Chapter 3

Notes on Chapter 3

Proposition 0.1

Prove two things,

1. $\limsup_{r \to R} \phi(r) = \lim_{\varepsilon \to 0} \sup_{0 < |r-R| < \varepsilon} \phi(r) = \inf_{\varepsilon > 0} \sup_{0 < |r-R| < \varepsilon} \phi(r)$,
2. $\lim_{r \to R} \phi(r) = c \iff \limsup_{r \to R} |\phi(r) - c| = 0$

Proof. ∎

Proposition 0.2

If $U \subseteq B(1,0) = \{|x| < 1\}$, and $U \in \mathbb{B}$, and if $m(U) > 0$, then the family of sets

$$E_r = \left\{ x + ry,\ y \in U \right\}$$

shrinks nicely to $x \in \mathbb{R}^n$.

Proof. Let $r > 0$ be fixed then $\forall z \in E_r \looparrowright z = x + ry$. Hence,

$$\begin{aligned} d(x,z) &= d(x, x+ry) \\ &= |r| d(0,y) < |r| \end{aligned}$$

by translation invariance. ■

Theorem 3.1

Proposition 1.1

Proof. Let ν be a signed measure, and fix any increasing sequence $E_j \nearrow E = \bigcup E_{j\geq 1}$ of sets. This induces a disjoint sequence in $\{F_n\}$. Define $F_1 = E_1$, and if $n \geq 2$,

$$F_n = E_n \setminus \bigcup E_{j\leq n-1}$$

and from this, the finite It is clear that $\bigcup F_{n\geq 1} = E$, and let us assume $\nu(E)$ is of finite measure.
By countable additivity, and the absolute convergence of the series $\sum_{j\leq n} \nu(F_j)$

$$\begin{aligned}\nu\left(\bigcup E_{j\geq 1}\right) &= \sum_{j\geq 1} \nu(F_j) \\ &= \lim_n \sum_{j\leq n} \nu(F_j) \\ &= \lim \nu(E_n)\end{aligned}$$

■

Theorem 3.2

Proposition 2.1

Proof. ∎

Theorem 3.3

Proposition 3.1

Proof. ■

Theorem 3.4

Proposition 4.1

Proof. ■

Theorem 3.5

Proposition 5.1

Proof. ∎

Theorem 3.6

Proposition 6.1

Proof. ■

Theorem 3.7

Proposition 7.1

Proof. ■

Theorem 3.8

Proposition 8.1

Proof. ■

Theorem 3.9

Proposition 9.1

Proof. ■

Theorem 3.10

Proposition 10.1

Proof. ∎

Theorem 3.11

Proposition 11.1

Proof. ■

Theorem 3.12

Proposition 12.1

Proof. ■

Theorem 3.13

Proposition 13.1

Proof. ∎

Theorem 3.14

Proposition 14.1

Proof. ∎

Theorem 3.15

Proposition 15.1

Proof. ■

Theorem 3.16

Proposition 16.1

Proof. ∎

Theorem 3.17

Proposition 17.1

Let the maximal function of any measurable $f \in \mathbb{B}_{\mathbb{R}^n}$ be denoted by $Hf(x)$, more precisely,

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

where $A_r|f|$ is the average of $|f|$ on a ball with radius $r > 0$ centered at $x \in \mathbb{R}^n$. In symbols,

$$A_r|f| = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

The maximal theorem makes two claims:

1. $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, and Hf is measurable for every $f \in L^1_{loc}$.
2. There exists a $C > 0$, for every $f \in L^1$

$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1$$

for every $\alpha > 0$.

Proof. Let $\alpha > 0$ and fix $z \in (Hf)^{-1}((\alpha, +\infty))$, so $Hf(z) > \alpha$ and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with $Hf(z) - \alpha > 0$, we get some $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some $r_0 > 0$. Since $A_r|f|$ (a function indexed by $r > 0$) is continuous in $x \in \mathbb{R}^n$, $(A_r|f|)^{-1}((\alpha, +\infty))$ is open, and Hf is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{ Hf > \alpha \right\} = \bigcup_{r>0} \{A_r|f| > \alpha\}$$

