# Chapter 1: Topological Manifolds

## Topological Manifolds

The study of differential geometry begins with tens of pages of definitions.

**Definition 1.1: Topological Manifold**

Let $M$ be a topological space. $M$ is a topological manifold of dimension $m$ if it is Hausdorff, second-countable, and locally homeomorphic to $\mathbb{R}^n$.

**Definition 1.2: Local homeomorphism**

$M$ locally homeomorphic to $\mathbb{R}^n$ if every point $x \in M$ an open set $U$, equipped with a homeomorphism which sends points in $U$ into an open subset of $\mathbb{R}^n$.

$$\phi : U \to \phi(U)$$

The tuple $(U, \phi)$ is called a coordinate chart.

**Definition 1.3: More on coordinate charts**

- A coordinate chart $(U, \phi)$ is centered at $p \in M$ if $p \in U$ and $\phi(p) = 0 \in \mathbb{R}^n$.
- We call $U$ the coordinate domain, and
- we call $\phi$ the coordinate map.
- If the choice of $(U, \phi)$ is unambiguous, then the local coordinates of $p$ are simply the coordinates of $\phi(p)$ in $\mathbb{R}^n$, and
- we sometimes also denote $\phi(U)$ by $\hat{U}$ if it is unambiguous to do so.
- If $\hat{U}$ is an open ball/cube, then $U$ is called a coordinate ball/cube.

The central theme of point-set topology (or even metric topology) is that of passing a topological argument to the basis or to a neighbourhood. Manifolds in particular have a nice basis.

**Proposition 1.1: Basis of precompact coordinate balls**

Every topological manifold has a countable basis of precompact coordinate balls.

**Proposition 1.2: Additional facts about topological manifolds**

If $M$ is a topological manifold,

- $M$ is locally compact. (Lee, Proposition 1.12)
- $M$ is paracompact, and every open cover has a refinement that is another countably locally finite open cover whose elements are chosen from an arbitrary (but fixed) basis of $M$. (Lee, Theorem 1.15)
- $M$ is locally-path connected.
- $M$ is connected iff it is path-connected.
- $M$ is metrizable. (Munkres Chapter 6)

## Smooth Manifolds

We wish to perform calculus on manifolds.

**Definition 2.1: Smooth function $F : \mathbb{R}^n \to \mathbb{R}^m$**

Let $F : \mathbb{R}^n \to \mathbb{R}^m$, replacing $\mathbb{R}^n$ and $\mathbb{R}^m$ with open subsets if necessary. $F$ is smooth its component functions has continuous partial derivatives of all orders. The set of smooth functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ is sometimes denoted by $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$. If $m = 1$, we sometimes write $C^\infty(\mathbb{R}^n)$, similar to a test function on the Swartz space.

**Definition 2.2: Transition map from $\phi$ to $\psi$**

Let $(U, \phi)$ and $(V, \psi)$ be coordinate charts on $M$. The composite function (whenever $U \cap V \neq \varnothing$)

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

is called the transition map. Notice $\psi \circ \phi^{-1}$ is by definition a homeomorphism.

**Definition 2.3: Smoothly compatiable**

Two coordinate charts on $M$, $(U, \phi)$ and $(V, \psi)$ are called smoothly compatible if either their domains are disjoint, or their transition map is a diffeomorphism on $\mathbb{R}^m$.

**Definition 2.4: Smooth atlas**

An atlas $\mathcal{A}$ of $M$ is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$ whose collection of coordinate domains $\{U_\alpha\}$ for an open cover of $M$.
It is called a smooth atlas if any two charts in the atlas are pairwise smoothly compatible.

**Definition 2.5: Smooth manifold**

A smooth atlas $\mathcal{A}$ on $M$ is maximal if it is not contained (properly) in any other smooth atlas as a subset. In other words, if $(U', \phi')$ is a chart on $M$ that is smoothly compatible with all elements in $\mathcal{A}$, then $(U', \phi') \in \mathcal{A}$ already.

This smooth atlas is often very large, it includes all translations of charts, dilations, and composition with diffeomorphisms in $\mathbb{R}^m$, restrictions onto open subsets, etc. A maximal smooth atlas is sometimes called a complete atlas, or a smooth manifold structure.

