

Chapter 3

Notes on Chapter 3

Proposition 0.1

Prove two things,

1. $\limsup_{r \rightarrow R} \phi(r) = \lim_{\varepsilon \rightarrow 0} \sup_{0 < |r-R| < \varepsilon} \phi(r) = \inf_{\varepsilon > 0} \sup_{0 < |r-R| < \varepsilon} \phi(r),$
2. $\lim_{r \rightarrow R} \phi(r) = c \iff \limsup_{r \rightarrow R} |\phi(r) - c| = 0$

Proof.



Proposition 0.2

If $U \subseteq B(1, 0) = \{|x| < 1\}$, and $U \in \mathbb{B}$, and if $m(U) > 0$, then the family of sets

$$E_r = \left\{ x + ry, y \in U \right\}$$

shrinks nicely to $x \in \mathbb{R}^n$.

Proof. Let $r > 0$ be fixed then $\forall z \in E_r \ni z = x + ry$. Hence,

$$\begin{aligned} d(x, z) &= d(x, x + ry) \\ &= |r|d(0, y) < |r| \end{aligned}$$

by translation invariance. ■

Definition 0.1: Signed measure

Let \mathcal{M} be a σ -algebra and $\nu : \mathcal{M} \rightarrow [-\infty, +\infty]$ be a set function on \mathcal{M} . It is a *signed measure* on \mathcal{M} if

- $\nu(\emptyset) = 0$,
- ν assumes at most one of the values $\pm\infty$,
- If $\{E_j\}_{j \geq 1}$ is a countable, disjoint sequence of sets, the expression

$$\sum_{j \geq 1} \nu(E_j) \quad \text{is unambiguous, and is equal to } \nu\left(\bigcup E_j\right)$$

More precisely,

- if $|\nu(\bigcup E_j)| < +\infty$, the series $\sum \nu(E_j)$ converges absolutely,
- if $\nu(\bigcup E_j) = \pm\infty$, the series $\sum \nu(E_j)$ diverges to $\pm\infty$ on every permutation.

Definition 0.2: Positive, negative, null sets

Let ν be a signed measure on \mathcal{M} . A measurable set $E \in \mathcal{M}$ is called *positive* (resp. *negative*, *null*) if every measurable subset $F \subseteq E$ satisfies $\nu(F) \geq 0$ (resp. $\nu(F) \leq 0$, $\nu(F)=0$).

Definition 0.3: Mutual singularity

Two signed measures, ν and μ on a common σ -algebra \mathcal{M} are *mutually singular*, denoted by $\nu \perp \mu$ if there exists disjoint, measurable sets E, F whose union is \mathbf{X} .

$$\mu \text{ is null on } E, \quad \text{and } \nu \text{ is null on } F$$

Proposition 0.3

Let ν be a signed measure on (X, \mathcal{M}) . If $\{E_j\}$ is an increasing sequence in \mathcal{M} , $\lim_{n \rightarrow +\infty} \nu(E_j) = \nu(\bigcup E_j)$. If $\{E_j\}$ is a decreasing sequence in \mathcal{M} , $\lim_{n \rightarrow +\infty} \nu(E_j) = \nu(\bigcap E_j)$ provided $\nu(E_1)$ is of finite measure.

Proof. Let ν be a signed measure, and fix any increasing sequence $E_j \nearrow E = \bigcup E_{j \geq 1}$ of sets. This induces a disjoint sequence in $\{F_n\}$. Define $F_1 = E_1$, and if $n \geq 2$,

$$F_n = E_n \setminus \bigcup_{j \leq n-1} E_j$$

Use σ -additivity of ν , where the sum is 'defined' to be non-ambiguous.

For the second part of the proof, notice if $A \subseteq B$ are measurable sets, if $\nu(A) = \pm\infty$, then $\nu(B) = \pm\infty$, because of the second property of ν . Indeed,

$$\nu(B) = \nu(A) + \nu(B \setminus A) = \pm\infty + c$$

where $c \in \mathbb{R} \cup \{\pm\infty\}$. Therefore $\nu(B) = \nu(A)$. By assumption $\nu(E_1) \in \mathbb{R}$, the contrapositive of the previous argument shows that the intersection $\bigcap E_j$ is of finite measure as well. We can produce an increasing sequence $G_n = E_1 \setminus E_n$ for $n \in \mathbb{N}^+$. Then

$$\bigcup G_n = \bigcup E_1 \setminus E_n = E_1 \cap \left[\bigcup E_n^c \right] = \left[\bigcap E_j \right]^c$$

We then write

$$E_1 = \left[\bigcup G_n \right] + \left[\bigcap E_n \right]$$

The finiteness of $\nu(E_1)$ on the left hand side implies all the terms in the union converge absolutely. Therefore

$$\begin{aligned} \nu(E_1) - \nu\left(\bigcap E_n\right) &= \lim_{n \rightarrow +\infty} \nu(G_n) \\ &= \lim_{n \rightarrow +\infty} \nu(E_1) - \nu(E_n) \\ &= \nu(E_1) - \lim_{n \rightarrow +\infty} \nu(E_n) \end{aligned}$$

Cancelling terms finishes the proof. ■

Proposition 0.4

Any measurable subset of a positive set is again positive, and any countable union of positive sets is again positive. Similarly for negative, and null sets.

