

Chapter 3

Notes on Chapter 3

Proposition 0.1. *Prove two things,*

1. $\limsup_{r \rightarrow R} \phi(r) = \lim_{\varepsilon \rightarrow 0} \sup_{0 < |r-R| < \varepsilon} \phi(r) = \inf_{\varepsilon > 0} \sup_{0 < |r-R| < \varepsilon} \phi(r),$
2. $\lim_{r \rightarrow R} \phi(r) = c \iff \limsup_{r \rightarrow R} |\phi(r) - c| = 0$

Proof. ■

Proposition 0.2. *If $U \subseteq B(1, 0) = \{|x| < 1\}$, and $U \in \mathbb{B}$, and if $m(U) > 0$, then the family of sets*

$$E_r = \left\{ x + ry, y \in U \right\}$$

shrinks nicely to $x \in \mathbb{R}^n$.

Proof. Let $r > 0$ be fixed then $\forall z \in E_r \leadsto z = x + ry$. Hence,

$$\begin{aligned} d(x, z) &= d(x, x + ry) \\ &= |r|d(0, y) < |r| \end{aligned}$$

by translation invariance. ■

Theorem 3.1**Proposition 1.1.**

Proof. Let ν be a signed measure, and fix any increasing sequence $E_j \nearrow E = \bigcup E_{j \geq 1}$ of sets. This induces a disjoint sequence in $\{F_n\}$. Define $F_1 = E_1$, and if $n \geq 2$,

$$F_n = E_n \setminus \bigcup E_{j \leq n-1}$$

and from this, the finite It is clear that $\bigcup F_{n \geq 1} = E$, and let us assume $\nu(E)$ is of finite measure.

By countable additivity, and the absolute convergence of the series $\sum_{j \leq n} \nu(F_j)$

$$\begin{aligned} \nu\left(\bigcup E_{j \geq 1}\right) &= \sum_{j \geq 1} \nu(F_j) \\ &= \lim_n \sum_{j \leq n} \nu(F_j) \\ &= \lim \nu(E_n) \end{aligned}$$

■

Theorem 3.2

Proposition 2.1.

Proof.



Theorem 3.3

Proposition 3.1.

Proof.



Theorem 3.4

Proposition 4.1.

Proof.



Theorem 3.5

Proposition 5.1.

Proof.



Theorem 3.6

Proposition 6.1.

Proof.



Theorem 3.7

Proposition 7.1.

Proof.



Theorem 3.8

Proposition 8.1.

Proof.



Theorem 3.9

Proposition 9.1.

Proof.



Theorem 3.10

Proposition 10.1.

Proof.



Theorem 3.11

Proposition 11.1.

Proof.



Theorem 3.12

Proposition 12.1.

Proof.



Theorem 3.13

Proposition 13.1.

Proof.



Theorem 3.14

Proposition 14.1.

Proof.



Theorem 3.15

Proposition 15.1.

Proof.



Theorem 3.16

Proposition 16.1.

Proof.



Theorem 3.17

Proposition 17.1. *Let the maximal function of any measurable $f \in \mathbb{B}_{\mathbb{R}^n}$ be denoted by $Hf(x)$, more precisely,*

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

where $A_r|f|$ is the average of $|f|$ on a ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$. In symbols,

$$A_r|f| = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

The maximal theorem makes two claims:

1. $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, and Hf is measurable for every $f \in L^1_{loc}$.
2. There exists a $C > 0$, for every $f \in L^1$

$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1$$

for every $\alpha > 0$.

