

## Notation

We will use the following notation to simplify computations with multilinear maps. Let  $E$  and  $F$  be sets, and  $v_1, \dots, v_k \in E$ .  $f : E \rightarrow F$ .

- Listing individual elements:  $v_{\underline{k}}$  means  $v_1, \dots, v_k$  as separate elements.
- Creating a  $k$ -list:  $(v_{\underline{k}}) = (v_1, \dots, v_k) \in \prod E_{j \leq k}$  if  $v_i \in E_i$  for  $i = \underline{k}$ .
- Double indices:  $(v_{\underline{n_k}}) = (v_{n_k}) = (v_{n_1}, \dots, v_{n_k})$ , and

$$(v_{\underline{n_k}}) \neq (v_{n_{(1, \dots, k)}})$$

- Closest bracket convention:

$$(v_{(n_k)}) = (v_{(n_1, \dots, n_k)}) \quad \text{and} \quad (v_{n_{(k)}}) = (v_{n_{(1, \dots, k)}})$$

- Underlining 0 means it is iterated 0 times:

$$(v_{\underline{0}}, a, b, c) = (a, b, c)$$

- Skipping an index:

$$(v_{\underline{i-1}}, v_{i+\underline{k-i}}) = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k) \quad (1)$$

for  $i = \underline{k}$ .

- Applying  $f$  to a particular index:

$$(v_{\underline{i-1}}, f(v_i), v_{i+\underline{k-i}}) = (v_1, \dots, v_{i-1}, f(v_i), v_{i+1}, \dots, v_k) \quad (2)$$

Of course, if  $i = 1$ , then the above expression reads  $(f(v_1), v_2, \dots, v_k)$  by the  $\underline{0}$  interpretation.

- If  $\wedge : E \times E \rightarrow F$  is any associative binary operation,

$$\bigcircled{\wedge}(v_{\underline{k}}) = v_1 \wedge \dots \wedge v_k$$

- In any list using this 'underline' notation, we can find the size of a list by summing over all the underlined numbers and the number of terms without an underline. We see eq. (1), eq. (2) have  $k-1$ ,  $k$  terms respectively.

### Remark 1.1: Preview of exterior calculus

We can write the cofactor expansion formula of the determinant of a  $\mathbb{R}^{k \times k}$  matrix in this notation. Suppose  $a_i \in \mathbb{R}$ , and  $b_i \in \mathbb{R}^{k-1}$  for  $i = \underline{k}$ .

$$M = \begin{bmatrix} a_1 & \cdots & a_k \\ | & & | \\ b_1 & \cdots & b_k \\ | & & | \end{bmatrix}$$

The determinant of  $M$ , can then be written as

$$\det(M) = \sum_{i=k} (-1)^{i-1} a_i \det(\underline{b_{i-1}}, \underline{b_{i+k-i}})$$

 $k$ -linear maps

### Definition 2.1: $k$ -linear maps

Let  $E_k, F$  be Banach spaces. A map  $\varphi : \prod E_k$  is  $k$ -linear if for every  $i = \underline{k}$ ,  $v_i \in E_i$ ,

$$\varphi(\cdot \frac{i-1}{}, v_i, \cdot \frac{k-i}{}) : \bigoplus (E_{i-1}, E_{i+k-i}) \rightarrow F \quad \text{is } (k-1)\text{-linear}$$

The following theorem should give confidence to the notation we have adopted to use.

### Proposition 2.1

Let  $E_{\underline{k}}$  and  $F$  be Banach spaces, a  $k$ -linear map  $\varphi : \prod E_{\underline{k}} \rightarrow F$  is continuous iff there exists a  $C > 0$ , such that for every  $x_i \in E_i, i = \underline{k}$

$$|\varphi(x_{\underline{k}})| \leq C \prod |x_{\underline{k}}|$$

*Proof.* Suppose  $\varphi$  is continuous, then it is continuous at the origin. Picking  $\varepsilon = 1$  induces a  $\delta > 0$  such that for  $|(x_{\underline{k}})| \leq \delta$ ,  $|\varphi(x_{\underline{k}})| \leq 1$ . The usual trick of normalizing an arbitrary vector  $(x_{\underline{k}}) \in \prod E_{\underline{k}}$  does the job:

$$\left| \varphi(x_k \cdot |x_{\underline{k}}|^{-1} \cdot \delta) \right| \leq 1 \implies \left| \varphi(x_{\underline{k}}) \right| \leq \delta^{-k} \prod |x_{\underline{k}}|$$

