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# Chapter 1

## Theorem 1.1

**WTS 1.1.** *Let  $\mathcal{M}(\mathcal{F})$  be the  $\sigma$ -algebra generated by  $\mathcal{F}$ , if  $\mathcal{E}$  is a subset of  $\mathbb{P}(X)$ , with  $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$ , then  $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$ .*

*Proof.* Notice that because  $\mathcal{E} \subseteq \mathcal{M}(\mathcal{F})$ ,

$$\mathcal{M}(\mathcal{F}) \in \{\mathcal{M}, \mathcal{E} \subseteq \mathcal{M}, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

Taking the intersection, noting that  $\mathcal{M}(\mathcal{E})$  is the intersection of all  $\sigma$ -algebras containing  $\mathcal{E}$  as a subset, we have

$$\bigcap \{\mathcal{M}(\mathcal{F})\} \supseteq \bigcap \{\mathcal{M}, \mathcal{E} \subseteq \mathcal{M}, \mathcal{M} \text{ is a } \sigma\text{-algebra}\}$$

And

$$\mathcal{M}(\mathcal{E}) \subseteq \mathcal{M}(\mathcal{F})$$

■



**Theorem 1.2**

**WTS 2.1.** *The Borel  $\sigma$ -algebra of  $\mathbb{R}$ ,  $\mathbb{B}$  is generated by the following*

- The family of open intervals  $\mathcal{E}_\infty = \{(-\infty, b), a < \infty\}$ ,
- The family of closed intervals  $\mathcal{E}_\epsilon = \{[a, b], a < b\}$ ,
- The family of half-open intervals  $\mathcal{E}_\ni = \{(-\infty, b), a < \infty\}$  or  $\mathcal{E}_\Delta = \{[a, b], a < b\}$
- The open rays  $\mathcal{E}_\nabla = \{(-\infty, +\infty), a \in \mathbb{R}\}$  or  $\mathcal{E}_\nabla = \{(-\infty, a), a \in \mathbb{R}\}$
- The closed rays  $\mathcal{E}_\lrcorner = \{[a, +\infty), a \in \mathbb{R}\}$  or  $\mathcal{E}_\lrcorner = \{(-\infty, a], a \in \mathbb{R}\}$

*Proof.* By definition,  $\mathbb{B}$  is generated by the family of all open sets in  $\mathbb{R}$ , but every open set is a countable union of open intervals. Therefore

$$\mathcal{T}_\mathbb{R} \subseteq \mathcal{M}(\mathcal{E}_\infty) \implies \mathbb{B} \subseteq \mathcal{M}(\mathcal{E}_\infty)$$

Conversely, every open interval is an open set, hence

$$\mathcal{E}_\infty \subseteq \mathcal{T}_\mathbb{R} \subseteq \mathbb{B} \implies \mathcal{M}(\mathcal{E}_\infty) \subseteq \mathbb{B}$$

Every closed interval can also be written as a countable intersection of open intervals, for every  $[a, b]$ , with  $a < b$ , we have

$$[a, b] = \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1}) \quad (1)$$

Indeed, fix any  $x \in [a, b]$  then for every  $n \geq 1$ ,

$$a - n^{-1} < a \leq x \leq b < b + n^{-1}$$

So  $x \in \bigcap_{n \geq 1} (a - n^{-1}, b + n^{-1})$ . If  $x$  an element of the left member, then for every  $n \geq 1$ ,

$$a - n^{-1} < x \implies a - x < n^{-1}$$

Similarly for  $x \leq b$ , therefore equation (1) is valid, and  $\mathcal{E}_\epsilon \subseteq \mathbb{B} = \mathcal{M}(\mathcal{E}_\infty)$ . To show the reverse estimate, every open interval can be written as a countable union of closed intervals,

$$(a, b) = \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \quad (2)$$

To show that the above estimate is indeed true, fix any  $x \in (a, b)$ , then

$$\begin{aligned} a < x < b &\iff a < a + n^{-1} \leq x \leq b - n^{-1} < b \\ &\iff x \in \bigcup_{n \geq 1} [a + n^{-1}, b - n^{-1}] \end{aligned}$$

So that equation (2) holds. By similar argumentation we have  $\mathcal{E}_\infty \subseteq \mathcal{M}(\mathcal{E}_\epsilon) \implies \mathcal{M}(\mathcal{E}_\epsilon) = \mathcal{M}(\mathcal{E}_\infty)$ .

