Chapter 1: Manifolds

Introduction

In this chapter, E and F will always denote Banach spaces, and all Banach spaces will be over $\mathbb{R}$. We sometimes say E (resp. F) is a space for brevity, and

- $\mathcal{L}(E, F)$ = linear maps between E and F,
- $L(E, F)$ = toplinear (continuous and linear) maps between E and F,
- $\mathrm{Topliso}(E, F)$ = toplinear isomorphisms between E and F,
- $\mathrm{Laut}(E)$ = toplinear automorphisms on E, which form a strongly open subset of $L(E, E)$.

We will be working in the category of C^p Banach spaces — where $p \geq 0$. The morphisms in the category of $\mathrm{Ban}_{\mathbb{R}}$ are called C^p morphisms, which are p-times continuously differentiable functions.

Definition 1.1: Morphisms between open subsets of Banach spaces

Let E and F be Banach spaces, and $U \subseteq E$, $V \subseteq F$ be open subsets. A mapping $f : E \to F$ is of class C^p if $f \in C(E, F)$ and eq. (1) holds.

$$D^{(i)} f : E \to L^i(E, F) \quad \text{exists and is continuous for} \quad i = \underline{p} \tag{1}$$

$C^p(E, F)$ denotes the vector space of C^p mappings between E and F. Sometimes, we restrict our attention to *open subsets* of E and F, in this case: $f \in C^p(U, V)$ if $f \in C(U, V)$ and eq. (2) holds.

$$D^{(i)} f : U \to L^i(E, F) \quad \text{exists and is continuous for} \quad i = \underline{p} \tag{2}$$

We sometimes write C^p for $C^p(E, F)$ when it is clear. A C^p *isomorphism* is a bijective C^p morphism whose inverse is also a morphism.

Remark 1.1: Implicit assumption

In eq. (2) we assumed that $f(U) \subseteq V$. This is a non-trivial part of the definition of C^p morphisms between E and F, we will come back to this in def. 3.1.

Let f_1 and f_2 be mappings, and X a non-empty set.

- We say they are *composable* if either one of $f_2 \circ f_1$ or $f_1 \circ f_2$ makes sense.
- We also write $f_2 f_1$ to refer to $f_2 \circ f_1$ if there is no ambiguity.
- If $U \subseteq X$ and $V \subseteq Y$, and $f : U \to V$ is a bijection — meaning $f(U) = V$ and f is injective, we say f is a bijection between U and V.
- With regards to inverse image notation, we allow ourselves to write

$$f_2^{-1} \circ f_1^{-1} \quad \text{is the same as} \quad f_2^{-1} f_1^{-1}$$

and inversion is never left associative.

$$f_2 f_1^{-1} = f_2 \circ f_1^{-1} \neq (f_2 \circ f_1)^{-1}$$

Composable C^p mappings are functors in the category of open subsets between Banach spaces. Few basic facts about C^p morphisms:

- If f is a toplinear mapping between E and F, then $f \in C^p(E,F)$ for all $p \geq 0$.
- If f is a bijective toplinear mapping, then it is a C^p isomorphism for all $p \geq 0$.
- However, a bijective C^p morphism need not be a C^p isomorphism.

Structure of a manifold

It is fruitful to *construct* the manifold rather than *define* it. We also insist on working with open sets of Banach spaces instead coordinate functions as our primary data.

Definition 2.1: Chart

Let X be a non-empty set. A *chart on X modelled on a Banach space E* is a tuple (U,φ), such that $U \subseteq X$, $\varphi(U) = \hat{U}$ is an *open* subset of E, and φ is a bijection onto $\hat{U}$.

Definition 2.2: Compatibility

Let (U,φ) and (V,ψ) be charts on X modelled on E, they are called C^p compatible (for $p \geq 0$) if $U \cap V = \varnothing$, or both of the following hold

- $\varphi(U \cap V)$ and $\psi(U \cap V)$ are *both* open subsets of E, and
- the *transition map* $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ is a C^p isomorphism between open subsets of E.

Definition 2.3: Atlas

Let X be a non-empty set and $p \geq 0$. A *C^p atlas on X modelled on E* is a pairwise C^p compatible collection of charts $\{(U_\alpha,\varphi_\alpha)\}$ whose union over the domains cover X.

We will assume hereinafter that atlases are of class C^p for $p \geq 0$. Let X be a non-empty set, equipped with an atlas $\{(U_\alpha,\varphi_\alpha)\}$ modelled on a space E. Suppose α, and β both index the atlas.

- We write $\hat{U}_\alpha$ to refer to $\varphi_\alpha(U_\alpha)$, and
- $\hat{p} = \varphi_\alpha(p)$ for $p \in U_\alpha$ when it is clear which chart we are using.
- $U_{\alpha\beta} = U_\alpha \cap U_\beta$, and if $U_{\alpha\beta} \neq \varnothing$: the *transition map from α to β* is defined in eq. (3).

$$\varphi_{\alpha\beta} \triangleq \varphi_\beta|_{U_{\alpha\beta}} \circ \left(\varphi_\alpha|_{U_{\alpha\beta}}\right)^{-1} : \varphi_\alpha(U_{\alpha\beta}) \to \varphi_\beta(U_{\alpha\beta}) \tag{3}$$

- We often suppress the restrictions of the two charts in the composition, and eq. (3) reads

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} = \varphi_\beta\varphi_\alpha^{-1} \tag{4}$$

Remark 2.1: Omissions of C^p

We might refer to two charts as *compatible* or *smoothly compatible*, implying they are C^p compatible. This comes from the perspective that, in the context of C^p manifolds, any smoothness exceeding C^p is deemed sufficiently smooth for our purposes. We also say C^p for C^p where $p \geq 0$.

Given that compatibility is an equivalence relation on the set of all charts on X that are modelled on E, it should not be surprising it descends into an equivalence relation among atlases. This is condensed in note 2.1.

Note 2.1: Descent of an equivalence relation

Let Ω be a non-empty set with an associated equivalence relation $\sim$. Suppose $A_i \subseteq \Omega$ is also a subset of the equivalence class $[A_i]$ where $i = \underline{2}$. We say the $A_1 \sim A_2$ if any of the following equivalent statements hold.

1. For every $(x, y) \in A_1 \times A_2$, we have $x \sim y$.
2. There exists $x \in A_i$, where $x \sim y$ for all $y \in A_{3-i}$.
3. $A_1 \cup A_2$ is a subset of an equivalence class over $\Omega / \sim$.
4. $A_j \subseteq [A_i]$ for $i, j = \underline{2}$.

It is not hard to see this defines an equivalence relation. And $[A_i]$ represents the largest superset of A_i that is contained within a single equivalence class.

Definition 2.4: Structure determined by an atlas

Let $\mathcal{A}$ be an atlas on X, the maximal atlas containing $\mathcal{A}$ is called the C^p *structure determined by* $\mathcal{A}$.

Definition 2.5: Manifold

A C^p *manifold modelled on* E is a non-empty set X with a C^p structure modelled on E. We refer to E as the *model space* of X.

Proposition 2.1: E is a manifold

The identity id_E defines an atlas on E, which determines a C^p structure called the *standard structure of* E for $p \geq 0$. We call (E, id_E) the *standard chart* on E.

Proposition 2.2: Topology is unique on a manifold

Let X be a C^p manifold modelled on E, it induces a unique topology such that the domain for each chart in its smooth structure is open, and each chart is a homeomorphism onto its range in the subspace topology.

Proof. We offer a sketch of the proof. Fix a chart (U, φ), it is clear that U has to be in the topology of X, and because $\varphi : U \to \hat{U}$ is required to be a homeomorphism, we duplicate all the open sets in $\hat{U}$ by using the inverse image through φ. The collection of all such inverse images form a sub-basis, thus defines a unique topology as is well known.

There is an alternate way constructing the above topology. It is well known of the existence of a unique coarsest topology on a chart domain U where all charts (V, φ) whose domains intersect U — when restricted onto U — are homeomorphisms onto their ranges. Stitching the weak topologies together, we obtain an ambient topology on X. ■

Remark 2.2: Not necessarily Hausdorff

The topology generated by prop. 2.2 is not necessarily Hausdorff, nor second countable. So a manifold X may not admit partitions of unity, but for our current purposes we will work with this general definition. Because of the uniqueness of the topology, we sometimes refer to the topology as being part of the *structure* of the manifold.

Remark 2.3: Omission of model space

For any of the objects we have defined in this section, that depend upon a model space or a morphism class (i.e C^p), we will say 'X is a manifold', rather than X is a manifold of class C^p modelled over E when it is convenient to do so. If the model space E is infinite (resp. finite) dimensional, we say X is infinite (resp. finite) dimensional. And a reminder: C^p should always be interpreted with $p \geq 0$.

