MATH 263: Section 003, Tutorial 8

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1 Introduction to Nonhomogeneous ODEs: Method of Undetermined Coefficients

To solve a nonhomogeneous linear ODE of the form:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = g(x) \neq 0$$

note that by superposition, if $y_1(x)$ and $y_2(x)$ solve the homogeneous ODE:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

and that some specific $Y(x)$ solves the nonhomogeneous ODE, then:

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + Y(x)$$

is the general solution to the nonhomogeneous ODE. To find $Y(x)$, we need to make an educated guess on the form of the solution: we want that guessed form of $Y(x)$ and/or its derivatives to be linearly dependent on $g(x)$. One also must make sure that their chosen $Y(x)$ is not $g(x)$, since it already solves the homogeneous equation. Then find $Y(x)$'s coefficients by plugging them in the ODE.

For the particular solution of $ay" + by' + cy = g(x)$, here's a table directly from Boyce and DiPrima, 10th edition (3.5, table 3.5.1, p.182):

$g(x)$	$Y(x)$
$P_n(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)$
$P_n(x)e^{\alpha x}$	$x^s(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}$
$P_n(x)e^{\alpha x}\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$x^s[(A_0x^n + A_1x^{n-1} + \cdots + A_n)e^{\alpha x}\cos\beta x + (B_0x^n + B_1x^{n-1} + \cdots + B_n)e^{\alpha x}\sin\beta x]$

Note: Here s is the smallest nonnegative integer (s = 0, 1, or 2) that will ensure that no term in $Y(x)$ is a solution of the corresponding homogeneous equation. Equivalently, for the three cases, **s is the number of times 0 is a root of the characteristic equation**, α **is a root of the characteristic equation**, and $\alpha + i\beta$ **is a root of the characteristic equation**, respectively.

Problem 1.1. Solve the IVP:

$$y"(x) + 4y(x) = 2\cos 2x,\ y(0) = 1,\ y'(0) = 0.$$

Problem 1.2. From Boyce and DiPrima, 10th edition (3.5, exercise 17, p.184):
Solve the IVP:

$$y"(t) - 2y'(t) + y(t) = te^t + 4,\ y(0) = 1,\ y'(0) = 1.$$