Proof. Trivial. ■

Proposition 0.5: Hahn Decomposition Theorem

Let ν be a signed measure on the measurable space $(\mathbf{X}, \mathcal{M})$, then there exists positive and negative sets $P, N \in \mathcal{M}$ where $P \cup N = \mathbf{X}$, and $P \cap N = \emptyset$. If P' and N' are another such decomposition,

$$P \Delta P' = N \Delta N' \text{ is } \nu\text{-null.}$$

Proof. There are multiple steps to this proof. Suppose ν does not attain $+\infty$. Define

$$m = \sup \left\{ \nu(P), P \text{ is a positive set} \right\}$$

By assumption $m < +\infty$, let $\{P_j\}$ be a sequence of positive sets with $\nu(P_j) \nearrow m$. We claim the supremum is attained. Indeed, if $P \triangleq \bigcup P_j$, then P is a positive set as well, by monotonicity $\nu(P) \geq \nu(P_j)$, taking the supremum on both sides reads $\nu(P) = m$.

Wanting to prove $N \triangleq \mathbf{X} \setminus P$ is a ν -negative set,

- Clearly N cannot contain any positive sets $A \subseteq N$ with a non-null measure, since

$$\nu(A) > 0 \implies \nu(A) + \nu(P) = \nu(A + P) > m$$

contradicting the supremum.

- Let us examine the properties of subsets of N with *positive measure*. Call this set $A \subseteq N$, where $\nu(A) > 0$.

The previous bullet point tells us A cannot be a ν -positive set. There exists a $B \subseteq A$ of strictly negative measure,

$$\nu(A \setminus B) + \nu(B) = \nu(A) \implies \nu(A \setminus B) > \nu(A)$$

Notice the assumption ν does not attain $+\infty$ allows us to subtract B over.

Summarizing,

existence of subset of positive measure \implies subset with even greater positive measure

We will use the above inductively to construct a measurable subset of N , that is 'small' but has 'large' positive measure at the same time.

- Suppose N is not ν -negative, so it admits a set of positive measure in $A_1 \subseteq N$.

Let $n_1 = \text{least} \left\{ n \in \mathbb{N}^+, \exists B \subseteq A_1, \nu(B) > \nu(A) + n^{-1} \right\}$, since n_1 is attained, it corresponds to some $A_2 \subseteq A_1$ with $\nu(A_2) > \nu(A_1) + n_1^{-1}$.

Repeating this process inductively, we see

$$\nu(A_k) > \nu(A_{k-1}) + n_k^{-1}$$

Let $A = \bigcap A_k$, this should be a set of large positive measure. A simple induction will show

$$\nu(A_k) > \nu(A_1) + \sum_{j=1}^k n_j^{-1} > \sum_{j=1}^k n_j^{-1}$$

However, $\nu(A) < +\infty$ by assumption. Upon taking limits and using the estimate above,

$$\sum_{j \geq 1} n_j^{-1} = \lim_{n \rightarrow \infty} \nu(A_n) = \nu(A) < +\infty$$

The sum on the left is finite, so its terms must converge to 0. Notice $\nu(A)$ is a subset of N of positive measure, it admits a subset $B \subseteq A$ with $\nu(B) > \nu(A) + n^{-1}$ for $n \geq 1$.

$n_j^{-1} \rightarrow 0$ implies $n_j \rightarrow \infty$. So $n < n_j$ for large j . Notice $B \subseteq A \subseteq A_j$, and $\nu(B) > \nu(A_j) + n^{-1}$. This contradicts our definition of n_j , stated below for convenience

$$n_j = \text{least} \left\{ n \in \mathbb{N}^+, \exists B \subseteq A_j, \nu(B) > \nu(A_j) + n^{-1} \right\}$$

This proves N is ν -negative.

To show this composition is ν -unique, let P' and N' be disjoint, measurable positive and negative sets of \mathbf{X} . Then

$$P \setminus P' \subseteq P \quad \text{and} \quad P \setminus P' \setminus \mathbf{X} \setminus P' \subseteq N'$$

So $P \setminus P'$ is at the same time a ν -positive and a ν -negative set, hence it is ν -null by Lemma 3.2.

Finally, the case for when ν attains $+\infty$ can be handled if we consider $-\nu$. P is positive for $-\nu$ iff it is negative for ν , and similarly for N . Relabelling P and N finishes the proof. \blacksquare

Theorem 3.4

Proposition 1.1

Proof.



Theorem 3.5

Proposition 2.1

Proof.



Theorem 3.6

Proposition 3.1

Proof.



Theorem 3.7

Proposition 4.1

Proof.



Theorem 3.8

Proposition 5.1

Proof.



Theorem 3.9

Proposition 6.1

Proof.



Theorem 3.10

Proposition 7.1

Proof.



Theorem 3.11

Proposition 8.1

Proof.



Theorem 3.12

Proposition 9.1

Proof.



Theorem 3.13

Proposition 10.1

Proof.



