overline{V})^c\right) = \left(f|_{\overline{V}}\right)^{-1}(E) \cup \left(f|_{\overline{V}^c}\right)^{-1}(E)$$

The above equalities are messy in print. They are but a simple consequence of disjoint decomposition of the pre-images, since

$$\overline{V} \cap f^{-1}(E) = \{x \in \overline{V} : f(x) \in E\} = f|_{\overline{V}}^{-1}(E)$$

Back to our main discussion, recall that for every $x \in \partial V$

$$f(x) = 0 \in f^{-1}(\{0\}) \subseteq f^{-1}|_{\overline{V}}(E)$$

Therefore $\partial V \subseteq f^{-1}|_{\overline{V}}(E)$, and $(\overline{V})^c = f^{-1}|_{(\overline{V})^c}(E)$ gives us (since V^c is closed),

$$\begin{aligned}
f^{-1}(E) &= f^{-1}|_{\overline{V}}(E) \cup \partial V \cup (\overline{V})^c \\
&= f^{-1}|_{\overline{V}}(E) \cup \overline{(V^c)} \cup (\overline{V})^c \\
&= f^{-1}|_{\overline{V}}(E) \cup (V^c \cup V^{co}) \\
&= f^{-1}|_{\overline{V}}(E) \cup V^c
\end{aligned}$$

Since $f^{-1}|_{\overline{V}}(E)$ and V^c are closed subsets of X, then $f^{-1}(E)$ is also closed, and $f \in C(X, [0, 1])$. $\square$

Theorem 4.33

WTS. *Every LCH space is completely regular (or $T_{3.5}$).*

Proof. Recall that a space X is completely regular if it is T_1 and every closed subset A and every $x \notin A$ there exists some

$$f \in C(X, [0,1]), \quad f(x) = 1,\ f|_A = 0$$

Fix a closed set $A \subseteq X$, then for every $x \in A^c$, there exists a compact $E_x \in \mathcal{N}_B(x)$ with $E_x \subseteq A^c$ (by Theorem 4.30).

Note that $E_x \subseteq A^c$ where E_x is compact and A^c is closed, then an application of Theorem 4.31 tell us that there exists an $f \in C(X, [0,1])$ such that for every $x \in E_x$, $f(x) = 1$ and for points $y \notin A^c$ (which means that $y \in A$), $f(y) = 0$. Therefore X is completely regular. $\square$

Theorem 4.34

WTS.

Proof. □

Theorem 4.35

WTS. *If X is a LCH space, we claim that*

$$\overline{\mathrm{C}_c(X)} = \mathrm{C}_0(X)$$

Proof. We begin by proving several things that are mentioned before this Theorem, namely

$$\mathrm{C}_c(X) \subseteq \mathrm{C}_0(X) \subseteq \mathrm{BC}(X)$$

Fix an $f \in \mathrm{C}_c(X)$, and for every $\varepsilon > 0$,

$$x \in |f|^{-1}([\varepsilon, +\infty)) \implies |f(x)| \geq \varepsilon > 0$$

Therefore $|f|^{-1}([\varepsilon, +\infty))$ is a closed subset of $\mathrm{supp}\,(f)$, since $(-\infty, \varepsilon)$ is open in $\mathbb{R}$, then $[\varepsilon, +\infty)$ is a closed set. And by continuity of $|\cdot| \circ f$ (a composition of two continuous functions), $|f|^{-1}([\varepsilon, +\infty))$ is closed. Using the fact that closed subsets of compact $\mathrm{supp}\,(f)$ are also compact, we get $f \in \mathrm{C}_0(X)$.

Next, we show that $\mathrm{C}_0(X) \subseteq \mathrm{BC}(X)$. Fix any element f of $\mathrm{C}_0(X)$ with an arbitrary $\varepsilon > 0$, then $E_\varepsilon = \{x \in X : |f(x)| \geq \varepsilon\}$ is compact. The continuity of f guarantees that the direct image of a compact set is another compact set (Theorem 4.26)

$$|f|(E_\varepsilon) \text{ is a compact subset of } \mathbb{R}$$

And therefore for every $x \in E_\varepsilon \implies |f(x)| \in f(E\varepsilon)$, then by Heine-Borel, there exists some $M \geq 0$ such that $|f(x)| \leq M$. If $x \notin E_\varepsilon$, then by definition of E_ε, implies that $|f(x)| < \varepsilon$. Then $|f(x)| \leq M + \varepsilon$ for every $x \in X$. Hence $f \in \mathrm{BC}(X)$.

