

**Theorem 7.2**

**WTS.** *The Riesz-Markov-Kakutani Representation Theorem. If (for every)  $I$  is a positive linear functional on  $C_c(X)$ , then there exists a unique Radon measure  $\mu$  on  $X$ , such that*

$$I(f) = \int f d\mu$$

for every  $f \in C_c(X)$ .  $\mu$  also satisfies, for every open  $U$ , and every compact  $K \subseteq X$

$$\mu(U) = \sup \{I(f), f \in C_c(X), f \prec U\} \quad (1)$$

$$\mu(K) = \inf \{I(f), f \in C_c(X), f \geq \chi_K\} \quad (2)$$

For the sake of completeness, we place the definitions for a Radon measure. Let  $X$  be a LCH space, and  $\mathbb{B}$  be its usual  $\sigma$ -algebra, a measure  $\nu$  is a Radon measure iff

- (i)  $\nu(K) < +\infty$  for every compact  $K$ .
- (ii)  $\nu$  is outer-regular on all Borel sets  $E$ ,

$$\nu(E) = \inf \{\nu(U), U \supseteq E, U \in \mathcal{T}\}$$

Intuition: approximation by open supersets.

- (iii)  $\nu$  is inner-regular on all open sets  $U \in \mathcal{T}$ ,

$$\nu(U) = \sup \{\mu(K), K \subseteq U, K \text{ compact}\}$$

Intuition: approximation by compact subsets

The main proof is extremely long, so we will divide it into several parts. Following Folland's argumentation closely, we will prove (in order)

- (a) If  $\mu_1, \mu_2$  are Radon measures on  $X$  such that for every  $f \in C_c(X)$

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

then  $\mu_1, \mu_2$  must satisfy (1), and  $\mu_1 = \mu_2$  on  $\mathbb{B}$ .

- (b) If we define, for every open set  $U$ , define  $\mu : \mathcal{T} \rightarrow [0, +\infty]$  such that

$$\mu(U) = \sup \{I(f), f \in C_c(X), f \prec U\} \quad (3)$$

Then  $\mu$  is countably subadditive, meaning for every  $U \in \mathcal{T}$ ,  $\{U_{j \geq 1}\} \subseteq \mathcal{T}$

$$U = \bigcup U_{j \geq 1} \implies \mu(U) \leq \sum \mu(U_{j \geq 1})$$

- (c)  $\mu(\emptyset) = 0$ ,  $\{\emptyset, X\} \subseteq \mathcal{T}$ , so that by Theorem 1.10  $\mu$  induces an outer-measure  $\mu^*$

$$\mu^*(E) = \inf \left\{ \sum \mu(U_{j \geq 1}), U_j \in \mathcal{T}, E \subseteq \bigcup U_{j \geq 1} \right\} \quad (4)$$

- (d) If  $\mu^*$  is as described above, then if  $\mu$  is countably subadditive on  $\mathcal{T}$ , then

$$\mu^*(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \} \quad (5)$$

Meaning the two definitions in (4) and (5) are equal.

- (e)  $\mu^*$  and  $\mu$  agree on all open sets, and  $\mu^*|_{\mathcal{T}} = \mu$ ,  
 (f) Using again the definition in (4) and (5), we show that every open set  $U \in \mathcal{T}_X$  is  $\mu^*$ -measurable, meaning for every  $E \subseteq X$ ,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

With this, since the set of all outer-measurable ( $\mu^*$ -measurable) sets,  $\mathcal{M}^*$  form a  $\sigma$ -algebra,

$$\mathcal{T} \subseteq \mathcal{M}^* \implies \mathbb{B} \subseteq \mathcal{M}^*$$

By Theorem 1.1, and define

$$\mu = \mu^*|_{\mathbb{B}} \quad (6)$$

is a Borel measure. And we note in passing that  $\mu$  is outer-regular on all  $E \in \mathbb{B}$ ,

$$\mu(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \} \quad (7)$$

(g) Using (6) for the definition of  $\mu$  on  $\mathbb{B}$ , we prove that

- $\mu$  is outer-regular on all Borel sets, and
- $\mu$  satisfies Equation (1).

