Chapter A: Review of Topology

Theorem Properties of Compact Spaces

Proposition 1.1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be topological spaces.*

(a) If $F \in C(\mathbf{X}, \mathbf{Y})$*, and* $\mathbf{X}$ *is compact, then* $F(\mathbf{X})$ *is compact.*

(b) If $\mathbf{X}$ *is compact and* $F \in C(\mathbf{X}, \mathbb{R})$*, then* $F(\mathbf{X})$ *is bounded, and* F *attains its supremum and infimum on* $\mathbf{X}$*.*

(c) A finite union of compact subspaces of $\mathbf{X}$ *is again compact.*

(d) If $\mathbf{X}$ *is Hausdorff, and* A*,* B *are disjoint, compact subspaces of* $\mathbf{X}$*, there exists open* U *and* V *st*

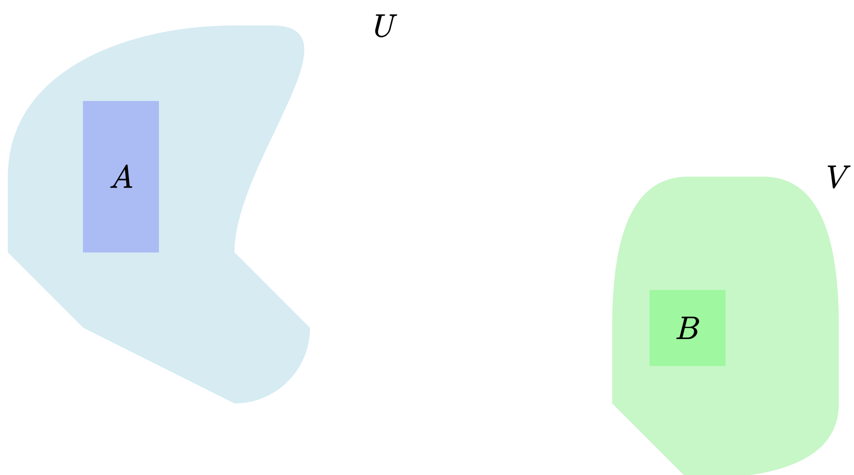

Figure 1: Closed sets A and B within open sets U and V, respectively.

(e) Every closed subset of a compact space is compact.

(f) Every compact subset of a Hausdorff space is closed.

(g) Every compact subset of a metric space is bounded.

(h) Every finite product of compact spaces is compact.

(i) Every quotient of a compact space is compact.

Proof of Proposition 1.1 Part A. Let $f \in C(\mathbf{X}, \mathbf{Y})$ with $\mathbf{X}$ compact. Fix an open cover of $f(\mathbf{X})$ in the relative topology,

$$\{U_\alpha \cap f(\mathbf{X})\}_{\alpha \in A} \text{ covers } \mathbf{X},\ U_\alpha \text{ open in } \mathbf{Y}$$

So that $\bigcup f^{-1}(U_\alpha) = \bigcup f^{-1}(U_\alpha \cap f(\mathbf{X})) = \mathbf{X}$. Since $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover for $\mathbf{X}$, this induces a finite subcollection of indices $\{\alpha_1, \ldots, \alpha_n\}$ with

$$\bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^{n} f^{-1}(U_\alpha \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^{n} f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^{n} U_{\alpha_j}$$

■

Proof of Proposition 1.1 Part B. Let $\mathbf{X}$ be compact, and $f \in C(\mathbf{X}, \mathbb{R})$, so that $f(\mathbf{X}) \subseteq \mathbb{R}$ is compact. Compact subsets are closed and bounded in $\mathbb{R}$, let $A = \sup f(\mathbf{X})$ and $B = \inf f(\mathbf{X})$. Both A and B are accumulation points of $f(\mathbf{X})$, so $A = f(x)$ and $B = f(y)$ for some x, y in $\mathbf{X}$. ■

Proof of Proposition 1.1 Part C. Let $\mathbf{X}$ be a topological space, and $K_1, \ldots K_n$ be compact subspaces. Denote $K = \bigcup_{j=1}^{n} K_j$. Let $\{U_\alpha \cap K\}_{\alpha \in A}$ be an open cover for K, where U_α is open in $\mathbf{X}$. We can pass the argument to each individual K_j as follows. Let $1 \leq j \leq n$, then $\{U_\alpha \cap K_j\}_{\alpha \in A}$ is an oepn cover for K_j, so there exists as finite subcollection of indices $I_j \subseteq A$, (a finite subset of A) whose open sets cover K_j. Repeat this process for each j and

$$I = \bigcup_{j=1}^{n} I_j \text{ is a finite subset of } A$$

with $K_j \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K)$. Taking the union over all K_j reads

$$K = \bigcup_{j=1}^{n} K_j \subseteq \bigcup_{j=1}^{n} \bigcup_{\alpha \in I_j} (U_\alpha \cap K) = \bigcup_{\alpha \in I} U_\alpha \cap I$$

■

Proof of Proposition 1.1 Part D. Let $\mathbf{X}$ be Hausdorff. We first prove that compact subspaces of $\mathbf{X}$ are closed. Indeed, if K is compact in $\mathbf{X}$, fix any $x \in K^c$. Let y range through the elements of K, then $x \neq y$ induces a pair of disjoint open sets U_y and V_y, such that

- $x \in U_y$
- $y \in V_y$
- $U_y \cap V_y = \varnothing$
- Picture below

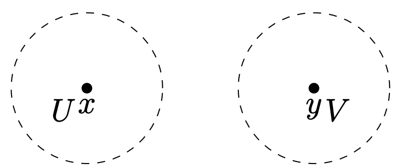

Figure 2: In a Hausdorff space, any two distinct points x and y can be separated by disjoint open neighbourhoods U and V.

