

Chapter 8

Theorem 8.1

Proposition 1.1

Proof.



Theorem 8.2

Proposition 2.1

Proof.



Theorem 8.3**Proposition 3.1**

If $f \in C^\infty$, then $f \in \mathcal{S}$ if and only if $x^\beta \partial^\alpha f$ is bounded for all multi-indices α, β

Proof.



Theorem 8.4

Proposition 4.1

Proof.



Theorem 8.5

Proposition 5.1

Proof.



Theorem 8.6

Proposition 6.1

Proof.



Theorem 8.7

Proposition 7.1

Proof.



Theorem 8.8

Proposition 8.1

Proof.



Theorem 8.9

Proposition 9.1

Proof.



Theorem 8.10

Proposition 10.1

Proof.



Theorem 8.11

Proposition 11.1

Proof.



Theorem 8.12

Proposition 12.1

Proof.



Theorem 8.13

Proposition 13.1

Proof.



Theorem 8.14**Proposition 14.1**

Suppose $\phi \in L^1$, and $\int \phi(x)dx = a$.

- (a) If $f \in L^p$, $p \in [1, +\infty]$, then $f * \phi_t \rightarrow af$ in the L^p norm as $t \rightarrow 0$.
- (b) If f is bounded and uniformly continuous, then $f * \phi_t \rightarrow af$ uniformly as $t \rightarrow 0$.
- (c) If $f \in L^\infty$ and f is continuous on an open set U , then $f * \phi_t \rightarrow af$ uniformly on compact subsets of U as $t \rightarrow 0$

Proof of Part A. First, the convolution $f * \phi_t$ is in L^p by Young's Inequality (Theorem 8.7). Furthermore,

$$f * \phi_t - af = \int_{y \in \mathbb{R}^n} f(x-y)t^{-n}\phi(t^{-1}y)dy - \int_{y \in \mathbb{R}^n} f(x)\phi(y)dy \quad (1)$$

Now apply Theorem 2.44, with $y \mapsto y/t$, and denote this invertible map by $T \in GL(n, \mathbb{R})$, so that $|\det(T)| = t^{-n}$, then $y = T(y)t$ for every $t > 0$. It follows that

$$\begin{aligned} (f * \phi_t)(x) &= |\det(T)| \cdot \int_{y \in \mathbb{R}^n} f(x - t \cdot Ty)\phi(T(y))dy \\ &= \int_{z \in \mathbb{R}^n} f(x - tz)\phi(z)dz \\ &= \int_{z \in \mathbb{R}^n} \tau_{tz}f(x)\phi(z)dz \end{aligned} \quad (2)$$

Next, $a = \int \phi$ so $af = \int f(x)\phi(z)dz$. Using Equations (1) and (2) we get

$$(f * \phi_t - af)(x) = \int_{z \in \mathbb{R}^n} (\tau_{tz}f - f)\phi(z)dz \quad (3)$$

We wish to apply Minkowski's Inequality for integrals, which states, roughly speaking:

The norm of an integral is less than the integral of the norm.

to Equation (3), and

$$\|f * \phi_t - af\|_p \leq \int_{z \in \mathbb{R}^n} \|(\tau_{tz}f - f)\phi(z)\|_p dz \quad (4)$$

The assumptions for Theorem 6.19 are satisfied by

1. Notice for every $z \in \mathbb{R}^{n'}$,

$$\|(\tau_{tz}f - f)\phi(z)\|_p = \left(\int_{x \in \mathbb{R}^n} |(\tau_{tz}f(x) - f(x))\phi(z)|^p dx \right)^{1/p} \leq |\phi(z)| \left(2\|f\|_p \right) < +\infty$$

Since $\|\phi\|_1 < +\infty$, $|\phi(z)| < +\infty$ almost everywhere.

2. Next, to show $z \mapsto \|\phi(z)(\tau_{tz}f - f)\|_p$ is in $L^1\mathbb{R}^n$, z . Reuse the last estimate in the previous bullet point, and

$$\|\phi(z)(\tau_{tz}f - f)\|_p \leq |\phi(z)| \left(2\|f\|_p \right)$$

Taking the integral in L^+ with respect to z , we get

$$\left\| \left(\|\phi(z)(\tau_{tz}f - f)\|_p \right) \right\|_1 < +\infty$$

so both assumptions are satisfied.

