Chapter A: Review of Topology

Proposition 0.1: Folland Theorem 4.14

Suppose that A and B are disjoint closed subsets of the normal space X, and let $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that

1. $A \subseteq U_r \subseteq B^c$ for every $r \in \Delta$,
2. $\overline{U_r} \subseteq U_s$ for $r < s$, and
3. For every $r < s$, $\overline{U}_r \subseteq U_s$

Proof. The goal of this proof is to show that for every $r \in \Delta$, there exists a open U_r that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some $N \geq 1$ that all previous U_r in previous layers have been constructed properly, meaning if $r = k/2^n$, then for every $1 \leq n \leq N - 1$, we have

$$r = \frac{k}{2^n},\ 1 \leq n \leq N - 1,\ 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each U_r,

- $A \subseteq U_r \subseteq B^c$ and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer $N \geq 1$, we only have to construct the $U_{k/2^N}$ for every odd k, this is because if k is an even number, then $k = 2j$ and $r = 2j/2^N = j/2^{N-1}$ and for this particular U_r is already constructed. So for every odd $k = 2j + 1$, the sets of the form $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every $k - 1 \neq 0$ and $k + 1 \neq 1$. (We will consider these cases later). We claim that for every pair of open sets, $E_1, E_2 \in \mathcal{T}_X$, then there exists some open set $G \in \mathcal{T}_X$ such that if $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$ where H is defined as the set

$$H = \left\{(E_1, E_2) \subseteq (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \varnothing\right\}$$

Then there exists some $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$ such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any $(E_1, E_2) \in H$, then this pair induces a pair of disjoint sets $\overline{E_1}$ and E_2^c since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \varnothing$$

And by normality, there exists disjoint open sets G_1, G_2 such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \varnothing \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since G_2^c is a closed set that contains G_1 as a subset, $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of G_2 (so we may discard the notion of G_2 in our minds). Let us now replace G with G_1 then it is an easy task to verify that $G = G_1 = \mathcal{J}(E_1, E_2)$ has the required properties.

Now define for every odd k, since $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (we note in passing that $\mathcal{J}$ is not a function as the set G may not be unique).

$$U_{k/2^N} = \mathcal{J}\left(U_{(k-1)/2^N}, U_{(k+1)/2^N}\right)$$

Then, if $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$ sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$ and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed' $U_{k/2^N}$ for every $k-1 \neq 0$ and $k+1 \neq 1$. Now let us deal with the remaining pathological cases.

If $k-1$ so happens to be 0, then no $r \in \Delta$ satisfies $r = 0/2^N$, and we substitute

$$\overline{U}_0 = A, \qquad \text{or alternatively, } U_0 = A^o$$

Then $U_0 \in \mathcal{T}_X$, $\overline{U}_0 = A \subseteq B^c$. It is at this point that we must mention that $0, 1 \notin \Delta$, so U_0 and U_1 do not have to obey the rules we have laid out for $U_{r\in\Delta}$.

Now if $k+1$ is equal to 2^N (this makes $r = (k+1)/2^N = 1$) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every $0 \leq m \leq 2^N - 1, U_{m/2^N}$ must staisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (even for when $N = 1$, since $A = \overline{U}_0 \subseteq U_1 = B^c$) and a corresponding $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$ such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when $N = 1$. We would only have to construct for $k = 1$, since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■

Proposition 0.2: Folland Theorem 4.15: Urysohn's Lemma

Urysohn's Lemma. Let X be a normal space, if A and B are disjoint closed subsets of X, then there exists a $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Proof. Let $r \in \Delta$ be as in Lemma 4.14, and set U_r accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0,1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic ε argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since A and B are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that x is not a member of any U_r, but we set $U_1 = X$. Since none of the $r \in (0,1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The ε argument follows: suppose for every $\varepsilon > 0$, $(1-\varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in X. So $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of Δ, there exists an r, $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r<\alpha} U_r$. So $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r<\alpha} U_r$.

Fix an element $x \in \bigcup_{r<\alpha} U_r$, this induces an r such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r<\alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$.

