Chapter 1: Topological Manifolds

The n-sphere as a topological manifold. Define

$$S^n = \left\{ x \in \mathbb{R}^{n+1},\ |x| = 1 \right\}$$

We claim that $\{U_i^{\pm}\}_{i=1}^{n+1}$ form an open cover, where

$$U_i^+ = \left\{ x \in S^n, x^i > 0 \right\} \quad U_i^- = \left\{ x \in S^n, x^i < 0 \right\}$$

Each $U_i^{\pm}$ is the inverse image of $\pi_i^{-1}((0, +\infty)) \cap S^n$ or $\pi_i^{-1}((0, -\infty)) \cap S^n$, hence open. For every $x \in S^n$, there exists at least some $1 \leq j \leq n+1$ that makes the j-th coordinate of x, $x^j \neq 0$. So

$$S^n = \bigcup_i U_i^{\pm}$$

Denote the unit ball $\left\{ x \in \mathbb{R}^n,\ |x| < 1 \right\}$ in $\mathbb{R}^n$ by $\mathbb{B}^n$.

Chapter 3: Tangent Spaces

We will go through the section on the Change of Coordinates, and how different coordinate charts change the representation of a derivation at $p \in M$, where M is some smooth manifold.

Proposition 0.1

Let M be a smooth manifold, and fix $p \in M$. If $\nu \in T_pM$ is given with respect to the bases

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^m}\bigg|_p \right\} \quad \text{and} \quad \left\{ \frac{\partial}{\partial y^1}\bigg|_p, \dots, \frac{\partial}{\partial y^m}\bigg|_p \right\}$$

Defined by

$$\frac{\partial}{\partial x^j}\bigg|_p \triangleq d\left(\phi^{-1}\Big|_{\phi(p)}\right)\left(\frac{\partial}{\partial x^j}\bigg|_{\phi(p)}\right) \quad \text{and} \quad \frac{\partial}{\partial y^j}\bigg|_p \triangleq d\left(\psi^{-1}\Big|_{\psi(p)}\right)\left(\frac{\partial}{\partial y^j}\bigg|_{\psi(p)}\right)$$

and we write ν in terms of the first basis

$$\nu = \nu^j \frac{\partial}{\partial x^j}\bigg|_p = \sum_{j=1}^m \nu^j \frac{\partial}{\partial x^j}\bigg|_p$$

and the second basis

$$\nu = \nu^j \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p = \sum_{k=1}^m \sum_{j=1}^m \nu^j \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p$$

If $f \in C^\infty(M)$, then

$$\nu(f) = \nu^j \frac{\partial}{\partial x^j}\bigg|_p f = \nu^j \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p f$$

Proof. Recall $\frac{\partial}{\partial x^j}\big|_p f \triangleq \frac{\partial}{\partial x^j}\big|_{\phi(p)} f \circ \phi^{-1}$, similarly for $\frac{\partial}{\partial y^j}\big|_p f$. Deriving f and p and by vector space operations on T_pM, the first basis expansion gives

$$\nu^j \frac{\partial}{\partial x^j}\bigg|_p f = \nu^j \frac{\partial}{\partial x^j}\bigg|_{\phi(p)} f \circ \phi^{-1} \tag{1}$$

and the second expression reads

$$\nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_p f = \nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_{\psi(p)} f \circ \psi^{-1} \tag{2}$$

Since $f \circ \phi^{-1} \in C^\infty(\mathbb{R}^m, \mathbb{R})$, we see the expressions are indeed equal. By the chain rule, if

$$\psi \circ \phi^{-1}(x^1, \dots x^m) = (y^1, \dots y^m)$$

then

$$D(\psi \circ \phi^{-1})(\phi(p)) = \begin{bmatrix} \left.\frac{\partial y^1}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^1}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^1}{\partial x^m}\right|_{\phi(p)} \\ \left.\frac{\partial y^2}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^2}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^2}{\partial x^m}\right|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial y^m}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^m}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^m}{\partial x^m}\right|_{\phi(p)} \end{bmatrix}$$

It follows from Proposition 3.6d) that the matrix $D(\psi \circ \phi^{-1})|_{\phi(p)}$ is invertible, as $\psi \circ \phi^{-1}$ is a diffeomorphism. ■

An important application of this is the following. We begin with the $\mathbb{R}^m \to \mathbb{R}^n$ case. We will see that if p and $F(p)$ are represented by another pair of coordinate charts (smoothly compatible with the previous pair), then the rank of dF_p does not change. So the rank of the differential is an invariant of the choice of coordinate chart.

