

# Chapter B: Abstract Algebra

## Groups

### Definition 1.1: Semigroups, Monoids

A non-empty set  $G$  equipped with an associative binary operation  $G \times G \rightarrow G$  is called a semigroup. For every  $a, b, c \in G$ , we have

$$a(bc) = (ab)c \quad (1)$$

A *monoid* is a semigroup  $G$  which contains a *two-sided identity* element  $e \in G$  such that  $ae = ea$  for all  $a \in G$ . (not necessarily unique)

Monoids admit unique two-sided identities.

### Lemma 1.1: Monoids: unique identity

Let  $e$  and  $i$  be two-sided identities for a monoid  $G$ , then

*Proof.*

$$e = ei = i$$

■

### Definition 1.2: Group

A semigroup  $G$  is a group if every element  $a \in G$  admits a two-sided inverse  $a^{-1}$ . (not necessarily unique)

$$aa^{-1} = a^{-1}a = e$$

### Proposition 1.1: Properties of Groups (Hungerford: Theorem 1.2)

Let  $G$  be a group with identity  $e$ , which is unique by lemma 1.1. Then

(i)  $c \in G$  and  $cc = c$  implies  $c = e$ .

(ii) Left/Right cancellation:

$$\begin{cases} ab = ac \implies b = c \\ ba = ca \implies b = c \end{cases}$$

(iii) If  $a \in G$ , its two-sided inverse is unique.

(iv) Let  $a \in G$ , then the inverse of its two-sided inverse (uniqueness guaranteed by iii), is  $a$  itself; or  $(a^{-1})^{-1} = a$ .

(v) If  $a, b \in G$ , then the following equations in  $x, y$  admit unique solutions

$$\begin{cases} ax = b \\ ya = b \end{cases}$$

*Proof of Proposition 1.1.*

Proof of Part (i):

$$cc = c \implies (cc)c^{-1} = cc^{-1} \implies c(cc^{-1}) = e \implies ce = c = e$$

Proof of Part (ii): First claim:

$$\begin{aligned} ab = ac &\implies a^{-1}(ab) = a^{-1}(ac) \\ &\implies (a^{-1}a)b = (a^{-1}a)c \implies eb = ec \implies b = c \end{aligned}$$

Second claim is the same, just cancel from the right using  $aa^{-1} = e$  and associativity.

Proof of Part (iii): Suppose  $b$  and  $c$  are two-sided inverse for  $a$ , it follows from Part ii that

$$ab = ac \implies b = c = a^{-1}$$

Proof of Part (iv): From Part iii, the two-sided inverses of group elements exist and are unique, and  $a^{-1}a = aa^{-1}$  so  $a$  is an inverse for  $a^{-1}$ , and it is the only inverse.

Proof of Part (v): First equation: write  $ax = b = a(a^{-1}b)$ , left-cancelling reads  $x = a^{-1}b$ , uniqueness follows from Part ii. Second equation is similar. ■

### Lemma 1.2: Group: equality lemma

For any pair of elements  $a, b \in G$ ,  $a = b \iff ab^{-1} = e$ .

*Proof.* ( $\implies$ ):  $a = b \implies ab^{-1} = bb^{-1} = e$ . ( $\impliedby$ ):  $ab^{-1} = e \implies a(b^{-1}b) = eb \implies a = eb = b$ . ■

### Proposition 1.2: Semigroup: upgrade to group I (Hungerford Proposition 1.3)

Let  $G$  be a semigroup,  $G$  is also a group iff both of the conditions below hold

- Existence of a left-identity: there exists  $e \in G$  for every  $a \in G$ ,  $ea = a$ .
- Existence of left-inverses: for every  $a \in G$ , there exists a  $a^{-1} \in G$  with  $a^{-1}a = e$ , where  $e$  is any left-identity element.