Proposition 2.3: Open subsets of manifolds

Let U be an open subset of a manifold X, then U is a manifold whose structure is determined by the atlas $\mathcal{A}$ in eq. (5).

$$\mathcal{A} = \Big\{ (V, \varphi) \text{ in the structure of } X, \text{ where } V \subseteq U \Big\} \tag{5}$$

Proof. The structure of X includes all possible restrictions to open sets; hence $\mathcal{A}$ in eq. (5) is an atlas, and a unique structure by def. 2.4. ■

Morphisms between manifolds

Definition 3.1: Morphisms between manifolds

A mapping $f : X \to Y$ between manifolds is a *morphism* (a C^p morphism to be precise) if for every $p \in X$, there exist charts $(U, \varphi) \in X$ and $(V, \psi) \in Y$ such that 1) the image $f(U)$ is contained in a the chart domain V, and 2)

$$f_{U,V} \triangleq \psi \circ f \circ \varphi^{-1} \in C^p(\hat{U}, \hat{V}) \quad \text{in the sense of def. 1.1.} \tag{6}$$

The map $f_{U,V}$ as defined in eq. (6) is called the *coordinate representation of f* with respect to the charts $(U, \varphi), (V, \psi)$.

Remark 3.1: Identifying X with its structure

If (U, φ) is a chart in the structure of X, we will simply say (U, φ) is in X.

Remark 3.2: Identifying charts with their domains

The scenario in eq. (6) occurs so often that we decide to simply write

$$f_{U,V} = \psi f \varphi^{-1} \tag{7}$$

to mean there exists charts (U, φ), $(V\psi)$ in the structure of X, Y with

$$f(U) \subseteq V \tag{8}$$

Consistent with the notation of putting hats on objects borrowed or pulled back from the model spaces, we write $\hat{f} = f_{U,V}$. Equation (9) gives an example of this.

$$\hat{f}(\hat{p}) = f_{U,V}(\hat{p}) = f_{U,V}(\varphi(p)) \tag{9}$$

for any morphism $f \in \mathrm{Mor}(X, Y)$, and charts that satisfy eq. (8). We refer to the map in eq. (9) as a *coordinate representation of f about p*, with the inference that $p \in (U, \varphi)$.

Definition 3.1 may leave one unsatisfied. Why do we require the image $f(U)$ be contained in another chart domain in Y? There are two reasons.

1. Suppose f is a map between E and F, and the restriction of f onto a family of open subsets $U_\alpha \subseteq E$ is C^p for $p \geq 0$. If $\{U_\alpha\}$ is an open cover for E, then f is continuous. Proposition 3.1 shows this equally holds for manifolds.

2. The definition of smoothness between open subsets of Banach spaces (see def. 1.1) is a purely local one. And let us recall: every chart domain U in a manifold X corresponds to an open subset $\hat{U} \subseteq E$ in the model space, and see remark 1.1 as well. Hence, **the necessity that the image $f(U)$ is contained in a single chart domain of Y is a relic of the original definition**. The astute reader will also see that the openness requirement of $\psi(U \cap V)$, and $\varphi(U \cap V)$ in def. 2.2 is completely natural as well, since C^p morphisms are defined between open subsets of Banach spaces.

Proposition 3.1: Properties of morphisms between manifolds

Every C^p morphism between manifolds is a continuous map, and the composition of C^p morphisms is again a morphism.

Proof. The first claim is proven if we show f is locally continuous. Using Equation (6), since p is arbitrary, choose any neighbourhood W of $f(p)$, by shrinking this neighbourhood, it suffices to assume it is a subset of the chart domain V. The charts on X and Y are homeomoprhisms, and unwinding the formula shows that $f|_U = \psi^{-1} f_{U,V} \varphi$, so that

$$U \cap f^{-1}(W) = (f|_U)^{-1}(W) \quad \text{is open in } X$$

To prove the second, let $X_{\underline{3}}$ be manifolds modelled over $E_{\underline{3}}$, and f_1, f_2 is smooth between X_i such that $f_2 \circ f_1$ makes sense. Since f_1 is smooth, there a pair of charts $(U_i, \varphi_i) \in X_i$ for $i = 1, 2$ about each $p \in X_1$

such that $(f_1)_{U_1,U_2}$ is C^p between open subsets.

$f_2(f_1(p))$ induces another pair of charts $(V_i, \psi_i) \in X_i$ for $i = 2, 3$. Since f_2 is smooth, it is continuous. $f_1^{-1} \circ f_2^{-1}(V_3)$ is open in X_1, and we can shrink all of our charts so that $f_2 f_1(U_1)$ is contained in V_3. Finally, because C^p morphisms between open subsets of Banach spaces is closed under composition, $f_{U_1 \cap f_1^{-1} f_2^{-1}(V_3), V_3}$ is smooth. ■

Remark 3.3: Morphisms between C^k, C^p manifolds

Let X be a C^k-manifold, and Y a C^p manifold, where $k, p \geq 0$. A morphism between X and Y is a map $f : X \to Y$ such that each point $p \in X$ admits a coordinate representation

$$f_{U,V} \in C^{\min(p,k)}(\hat{U}, \hat{V}) \tag{10}$$

If $\min(p, k) \geq 1$, then we define its differential as in def. 4.4 by treating both X and Y as $C^{\min(k,p)}$ manifolds.

Tangent spaces

In this section, all manifolds will be of class C^p for $p \geq 1$. The next question that we will address is taking derivatives of smooth maps between manifolds. There is no reason to demand C^p smoothness between maps, or even a C^p category of manifolds if we cannot borrow something more other than the morphisms on open sets.

Suppose U is an open subset of E and $f : U \to Y$ is C^p. The derivative $Df(x)$ is a linear map $E \to F$, not from U to F (U might not even be a vector space). This suggests the 'derivative' of a morphism $F : X \to Y$ between manifolds can in some sense be interpreted as the *ordinary derivative* of its coordinate representation $DF_{U,V}(\hat{p})$, adhering to our principle of using open sets.

But there is a problem with this 'derivative': it gives different values for different charts. With infinitely many charts in X and Y, this definition becomes useless. To see this, let X be a manifold modelled on E and $p \in X$. If $f : X \to Y$ is a morphism, and (U_1, φ_1), (U_2, φ_2) are charts defined about p such that the representations $f_{U_1,V}$ and $f_{U_2,V}$ are morphisms. Writing $p_i = \varphi_i(p)$, $U_{12} = U_1 \cap U_2$ and

$$\varphi_{12} = \varphi_2 \varphi_1^{-1} : \varphi_1(U_{12}) \to \varphi_2(U_{12}) \tag{11}$$

(because the map in eq. (11) goes from the domain U_1 to U_2), a simple computation yields eq. (12).

$$\begin{aligned} Df_{U_1,V}(p_1)(v) &= D(\psi f \varphi_2^{-1} \varphi_2 \varphi_1^{-1})(p_1)(v) \\ &= Df_{U_2,V}(p_2)\Big(D\varphi_{12}(p_1)(v)\Big) \\ &= Df_{U_2,V}(p_2) \circ D\varphi_{12}(p_1) \cdot (v) \end{aligned} \tag{12}$$

where $\cdot(v)$ denotes the evaluation at $v \in E$, and is assumed to be left associative over composition. The computation in eq. (12) suggests that interpreting the derivative by pre-conjugation is dependent on the chart being used to interpret the derivative. In fact, $D\varphi_{12}(p_1)$ can be replaced with any toplinear isomorphism on E (relabel $\varphi_2 = A\varphi_1$ where A is any linear automorphism on E), so the right hand side of eq. (12) can be interpreted as $Df_{U_2,V}(p_2)(w)$ where w is any vector in E.

Definition 4.1: Concrete tangent vector

Suppose $k \geq 1$, X a C^k-manifold on E, and $p \in X$. If (U, φ) is any chart containing p, for each $v \in E$ we call (U, φ, p, v) a *concrete tangent vector at p* that is *interpreted* with respect to the chart (U, φ). The disjoint union of concrete tangent vectors, as shown in eq. (13)

$$T_{(U,\varphi,p)}X = \bigcup_{v \in E} \{(U, \varphi, p, v)\} \cong E \tag{13}$$

is called the *concrete tangent space at p* interpreted with respect to (U, φ); and it inherits a TVS structure from E.