Definition 0.1

Let $F \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$, and $p \in \mathbb{R}^m$ induces two charts $p \in (U, \mathrm{id}_{\mathbb{R}^m})$ and $F(p) \in (V\ \mathrm{id}_{\mathbb{R}^n})$., where $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. The matrix representation of the differential at p, $dF_p : T_p\mathbb{R}^m \to T_{F(p)}\mathbb{R}^n$ is nothing but the Jacobian matrix of F at p.

$$\mathcal{M}\{dF_p\} = DF(p) = \begin{bmatrix} \left.\frac{\partial F^1}{\partial x^1}\right|_p & \left.\frac{\partial F^1}{\partial x^2}\right|_p & \cdots & \cdots & \left.\frac{\partial F^1}{\partial x^m}\right|_p \\ \left.\frac{\partial F^2}{\partial x^1}\right|_p & \left.\frac{\partial F^2}{\partial x^2}\right|_p & \cdots & \cdots & \left.\frac{\partial F^2}{\partial x^m}\right|_p \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial F^n}{\partial x^1}\right|_p & \left.\frac{\partial F^n}{\partial x^2}\right|_p & \cdots & \cdots & \left.\frac{\partial F^n}{\partial x^m}\right|_p \end{bmatrix} \tag{3}$$

Definition 0.2

Let $F \in C^\infty(M, N)$, and $p \in M$ induces two charts $p \in (U, \phi)$ and $F(p) \in (V\,\psi)$. The matrix representation of the differential at p, $dF_p : T_pN \to T_{F(p)}N$ is nothing but the Jacobian matrix of the coordinate representation at p.

$$\mathcal{M}\{dF_p\} = \begin{bmatrix} \left.\frac{\partial \hat{F}^1}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial \hat{F}^1}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial \hat{F}^1}{\partial x^m}\right|_{\phi(p)} \\ \left.\frac{\partial \hat{F}^2}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial \hat{F}^2}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial \hat{F}^2}{\partial x^m}\right|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial \hat{F}^n}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial \hat{F}^n}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial \hat{F}^n}{\partial x^m}\right|_{\phi(p)} \end{bmatrix} \tag{4}$$

Alternately, if we write $\hat{p} = \phi(p)$ as the $\mathbb{R}^m$ coordinates at p, then

$$\mathcal{M}\{dF_p\} = \begin{bmatrix} \left.\frac{\partial \hat{F}^1}{\partial x^1}\right|_{\hat{p}} & \left.\frac{\partial \hat{F}^1}{\partial x^2}\right|_{\hat{p}} & \cdots & \cdots & \left.\frac{\partial \hat{F}^1}{\partial x^m}\right|_{\hat{p}} \\ \left.\frac{\partial \hat{F}^2}{\partial x^1}\right|_{\hat{p}} & \left.\frac{\partial \hat{F}^2}{\partial x^2}\right|_{\hat{p}} & \cdots & \cdots & \left.\frac{\partial \hat{F}^2}{\partial x^m}\right|_{\hat{p}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial \hat{F}^n}{\partial x^1}\right|_{\hat{p}} & \left.\frac{\partial \hat{F}^n}{\partial x^2}\right|_{\hat{p}} & \cdots & \cdots & \left.\frac{\partial \hat{F}^n}{\partial x^m}\right|_{\hat{p}} \end{bmatrix} \tag{5}$$

Proposition 0.2

Let F be a smooth map between M and N, at every $p \in M$, rank dF_p is an invariant over (smoothly compatible) pairs of charts in M and N.

Proof. Let $p \in (U_1, \phi_1) \cap (U_2, \phi_2)$, and $F(p) \in (V_1, \psi_1) \cap (V_2, \psi_2)$. Where all charts are smoothly compatible if it makes sense to talk about it. Both $\phi_2 \circ \phi_1^{-1}$ and $\psi_2 \circ \psi_1^{-1}$ are diffeomorphisms, and the change of basis matrices $D(\phi_2 \circ \phi_1^{-1})\big|_{\phi_1(p)}$ and $D(\psi_2 \circ \psi_1^{-1})\big|_{\psi_1(F(p))}$ are invertible by Proposition 3.6d) again, so the ranks dF_p with respect to any of the two charts are equal.

$$\underbrace{D(\psi_2 \circ \psi_1^{-1})\Big|_{\psi_1(F(p))}}_{\text{invertible}} \left(\mathcal{M}\{dF_p\}\right) \underbrace{D(\phi_2 \circ \phi_1^{-1})\Big|_{\phi_1(p)}}_{\text{invertible}}$$

■