MATH 263: Section 003, Tutorial 4

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1 Exact ODEs (review)

An **exact ODE** is of the form:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$
$$M(x,y)\ dx + N(x,y)\ dy = 0$$

where

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

Then, we can define some $F(x,y)$ such that:

$$d(F(x,y)) = M(x,y)\ dx + N(x,y)\ dy = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$$

Then, find $F(x,y)$, and the relation $F(x,y) = \int 0\ dx = C$ is the solution.
Note: $d(F(x,y)) = \frac{\partial F}{\partial x}\ dx + \frac{\partial F}{\partial y}\ dy$ is a form of the multivariable chain rule.

Problem 2.2a. From Boyce and DiPrima, 10th edition (2.6, exercise 5, p.101):
Check whether the differential equation:

$$\frac{dy}{dx} = -\frac{ax+by}{bx+cy}$$

is exact. If so, find the general solution.

Solution:

$$(ax+by)\ dx + (bx+cy)\ dy = 0$$

Let $M(x,y) = ax+by$ and $N(x,y) = bx+cy$. Check for exactness:

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = b.$$

Therefore, the ODE is exact.

$$M(x,y) = \frac{\partial f}{\partial x} = ax+by$$

Then integrate. With a multivariable function, indefinite integrals must have a function of integration (similar to adding a constant of integration):

$$f = \int ax+by\ dx = \frac{a}{2}x^2 + bxy + h(y)$$

$$N(x,y) = \frac{\partial f}{\partial y} = bx + h'(y) = bx+cy$$

$$h'(y) = cy$$

$$h(y) = \frac{c}{2}y^2 + C_0$$

Therefore,

$$f(x,y) = \frac{a}{2}x^2 + bxy + \frac{c}{2}y^2 + C_0 = 0$$

Let $d = -2C_0$:

$$ax^2 + 2bxy + cy^2 = d$$

Note: when the ODE is not exact, an integrating factor may make it exact. Let it be $\mu(x,y)$:

$$(\mu(x,y)M(x,y)) + (\mu(x,y)N(x,y))\frac{dy}{dx} = 0$$

To make the ODE exact,

$$\frac{\partial}{\partial x}(\mu N) = \frac{\partial}{\partial y}(\mu M)$$

$$\frac{\partial N}{\partial x}\mu + N\frac{\partial \mu}{\partial x} = \frac{\partial M}{\partial y}\mu + M\frac{\partial \mu}{\partial y}$$

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu + N\frac{\partial \mu}{\partial x} - M\frac{\partial \mu}{\partial y} = 0$$

Instead of solving a PDE, consider two specific cases:
1. $\mu(x,y) = \mu(x)$:

$$(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x})\mu = N\frac{d\mu}{dx}$$

$$\mu(x) = \exp[\int \frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N}\ dx]$$

2. $\mu(x,y) = \mu(y)$:

$$(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y})\mu = M\frac{d\mu}{dy}$$

$$\mu(y) = \exp[\int \frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M}\ dy]$$

Try both cases if needed, your integrating factor must be single-variable for it to work. In the general case, both or none may work.

Problem 2.2b. From Boyce and DiPrima, 10th edition (2.6, modified exercise 27, p.102):
Find the solution $x(t)$ of:

$$dt + (\frac{t}{x} - \sin x)\ dx = 0.$$

With IVP $x(0) = -\pi$.

Solution: Let $M(t,x) = 1$ and $N(t,x) = \frac{t}{x} - \sin x$. Check for exactness:

$$\frac{\partial N}{\partial t} = \frac{1}{x} \neq \frac{\partial M}{\partial x} = 0.$$