(h)  $\mu$  satisfies Equation (2)

(i)  $\mu$  is finite on all compact sets.

(j)  $\mu$  is inner-regular on all open sets.

(k) For every  $f \in C_c(X, [0, 1])$ ,

$$I(f) = \int f d\mu \quad (8)$$

(l) For every  $f \in C_c(X)$ ,

$$I(f) = \int f d\mu \quad (9)$$

A small lemma needs to be made before proceeding, that concerns the 'monotonicity' of  $I$  on  $C_c X$ .

**Lemma 1.1** *Suppose that  $f, g \in C_c(X)$ , and  $f \geq g \geq 0$  for every  $X$ , then  $f - g \in C_c(X)$  and  $I(f) \geq I(g)$*

*Proof.* Suppose that  $x \in X$  where  $f(x) = 0$ , then

$$f(x) - g(x) = -g(x) \geq 0 \implies g(x) = 0 \implies f - g = 0$$

Hence

$$\begin{aligned} \{x, f(x) = 0\} &\subseteq \{x, f(x) - g(x) = 0\} \implies \{x, f(x) - g(x) \neq 0\} \subseteq \{x, f(x) \neq 0\} \\ &\implies \text{supp}(f - g) \subseteq \text{supp}(f) \end{aligned}$$

Since  $\text{supp}(f)$  is compact, and  $\text{supp}(f - g)$  is a closed subset of  $\text{supp}(f)$ , yields  $f - g \in C_c(X)$ . And if  $I$  is any positive linear functional on  $C_c(X)$ , then

$$\begin{aligned} f - g \geq 0 &\implies I(f - g) \geq 0 \\ &\implies I(f) \geq I(g) \geq 0 \end{aligned}$$

■

**Remark 1.1** If  $f \prec U$  and  $g \prec U$  for some open subset  $U \subseteq X$ , then clearly  $\text{supp}(f - g) \subseteq \text{supp}(f) \subseteq U$ , and  $1 \geq f \geq f - g \geq 0$  means that  $f - g \prec U$  as well.

### Part a

*Proof.* Suppose that  $\mu_1$  and  $\mu_2$  are Radon measures on  $X$ , and for every  $f \in C_c(X)$ ,

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

We first prove (1). Without loss of generality, by monotonicity of  $L^+$ , if  $f \prec U$  for some open  $U$ , then  $0 \leq f \leq \|f\|_u \chi_U = \chi_U$  for all  $x$  and

$$\int f d\mu_1 \leq \int \|f\|_u \chi_U d\mu_1 \leq \mu_1(U)$$

Therefore  $\mu_1(U)$  (resp.  $\mu_2(U)$ ) is an upper-bound for the set

$$\{I(f), f \in C_c(X), f \prec U\}$$

Since  $\mu_1$  is inner-regular on  $U \in \mathcal{T}$ , for every  $\varepsilon > 0$  we can find some compact  $K \subseteq U$  where

$$\mu_1(U) - \varepsilon < \mu_1(K)$$

By Urysohn's Lemma (Theorem 4.32), there exists some  $g \in C_c(X)$  with

- $g \in C_c(X, [0, 1])$ ,
- $g = 1$  on  $K \subseteq U$ ,
- $g = 0$  outside some  $\bar{V} \subseteq U$ , and
- $g \prec U$ .