For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an r (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) — we have $x \notin \overline{U_s}$, but $\left(\overline{U_s}\right)^c$ is open in X. It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq \left(\overline{U_s}\right)^c \subseteq \bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$. To show the reverse, fix an element x in the union, then this induces some $x \in \left(\overline{U_s}\right)^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of W. This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen s is a lower bound for W, and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in X, using Theorem 4.9 finishes the proof. ■

Notes on the construction of the countable 'onion' sequence within a normal space **X**.

If **X** is a normal space, and A and B are disjoint closed subsets, then we can easily find an open U with

$$A \subseteq U \subseteq \overline{U} \subseteq B^c \tag{1}$$

We say that U hides in B^c if the closure of U is contained in B^c. Define $\Delta_n = \left\{k2^{-n},\ 1 < k < 2^n\right\}$, so that $\Delta_n \subseteq (0, 1)$ for all $n \geq 1$. Notice

$$\Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

and the even indices for Δ_{n+1} are contained in Δ_n. Suppose Δ_n is well defined, it suffices to choose the odd indices for Δ_{n+1}. If $r = j2^{-(n+1)}$, where j is odd, then r sits in between precisely two elements in $\Delta_n \cup \{0, 1\}$. If r sits between an endpoint, then define $\overline{U_0} = A$, and $B^c = U_1$. And denote the closest left and neighbours by s, t respectively. If $s < r < t$, it is clear that $\overline{U_s}$ and U_t^c are disjoint closed sets.

Use the 'normal space' construction to obtain an superset of $\overline{U_s}$ that hides in U_t, denote this open set by U_r, and similar to Equation (1)

$$\overline{U_s} \subseteq U_r \subseteq \overline{U_r} \subseteq U_t$$

Now that the construction of this sequence is complete, we wish to prove Urysohn's Lemma. Let A and B be disjoint closed sets. And define

$$f(x) = \inf\left\{r \in \Delta \cup \{1\},\ x \in U_r\right\}$$

where $U_1 = \mathbf{X}$. So that $0 \le f(x) \le 1$ is immediate. If $x \in A$, then x is in all U_r, and by density of $\Delta \subsetneq (0,1)$, we have $f(x) = 0$. Conversely, if $x \in B$ then $x \notin U_r$ for all $r \in \Delta$, if E denotes the indices in Δ where $x \in U_s$ when $s \in E$,

$$(-\infty,\, r) \subseteq E^c \iff E \subseteq [r,\, +\infty) \iff \inf(E) \ge r \tag{2}$$

Send $r \to 1$ and $f(x) = 1$. Thus $f|_A = 0$ and $f|_B = 1$.

To show continuity, it suffices to show that the inverse images of the open half $\left\{(x > \alpha),\ (x < \alpha)\right\}_{\alpha \in \mathbb{R}}$ lines are indeed open in $\mathbf{X}$. Let α be fixed. And if $x \in \{f < \alpha\}$, we can 'wiggle' the infimum towards the right (towards α), and using density of Δ within $(0,1)$, there exists a $r \in E$ that satisfies $f(x) < r < \alpha$. This is equivalent to

$$x \in \bigcup_{r<\alpha} U_r$$

If there exists an $r < \alpha$ st x belongs to U_r as an element, then $f(x) \le r < \alpha$.

If $f(x) > \alpha$, then $(-\infty,\, \alpha) \subsetneq E^c$, by Equation (2). Suppose $\alpha < 1$, otherwise $\{f > \alpha\} = \varnothing$. Wiggle $f(x)$ to the left and obtain an $r \in \Delta$, $\alpha < r < f(x)$ with $x \notin U_r$. By density again, take any $s < r$ by a small amount (st $s > \alpha$, $s \in \Delta$), and

$$\overline{U}_s \subseteq U_r \iff U_r^c \subseteq \overline{U}_s$$

so that $x \in \overline{U}_s^c$ for some $s > \alpha$. This is equivalent to

$$x \in \bigcup_{s>\alpha} \overline{U}_s^c$$

Conversely, if $x \notin \overline{U}_s^c$ for some $s > \alpha$, since $\{U_r\}$ (thus $\{\overline{U}_r\}$) is increasing, and $x \notin U_r$ for every $r \le s$. Hence,

$$(-\infty,\, s] \subseteq E^c \iff E \subseteq (s,\, +\infty) \iff f(x) \ge s > \alpha$$

Compactness

Compactness is one of the most important concepts in topology and analysis.

