Chapter A: Review of Topology

Set Operations

This section is meant for reference.

Proposition 1.1: Direct and Inverse Images of Maps

Let $f : \mathbf{X} \to \mathbf{Y}$, where $\mathbf{X}$ and $\mathbf{Y}$ are sets. If $A \subseteq \mathbf{X}$, $B \subseteq \mathbf{Y}$, and $\{E_\alpha\}$ is an indexed collection of subsets of $\mathbf{X}$, $\{G_\beta\}$ is an indexed collection of subsets of $\mathbf{Y}$, then

Direct images

$$f\Big(\bigcap E_\alpha\Big) \subseteq \bigcap f(E_\alpha) \quad \text{equality if injective} \tag{1}$$

$$f\Big(\bigcup E_\alpha\Big) = \bigcup f(E_\alpha) \tag{2}$$

Estimates

$$f\Big(f^{-1}(B)\Big) \subseteq B \quad \text{equality if surjective} \tag{3}$$

$$A \subseteq f^{-1}(f(A)) \quad \text{equality if injective} \tag{4}$$

Inverse images

$$f^{-1}\Big(\bigcup G_\beta\Big) = \bigcup f^{-1}(G_\beta) \tag{5}$$

$$f^{-1}\Big(\bigcap G_\beta\Big) = \bigcap f^{-1}(G_\beta) \tag{6}$$

$$f^{-1}(B^c) = \Big(f^{-1}(B)\Big)^c \tag{7}$$

Proposition 1.2: Composition of Maps

Let $h = g \circ f$, we assume this composition is well defined.

- If h is a surjection, then g is a surjection,
- If h is an injection, then f is an injection.

Proof. Take the contrapositive. ■

Proposition 1.3: Left and Right inverses

Let $F : \mathbf{X} \to \mathbf{Y}$,

- F is surjective if and only if there exists right inverse $G : \mathbf{Y} \to \mathbf{X}$,

  $$F \circ G = \mathrm{id}_{\mathbf{Y}}$$

  if $A \subseteq \mathbf{X}$,

  $$G^{-1}(A) \subseteq F(A)$$

- F is injective if and only if there exists a left inverse $H : F(\mathbf{X}) \to \mathbf{X}$

$$H \circ F = \mathrm{id}_{\mathbf{X}}$$

and if $B \subseteq \mathbf{Y}$,

$$F^{-1}(B) \subseteq H(B)$$

Topological Spaces

This section will roughly follow Munkres text on General Topology, in particular we hope to cover Chapters 2, 3, 4 and 9. The rest of the Chapters should be covered proper by the subsequent section.

Definition 2.1: Topology

Let $\mathbf{X}$ be a non-empty set. A topology $\mathcal{T}$ on $\mathbf{X}$, sometimes denoted by $\mathcal{T}_{\mathbf{X}}$ is a family of subsets of $\mathbf{X}$,

- $\{\varnothing, \mathbf{X}\} \subseteq \mathcal{T}$,
- If U_1 and U_2 are elements of $\mathcal{T}$, so is their intersection.
- If $\{U_\alpha\}$ is an arbitrary family of sets in $\mathcal{T}$, their union is also contained in $\mathcal{T}$ as an element.

We call the elements of $\mathcal{T}$ open sets. The complements of elements in $\mathcal{T}$ are closed sets.

Basis of a Topology

Definition 3.1: Basis of a topology

A basis $\mathbb{B}$ is a family of subsets of $\mathbf{X}$, that satisfies:

- Every $x \in \mathbf{X}$ belongs (as an element) in some $V \in \mathbb{B}$.
- If B_1 and B_2 are basis elements, such that their intersection is non-empty. Then every $x \in B_1 \cap B_2$ induces a $B_3 \in \mathbb{B}$ with

$$x \in B_3 \subseteq B_1 \cap B_2$$

This roughly means a basis is 'finitely' fine at every point in x.

If $\mathbb{B}$ is a basis, it 'generates' a topology $\mathcal{T}$ through

$$\mathcal{T} = \left\{ U \subseteq \mathbf{X},\ \forall x \in U,\ x \in B \subseteq U \text{ for some } B \in \mathbb{B} \right\} \tag{8}$$

Notice this is equivalent to $\mathcal{T}$ is the collection of all unions of basis elements in $\mathbb{B}$.

> **Proposition 3.1**
>
> Let $\mathbb{B}$ be a basis as defined in Definition 3.1, then $\mathcal{T}$ as defined in Equation (8) is a valid topology on $\mathbf{X}$. And every member of $\mathcal{T}$ is and is precisely the union of elements in $\mathbb{B}$.

Proof. Every point in $\mathbf{X}$ belongs in some basis element, so $\mathbf{X} \in \mathcal{T}$, so does $\varnothing$. Next, if U_1 and U_2 are in $\mathcal{T}$, then

$$\begin{cases} x \in U_1 \looparrowright x \in B_1 \subseteq U_1 \\ x \in U_2 \looparrowright x \in B_2 \subseteq U_2 \end{cases} \implies x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

for some $B_3 \in \mathbb{B}$, so $\mathcal{T}$ is closed under finite intersections (perhaps after a standard induction argument).

If $\{U_\alpha\} \subseteq \mathcal{T}$, and x belongs in the union of all U_α, then $x \in B_\alpha \subseteq U_\alpha$, which is a subset of the entire union. So the union over U_α is again contained in $\mathcal{T}$, and $\mathcal{T}$ is a topology on $\mathbf{X}$.

It is worth noting that $\mathbb{B} \subseteq \mathcal{T}$. Finally, if $U \in \mathcal{T}$,

$$U = \bigcup_{x \in U} B_x$$

where B_x is the basis element taken to satisfy $x \in B_x \subseteq U$. Every point in U is included in some B_x, and hence is included in the union. For the reverse inclusion, notice the union of subsets of U is again a subset of U.