Fix a point p in a manifold X. Suppose (U_i, φ_i) are charts containing p, from eq. (12) there exists a natural (toplinear) isomorphism between the concrete tangent spaces, namely

$$(U_1, \varphi_1, p, v_1) \sim (U_2, \varphi_2, p, v_2) \quad \text{iff} \quad v_2 = D\varphi_{12}(p_1)(v_1) \tag{14}$$

where $p_i = \varphi_i(p)$. The right member of eq. (14) is the derivative of a transition map — which is a toplinear automorphism on E. Hence $D\varphi_{12}(p_1)$ defines a toplinear isomorphism between $T_{(U_1,\varphi_1,p)}X$ and $T_{(U_2,\varphi_2,p)}X$. With this, we define the primary object of our study.

Definition 4.2: Tangent vector

A *tangent vector* (or an *abstract* tangent vector) at p is defined as an equivalence class of concrete tangent vectors at p, under the relation in eq. (14).

Definition 4.3: Tangent space

The *tangent space* at p, denoted by T_pX is the set of all tangent vectors at p. It is toplinearly isomorphic to the model space E.

Definition 4.4: Differential of a morphism

Let X and Y be modelled on the spaces E and F. If f be a morphism between X and Y, and fix $p \in X$. We define a linear map, called the *differential of f at p* shown in eq. (15).

$$df(p) : T_pX \to T_{f(p)}Y \tag{15}$$

Whose action on tangent vectors is characterized by

- if (U, φ) and (V, ψ) are any pair of charts that satisfy the morphism condition in eq. (6) about p, and suppose
- $v \in T_pM$ is represented by $\Big(U, \varphi, p, \hat{v}\Big)$
- then $df(p)(v) \in T_{f(p)}Y$ is represented by $\Big(V, \psi, f(p), Df_{U,V}(\hat{p})(\hat{v})\Big)$

Alternatively, the diagram shown in fig. 1 commutes. We also write $df_p = df(p)$.

$$\begin{tikzcd} T_pX \arrow[r] \arrow[d, "df(p)"] & T_{(U,\varphi,p)}X \arrow[d, "Df_{U,V}(\hat{p})"] \\ T_{f(p)}Y \arrow[r] & T_{(V,\psi,f(p))}Y \end{tikzcd}$$

Figure 1: Differential of a morphism

Velocities

In the previous section, we motivated the definition of T_pX using the computation of the derivative of a morphism from X. Dually, the tangent space allows us compute the derivatives of morphisms into X in a coordinate independent manner.

Definition 5.1: Curve

Let $J_\varepsilon = (-\varepsilon, +\varepsilon)$ be an open interval in $\mathbb{R}$ containing the origin. Proposition 2.3 tells us J_ε is a manifold. A morphism $\gamma : J_\varepsilon \to X$ is called a *curve in* X, and $\gamma(0)$ is called the *starting point of* γ.

Remark 5.1: Omission of chart in concrete representation

If p is a point on a manifold X, and $v \in T_pX$ is represented by $(U, \varphi, p, \hat{v})$, we write

$$(U, \hat{v}) = (\hat{p}, \hat{v}) = \hat{v} = (U, \varphi, p, \hat{v}) \tag{16}$$

Remark 5.2: Standard representation of tangent vectors

If X is an open subset of E, and $p \in X$, we identify a tangent vector $v \in T_pX$ by its *standard representation*. Instead of using a $\hat{v}$, we use $\overline{v}$.

$$(X, \mathrm{id}_X, p, \overline{v}) = (X, \overline{v}) = (X, \hat{v}) \quad \text{is a representation of } v \in T_pX \tag{17}$$

Definition 5.2: Velocity of a curve

Let γ be a curve in X and $t \in J_\varepsilon$. We denote the *velocity* of a curve γ at $t = t_0$ by $\gamma'(t_0)$; which is defined in eq. (18).

$$\gamma'(t_0) = \left[D\gamma_{J_\varepsilon,V}(t_0)(\overline{1})\right] \tag{18}$$

where $(J_\varepsilon, \mathrm{id}_{J_\varepsilon}, t_0, \overline{1})$ is a concrete tangent vector within $T_{t_0}J_\varepsilon$.

Equation (18) might seem arbitrary at first, but we must emphasize that this is the most natural interpretation of a velocity, encodes much of the geometric information of a tangent vector. We will revisit this topic when we discuss exterior differentiation.

> **Proposition 5.1: Tangent vectors are velocities**
>
> Let p be a point on a manifold X. For every tangent vector $v \in T_pX$, there exists a curve starting at p whose velocity is v.

Proof. Find a chart (U) in X where $\hat{p} = 0$. Such a chart exists, because translations and dilations are C^p isomorphisms. If the tangent vector v has interpretation $\hat{v}$ in U, there exists $\varepsilon > 0$ so small that the range of $\hat{\gamma}$, as defined eq. (19), lies in $\hat{U}$

$$\hat{\gamma} : J_\varepsilon \to \hat{U} \quad \gamma(t) = \int_0^t \hat{v} dt \tag{19}$$

$\hat{\gamma}$ is a curve in $\hat{U}$ starting at $\hat{p}$ with velocity $\hat{v}$. Defining γ as the composition of $\hat{\gamma}$ with the chart inverse finishes the proof. ∎

Splitting

Recall: if W is a vector space and W_1, W_2 are linear subspaces of V. W_2 is the vector space complement of W_1 (resp. with the indices reversed) if

$$W_1 + W_2 = W, \quad \text{and} \quad W_1 \cap W_2 = 0$$

We sometimes refer to the vector space complement of W_1 as its *linear complement*.

> **Definition 6.1: Splitting in E**
>
> A linear subspace E_1 splits in E if both E_1 and its vector space complement E_2 are closed, and the addition map $\theta : E_1 \times E_2 \to E$ given by
>
> $$\theta(x, y) = x + y \quad \text{is a toplinear isomorphism.}$$

> **Definition 6.2: Splitting in $L(E, F)$**
>
> A continuous, injective linear map $\lambda \in L(E, F)$ *splits* iff its range splits in F.

Every finite dimensional or finite codimensional linear subspace of E splits. And if E itself is finite dimensional, then every linear subspace of E splits. An alternative definition of def. 6.2 is as follows: an map $\lambda \in L(E, F)$ splits iff there exists a toplinear isomorphism $\theta : F \to F_1 \times F_2$ such that λ composed with α induces a toplinear isomorphism from E onto $F_1 \times 0$ — which we identify with F_1.

If E and F are finite dimensional (so $E = \mathbb{R}^n$ and $F = \mathbb{R}^m$ respectively), def. 6.2 refers to the familiar matrix canonical form in eq. (20). Definitions 6.3 and 6.4 are the infinite-dimensional, manifold analogues of eqs. (20) and (21).

$$A_{injective} = \begin{bmatrix} \mathrm{id}_{m \times m} \\ 0_{n-m \times m} \end{bmatrix} \tag{20}$$

$$A_{surjective} = \begin{bmatrix} \mathrm{id}_{n \times n} & 0_{n \times m-n} \end{bmatrix} \tag{21}$$

Definition 6.3: Immersion

A morphism $f \in \mathrm{Mor}(X, Y)$ is an *immersion at p* if there exists a coordinate representation about $f_{U,V}$ such that

$$Df_{U,V}(\hat{p}) \quad \text{is injective and splits.} \tag{22}$$

The morphism f is called an immersion if eq. (22) holds at every p.

Definition 6.4: Submersion

A morphism $f \in \mathrm{Mor}(X, Y)$ is a *submersion at p* if there exists a coordinate representation about $f_{U,V}$ such that

$$Df_{U,V}(\hat{p}) \quad \text{is surjective and its kernel splits.} \tag{23}$$

The morphism f is called a submersion if eq. (23) holds at every p.

Definition 6.5: Embedding

A morphism $f \in \mathrm{Mor}(X, Y)$ is an *embedding* if it is an immersion and a homeomorphism onto its range.

Definition 6.6: Toplinear subspace

Let E be a Banach space, a *toplinear subspace of E* is a closed linear subspace E_1 which splits in E.

Submanifolds

Before we state the definition of a submanifold, it is important to recapitulate the construction of a manifold X.

1. Given a non-empty set X and an atlas modelled on a space E.
2. The purpose of each chart in the atlas is to borrow open subsets $\hat{U} \overset{\circ}{\subseteq} E$. If we single out a single chart, **the construction is entirely topological**. It is of little importance *how* the individual chart domains U are mapped onto $\hat{U}$,
3. Each chart is in **bijection with its range**, which is an open subset of E, and
4. the transition maps $\varphi_{\alpha\beta} = \varphi_\beta \varphi_\alpha^{-1}$ are **morphisms between open subsets of E**.

If $(U, \varphi) \in X$ is a chart whose domain intersects S, the question then becomes: Is it possible to modify (U, φ) so that it becomes a chart modelled on E_1? If we restrict φ onto $U \cap S$, its range is still an open subset of E. We can assume $\varphi(U \cap S) \subseteq E$ is constant on the linear complement of E_1, that way $\varphi|_{U \cap S}$ will be a bijection.

