MATH 263: Section 003, Tutorial 13

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1 (Review) The Laplace Transform

The **Laplace Transform** is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st}\ f(t)\ dt$$

where the integral is defined.

Note that the **Laplace Transform** is an **integral Transform**, meaning that it is linear (due to the properties of the integral). Therefore, for all $a, b \in \mathbb{R}$:

$$\mathcal{L}\{a\ f(t) + b\ g(t)\} = a\ \mathcal{L}\{f(t)\} + b\ \mathcal{L}\{g(t)\}$$

Example: Find $\mathcal{L}\{\cos t\}$ in two ways:
Using Taylor Series:

$$\mathcal{L}\{\cos t\} = \mathcal{L}\{\sum_{n=0}^{\infty}(-1)^n \frac{t^{2n}}{(2n)!}\}$$

From linearity:

$$\mathcal{L}\{\cos t\} = \sum_{n=0}^{\infty} \mathcal{L}\{(-1)^n \frac{t^{2n}}{(2n)!}\}$$

$$\mathcal{L}\{\cos t\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \mathcal{L}\{t^{2n}\}$$

$$\mathcal{L}\{\cos t\} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \frac{(2n)!}{s^{2n+1}}$$

$$\mathcal{L}\{\cos t\} = \frac{1}{s} \sum_{n=0}^{\infty} (\frac{-1}{s^2})^n$$

This is a geometric series, so

$$\mathcal{L}\{\cos t\} = \frac{s}{s^2} \frac{1}{1 - \frac{-1}{s^2}}$$

$$\mathcal{L}\{\cos t\} = \frac{s}{1+s^2}$$

Another way is by using the definition and the periodicity of $\cos t$: $\cos t = \cos t + 2\pi$:

$$\mathcal{L}\{\cos t\} = \int_0^\infty e^{-st} \cos t\ dt$$

$$\mathcal{L}\{\cos t\} = \int_0^{2\pi} e^{-st} \cos t\ dt + \int_{2\pi}^{4\pi} e^{-st} \cos t\ dt + \int_{4\pi}^{6\pi} e^{-st} \cos t\ dt + \ldots$$

$$\mathcal{L}\{\cos t\} = \int_0^{2\pi} e^{-st}\cos t\ dt + \int_0^{2\pi} e^{-s(t+2\pi)}\cos{(t+2\pi)}\ dt + \int_0^{2\pi} e^{-s(t+4\pi)}\cos{(t+4\pi)}\ dt + \dots$$

$$\mathcal{L}\{\cos t\} = \int_0^{2\pi} e^{-st}e^{0\pi s}\cos t\ dt + \int_0^{2\pi} e^{-st}e^{-2\pi s}\cos t\ dt + \int_0^{2\pi} e^{-st}e^{-4\pi s}\cos t\ dt + \dots$$

$$\mathcal{L}\{\cos t\} = \int_0^{2\pi} e^{-st}\cos t\ dt + \int_0^{2\pi} e^{-st}e^{-2\pi s}\cos t\ dt + \int_0^{2\pi} e^{-st}e^{-4\pi s}\cos t\ dt + \dots$$

$$\mathcal{L}\{\cos t\} = (\int_0^{2\pi} e^{-st}\cos t\ dt)\sum_{n=0}^{\infty} e^{-2\pi ns} = (\int_0^{2\pi} e^{-st}\cos t\ dt)\sum_{n=0}^{\infty}(e^{-2\pi s})^n$$

By solving the definite integral and the geometric series, we get:

$$\mathcal{L}\{\cos t\} = \frac{s(1-e^{-2\pi s})}{1+s^2}\frac{1}{1-(e^{-2\pi s})} = \frac{s}{1+s^2}$$

2 (Review) Solution of Initial Value Problems using The Laplace Transform

Using integration by parts, we can find the Laplace Transform of a derivative:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

note that this Laplace Transform is also in terms of the function's initial conditions. Now, from induction, we can find the Laplace Transform of the n'th derivative of a function:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\mathcal{L}\{f(t)\} - s^{n-1}f(0) - \dots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

This Laplace Transform has $n-1$ initial conditions. Note that this way, we can solve first order constant coefficient ODE of the form:

$$ay"(t) + by'(t) + cy(t) = g(t)$$

By taking the Laplace Transform on both sides, using the formula to take the Laplace Transform of a derivative, solve for the solution $Y(s)$, and apply the Inverse Laplace Transform ($\mathcal{L}^{-1}\{F(s)\}$) to find the solution to the IVP. To find inverse Laplace Transforms in this course, one should decompose $Y(s)$ into simpler terms (usually using partial fractions) and use a Laplace Transform table to find the solution $y(t)$.