Now, if $E \subseteq \mathbf{X}$ is the union of basis elements in $\mathbb{B}$, if E is non-empty, then every point $x \in E$ belongs in some B_x. Recycling the previous argument, and we see that E is open in $\mathcal{T}$. If E is empty, we define the 'union' of no sets as the empty set. So $\mathcal{T}$ is precisely the collection of all unions of basis elements $\mathbb{B}$. ■

We are now in a position to compare the relative 'fineness' of topologies.

> **Definition 3.2: Fineness of topologies**
>
> If $\mathcal{T}'$ and $\mathcal{T}$ are both topologies on some non-empty set $\mathbf{X}$. We say $\mathcal{T}'$ is finer than $\mathcal{T}$, or $\mathcal{T}$ is coarser than $\mathcal{T}'$ if
>
> $$\mathcal{T}' \supseteq \mathcal{T}$$

Proposition 3.2

If $\mathbb{B}$ and $\mathbb{B}'$ are bases for $\mathcal{T}'$ and $\mathcal{T}$, the following are equivalent:

- $\mathcal{T}'$ is finer than $\mathcal{T}$,
- If B is an arbitrary basis element in $\mathbb{B}$, then every point $x \in B$ induces a basis element in $\mathbb{B}'$ with
$$x \in \mathbb{B}' \subseteq \mathbb{B}$$

Proof. Suppose $\mathcal{T}'$ is finer than $\mathcal{T}$. Notice $\mathbb{B} \subseteq \mathcal{T}'$ as well. By Equation (8), each $x \in B$ induces a $B' \in \mathbb{B}'$

$$x \in B' \subseteq B$$

Conversely, fix any open set $U \in \mathcal{T}$, and for each $x \in U$,

$$x \in B' \subseteq B \subseteq U$$

Applying Definition 3.1 tells us U is open in $\mathcal{T}'$. ■

The last of the big three 'generating' definitions for topologies will be the sub-basis. It simply means the first condition (but not necessarily) the second, is satisfied in Definition 3.1

Definition 3.3: Sub-basis of a topology

A sub-basis $S \in \mathbb{P}(\mathbf{X})$ is a family of subsets of $\mathbf{X}$ that satisfies one property. Any point x in $\mathbf{X}$ belongs to at least one member of S.

A sub-basis can be upgraded to a basis by collecting all of its finite intersections.

Proposition 3.3

Let S be a sub-basis of $\mathbf{X}$, then the collection of all finite intersections of S forms a basis $\mathbb{B}$ of $\mathbf{X}$.

Proof. Every point in $\mathbf{X}$ lies in some element of S, hence in some element of $\mathbb{B}$. The second basis property is immediate, since $\mathbb{B}$ is closed under finite intersections. ■

Product Topology

We will start with products of a finite collection of topological spaces.

Definition 4.1: Finite Product of Topological Spaces

Let $(\mathbf{X}, \mathcal{T}_{\mathbf{X}})$ and $(\mathbf{Y}, \mathcal{T}_{\mathbf{Y}})$ be topological spaces. The product topology (denoted by $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$) on $X \times Y$ is defined as the topology generated by the basis

$$\mathbb{B}_{\mathbf{X}\times\mathbf{Y}} = \left\{ U \times V,\ (U,V) \in \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}} \right\} \tag{9}$$

Since bases are easier to describe than topologies, we have the following statement concerning the basis of the product topology.

Proposition 4.1

If $\mathbb{B}_{\mathbf{X}}$ and $\mathbb{B}_{\mathbf{Y}}$ are bases for $\mathcal{T}_{\mathbf{X}}$ and $\mathcal{T}_{\mathbf{Y}}$, then the product topology (as described in Definition 4.1) is also generated by

$$\mathcal{M} = \left\{ U \times V,\ (U,V) \in \mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \right\} \tag{10}$$

Proof. We will introduce (and use) the technique of 'double inclusion' by proving that the topologies generated are both finer than the other. Let us denote the topology generated by $\mathcal{M}$ in Equation (10) by $\mathcal{T}_{\mathcal{M}}$.

Since $\mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \subseteq \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}$, if $U \times V \in \mathcal{M}$ as in Equation (10), then we can pick the same 'open rectangle' again. We trivially have

$$x \in \underbrace{U \times V}_{\text{member of } \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}} \subseteq U \times V$$

and by Proposition 3.2, $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$ is finer than $\mathcal{T}_{\mathcal{M}}$.

Fix any set $U \times V \in \mathbb{B}_{\mathbf{X}\times\mathbf{Y}}$, and if $(p,q) \in U \times V$, each coordinate induces basis elements from $\mathbb{B}_{\mathbf{X}}$ and By, more precisely:

$$\begin{cases} p \in U \implies p \in \text{Basis element of } \mathbb{B}_{\mathbf{X}} \subseteq U \\ q \in V \implies q \in \text{Basis element of } \mathbb{B}_{\mathbf{Y}} \subseteq V \end{cases} \implies (p,q) \in \underbrace{}_{\text{in } \mathbb{B}_{\mathbf{X}}} \times \underbrace{}_{\text{in } \mathbb{B}_{\mathbf{Y}}} \subseteq U \times V$$

by Proposition 3.2, $\mathcal{T}_{\mathcal{M}}$ is finer than $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}}$ and $\mathcal{T}_{\mathbf{X}\times\mathbf{Y}} = \mathcal{T}_{\mathcal{M}}$. ■

Quotient Topology

Product Topology

The Cartesian Product of an arbitrary family of topological spaces, if equipped with the product topology, preserves a lot of the structure. If $\{X_\alpha\}_{\alpha \in A}$ is a family of topological spaces which are _____, then $\prod X_\alpha$ is _____. Replace _____ with:

1. Hausdorff, (Folland)
2. Regular,

3. Connected, (Munkres chp23, exercise 10)
4. First countable, if A is countable,
5. Second countable, if A is countable,
6. Compact (Tynchonoff's Theorem, Folland)

Connectedness

Definition 7.1: Connectedness

A topological space $\mathbf{X}$ is connected if U and V are disjoint open subsets whose union is $\mathbf{X}$, then at least one of U or V is empty.