The range of the restricted chart is still a subset of E, and not E_1. An easy fix to this would be to require E_1 **to split in** E (and shrinking U using a basis argument). Let θ be a toplinear isomorphism between E and $E_1 \times E_2$, and we obtain eq. (24).

$$\theta\varphi(S \cap U) = \hat{U}_1 \times a_2 \quad \text{where} \quad \hat{U}_1 \overset{\circ}{\subseteq} E_1 \text{ and } a_2 \in E_2 \tag{24}$$

Identifying $\hat{U}$ with $\theta(\hat{U})$, and requiring $U_1 \times a_2$ to be in $\theta(\hat{U})$, we arrive at the following definition.

Definition 7.1: Submanifold

Let X be a manifold, and S a subset of X. We call S a *submanifold* of X if there exist split subspaces E_1, E_2 of E; such that, every $p \in S$ is contained in the domain of some chart (U, φ) in X. Where

$$\varphi : U \to \hat{U} \cong \hat{U}_1 \times \hat{U}_2, \quad \text{where} \quad \hat{U}_i \mathring{\subseteq} E_i \quad \text{for} \quad i = \underline{2}. \tag{25}$$

and there exists an element $a_2 \in \hat{U}_2$

$$\varphi(U \cap S) = \hat{U}_1 \times a_2 \tag{26}$$

We call a chart satisfying eqs. (25) and (26) a *slice chart* of S; to simplify what follows, we write $\varphi^i = \mathrm{proj}_i \varphi$ for $i = \underline{2}$ for any slice chart (U). Given that proj_i is a morphism between open subsets of Banach spaces, φ^i is again a morphism. In particular, φ^1 is a bijection from $U^s = U \cap S$ onto $\hat{U}_1$; the latter being an open subset of E_1. To show S is indeed a manifold it remains to show the collection of charts in eq. (27) forms a C^p atlas modelled E_1, which we will prove in prop. 7.1

$$\mathcal{A} = \Big\{ (U^s, \varphi^s) = (U^s, \varphi^1),\ (U, \varphi) \text{ is a slice chart of } S \Big\} \tag{27}$$

Proposition 7.1: Structure of a submanifold

If S is a submanifold of X, eq. (27) defines a C^p atlas over the space E_1. The manifold S has a topology that coincides with the subspace topology. Furthermore, the inclusion map $\iota_S : S \to X$ is a morphism and an embedding.

Proof. Each of the charts in eq. (27) is in bijection with an open subset of E_1. Let $(U^s_\alpha, \varphi^s_\alpha)$ and $(U^s_\beta, \varphi^s_\beta)$ be overlapping charts in $\mathcal{A}$. Using θ as our toplinear isomorphism from E onto $E_1 \times E_2$ as usual.

- By eq. (25), $(U^s_\alpha, \varphi^s_\alpha)$ is induced by a chart $(U_\alpha, \varphi_\alpha) \in X$.

$$\varphi_\alpha : U_\alpha \to \hat{U}_\alpha \mathring{\subseteq} E \quad \text{which splits into} \quad \theta(\hat{U}_\alpha) = \hat{U}^s_\alpha \times \hat{U}_{2,\alpha}$$

  such that $\hat{U}^s_\alpha \mathring{\subseteq} E_1$ and $\hat{U}_{2,\alpha} \mathring{\subseteq} E_2$. Similarly for β as well.

- There exists elements $a_2 \in \hat{U}_{2,\alpha}$, (resp. $b_2 \in \hat{U}_{2,\beta}$) where

$$\theta\varphi_\alpha(U^s_\alpha) = \hat{U}^s_\alpha \times a_2 \quad \text{resp. } \beta.$$

Note 7.1

Let us define $U^s_{\alpha\beta} = U^s_\alpha \cap U^s_\beta$, we will show lem. 7.1.

Lemma 7.1

Both $\varphi^s_\alpha(U^s_{\alpha\beta})$ and $\varphi^s_\beta(U^s_{\alpha\beta})$ are open subsets of E_1.

Proof of lem. 7.1. We can factor $U^s_{\alpha\beta} = (U^s \cap U_\alpha) \cap U_{\alpha\beta}$, and because φ_α is a bijection, we have

$\varphi^s_\alpha(U^s_{\alpha\beta}) = \mathrm{proj}_1\, \theta\Big(\varphi_\alpha(U^s \cap U_\alpha) \cap \varphi_\alpha(U_{\alpha\beta})\Big)$.

θ and proj_1 are both open maps, and because $W \triangleq \varphi_\alpha(U_{\alpha\beta})$ is open in E: $\theta(\varphi_\alpha(U^s \cap U_\alpha) \cap W)$ splits into a subset of $\hat{U}^s_\alpha \times a_2$,

$$\mathrm{proj}_1\, \theta(\varphi_\alpha(U^s \cap U_\alpha) \cap W) = \mathrm{proj}_1(\text{Open subset of } E_1 \times a_2)$$

which is open in E_1. ■

The diagram in fig. 2 provides a summary.

$$\begin{tikzcd}
U^s_{\alpha\beta} \arrow[r, "\varphi_\alpha"] & \varphi_\alpha(U^s_{\alpha\beta}) \arrow[r, "\theta"] \arrow[d, "\varphi_{\alpha\beta}"] & \varphi_\alpha(U^s_{\alpha\beta})_1 \times a_2 \arrow[r, "\mathrm{proj}_1"] \arrow[d, "\theta\varphi_{\alpha\beta}\theta^{-1}"] & \varphi^s_\alpha(U^s_{\alpha\beta}) \\
U^s_{\alpha\beta} \arrow[r, "\varphi_\beta"] & \varphi_\beta(U^s_{\alpha\beta}) \arrow[r, "\theta"] & \varphi_\beta(U^s_{\alpha\beta})_1 \times b_2 \arrow[r, "\mathrm{proj}_1"] & \varphi^s_\beta(U^s_{\alpha\beta})
\end{tikzcd}$$

Figure 2: Overlap of slice charts

Identifying a_2 (resp. b_2) with the constant function $(p \mapsto a_2)$ for $p \in U^s_\alpha$, we get eq. (28).

$$\varphi^s_\alpha \times a_2 = \theta \circ \varphi_\alpha \quad \text{resp. } \beta \tag{28}$$

Suppressing the restrictions onto domains, the transition map is given by the composition of maps in eq. (29).

$$\varphi^s_\beta \circ (\varphi^s_\alpha)^{-1} = \mathrm{proj}_1\, \theta\varphi_\beta\varphi_\alpha^{-1}\theta^{-1}\, \mathrm{proj}_1^{-1} : \varphi^s_\alpha(U^s_{\alpha\beta}) \to \varphi^s_\beta(U^s_{\alpha\beta}) \tag{29}$$

which is clearly a bijection. It suffices to show eq. (29) is a morphism between open subsets of E_1. Let $a_2 : \varphi^s_\alpha(U^s_{\alpha\beta}) \to \hat{U}_{2,\alpha}$, which is the constant function a_2 and hence a morphism.

The product $(\mathrm{id}_{\varphi^s_\alpha(U^s_{\alpha\beta})} \times a_2) = \mathrm{proj}_1^{-1}$ is a morphism into $\varphi^s_\alpha(U^s_{\alpha\beta}) \times \hat{U}_{2,\alpha}$. The inverse of θ is an open morphism, and the terms $\varphi_\beta\varphi_\alpha^{-1}$ combine into the transition map $\varphi_{\alpha\beta}$ in X (up to a restriction on an open set). Equation (29) then reads

$$\varphi^s_\beta \circ (\varphi^s_\alpha)^{-1} = \mathrm{proj}_1\, \theta\varphi_{\alpha\beta}\theta^{-1}(\mathrm{id}_{\varphi^s_\alpha(U^s_{\alpha\beta})} \times a_2) \tag{30}$$

which is a morphism between open subsets. Reversing the roles of α, β shows that eq. (29) is an isomorphism. Therefore the collection of charts in eq. (27) forms an atlas of S.

Let us use $\iota_S : S \to X$ to represent the inclusion map and consider a point $p \in S$. It is always possible to identify a slice chart (U, φ) within X that contains $p = \iota_S(p)$ in its domain. By definition of the atlas on S, this induces a truncated chart (U^s, φ^s).

Observing that $\iota_S(U^s) = \iota_S(U \cap S)$ lies within (U, φ), the morphism criteria in eq. (6) is satisfied. Computing the coordinate representation of ι_S, we obtain eq. (31).

$$(\iota_S)_{U^s, U} = \varphi\iota_S(\varphi^s)^{-1} = \mathrm{id}_{\hat{U}_1} \times a_2 \tag{31}$$

Equation (31) shows that the coordinate representation of ι_S is a local isomorphism. Since the inclusion map is a bijection and continuous, and the coordinate representation of ι_S^{-1} is simply the inverse eq. (31); ι_S^{-1} is a morphism and therefore continuous. The manifold topology of S coincides with its subspace topology.