*Proof.* ( $\impliedby$ ) is trivial. Suppose both conditions hold, notice the proof for Proposition 1.1 Part (i) we only used left-cancellation.  $cc = c \implies e$ . To prove  $a^{-1}$  is also a right-inverse for  $a$ , we can force it as follows:

$$(aa^{-1})(aa^{-1}) = a(a^{-1}a)a^{-1} = aea^{-1} = e \implies aa^{-1} = e$$

and  $a^{-1}$  is also a right-inverse, so every element  $a \in G$  admits a two-sided inverse denoted by  $a^{-1}$ . To show  $e$  is also a right-identity for any arbitrary element  $a \in G$ ,

$$\begin{aligned} ae &= a(a^{-1}a) && \text{left inverse} \\ &= (aa^{-1})a && \text{associativity} \\ &= ea && \text{right inverse} \\ &= a && \text{left identity} \end{aligned}$$

■

**Proposition 1.3: Semigroup: upgrade to group II (Hungerford Proposition 1.4)**

Let  $G$  be a semigroup,  $G$  is a group iff for every pair of elements  $a, b \in G$ , the equations in  $x$  and  $y$

$$\begin{cases} ax = b \\ ya = b \end{cases} \quad (2)$$

have solutions (not necessarily unique).

*Proof.* If  $G$  is a group, the existence of the solutions to eq. (2) follow from Proposition 1.1. We will attempt the contrapositive. Suppose  $G$  has no left identity, for every  $e \in G$  we can always find an element  $a \in G$  such that  $ea \neq a$ , but this is precisely the (first) equation for  $a = a$  and  $b = a$ .

Now suppose  $G$  has a left identity element (not necessarily unique). Fix  $e \in G$  as any left-identity, and suppose there is an element  $a \in G$  with no left inverse, so for every  $b \in G$ ,  $ba \neq e$ . But  $b$  is precisely the solution to the (second) equation with parameters  $a = a$  and  $b = e$ . The negation of Proposition 1.2 is precisely the negation of Proposition 1.3, and the proof is complete. ■

**Proposition 1.4: Hungerford Theorem 1.5**

Let  $R/\sim$  be an equivalence relation on a group  $G$ , such that it 'preserves' the group multiplication. More precisely,

$$\begin{cases} a_1 \sim a_2 \\ b_1 \sim b_2 \end{cases} \implies a_1 b_1 \sim a_2 b_2$$

Then the set  $G/R$  of all equivalence classes of  $G$  under  $R$  is a monoid under the binary operation defined by

$$(\bar{a})(\bar{b}) = \overline{ab} \quad \text{reads: the product of two classes is the class containing the product of any pair of elements from the two classes} \quad (3)$$

where  $\bar{a}$  denotes the equivalence class containing  $a$ . If  $G$  is a group, so is  $G/R$ , if  $G$  is an abelian group, so is  $G/R$ .

*Proof.* First, notice the binary operation in Equation (3) is well defined. It is independent of the equivalence class representatives chosen, as we have restriction on  $R$  that 'forces' the operation on  $G/R$  to be well defined. Indeed, let  $\bar{a}$  and  $\bar{b}$  be elements of  $G/R$ , if  $a_1, a_2 \in \bar{a}$ , and  $b_1, b_2 \in \bar{b}$ , by definition of  $R$ :

$$a_1 \sim a_2 \quad \text{and} \quad b_1 \sim b_2$$

by Equation (3),  $a_1 b_1 \sim a_1 b_2 \implies \overline{a_1 b_1} = \overline{a_1 b_2}$ .

Associativity is proven similarly, fix  $\bar{a}, \bar{b}, \bar{c} \in G/R$ , we pass the argument to any of the representatives of the three classes, so

$$(\bar{a}\bar{b})\bar{c} \triangleq \overline{ab\bar{c}} = \overline{(ab)c} = \overline{a(bc)} \triangleq \overline{a\bar{b}c} = \bar{a}(\bar{b}\bar{c})$$

Pass the argument to the representatives, let  $e$  denote the identity element in  $G$ , it is easily shown that  $\bar{e}$  is the identity element in  $G/R$ , similarly for two-sided inverses and commutativity of the binary operation. ■

## Homomorphisms

### Definition 2.1: Homomorphism

Let  $G$  and  $H$  be semigroups,  $f : G \rightarrow H$  is a semi-group *homomorphism* if for all  $a, b \in G$ ,

$$f(ab) = f(a)f(b) \quad (4)$$

### Definition 2.2: Monomorphism

Injective homomorphism.

