MATH 263: Section 003, Tutorial 2

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1 Review of the Material from Week 1

1.1 Ordinary and Partial Differential Equations

Ordinary Differential Equations (ODE's) are differential equations involving a single variable function and its derivatives. For example:

$$y''(x) + y(x) = \cos x$$

Partial Differential Equations (PDE's) are differential equations involving a multi-variable function and its partial derivatives. For example:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

1.2 Order of a Differential Equation (DE)

The **order of a DE** corresponds to the highest derivative it contains. For example,

$$y^{(69)}(x) + y(x)^2 = \sin x$$

is a 69^{th} order ODE.

1.3 Verify Whether a Function Solves a DE

Given a solution to verify, one simply needs to compute its derivatives and substitute them in the differential equation.

1.4 Initial and Boundary Value Problems and Conditions

An **initial value problem** (IVP) is a differential equation with initial value conditions. Those conditions are restrictions on the solution's value and derivatives at a point, such as $y(0) = 1,\ y'(1) = 0$.
A **boundary value problem** (BVP) uses boundary value conditions, which are multiple restrictions on the solution's value, such as $y(0) = 1,\ y(1) = -1,\ y(2) = 7$. In general, an n^{th} order ODE will require n initial conditions to produce a unique solution.

1.5 Autonomous ODE's

Autonomous ODE's only contain the dependent variable, they are of the form:

$$y^{(n)} = f(y, y', y'', \ldots, y^{(n-1)})$$

1.6 Linear and Non-Linear ODE's

A **linear ODE** can be written as a linear combination of y and its derivatives as such:

$$\sum_{k=0}^{n} a_k(x)\ y^{(k)} = g(x)$$

An example would be:

$$x^2\ y''(x) + 2x\ y'(x) - y(x) = \cos x$$

Otherwise, the ODE is **non-linear**.
Note: when the right hand side $g(x)$ is 0, the ODE is also **homogeneous**.

1.7 Slope Fields

A **slope field** is a graphical representation of a family of functions satisfying $y' = f(x, y)$. For some point (x, y), one draws the slope $y' = f(x, y)$ to qualitatively represent the solutions. Given a slope field, starting at an initial condition and tracing along the field sketches the particular solution.

2 Tutorial 2

2.1 Separable ODE's

A **separable ODE** is of the form:

$$\frac{dy}{dx} = f(x)g(y)$$

Problem 2.1. Solve the IVP:

$$\frac{dy}{dx} = \frac{x}{y}\sqrt{1 + x^2}$$

for $y(0) = -\sqrt{\frac{5}{3}}$.

2.2 Solving First Order Linear ODE's: Integrating Factors

A **first order linear ODE** is of the form:

$$y' + p(x)y = q(x)$$

Problem 2.2a. Determine the general solution of:

$$xy' + 2y = e^{-x}$$

Then, determine the solution's long term behaviour.

Problem 2.2b. Solve the IVP:

$$\cos x\ y' + \sin x\ y = \tan x$$

for $y(x_0) = 1,\ 0 \le x_0 \le \frac{\pi}{2}$. For which value(s) of x_0 does the IVP have no solution?

2.3 Homogeneous First Order ODE's

A **homogeneous ODE** is of the form:

$$\frac{dy}{dx} = F(\frac{y}{x})$$

Note: **not** the same as the definition given in 1.6.
Let $v = \frac{y}{x} \Rightarrow y = vx \Rightarrow y' = xv' + v$. Then substitute and solve for v to find y.

Note: Other types of substitution to solve ODE's exist, such as $v = y'(x) \; or \; v = ax + by$.

Problem 2.3. Determine the general solution of:

$$xy' = y + x\ e^{\frac{y}{x}}$$

2.4 Bernoulli Equations

A **Bernoulli equation** is of the form:

$$\frac{dy}{dx} + P(x)y = Q(x)y^n$$

When $n \notin \{0, 1\}$, we can let $v = y^{1-n}$, making the ODE linear for v.

Problem 2.4. Solve the IVP:

$$y' + \frac{y}{x} = xy^3$$

for $x > 0$ and $y(1) = \frac{1}{2}$.