Therefore Equation (4) holds. Next, fix any sequence of $t_n > 0$ with $t_n \rightarrow 0$. The Dominated Convergence Theorem gives, since $|\phi(z)|\|\tau_{t_n z}f - f\|_p$ is dominated by $|\phi(z)| \cdot 2\|f\|_p \in L^1 \cap L^+$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{z \in \mathbb{R}^n} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz &= \int_{z \in \mathbb{R}^n} \lim_{n \rightarrow \infty} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz \\ &= \int_{z \in \mathbb{R}^n} 0 dz \\ &= 0 \end{aligned}$$

The second last equality is from Lemma 8.4, as translation is continuous in the L^p norm for $p \in [1, +\infty)$. So almost every $z \in \mathbb{R}^n$ (since again, $|\phi(z)|$ can be infinite on a null set),

$$\|\tau_{t_n z}f - f\|_p \rightarrow 0 \implies \|\tau_{t_n z}f - f\|_p |\phi(z)| \rightarrow 0$$

as $n \rightarrow +\infty$. It follows that

$$\lim_{n \rightarrow \infty} \|f * \phi_{t_n} - af\|_p = \lim_{n \rightarrow \infty} \left\| \int_{z \in \mathbb{R}^n} [\tau_{t_n z}f(x) - f(x)] \phi(z) dz \right\|_p = 0$$

Since the sequence $t_n \rightarrow 0$ is arbitrary, we conclude that the function $t \mapsto \|f * \phi_t - af\|_p$ has a limit of 0 as $t \rightarrow 0$. ■

Proof of Part B. Suppose $f \in \text{UBC}(\mathbb{R}^n)$, so that f is uniformly continuous and bounded. We wish to show $f * \phi_t \rightarrow af$ uniformly as $t \rightarrow 0$. In symbols,

$$g : t \mapsto \|f * \phi_t - af\|_u, \quad g \rightarrow 0, \text{ as } t \rightarrow 0$$

The convolution between f and ϕ_t makes sense at every $x \in \mathbb{R}^n$, as

$$\int |\tau_y f(x)| |\phi(y)| dy \leq \|f\|_u \cdot \|\phi\|_1 < +\infty$$

Taking the supremum norm on both sides of Equation (3), we get

$$\begin{aligned}
\|f * \phi_t - af\|_u &= \sup_{x \in \mathbb{R}^n} \left| \int_{z \in \mathbb{R}^n} (\tau_{tz}f - f) \cdot \phi(z) dz \right| \\
&\leq \sup_{x \in \mathbb{R}^n} \int_{z \in \mathbb{R}^n} |\tau_{tz}f - f| \cdot |\phi(z)| dz \\
&\leq \int_{z \in \mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |\tau_{tz}f - f| \cdot |\phi(z)| dz \\
&= \int_{z \in \mathbb{R}^n} \|\tau_{tz}f - f\|_u \cdot |\phi(z)| dz
\end{aligned} \tag{5}$$

the last equality is a simple consequence of the monotonicity of the integral in L^+ , indeed. For every $x \in \mathbb{R}^n$, the following holds pointwise for almost every z

$$|\tau_{tz}f - f| \leq \|\tau_{tz}f - f\|_u \implies \sup_{x \in \mathbb{R}^n} |\tau_{tz}f - f| \leq \|\tau_{tz}f - f\|_u$$

Apply the Dominated Theorem to the right member of (5), noting that it is dominated by $|\phi(z)| \cdot 2\|f\|_u \in L^1 \cap L^+$ as we have done for Part A of the proof. Since this holds for every sequence $t_n \rightarrow 0$, the proof is complete. \blacksquare

Proof of Part C. Next, suppose that $f \in L^\infty$, and $f \in C(U)$, where U is open in \mathbb{R}^n . We claim that

$$f * \phi_t \rightarrow af$$

within the topology of uniform convergence on compact subsets of U . So that for every $K \in \mathfrak{J}$, $K \subseteq U$

$$\sup_{x \in K} |f * \phi_t - af| \rightarrow 0, \text{ as } t \rightarrow 0$$

First, a small technical Lemma.