Conversely, fix a sequence (indexed by  $n$ , in  $k$  elements in the product space  $\prod E_k$ ), so

$$(x_n^k) \rightarrow (x^k) \quad \text{as } n \rightarrow +\infty \quad (3)$$

To proceed any further, we need to prove eq. (4) that expresses the difference of two values of  $\varphi$  in terms of its arguments.

$$\varphi(b^k) - \varphi(a^k) = \sum_{i=k} \varphi(b^{i-1}, \Delta_i, a^{i+k-i}) \quad (4)$$

where  $(b^{\underline{k}})$  and  $(a^{\underline{k}})$  are elements in  $\prod E_k$ , and  $\Delta_i = b^i - a^i$  for  $i = \underline{k}$ . The proof is contained in the following note, which is in more detail than usual - to help the reader ease into the new notation.

### Note 2.1

We proceed by induction, and eq. (4) follows by setting  $m = k$  in

$$\varphi(a^k) = \varphi(b^m, a^{m+k-m}) - \sum_{i=m}^k \varphi(b^{i-1}, \Delta_i, a^{i+k-i}) \quad (5)$$

Base case: set  $m = 1$ , by definition of  $k$ -linearity (def. 2.1) of  $\varphi$ . Since  $a^1 = b^1 - \Delta_1$ ,

$$\varphi(a^{\underline{k}}) = \varphi(b^1 - \Delta_1, a^{1+\underline{k-1}}) = \varphi(b^1, a^{1+\underline{k-1}}) - \varphi(\Delta_1, a^{1+\underline{k-1}})$$

Induction hypothesis: suppose eq. (5) holds for a fixed  $m$ . Since  $a^{m+1} = b^{m+1} - \Delta_{m+1}$ ,

$$\begin{aligned}\varphi(a^{\underline{k}}) &= \varphi(b^{\underline{m}}, a^{m+\underline{k}-m}) - \sum_{i=\underline{m}} \varphi(b^{i-1}, \Delta_i, a^{i+\underline{k}-i}) \\ &= \varphi(b^{\underline{m}}, a^{m+1}, a^{(m+1)+\underline{k}-(m+1)}) - \sum_{i=\underline{m}} \varphi(b^{i-1}, \Delta_i, a^{i+\underline{k}-i}) \\ &= \varphi(b^{m+1}, a^{(m+1)+\underline{k}-(m+1)}) - \varphi(b^{m+1}, \Delta_{m+1}, a^{(m+1)+\underline{k}-(m+1)}) - \sum_{i=\underline{m}} \varphi(b^{i-1}, \Delta_i, a^{i+\underline{k}-i})\end{aligned}$$

and this proves eq. (4)

We substitute  $a^i = x^i$ , and  $b^i = x_n^i$  for  $i = \underline{k}$ , and eq. (4) becomes eq. (6)

$$\varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) = \sum_{i=\underline{k}} \varphi(x_n^{i-1}, x_n^i - x^i, x^{i+\underline{k}-i}) \quad (6)$$

Then the triangle inequality reads

$$\begin{aligned}\left| \varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) \right| &\leq \sum_{i=\underline{k}} \left| \varphi(x_n^{i-1}, x_n^i - x^i, x^{i+\underline{k}-i}) \right| \\ &\leq \sum_{i=\underline{k}} |\varphi| \cdot \bigoplus \left( x_n^{i-1}, \Delta_i, x^{i+\underline{k}-i} \right) \\ &\leq \sum_{i=\underline{k}} |\varphi| \cdot \left| x_n^i - x^i \right| \bigoplus \left( x_n^{i-1}, x^{i+\underline{k}-i} \right) \\ &\lesssim_n |\varphi| \sup_{i=\underline{k}} |x_n^i - x^i| \rightarrow 0\end{aligned}$$

where we identify the product  $\bigoplus(v^{\underline{k}})$  with the product of their norms  $\bigoplus(|v^{\underline{k}}|)$ . ■