MATH 263: Section 003, Tutorial 7

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1 Higher Order Homogeneous Equations and the Wronskian

Given n solutions to an n^{th} order linear ODE, showing their independence can also be shown by their **Wronskian**, which must be nonzero for linear independence. In general, it is of the form:

$$W(y_1,\ y_2,\ y_3,\ \dots\ y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ y_1' & y_2' & y_3' & \dots & y_n' \\ y_1" & y_2" & y_3" & \dots & y_n" \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & y_3^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

Problem 1. From Boyce and DiPrima, 10th edition (4.2, exercise 32, p.234):
Find the general solution to:

$$y''' - y'' + y' - y = 0$$

Find the Wronskian of the fundamental solutions: are the fundamental solutions independent? Then, solve the IVP: for $y(0) = 2,\ y'(0) = -1,\ y''(0) = -2$.

Solution: Let $y = e^{\lambda x}$, the characteristic equation is then:

$$\lambda^3 - \lambda^2 + \lambda - 1 = 0$$
$$\lambda^2(\lambda - 1) + (\lambda - 1) = 0$$
$$(\lambda^2 + 1)(\lambda - 1) = 0$$
$$(\lambda + i)(\lambda - i)(\lambda - 1) = 0$$
$$\lambda_1 = 1,\ \lambda_2 = i,\ \lambda_3 = -i.$$

By Euler's Formula, the fundamental set of solutions are then:

$$y_1(x) = e^x,\ y_2(x) = \cos x,\ y_3(x) = \sin x.$$

To show their independence, find their Wronkskian:

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & \cos x & \sin x \\ e^x & -\sin x & \cos x \\ e^x & -\cos x & -\sin x \end{vmatrix} = 2e^x \neq 0$$

for all $x \in \mathbb{R}$. Note: **Abel's Theorem** also can show the Wronskian for higher order ODEs. For an n^{th} order linear ODE of the form:

$$y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \cdots + p_1(x)y'(x) + p_0(x)y(x) = 0$$

Given n fundamental solutions $y_1,\ y_2,\ y_3,\ \dots\ y_n$, **Abel's Theorem** states that:

$$W(y_1,\ y_2,\ y_3,\ \dots\ y_n) = \begin{vmatrix} y_1 & y_2 & y_3 & \dots & y_n \\ y_1' & y_2' & y_3' & \dots & y_n' \\ y_1" & y_2" & y_3" & \dots & y_n" \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & y_3^{(n)} & \dots & y_n^{(n)} \end{vmatrix} = C\ \exp[-\int p_{n-1}(x)\ dx]$$

For this ODE, it would be:

$$W(y_1, y_2, y_3) = C\ e^{\int dx} = Ce^x.$$

Therefore, by superposition, the general solution is:

$$y(x) = c_1e^x + c_2 \cos x + c_3 \sin x.$$

Now, find the solution to the IVP $y(0) = 2,\ y'(0) = -1,\ y''(0) = -2$. To do so, we must first find the solution's first two derivatives:

$$y'(x) = c_1e^x - c_2 \sin x + c_3 \cos x$$

$$y''(x) = c_1e^x - c_2 \cos x - c_3 \sin x.$$

Therefore,

$$y(0) = c_1e^0 + c_2 \cos 0 + c_3 \sin 0 = c_1 + c_2 = 2$$

$$y'(0) = c_1e^0 - c_2 \sin 0 + c_3 \cos 0 = c_1 + c_3 = -1$$

$$y''(0) = c_1e^0 - c_2 \cos 0 - c_3 \sin 0 = c_1 - c_2 = -2$$

To find the constants, solve the system of equations using any method (substitution, Gaussian elimination, inverting the matrix).

$$c_1 = 0,\ c_2 = 2,\ c_3 = -1.$$

$$y(x) = 2 \cos x - \sin x.$$

2 Existence and Uniqueness Theorem

Given the IVP:

$$y" + p(x)y' + q(x)y = g(x),\ y(x_0) = y_0,\ y'(x_0) = y_0$$

if p, q, and g are continuous on the open interval I that contains the point x_0, then the solution to the IVP is **unique, differentiable, and exists** on the interval I.

Problem 2. From Boyce and DiPrima, 10th edition (3.2, exercise 10, p.155):
Determine the longest interval in which the initial value problem:

$$y"(x) + \cos x\ y'(x) + 3 \ln|x|\ y(x) = 0,\ y(2) = 3,\ y'(2) = 1$$

is certain to have a unique twice-differentiable solution.

Solution: $p(x) = \cos x$ and $g(x) = 0$ are continuous for all x, and $q(x) = 3 \ln|x|$ is continuous for all $x \neq 0$. Therefore, general solutions can be continuous in $x \in (-\infty, 0) \cup (0, +\infty)$. Since $x_0 = 2$, the interval of existence of the unique solution to the IVP is $x > 0$.

