MATH 263: Section 003, Tutorial 9

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1 Nonhomogeneous ODEs: Variation of Parameters

To solve a nonhomogeneous linear ODE of the form:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = g(x) \neq 0$$

the idea is to guess a solution of the form:

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

Plugging it in the ODE, we can derive formulas for $u_1(x)$ and $u_2(x)$:

$$u_1(x) = \int \frac{-y_2(x)g(x)}{W(y_1, y_2)(x)} dx + c_1$$
$$u_2(x) = \int \frac{y_1(x)g(x)}{W(y_1, y_2)(x)} dx + c_2$$

This method is more systematic and works for methods that the method of undetermined coefficients cannot solve, but it requires computing integrals, which may be tedious or impossible to solve. Note: a similar, although more complex derivation can be used with higher order linear ODEs. Again, using this method for higher order ODEs can become rather convoluted.

Problem 1.1. Problem from Assignment 1 (question 5) solved another way, find the general solution of:

$$x"(t) + 3x'(t) = 2te^{-3t}$$

Solution: First solve the homogeneous ODE:

$$x"(t) + 3x'(t) = 0$$

The right-hand side is

$$g(t) = 2te^{-3t}$$

The characteristic equation is:

$$r^2 + 3r = 0$$

The roots being

$$r_1 = 0, r_2 = -3$$

Therefore, the set of fundamental solutions is:

$$x_1(t) = 1,\ x_2(t) = e^{-3t}$$
$$x_1'(t) = 0,\ x_2'(t) = -3e^{-3t}$$

$$W(x_1,x_2) = \begin{pmatrix} x_1 & x_2 \\ x_1' & x_2' \end{pmatrix} = \begin{pmatrix} 1 & e^{-3t} \\ 0 & -3e^{-3t} \end{pmatrix} = -3e^{-3t}$$

$$x(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$$

Using integration by parts, we get:

$$u_1(t) = \int \frac{-x_2(t)g(t)}{W(x_1,x_2)(t)} dt + c_1 = \int \frac{-e^{-3t}\ 2te^{-3t}}{-3e^{-3t}} dt + c_1 = \int \frac{2}{3} te^{-3t}\ dt + c_1$$

$$u_1(t) = c_1 - \frac{2}{9}te^{-3t} - \frac{2}{27}e^{-3t}$$

Similarly,

$$u_2(t) = \int \frac{x_1(t)g(t)}{W(x_1,x_2)(t)} = \int \frac{2te^{-3t}}{-3e^{-3t}} dt + c_2 = \int \frac{-2t}{3} dt + c_2$$

$$u_2(t) = c_2 - \frac{1}{3}t^2$$

Therefore,

$$x(t) = u_1(t)x_1(t) + u_2(t)x_2(t)$$

$$x(t) = (c_1 - \frac{2}{9}te^{-3t} - \frac{2}{27}e^{-3t}) + (c_2 - \frac{1}{3}t^2)e^{-3t}$$

$$x(t) = c_1 + (c_2 - \frac{2}{27})e^{-3t} - \frac{1}{3}t^2e^{-3t} - \frac{2}{9}te^{-3t}$$

$$x(t) = c_1 + c_3e^{-3t} - \frac{1}{3}t^2e^{-3t} - \frac{2}{9}te^{-3t} = c_1 - \frac{1}{3}(t^2 + \frac{2}{3}t + c_4)e^{-3t},$$

Where $c_3 = -c_4 = c_2 - \frac{2}{27}$.

Problem 1.2. Find the general solution of:

$$y"(x) + a^2y(x) = g(x),\ a \neq 0$$

Solution: First solve the homogeneous ODE:

$$y"(x) + a^2y(x) = 0$$

The characteristic equation is:

$$r^2 + a^2 = 0$$

The double roots being

$$r_{1,2} = \pm ai.$$

From Euler's Formula , the solution to the homogeneous ODE is:

$$y_1(x) = \cos(ax),\ y_2(x) = \sin(ax)$$

$$y_1'(x) = -a\sin(ax),\ y_2'(x) = a\cos(ax)$$

$$W(y_1,y_2) = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = \begin{pmatrix} \cos(ax) & \sin(ax) \\ -a\sin(ax) & a\cos(ax) \end{pmatrix} = a\cos^2(ax) + a\sin^2(ax) = a$$

$$u_1(x) = \int -\frac{1}{a}\sin(ax)g(x)dx + c_1$$

$$u_2(x) = \int \frac{1}{a}\cos(ax)g(x)dx + c_2$$

$$y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$
$$y(x) = (\int -\frac{1}{a}\sin(ax)g(x)dx + c_1)\cos(ax) + (\int \frac{1}{a}\cos(ax)g(x)dx + c_2)\sin(ax)$$
$$y(x) = c_1\cos(ax) + c_2\sin(ax) + [\frac{1}{a}\sin(ax)\int \cos(ax)g(x)\ dx] - [\frac{1}{a}\cos(ax)\int \sin(ax)g(x)\ dx].$$

2 Review of Power Series

A **Power Series** centered at $x = x_0$ is of the form:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

which is well defined on a **radius of convergence** on which the series converges. The two main ways to find the radius of convergence of a Power Series:

The **Ratio Test**:
If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty} |\frac{a_{n+1}(x - x_0)^{n+1}}{a_n(x - x_0)^n}| = |x - x_0| \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |x - x_0|L,$$

then the power series converges absolutely at that value of x if $|x - x_0|L < 1$ and diverges if $|x - x_0|L > 1$. If $|x - x_0|L = 1$, the test is inconclusive.