See Folland Exercise 4.10 for more properties.

Definition 7.2: Path-connectedness

A topological space $\mathbf{X}$ is path-connected if for any two pair of points $x,\ y \in \mathbf{X}$. There exists a continuous function $f : [a, b] \to \mathbf{X}$, with $f(a) = x$ and $f(b) = y$.

Definition 7.3: Connected component

The connected components of $\mathbf{X}$ is the family of equivalence classes on $\mathbf{X}$, where $x \sim y$ if there is a connected subspace of $\mathbf{X}$ that contains both of them.

Proposition 7.1

Continuous functions map connected spaces to connected spaces (in the subspace topology).

Proof. Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces and $f : \mathbf{X} \to \mathbf{Y}$ be continuous. If $f(\mathbf{X})$ is disconnected, then we can find U and V, open and disjoint in $\mathcal{T}_{f(\mathbf{X})}$ such that

$$U \cup V = f(\mathbf{X}) \implies f^{-1}(U) \cup f^{-1}(V) = \mathbf{X}$$

where $f^{-1}(f(\mathbf{X})) = \mathbf{X}$. Both $f^{-1}(U)$ and $f^{-1}(V)$ are open, non-empty, and are pairwise disjoint. So $\mathbf{X}$ is separated. ∎

Proposition 7.2

Let $(\mathbf{X}_\alpha,\ \mathcal{T}_\alpha)$ be a family of connected topological spaces indexed by $\alpha \in A$. Then

$\prod_{\alpha \in A} \mathbf{X}_\alpha$ is disconnected in the product topology.

Proof. We will attempt the contrapositive. Suppose $\prod_{\alpha \in A} \mathbf{X}_\alpha$ is disconnected, then ■

Interiors and closures

Definition 8.1: Interior of a set

A^o is defined to be the largest open subset of A,

$$A^o = \bigcup_{\substack{U \text{ open,} \\ U \subseteq A}} U$$

Corollary 8.1

The union of subsets of A is again a subset of A, therefore Corollary 8.1 implies $A^o \subseteq A$ for any $A \subseteq \mathbf{X}$.

Definition 8.2: Closure of a set

and $\overline{A}$ is the smallest closed superset of A,

$$\overline{A} = \bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K$$

Proposition 8.1

The complement of the closure is the interior of the complement, or equivalently: $(\overline{A})^c = A^{co}$

Proof. Taking complements, and the substitution $U = K^c$ reads

$$(\overline{A})^c = \left[\bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K \right]^c$$

$$= \bigcup_{\substack{K \text{ closed,} \\ K^c \subseteq A^c}} K^c$$

$$= \bigcap_{\substack{U \text{ open,} \\ U \subseteq A^c}} U$$

$$= A^{co}$$

■

Remark 8.1

Personally, I remember this as pushing the complement inside and flipping the bar to a c!

Neighbourhoods

The concept of a neighbourhood allows us to characterize the interior of a set 'locally'.

Definition 9.1: Neighbourhood (not necessarily open)

A neighbourhood of $x \in \mathbf{X}$ is a set $U \subseteq \mathbf{X}$ where $x \in U^o$. The set of neighbourhoods for a point $x \in \mathbf{X}$ will sometimes be denoted by $\mathcal{N}(()x)$.

Proposition 9.1: Characterization of the interior

If $W = \left\{ x \in \mathbf{X}, \text{ there exists a neighbourhood } U \text{ of } x, U \subseteq A \right\}$, then $W = A^o$.

Proof. If $x \in A^o$, then A is a neighbourhood of x, and $A \subseteq A$, so $x \in W$. Conversely, if x is a member of W, it has a neighbourhood $U \subseteq A$ (not necessarily open). By monotonicity of the interior,

$$x \in U^o \subseteq A^o$$

and $x \in A^o$. ■

It is easy to see that A is open $\iff A^o = A \iff A$ is a neighbourhood of itself.

- The first equivalence follows from:

$$E \subseteq \mathbf{X} \implies E^o \subseteq E$$

  and if A is an open set, it is an open subset of itself, by Corollary 8.1 $A \subseteq A^o$. If $A^o = A$, then it suffices to show that A^o is open. Which it is, since it is the arbitrary union of open sets.

- To prove the second equivalence: suppose $A^o = A$, then each $x \in A$ has a neighbourhood contained (as a subset) in A, namely A itself. (This statement is hard to parse, the reader is encouraged to really work through this and be honest).

$$x \in A^o \subseteq A \implies A \subseteq A^o$$

  so A is a neighbourhood of itself. Conversely, if $A \subseteq A^o$, then $A = A^o$, since the reverse inclusion follows immediately from Corollary 8.1.

Adherent points

Similar to the neighbourhood, the concept of an adherent point of a set allows us to speak of the closure in more concrete terms. The following definition is key in understanding the relationship between the closure, interior, and the boundary.

Definition 10.1: Adherent point of a set

Let $A \subseteq \mathbf{X}$, $x \in \mathbf{X}$ is an adherent point of A if every neighbourhood U of x intersects A. In symbols,

$$U \cap A \neq \varnothing, \quad \forall U \in \mathcal{N}(x)$$

Proposition 10.1: Characterization of the closure

Let $A \subseteq X$, and let W be the set of adherent points of A, then $\overline{A} = W$

Proof. Suppose $x \notin W$, then there exists a neighbourhood U of x where

$$U \cap A = \varnothing \iff U \subseteq A^c$$

this is exactly the definition of the interior of A^c, so $x \in A^{co}$ and recall (from Proposition 8.1) that $(\overline{A})^c = A^{co}$, so $x \notin \overline{A}$. For the reverse inclusion, read the proof backwards, by flipping $\forall \to \exists$ within the set, and we see that

$$W^c = A^{co} = (\overline{A})^c$$

■

Dense and nowhere dense subsets

Definition 11.1: Dense subset

A subset of a topological space $E \subseteq \mathbf{X}$ is dense if $\overline{E} = \mathbf{X}$.