Hence for every  $x \in K$ ,  $g \geq \chi_K$ . If  $x \notin K$  then  $g \geq 0 = \chi_K$ ; so  $g - \chi_K \geq 0$  for every  $x \in X$ . Since  $\chi_K \prec U$ , using Lemma 1.1, we get

$$\mu_1(K) \leq \int \chi_K d\mu_1 = I(\chi_K) \leq I(g)$$

So for every  $\varepsilon > 0$ , there exists a  $g \in C_c(X)$ , and  $g \prec U$  where

$$\mu_1(U) - \varepsilon < \mu_1(K) \leq I(g)$$

Therefore  $\mu_1(U) = \sup \{I(f), f \in C_c(X), f \prec U\}$ , and the first claim in (a) is proven. To show that  $\mu$  is indeed unique, since for every open set  $U$ , we must have  $\mu_1(U) = \mu_2(U)$ , and if  $E \in \mathbb{B}$  is any Borel set, and by outer-regularity,

$$\mu_1(E) = \inf \{\mu_1(U), U \supseteq E, U \in \mathcal{T}\} = \inf \{\mu_2(U), U \supseteq E, U \in \mathcal{T}\} = \mu_2(E)$$

Therefore this measure is unique. ■

### Part b

*Proof.* To show countable subadditivity for  $\mu$  with equation (3), fix any  $U \in \mathcal{T}$  and a sequence  $\{U_{j \geq 1}\} \subseteq \mathcal{T}$  with  $U = \bigcup U_{j \geq 1}$ . It suffices to show that the partial sum of  $\sum \mu(U_{j \leq n})$  is greater than  $I(f)$  for any  $f \in C_c(X)$ ,  $f \prec U$  (hence it is an upper bound).

Fix any  $f$ , then denote  $K = \text{supp}(f) \subseteq U$ , and since  $\{U_{j \geq 1}\}$  is an open cover for  $K$ , there exists a finite subcollection,  $B \subseteq \mathbb{N}^+$  such that

$$K \subseteq \bigcup_{j \in B} U_j$$

Using Theorem 4.41 on this finite cover of  $K$ , there exists a partition of unity in  $\{g_{j \leq n}\}$  where

- $g_j \in C_c(X, [0, 1])$ ,
- $g_j \prec U_j \subseteq U$  for every  $j \leq n$ , and
- $\sum g_j = 1$  on  $K$ ,

And notice for every  $j \leq n$ ,

$$\begin{aligned} \{f = 0\} \cup \{g_j = 0\} &\subseteq \{f \cdot g_j = 0\} \implies \{f \cdot g_j \neq 0\} \subseteq \{f \neq 0\} \cap \{g_j \neq 0\} \\ &\implies \text{supp}(f \cdot g_j) \subseteq \text{supp}(f) \cap \text{supp}(g_j) \\ &\implies \text{supp}(f \cdot g_j) \subseteq U_j \subseteq U \end{aligned}$$

Hence  $f \cdot g_j \prec U$  and  $f \cdot g_j \in C_c(X, [0, 1])$  for every  $1 \leq j \leq n$ . Moreover, if we take the sum over a finite  $n$ , we obtain  $f = \sum f \cdot g_{j \leq n}$ , this is because for every  $x \in X$ , so we have

$$\sum_{j \leq n} f(x) \cdot g_j(x) = f(x) \cdot \sum_{j \leq n} g_j(x) = f(x)$$

Then  $I(f) = I(\sum f \cdot g_j) = \sum I(f \cdot g_j)$ . And by definition of  $\mu(U_j)$ , since it is the supremum over all  $I(h_j)$ , where  $h_j \in C_c(X, [0, 1])$  and  $h_j \prec U_j$

$$I(f \cdot g_j) \leq \mu(U_j), \quad \forall j \leq n$$

Hence

$$I(f) \leq \sum_{j \leq n} \mu(U_j) \leq \sum_{j \geq 1} \mu(U_j)$$

Where for the last estimate we used the fact that  $\mu$  is non-negative, and since this holds for any  $f$ , we can conclude that  $\mu(U) \leq \sum_{j \geq 1} \mu(U_j)$ . ■

