

Chapter A: Review of Topology

Theorem Properties of Compact Spaces

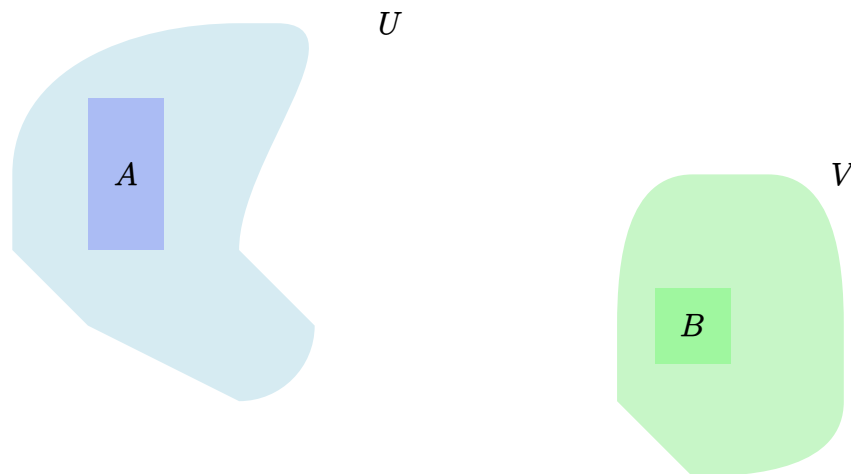


Figure 1: Closed sets A and B within open sets U and V , respectively.

Proposition 1.1

Let \mathbf{X} and \mathbf{Y} be topological spaces.

- (a) If $F \in C(\mathbf{X}, \mathbf{Y})$, and \mathbf{X} is compact, then $F(\mathbf{X})$ is compact.
- (b) If \mathbf{X} is compact and $F \in C(\mathbf{X}, \mathbb{R})$, then $F(\mathbf{X})$ is bounded, and F attains its supremum and infimum on \mathbf{X} .
- (c) A finite union of compact subspaces of \mathbf{X} is again compact.
- (d) If \mathbf{X} is Hausdorff, and A, B are disjoint, compact subspaces of \mathbf{X} , there exists open U and V , (see fig. 1).
- (e) Every closed subset of a compact space is compact.
- (f) Every compact subset of a Hausdorff space is closed.
- (g) Every compact subset of a metric space is bounded.
- (h) Every finite product of compact spaces is compact.

(i) Every quotient of a compact space is compact.

Proof of Proposition 1.1 Part A. Let $f \in C(\mathbf{X}, \mathbf{Y})$ with \mathbf{X} compact. Fix an open cover of $f(\mathbf{X})$ in the relative topology,

$$\{U_\alpha \cap f(\mathbf{X})\}_{\alpha \in A} \text{ covers } \mathbf{X}, U_\alpha \text{ open in } \mathbf{Y}$$

So that $\bigcup f^{-1}(U_\alpha) = \bigcup f^{-1}(U_\alpha \cap f(\mathbf{X})) = \mathbf{X}$. Since $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover for \mathbf{X} , this induces a finite subcollection of indices $\{\alpha_1, \dots, \alpha_n\}$ with

$$\bigcup_{j=1}^n f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^n f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^n U_{\alpha_j}$$

■

Proof of Proposition 1.1 Part B. Let \mathbf{X} be compact, and $f \in C(\mathbf{X}, \mathbb{R})$, so that $f(\mathbf{X}) \subseteq \mathbb{R}$ is compact. Compact subsets are closed and bounded in \mathbb{R} , let $A = \sup f(\mathbf{X})$ and $B = \inf f(\mathbf{X})$. Both A and B are accumulation points of $f(\mathbf{X})$, so $A = f(x)$ and $B = f(y)$ for some x, y in \mathbf{X} . ■

Proof of Proposition 1.1 Part C. Let \mathbf{X} be a topological space, and K_1, \dots, K_n be compact subspaces. Denote $K = \bigcup_{j=1}^n K_j$. Let $\{U_\alpha \cap K\}_{\alpha \in A}$ be an open cover for K , where U_α is open in \mathbf{X} . We can pass the argument to each individual K_j as follows. Let $1 \leq j \leq n$, then $\{U_\alpha \cap K_j\}_{\alpha \in A}$ is an open cover for K_j , so there exists a finite subcollection of indices $I_j \subseteq A$, (a finite subset of A) whose open sets cover K_j . Repeat this process for each j and

$$I = \bigcup_{j=1}^n I_j \text{ is a finite subset of } A$$

with $K_j \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K)$. Taking the union over all K_j reads

$$K = \bigcup_{j=1}^n K_j \subseteq \bigcup_{j=1}^n \bigcup_{\alpha \in I_j} (U_\alpha \cap K) = \bigcup_{\alpha \in I} U_\alpha \cap K$$

■

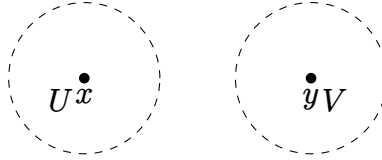


Figure 2: In a Hausdorff space, any two distinct points x and y can be separated by disjoint open neighbourhoods U and V .