Therefore, the equation is not exact without an integrating factor. By observing the two formulas given, one can see that the expression for $\mu(x)$ will be single-variable:

$$\mu(x) = \exp[\int \frac{\frac{\partial N}{\partial t} - \frac{\partial M}{\partial x}}{M}\ dx] = \exp[\int \frac{dx}{x}]$$

$$\mu(x) = e^{\ln|x|} \Rightarrow \mu(x) = x.$$

Multiplying by the integrating factor:

$$x\ dt + (t - x\ \sin x)\ dx = 0.$$

Let $M^*(t,x) = x$ and $N^*(t,x) = t - x\ \sin x$. Check for exactness:

$$\frac{\partial N^*}{\partial t} = \frac{\partial M^*}{\partial x} = 1.$$

$$M^* = \frac{\partial f}{\partial t} = x$$

$$f = \int x\ dt = tx + h(x)$$

$$N^* = \frac{\partial f}{\partial x} = t + h'(x) = t - x\ \sin x$$

$$h'(x) = -x\ \sin x$$

Using integration by parts, we get:

$$h(x) = x\cos x - \sin x - C$$

$$f(t,x) = tx + x\cos x - \sin x - C = 0$$

$$tx + x\cos x - \sin x = C.$$

IVP: $x(0) = -\pi$:

$$C = 0 \cdot (-\pi) + (-\pi)\cos(-\pi) - \sin(-\pi) = \pi.$$

Therefore,

$$tx + x\cos x - \sin x = \pi.$$

2 Autonomous Equations

(From Tutorial 2): **Autonomous ODE's** only contain the dependent variable, they are of the form:

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)})$$

(From Tutorial 2): A **slope field** is a graphical representation of a family of functions satisfying $y' = f(x,y)$. For some point (x,y), one draws the slope $y' = f(x,y)$ to qualitatively represent solutions. Given a slope field, starting at an initial condition and tracing along the field sketches the particular solution.

Problem 2.4. From Boyce and DiPrima, 10th edition (2.5, exercise 12, p.89):
Consider the following ODE:

$$\frac{d\theta}{dt} = -\theta^2(\theta^2 - 4)$$

Sketch the graph of $f(\theta)$ versus θ, determine and classify the equilibrium points. Then find the general solution.

Solution:

$$\frac{d\theta}{dt} = f(\theta) = -\theta^2(\theta^2 - 4) = -\theta^2(\theta - 2)(\theta + 2)$$

Seeing that $f(\theta < -2) < 0$, $f(\theta > 2) < 0$, $f(0 < \theta < 2) > 0$, and $f(-2 < \theta < 0) > 0$, one can sketch the graph as such:

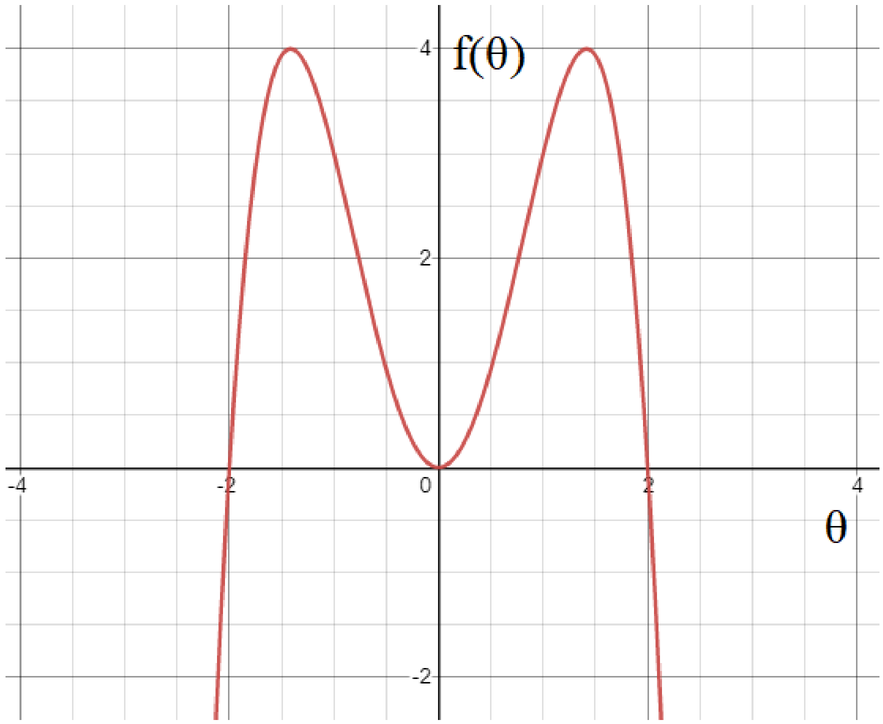

Figure 1: Graph of $f(\theta) = -\theta^2(\theta^2 - 4) = -\theta^2(\theta - 2)(\theta + 2)$.