Definition 1.1: Compact topological space

A topological space $\mathbf{X}$ is compact if every open covering $\{U_\alpha\}$ contains a finite subcover. That is, if $\{U_\alpha\}$ is an arbitrary collection of open sets, then

$$\mathbf{X} = \bigcup U_{\alpha \in A} \implies \bigcup_{j \leq n} U_{\alpha_j}$$

Definition 1.2: Compact set

$E \subseteq \mathbf{X}$ is compact if it is compact in the subspace topology.

Definition 1.3: Precompact set

$E \subseteq \mathbf{X}$ is precompact if its closure is compact (as a subset).

Definition 1.4: Paracompact space

A topological space $\mathbf{X}$ is paracompact if every open covering of $\mathbf{X}$ has a locally finite open refinement that covers $\mathbf{X}$.

Definition 1.5: Locally finite collection of sets

Let $\mathcal{A}$ be a collection of subsets of $\mathbf{X}$. It is called locally finite, if at every point $p \in \mathbf{X}$, we can find a neighbourhood U of p (not necessarily open), that intersects only finitely many members of $\mathcal{A}$. In symbols,

$$U \cap E = \varnothing \quad \text{for all but finitely many} \quad E \in \mathcal{A}$$

We do not require $\mathcal{A}$ to be a cover of $\mathbf{X}$, nor do we require $\mathcal{A}$ to be a collection of open sets.

Definition 1.6: Countably locally finite

A collection $\mathbb{B}$ is countably locally finite if it is the countable union of locally finite families.

$$\mathbb{B} = \overbrace{\bigcup_{\mathbb{N}}}^{\text{countable union}} \mathbb{B}_n, \quad \text{where each } \mathbb{B}_n \text{ is a locally finite collection}$$

Definition 1.7: Refinement

If $\mathcal{A}$ is a collection of sets, $\mathbb{B}$ is a refinement of $\mathcal{A}$ if every element $B \in \mathbb{B}$, induces an element $A \in \mathcal{A}$, such that $B \subseteq A$.

Remark 1.1: Intuition for refinements

If $\mathbb{B}$ is a refinement of $\mathcal{A}$, we can use the 'absolute continuity' muscle. For each element in $\mathbb{B}$ is dominated by some element (through subset inclusion) in $\mathcal{A}$. Recall, if ν and μ are non-negative measures, then $\nu \ll \mu$ if for every measurable set $E \in \mathcal{M}$, $\mu(E) = 0 \implies \nu(E) = 0$.

A refinement of a family of sets is another family of sets, whose elements are dominated by some other element in the un-refined family. *Refining families makes them 'smaller', cover less area.*

Proposition 1.1

Compact Hausdorff spaces are normal, compact subsets of Hausdorff spaces are closed, and closed subsets of compact sets are again compact.

Locally Compact Hausdorff Spaces

Compactness is an intrinsic topological property (in the subspace topology). We see from Proposition 4.25 that compact Hausdorff spaces are normal, which gives a sufficient condition for us to approximate and extend any continuous function; and allows us to extend certain 'local' properties to 'global' properties.

If given a Hausdorff space, not necessarily compact, the natural question is to ask 1) whether a topological space has 'enough' compact subsets to work with, and 2) whether we can embed a given topological space in a larger one to force it to be compact.

Definition 2.1: LCH space

Let $\mathbf{X}$ be a Hausdorff space. We call $\mathbf{X}$ a LCH space if every point $p \in \mathbf{X}$ admits a compact neighbourhood. That is, a compact set K whose interior contains p.

