MATH 263: Section 003, Tutorial 11

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1 Series Solutions Near a Regular Singular Point, Part I and II

Consider the general second order linear ODE:

$$P(x)y"(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$

Where $P(x_0) = 0$, meaning that $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are not analytical at $x = x_0$. $x = x_0$ is then a **singular point**.

Consider the case of **regular singular points**, where $(x - x_0)p(x) = (x - x_0)\frac{Q(x)}{P(x)}$ and $(x - x_0)^2 q(x) = (x-x_0)^2 \frac{R(x)}{P(x)}$ are analytic at $x = x_0$. We can write them as: $(x-x_0)p(x) = (x-x_0)\frac{Q(x)}{P(x)}$ and $(x-x_0)^2 q(x) = (x - x_0)^2 \frac{R(x)}{P(x)}$ are

$$(x - x_0)p(x) = \sum_{n=0}^{\infty} p_n (x - x_0)^n,$$

and

$$(x - x_0)^2 q(x) = \sum_{n=0}^{\infty} q_n (x - x_0)^n,$$

Plugging them in the ODE, we get

$$(x - x_0)^2 y" + (x - x_0)[(x - x_0)p(x)]y' + [(x - x_0)^2 q(x)]y = 0$$

$$(x-x_0)^2 y" + (x-x_0)(p_0 + p_1(x-x_0) + \cdots + p_n(x-x_0)^n + \dots)y' + (q_0 + q_1(x-x_0) + \cdots + q_n(x-x_0)^n + \dots)y = 0.$$

As x approaches x_0, the ODE behaves as an Euler equation as such:

$$(x - x_0)^2 y" + p_0(x - x_0)y' + q_0 y = 0.$$

where $p_0 = \lim_{x \to x_0} \frac{Q(x)}{P(x)}$ and $q_0 = \lim_{x \to x_0} \frac{R(x)}{P(x)}$

The solutions will be of the form of Euler solutions times a power series as such:

$$y(x) = x^r \sum_{n=0}^{\infty} a_n (x - x_0)^n,\ a_0 \neq 0$$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^{r+n}$$

Then plug it in the ODE. The Euler characteristic equation will arise along with a recurrence relation for a_n that depends on r.

When these roots are identical or differ by an integer, one can only find one fundamental solution with this method.

Let $x_0 = 0$. Given a double root r_1, the second solution will be of the form:

$$y_2(x) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} b_n(r_1) x^n$$

Given roots r_1 and r_2 that differ by an integer (in other words $r_1 - r_2 = N > 0$, where $N \in \mathbb{N}$), the second solution will be of the form:

$$y_2(x) = C y_1(x) \ln x + x^{r_2} [1 + \sum_{n=1}^{\infty} c_n(r_2) x^n]$$

In each case, we plug this equation back in the ODE to find all constants and coefficients of the second solution. Note: $r_1 > r_2$. Given roots r_1 and r_2 that do not differ by an integer, the general solution will be of the form:

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

$$y(x) = c_1 x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n + c_2 x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n.$$

Problem 1. From Boyce and DiPrima, 10th edition (5.5, exercise 3, p.286):
Find one fundamental solution of:

$$xy\text{''} + y = 0.$$