At last, the inclusion map ι_S has coordinate representation eq. (31). Computing its ordinary derivative we obtain eq. (32).

$$D(\iota_S)_{U^s,U}(\hat{p}) : T_{(U^s,\varphi^s,p)} \longrightarrow T_{(U,\varphi,p)} \quad \text{and} \quad D(\iota_S)_{U^s,U}(\hat{p}) = \mathrm{id}_{E_1} \times 0 \tag{32}$$

which is a toplinear morphism between concrete tangent spaces and has a simple representation of 'adding zeroes' (see def. 6.2) at the end of a vector $\hat{v} \in E_1$ — which is to say: **the differential of ι_S is injective and splits in E**. Therefore ι_S is an embedding. ■

Remark 7.1: Pairs of slice charts

Proposition 7.1 shows every point $p \in S$ is in the domain of a slice chart in S, and the domain of the chart in X that induces the slice chart — whose inclusion map satisfies eqs. (31) and (32). If p is a point on a submanifold S, we refer to a *pair of slice charts* containing p as the pair (U^s, φ^1) and (U, φ) in the structure of S and X.

Definition 7.2: Exterior tangent space of S

The *exterior tangent space* of a point $p \in S$ is the image of T_pS under $d\iota_S(p)$,

$$T_p^{ext}S = d\iota_S(p)(T_pS) \tag{33}$$

which is a toplinear subspace of T_pX.

Chapter 2: Vector Bundles

Vector Bundles

Let X be a class C^p manifold modelled on a space E, and F another Banach space. Our goal in this section is to construct the vector bundle of a manifold, which has the following desirable properties.

- The vector bundle W embeds X into itself as a submanifold.
- At each point $p \in X$, we attach a F space structure exclusive to each p like the tangent space T_pX.
- W locally isomorphic to the product space $U \times F$, where $U \overset{\circ}{\subseteq} X$, and
- a subset of the morphisms $A : X \to W$ locally resemble morphisms $U \to U \times F$.

Definition 1.1: Coproduct of fibers

Suppose for each p, the set W_p is toplinearly isomorphic to F at for each p, then we call W_p an *F-fiber* at p. The set-theoretic coproduct of all such W_p as in eq. (34) is called a *coproduct of F-fibers modelled over X*.

$$W = \coprod_{p \in X} W_p \quad \text{comes with} \quad \pi : W \to X, \quad \pi^{-1}(p) = W_p \tag{34}$$

where π is a surjection onto X and is called the *canonical projection*.

It turns out the natural way of making W a manifold would be to steal open sets from *both* E and F — in this case, sets of the form $\hat{U} \times F$. We sometimes write $\widetilde{U}$ instead of $\pi^{-1}(U)$ for brevity, and $\widetilde{p}$ in place of $\pi^{-1}(p)$. The next few definitions should feel familiar.

Definition 1.2: Local trivialisation

Let W be as in eq. (34). A *local trivialisation* of W is a tuple $(\widetilde{U}, \Phi)$, such that the diagram in fig. 3 commutes, and

- $U \subseteq X$ is open in X, and for each $p \in U$,
- $\Phi|_{\widetilde{p}}$ is in bijection with $W_p = F$.

Definition 1.3: Compatibility between trivialisations

Let $(\widetilde{U}, \Phi)$ and $(\widetilde{V}, \Psi)$ be local trivialisations of W, they are called C^k-compatible if $U \cap V = \varnothing$, or both of the following hold:

- for each $p \in U \cap V$ — the restriction of $\Psi \circ \Phi^{-1}$ onto the fiber of p — $(\Psi \circ \Phi^{-1})|_{\widetilde{p}}$ is a toplinear isomorphism, and
- the map $\theta : U \cap V \to L(F, F)$ as defined by eq. (35), is a C^k morphism into the Banach space $L(F, F)$.

$$\theta(p) = (\Psi \circ \Phi^{-1})|_{\widetilde{p}} \tag{35}$$

(equivalently, we can require θ be a C^k morphism into the open submanifold $\operatorname{Laut}(F)$).

Note: we assume that $0 \leq k \leq p$.

Definition 1.4: Trivialisation covering

Let W be a coproduct of F-fibers over X. A C^k *trivialisation covering of* W is a collection of pairwise C^k-compatible local trivialisations $\{(\widetilde{U}_\alpha, \Phi_\alpha)\}$ where $\{U_\alpha\}$ is an open cover of X.

Definition 1.5: Vector bundle

Let X be a C^p manifold over E, and let F be a Banach space. An *F-vector bundle of rank k over X* is a coproduct of F-fibers modelled over X equipped with a **maximal C^k trivialisation covering**.

Remark 1.1: Maximality of trivialisation covering

One can easily verify the compatibility condition defines an equivalence relation, thus any C^k- trivialisation covering *determines* a maximal one.

Remark 1.2: Omissions for vector bundles

We say W is a *bundle over* X when it is unambiguous to do so.

$$\begin{tikzcd}
\widetilde{U} \arrow[r, "\Phi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\
U &
\end{tikzcd}$$

Figure 3: Local Trivialisation

The above definitions calls for some commentary, our end goal is to make an arbitrary rank C^k vector bundle W a C^k manifold. Open sets will still be our primary topological data. To ensure that W is as similar to X as possible, the eventual manifold structure we will put on W will **embed the structure of X into W**. We are repeating (essentially) the same argument as in the submanifold case but with the roles of X and the submanifold S reversed.

Suppose we have a structure on W, then $X = \bigcup_{p \in X} \{p\} \times 0$ is a submanifold of the W as E splits in the product space $E \times F$. Let us motivate a couple of the requirements above.

Definition 1.2

- U is required to be open because W inherits part of the topology, and hence the charts in E whose domain is a subset of U,
- The second requirement implies **each Φ is in bijection with $\Phi(\widetilde{U}) = U \times F$, which is open in** $E \times F$, which will allow us to construct bijections with open subsets of the model space $E \times F$. Furthermore, eq. (36) holds for an arbitrary $V \subseteq X$.

$$\Phi|_{\pi^{-1}(U \cap V)} \quad \text{is a bijection onto } U \cap V \times F \tag{36}$$

Definition 1.3

- The overlap restricts to a toplinear isomorphism on each fiber because, it allows us **to quotient out the effects of the trivialisation transitions**, by rehearsing the same 'coproduct and quotient' argument in Definitions 4.1 to 4.3.
- The requirement that the mapping eq. (35) is a morphism is because we wish to **have control over the smoothness of morphisms $X \to W$**.

Suppose W is an F-vector bundle over X with the trivialisation covering $\{(\widetilde{U}^\alpha, \Phi_\alpha)\}$. For each α, we can cover U^α using chart domains $(U^\alpha_\beta, \varphi^\alpha_\beta)$ in X — without loss of generality, we can assume $U^\alpha_\beta \subseteq U^\alpha$ by restricting the chart domain and relabelling.

Similar to the construction of the induced atlas of a submanifold, given a 'piece' of the original manifold X — **instead of dropping the coordinates that correspond to E_2, we add an F-component to construct a bijection with an open subset of $E \times F$**. This is shown in eq. (37)

$$\widetilde{\varphi}^\alpha_\beta : \widetilde{U}^\alpha_\beta \longrightarrow \hat{U}^\alpha_\beta \times F \quad \text{defined by} \quad \widetilde{\varphi}^\alpha_\beta = \left(\varphi^\alpha_\beta \times \mathrm{id}_F\right) \circ \Phi_\alpha|_{\widetilde{U}^\alpha_\beta} \tag{37}$$

Remark 1.3: Hats and wiggles

Here, $\widetilde{U}^\alpha_\beta$ should be interpreted as the inverse image of the open set U^α_β through π. Similarly, $\hat{U}^\alpha_\beta$ is the image of U^α_β through φ^α_β.