Definition 11.2: Nowhere dense subset

A subset of a topological space $E \subseteq \mathbf{X}$ is nowhere dense if $\overline{E}^o = \varnothing$.
This means E is dense in none of the (non-trivial) open subspaces of $\mathbf{X}$.

Proposition 11.1

E is dense in $\mathbf{X}$ iff for every non-empty, open set $U \subseteq \mathbf{X}$, $U \cap E \neq \varnothing$.

Proof of Proposition 11.1. Suppose E is dense, then $\overline{E} = \mathbf{X}$. Every point of $\mathbf{X}$ is an adherent point of E. Let $U \subseteq \mathbf{X}$ be a non-empty open set. If $x \in U$ then U is a neighbourhood of x, thus U intersects E. Conversely, suppose every non-empty open set U intersects E. Fix any point $x \in \mathbf{X}$, and any neighbourhood U of x. U has a non-empty interior (because it must contain x). But U^o is a non-empty open set, therefore $\varnothing \neq U^o \cap E \subseteq U \cap E$ ■

Proposition 11.2

Let $f : \mathbf{X} \to \mathbf{X}$ be a homeomorphism. E is nowhere dense iff $f(E)$ is nowhere dense.

Proof. Since f^{-1} is a homeomorphism, suppose $\overline{f^{-1}(E)}^o \neq \varnothing$, there exists a non-empty, open subset $U \subseteq \mathbf{X}$ with

$$\overline{f^{-1}(E)} \cap U = U$$

The direct image yields

$$f\left((\overline{f^{-1}(E)}) \cap U\right) = f(U)$$

since f is a bijection (injectivity is necessary here), it commutes with intersections.

$$f(\overline{f^{-1}(E)}) \cap f(U) = f\left((\overline{f^{-1}(E)}) \cap U\right) = f(U) \tag{11}$$

and f is continuous, so $f(\overline{A}) \subseteq \overline{f(A)}$ for any $A \subseteq \mathbf{X}$. For the reverse inclusion, f is a closed map, so $f(\overline{A})$ is a closed superset of $f(A)$ so

$$f(\overline{A}) = \overline{f(A)}$$

Take $A = f^{-1}(E)$, and $f(\overline{f^{-1}(E)}) = \overline{f(f^{-1}(E))} = \overline{E}$. From eq. (11), we see that

$$\overline{E} \cap f(U) = f(U)$$

$f(U)$ is a non-empty open subset of $\mathbf{X}$, since f is an open map, so E is not no-where dense. The reverse implication can be proven by replacing f with f^{-1}. ■

Urysohn's Lemma

Proposition 12.1: Folland Theorem 4.14

Suppose that A and B are disjoint closed subsets of the normal space X, and let $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$ be the set of dyadic rationals in $(0, 1)$. There is a family $\{U_r : r \in \Delta\}$ of open sets such that

1. $A \subseteq U_r \subseteq B^c$ for every $r \in \Delta$,

2. $\overline{U_r} \subseteq U_s$ for $r < s$, and

3. For every $r < s$, $\overline{U}_r \subseteq U_s$

Proof. The goal of this proof is to show that for every $r \in \Delta$, there exists a open U_r that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some $N \geq 1$ that all previous U_r in previous layers have been constructed properly, meaning if $r = k/2^n$, then for every $1 \leq n \leq N-1$, we have

$$r = \frac{k}{2^n},\ 1 \leq n \leq N-1,\ 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each U_r,

- $A \subseteq U_r \subseteq B^c$ and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer $N \geq 1$, we only have to construct the $U_{k/2^N}$ for every odd k, this is because if k is an even number, then $k = 2j$ and $r = 2j/2^N = j/2^{N-1}$ and for this particular U_r is already constructed. So for every odd $k = 2j+1$, the sets of the form $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every $k-1 \neq 0$ and $k+1 \neq 1$. (We will consider these cases later). We claim that for every pair of open sets, $E_1, E_2 \in \mathcal{T}_X$, then there exists some open set $G \in \mathcal{T}_X$ such that if $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$ where H is defined as the set

$$H = \left\{(E_1, E_2) \subseteq (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \varnothing\right\}$$

Then there exists some $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$ such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any $(E_1, E_2) \in H$, then this pair induces a pair of disjoint sets $\overline{E_1}$ and E_2^c since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \varnothing$$

And by normality, there exists disjoint open sets G_1, G_2 such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$

- $G_1 \cap G_2 = \varnothing \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since G_2^c is a closed set that contains G_1 as a subset, $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of G_2 (so we may discard the notion of G_2 in our minds). Let us now replace G with G_1 then it is an easy task to verify that $G = G_1 = \mathcal{J}(E_1, E_2)$ has the required properties.

Now define for every odd k, since $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (we note in passing that $\mathcal{J}$ is not a function as the set G may not be unique).

$$U_{k/2^N} = \mathcal{J}\left(U_{(k-1)/2^N}, U_{(k+1)/2^N}\right)$$

Then, if $U_{(k-1)/2^N}$ and $U_{(k+1)/2^N}$ is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$ sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$ and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed' $U_{k/2^N}$ for every $k-1 \neq 0$ and $k+1 \neq 1$. Now let us deal with the remaining pathological cases.

If $k-1$ so happens to be 0, then no $r \in \Delta$ satisfies $r = 0/2^N$, and we substitute

$$\overline{U}_0 = A, \qquad \text{or alternatively, } U_0 = A^o$$

Then $U_0 \in \mathcal{T}_X$, $\overline{U}_0 = A \subseteq B^c$. It is at this point that we must mention that $0, 1 \notin \Delta$, so U_0 and U_1 do not have to obey the rules we have laid out for $U_{r\in\Delta}$.