Theorem 3.14

Proposition 11.1

Proof.



Theorem 3.15

Proposition 12.1

Proof.



Theorem 3.16

Proposition 13.1

Proof.



Theorem 3.17**Proposition 14.1**

Let the maximal function of any measurable $f \in \mathbb{B}_{\mathbb{R}^n}$ be denoted by $Hf(x)$, more precisely,

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r, x))} \int_{B(r, x)} f(y) dy$$

where $A_r|f|$ is the average of $|f|$ on a ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$. In symbols,

$$A_r|f| = \frac{1}{m(B(r, x))} \int_{B(r, x)} f(y) dy$$

The maximal theorem makes two claims:

1. $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, and Hf is measurable for every $f \in L^1_{loc}$.
2. There exists a $C > 0$, for every $f \in L^1$

$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1$$

for every $\alpha > 0$.

Proof. Let $\alpha > 0$ and fix $z \in (Hf)^{-1}((\alpha, +\infty))$, so $Hf(z) > \alpha$ and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with $Hf(z) - \alpha > 0$, we get some $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some $r_0 > 0$. Since $A_r|f|$ (a function indexed by $r > 0$) is continuous in $x \in \mathbb{R}^n$, $(A_r|f|)^{-1}((\alpha, +\infty))$ is open, and Hf is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{ Hf > \alpha \right\} = \bigcup_{r>0} \{A_r|f| > \alpha\}$$

Let $x \in E_\alpha$, this induces a $r_x > 0$ where $x \in \{A_{r_x}|f| > \alpha\}$. Rearranging gives

$$\left(\frac{1}{\alpha} \int_{B(r, x)} |f| dz \right) < m(B(r, x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each $x \in E_\alpha \mapsto r_x > 0 \mapsto A_{r_x}|f|$,
- If $U = \bigcup_{x \in E_\alpha} B(r_x, x)$, then $E_\alpha \subseteq U$,
- Choose $c < m(E_\alpha) \leq m(U)$ (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

$$x_1, \dots, x_N \in E_\alpha$$

so that $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$, and $c < 3^n \sum_{j \leq k} m(B_j)$

- Define $B_j = B(r_{x_j}, x_j)$ for all $j \leq k$, and

$$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

by finite additivity,

$$c3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

and finally

$$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of m on \mathbb{B} , since

$$m(E_\alpha) = \sup \left\{ m(K), K \in \mathcal{J}_{\mathbb{R}^n}, K \subseteq E_\alpha \right\}$$

for any $K \in \mathcal{J}_{\mathbb{R}^n}$, $K \subseteq E_\alpha$, we have $m(K) < +\infty$, $m(K) \leq m(E_\alpha)$ and

$$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

Remark 14.1

We used the properties of a Radon Measure here, without relying on the phrase ‘sending $c \rightarrow E_\alpha$ ’, which would require us to deal with two cases $m(E_\alpha) < +\infty$ and $m(E_\alpha) = +\infty$.

■

Theorem 3.18

Proposition 15.1

Proof.



Theorem 3.19

Proposition 16.1

Proof.



Theorem 3.20

Proposition 17.1

Proof.



Theorem 3.21**Proposition 18.1**

The Lebesgue Differentiation Theorem. Suppose $f \in L^1_{loc}$, and for every $x \in \mathcal{L}_f$, (so that $x \in \mathbb{R}^n$ a.e). We have

1. $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0,$
2. $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x),$

For every family $\{E_r\}_{r>0}$ that shrinks nicely to $x \in \mathbb{R}^{n'}$.

Proof. Since the family $\{E_r\}_{r>0}$ shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r, x)) \implies m(E_r) > \alpha \cdot m(B(r, x))$$

for some $\alpha > 0$, independent on r . Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r, x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r, x)} |f(y) - f(x)| dy$$

Combining the last two results, for every $\varepsilon > 0$, if $0 < r < \varepsilon$, then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0 < r < \varepsilon} m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

and sending $\varepsilon \rightarrow 0$, proves the first claim. The second claim is immediate upon applying the L^1 inequality.

Fix any $\varepsilon > 0$, and

$$\begin{aligned} \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} f(y) dy = f(x) &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y) dy - f(x) \right| \\ &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)] dy \right| \\ &\leq \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\ &= \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\ &= 0 \end{aligned}$$

■

Theorem 3.22

Proposition 19.1

Proof.



Theorem 3.23

Proposition 20.1

Proof.



Theorem 3.24

Proposition 21.1

Proof.



Theorem 3.25

Proposition 22.1

Proof.



Theorem 3.26

Proposition 23.1

Proof.



Theorem 3.27

Proposition 24.1

Proof.



Theorem 3.28

Proposition 25.1

Proof.



Theorem 3.29

Proposition 26.1

Proof.



Theorem 3.30

Proposition 27.1

Proof.



Theorem 3.31

Proposition 28.1

Proof.



Theorem 3.32

Proposition 29.1

Proof.



Theorem 3.33

Proposition 30.1

Proof.



Theorem 3.34

Proposition 31.1

Proof.



Theorem 3.35

Proposition 32.1

Proof.



Theorem 3.36

Proposition 33.1

Proof.