Lemma 14.1

If $\phi \in L^1(\mathbb{R}^n)$, then for every $\varepsilon > 0$, there exists $E \in \mathfrak{J}$, with

$$\int_{E^c} |\phi| = \|\phi \chi_{E^c}\|_1 < +\varepsilon$$

Proof. Assume that $\phi \geq 0$, if not, replace ϕ by $|\phi|$. Since $C_c(\mathbb{R}^n)$ is dense in L^1 for every $\varepsilon 2^{-1} > 0$ there exists some $\psi \in C_c(\mathbb{R}^n)$ with $\|\psi - \phi\|_1 < \varepsilon^{-1}$, and denote $E = \text{supp}(\psi) \in \mathfrak{J}$, then

$$\|\psi - \phi\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

So we can assume $\psi \geq 0$ as well, perhaps by relabelling ψ by $|\psi|$. Then,

$$\|\psi - \chi_E \phi\|_1 = \|\chi_E(\psi - \phi)\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

by monotonicity in L^+ . The Triangle Inequality in L^1 gives

$$\|\chi_{E^c} \phi\|_1 = \|\phi - \chi_E \phi\|_1 = \|\phi(1 - \chi_E)\|_1 \leq \|\phi - \psi\|_1 + \|\psi - \chi_E \phi\|_1 < \varepsilon$$

■

Back to the main proof of Part C, fix any $\varepsilon > 0$, then by Lemma 14.1, ϕ induces some $E \in \mathcal{J}$ with $\|\chi_{E^c} \phi\|_1 < +\varepsilon$. By Lemma 8.4, $\chi_K f \in C_c(\mathbb{R}^n) \subseteq \text{UBC}(\mathbb{R}^n)$. Uniform continuity of $\chi_K f$ gives us the continuity of translations. Now for the same $\varepsilon > 0$, there exists $r > 0$, for every $w \in \mathbb{R}^n$,

$$|w| < r \implies \|\tau_w \chi_K f - \chi_K f\|_u < +\varepsilon \quad (6)$$

Since $E \in \mathcal{J}$, it is bounded, and let t be a small positive number such that for every $z \in E$,

$$|tz| < t \cdot (1 + \sup_{z \in E} |z|) < r$$

There exists such a t , namely $t = r 2^{-1} (1 + \sup_{z \in E} |z|)^{-1}$. And for this $t > 0$, it follows that for every $z \in E$,

$$\sup_{x \in K} |\tau_{tz} f - f| < +\varepsilon$$

Since this holds for every $z \in E$, we write

$$\sup_{x \in K, z \in E} |\tau_{tz} f - f| < +\varepsilon$$

And

$$|\phi(z)| \left[\sup_{x \in K, z \in E} |\tau_{tz} f - f| \right] < |\phi(z)| \varepsilon$$

Monotonicity in $L^+(E, z)$ reads, for every $x \in K$,

$$\int_{z \in E} |\phi(z)(\tau_{tz} f - f)| dz \leq \int_{z \in E} |\phi| \varepsilon dz = \varepsilon \|\chi_E \phi\|_1 \leq \varepsilon \|\phi\|_1$$

Since this holds for every $x \in \mathbb{R}^n$,

$$\sup_{x \in K} \left\{ \int_{z \in E} |\phi(z)| \cdot |\tau_{tz} f - f| dz \right\} \leq \varepsilon \|\phi\|_1 \quad (7)$$

Next, notice for every t, z , we have

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f\|_u + \|f\|_u \leq 2 \cdot \|f\|_u$$

And the following holds $z \in E^c$ a.e,

$$|\phi(z)| \cdot |\tau_{tz} f - f| \leq |\phi(z)| \cdot 2 \|f\|_u$$

Taking the integral, and applying the condition we imposed on E from Lemma (14.1), so that

$$\int_{z \in E^c} |\phi(z)| \cdot |\tau_{tz}f - f| dz \leq 2\|f\|_u \int_{z \in E^c} |\phi(z)| dz \leq 2\|f\|_u \varepsilon$$

Taking the supremum of the above estimate, so

$$\sup_{x \in K} \left\{ \int_{z \in E^c} |\phi(z)(\tau_{tz}f - f)| dz \right\} \leq 2\|f\|_u \varepsilon \quad (8)$$

Combining Equations (7) and (8). Applying the additivity of the supremum (of $x \in K$), since both members are finite,

$$\sup_{x \in K} \left\{ \int_E |\phi(z)|(\tau_{tz}f - f) dz + \int_{E^c} |\phi(z)|(\tau_{tz}f - f) dz \right\} < \varepsilon(2\|f\|_u + \|\phi\|_1)$$

The left member above is equal to $\sup_{x \in K} |f * \phi_t - af|$. Since $\varepsilon > 0$ is arbitrary, this completes the proof of Part C. ■

Theorem 8.15**Proposition 15.1**

If $|\phi(x)| \leq C(1 + |x|)^{-n-\varepsilon}$, where $\varepsilon > 0$, and if $f \in L^p$, for $p \in [1, +\infty)$, then

$$f * \phi_t \rightarrow af$$

pointwise for every x in the Lebesgue set of f ,

$$\mathcal{L}_f = \left\{ x \in \mathbb{R}^n, \quad \lim_{r \rightarrow 0} \frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy = 0 \right\}$$

We also claim that $m(\mathcal{L}_f^c) = 0$, and $x \in \mathcal{L}_f$ at every continuous $f(x)$.