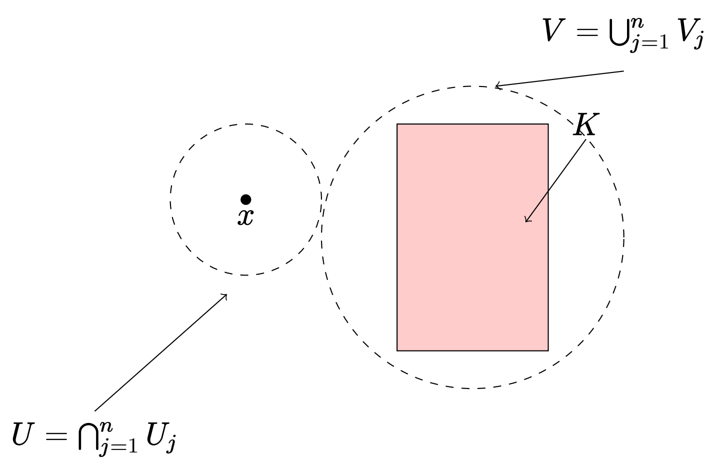

Figure 3: Compact sets are closed in Hausdorff spaces

Let V_y range through all possible $y \in K$, So that $\{V_y\}_{y \in K}$ is an open cover. There exists a finite subcollection of 'anchor points' of K, $y_1, \ldots, y_n$ that corresponds with $\{V_{y_j}\}_{j=1}^{n}$.

A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^{n} U_{y_j} \text{ is open}$$

Define $V = \bigcup_{j=1}^{n} V_{y_j}$, so $V \subseteq K$ and $U \cap V = \varnothing$ and $x \in U \subseteq K^c$ (see fig. 3). Therefore K is closed.

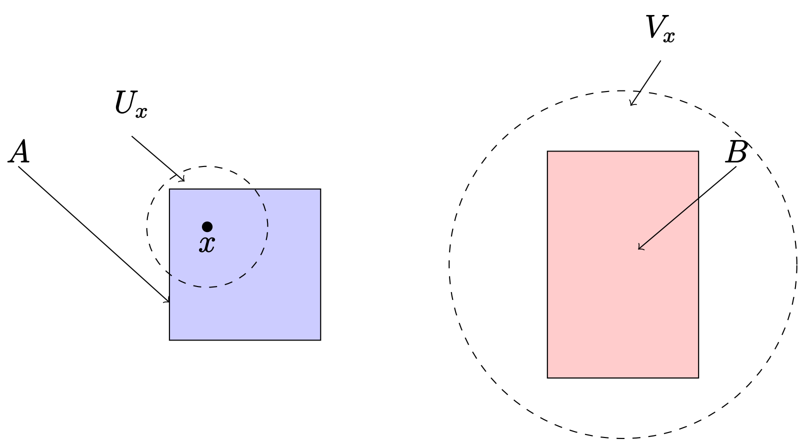

Figure 4: Closed sets A and B, point x in A, and disjoint neighbourhoods U around x and V around B.

Finally, if A and B are disjoint compact sets, then each $x \in A \subseteq B^c$ induces neighbourhoods $x \in U_x$, and $B \subseteq V_x$ (see fig. 4), let x range through all the elements of A. By compactness of A, this produces a finite subcover, and

$$U = \bigcup_{j=1}^{n} U_{x_j} \quad V = \bigcap_{j=1}^{n} V_{x_j}$$

are disjoint open sets that contain A and B respectively.

■

Proof of Proposition 1.1 Part E. Let $K \subseteq \mathbf{X}$ be a closed set of a compact space. Let $\{U_\alpha \cap K\}$ be an open cover for K, where each U_α is open in $\mathbf{X}$. We can append an extra set K^c which is open in $\mathbf{X}$. The collection

$$W = \{U_\alpha\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of $W_1, \ldots, W_n$ that cover $\mathbf{X}$ (since $\mathbf{X}$ is comapct by itself). Remove K^c from this finite subcollection if it exists, and take the intersection with K for each element W_j, and

$$\{W_1 \cap K, \ldots, W_n \cap K\} = \{U_1 \cap K, \ldots, U_n \cap K\} \text{ covers } K$$

so K is compact. ■

Proof of Proposition 1.1 Part F. Proven in Part D. ■

Proof of Proposition 1.1 Part G. let $K \subseteq \mathbf{X}$ be a compact subset of the metric space $(\mathbf{X}, d)$. Compact subsets of $\mathbf{X}$ are totally bounded, and hence bounded. ■

Proof of Proposition 1.1 Part H. See Tynchonoff's Theorem in Folland Chapter 4. ■

Proof of Proposition 1.1 Part I. Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces and $\pi : \mathbf{X} \to \mathbf{Y}$ be a quotient map. So that $\mathbf{Y}$ is endowed with the quotient topology. So that π is a surjective continuous map. and $\pi(\mathbf{X}) = \mathbf{Y}$. Apply Part A, and we see that $\mathbf{Y}$ is compact. ■