### Part c

*Proof.* By definition of a topology,  $\{\emptyset, X\} \subseteq \mathcal{T}$ , and  $\mu(\emptyset) = \sup\{I(f), f \in C_c(X), f \prec \emptyset\}$ , so  $\text{supp}(f) = \emptyset$ , and  $\{x, f(x) \neq 0\} \subseteq \emptyset$ , so the set contains one element, namely  $I(0) = 0$  by linearity. So  $\mu(\emptyset) = 0$ . The assumptions for Theorem 1.10 are satisfied and (4) is indeed an outer-measure. ■

### Part d

*Proof.* Denote the right members of (4) and (5) by  $W_1$  and  $W_2$ , we wish to show that  $\inf W_1 = \inf W_2$ . Clearly  $\inf W_1 \leq \inf W_2$ , since  $W_2 \subseteq W_1$ . Now, if  $\mu$  is countably additive, then for every  $\omega \in W_1$  induces a sequence of open sets  $\{U_{j \geq 1}\}$  such that  $E \subseteq \bigcup U_{j \geq 1}$ . Denote the union over  $\{U_{j \geq 1}\}$  by  $U$ , which is also another open set,

$$\inf W_2 \leq \mu(U) \leq \sum \mu(U_{j \geq 1}) = \omega$$

Since  $\omega$  is arbitrary, we conclude that  $\inf W_2 = \inf W_1$ , and this proves (d). ■

### Part e

*Proof.* If  $U$  and  $V$  are open subsets of  $X$ , and if  $U \subseteq V$ , then

$$\begin{aligned} U \subseteq V &\implies \{f \in C_c(X), f \prec U\} \subseteq \{f \in C_c(X), f \prec V\} \\ &\implies \{I(f), f \in C_c(X), f \prec U\} \subseteq \{I(f), f \in C_c(X), f \prec V\} \end{aligned}$$

Hence  $\mu(U) \leq \mu(V)$ . Now by equation (5),  $\mu^*(U) \leq \mu(U)$ . To show the reverse inequality, suppose by contradiction that  $\mu^*(U) < \mu(U)$ .

Since  $\mu^*(U)$  is an infimum, then for every  $\varepsilon > 0$  there exists some  $V \supseteq U$  where if we write  $\mu^*(U) + \varepsilon = \mu(U)$

$$\mu(V) < \mu^*(U) + \varepsilon = \mu(U) \implies \mu(V) < \mu(U), U \subseteq V$$

This contradicts what we have just proven, and therefore  $\mu^*(U) = \mu(U)$  for every open set  $U$ . ■

### Part f

*Proof.* We wish to show that every open set  $U$  is  $\mu^*$ -measurable. By Theorem 1.10, it suffices to show that for every  $E \subseteq X$

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U) \quad (10)$$

because the reverse inequality is given by subadditivity of  $\mu^*$ , and we can also assume that  $\mu^*(E) < +\infty$ . Let us assume that  $E$  is open, we wish to find some function  $h \in C_c(X)$ ,  $h \prec E$  with

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

The above formula is fussy, but the liberty is taken to show it beforehand to avoid any potential confusion that follows. Since  $E \cap U$  is an open subset of  $X$ , the definition of  $\mu(E \cap U) = \mu^*(E \cap U)$  in (3) tells us that every  $\varepsilon > 0$  induces some  $f \in C_c(X)$ ,  $f \prec E \cap U$  where

$$I(f) > \mu(E \cap U) - \varepsilon = \mu^*(E \cap U) - \varepsilon \quad (11)$$

Also,  $\text{supp}(f)$  is a closed set (compact subsets of Hausdorff spaces are closed), therefore  $E \setminus \text{supp}(f)$  is an open set. We make a small diversion from the current part of the proof and turn our attention to the fact that

$$\begin{aligned} \text{supp}(f) \subseteq U &\implies U^c \subseteq (\text{supp}(f))^c \\ &\implies E \setminus U \subseteq E \setminus \text{supp}(f) \end{aligned}$$