The proof is long, and will be divided into several parts. Let us start with a couple of Lemmas about the Lebesgue Set of f , and several pointwise estimates that will be of use.

Lemma 15.1

If $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$, and

$$|\phi(x)| \leq C(1 + |x|)^{n-\varepsilon}, \quad \varepsilon > 0 \tag{9}$$

then $\phi \in L^1$. Furthermore, $\phi_t \in L^1$ for every $t > 0$.

Proof of 15.1. If $x \neq 0$, then

$$|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)} \leq C \cdot |x|^{-(n+\varepsilon)}$$

on some B^c as defined in Theorem 2.52, so $\phi \in L^1(B^c)$. Next,

$$n + \varepsilon > n > n/2 = a$$

and by monotonicity,

$$|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)} \leq C \cdot (1 + |x|)^{-(n/2)}$$

so $\phi \in L^1(\mathbb{R}^n)$. Next, if $\phi \in L^1$, then

$$|\phi_t(x)| = t^{-n} |\phi(t^{-1}x)|$$

taking the integral in L^+ , and applying Theorem 2.44, with $T : x \mapsto t^{-1}x$, and $\det(T) = t^{-n}$, so that

$$\int |\phi_t|(x) dx = |\det(T)| \int |\phi| \circ T(x) dx = \int |\phi|(x) dx < +\infty$$

This completes the Lemma. ■

Lemma 15.2

If $f : \mathbb{R}^n \rightarrow \mathbb{C}$, and if $f \in C(\mathbb{R}^n)$, then $\mathcal{L}_f = \mathbb{R}^n$.

Proof of 15.2. Let $x \notin \mathcal{L}_f$, and there exists a sequence $r_k \rightarrow 0$ and $\varepsilon_0 > 0$ but

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy \geq \varepsilon_0$$

We claim that for every $k \geq 1$, we can find a $y_k \in B(r_k, x) \setminus \{x\}$ with

$$|f(x) - f(y_k)| \geq \varepsilon_0$$

Indeed, suppose by contradiction that no such y_k exists, and by monotonicity,

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy < \frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} \varepsilon_0 dy = \varepsilon_0$$

So choose y_k as above, and it is clear that $y_k \rightarrow x$ as $k \rightarrow \infty$, but $f(y_k) \not\rightarrow f(x)$. Therefore f is not continuous at x . ■

Lemma 15.3

If $x \in \mathcal{L}_f$, then for every $\delta > 0$ there exists a $\eta > 0$, with

$$r \leq \eta \implies \int_{|y| < r} |f(x - y) - f(x)| dy \leq \delta \cdot r^n$$

Proof of 15.3. We will start with something trivial.

$$m(B(r)) = r^n m(B(1)) \tag{10}$$

where $B(r) = \{x \in \mathbb{R}^n, |x| < r\}$. By Theorem 2.44,

$$\begin{aligned} m(B(r)) &= \int \chi_{B(r)}(x) dx \\ &= |\det(T)|^{-1} \int \chi_B(x) dx \\ &= r^n m(B(1)) \end{aligned}$$

where $T : x \mapsto x/r$ and $\det(T) = r^{-n}$. Fix $x \in \mathcal{L}_f$, and take $\varepsilon = \delta/m(B(1)) > 0$, and by definition this induces some $\eta > 0$, and for every $r \leq \eta$

$$\frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy \leq \varepsilon$$

By translation invariance of m ,

$$m(B(r, x)) = m(B(r)) = r^n \cdot m(B(1))$$

and apply the map $y \mapsto x - y$, which is a composition a rotation by $|-1|$ and a translation by $x \in \mathbb{R}^n$. By Theorems 2.44 and 2.42,

$$\int_{|y| \in B(r)} |f(x) - f(x - y)| dy = \int_{y \in B(r, x)} |f(x) - f(y)| dy < \varepsilon m(B(1)) \cdot r^n = \delta r^n$$

where we used the fact that

$$\begin{aligned} d(x - y, x) < r &\iff d(-y, 0) < r \\ &\iff d(y, 0) < r \end{aligned}$$

hence

$$\chi_{B(r, x)}(x - y) = \chi_{B(r, 0)}(y)$$

■

Lemma 15.4

Let $A_j = \left\{ |y| \in [2^{-j}\eta, 2^{1-j}\eta) \right\}$, and if Equation (9) holds for ϕ then ϕ_t satisfies

$$|\phi_t| \leq C \cdot t^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \quad (11)$$

on A_j for every $t > 0$, where $\alpha = t^{-1}\eta$ for some $\eta > 0$.