### Definition 2.3: Epimorphism

Surjective homomorphism.

### Definition 2.4: Isomorphism

Bijjective homomorphism.

### Definition 2.5: Endomorphism

Homomorphism for which the domain and codomain (not the range) are equal; i.e  $H = G$ .

### Definition 2.6: Automorphism

Bijjective endomorphism.

### Definition 2.7: Kernel of a homomorphism

The kernel of  $f \in \text{Hom}(G, H)$  is defined

$$\text{Ker } f = \left\{ a \in G, f(a) = e \in H \right\} \quad (5)$$

as the set of elements in  $G$  that get sent to the identity of  $H$ .

### Proposition 2.1: Hungerford Theorem 2.3

Let  $G$  and  $H$  be groups and let  $f \in \text{Hom}(G, H)$ . Denote the identity elements of  $G$  and  $H$  by  $e_G$  and  $e_H$

(i)  $f(e_G) = e_H$ ,

(ii)  $f(a^{-1}) = (f(a))^{-1}$  for every  $a \in G$ .

(iii)  $f$  is a monomorphism iff  $\ker f = \{e_G\}$ ,

(iv)  $f$  is an isomorphism iff there exists a homomorphism  $f^{-1} : H \rightarrow G$  that is also a two-sided inverse for  $f$ . In symbols:

$$f \circ f^{-1} = \text{id}_H \quad \text{and} \quad f^{-1} \circ f = \text{id}_G \quad (6)$$

*Proof of Proposition 2.1.*

Proof of Part (i): We will use Proposition 1.1 (i). Since  $f(e_G) = f(e_G e_G) = f(e_G)f(e_G)$  in  $H$ , we see that  $f(e_G) = H$  and  $e_G \in \text{Ker } f$

Proof of Part (ii): Let  $a \in G$  be arbitrary, using Part (i), we can 'pass the multiplication' between  $f(a)$  and  $f(a^{-1})$  into  $G$ ,

$$f(a)f(a^{-1}) = f(e_G) = e_H \implies f(a^{-1}) = (f(a))^{-1}$$

Proof of Part (iii): Suppose  $\ker f = e_G$ . Let  $a, b \in G$  such that  $f(a) = f(b)$ . The equality lemma Lemma 1.2 tells us  $(f(a))^{-1} = f(b)$  and  $b = a^{-1}$ , so  $a = b$  by the Lemma again;  $f$  is injective.

Conversely, suppose  $f$  is injective, Part (i) tell us  $\{e_G\} \subseteq \ker f$ . Suppose  $a \in \ker f \subseteq G$ , but  $e_G \in \ker f$ , so  $f(a) = f(e_G) = e_H$  forces  $a = e_G$ , and  $\ker f = \{e_G\}$ .

Proof of Part (iv): ( $\Leftarrow$ ) is trivial since the existence of a (functional) two-sided inverse is equivalent to bijectivity. Suppose  $f$  is an isomorphism, and define  $f^{-1}$  as its two-sided (functional) inverse, it suffices to show that  $f^{-1} \in \text{Hom}(H, G)$ . Fix  $f(a)$  and  $f(b)$  as arbitrary elements in  $H$ . We can do this because  $f$  is a bijection, so every element in  $H$  has a unique 'representative' in  $G$ .

$$f^{-1}(f(a)) f^{-1}(f(b)) = ab = f^{-1}(f(ab)) = f^{-1}(f(a)f(b))$$

■

### Definition 2.8: Subgroup $H < G$

Let  $G$  be a group. If  $H$  is a non-empty subset of  $G$  that is closed under group operations, then it is a *subgroup* of  $G$ . We write  $H < G$ .

The *trivial subgroup* of  $G$  is  $\{e\}$  and consists of one element.  $H$  is called a *proper subgroup* if  $H \neq G$  and  $H \neq \{e\}$ .