A smooth manifold is the tuple $(M, \mathcal{A})$, where $\mathcal{A}$ is some smooth atlas. It can happen if $M$ is originally a topological manifold with a huge number of charts, some of which are smoothly compatible with others, that $\mathcal{A}$ is a strict subset, and both $\mathcal{A}_1$ and $\mathcal{A}_2$ are maximal smooth atlases on $M$, but $\mathcal{A}_1 \neq \mathcal{A}_2$. We often omit $\mathcal{A}$ and write $M$ if the smooth atlas is understood or not of importance.

**Definition 2.6: Smooth coordinate terminologies**

Let $(M, \mathcal{A})$ be a smooth manifold.

- Any coordinate chart $(U, \phi) \in \mathcal{A}$ is called a smooth chart, similar to definition 1.3
- We call $U$ the *smooth coordinate domain* or *smooth coordinate neighbourhood* of any $p \in U$, and
- we call $\phi$ the *smooth coordinate map.*
- The terms *smooth coordinate ball* and *smooth coordinate cube* are used similarly.
- A set $B \subseteq M$ is a *regular coordinate ball* if its image is a smooth coordinate ball centered at the origin; and the closure of this ball in $\mathbb{R}^m$ is a subset of the image of another smooth coordinate ball, centered at the origin.

**Definition 2.7: Standard smooth structure on $\mathbb{R}^n$**

The maximal smooth atlas containing $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$ is called the *standard smooth structure on* $\mathbb{R}^n$.

Manifolds with boundary are not that important as regular manifolds, but they are worth mentioning.

**Definition 2.8: Closed n-dimensional upper half-plane $\mathbb{H}^n \subseteq \mathbb{R}^n$**

We define the following symbols for the upper half plane.

- $\mathbb{H}^n = \left\{x \in \mathbb{R}^n,\ x^n \geq 0\right\}$,
- $\operatorname{Int}\mathbb{H}^n = \left\{x \in \mathbb{R}^n,\ x^n > 0\right\}$,
- $\partial\mathbb{H}^n = \left\{x \in \mathbb{R}^n,\ x^n = 0\right\}$

**Definition 2.9: Manifolds with boundary**

A topological space $M$ is called a manifold with boundary if it is Hausdorff, second-countable, and locally homeomorphic to an open subset of $\mathbb{H}^n$ (endowed with the subspace topology from $\mathbb{R}^n$).

A chart $(U, \phi)$ is an *interior chart* if its coordinate image is disjoint from the 'boundary' of the upper-half plane. This means $\phi(U) \cap \partial\mathbb{H}^n = \varnothing$. Similarly, $(V, \psi)$ is a *boundary chart* if its range contains a point in $\partial\mathbb{H}^n$; so $\psi(V) \cap \partial\mathbb{H}^n \neq \varnothing$.

Similar to definition 1.3 and definition 2.6, we use the terms *coordinate half-ball*, *coordinate half-cube*, *regular coordinate half-ball*.

Let $p \in M$, it is called an *interior point of* $M$ (not to be confused with the topological interior) if it is in the domain of some interior chart, and $p$ is called a *boundary point of* $M$ if there exists a boundary chart that sends $p$ into $\partial\mathbb{H}^n$. The set of interior points and boundary points of $M$ will be denoted by $\operatorname{Int} M$ and $\partial M$.

The $n$-sphere as a topological manifold. Define

$$S^n = \left\{x \in \mathbb{R}^{n+1},\ |x| = 1\right\}$$

We claim that $\{U_i^\pm\}_{i=1}^{n+1}$ form an open cover, where

$$U_i^+ = \left\{x \in S^n, x^i > 0\right\} \quad U_i^- = \left\{x \in S^n, x^i < 0\right\}$$

Each $U_i^\pm$ is the inverse image of $\pi_i^{-1}((0, +\infty)) \cap S^n$ or $\pi_i^{-1}((0, -\infty)) \cap S^n$, hence open. For every $x \in S^n$, there exists at least some $1 \leq j \leq n+1$ that makes the $j$-th coordinate of $x$, $x^j \neq 0$. So

$$S^n = \bigcup_i U_i^\pm$$

Denote the unit ball $\left\{x \in \mathbb{R}^n,\ |x| < 1\right\}$ in $\mathbb{R}^n$ by $\mathbb{B}^n$.