And because the outer-measure  $\mu^*$  is monotone,

$$\mu^*(U) \leq \mu^*(E \setminus \text{supp}(f)) \quad (12)$$

Now, using the definition of  $\mu(E \setminus \text{supp}(f))$  (recall that  $E \setminus \text{supp}(f)$  is an open set), for every  $\varepsilon > 0$ , there exists some  $g \in C_c(X)$ ,  $g \prec E \setminus \text{supp}(f)$  with

$$I(g) > \mu(E \setminus \text{supp}(f)) - \varepsilon = \mu^*(E \setminus \text{supp}(f)) - \varepsilon \quad (13)$$

It is at this part of the proof where we wish to define  $h = f + g$ , but first we must verify

- $f + g \in C_c(X, [0, 1])$ ,
- $f + g \prec E$

The sum of two non-negative functions is non-negative, and for every  $x \in \text{supp}(f)$ ,  $f \leq 1$ . Also

$$\begin{aligned} \text{supp}(g) \subseteq (\text{supp}(f))^c &\implies \text{supp}(f) \subseteq (\text{supp}(g))^c \\ &\implies \text{supp}(f) \subseteq \{g = 0\} \end{aligned}$$

The last implication comes from taking complements on both sides of  $\{g \neq 0\} \subseteq \text{supp}(g)$ . So  $x \in \text{supp}(f) \implies f + g \leq 1$ . Now if  $x \notin \text{supp}(f)$ , then  $f + g = g \leq 1$ . Furthermore,  $\text{supp}(f + g)$  is a closed subset of compact  $\text{supp}(f) \cup \text{supp}(g)$ . This is because  $\{f + g \neq 0\} \subseteq \{f \neq 0\} \cup \{g \neq 0\}$ , and the finite union of two compact sets is again compact.

A moment's thought should yield the fact that the last estimate should be an equality, but it is a needless distraction. Therefore  $\text{supp}(f + g)$  is compact and  $f + g \in C_c(X, [0, 1])$ .

Now both bullet points are satisfied, and we can set  $h = f + g$ . Adding equation (13) with (11) gives us

$$I(h) = I(f) + I(g) > \mu^*(E \cap U) + \mu^*(E \setminus \text{supp}(f)) - 2\varepsilon$$

Upon applying (12) to the right member of the above estimate, we have

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

But this particular  $h \in C_c(X) \cap \{f \prec E\}$ , therefore

$$\mu^*(E) \geq I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Since  $\varepsilon > 0$  is arbitrary, equation (10) holds for every open  $E$ . Now for any general  $E \subseteq X$ , fix any  $\varepsilon > 0$  and by how we defined  $\mu^*(E)$ , there exists some open  $V \supseteq E$  — recall that  $\mu^*(E)$  is the infimum over the set of  $\mu(V)$  where  $V$  is an open superset of  $E$  — hence

$$\mu^*(E) + \varepsilon > \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$



By monotonicity (twice) of the outer-measure  $\mu^*$ , we have

$$\mu^*(E) + \varepsilon > \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Let  $\varepsilon \rightarrow 0$ , and we get

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Therefore every open  $U \subseteq X$  is  $\mu^*$ -measurable. So  $\mu = \mu^*|_{\mathbb{B}}$  is a Borel measure on  $X$ . ■

### Part g

*Proof.* To show outer-regularity, fix any  $E \in \mathbb{B}$ , then by definition,

$$\mu(E) = \mu^*(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \}$$

And for every open  $U$ , (1) follows from Equation (3). ■

### Part h

*Proof.* We want to show that for every compact  $K$ , Equation (2) holds. To reduce the notational baggage that follows, we agree to define

$$\{I(f), f \in C_c(X), f \prec U\} = \{I(f), f \prec U\}$$

Similarly for  $\{I(f), f \geq \chi_K\}$ . If  $\mu(K) = 0$ , then  $\mu(K)$  is obviously a lower bound, since  $f \geq \chi_K \geq 0$  means that  $I(f) \geq 0$ , for every  $f \geq \chi_K$ . So we can suppose  $\mu(K) > 0$ .