Moreover, if $A_0 = \left\{ |y| < 2^{-K}\eta \right\}$, where $K \geq 0$, then

$$|\phi_t(y)| \leq C \cdot t^{-n} \quad (12)$$

on A_0

Proof of 15.4. Notice that

$$t^{-1}y \in [2^{-j} \cdot \eta/t, 2^{1-j} \cdot \eta/t) = [2^{-j} \cdot \alpha, 2^{1-j} \cdot \alpha)$$

And

$$1 + |t^{-1}y| \geq |t^{-1}y| \geq 2^{-j}\alpha$$

Therefore

$$C \cdot t^{-n} (1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)}$$

and applying Equation (9) establishes the first claim.

The second claim follows from Equation (9),

$$|\phi_t(y)| \leq C \cdot t^{-n} (1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}$$

■

Main Proof of Theorem 8.15. The outline of the proof is as follows,

1. $|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)}$ for $\varepsilon > 0$ and
2. $f \in L^p$ for $p \in [1, +\infty)$,
3. for any $x \in \mathcal{L}_f$, we wish to show

$$|f * \phi_t - af|(x) \rightarrow 0, \quad \text{as } t \rightarrow 0$$

4. To prove this, we fix some $\beta > 0$ and show that

$$|f * \phi_t - af|(x) < \beta$$

since β is arbitrary, the proof will be complete.

5. By Lemma 15.3, for every $\delta > 0$ there exists a $\eta > 0$ where $r \leq \eta$ implies

$$\int_{|y| < r} |f(x) - f(x - y)| dy \leq \delta \cdot r^n$$

and using the L^1 inequality,

$$\begin{aligned} |f * \phi_t - af|(x) &= \left| \int [f(x - y) - f(x)] \cdot \phi_t(y) dy \right| \\ &\leq \int |f(x - y) - f(x)| \cdot |\phi_t(y)| dy \\ &= \int_{|y| < \eta} |f(x - y) - f(y)| \cdot |\phi_t(y)| dy + \int_{|y| \geq \eta} |f(x - y) - f(y)| \cdot |\phi_t(y)| dy \\ &= I_1 + I_2 \end{aligned}$$

6. Let $\delta = \beta(2A)^{-1}$, where

$$A = 2^n \cdot C \left[\frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$$

we make the claim that this choice of δ will give us $I_1 < \beta/2$

7. After choosing $\delta > 0$, (which induces $\eta > 0$), we will show that $I_2 < \beta/2$ (for a fixed $\eta > 0$) for t sufficiently small, and applying the Triangle Inequality finishes the proof.

Let η be as above, and for $t > 0$ and suppose we can find a $K \in \mathbb{N}^+$ with

$$2^K \leq \eta/t \leq 2^{K+1} \tag{13}$$

and define $\alpha = \eta/t$ for convenience.

Notice for any $K \geq 1$, the interval $[0, 1)$ can be partitioned in the following manner

$$[0, 1) = [0, 2^{-K}) \cup \left(\bigcup_{j=1}^K [2^{-j}, 2^{1-j}) \right)$$

and let us define

$$A_j = \left\{ |y| \in [2^{-j}\eta, 2^{1-j}\eta) \right\}, \quad A_0 = \left\{ |y| \in [0, 2^{-K}\eta) \right\}$$

If no such K exists, then let $A_j = \emptyset$ and set $A_0 = \{|y| \in [0, \eta)\}$. The disjoint union of all $A_{j \geq 0}$ is the open ball $\{|y| \in [0, \eta)\}$. By Lemma 15.4 and Lemma 15.3 each $j \geq 0$,

$$\begin{aligned} I_1 &= \sum_{j=0}^K \int_{y \in A_j} |f(x-y) - f(y)| |\phi_t(y)| dy \\ &\leq Ct^{-n} \delta(2^{-K}\eta)^n + \sum_{j=1}^K \int_{y \in A_j} |f(x-y) - f(y)| |\phi_t(y)| dy \\ &\leq Ct^{-n} \delta(2^{-K}\eta)^n + \sum_{j=1}^K Ct^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \delta(2^{1-j}\eta)^n \end{aligned}$$