# Chapter 2: Smooth Maps

# Chapter 3: Tangent Spaces

We will go through the section on the Change of Coordinates, and how different coordinate charts change the representation of a derivation at $p \in M$, where $M$ is some smooth manifold.

**Proposition 0.1**

Let $M$ be a smooth manifold, and fix $p \in M$. If $\nu \in T_pM$ is given with respect to the bases

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^m}\bigg|_p \right\} \quad \text{and} \quad \left\{ \frac{\partial}{\partial y^1}\bigg|_p, \dots, \frac{\partial}{\partial y^m}\bigg|_p \right\}$$

Defined by

$$\frac{\partial}{\partial x^j}\bigg|_p \triangleq d\left(\phi^{-1}\Big|_{\phi(p)}\right)\left(\frac{\partial}{\partial x^j}\bigg|_{\phi(p)}\right) \quad \text{and} \quad \frac{\partial}{\partial y^j}\bigg|_p \triangleq d\left(\psi^{-1}\Big|_{\psi(p)}\right)\left(\frac{\partial}{\partial y^j}\bigg|_{\psi(p)}\right)$$

and we write $\nu$ in terms of the first basis

$$\nu = \nu^j \frac{\partial}{\partial x^j}\bigg|_p = \sum_{j=1}^m \nu^j \frac{\partial}{\partial x^j}\bigg|_p$$

and the second basis

$$\nu = \nu^j \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p = \sum_{k=1}^m \sum_{j=1}^m \nu^j \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p$$

If $f \in C^\infty(M)$, then

$$\nu(f) = \nu^j \frac{\partial}{\partial x^j}\bigg|_p f = \nu^j \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p f$$

*Proof.* Recall $\frac{\partial}{\partial x^j}\big|_p f \triangleq \frac{\partial}{\partial x^j}\big|_{\phi(p)} f \circ \phi^{-1}$, similarly for $\frac{\partial}{\partial y^j}\big|_p f$. Deriving $f$ and $p$ and by vector space operations on $T_pM$, the first basis expansion gives

$$\nu^j \frac{\partial}{\partial x^j}\bigg|_p f = \nu^j \frac{\partial}{\partial x^j}\bigg|_{\phi(p)} f \circ \phi^{-1} \tag{1}$$

and the second expression reads

$$\nu^j \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_p f = \nu^j \frac{\partial y^k}{\partial x^j}\bigg|_{\phi(p)} \frac{\partial}{\partial y^k}\bigg|_{\psi(p)} f \circ \psi^{-1} \tag{2}$$

Since $f \circ \phi^{-1} \in C^\infty(\mathbb{R}^m, \mathbb{R})$, we see the expressions are indeed equal. By the chain rule, if

$$\psi \circ \phi^{-1}(x^1, \dots x^m) = (y^1, \dots y^m)$$

then

$$D(\psi\circ\phi^{-1})(\phi(p)) = \begin{bmatrix} \frac{\partial y^1}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial y^1}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial y^1}{\partial x^m}\Big|_{\phi(p)} \\ \frac{\partial y^2}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial y^2}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial y^2}{\partial x^m}\Big|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y^m}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial y^m}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial y^m}{\partial x^m}\Big|_{\phi(p)} \end{bmatrix}$$

It follows from Proposition 3.6d) that the matrix $D(\psi\circ\phi^{-1})|_{\phi(p)}$ is invertible, as $\psi\circ\phi^{-1}$ is a diffeomorphism. ■

An important application of this is the following. We begin with the $\mathbb{R}^m \to \mathbb{R}^n$ case. We will see that if $p$ and $F(p)$ are represented by another pair of coordinate charts (smoothly compatible with the previous pair), then the rank of $dF_p$ does not change. So the rank of the differential is an invariant of the choice of coordinate chart.

