

# Chapter 6

**Theorem 6.1**

**Proposition 1.1.** *For every  $a, b \geq 0$ , and  $0 < \lambda < 1$ , then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b$$

**Theorem 6.2**

**Proposition 2.1.**

*Proof.*



**Theorem 6.3**

**Proposition 3.1.**

*Proof.*



**Theorem 6.4**

**Proposition 4.1.**

*Proof.*



**Theorem 6.5**

**Proposition 5.1.**

*Proof.*



**Theorem 6.6**

**Proposition 6.1.**

**Theorem 6.7**

**Proposition 7.1.**



**Theorem 6.8**

**Proposition 8.1.**

**Theorem 6.9**

**Proposition 9.1.**

*Proof.*



**Theorem 6.10**

**Proposition 10.1.**

*Proof.*



**Theorem 6.11**

**Proposition 11.1.**

*Proof.*



**Theorem 6.12**

**Proposition 12.1.**

*Proof.*



**Theorem 6.13**

**Proposition 13.1.**

**Theorem 6.14**

**Proposition 14.1.**

**Theorem 6.15****Proposition 15.1.**

*Proof.* First suppose that  $(X, \mathcal{M}, \mu)$  is finite measure space. If  $\mu(X) < +\infty$ , then for every  $E \in \mathcal{M}$ , by monotonicity  $E \subseteq X$  yields  $\mu(E) \leq \mu(X) < +\infty$ . Next, for any  $p < +\infty$ ,  $\|\chi_E\|_p^p < +\infty$  and  $\|\chi_E\|_{+\infty} \leq 1 < +\infty$ . So all indicator functions are in  $L^p$ .

It follows that every simple function is also in  $L^p$ , since it is a finite linear combination of indicators. We now define  $\nu(E) = \phi(\chi_E)$ , we wish to show that  $\nu : \mathcal{M} \rightarrow \mathbb{C}$  is a complex measure which is absolutely continuous with respect to  $\mu$ .

To show  $\sigma$ -additivity, fix any disjoint sequence  $\{E_j\}_{j \geq 1} \subseteq \mathcal{M}$ . Where we also note that  $\mu(E) = \mu(\cup E_j) < +\infty$ . Now suppose that  $p < +\infty$ , then the following converges in the  $p$ -norm

$$\chi_E = \sum_{j \geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left( \bigcup_{j \leq n} E_j \right) = \left( \bigcup_{j \geq 1} E_j \right) \setminus \left( \bigcup_{j \leq n} E_j \right) = \bigcup_{j \geq n+1} E_j$$

and define  $F_{n+1}$  as the rightmost member above. Then  $\{F_{n \geq 1}\}$  is a decreasing sequence of sets. All sets are of finite measure, hence  $\mu(E) - \mu(\cup_{j \leq n} E_j) = \mu(F_{n+1}) \rightarrow 0$ .

Now, for any fixed  $n \geq 1$ ,

$$\left| \chi_E - \sum \chi_{E_{j \leq n}} \right| = \left| \sum \chi_{E_{j \geq n+1}} \right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the  $p$ th power does not change pointwise, and

$$\left| \sum \chi_{E_{j \geq n+1}} \right|^p = \left| \sum \chi_{E_{j \geq n+1}} \right| = \sum \chi_{E_{j \geq n+1}}$$

Convergence in  $p$ -norm is given by



$$\left\| \chi_E - \sum \chi_{E_{j \leq n}} \right\| = \left\| \sum \chi_{E_{j \geq n+1}} \right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our  $\phi \in L^{p*}$

$$\begin{aligned} \nu(E) &= \phi(\chi_E) \\ &= \phi\left(\lim_{n \rightarrow \infty} \sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \phi\left(\sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \sum \phi\left(\chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \sum \nu(E_{j \leq n}) \end{aligned}$$

To show absolute convergence, recall that for any  $\phi(\chi_{E_j}) \in \mathbb{C}$ , define  $\beta_j = \overline{\text{sgn}(\|\phi(\chi_{E_j})\|)}$  then multiplication yields

$$\|\phi(\chi_{E_j})\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the  $p$ -norm.

$$\left\| \sum_{j \geq 1} \beta_j \chi_{E_j} - \sum_{j \leq n} \beta_j \chi_{E_j} \right\|_p = \left\| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right\|_p$$