We note in passing that the above definition differs slightly from the usual 'local' definitions.

Definition 2.2: Locally connected

Let $\mathbf{X}$ be a topological space, it is locally connected if for every $x \in \mathbf{X}$, and open neighbourhood U containing x, there exists a connected, open neighbourhood V of x such that $x \in V \subseteq U$.

Definition 2.3: Locally path-connected

Let $\mathbf{X}$ be a topological space, it is locally connected if for every $x \in \mathbf{X}$, and open neighbourhood U containing x, there exists a path-connected, open neighbourhood V of x such that $x \in V \subseteq U$.

Definition 2.4: Local homeomorphism

$\mathbf{X}$ locally homeomorphic to $\mathbb{R}^n$ if every point $x \in \mathbf{X}$ belongs to a coordinate chart (U, ϕ), where U is an open neighbourhood of x and ϕ is a homeomorphism from $U \to \phi(U) \subseteq \mathbb{R}^n$.

Definition 2.5: Local diffeomorphism

Let M be a smooth manifold and $F \in C^\infty(M, N)$. F is a local diffeomorphism if every $p \in M$ in its domain induces a neighbourhood $U \subseteq M$ with $F|U : U \to F(U)$ is a diffeomorphism (in the sense of two open sub-manifolds).

Properties of Compact Spaces

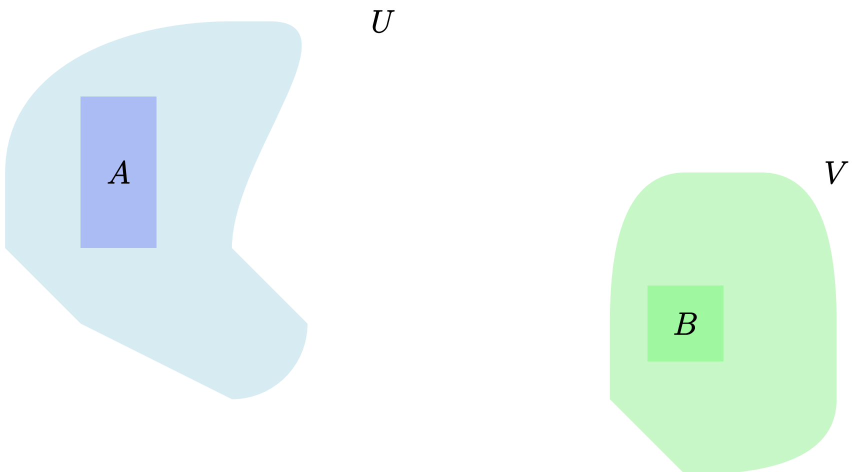

Figure 1: Closed sets A and B within open sets U and V, respectively.

Proposition 3.1

Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces.

(a) If $F \in C(\mathbf{X}, \mathbf{Y})$, and $\mathbf{X}$ is compact, then $F(\mathbf{X})$ is compact.

(b) If $\mathbf{X}$ is compact and $F \in C(\mathbf{X}, \mathbb{R})$, then $F(\mathbf{X})$ is bounded, and F attains its supremum and infimum on $\mathbf{X}$.

(c) A finite union of compact subspaces of $\mathbf{X}$ is again compact.

(d) If $\mathbf{X}$ is Hausdorff, and A, B are disjoint, compact subspaces of $\mathbf{X}$, there exists open U and V, (see fig. 1).

(e) Every closed subset of a compact space is compact.

(f) Every compact subset of a Hausdorff space is closed.

(g) Every compact subset of a metric space is bounded.

(h) Every finite product of compact spaces is compact.

(i) Every quotient of a compact space is compact.