Proof. Let $\alpha > 0$ and fix $z \in (Hf)^{-1}((\alpha, +\infty))$, so $Hf(z) > \alpha$ and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with $Hf(z) - \alpha > 0$, we get some $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some $r_0 > 0$. Since $A_r|f|$ (a function indexed by $r > 0$) is continuous in $x \in \mathbb{R}^n$, $(A_r|f|)^{-1}((\alpha, +\infty))$ is open, and Hf is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{ Hf > \alpha \right\} = \bigcup_{r>0} \{A_r |f| > \alpha\}$$

Let $x \in E_\alpha$, this induces a $r_x > 0$ where $x \in \left\{ A_{r_x} |f| > \alpha \right\}$. Rearranging gives

$$\left(\frac{1}{\alpha} \int_{B(r,x)} |f| dz \right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each $x \in E_\alpha \mapsto r_x > 0 \mapsto A_{r_x} |f|$,
- If $U = \bigcup_{x \in E_\alpha} B(r_x, x)$, then $E_\alpha \subseteq U$,
- Choose $c < m(E_\alpha) \leq m(U)$ (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

$$x_1, \dots, x_N \in E_\alpha$$

so that $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$, and $c < 3^n \sum_{j \leq k} m(B_j)$

- Define $B_j = B(r_{x_j}, x_j)$ for all $j \leq k$, and

$$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

by finite additivity,

$$c 3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

and finally

$$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of m on \mathbb{B} , since

$$m(E_\alpha) = \sup \left\{ m(K), K \in \mathcal{J}_{\mathbb{R}^n}, K \subseteq E_\alpha \right\}$$

for any $K \in \mathcal{J}_{\mathbb{R}^n}$, $K \subseteq E_\alpha$, we have $m(K) < +\infty$, $m(K) \leq m(E_\alpha)$ and

$$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

Remark 17.1 *We used the properties of a Radon Measure here, without relying on the phrase ‘sending $c \rightarrow E_\alpha$ ’, which would require us to deal with two cases $m(E_\alpha) < +\infty$ and $m(E_\alpha) = +\infty$.*

■

Theorem 3.18

Proposition 18.1.

Proof.



Theorem 3.19

Proposition 19.1.

Proof.



Theorem 3.20

Proposition 20.1.

Proof.



Theorem 3.21

Proposition 21.1. *The Lebesgue Differentiation Theorem. Suppose $f \in L^1_{loc}$, and for every $x \in \mathcal{L}_f$, (so that $x \in \mathbb{R}^n$ a.e). We have*

1. $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0,$
2. $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x),$

For every family $\{E_r\}_{r>0}$ that shrinks nicely to $x \in \mathbb{R}^{n'}$.

Proof. Since the family $\{E_r\}_{r>0}$ shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r, x)) \implies m(E_r) > \alpha \cdot m(B(r, x))$$

for some $\alpha > 0$, independent on r . Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r, x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r, x)} |f(y) - f(x)| dy$$

Combining the last two results, for every $\varepsilon > 0$, if $0 < r < \varepsilon$, then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1} B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0 < r < \varepsilon} m^{-1} B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

and sending $\varepsilon \rightarrow 0$, proves the first claim. The second claim is immediate upon applying the L^1 inequality.

Fix any $\varepsilon > 0$, and

$$\begin{aligned}
 \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} f(y) dy = f(x) &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y) dy - f(x) \right| \\
 &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)] dy \right| \\
 &\leq \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
 &= \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
 &= 0
 \end{aligned}$$

■

Theorem 3.22

Proposition 22.1.

Proof.



Theorem 3.23

Proposition 23.1.

Proof.



Theorem 3.24

Proposition 24.1.

Proof.



Theorem 3.25

Proposition 25.1.

Proof.



Theorem 3.26

Proposition 26.1.

Proof.



Theorem 3.27

Proposition 27.1.

Proof.



Theorem 3.28

Proposition 28.1.

Proof.



Theorem 3.29

Proposition 29.1.

Proof.



Theorem 3.30

Proposition 30.1.

Proof.



Theorem 3.31

Proposition 31.1.

Proof.



Theorem 3.32

Proposition 32.1.

Proof.



Theorem 3.33

Proposition 33.1.

Proof.



Theorem 3.34

Proposition 34.1.

Proof.



Theorem 3.35

Proposition 35.1.

Proof.



Theorem 3.36

Proposition 36.1.

Proof.