Fix an arbitrary  $f \geq \chi_K$ , then this particular  $f$  induces an open set  $U_\alpha = \{f > 1 - \alpha\}$ , where  $\alpha > 0$ . Notice also that

$$K \subseteq \{f \geq 1\} \subseteq \{f > 1 - \alpha\} = U_\alpha$$

Since  $U_\alpha$  is an open superset of  $K$ , by Equation (7),  $\mu(K) \leq \mu(U_\alpha)$ , but  $\mu(U_\alpha)$  is simply the supremum of  $\{I(g), g \prec U_\alpha\}$ . If we wish to show that  $\mu(K) \leq \mu(U_\alpha) \leq I(f)$ , it suffices to show that  $I(f)$  is an upper-bound for  $\{I(g), g \prec U_\alpha\}$ .

Fix any  $I(g) \in \{I(g), g \prec U_\alpha\}$ , note that  $1 - \alpha \neq 0$  for any  $\alpha$  small enough, then

- $f/(1 - \alpha) > 1$  on  $U_\alpha$ ,
- $1 \geq g \geq 0$  on  $U_\alpha$ , in particular,  $f/(1 - \alpha) - g \geq 0$  on  $U_\alpha$ ,
- If  $x \notin U_\alpha$ , then  $f/(1 - \alpha) - g = f(1 - \alpha) \geq 0$ .
- Therefore  $f/(1 - \alpha) - g \geq 0$  for any  $x$ , and by Lemma 1.1,

$$I(f/(1 - \alpha)) \geq I(g) \quad \forall g \prec U_\alpha$$

Combining the above estimate with  $\mu(K) \leq \mu(U_\alpha)$  gives us

$$\mu(K) \leq \frac{1}{1 - \alpha} I(f)$$

Now write  $\varepsilon = \alpha/\mu(K) > 0$  and for every  $\varepsilon > 0$  we get

$$\mu(K) - I(f) \leq \alpha\mu(K) = \varepsilon$$

Send  $\varepsilon \rightarrow 0$  and  $\mu(K) \leq I(f)$  for every  $f \geq \chi_K$ .

To show that  $\mu(K)$  is indeed the infimum for  $\{I(f), f \geq \chi_K\}$ , notice that for every  $\varepsilon > 0$  we can obtain some open superset  $U \supseteq K$  (by outer-regularity) where  $\mu(U) < \mu(K) + \varepsilon$ . By Urysohn's Lemma, there exists some  $g \prec U$ ,  $g(x) = 1$  for every  $x \in K$ .

$$g \in \{I(f), f \prec U\} \cap \{I(f), f \geq \chi_K\}$$

Therefore  $I(g) \leq \mu(U) < \mu(K) + \varepsilon$  as desired, and Equation (2) holds. ■

### Part i

*Proof.*  $\mu(K) < +\infty$  for every compact  $K$ . Indeed, since  $I(\chi_K) \in \{I(f), f \geq \chi_K\}$ , then by Theorem 7.1, there exists a constant  $C_K \geq 0$  that bounds

$$\mu(K) \leq |I(\chi_K)| = I(\chi_K) \leq C_K \cdot \|\chi_K\| = C_K < +\infty$$

■

**Part j**

*Proof.* Fix any open set  $U$ , then for every  $\varepsilon > 0$ , there exists some  $f \prec U$  with  $\mu(U) - \varepsilon < I(f)$ . Then denote  $K = \text{supp}(f) \subseteq U$ . If we take any  $I(h) \in \{I(h), h \geq \chi_K\}$ , then  $h \geq f$  gives us  $I(h) \geq I(f)$  by Lemma 1.1. So  $I(f)$  is a lower bound of  $\{I(h), h \geq \chi_K\}$ , therefore

$$\mu(U) - \varepsilon \leq I(f) \leq \mu(K)$$

Since  $\text{supp}(f) = K \subseteq U$ , this proves inner-regularity of  $\mu$  on open sets. ■