Now if $k+1$ is equal to 2^N (this makes $r = (k+1)/2^N = 1$) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every $0 \leq m \leq 2^N - 1, U_{m/2^N}$ must staisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$ (even for when $N = 1$, since $A = \overline{U}_0 \subseteq U_1 = B^c$) and a corresponding $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$ such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when $N = 1$. We would only have to construct for $k = 1$, since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■

Proposition 12.2: Folland Theorem 4.15: Urysohn's Lemma

Urysohn's Lemma. Let X be a normal space, if A and B are disjoint closed subsets of X, then there exists a $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Proof. Let $r \in \Delta$ be as in Lemma 4.14, and set U_r accordingly except for $U_1 = X$. Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write $W = \{k : x \in U_k\}$, Then for every $x \in A$ we have $f(x) = 0$, since by the construction of the 'onion' function in Lemma 4.14, for each $r \in \Delta \cap (0,1)$,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since $r > 0$ is arbitrary, and $0 \in W$, we can use a classic ε argument. If $f(x) > 0$ then there exists some $0 < r < f(x)$ by density of the dyadic rationals on the line, if $f(x) < 0$ then this implies that there exists some $f(x) < r < 0$ such that $x \in U_r$, but no $r \in \Delta$ can be negative, hence $f(x) = 0$.

Now, for every $x \in B$, since A and B are disjoint, and $A \subseteq U_r \subseteq B^c$, then for every $x \in B$ means that x is not a member of any U_r, but we set $U_1 = X$. Since none of the $r \in (0,1)$ is a member of the set we are taking the infimum, and $x \in U_1 = X$. The ε argument follows: suppose for every $\varepsilon > 0$, $(1 - \varepsilon) \notin W$, and $1 \in W$, then $f(x) = 1$.

Since $x \in U_1 = X$, for every $x \in X$, $f(x) \leq 1$, and $f(x)$ cannot be negative as $r > 0$ for every $r \in \Delta$. So $0 \leq f(x) \leq 1$. Now we have to show that this $f(x)$ is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in X. So $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$ and $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$.

Suppose that $f(x) < \alpha$, so $\inf W < \alpha$, and using the density of Δ, there exists an r, $f(x) < r < \alpha$ such that $x \in U_r$ such that $x \in \bigcup_{r<\alpha} U_r$. So $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r<\alpha} U_r$.

Fix an element $x \in \bigcup_{r<\alpha} U_r$, this induces an r such that $\inf W \leq r < \alpha$ therefore $f(x) < \alpha$, and $\bigcup_{r<\alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$.
For the second case, suppose that $f(x) > \alpha$, then $\inf W > \alpha$, and there exists an r (by density) such that $\inf W > r > \alpha$ such that for every $k \in W$, $k \neq r$. Therefore $x \notin U_r$, but by density again, and using the property of the onion function: for every $s < r$ we

get $\overline{U_s} \subseteq U_r$, taking complements (which reverses the estimate) — we have $x \notin \overline{U_s}$, but $\left(\overline{U_s}\right)^c$ is open in X. It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq \left(\overline{U_s}\right)^c \subseteq \bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$$

So $f^{-1}((\alpha, +\infty))$ is a subset of $\bigcup_{s>\alpha} \left(\overline{U_s}\right)^c$. To show the reverse, fix an element x in the union, then this induces some $x \in \left(\overline{U_s}\right)^c \subseteq (U_s)^c$. Then for this $s > \alpha$, $(-\infty, s)$ contains no elements of W. This is because for every $p < s$ implies that $(U_s)^c \subseteq (U_p)^c$, so $p \notin W$. Our chosen s is a lower bound for W, and $\alpha < s \leq \inf W = f(x)$.

Since all of the inverse images from the generating set of $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$ are open in X, using Theorem 4.9 finishes the proof. ■

Notes on the construction of the countable 'onion' sequence within a normal space $\mathbf{X}$.

If $\mathbf{X}$ is a normal space, and A and B are disjoint closed subsets, then we can easily find an open U with

$$A \subseteq U \subseteq \overline{U} \subseteq B^c \tag{12}$$

We say that U hides in B^c if the closure of U is contained in B^c. Define $\Delta_n = \left\{k2^{-n},\ 1 < k < 2^n\right\}$, so that $\Delta_n \subseteq (0, 1)$ for all $n \geq 1$. Notice

$$\Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

and the even indices for Δ_{n+1} are contained in Δ_n. Suppose Δ_n is well defined, it suffices to choose the odd indices for Δ_{n+1}. If $r = j2^{-(n+1)}$, where j is odd, then r sits in between precisely two elements in $\Delta_n \cup \{0, 1\}$. If r sits between an endpoint, then define $\overline{U_0} = A$, and $B^c = U_1$. And denote the closest left and neighbours by s, t respectively. If $s < r < t$, it is clear that $\overline{U_s}$ and U_t^c are disjoint closed sets.

Use the 'normal space' construction to obtain an superset of $\overline{U_s}$ that hides in U_t, denote this open set by U_r, and similar to Equation (12)

$$\overline{U_s} \subseteq U_r \subseteq \overline{U_r} \subseteq U_t$$

Now that the construction of this sequence is complete, we wish to prove Urysohn's Lemma. Let A and B be disjoint closed sets. And define

$$f(x) = \inf\left\{r \in \Delta \cup \{1\},\ x \in U_r\right\}$$

where $U_1 = \mathbf{X}$. So that $0 \leq f(x) \leq 1$ is immediate. If $x \in A$, then x is in all U_r, and by density of $\Delta \subseteq (0, 1)$, we have $f(x) = 0$. Conversely, if $x \in B$ then $x \notin U_r$ for all $r \in \Delta$, if E denotes the indices in Δ where $x \in U_s$ when $s \in E$,

$$(-\infty,\ r) \subseteq E^c \iff E \subseteq [r,\ +\infty) \iff \inf(E) \geq r \tag{13}$$

Send $r \to 1$ and $f(x) = 1$. Thus $f|_A = 0$ and $f|_B = 1$.