Proof of Proposition 3.1 Part A. Let $f \in C(\mathbf{X}, \mathbf{Y})$ with $\mathbf{X}$ compact. Fix an open cover of $f(\mathbf{X})$ in the relative topology,

$$\{U_\alpha \cap f(\mathbf{X})\}_{\alpha \in A} \text{ covers } \mathbf{X},\ U_\alpha \text{ open in } \mathbf{Y}$$

So that $\bigcup f^{-1}(U_\alpha) = \bigcup f^{-1}(U_\alpha \cap f(\mathbf{X})) = \mathbf{X}$. Since $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover for $\mathbf{X}$, this induces a finite subcollection of indices $\{\alpha_1, \ldots, \alpha_n\}$ with

$$\bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^{n} f^{-1}(U_\alpha \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^{n} f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^{n} U_{\alpha_j}$$

■

Proof of Proposition 3.1 Part B. Let $\mathbf{X}$ be compact, and $f \in C(\mathbf{X}, \mathbb{R})$, so that $f(\mathbf{X}) \subseteq \mathbb{R}$ is compact. Compact subsets are closed and bounded in $\mathbb{R}$, let $A = \sup f(\mathbf{X})$ and $B = \inf f(\mathbf{X})$. Both A and B are accumulation points of $f(\mathbf{X})$, so $A = f(x)$ and $B = f(y)$ for some x, y in $\mathbf{X}$. ■

Proof of Proposition 3.1 Part C. Let $\mathbf{X}$ be a topological space, and $K_1, \ldots K_n$ be compact subspaces. Denote $K = \bigcup_{j=1}^{n} K_j$. Let $\{U_\alpha \cap K\}_{\alpha \in A}$ be an open cover for K, where U_α is open in $\mathbf{X}$. We can pass the argument to each individual K_j as follows. Let $1 \leq j \leq n$, then $\{U_\alpha \cap K_j\}_{\alpha \in A}$ is an oepn cover for K_j, so there exists as finite subcollection of indices $I_j \subseteq A$, (a finite subset of A) whose open sets cover K_j. Repeat this process for each j and

$$I = \bigcup_{j=1}^{n} I_j \text{ is a finite subset of } A$$

with $K_j \subseteq \bigcup_{\alpha \in I_j}(U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j}(U_\alpha \cap K)$. Taking the union over all K_j reads

$$K = \bigcup_{j=1}^{n} K_j \subseteq \bigcup_{j=1}^{n} \bigcup_{\alpha \in I_j} (U_\alpha \cap K) = \bigcup_{\alpha \in I} U_\alpha \cap I$$

■

Proof of Proposition 3.1 Part D. Let $\mathbf{X}$ be Hausdorff. We first prove that compact subspaces of $\mathbf{X}$ are closed. Indeed, if K is compact in $\mathbf{X}$, fix any $x \in K^c$. Let y range through the elements of K, then $x \neq y$ induces a pair of disjoint open sets U_y and V_y, such that

- $x \in U_y$

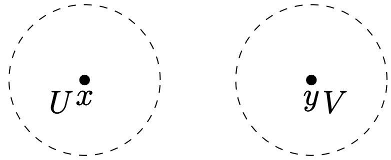

Figure 2: In a Hausdorff space, any two distinct points x and y can be separated by disjoint open neighbourhoods U and V.

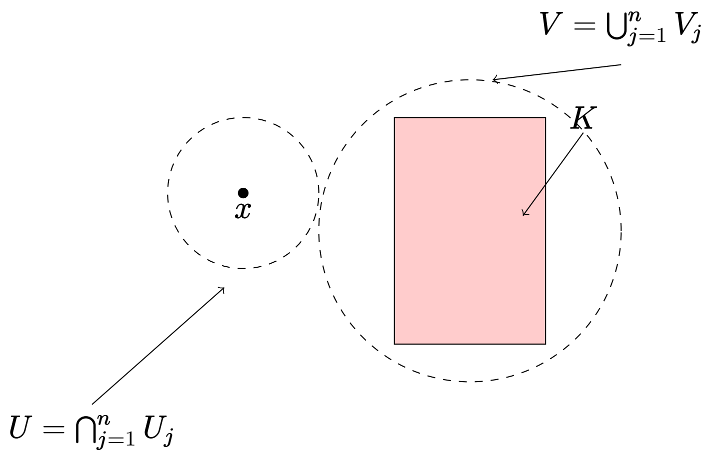

Figure 3: Compact sets are closed in Hausdorff spaces

- $y \in V_y$
- $U_y \cap V_y = \varnothing$
- See fig. 2

Let V_y range through all possible $y \in K$, So that $\{V_y\}_{y\in K}$ is an open cover. There exists a finite subcollection of 'anchor points' of K, $y_1, \ldots, y_n$ that corresponds with $\{V_{y_j}\}_{j=1}^n$. A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^n U_{y_j} \text{ is open}$$