**Part k**

*Proof.* Suppose  $f \in C_c(X, [0, 1])$ , we first show that Equation (8) holds. We divide the interval  $[0, 1]$  into  $N \geq 1$  chunks by writing

$$K_j = \{f \geq j/N\}$$

for every  $1 \geq j \geq N$ . And define  $K_0 = \text{supp}(f)$ . Each  $K_j$  is a closed subset of  $\text{supp}(f)$ , and therefore compact. More is true,

- $K_{j-1} \supseteq K_j$  for every  $1 \leq j \leq N$ .
- $x \in K_j$  iff  $f(x) \in [\frac{j}{N}, 1]$ ,
- $x \notin K_j$  iff  $f(x) \in [0, \frac{j}{N})$ , and
- $x \in (K_{j-1} \setminus K_j)$  iff  $f(x) \in [\frac{j-1}{N}, \frac{j}{N})$

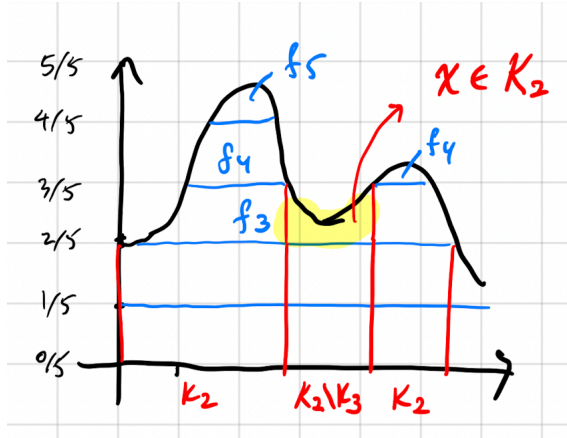
Folland constructs a finite sequence of compactly supported functions,  $\{f_j\}$ , where  $1 \leq j \leq N$  such that

- Each  $0 \leq f_j \leq 1/N$ ,
- If  $x \in (K_m \setminus K_{m+1})$  iff  $f(x) \in [\frac{m}{N}, \frac{m+1}{N})$  means that  $f_j = 1$  for all  $1 \leq j \leq m$ , and
- $f_{m+1} = f - m/N$  on  $K_m$ , such that

$$f(x) = \left(\sum f_{j \leq m}(x)\right) + \left(f(x) - \frac{m}{N}\right) = \frac{m}{N} + \left(f(x) - \frac{m}{N}\right)$$

- And for every  $m < j \leq N$ ,  $f_j = 0$ .
- If  $x \notin K_m$  iff  $f(x) \in [0, \frac{m}{N})$  then for every  $m + 1 \leq j \leq N$ ,  $f_j = 0$ .

The illustration for when  $N = 5$  below should make things clearer.



It is also trivial to verify that

- For every  $x \in K_j$ ,  $f_j = N^{-1}$ , and

$$\chi_{K_j} N^{-1} \leq f_j \quad (14)$$

Also, if  $x \notin K_j$  then  $f_j \geq 0$ , therefore  $f_j \geq \chi_{K_j} N^{-1}$  at every  $x$ .

- If  $x \notin K_{j-1}$  then  $f_j = 0 \leq \chi_{K_{j-1}} \cdot N^{-1}$ . If  $x$  is in  $K_{j-1}$  then  $f_j \leq N^{-1}$  by construction and therefore

$$f_j \leq \chi_{K_{j-1}} N^{-1} \quad (15)$$

for all  $x$ .

- $f_j \in C_c(X)$ , since  $\text{supp}(f_j) \subseteq \text{supp}(f)$ .