To show continuity, it suffices to show that the inverse images of the open half $\left\{(x > \alpha),\ (x < \alpha)\right\}_{\alpha \in \mathbb{R}}$ lines are indeed open in $\mathbf{X}$. Let α be fixed. And if $x \in \{f < \alpha\}$, we can 'wiggle' the infimum towards the right (towards α), and using density of Δ within $(0, 1)$, there exists a $r \in E$ that satisfies $f(x) < r < \alpha$. This is equivalent to

$$x \in \bigcup_{r < \alpha} U_r$$

If there exists an $r < \alpha$ st x belongs to U_r as an element, then $f(x) \leq r < \alpha$.

If $f(x) > \alpha$, then $(-\infty,\ \alpha) \subseteq E^c$, by Equation (13). Suppose $\alpha < 1$, otherwise $\{f > \alpha\} = \varnothing$. Wiggle $f(x)$ to the left and obtain an $r \in \Delta$, $\alpha < r < f(x)$ with $x \notin U_r$. By density again, take any $s < r$ by a small amount (st $s > \alpha$, $s \in \Delta$), and

$$\overline{U}_s \subseteq U_r \iff U_r^c \subseteq \overline{U}_s$$

so that $x \in \overline{U}_s^c$ for some $s > \alpha$. This is equivalent to

$$x \in \bigcup_{s > \alpha} \overline{U}_s^c$$

Conversely, if $x \notin \overline{U}_s^c$ for some $s > \alpha$, since $\{U_r\}$ (thus $\{\overline{U}_r\}$) is increasing, and $x \notin U_r$ for every $r \leq s$. Hence,

$$(-\infty,\ s] \subseteq E^c \iff E \subseteq (s,\ +\infty) \iff f(x) \geq s > \alpha$$

Compactness

Compactness is one of the most important concepts in topology and analysis.

Definition 13.1: Compact topological space

A topological space $\mathbf{X}$ is compact if every open covering $\{U_\alpha\}$ contains a finite subcover. That is, if $\{U_\alpha\}$ is an arbitrary collection of open sets, then

$$\mathbf{X} = \bigcup U_{\alpha \in A} \implies \bigcup_{j \leq n} U_{\alpha_j}$$

Definition 13.2: Compact set

$E \subseteq \mathbf{X}$ is compact if it is compact in the subspace topology.

Definition 13.3: Precompact set

$E \subseteq \mathbf{X}$ is precompact if its closure is compact (as a subset).

Definition 13.4: Paracompact space

A topological space $\mathbf{X}$ is paracompact if every open covering of $\mathbf{X}$ has a locally finite open refinement that covers $\mathbf{X}$.

Definition 13.5: Locally finite collection of sets

Let $\mathcal{A}$ be a collection of subsets of $\mathbf{X}$. It is called locally finite, if at every point $p \in \mathbf{X}$, we can find a neighbourhood U of p (not necessarily open), that intersects only finitely many members of $\mathcal{A}$. In symbols,

$$U \cap E = \varnothing \quad \text{for all but finitely many} \quad E \in \mathcal{A}$$

We do not require $\mathcal{A}$ to be a cover of $\mathbf{X}$, nor do we require $\mathcal{A}$ to be a collection of open sets.

Definition 13.6: Countably locally finite

A collection $\mathbb{B}$ is countably locally finite if it is the countable union of locally finite families.

$$\mathbb{B} = \overbrace{\bigcup_{\mathbb{N}}}^{\substack{\text{countable}\\ \text{union}}} \mathbb{B}_n, \quad \text{where each } \mathbb{B}_n \text{ is a locally finite collection}$$

Definition 13.7: Refinement

If $\mathcal{A}$ is a collection of sets, $\mathbb{B}$ is a refinement of $\mathcal{A}$ if every element $B \in \mathbb{B}$, induces an element $A \in \mathcal{A}$, such that $B \subseteq A$.

Remark 13.1: Intuition for refinements

If $\mathbb{B}$ is a refinement of $\mathcal{A}$, we can use the 'absolute continuity' muscle. For each element in $\mathbb{B}$ is dominated by some element (through subset inclusion) in $\mathcal{A}$. Recall, if ν and μ are non-negative measures, then $\nu \ll \mu$ if for every measurable set $E \in \mathcal{M}$, $\mu(E) = 0 \implies \nu(E) = 0$.

A refinement of a family of sets is another family of sets, whose elements are dominated by some other element in the un-refined family. *Refining families makes them 'smaller', cover less area.*

Proposition 13.1

Compact Hausdorff spaces are normal, compact subsets of Hausdorff spaces are closed, and closed subsets of compact sets are again compact.

Properties of Compact Spaces

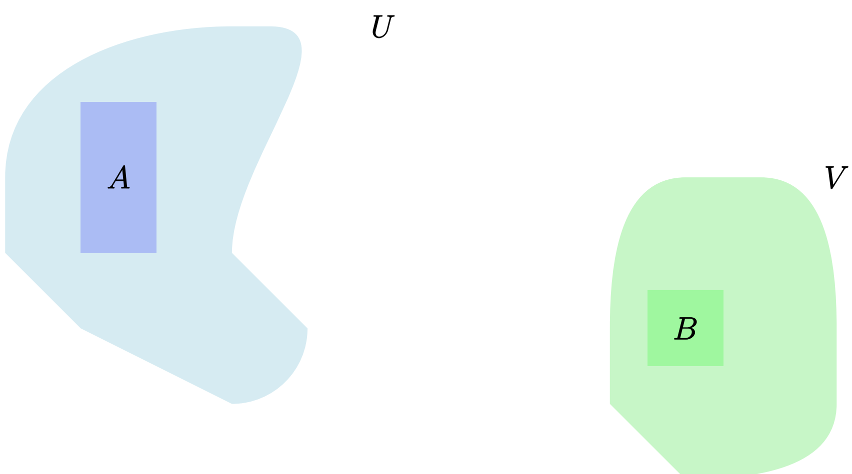

Figure 1: Closed sets A and B within open sets U and V, respectively.