3 Reduction of Order

Recall in **Tutorial 5** that given the ODE:

$$y" + p(x)y'(x) + q(x)y(x) = 0$$

and one solution $y_1(x)$, we can find **a general solution of the form** $y(x) = v(x)y_1(x)$. Then, find y"s derivatives and substitute them in the ODE to find v and y:

$$y'(x) = v'(x)y_1(x) + v(x)y_1'(x)$$
$$y_1 v" + (2y_1' + py_1)v' + (y_1" + py_1' + qy_1)v = 0$$
$$y_1 v" + (2y_1' + py_1)v' = 0.$$

Problem 3. From Boyce and DiPrima, 10th edition (3.4, exercise 27, p.174):
Use reduction of order to find a general solution to:

$$xy" - y' + 4x^3 y = 0,\ x > 0$$

given one solution $y_1(x) = \sin x^2$.

Solution: Divide by x, since $x \neq 0$:

$$y" - \frac{1}{x}y' + 4x^2 y = 0,\ x > 0$$

therefore, $p(x) = \frac{-1}{x}$. Use reduction of order to solve for v and y, where $y(x) = v(x)y_1(x) = v(x)\sin x^2$.

$$y_1' = 2x\cos x^2$$

$$y_1 v" + (2y_1' + py_1)v' = \sin x^2 v" + (4x\cos x^2 - \frac{1}{x}\sin x^2)v' = 0$$

Let $\psi(x) = v'$, meaning that $v(x) = C + \int \psi(x)\ dx$, and $v" = \frac{d\psi}{dx}$:

$$\sin x^2 \frac{d\psi}{dx} + (4x\cos x^2 - \frac{1}{x}\sin x^2)\psi = 0$$

$$x\sin x^2 \frac{d\psi}{dx} = (\sin x^2 - 4x^2\cos x^2)\psi$$

$$\frac{d\psi}{\psi} = \frac{\sin x^2 - 4x^2\cos x^2}{x\sin x^2}\ dx$$

$$\int \frac{d\psi}{\psi} = \ln|\psi| = \int \frac{1}{x} - 4x\cot x^2\ dx$$

Let $u = x^2$, $du = 2x\ dx$:

$$\ln|\psi| = C + \ln|x| - \int 2\cot u\ du = C + \ln|x| - \int 2\frac{\cos u}{\sin u}\ du$$

Let $t = \sin u$, $dt = \cos u\ dt$:

$$\ln|\psi| = C + \ln|x| - \int \frac{2}{t}\ dt$$

Since $x > 0,\ |x| = x$:

$$\ln|\psi| = C + \ln x - \ln|t| = C + \ln x - 2\ln|\sin x^2|$$

Let $K = \pm e^C$:

$$\psi = Kx\exp[-2\ln|\sin x^2|] = Kx\csc^2(x^2)$$

Now find $v(x)$:

$$v(x) = \int Kx \csc^2(x^2)\ dx$$

Let $u = x^2$, $du = 2x\ dx$:

$$v(x) = \int \frac{K}{2} \csc^2(u)\ du = K_1 - \frac{K}{2} \cot u$$

Let $K_2 = \frac{-K}{2}$:

$$v(x) = \int \frac{K}{2} \csc^2(u)\ du = K_1 + K_2 \cot x^2.$$

Finally,

$$y(x) = v(x)y_1(x) = (K_1 + K_2 \cot x^2) \sin x^2$$
$$y(x) = K_1 \sin x^2 + K_2 \cos x^2,\ x > 0.$$

4 Euler's Equations: Change of Variables

More general Euler Equations of the form:

$$a(x - x_0)^2 y" + b(x - x_0)y' + cy = 0$$

can solved by simply letting $t = x - x_0,\ dt = dx$.

Then, the solution is solved the same way as done in **Tutorial 6** to find in general:

$$y = y(t) = y(x - x_0),\ x \neq x_0.$$

Problem 4. From Boyce and DiPrima, 10th edition (5.3, exercise 10, p.280):
Find the general solution of:

$$(x - 2)^2 y" + 5(x - 2)y' + 8y = 0,\ x \neq 2.$$

Solution: let $t = x - 2$:

$$t^2 y" + 5ty' + 8y = 0,\ t \neq 0$$

Let $y = |t|^r$, the characteristic equation is:

$$r^2\ + 4r + 8 = 0.$$

The roots are then:

$$r_1 = -2 + 2i,\ r_2 = -2 - 2i.$$

Therefore,

$$y(t) = \frac{1}{t^2}[c_1 \cos(2 \ln |t|) + c_2 \sin(2 \ln |t|)].$$

Plugging in back $t = x - 2$, we get:

$$y(x) = \frac{1}{(x - 2)^2}[c_1 \cos(2 \ln |x - 2|) + c_2 \sin(2 \ln |x - 2|)],\ x \neq 2.$$