The collection of charts in eq. (38) cover W with their chart domains, and each chart is in bijection with an open subset of $E \times F$.

$$\mathcal{A} = \left\{(\widetilde{U}^\alpha_\beta,\ \widetilde{\varphi}^\alpha_\beta),\ (\widetilde{U}^\alpha, \Phi_\alpha) \text{ is in the trivialisation covering of } W.\right\} \tag{38}$$

Proposition 1.1: Structure of a vector bundle

Let X be a C^p manifold modelled over E. If W is a C^k vector bundle modelled on F over the manifold X, then W is a C^k manifold modelled on the product space $E \times F$. Furthermore:

1. The *canonical projection* $\pi : W \to X$ is a morphism and a submersion.
2. X is C^k isomorphic to a submanifold of W

Proof. Suppose we are given two charts in eq. (38), $(\widetilde{U}^{\alpha_1}_{\beta_1})$, and $(\widetilde{U}^{\alpha_2}_{\beta_2}, \widetilde{\varphi}^{\alpha_2}_{\beta_2})$. We first prove that $\widetilde{\varphi}^{\alpha_1}_{\beta_1}(\widetilde{U}^{\alpha_1}_{\beta_1} \cap \widetilde{U}^{\alpha_2}_{\beta_2})$ is open in $E \times F$.

$$\begin{aligned}
\widetilde{\varphi}^{\alpha_1}_{\beta_1}(\widetilde{U}^{\alpha_1}_{\beta_1} \cap \widetilde{U}^{\alpha_2}_{\beta_2}) &= \left[(\varphi^{\alpha_1}_{\beta_1} \times \mathrm{id}_F) \circ \Phi_{\alpha_1}\right] (\widetilde{U}^{\alpha_1}_{\beta_1} \cap \widetilde{U}^{\alpha_2}_{\beta_2}) \\
&= \left[(\varphi^{\alpha_1}_{\beta_1} \times \mathrm{id}_F) \circ \Phi_{\alpha_1}\right] (\pi^{-1}(U^{\alpha_1}_{\beta_1} \cap U^{\alpha_2}_{\beta_2})) \\
&= (\varphi^{\alpha_1}_{\beta_1} \times \mathrm{id}_F)\Big((U^{\alpha_1}_{\beta_1} \cap U^{\alpha_2}_{\beta_2}) \times F\Big) && \text{by eq. (36)}
\end{aligned}$$

Suppressing restrictions and computing the chart transistions in eq. (39),

$$\widetilde{\varphi}^{\alpha_2}_{\beta_2}\left(\widetilde{\varphi}^{\alpha_1}_{\beta_1}\right)^{-1} = (\varphi^{\alpha_2}_{\beta_2} \times \mathrm{id}_F) \circ \Phi_{\alpha_2}\Phi^{-1}_{\alpha_1} \circ \left((\varphi^{\alpha_1}_{\beta_1})^{-1} \times \mathrm{id}_F\right) \tag{39}$$

which is clearly a bijection. And it is not hard to see that eq. (39) can be factored into

$$\widetilde{\varphi}^{\alpha_2}_{\beta_2}\left(\widetilde{\varphi}^{\alpha_1}_{\beta_1}\right)^{-1}(x,v) = \left(\varphi^{\alpha_1\alpha_2}_{\beta_1\beta_2}(x),\ \left[\theta\circ(\varphi^{\alpha_1}_{\beta_1})^{-1}\right](x)(v)\right) \tag{40}$$

for any $x \in \varphi^{\alpha_1}_{\beta_1}(U^{\alpha_1\alpha_2}_{\beta_1\beta_2})$ and $v \in F$. **From eq. (40), it should now be clear why we demand $k \leq p$** . The mapping in the second coordinate within eq. (40) can be reduced to a composition with the evaluation map $\mathbf{E} : \mathrm{Laut}(F) \times F \to F$.

$$\left[\theta\circ(\varphi^{\alpha_1}_{\beta_1})^{-1}\right](x)(v) = \mathbf{E}\circ\left([\theta\circ(\varphi^{\alpha_1}_{\beta_1})^{-1}]\times \mathrm{id}_F\right) \tag{41}$$

Since $\mathbf{E}$ is continuous and bilinear, eq. (41) and hence eq. (39) describes a C^k mapping between open subsets of Banach spaces. It is a morphism, and reversing the roles of the two charts proves its inverse is again a morphism.

To prove π is a submersion, recall W is the set-theoretic disjoint union of F-fibers. Every element in W can be represented by $(x,v) \in X \times F$. **We will identify elements of W as elements in $X \times F$. However, this is not a manifold isomorphism**.

Fix $(x,v) \in W$, it is in the domain of some chart $(\widetilde{U}^{\alpha}_{\beta}, \widetilde{\varphi}^{\alpha}_{\beta})$. The π-image of the chart domain is $\pi\pi^{-1}(U^{\alpha}_{\beta}) = U^{\alpha}_{\beta}$ because π is surjective. Using eq. (37) and the diagram found in fig. 3, the coordinate representation of π becomes

$$\begin{aligned}\pi_{(\widetilde{U}^{\alpha}_{\beta}, U^{\alpha}_{\beta})} &= \varphi^{\alpha}_{\beta}\circ\pi\circ\Phi^{-1}_{\alpha}\circ\left((\varphi^{\alpha}_{\beta})^{-1}\times\mathrm{id}_F\right)\\ &= \varphi^{\alpha}_{\beta}\circ\mathrm{proj}_1\circ\left((\varphi^{\alpha}_{\beta})^{-1}\times\mathrm{id}_F\right)\\ &= \mathrm{proj}_1(\mathrm{id}_{\hat{U}^{\alpha}_{\beta}}\times\mathrm{id}_F)\end{aligned} \tag{42}$$

We can differentiate both sides of eq. (42) and if we write $\hat{U} = \hat{U}^{\alpha}_{\beta}$, we obtain eq. (43).

$$D\,\mathrm{proj}_1(\mathrm{id}_{\hat{U}}\times\mathrm{id}_F)(x,v) = \mathrm{proj}_1 \in L(E\times F; E) \quad \forall x\in\hat{U},\ v\in F \tag{43}$$

which means π submersion.

Finally, the subset $X \times 0 \subseteq W$ is easily shown to be a submanifold of W, and is isomorphic to X by dropping the F coordinate and retracing the argument we made in constructing the structure of W. ■

Remark 1.4: Pair of bundle charts

If X is a manifold and W a vector bundle over X, the charts realizing the representations of π in eqs. (42) and (43) are called *bundle charts*.

Definition 1.6: Section of a vector bundle

Let W be a bundle over a manifold X. A *section* of W is a morphism $\sigma \in \mathrm{Mor}(X, W)$ such that the diagram in fig. 4a commutes, which is synonymous with $\pi\sigma = \mathrm{id}_X$. A *local section* of W is a morphism $\sigma : U \to W$ where $U \overset{\circ}{\subseteq} X$ is viewed as a submanifold and $\pi\sigma = \mathrm{id}_U$. We denote the sections of W by

$\Gamma(W)$.

$$\Gamma(W) = \Big\{\sigma : X \to W,\ \sigma \text{ is a section of } W.\Big\} \tag{44}$$

The *zero section* of W is the section $\sigma(p) = 0 \in W_p$ for every $p \in X$. If σ is a section of W, $\operatorname{supp}(\sigma)$ refers to the *support* of σ, and is defined in eq. (45).

$$\operatorname{supp}(\sigma) = \overline{\{p \in X,\ \sigma(p) \neq 0\}} \tag{45}$$

Remark 1.5: Bundle coordinates

Let X and W be as in def. 1.6, and suppose σ is a section on W. Using a pair of bundle charts, $(U) \in X$ and $(\widetilde{U}) \in W$, we define the *bundle coordinates of* σ

$$\sigma_{U,\widetilde{U}} = \widetilde{\varphi} \circ \sigma \circ \varphi^{-1} \tag{46}$$

expanding the induced chart on W within eq. (46) reads

$$\sigma_{U,\widetilde{U}} = (\varphi \times \mathrm{id}_F) \circ \Phi \circ \sigma \circ \varphi^{-1} \tag{47}$$

Refer to the diagram in fig. 4b. We will always use bundle charts when discussing the coordinate representation of a section, and we write

$$\sigma_U = \sigma_{U,\widetilde{U}} = \hat{\sigma}$$

Sections are precisely the morphisms into W whose coordinate representation resembles that of a graph: $\hat{\sigma} : \hat{U} \to \hat{U} \times F$ and because of this: we identify $\hat{\sigma}(\hat{p}) = (\hat{p}, v)$ with $v \in F$.

$$\begin{tikzcd} W \arrow[d, "\pi"] \\ X \arrow[u, bend left=60, "\sigma"] \end{tikzcd}$$

(a) Section of a bundle

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\Phi"] \arrow[d, "\pi"] & U \times F \arrow[dl, "\mathrm{proj}_1"] \arrow[d, "\widetilde{\varphi}"] \\ U \arrow[u, bend left=60, "\sigma"] \arrow[d, "\varphi"] & \hat{U} \times F \\ \hat{U} \arrow[ur, "\sigma_{U,\widetilde{U}}"] & \end{tikzcd}$$

(b) Local coordinates of a bundle section

Figure 4: Diagrams for bundle section and its local representation

Functorality

Let X and E_i be Banach spaces for $i = \underline{2}$. We discussed the difference in the role that a toplinear mapping $f \in L(E_1, E_2)$ plays in pushing points from $E_1 \to E_2$, and the role it plays from pushing *incoming maps* with source X from $L(X, E_1) \to L(X, E_2)$ by composing the incoming map with itself.

$$f : E_1 \to E_2 \quad \text{and} \quad f_c : L(X, E_1) \to L(X, E_2) \tag{48}$$

where the c within f_c stands for composition. This is summarized in fig. 5

$$\begin{tikzcd} X \arrow[r, "\lambda"] \arrow[dr, "\lambda_c(f)"'] & E_2 \arrow[d, "f"] \\ & E_2 \end{tikzcd}$$

Figure 5: Functorality through post-composition

The two maps f and f_c often thought of as the same, and we can identify f with two separate actions. Here, it is not so obvious why we need the concept of functorality, a closer look at the **different roles that the same mathematical object can play** will surely motivate the above discussion.