And because  $\left| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right|$  is pointwise equal to  $\left| \sum_{j \geq n+1} \chi_{E_j} \right|$ , since  $|\beta_j| = 1$  for every  $j \geq 1$ . We can reuse the same continuity and linearity argument. We also note that  $\sum_{j \geq 1} \beta_j \chi_{E_j} \in L^p$  since its  $p$ -norm is equal to  $\mu(E)^{1/p}$ .

$$\begin{aligned}
\sum_{j \geq 1} |\nu(E_j)| &= \sup_{n \geq 1} \sum_{j \leq n} \|\nu(E_{j \leq n})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \|\phi(\chi_{E_j})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \phi(\chi_{E_j}) \\
&= \lim_{n \rightarrow \infty} \phi \left( \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&= \phi \left( \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&\leq \|\phi\| \left\| \sum_{j \geq 1} \beta_j \chi_{E_j} \right\|_p \\
&< +\infty
\end{aligned}$$

Assuming the above estimate holds, then we only need  $\nu(E) = \phi(\chi_E) = \mu(E) = 0$  ( $\nu$  is now a measure and  $\nu \ll \mu$ ). As the indicator of a null set is equal to the zero element in  $L^p$ . Then by Radon-Nikodym we can have some  $g \in L^1(\mu)$  such that

$$d\nu = g d\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function  $g$ . For every  $\chi_E$  measurable,  $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$ , by monotonicity of the integral in  $L^+$ . So any simple function,  $\alpha = \sum a_j \cdot \chi_{E_j}$  means that  $\alpha g$  is in  $L^1(\mu)$ , and

$$\phi(\alpha) = \int \alpha g d\mu$$

If  $\|\alpha\|_p = 1$ , then

$$\left| \int \alpha g \right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$$M_q(g) = \sup \left\{ \left| \int \alpha \cdot g \right|, \|\alpha\|_p = 1, \text{ and } \alpha \text{ is simple, and vanishes out-} \right. \\ \left. \text{side a set of finite measure.} \right\} < \infty$$

Since  $S_g = \{x \in X, g(x) \neq 0\}$  is  $\sigma$ -finite, an application of Theorem 6.14 tells us that  $g \in L^q$ , and  $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$ . Now that we know  $g$  is in  $L^q$  we can use the density of  $\alpha$  in  $L^p$  to show, for every single  $f \in L^p$

$$\phi(f) = \int f g d\mu$$

Conjure a sequence of  $\alpha$ 's, and call them  $\{f_n\} \rightarrow f$  p.w.a.e, then each  $f_n \cdot g \in L^1$ . An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n g d\mu = \int f g d\mu = \phi(f)$$

This completes the proof for when  $\mu$  is finite.

Let us upgrade our  $\mu$  into a  $\sigma$ -finite measure. Then there exists an increasing sequence  $\{E_n\} \nearrow X$  such that each  $E_n$  is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in  $P_n$  vanishes outside a set of finite measure and is also in  $L^p$ . And  $Q_n$  is defined in a similar manner. Now, fix our  $\phi \in L^{p*}$ , and for each  $f \in P_n$ , there exists a corresponding  $g_n \in Q_n$ . Then  $p \in [1, +\infty)$  tells us that  $q \in (1, +\infty]$ , and the assumptions for Theorem 6.13 all hold. Therefore for each  $g_n \in Q_n$ , there is a corresponding bounded linear operator  $\phi_{g_n} \in (P_n)^*$  such that

$$\phi(f) = \phi|_{P_n}(f) = \int f g_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of  $g_n$  towards some  $g \in L^q$ . We claim that this limit makes sense. As for any  $n < m$ , such that  $E_n \subseteq E_m$  then  $g_n = g_m$  on  $E_n$  pointwise. The proof is simple since each the restriction of our  $\phi \in L^{p*}$  onto  $E_n$  and  $E_m$  spawns two functions  $g_n$  and  $g_m \in L^1$ . To verify, take any subset  $Z \subseteq E_n$  then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So  $g_n = g_m$  pointwise a.e on  $E_n$ . Now we define  $g$  measurable such that  $g|_{E_n} = g_n$  for every  $n$ . And