**Definition 0.10: Matrix representation of the differential of $F : \mathbb{R}^m \to \mathbb{R}^n$**

Let $F \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$, and $p \in \mathbb{R}^m$ induces two charts $p \in (U, \mathrm{id}_{\mathbb{R}^m})$ and $F(p) \in (V\ \mathrm{id}_{\mathbb{R}^n})$, where $U \subseteq \mathbb{R}^m$ and $V \subseteq \mathbb{R}^n$. The matrix representation of the differential at $p$, $dF_p : T_p\mathbb{R}^m \to T_{F(p)}\mathbb{R}^n$ is nothing but the Jacobian matrix of $F$ at $p$.

$$\mathcal{M}\{dF_p\} = DF(p) = \begin{bmatrix} \frac{\partial F^1}{\partial x^1}\Big|_p & \frac{\partial F^1}{\partial x^2}\Big|_p & \cdots & \cdots & \frac{\partial F^1}{\partial x^m}\Big|_p \\ \frac{\partial F^2}{\partial x^1}\Big|_p & \frac{\partial F^2}{\partial x^2}\Big|_p & \cdots & \cdots & \frac{\partial F^2}{\partial x^m}\Big|_p \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial F^n}{\partial x^1}\Big|_p & \frac{\partial F^n}{\partial x^2}\Big|_p & \cdots & \cdots & \frac{\partial F^n}{\partial x^m}\Big|_p \end{bmatrix} \tag{3}$$

**Definition 0.11: Matrix representation of the differential of $F : M \to N$**

Let $F \in C^\infty(M, N)$, and $p \in M$ induces two charts $p \in (U, \phi)$ and $F(p) \in (V\ \psi)$. The matrix representation of the differential at $p$, $dF_p : T_pN \to T_{F(p)}N$ is nothing but the Jacobian matrix of the coordinate representation at $p$.

$$\mathcal{M}\{dF_p\} = \begin{bmatrix} \frac{\partial \hat{F}^1}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial \hat{F}^1}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^1}{\partial x^m}\Big|_{\phi(p)} \\ \frac{\partial \hat{F}^2}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial \hat{F}^2}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^2}{\partial x^m}\Big|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \hat{F}^n}{\partial x^1}\Big|_{\phi(p)} & \frac{\partial \hat{F}^n}{\partial x^2}\Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^n}{\partial x^m}\Big|_{\phi(p)} \end{bmatrix} \tag{4}$$

Alternately, if we write $\hat{p} = \phi(p)$ as the $\mathbb{R}^m$ coordinates at $p$, then

$$\mathcal{M}\{dF_p\} = \begin{bmatrix} \frac{\partial \hat{F}^1}{\partial x^1}\Big|_{\hat{p}} & \frac{\partial \hat{F}^1}{\partial x^2}\Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^1}{\partial x^m}\Big|_{\hat{p}} \\ \frac{\partial \hat{F}^2}{\partial x^1}\Big|_{\hat{p}} & \frac{\partial \hat{F}^2}{\partial x^2}\Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^2}{\partial x^m}\Big|_{\hat{p}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \hat{F}^n}{\partial x^1}\Big|_{\hat{p}} & \frac{\partial \hat{F}^n}{\partial x^2}\Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^n}{\partial x^m}\Big|_{\hat{p}} \end{bmatrix} \tag{5}$$

**Proposition 0.2**

Let $F$ be a smooth map between $M$ and $N$, at every $p \in M$, rank $dF_p$ is an invariant over (smoothly compatible) pairs of charts in $M$ and $N$.

*Proof.* Let $p \in (U_1, \phi_1) \cap (U_2, \phi_2)$, and $F(p) \in (V_1, \psi_1) \cap (V_2, \psi_2)$. Where all charts are smoothly compatible if it makes sense to talk about it. Both $\phi_2 \circ \phi_1^{-1}$ and $\psi_2 \circ \psi_1^{-1}$ are diffeomorphisms, and the change of basis matrices $D(\phi_2 \circ \phi_1^{-1})\Big|_{\phi_1(p)}$ and $D(\psi_2 \circ \psi_1^{-1})\Big|_{\psi_1(F(p))}$ are invertible by Proposition 3.6d) again, so the ranks $dF_p$ with respect to any of the two charts are equal.

$$\underbrace{D(\psi_2 \circ \psi_1^{-1})\Big|_{\psi_1(F(p))}}_{\text{invertible}} \Big(\mathcal{M}\{dF_p\}\Big) \underbrace{D(\phi_2 \circ \phi_1^{-1})\Big|_{\phi_1(p)}}_{\text{invertible}}$$

■