Proposition 14.1

Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces.

(a) If $F \in C(\mathbf{X}, \mathbf{Y})$, and $\mathbf{X}$ is compact, then $F(\mathbf{X})$ is compact.

(b) If $\mathbf{X}$ is compact and $F \in C(\mathbf{X}, \mathbb{R})$, then $F(\mathbf{X})$ is bounded, and F attains its supremum and infimum on $\mathbf{X}$.

(c) A finite union of compact subspaces of $\mathbf{X}$ is again compact.

(d) If $\mathbf{X}$ is Hausdorff, and A, B are disjoint, compact subspaces of $\mathbf{X}$, there exists open U and V, (see fig. 1).

(e) Every closed subset of a compact space is compact.

(f) Every compact subset of a Hausdorff space is closed.

(g) Every compact subset of a metric space is bounded.

(h) Every finite product of compact spaces is compact.

(i) Every quotient of a compact space is compact.

Proof of Proposition 14.1 Part A. Let $f \in C(\mathbf{X}, \mathbf{Y})$ with $\mathbf{X}$ compact. Fix an open cover of $f(\mathbf{X})$ in the relative topology,

$$\{U_\alpha \cap f(\mathbf{X})\}_{\alpha \in A} \text{ covers } \mathbf{X},\ U_\alpha \text{ open in } \mathbf{Y}$$

So that $\bigcup f^{-1}(U_\alpha) = \bigcup f^{-1}(U_\alpha \cap f(\mathbf{X})) = \mathbf{X}$. Since $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$ is an open cover for $\mathbf{X}$, this induces a finite subcollection of indices $\{\alpha_1, \ldots, \alpha_n\}$ with

$$\bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^{n} f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^{n} f^{-1}(U_\alpha \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^{n} f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^{n} U_{\alpha_j}$$

■

Proof of Proposition 14.1 Part B. Let $\mathbf{X}$ be compact, and $f \in C(\mathbf{X}, \mathbb{R})$, so that $f(\mathbf{X}) \subseteq \mathbb{R}$ is compact. Compact subsets are closed and bounded in $\mathbb{R}$, let $A = \sup f(\mathbf{X})$ and $B = \inf f(\mathbf{X})$. Both A and B are accumulation points of $f(\mathbf{X})$, so $A = f(x)$ and $B = f(y)$ for some x, y in $\mathbf{X}$. ■

Proof of Proposition 14.1 Part C. Let $\mathbf{X}$ be a topological space, and $K_1, \ldots K_n$ be compact subspaces. Denote $K = \bigcup_{j=1}^{n} K_j$. Let $\{U_\alpha \cap K\}_{\alpha \in A}$ be an open cover for K, where U_α is open in $\mathbf{X}$. We can pass the argument to each individual K_j as follows. Let $1 \leq j \leq n$, then $\{U_\alpha \cap K_j\}_{\alpha \in A}$ is an oepn cover for K_j, so there exists as finite subcollection of indices $I_j \subseteq A$, (a finite subset of A) whose open sets cover K_j. Repeat this process for each j and

$$I = \bigcup_{j=1}^{n} I_j \text{ is a finite subset of } A$$

with $K_j \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K)$. Taking the union over all K_j reads

$$K = \bigcup_{j=1}^{n} K_j \subseteq \bigcup_{j=1}^{n} \bigcup_{\alpha \in I_j} (U_\alpha \cap K) = \bigcup_{\alpha \in I} U_\alpha \cap I$$

■

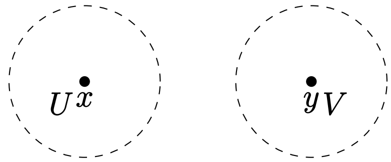

Figure 2: In a Hausdorff space, any two distinct points x and y can be separated by disjoint open neighbourhoods U and V.

Proof of Proposition 14.1 Part D. Let $\mathbf{X}$ be Hausdorff. We first prove that compact subspaces of $\mathbf{X}$ are closed. Indeed, if K is compact in $\mathbf{X}$, fix any $x \in K^c$. Let y range through the elements of K, then $x \neq y$ induces a pair of disjoint open sets U_y and V_y, such that

- $x \in U_y$
- $y \in V_y$
- $U_y \cap V_y = \varnothing$
- See fig. 2

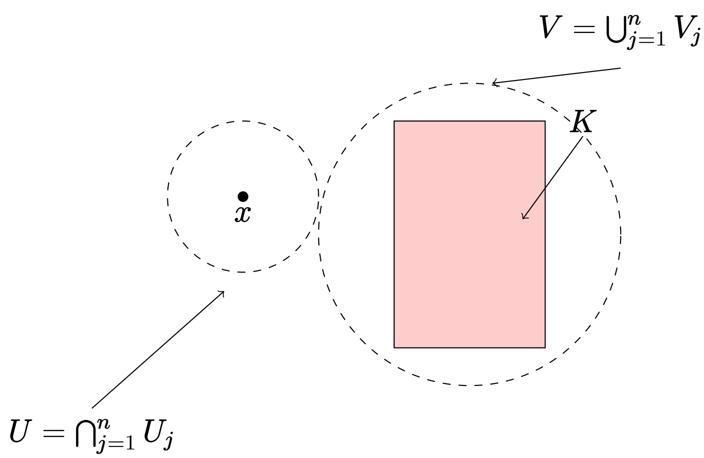

Figure 3: Compact sets are closed in Hausdorff spaces

Let V_y range through all possible $y \in K$, So that $\{V_y\}_{y \in K}$ is an open cover. There exists a finite subcollection of 'anchor points' of K, $y_1, \ldots, y_n$ that corresponds with $\{V_{y_j}\}_{j=1}^n$. A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^n U_{y_j} \text{ is open}$$

Define $V = \bigcup_{j=1}^n V_{y_j}$, so $V \subseteq K$ and $U \cap V = \varnothing$ and $x \in U \subseteq K^c$ (see fig. 3). Therefore K is closed.