### Proposition 2.2: Hungerford Exercise 9

Let  $f \in \text{Hom}(G, H)$ ,  $A < G$  and  $B < H$ ,

(i)  $\text{Ker } f$  is a subgroup of  $G$ ,

- (ii)  $f(A)$  is a subgroup of  $H$ ,
- (iii)  $f^{-1}(B)$  is a subgroup of  $G$ .

## Subgroups

### Proposition 3.1: Subgroup criteria (Hungerford Theorem 2.5)

A non-subset  $H \subseteq G$  is a subgroup iff for any  $a, b$  in  $H$ ,  $ab^{-1} \in H$ .

*Proof.* ( $\Leftarrow$ ): Choose  $a = b$ , then  $aa^{-1} = e \in H$  acts as the two-sided identity in  $H$ , and  $ea^{-1} = a^{-1} \in H$  for every  $a \in H$ . So  $H$  is a subgroup by Proposition 1.2. ( $\Rightarrow$ ): If  $H$  is a subgroup, then Proposition 1.2 tells us  $b^{-1} \in H$  for every  $b \in H$ , hence  $ab^{-1} \in H$  for elements  $a, b \in H$ . ■

### Corollary 3.1: Hungerford Corollary 2.6

If  $G$  is a group and  $\{H_i, i \in I\}$  is a nonempty family of subgroups, then their intersection  $H \triangleq \bigcap_{i \in I} H_i$  is again a subgroup in  $G$ .

*Proof.* Let  $a, b \in H$ , then  $ab^{-1} \in H_i$  for every  $i \in I$ , hence  $ab^{-1} \in \bigcap_{i \in I} H_i = H$ , and  $H$  is a subgroup by Proposition 3.1. ■

### Definition 3.1: Subgroup generated by $A \subseteq G$

Let  $A$  be a subset of  $G$ , the *subgroup generated by  $A$*  is the smallest subgroup  $H < G$  that contains  $A$  as a subset, denoted by  $\langle A \rangle$ .

Proposition 3.1 gives us an explicit formula for  $H$ ,

$$H = \bigcap_{\substack{H_i < G \\ A \subseteq H_i}} H_i$$

If  $A$  is finite, and  $H = \langle A \rangle$  is said to be *finitely generated*. We also write

$$H = \langle a_1, \dots, a_n \rangle = \langle \{a_1, \dots, a_n\} \rangle$$

If  $A$  consists of one element,  $\{a\} = A$ , then  $\langle a \rangle = \langle \{a\} \rangle$  is called the *cyclic group generated by  $a$* .

### Proposition 3.2: Hungerford Theorem 2.8

If  $G$  is a group and  $A \subseteq G$  is a non-empty subset, the subgroup generated by  $A$  is precisely the collection of all finite products (powers included), or

$$\langle A \rangle = \left\{ a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}, a_i \in A, n_i \in \mathbb{Z} \right\}$$

If  $A = \{a\}$ , then  $\langle A \rangle = \{a^k, k \in \mathbb{Z}\}$ .

*Proof.* Let  $W = \{a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}, a_i \in A, n_i \in \mathbb{Z}\}$ . We first show  $W$  is a subgroup of  $G$ . Indeed, if  $n_1 = n_2 = \cdots = n_t = 0$ , then  $a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t} = e$ . And for any  $b \in W$ ,

$$b = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t} \implies b^{-1} = a_t^{-n_t} \cdots a_2^{-n_2} a_1^{-n_1}$$

where each  $-n_i \in \mathbb{Z}$ , so  $b^{-1} \in W$  as well.  $\langle A \rangle$  is the smallest subgroup containing  $A$ , so  $\langle A \rangle \subseteq W$ . Conversely, fix an element  $b \in W$ , so  $b$  has the form

$$b = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}, a_i \in A, n_i \in \mathbb{Z}$$

and a simple induction will show each  $a_i^{n_i} \in \langle A \rangle$  for  $1 \leq i \leq t$ , so  $b \in \langle A \rangle$  and  $\langle A \rangle = W$ . ■

### Definition 3.2: Lattice of subgroups

Let  $\{H_i\}_{i \in I}$  be a collection of subgroups of  $G$ , then

$$\text{glb}\{H_i\} = \bigcap_{i \in I} H_i, \quad \text{lub} = \left\langle \bigcup_{i \in I} H_i \right\rangle$$

The collection of subgroups of  $G$  is a *complete lattice*.