Here I wish to offer an alternate proof for $\mathrm{C}_0(X) \subseteq \mathrm{BC}(X)$, we begin by constructing an open cover for $\mathrm{supp}\,(f)$ such that

$$\{U_n\}_{n>0} = \{x \in X | f(x)| < n\}$$

Then there exists a finite subcollection of $\{U_n\}_{n \in B}$ where B is a finite set, then define $M = 1 + \sum_{n \in B} n$ and for every $x \in \mathrm{supp}\,(f)$ we have $|f(x)| < n$ and since $n > 0$ this holds for every $x \in X$ too. Therefore $f \in \mathrm{BC}(X)$.

For the main proof of Theorem 4.35, since $\mathrm{BC}(X)$ is endowed with the uniform metric, it is also first countable, and therefore by Theorem 4.6, it suffices to show that every sequence $\{f_n\}_{n\geq 1} \subseteq \mathrm{C}_c(X)$ converges in $\mathrm{C}_0(X)$. And every element $f \in \mathrm{C}_0(X)$ has a convergence sequence in $\mathrm{C}_c(X)$.

Fix a convergent sequence $\{f_n\}_{n\geq 1} \subseteq \mathrm{C}_c(X)$ that converges uniformly to some $f \in \mathrm{BC}(X)$ (since $\mathrm{BC}(X)$ is a closed subset of $C(X)$ with respect to the uniform norm), then for every $\varepsilon > 0$, there exists some $n \geq 1$ with

$$\|f_n - f\|_u < \varepsilon$$

We aim to show that $(\mathrm{supp}\,(f_n))^c \subseteq |f|^{-1}((-\infty, \varepsilon))$, so fix any $x \notin \mathrm{supp}\,(f_n)$, then

$$|f(x) - f_n(x)| = |f(x)| \leq \|f - f_n\|_u < \varepsilon$$

This establishes the estimate, and taking complements

$$|f|^{-1}([\varepsilon, +\infty)) \subseteq \mathrm{supp}\,(f_n)$$

Therefore for any arbitrary $\varepsilon > 0, \{x \in X,\ |f(x)| \geq \varepsilon\}$ is compact, and $\overline{\mathrm{C}_c(X)} \subseteq \mathrm{C}_0(X)$. Conversely, fix any $f \in \mathrm{C}_0(X)$, and for every $n \geq 1$, define

$$K_n = \{x \in X,\ |f(x)| \geq 1/n\}$$

Using Urysohn's Lemma for our LCH space X, there exists some g_n that has a compact support, and $g_n(x) = 1$ for every $x \in K_n$. We then write $f_n = g_n \cdot f \in \mathrm{C}_c(X)$. We wish to show that $f_n \to f$ uniformly. Notice that for any fixed $n \geq 1$, if $x \in K_n$ then

$$f_n(x) = f(x) \implies |f_n - f|(x) = 0$$

If $x \notin K_n$, $|f(x)| < 1/n$ (recall what K_n does), and $f_n = g_n \cdot f \in [0, 1]$ by definition of g_n from Theorem 4.32, hence

$$|f_n(x) - f(x)| = |f(x)| \cdot |1 - g_n| \leq |f(x)| < 1/n$$

Taking the supremum over $x \in X$, we have

$$\|f_n - f\|_u < 1/n \to 0$$

As we send n to $+\infty$, and $f_n \to f$ uniformly. This completes the proof. □

Theorem 4.36

WTS.

Proof. □

Theorem 4.37

WTS. *If X is an LCH space and $E \subseteq X$. E is closed if and only if $E \cap K$ is closed for every compact $K \subseteq X$.*

Proof. Suppose that E is closed, then $E \cap K$ is closed, since compact subsets of Hausdorff spaces are closed, and $E \cap K \subseteq K$ tells us that $E \cap K$ is indeed compact.

Now suppose that E is not closed, by Theorem 4.1, $E \neq \overline{E}$, so pick some $x \in (\overline{E} \setminus E) = \operatorname{acc}(E) \cap E^c$, since X is locally compact, let K_x be a compact neighbourhood of x, then for every neighbourhood $U \in \mathcal{N}_B(x)$, we have

$$x \in U^o,\ x \in K_x^o, \implies x \in (U^o \cap K_x^o) \subseteq (U \cap K_x)^o$$

Since $(U^o \cap K_x^o)$ is an open subset of $(U \cap K_x)$, then $(U \cap K_x) \in \mathcal{N}_B(x)$, and recall that $x \in \operatorname{acc}(E)$, therefore

$$(U \cap K_x) \cap E \setminus \{x\} = U \cap (K_x \cap E) \neq \varnothing$$

But $x \notin E \implies x \notin E \cap K_x$. So x is an accumulation point of $E \cap K_x$ that is not in $E \cap K_x$. Therefore there exists some $E \cap K_x$ (with K_x compact) that is not closed. $\square$

Theorem 4.38

WTS.

Proof. □

Theorem 4.39

WTS.

Proof. □

Theorem 4.40

WTS.

Proof. □

Theorem 4.41

WTS.

Proof. □