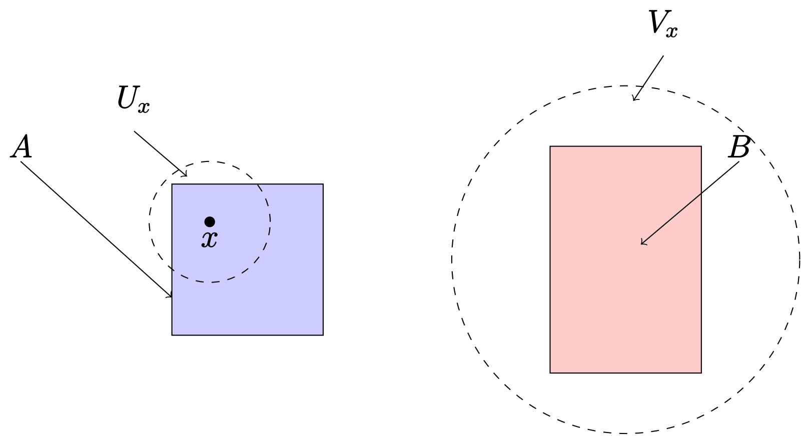

Figure 4: Closed sets A and B, point x in A, and disjoint neighbourhoods U around x and V around B.

Finally, if A and B are disjoint compact sets, then each $x \in A \subseteq B^c$ induces neighbourhoods $x \in U_x$, and $B \subseteq V_x$ (see fig. 4), let x range through all the elements of A. By compactness of A, this produces a finite subcover, and

$$U = \bigcup_{j=1}^{n} U_{x_j} \quad V = \bigcap_{j=1}^{n} V_{x_j}$$

are disjoint open sets that contain A and B respectively.

∎

Proof of Proposition 14.1 Part E. Let $K \subseteq \mathbf{X}$ be a closed set of a compact space. Let $\{U_\alpha \cap K\}$ be an open cover for K, where each U_α is open in $\mathbf{X}$. We can append an extra set K^c which is open in $\mathbf{X}$. The collection

$$W = \{U_\alpha\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of $W_1, \ldots, W_n$ that cover $\mathbf{X}$ (since $\mathbf{X}$ is compact by itself). Remove K^c from this finite subcollection if it exists, and take the intersection with K for each element W_j, and

$$\{W_1 \cap K, \ldots, W_n \cap K\} = \{U_1 \cap K, \ldots, U_n \cap K\} \text{ covers } K$$

so K is compact. ∎

Proof of Proposition 14.1 Part F. Proven in Part D. ∎

Proof of Proposition 14.1 Part G. let $K \subseteq \mathbf{X}$ be a compact subset of the metric space $(\mathbf{X}, d)$. Compact subsets of $\mathbf{X}$ are totally bounded, and hence bounded. ∎

Proof of Proposition 14.1 Part H. See Tynchonoff's Theorem in Folland Chapter 4. ∎

Proof of Proposition 14.1 Part I. Let $\mathbf{X}$ and $\mathbf{Y}$ be topological spaces and $\pi : \mathbf{X} \to \mathbf{Y}$ be a quotient map. So that $\mathbf{Y}$ is endowed with the quotient topology. So that π is a surjective continuous map. and $\pi(\mathbf{X}) = \mathbf{Y}$. Apply Part A, and we see that $\mathbf{Y}$ is compact. ■

Locally Compact Hausdorff Spaces

Compactness is an intrinsic topological property (in the subspace topology). We see from Proposition 4.25 that compact Hausdorff spaces are normal, which gives a sufficient condition for us to approximate and extend any continuous function; and allows us to extend certain 'local' properties to 'global' properties.

If given a Hausdorff space, not necessarily compact, the natural question is to ask 1) whether a topological space has 'enough' compact subsets to work with, and 2) whether we can embed a given topological space in a larger one to force it to be compact.

Definition 15.1: LCH space

Let $\mathbf{X}$ be a Hausdorff space. We call $\mathbf{X}$ a LCH space if every point $p \in \mathbf{X}$ admits a compact neighbourhood. That is, a compact set K whose interior contains p.

We note in passing that the above definition differs slightly from the usual 'local' definitions.

Definition 15.2: Locally connected

Let $\mathbf{X}$ be a topological space, it is locally connected if for every $x \in \mathbf{X}$, and open neighbourhood U containing x, there exists a connected, open neighbourhood V of x such that $x \in V \subseteq U$.

Definition 15.3: Locally path-connected

Let $\mathbf{X}$ be a topological space, it is locally connected if for every $x \in \mathbf{X}$, and open neighbourhood U containing x, there exists a path-connected, open neighbourhood V of x such that $x \in V \subseteq U$.

Definition 15.4: Local homeomorphism

$\mathbf{X}$ locally homeomorphic to $\mathbb{R}^n$ if every point $x \in \mathbf{X}$ belongs to a coordinate chart (U, ϕ), where U is an open neighbourhood of x and ϕ is a homeomorphism from $U \to \phi(U) \subseteq \mathbb{R}^n$.

Definition 15.5: Local diffeomorphism

Let M be a smooth manifold and $F \in C^\infty(M, N)$. F is a local diffeomorphism if every $p \in M$ in its domain induces a neighbourhood $U \subseteq M$ with $F|U : U \to F(U)$ is a diffeomorphism (in the sense of two open sub-manifolds).