MATH 263: Section 003, Tutorial 13

Mohamed-Amine Azzouz
mohamed-amine.azzouz@mail.mcgill.ca

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1 (Review) The Laplace Transform

The **Laplace Transform** is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt$$

where the integral is defined.

Note that the **Laplace Transform** is an **integral Transform**, meaning that it is linear (due to the properties of the integral). Therefore, for all $a, b \in \mathbb{R}$:

$$\mathcal{L}\{a\ f(t) + b\ g(t)\} = a\ \mathcal{L}\{f(t)\} + b\ \mathcal{L}\{g(t)\}$$

Example: Find $\mathcal{L}\{\cos t\}$ in two ways:
Using Taylor Series:
(*Will be discussed in the tutorial*)
Another way is by using the definition and the periodicity of $\cos t$: $\cos t = \cos t + 2\pi$:
(*Will be discussed in the tutorial*)

2 (Review) Solution of Initial Value Problems using The Laplace Transform

Using integration by parts, we can find the Laplace Transform of a derivative:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

note that this Laplace Transform is also in terms of the function's initial conditions. Now, from induction, we can find the Laplace Transform of the n'th derivative of a function:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

This Laplace Transform has $n-1$ initial conditions. Note that this way, we can solve first order constant coefficient ODE of the form:

$$ay"(t) + by'(t) + cy(t) = g(t)$$

By taking the Laplace Transform on both sides, using the formula to take the Laplace Transform of a derivative, solve for the solution $Y(s)$, and apply the Inverse Laplace Transform ($\mathcal{L}^{-1}\{F(s)\}$) to find the solution to the IVP. To find inverse Laplace Transforms in this course, one should decompose $Y(s)$ into simpler terms (usually using partial fractions) and use a Laplace Transform table to find the solution $y(t)$.

3 Solution of Initial Value Problems using The Laplace Transform

Unit step function:

$$u(x - x_0) = \begin{cases} 1 \; when \; x \geq x_0 \\ 0 \; when \; x < x_0 \end{cases}$$

Dirac-Delta function:

$$\delta(x - x_0) = \begin{cases} \infty \; when \; x = x_0 \\ 0 \; when \; x \neq x_0 \end{cases}$$

Convolution integral:

$$f(t) * g(t) = \int_0^t f(t - \tau) g(\tau) d\tau$$

, where

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

Problem 3a. From Boyce and DiPrima, 10th edition (6.4, exercise 9, p.340):
Solve this IVP using the Laplace Transform:

$$y\text{"} + y = g(t); \; y(0) = 0, \; y'(0) = 1,$$

where

$$g(t) = \begin{cases} \frac{t}{2} \; when \; 0 \leq t < 6 \\ 3 \; when \; t \geq 6 \end{cases}$$

Problem 3b. From Boyce and DiPrima, 10th edition (6.5, exercise 4, p.348):
Solve this IVP using the Laplace Transform:

$$y\text{"} - y = -20\delta(t - 3); \; y(0) = 1, \; y'(0) = 0,$$