Notation

We will use the following notation to simplify computations with multilinear maps. Let E and F be sets, and $v_1, \ldots, v_k \in E$. $f : E \to F$.

- Listing individual elements: $v_{\underline{k}}$ means $v_1, \ldots, v_k$ as separate elements.

- Creating a k-list: $(v_{\underline{k}}) = (v_1, \ldots, v_k) \in \prod E_{j \leq k}$ if $v_i \in E_i$ for $i = \underline{k}$.

- Double indices: $(v_{n_{\underline{k}}}) = (v_{n_{\underline{k}}}) = (v_{n_1}, \ldots, v_{n_k})$, and

$$(v_{\underline{n_k}}) \neq (v_{n_{(1,\ldots,k)}})$$

- Closest bracket convention:

$$(v_{(n_{\underline{k}})}) = (v_{(n_1,\ldots,n_k)}) \quad \text{and} \quad (v_{n_{(\underline{k})}}) = (v_{n_{(1,\ldots,k)}})$$

- Underlining 0 means it is iterated 0 times:

$$(v_{\underline{0}}, a, b, c) = (a, b, c)$$

- Skipping an index:
$$(v_{\underline{i-1}}, v_{i+\underline{k-i}}) = (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k) \tag{1}$$

  for $i = \underline{k}$.

- Applying f to a particular index:

$$(v_{\underline{i-1}}, f(v_i), v_{i+\underline{k-i}}) = (v_1, \ldots, v_{i-1}, f(v_i), v_{i+1}, \ldots, v_k) \tag{2}$$

  Of course, if $i = 1$, then the above expression reads $(f(v_1), v_2 \ldots, v_k)$ by the $\underline{0}$ interpretation.

- If $\wedge : E \times E \to F$ is any associative binary operation,

$$\bigwedge(v_{\underline{k}}) = v_1 \wedge \cdots \wedge v_k$$

- In any list using this 'underline' notation, we can find the size of a list by summing over all the underlined numbers and the number of terms without an underline. We see eq. (1), eq. (2) have $k - 1$, k terms respectively.

Remark 1.1: Preview of exterior calculus

We can write the cofactor expansion formula of the determinant of a $\mathbb{R}^{k \times k}$ matrix in this notation. Suppose $a_i \in \mathbb{R}$, and $b_i \in \mathbb{R}^{k-1}$ for $i = \underline{k}$.

$$M = \begin{bmatrix} a_1 & \cdots & a_k \\ | & & | \\ b_1 & \cdots & b_k \\ | & & | \end{bmatrix}$$

The determinant of M, can then be written as

$$\det(M) = \sum_{i=\underline{k}}(-1)^{i-1}a_i\det\left(b_{\underline{i-1}}, b_{i+\underline{k-i}}\right)$$

k-linear maps

> **Definition 2.1: k-linear maps**
>
> Let $E_{\underline{k}}$, F be Banach spaces. A map $\varphi : \prod E_{\underline{k}}$ is k-linear if for every $i = \underline{k}$, $v_i \in E_i$,
>
> $$\varphi(\cdot\underline{^{i-1}}, v_i, \cdot\underline{^{k-i}}) : \textstyle\bigcirc\!\!\!\prod (E_{\underline{i-1}}, E_{i+\underline{k-i}}) \to F \quad \text{is } (k-1)\text{-linear}$$

The following theorem should give confidence to the notation we have adopted to use.

> **Proposition 2.1**
>
> Let $E_{\underline{k}}$ and F be Banach spaces, a k-linear map $\varphi : \prod E_{\underline{k}} \to F$ is continuous iff there exists a $C > 0$, such that for every $x_i \in E_i$, $i = \underline{k}$
> $$\left|\varphi(x_{\underline{k}})\right| \le C\prod\left|x_{\underline{k}}\right|$$

Proof. Suppose φ is continuous, then it is continuous at the origin. Picking $\varepsilon = 1$ induces a $\delta > 0$ such that for $\left|(x_{\underline{k}})\right| \le \delta$, $\left|\varphi(x_{\underline{k}})\right| \le 1$. The usual trick of normalizing an arbitrary vector $(x_{\underline{k}}) \in \prod E_{\underline{k}}$ does the job:

$$\left|\varphi(x_k \cdot \left|x_{\underline{k}}\right|^{-1} \cdot \delta)\right| \le 1 \implies \left|\varphi(x_{\underline{k}})\right| \le \delta^{-k}\prod\left|x_{\underline{k}}\right|$$