$$\begin{aligned}
|g_n| &= \chi_{E_n} \cdot |g_m| \implies \\
|g_n| &\leq |g_{n+1}| \implies \\
\|g_n\|_q &\leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty
\end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with  $(P_n \subseteq P_{n+1})$ . If  $q = +\infty$  then  $g \in L^\infty$  is trivial, but for any  $q < +\infty$ . We wish to show that  $g \in L^q$ . Since  $|g_n| \leq |g|$  pointwise for every  $n$ , and for each  $x \in X$ , there exists a  $N$ , where  $n \geq N$  implies  $|g(x)| = |g_n(x)|$ , so  $|g(x)|$  is an upperbound that is actually attained by the sequence  $|g_n(x)|$ . So,  $|g(x)| = \sup_{n \geq 1} \{|g_n(x)|\}$ .

Using the Monotone Convergence Theorem on  $|g_n|$ ,

$$\begin{aligned}
\int \lim_{n \rightarrow \infty} |g_n|^q d\mu &= \int \sup_{n \geq 1} |g_n|^q d\mu \\
&= \int |g|^q d\mu \\
&= \lim \int |g_n|^q d\mu
\end{aligned}$$

Which yields  $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$ . It follows that  $g \in L^q$ .

Finally, we will show that  $\phi(f) = \int f g$  for every  $f \in L^p$ . Redefine  $f_n = f \cdot \chi_{E_n} \in P_n$  for every  $n \geq 1$ . We claim that  $f_n \rightarrow f$  in the  $p$ -norm.

$$\begin{aligned}
|f_n - f| &\leq |f_n| + |f| \\
&\leq |f| + |f| \\
&\leq 2|f|
\end{aligned}$$

And  $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$ . Now it is permissible to apply the Dominated Theorem, and we will do so.

$$\begin{aligned}
\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\
\lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\
&= 0
\end{aligned}$$

And we have  $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}
 \phi(f) &= \lim \phi|_{P_n}(f_n) \\
 &= \lim \int f_n \cdot g_n \\
 &= \lim \int f \cdot g \cdot \chi_{E_n} \\
 &= \int \lim (fg \cdot \chi_{E_n}) \\
 &= \int fg
 \end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of  $fg\chi_{E_n} \rightarrow fg$  pointwise and Holder's Inequality. This completes the proof for when  $\mu$  is of  $\sigma$ -finite measure, and  $p \in [1, +\infty)$ .

Suppose now  $\mu$  is arbitrary, and  $p \in (1, +\infty)$ , then  $q < +\infty$ . Now let us agree to define, for every  $\sigma$ -finite  $E \subseteq X$

$$P_E = \{L^p, |f| = |f| \cdot \chi_E\}$$

Where  $Q_E$  does not hold any surprises. Then for each  $E$  we have a  $\phi|_E$  which induces a  $g_E$  that vanishes outside  $E$ . We are ready for the final part of the proof.

First, if  $E \subseteq F$  and both  $E$  and  $F$  are  $\sigma$ -finite, then  $\|g_E\|_q \leq \|g_F\|_q$ . This is a simple consequence of monotonicity in  $L^+$  if we take  $|g_E|^q \leq |g_F|^q$ .

Second, we define

$$W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}$$

Let  $M$  be the supremum of  $W$ , then there exists a sequence of  $\sigma$ -finite sets,  $\{E_n\}$  where  $\|g_{E_n}\|_q \rightarrow M \leq \|\phi\|_{p*}$ . Take a set  $F = \cup E_{n \geq 1}$ , which is also  $\sigma$ -finite, so that  $\|g_F\|_q = M$ . Now assume there exists another  $\sigma$ -finite superset of  $F$ , let us call it  $A$ . Then

$$\int |g_F|^q + \int |g_{A \setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have  $g_{A \setminus F} = 0$  pointwise a.e. For any  $f \in L^p$ , the spots where  $f$  does not vanish is  $\sigma$ -finite. This comes from  $\int |f|^p < +\infty$ . So it suffices to integrate over this  $\sigma$ -finite set. But we already know, even if this set  $A$  contains  $F$  as a subset,  $\int f g_F = \int f g_A$ .

We now define  $g = g_F$ , and the proof is complete. As for every  $\phi \in L^{p*}$ , there exists a  $g \in L^q$  such that the evaluation of any  $f \in L^p$  is given by integrating  $f$  with  $g$ . ■