## Cyclic groups

The proof for the following is straight-forward, the book separates the case into  $H = \langle 0 \rangle$  and  $H \neq \langle 0 \rangle$ , then  $H$  contains a non-zero element  $h \neq 0$ , then  $|h| \in \mathbb{N}^+$  is an element in  $H$  as well, so  $H$  contains a least positive element by invoking the Well Ordering Property. For the second half of the proof, we force  $r = 0$  by the Division Algorithm.

### Definition 4.1: Order of a subgroup $H < G$

The order of a subgroup  $H$  is its cardinality  $|H|$ .

### Definition 4.2: Order of an element $a \in G$

The order of an element  $a \in G$  is the order of  $|\langle a \rangle|$ .

### Proposition 4.1: Hungerford Theorem 3.1

Let  $\mathbb{Z}$  be equipped with its usual addition operation. Then every subgroup  $H < \mathbb{Z}$  is cyclic, either  $H = \langle 0 \rangle$  or  $H = \langle m \rangle$ . With  $m$  being the least positive integer in  $H$ . If  $H \neq \langle 0 \rangle$ , then  $H$  is infinite.



**Proposition 4.2: Hungerford Theorem 3.2**

Every infinite cyclic group  $G = \langle a \rangle$  is isomorphic to the additive group  $\mathbb{Z}$ , and every finite cyclic group of order  $0 < m < +\infty$  is isomorphic to the additive group  $\mathbb{Z}_m$ .

*Proof of Proposition 4.2.* Let  $f : \mathbb{Z} \rightarrow G$  and  $f(k) = a^k$ . By Proposition 3.2,

$$G = \left\{ a^k, k \in \mathbb{Z} \right\}$$

$f$  is clearly surjective and  $f \in \text{Hom}(\mathbb{Z}, G)$  is an easy exercise to verify. The proof splits into two parts

- (i)  $\text{Ker } f$  is trivial: By Proposition 2.1,  $f$  is an isomorphism from  $\mathbb{Z}$  to  $G$ , and  $G \cong \mathbb{Z} \implies |G| = |\mathbb{Z}|$ .
- (ii)  $\text{Ker } f$  is not trivial: Arguing as in Proposition 4.2,  $\text{Ker } f$  is a non-trivial subgroup of  $\mathbb{Z}$ , so it contains a least positive element  $m$ , and  $\text{Ker } f = \langle m \rangle$ .  $m$  is an element of  $\text{ker } f$ , so

$$f(m) = a^m = e \implies f(jm) = a^{jm} = \prod_{l=1}^j a^m = e, j \in \mathbb{Z} \quad (7)$$

Now suppose  $r$  and  $s$  are integers with  $f(r) = f(s)$ ,

$$\begin{aligned} a^r = a^s &\iff a^{r-s} = e \\ &\iff r - s \in \text{Ker } f \\ &\iff r - s \in \langle m \rangle \\ &\iff r \equiv s \pmod{m} \\ &\iff \bar{r} = \bar{s} \end{aligned}$$

where  $\bar{r}$  denotes the  $\mathbb{Z}_m$  equivalence class of  $r$ . Let  $\beta$  be a map from  $\mathbb{Z}_m \rightarrow G$ , such that

$$\beta(\bar{k}) = f(k) = a^k$$

This is well defined, since  $\beta$  is an invariant on each equivalence class, if  $r - s$  differ by a multiple of  $m$ , then Equation (7) states that  $\beta(\bar{r}) = \beta(\bar{s})$ .  $G$  is finite, as

$$G = \langle a \rangle = \left\{ a^k, k \in \mathbb{Z}, m < k < m \right\}$$

and the kernel of  $\beta$  is trivial, it is an isomorphism and  $\mathbb{Z}_m \cong G$ .

■

Cosets

Normal Subgroups

Isomorphism Theorems