Conversely, fix a sequence (indexed by n, in k elements in the product space $\prod E_{\underline{k}}$), so

$$(x_{\overline{n}}^{\underline{k}}) \to (x^{\underline{k}}) \quad \text{as } n \to +\infty \tag{3}$$

To proceed any further, we need to prove eq. (4) that expresses the difference of two values of φ in terms its arguments.

$$\varphi(b^{\underline{k}}) - \varphi(a^{\underline{k}}) = \sum_{i=\underline{k}}\varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}}) \tag{4}$$

where $(b^{\underline{k}})$ and $(a^{\underline{k}})$ are elements in $\prod E_{\underline{k}}$, and $\Delta_i = b^i - a^i$ for $i = \underline{k}$. The proof is contained in the following note, which is in more detail than usual - to help the reader ease into the new notation.

> **Note 2.1**
>
> We proceed by induction, and eq. (4) follows by setting $m = k$ in
>
> $$\varphi(a^{\underline{k}}) = \varphi(b^{\underline{m}}, a^{m+\underline{k-m}}) - \sum_{i=\underline{m}}\varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}}) \tag{5}$$
>
> Base case: set $m = 1$, by definition of k-linearity (def. 2.1) of φ. Since $a^1 = b^1 - \Delta_1$,
>
> $$\varphi(a^{\underline{k}}) = \varphi(b^1 - \Delta_1, a^{1+\underline{k-1}}) = \varphi(b^1, a^{1+\underline{k-1}}) - \varphi(\Delta_1, a^{1+\underline{k-1}})$$

Induction hypothesis: suppose eq. (5) holds for a fixed m. Since $a^{m+1} = b^{m+1} - \Delta_{m+1}$,

$$\varphi(a^{\underline{k}}) = \varphi(b^{\underline{m}}, a^{m+\underline{k-m}}) - \sum_{i=\underline{m}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}})$$

$$= \varphi(b^{\underline{m}}, a^{m+1}, a^{(m+1)+\underline{k-(m+1)}}) - \sum_{i=\underline{m}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}})$$

$$= \varphi(b^{\underline{m+1}}, a^{(m+1)+\underline{k-(m+1)}}) - \varphi(b^{\underline{m+1}}, \Delta_{m+1}, a^{(m+1)+\underline{k-(m+1)}}) - \sum_{i=\underline{m}} \varphi(b^{\underline{i-1}}, \Delta_i, a^{i+\underline{k-i}})$$

and this proves eq. (4)

We substitute $a^i = x^i$, and $b^i = x_n^i$ for $i = \underline{k}$, and eq. (4) becomes eq. (6)

$$\varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) = \sum_{i=\underline{k}} \varphi(x_n^{\underline{i-1}}, x_n^i - x^i, x^{i+\underline{k-i}}) \tag{6}$$

Then the triangle inequality reads

$$\left| \varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) \right| \leq \sum_{i=\underline{k}} \left| \varphi(x_n^{\underline{i-1}}, x_n^i - x^i, x^{i+\underline{k-i}}) \right|$$

$$\leq \sum_{i=\underline{k}} |\varphi| \cdot \bigcirc\!\!\!\!\prod \left(x_n^{\underline{i-1}}, \Delta_i, x^{i+\underline{k-i}} \right)$$

$$\leq \sum_{i=\underline{k}} |\varphi| \cdot \left| x_n^i - x^i \right| \bigcirc\!\!\!\!\prod \left(x_n^{\underline{i-1}}, x^{i+\underline{k-i}} \right)$$

$$\lesssim_n |\varphi| \sup_{i=\underline{k}} \left| x_n^i - x^i \right| \to 0$$

where we identify the product $\bigcirc\!\!\!\!\prod \left(v^{\underline{k}} \right)$ with the product of their norms $\bigcirc\!\!\!\!\prod \left(\left| v^{\underline{k}} \right| \right)$. ∎