Let E_i be Banach spaces, $f \in C^p(E_1, E_2)$ and $\lambda \in L(E_2, E_3)$.

- The derivative of f is a continuous map $Df : E_1 \to L(E_1, E_2)$.
- The derivative of λ (now 'identified with $\lambda \in C^p(E_2, E_3)$), is continuous constant map from E_2 to $L(E_2, E_3)$

$$D(\lambda \circ f) = \lambda \circ Df(x) \tag{49}$$

but what if we have a multi-linear map whose destination is E_1, and what about symmetric/alternating multi-linear maps, continuous maps, C^p morphisms? Should we let f take on all of those roles as well? Should we identify f with its adjoint as well? This is the first of the many problems.

The problem becomes even clearer when we look at maps between F-fibers. Fix a manifold X and F_i-bundles W^i over X for $i = \underline{2}$. Suppose $A : X \to W^1$ is a section on W^1, and $\lambda \in L(F_1, F_2)$.

At each point $p \in X$, our linear map λ can be identified with the linear map that acts between the fibers.

$$\lambda_p : W_p^1 \to W_p^2$$

which is toplinear hence a morphism. Our main problem is concerned with the conditions under which the composition λA — as defined in eq. (50) — is a morphism.

$$\lambda A : X \to W^2 \quad \text{and} \quad (\lambda A)(p) = \lambda_p(A_p) \in W_p^2 \tag{50}$$

Under what conditions can a morphism take on additional roles? The mapping λ_p on each tangent space is a C^p morphism in the manifold sense, and the morphisms that preserve the C^p smoothness of sections are called VB morphisms. Which we will define after some more category theory.

For the remainder of this section, let C_1 and C_2 be categories. We denote the objects of C_1 by E_i, and the objects of C_2 by F_i.

Definition 2.1: Functor

A correspondence θ between C_1 and C_2 is called a *functor* — which we denote by $\theta : C_1 \rightrightarrows C_2$ — if all of the following rules satisfied.

Ob1: θ maps objects in C_1 to objects in C_2. We write

$$\forall E \in Ob(C_1), \quad E^\theta = \theta(E) \in Ob(C_2) \tag{51}$$

Mor1: θ associates morphisms in C_1 to morphisms in C_2 that respects Ob1.

$$\forall f \in \mathrm{Mor}_{C_1}(E_1, E_2), \quad \theta(f) \in \mathrm{Mor}_{C_2}(E_1^\theta, E_2^\theta) \tag{52}$$

Mor2: Identity is associated with identity: $\theta(\mathrm{id}_E) = \theta(\mathrm{id}_{\theta(E)})$.

Mor3: Commutes with inversion: $\theta(f^{-1}) = \theta(f)^{-1}$ if the inverse of f exists.

Mor4: Functoral: $\theta(g \circ f) = \theta(g) \circ \theta(f)$.

Note 2.1: The Hom_X functor

We continue our discussion from fig. 5. Recall $\mathrm{Ban}_{\mathbb{R}}$ is the category of Banach spaces over $\mathbb{R}$, and we will refer to toplinear morphisms as morphisms for brevity. If X is an object in $\mathrm{Ban}_{\mathbb{R}}$, the Hom_X *functor* is a covariant functor between $\mathrm{Ban}_{\mathbb{R}}$ and $L(X, \cdot)$ — the space of toplinear mappings with source X such that

1. to each $E_i \in \mathrm{Ob}(\mathrm{Ban}_{\mathbb{R}})$ $\mathrm{Hom}_X(E_i) = L(X, E_i)$ — **the space of incoming morphisms with source X**, and
2. to each morphism $f \in L(E_1, E_2)$ another morphism between $L(X, E_1)$ and $L(X, E_2)$ — denoted by $(\mathrm{Hom}_X)f$.
3. The functor Hom_X converts **outgoing morphisms from E_1 to the redirection morphism of incoming morphisms with source X**.

Notice this is precisely what the diagram in fig. 5 describes.

Proposition 2.1: Hom_X functor is a functor

The Hom_X functor as defined in note 2.1 is a covariant functor.

Proof. Postponed. ■

Note 2.2: The tangent space functor

Let X be a C^1 manifold, we call the tuple (p, X) for $p \in X$ the *centering of X centered at p*. The category of pointed manifolds, denoted by Man_*. Its objects consist of all centerings across C^1 manifolds, and the morphisms in Man_* are called *pointed morphisms*.

If (q, Y) is another object in Man_*, a pointed morphism between (p, X) and (q, Y) is a tuple (p, f); where f is a manifold morphism between X and Y and $f(p) = q$. We sometimes write $f_p = (p, f)$ when it is clear.

The *tangent space functor*, denoted by $T : \mathrm{Man}_* \rightrightarrows \mathrm{Ban}_{\mathbb{R}}$ is a covariant functor where

- we define $T(p, X) = T_p X$ that takes a pointed C^1 manifold to its tangent space, and
- to each pointed C^1 morphism $f_p \in \mathrm{Mor}_{\mathrm{Man}_*}((p, X)\,(q, Y))$ we associate with the toplinear mapping
$$T f_p = df(p) : T_p X \to T_q Y \tag{53}$$

Proposition 2.2: Tangent space functor is a functor

The tangent space functor as defined in note 2.2 is a covariant functor.

Proof. Postponed. ■

We leave the verification that T satisfies the properties in def. 2.1 as an exercise. Dual to the concept of a functor is that of the cofunctor, which — for our purposes — captures the idea of the toplinear adjoint.

Definition 2.2: Cofunctor

Let C_1 and C_2 be categories. A correspondence $\eta : C_1 \leftrightarrows C_2$ is called a *cofunctor* (or a contravariant functor) if all of the following rules are satisfied.

Ob: η maps objects in C_1 to objects in C_2. We write $E^\lambda = \lambda(E) \in \mathrm{Ob}(C_2)$ for every $E \in \mathrm{Ob}(C_1)$.

Mor1: η associates morphisms in C_1 to morphisms in C_2 that respects Ob1.

$$\forall f \in \mathrm{Mor}_{C_1}(E_1, E_2), \quad \eta(f) \in \mathrm{Mor}_{C_2}(E_2^\eta, E_1^\eta) \tag{54}$$

Mor2: Identity is associated with identity: $\eta(\mathrm{id}_E) = \eta(\mathrm{id}_{\eta(E)})$.

Mor3: Commutes with inversion: $\eta(f^{-1}) = \eta(f)^{-1}$ if the inverse of f exists.

Mor4: Cofunctoral: $\eta(g \circ f) = \eta(f) \circ \eta(g)$.

Remark 2.1: Cofunctors are opposite to functors

The cofunctor η reverses the arrows a morphism f. Refer to fig. 6 for a comparison between eq. (54) and eq. (52).

Note 2.3: The Hom^X cofunctor

Let $X \in \mathrm{Ob}(\mathrm{Ban}_{\mathbb{R}})$, it defines a cofunctor from $\mathrm{Ban}_{\mathbb{R}} \leftrightarrows L(\cdot, X)$ where $L(\cdot, X)$ is the space of toplinear mappings whose destination is X.