Let $x \in E_\alpha$, this induces a $r_x > 0$ where $x \in \left\{ A_{r_x}|f| > \alpha \right\}$. Rearranging gives

$$\left(\frac{1}{\alpha} \int\limits_{B(r,x)} |f| dz \right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each $x \in E_\alpha \looparrowright r_x > 0 \looparrowright A_{r_x}|f|$,
- If $U = \bigcup\limits_{x \in E_\alpha} B(r_x, x)$, then $E_\alpha \subseteq U$,
- Choose $c < m(E_\alpha) \leq m(U)$ (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

  $$x_1, \ldots, x_N \in E_\alpha$$

  so that $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$, and $c < 3^n \sum_{j \leq k} m(B_j)$
- Define $B_j = B(r_{x_j}, x_j)$ for all $j \leq k$, and

  $$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

  by finite additivity,

  $$c3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

  and finally

  $$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of m on $\mathbb{B}$, since

  $$m(E_\alpha) = \sup \left\{ m(K),\ K \in \text{♪}_{\mathbb{R}^n},\ K \subseteq E_\alpha \right\}$$

  for any $K \in \text{♪}_{\mathbb{R}^n}$, $K \subseteq E_\alpha$, we have $m(K) < +\infty$, $m(K) \leq m(E_\alpha)$ and

  $$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

Remark 17.1

We used the properties of a Radon Measure here, without relying on the phrase 'sending $c \to E_\alpha$', which would require us to deal with two cases $m(E_\alpha) < +\infty$ and $m(E_\alpha) = +\infty$.

■

Theorem 3.18

Proposition 18.1

Proof. ■

Theorem 3.19

Proposition 19.1

Proof. ∎

Theorem 3.20

Proposition 20.1

Proof. ■

Theorem 3.21

Proposition 21.1

The Lebesgue Differentiation Theorem. Suppose $f \in L^1_{loc}$, and for every $x \in \mathcal{L}_f$, (so that $x \in \mathbb{R}^n$ a.e). We have

1. $\lim_{r\to 0} \dfrac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0$,
2. $\lim_{r\to 0} \dfrac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x)$,

For every family $\{E_r\}_{r>0}$ that shrinks nicely to $x \in \mathbb{R}^{n\prime}$.

Proof. Since the family $\{E_r\}_{r>0}$ shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r,x)) \implies m(E_r) > \alpha \cdot m(B(r,x))$$

for some $\alpha > 0$, independent on r. Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r,x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r,x)} |f(y) - f(x)| dy$$

Combining the last two results, for every $\varepsilon > 0$, if $0 < r < \varepsilon$, then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1} B(r,x) \int_{B(r,x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0<r<\varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0<r<\varepsilon} m^{-1} B(r,x) \int_{B(r,x)} |f(y) - f(x)| dy$$

and sending $\varepsilon \to 0$, proves the first claim. The second claim is immediate upon applying the L^1 inequality.

Fix any $\varepsilon > 0$, and

$$\begin{aligned}
\lim_{r\to 0} m^{-1}(E_r) \int_{E_r} f(y) dy = f(x) &\iff \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y) dy - f(x) \right| \\
&\iff \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)] dy \right| \\
&\leq \lim_{\varepsilon\to 0} \sup_{0<r<\varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
&= \lim_{r\to 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
&= 0
\end{aligned}$$

■

Theorem 3.22

Proposition 22.1

Proof. ■

Theorem 3.23

Proposition 23.1

Proof. ∎

Theorem 3.24

Proposition 24.1

Proof. ■

Theorem 3.25

Proposition 25.1

Proof. ■

Theorem 3.26

Proposition 26.1

Proof. ∎

Theorem 3.27

Proposition 27.1

Proof. ∎

Theorem 3.28

Proposition 28.1

Proof. ■

Theorem 3.29

Proposition 29.1

Proof. ■

Theorem 3.30

Proposition 30.1

Proof. ■

Theorem 3.31

Proposition 31.1

Proof. ■

Theorem 3.32

Proposition 32.1

Proof. ∎

Theorem 3.33

Proposition 33.1

Proof. ■

Theorem 3.34

Proposition 34.1

Proof. ■

Theorem 3.35

Proposition 35.1

Proof. ■

Theorem 3.36

Proposition 36.1

Proof. ∎