Combining Equations (14) with (15), and by monotonicity in  $L^+(X, \mathbb{B}, \mu)$ , since  $f_j \in L^+$

$$\int \frac{1}{N} \chi_{K_j} d\mu \leq \int f_j d\mu \leq \int \frac{1}{N} \chi_{K_{j-1}} d\mu$$

And for every  $1 \leq j \leq N$ ,

$$\frac{1}{N} \mu(K_j) \leq \int f_j d\mu \leq \frac{1}{N} \mu(K_{j-1}) \quad (16)$$

Furthermore, from Equation (14), since  $Nf_j \geq \chi_{K_j}$  then by Equation (2),

$$\mu(K_j) \leq I(Nf_j) \implies \frac{1}{N}\mu(K_j) \leq I(f_j)$$

Now for any arbitrary  $I(h) \in \{I(h), h \geq \chi_{K_{j-1}}\}$ , since

$$h \geq \chi_{K_{j-1}} \geq Nf_j \implies I(h) \geq I(Nf_j)$$

So  $NI(f_j)$  is a lower bound for  $\{I(h), h \geq \chi_{K_{j-1}}\}$  and

$$I(f_j) \leq \frac{1}{N}\mu(K_{j-1})$$

Combining the last two results, with  $I(f_j)$ , we get

$$\frac{1}{N}\mu(K_j) \leq I(f_j) \leq \frac{1}{N}\mu(K_{j-1}) \quad (17)$$

Taking the sum over  $1 \leq j \leq N$  for Equations (16) and (17). Define  $A = N^{-1} \sum_0^{N-1} \mu(K_j)$ , and  $B = N^{-1} \sum_1^N \mu(K_j)$

$$B \leq \int f d\mu \leq A$$

And also

$$B \leq I(f) \leq A$$

This is because of finite additivity of both  $I$  and the integral, and  $f = \sum f_j$  on  $K_0 = \text{supp}(f)$ . Subtracting the two equations (keeping in mind that  $\mu(K_j) < +\infty$  for any compact  $K_j$ ), we get

$$(-1)(A - B) \leq \left( \int f d\mu - I(f) \right) \leq A - B \implies \left| \int f d\mu - I(f) \right| \leq A - B$$

It is trivial to verify that

$$0 \leq A - B = N^{-1}(\mu(K_0) - \mu(K_N)) \leq N^{-1}\mu(K_0)$$

as  $K_N \subseteq K_0$ . Let  $N \rightarrow \infty$  and

$$\int f d\mu = I(f)$$

Equation (8) holds as desired. ■

**Part 1**

*Proof.* Now for any general  $f \in C_c(X)$ ,  $f$  must be bounded on the plane since  $C_c(X) \subseteq BC(X)$ , and  $|f| \leq M_0$  for some  $M_0 \geq 0$ . Since  $\text{supp}(f)$  is compact, we know that

$$\int |f| d\mu \leq \int M_0 \chi_{\text{supp}(f)} d\mu \leq M_0 \mu(\text{supp}(f)) < +\infty$$

And  $C_c(X) \subseteq L^1(\mu)$ . Furthermore,

$$\frac{1}{2}(|\text{Re } f| + |\text{Im } f|) \leq |f| \leq M_0$$

So that  $\text{Re } f$  and  $\text{Im } f$  are in  $C_c(X)$ . Without loss of generality, we may assume that  $f$  is real. Define  $f_1 = \text{Re } f^+/M_0$  and  $f_2 = \text{Re } f^-/M_0$  and it immediately follows that  $f_1, f_2 \in C_c(X, [0, 1])$ .

By linearity of  $I$  on  $C_c(X)$  and the integral in  $L^1(\mu)$ ,

$$I(f_1 - f_2) = I(f) = \int f d\mu = \int f_1 d\mu - \int f_2 d\mu$$

Then we may apply the above to the real and imaginary parts of a general  $f \in C_c(X)$ , and this completes the proof. ■