Proof of Proposition 1.1 Part D. Let \mathbf{X} be Hausdorff. We first prove that compact subspaces of \mathbf{X} are closed. Indeed, if K is compact in \mathbf{X} , fix any $x \in K^c$. Let y range through the elements of K , then $x \neq y$ induces a pair of disjoint open sets U_y and V_y , such that

- $x \in U_y$
- $y \in V_y$
- $U_y \cap V_y = \emptyset$
- See fig. 2

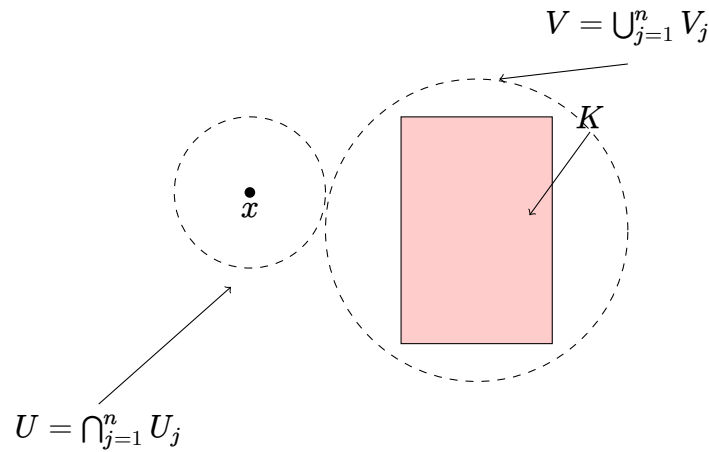


Figure 3: Compact sets are closed in Hausdorff spaces

Let V_y range through all possible $y \in K$, So that $\{V_y\}_{y \in K}$ is an open cover. There exists a finite subcollection of 'anchor points' of K , y_1, \dots, y_n that corresponds with $\{V_{y_j}\}_{j=1}^n$.

A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^n U_{y_j} \text{ is open}$$

Define $V = \bigcup_{j=1}^n V_{y_j}$, so $V \subseteq K$ and $U \cap V = \emptyset$ and $x \in U \subseteq K^c$ (see fig. 3). Therefore K is closed.

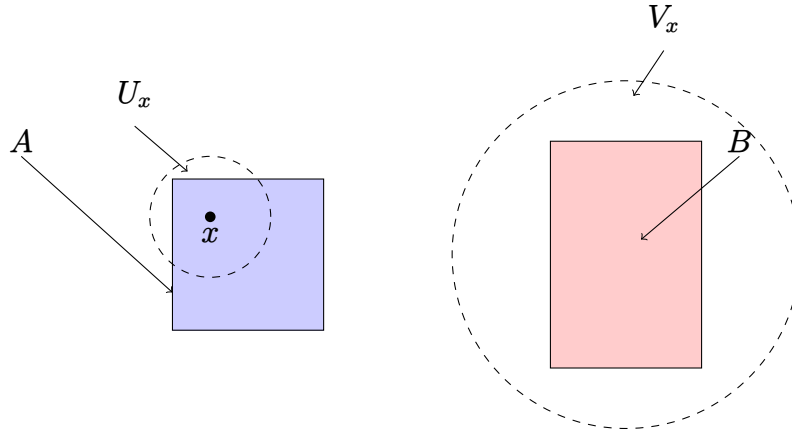


Figure 4: Closed sets A and B , point x in A , and disjoint neighbourhoods U around x and V around B .

Finally, if A and B are disjoint compact sets, then each $x \in A \subseteq B^c$ induces neighbourhoods $x \in U_x$, and $B \subseteq V_x$ (see fig. 4), let x range through all the elements of A . By compactness of A , this produces a finite subcover, and

$$U = \bigcup_{j=1}^n U_{x_j} \quad V = \bigcap_{j=1}^n V_{x_j}$$

are disjoint open sets that contain A and B respectively. ■

Proof of Proposition 1.1 Part E. Let $K \subseteq \mathbf{X}$ be a closed set of a compact space. Let $\{U_\alpha \cap K\}$ be an open cover for K , where each U_α is open in \mathbf{X} . We can append an extra set K^c which is open in \mathbf{X} . The collection

$$W = \{U_\alpha\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of W_1, \dots, W_n that cover \mathbf{X} (since \mathbf{X} is compact by itself). Remove K^c from this finite subcollection if it exists, and take the intersection with K for each element W_j , and

$$\{W_1 \cap K, \dots, W_n \cap K\} = \{U_1 \cap K, \dots, U_n \cap K\} \text{ covers } K$$

so K is compact. ■

Proof of Proposition 1.1 Part F. Proven in Part D. ■

Proof of Proposition 1.1 Part G. let $K \subseteq \mathbf{X}$ be a compact subset of the metric space (\mathbf{X}, d) . Compact subsets of \mathbf{X} are totally bounded, and hence bounded. ■

Proof of Proposition 1.1 Part H. See Tychonoff's Theorem in Folland Chapter 4. ■

Proof of Proposition 1.1 Part I. Let \mathbf{X} and \mathbf{Y} be topological spaces and $\pi : \mathbf{X} \rightarrow \mathbf{Y}$ be a quotient map. So that \mathbf{Y} is endowed with the quotient topology. So that π is a surjective continuous map. and $\pi(\mathbf{X}) = \mathbf{Y}$. Apply Part A, and we see that \mathbf{Y} is compact. ■