The **Root Test**: If, for a fixed value of x,

$$\lim_{n\to\infty} \sqrt[n]{|a_n(x - x_0)^n|} = |x - x_0| \lim_{n\to\infty} \sqrt[n]{|a_n|} = |x - x_0|L,$$

then the power series converges absolutely at that value of x if $|x - x_0|L < 1$ and diverges if $|x - x_0|L > 1$. If $|x - x_0|L = 1$, the test is inconclusive.

In particular, a **Taylor Series** centered at $x = x_0$ is of the form:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

Problem 2. From Boyce and DiPrima, 10th edition (5.1, exercise 2, p.253):
Find the radius of convergence of the Power Series:

$$\sum_{n=0}^{\infty} \frac{n}{2^n}x^n$$

Solution: Use the ratio test:

$$|x| \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |x| \lim_{n\to\infty} |\frac{\frac{n+1}{2\cdot 2^n}}{\frac{n}{2^n}}| = \frac{1}{2}|x| < 1$$

$$|x| < 2$$

One can check at $x = 2$ and $x = -2$ that the series diverges, so those values are not in the radius of convergence. Therefore,

$$x \in (0 - 2, 0 + 2) = (-2, 2),$$

Where the radius of convergence is $R = 2$.

3 Series Solutions Near an Ordinary Point, Part I

Consider the general second order linear ODE:

$$P(x)y"(x) + Q(x)\ y'(x) + R(x)\ y(x) = 0$$

Where $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$ are analytical around $x = x_0$. Such a point where $P(x_0) \neq 0$ is called an **ordinary point**. We can divide both sides by and get: $P(x)$ and get:

$$y"(x) + p(x)\ y'(x) + q(x)\ y(x) = 0$$

In that case, one can solve it by plugging in the power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

and finding the coefficients a_n, usually through a recurrence relation. It usually cannot be solved and one may only give the few first terms of the solution.

Problem 3. From Boyce and DiPrima, 10th edition (5.2, exercise 2, p.263):
Find the general solution of:

$$y" - xy' - y = 0.$$

Solution: Choosing $x_0 = 0$, let the solution be of the form:

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Take the first two derivatives (term by term):

$$y'(x) = \sum_{n=1}^{\infty} na_n\ x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}\ x^n$$

$$y"(x) = \sum_{n=2}^{\infty} n(n-1)a_n\ x^{n-2} = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\ x^n$$

Note that taking derivatives changes the initial index and that reindexing is required. Now, plug these series into the ODE:

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\ x^n - x\sum_{n=0}^{\infty}(n+1)a_{n+1}\ x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

Distribute the x:

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}\ x^n - \sum_{n=0}^{\infty}(n+1)a_{n+1}\ x^{n+1} - \sum_{n=0}^{\infty} a_n x^n = 0$$

Now we want all the sums to have the same coefficients and start at the same index. Therefore, we take out one term from the first and third series and reindex the second one:

$$2a_2 + \sum_{n=1}^{\infty}(n+1)(n+2)a_{n+2}\ x^n - [\sum_{n=1}^{\infty} na_n\ x^n] - a_0 - [\sum_{n=1}^{\infty} a_n x^n] = 0$$

Collect terms:

$$2a_2 - a_0 + \sum_{n=1}^{\infty}((n+1)(n+2)a_{n+2} - na_n - a_n)x^n = 0.$$

$$2a_2 - a_0 + \sum_{n=1}^{\infty}((n+1)(n+2)a_{n+2} - (n+1)a_n)x^n = 0.$$

Now, since $1, x, x^2, x^3, \ldots$ are linearly independent from each other we can check the constants alone, which need to amount to 0:

$$2a_2 - a_0 = 0$$

$$a_2 = \frac{1}{2}a_0$$

Furthermore, for all other x^n:

$$(n+1)(n+2)a_{n+2} - (n+1)a_n = 0$$

$$a_{n+2} = \frac{a_n}{n+2}$$

For even coefficients we have:

$$a_2 = \frac{a_0}{2}$$

$$a_4 = \frac{a_2}{4} = \frac{a_0}{2^2\ 2!}$$

$$a_6 = \frac{a_4}{6} = \frac{a_0}{2^3\ 3!}$$

In general, for $n = 2k$:

$$a_n = \frac{a_0}{2^k\ k!}$$

For odd coefficients we have:

$$a_3 = \frac{a_1}{3} = \frac{2a_1}{2 \cdot 3}$$

$$a_5 = \frac{a_3}{5} = \frac{2 \cdot 4 \cdot a_1}{5!}$$

$$a_7 = \frac{a_5}{7} = \frac{2 \cdot 4 \cdot 6 \cdot a_1}{7!}$$

In general, for $n = 2k+1$:

$$a_n = \frac{2^k k!\ a_1}{(2k+1)!}$$

By superposition, the solution is:

$$y(x) = a_0(1 + \frac{1}{2}x^2 + \frac{1}{8}x^4 + \ldots) + a_1(x + \frac{1}{3}x^3 + \frac{1}{15}x^5 + \ldots)$$

In this case, we can give a closed form solution, which is:

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n\ n!} + a_1 \sum_{n=0}^{\infty} \frac{2^n n!\ x^{2n+1}}{(2n+1)!}.$$