The left member reads,

$$\begin{aligned} Ct^{-n} \delta(2^{-K}\eta)^n &\leq C\delta\alpha^n 2^{-Kn} \\ &\leq C\delta 2^{n(K+1)} 2^{-Kn} \\ &= C\delta 2^n \end{aligned}$$

and termwise for the right,

$$\begin{aligned} Ct^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \delta(2^{1-j}\eta)^n &= C\delta \cdot t^\varepsilon \cdot 2^{j\varepsilon+n} \eta^{-\varepsilon} \\ &= (C\delta 2^n \alpha^{-\varepsilon}) \cdot 2^{j\varepsilon} \end{aligned}$$

Summing over the geometric series,

$$\begin{aligned} \sum_{j=1}^K 2^{j\varepsilon} &= 2^\varepsilon \sum_{j=0}^{K-1} 2^{j\varepsilon} \\ &= \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} \end{aligned}$$

using the estimate for α in Equation (13)

$$\alpha \in [2^K, 2^K + 1) \implies \alpha^{-\varepsilon} \in [2^{-\varepsilon(K+1)}, 2^{-\varepsilon K})$$

and combining the last few equations, the right member becomes

$$\begin{aligned} (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} &\leq (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)}}{2^\varepsilon - 1} \\ &\leq (C\delta 2^n) \cdot \frac{2^\varepsilon}{2^\varepsilon - 1} \end{aligned}$$

Finally, $I_1 \leq (C\delta 2^n) \left[\frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$, and by Step 6, $I_1 \leq \beta/2$.

Obtaining an estimate for I_2 is another laborious enterprise. Let us define $W = \{|y| \geq \eta\}$, and

- By Holder's Inequality,

$$I_2 \leq \|f\|_p \|\chi_W \cdot \phi_t\|_q + |f(x)| \|\chi_W \cdot \phi_t\|_1$$

where q is the conjugate exponent to p . Since $p \in [1, +\infty)$, it suffices to show $\|\chi_W \cdot \phi_t\|_q \rightarrow 0$ as $t \rightarrow 0$ for $q \in [1, +\infty]$.

- Suppose $q = +\infty$,

$$y \in W \iff |y| \geq \eta \iff |t^{-1}y| \geq \alpha$$

$$\text{then } \|\chi_W \cdot \phi_t\|_\infty \leq Ct^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq Ct^\varepsilon \eta^{-(n+\varepsilon)}$$

- Now suppose $q \in [1, +\infty)$, by polar integration and Theorems 2.51, 2.52 (brace yourselves):

$$\begin{aligned} \|\chi_W \cdot \phi_t\|_q^q &= t^{-nq} \cdot \int_{y \in W} C^q \cdot |t^{-1}y|^{-q \cdot (n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \int_{|y| \geq \eta} |y|^{-q \cdot (n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \sigma(S^{n-1}) \int_{r \geq \eta} r^{n-1} \cdot r^{-q \cdot (n+\varepsilon)} dr \\ &= \frac{C^q t^{\varepsilon q}}{n - q \cdot (n + \varepsilon)} r^{n - q \cdot (n + \varepsilon)} \Big|_\eta^\infty \\ &= \frac{C^q t^{\varepsilon q}}{q \cdot (n + \varepsilon) - n} \eta^{n - q \cdot (n + \varepsilon)} \\ \|\chi_W \cdot \phi_t\|_q &= \left[\frac{C}{(q \cdot (n + \varepsilon) - n)^{1/q}} \left(\eta^{n - q \cdot (n + \varepsilon)} \right)^{1/q} \right] t^\varepsilon \\ &= C_3(q) t^\varepsilon \end{aligned}$$

- Find a t sufficiently small so that

$$t^\varepsilon < \min \left\{ \beta(4C_3(1)|f(x)|)^{-1}, \beta(4C_3(q)\|f\|_p)^{-1}, \beta(4C \cdot \eta^{-(n+\varepsilon)})^{-1} \right\}$$

- Therefore $I_2 < \beta/2$, and the proof is complete upon sending $\beta \rightarrow 0$.

■

Theorem 8.16**Proposition 16.1**

See Theorem 8.15

Proof.



Theorem 8.17

Proposition 17.1

Proof.



Theorem 8.18

Proposition 18.1

Proof.



Theorem 8.19

Proposition 19.1

Proof.



Theorem 8.20

Proposition 20.1

Proof.