3 Solution of Initial Value Problems using The Laplace Transform

Unit step function:

$$u(x-x_0) = \begin{cases} 1\ when\ x \geq x_0 \\ 0\ when\ x < x_0 \end{cases}$$

Dirac-Delta function:

$$\delta(x-x_0) = \begin{cases} \infty\ when\ x = x_0 \\ 0\ when\ x \neq x_0 \end{cases}$$

Convolution integral:

$$f(t) * g(t) = \int_0^t f(t-\tau)g(\tau)d\tau$$

, where

$$\mathcal{L}\{f(t) * g(t)\} = F(s)G(s)$$

Problem 3a. From Boyce and DiPrima, 10th edition (6.4, exercise 9, p.340):
Solve this IVP using the Laplace Transform:

$$y" + y = g(t);\ y(0) = 0,\ y'(0) = 1,$$

where

$$g(t) = \begin{cases} \frac{t}{2} \ when\ 0 \leq t < 6 \\ 3\ when\ t \geq 6 \end{cases}$$

Solution: Note that

$$g(t) = \frac{t}{2}u(t) - \frac{t}{2}u(t-6) + 3u(t-6)$$

$$g(t) = \frac{t}{2}u(t) + (3 - \frac{t}{2})u(t-6) = \frac{t}{2}u(t) - \frac{1}{2}(t-6)u(t-6)$$

$$\mathcal{L}\{y" + y\} = \mathcal{L}\{\frac{t}{2}u(t) - \frac{1}{2}(t-6)u(t-6)\}$$

Now, take the Laplace Transform:

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = \mathcal{L}\{\frac{t}{2}u(t) - \frac{1}{2}(t-6)u(t-6)\}$$

$$(s^2+1)Y(s) - 1 = \frac{1}{2s^2} - \frac{e^{-6s}}{2s^2}\}$$

$$Y(s) = \frac{1}{1+s^2} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}$$

Find $\frac{1/2}{s^2(s^2+1)}$ using partial fractions:

$$\frac{1/2}{s^2(s^2+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{Cs+D}{s^2+1}$$

$$\frac{1/2}{s^2(s^2+1)} = \frac{As(s^2+1) + B(s^2+1) + Cs^3 + Ds^2}{s^2(s^2+1)}$$

$$\frac{1/2}{s^2(s^2+1)} = \frac{(A+C)s^3 + (B+D)s^2 + As + B}{s^2(s^2+1)}$$

Therefore,

$$s^3 : A + C = 0$$

$$s^2 : B + D = 0$$

$$s : A = 0$$

$$1 : B = \frac{1}{2}$$

Meaning that $A = 0, B = \frac{1}{2}, C = 0, D = -\frac{1}{2}$.

$$Y(s) = \frac{1}{1+s^2} + \frac{1}{2}(\frac{1}{s^2} - \frac{1}{1+s^2}) - \frac{e^{-6s}}{2}(\frac{1}{s^2} - \frac{1}{1+s^2})$$

$$Y(s) = \frac{1}{2}(\frac{1}{s^2} + \frac{1}{1+s^2}) - \frac{e^{-6s}}{2}(\frac{1}{s^2} - \frac{1}{1+s^2})$$

Taking the inverse Laplace Transform:

$$y(t) = \frac{1}{2}(t + \sin t) - \frac{1}{2}((t-6) + \sin(t-6))u(t-6)$$

Problem 3b. From Boyce and DiPrima, 10th edition (6.5, exercise 4, p.348):
Solve this IVP using the Laplace Transform:

$$y\text{''} - y = -20\delta(t-3);\ y(0) = 1,\ y'(0) = 0,$$

Solution:

$$\mathcal{L}\{y\text{''} - y\} = s^2Y(s) - sy(0) - y'(0) - Y(s) = \mathcal{L}\{-20\delta(t-3)\}$$

$$\mathcal{L}\{y\text{''} - y\} = s^2Y(s) - sy(0) - y'(0) - Y(s) = \mathcal{L}\{-20\delta(t-3)\}$$

$$(s^2-1)Y(s) - s = -20e^{-3s}\}$$

$$Y(s) = \frac{s}{s^2-1} - 20\frac{e^{-3s}}{s^2-1}$$

Taking the inverse Laplace Transform, we get

$$y(t) = \cosh t - 20\sinh(t-3)u(t-3)$$