1. to each $E_i \in \mathrm{Ob}(\mathrm{Ban}_{\mathbb{R}})$ $\mathrm{Hom}^X(E_i) = L(E_i, X)$ — **the space of outgoing morphisms with destination X**, and
2. to each morphism $f \in L(E_1, E_2)$ another morphism between $L(E_1, X)$ and $L(E_2, X)$ — denoted by $(\mathrm{Hom}^X)f$.
3. The functor Hom^X converts **outgoing morphisms from E_1 to the precomposition morphism which acts on morphisms with destination X**.

$$\begin{tikzcd}
\eta(E_1) & E_1 \arrow[l, "\eta"'] \arrow[r, "\theta"] \arrow[d, "f"] & \theta(E_1) \arrow[d, "\theta(f)"] \\
\eta(E_2) \arrow[u, "\eta(f)"] & E_2 \arrow[l, "\eta"'] \arrow[r, "\theta"] & \theta(E_2)
\end{tikzcd}$$

Figure 6: Functor θ vs. cofunctor η comparison

Tangent Bundle

Definition 3.1: Tangent Bundle

Definition 3.2: Cotangent bundle

Note 3.1 provides an example of a tangent bundle.

Note 3.1: Tangent Bundle

Let X be a C^p manifold with $p \geq 1$, so that the tangent space at every point is defined. If $p \in (U_i, \varphi_i)$ for $i = 1, 2$. Then φ_{12} is a C^p isomorphism between $\varphi_1(U_{12})$ and $\varphi_2(U_{12})$; **whose derivative is a C^{p-1} map into** $\mathrm{Laut}(E)$ **that encodes the transformation between the concrete tangent spaces**. In the notation of eq. (11), this means

$$x \mapsto D\varphi_{12}(x) \quad \text{is in } C^{p-1}(\hat{U}_{12}, \mathrm{Laut}(E))$$

In fact, the tangent bundle $TX \triangleq \coprod_{p \in X} T_pX$ is a C^{p-1} vector bundle (modelled on E) over X. If (U, φ) is a chart in X, it induces a local trivialisation on TX by taking each tangent vector $v \in T_pX$ to its concrete representation $(p, \hat{v}) \in X \times E$.

$$\Phi : \widetilde{U} \to U \times E \quad \text{and} \quad \Phi(v) = (p, \hat{v}) \tag{55}$$

where $(U, \varphi, p, \hat{v})$ is a concrete representation of $v \in T_pX$.

Chapter 3: Coordinates

Introduction

In the previous chapters, a chart (U, φ) was often equated with its domain. We will now express a concrete tangent vector as $(\hat{p}, \hat{v})$, omitting any reference to the chart or its domain.
Let X be a manifold and F a Banach space. Consider a morphism $f \in \text{Mor}(X, F)$ and fix a point $p \in X$, and write $q = f(p)$. By adopting the canonical interpretation $\overline{w}$ for a tangent vector $w \in T_qF$ (as discussed in remark 5.1), we

- reinterpret the differential at p df_p as a linear map from T_pX to F,
- always use the standard chart (id_F, F) so that $\hat{f} = f_{U,F}$.

In this context, morphisms into $\mathbb{R}$ almost serve as test functions in the framework of distribution theory. This requires a definition.

Definition 1.1: Function on X

Let X be a manifold of class C^p over $\mathbb{R}^n$ for $n, p \geq 1$. A *function* on X is a morphism $f : X \to \mathbb{R}$, where $\mathbb{R}$ should be interpreted as a manifold. We denote the commutative ring of functions on X by $C^p(X, \mathbb{R})$ or $C^p(X)$. If U is an open subset of X, its functions are denoted by $C^p(U, \mathbb{R})$ or $C^p(U)$.

For the rest of this chapter, assume all manifolds to be C^p-manifolds over $\mathbb{R}^n$, where $n, p \geq 1$.

Derivations

Let E and F be Banach spaces and $U \mathring{\subseteq} E$, suppose f is a morphism from U to F. If p is a point in U, $Df(p)$ is of course a linear map from E to F; this suggests a natural pairing $\hat{\mathcal{D}}$ of f with and $(p, v) \in U \times E$ as shown in eq. (56).

$$\hat{\mathcal{D}} : (U \times E) \times C^p(U, F) \longrightarrow F : \quad \Big((p, v), f\Big) \mapsto Df(p)(v) \in F \tag{56}$$

Suppose $F = \mathbb{R}$ and denote pointwise multiplication on $\mathbb{R}$ by m. The above pairing trivially satisfies the product rule displayed in eq. (57).

$$Dm(f_{\underline{k}})(p)(v) = \sum_{i=\underline{k}} m(f_{\underline{i-1}}(p), Df_i(p)(v), f_{i+\underline{k-i}}(p)) \tag{57}$$

where $f_{\underline{k}} \in C^p(U, \mathbb{R})$. Next, if f is a function (from a manifold X) defined on an open neighbourhood U of p. If $v \in T_pX$, the commentary in the introduction suggests a 'duality pairing' between f and (p, v) in the form of eq. (58).

$$\mathcal{D} : (U \times E) \times C^p(U, F) \longrightarrow F \quad \text{and} \quad \mathcal{D}\Big((p, v), f\Big) = df_p(v) \tag{58}$$

By definition of the differential df_p, the right hand side of eq. (58) is representation independent, hence

$$\mathcal{D}((p, v), f) = D\hat{f}(\hat{p})(\hat{v}), \quad \text{where the right member is an ordinary derivative} \tag{59}$$

for any representation $(\hat{p}, \hat{v})$, $\hat{f}$. We also see that $\mathcal{D}((p, v), f) = \hat{\mathcal{D}}((\hat{p}, \hat{v}), \hat{f})$, which shows functions defined on U are dual to T_pX for each $p \in U$. We will make this notion precise when we introduce covectors.

Definition 2.1: Derivation at p

A *derivation at* p is a **linear functional** v on $C^p(U,\mathbb{R})$, where U is any neighbourhood of p; such that for $f_{\underline{k}} \in C^p(U)$, eq. (60) holds.

$$v\Big(m(f_{\underline{k}})\Big) = \sum_{i=\underline{k}} m(f_{\underline{i-1}}(x), v(f_i), f_{i+\underline{k-i}}(x)) \tag{60}$$

We will denote the space of derivations at p by $\mathcal{D}_p(X)$, and if $v \in \mathcal{D}_p(X)$, we say v *derives* f for any function f defined about p.

We have shown every tangent vector is a derivation, since the product rule descends from eq. (57) and its computation in coordinates in eq. (59). If X is finite-dimensional, prop. 2.1 shows derivations at a point $p \in X$ are uniquely represented by a tangent vector.

Proposition 2.1: T_pX is isomorphic to $\mathcal{D}_p(X)$

Let p be a point on a manifold X, then its tangent space is isomorphic to the vector space of derivations. If $(\hat{p}, \hat{v})$ is a concrete tangent vector, its derivation of f computed using eq. (59).

Proof. Postponed. ■

Boundary

Exterior Derivative

Let X be a manifold, and $f \in C^p(X)$. If $\gamma : (-\delta, +\delta) \to X$ is a curve starting at $x_0 \in X$ with velocity v, the composition $f \circ \gamma$ is a morphism. Let us write

$$F : (-\delta, +\delta) \to \mathbb{R}, \quad F(\varepsilon) = f \circ \gamma(\varepsilon) - f \circ \gamma(0) \tag{61}$$

Suppose we wish to measure the rate at which f moves in the direction of v, then we can simply take the derivative of eq. (61). We define the *exterior derivative of f at x_0*, denoted by $df(x_0) : T_{x_0}X \to \mathbb{R}$ by eq. (62)

$$df(x_0)(v) = DF(0)(\overline{1}) \quad \text{where} \quad F = f \circ \gamma \tag{62}$$

for any curve starting at x_0 with velocity v.

Let E be a Banach space, and suppose ω is a k-form on E, and $x_0 \in E$ with $k+1$ tangent vectors $v_{\underline{k+1}}$. The parallelopiped defined by the $k+1$ vectors is

$$P_{x_0}(v_{\underline{k+1}}) = \left\{ x_0 + \sum_{i=\underline{k+1}} t_i v_i,\ 0 \le t_i \le 1\ \forall i = \underline{k+1} \right\}$$

As with eq. (61), we can integrate over the boundary defined by $P_{x_0}(v_{\underline{k+1}})$, and obtain a new function. We can shrink each v_i by ε_i, and we define

$$P_{x_0}^{\varepsilon_{\underline{k+1}}}(v_{\underline{k+1}}) = \left\{ x_0 + \sum_{i=\underline{k+1}} t_i v_i,\ 0 \le t_i \le \varepsilon_i\ \forall i = \underline{k+1} \right\} \tag{63}$$

(Note: Perhaps after shrinking the domain of F, here we should replace everything by their coordinate representations).

$$F : (-\delta, +\delta)^{k+1} \to \mathbb{R}, \quad F(\varepsilon_{\underline{k+1}}) = \int_{\partial P_{x_0}^{\varepsilon_{\underline{k+1}}}(v_{\underline{k+1}})} \omega \tag{64}$$

We define the exterior derivative of a k-form by the map $DF(0)(1^{(k+1)})$