Note: This graph shows the slope of the solutions, since $f(\theta) = \frac{d\theta}{dt}$. Therefore, the slope field can be drawn as such:

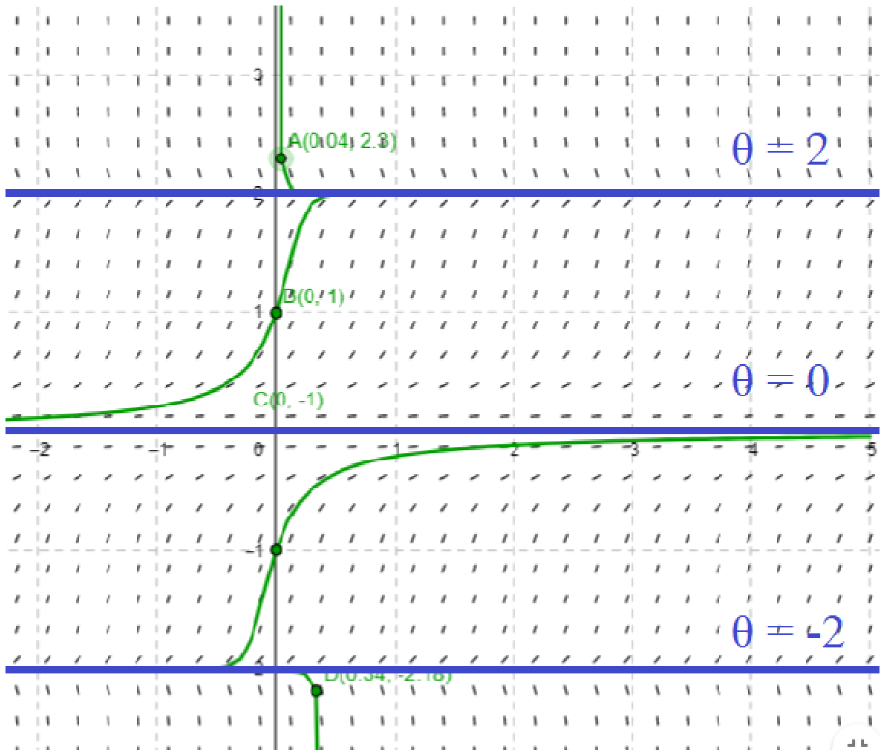

Figure 2: Slope field of $\frac{d\theta}{dt} = -\theta^2(\theta^2 - 4)$ with four solutions in green. Looking at equilibrium points, $\theta = 2$ is **stable**, $\theta = -2$ is **unstable**, $\theta = 0$ is **semistable**.

Now, find the general solution to

$$\frac{d\theta}{dt} = -\theta^2(\theta - 2)(\theta + 2)$$

$$\int -\frac{d\theta}{\theta^2(\theta - 2)(\theta + 2)} = \int dx = x + C_0$$

Using partial fractions to integrate the left hand side:

$$\int \frac{0}{\theta} + \frac{1/4}{\theta^2} + \frac{1/16}{\theta + 2} - \frac{1/16}{\theta - 2} = x + C_0$$

$$-\frac{1}{4\theta} + \frac{1}{16} \ln|\frac{\theta + 2}{\theta - 2}| = x + C_0$$

Let $C = 16C_0$:

$$-\frac{4}{\theta} + \ln|\frac{\theta + 2}{\theta - 2}| = 16x + C.$$