Define $V = \bigcup_{j=1}^n V_{y_j}$, so $V \subseteq K$ and $U \cap V = \varnothing$ and $x \in U \subseteq K^c$ (see fig. 3). Therefore K is closed.

Finally, if A and B are disjoint compact sets, then each $x \in A \subseteq B^c$ induces neighbourhoods $x \in U_x$, and $B \subseteq V_x$ (see fig. 4), let x range through all the elements of A. By compactness of A, this produces a finite subcover, and

$$U = \bigcup_{j=1}^n U_{x_j} \quad V = \bigcap_{j=1}^n V_{x_j}$$

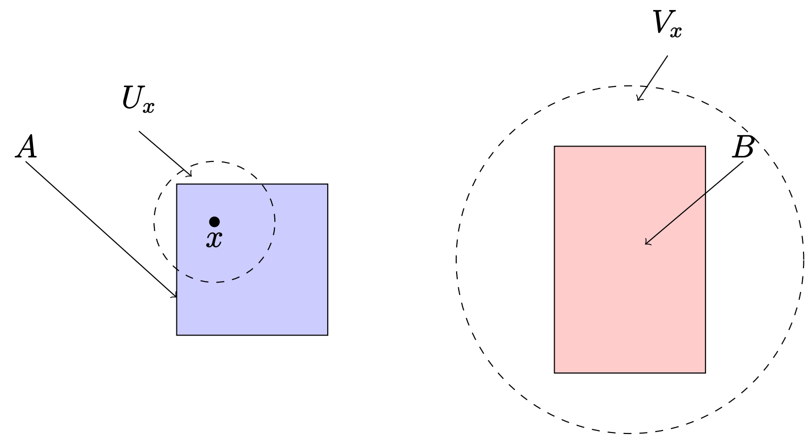

Figure 4: Closed sets A and B, point x in A, and disjoint neighbourhoods U around x and V around B.

are disjoint open sets that contain A and B respectively.

■

Proof of Proposition 3.1 Part E. Let $K \subseteq \mathbf{X}$ be a closed set of a compact space. Let $\{U_\alpha \cap K\}$ be an open cover for K, where each U_α is open in $\mathbf{X}$. We can append an extra set K^c which is open in $\mathbf{X}$. The collection

$$W = \{U_\alpha\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of $W_1, \ldots, W_n$ that cover $\mathbf{X}$ (since $\mathbf{X}$ is compact by itself). Remove K^c from this finite subcollection if it exists, and take the intersection with K for each element W_j, and

$$\{W_1 \cap K, \ldots, W_n \cap K\} = \{U_1 \cap K, \ldots, U_n \cap K\} \text{ covers } K$$

so K is compact. ■

Proof of Proposition 3.1 Part F. Proven in Part D. ■

Proof of Proposition 3.1 Part G. let $K \subseteq \mathbf{X}$ be a compact subset of the metric space $(\mathbf{X}, d)$. Compact subsets of $\mathbf{X}$ are totally bounded, and hence bounded. ■

Proof of Proposition 3.1 Part H. See Tynchonoff's Theorem in Folland Chapter 4. ■

Proof of Proposition 3.1 Part I. Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces and $\pi : \mathbf{X} \to \mathbf{Y}$ be a quotient map. So that $\mathbf{Y}$ is endowed with the quotient topology. So that π is a surjective continuous map. and $\pi(\mathbf{X}) = \mathbf{Y}$. Apply Part A, and we see that $\mathbf{Y}$ is compact. ■