For  $\mathcal{E}_\exists$ ,  $\mathcal{E}_\Delta$

- $(a, b] = \bigcap_{n \geq 1} (a, b + n^{-1})$ , proves  $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_1)$ ,
- $(a, b) = \bigcup_{n \geq 1} (a, b - n^{-1}]$ , proves  $\mathcal{M}(\mathcal{E}_1) \subseteq \mathcal{M}(\mathcal{E}_3)$ ,
- $[a, b) = \bigcup_{n \geq 1} [a, b - n^{-1}]$ , proves  $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_2)$ ,
- $[a, b] = \bigcap_{n \geq 1} [a, b + n^{-1})$ , proves  $\mathcal{M}(\mathcal{E}_2) \subseteq \mathcal{M}(\mathcal{E}_4)$

So that  $\mathcal{M}(\mathcal{E}_1) = \mathcal{M}(\mathcal{E}_2) = \mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_4) = \mathbb{B}$ . By taking complements of each element we get  $\mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8)$  and  $\mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7)$ . Notice also that

- $(a, b] = (a, +\infty) \cap (-\infty, b]$ , proves  $\mathcal{E}_\exists \subseteq \mathcal{M}(\mathcal{E}_\nabla)$ , and  $\mathcal{M}(\mathcal{E}_3) \subseteq \mathcal{M}(\mathcal{E}_5)$ .
- $(a, +\infty) = \bigcup_{n \geq 1} (a, a + n]$ , proves  $\mathcal{E}_\nabla \subseteq \mathcal{M}(\mathcal{E}_\exists)$ , and  $\mathcal{M}(\mathcal{E}_5) \subseteq \mathcal{M}(\mathcal{E}_3)$ .
- $[a, b) = [a, +\infty) \cap (-\infty, b)$ , proves  $\mathcal{E}_\Delta \subseteq \mathcal{M}(\mathcal{E})$ , and  $\mathcal{M}(\mathcal{E}_4) \subseteq \mathcal{M}(\mathcal{E}_7)$ ,
- $[a, +\infty) = \bigcup_{n \geq 1} [a, a + n)$ , proves  $\mathcal{E}_\Delta \subseteq \mathcal{M}(\mathcal{E}_\Delta)$ , and  $\mathcal{M}(\mathcal{E}_7) \subseteq \mathcal{M}(\mathcal{E}_4)$ .

Finally,  $\mathcal{M}(\mathcal{E}_3) = \mathcal{M}(\mathcal{E}_5) = \mathcal{M}(\mathcal{E}_8) = \mathbb{B}$  and  $\mathcal{M}(\mathcal{E}_4) = \mathcal{M}(\mathcal{E}_6) = \mathcal{M}(\mathcal{E}_7) = \mathbb{B}$ . ■

**Theorem 1.3**

**WTS 3.1.** *If  $A$  is countable, then  $\otimes_{\alpha \in A} \mathcal{M}_\alpha$  is the  $\sigma$ -algebra generated by*

$$W := \left\{ \prod_{\alpha \in A} E_\alpha, E_\alpha \in \mathcal{M}_\alpha \right\}$$

*Proof.* We agree to define

$$V := \left\{ \pi_\alpha^{-1}(E_\alpha), E_\alpha \in \mathcal{M}_\alpha \right\}$$

By definition,  $V$  generates  $\otimes_{\alpha \in A} \mathcal{M}_\alpha$ . Fix any element in  $x = \pi_\alpha^{-1}(E_\alpha) \in V$ , then

$$\pi_\alpha(x) \in E_\alpha, \pi_{\beta \neq \alpha}(x) \in X_\beta$$

Then  $x \in W$  if we choose  $x = \prod_{c \in A} E_c$ , for  $E_c = E_\alpha$  if  $c = \alpha$ , and  $E_c = X_c$  if  $c \neq \alpha$ . ■

**Theorem 1.4**

**WTS 4.1.**

*Proof.*



**Theorem 1.5****WTS 5.1.***Proof.*

**Theorem 1.6**

**WTS 6.1.**

*Proof.*



**Theorem 1.7**

**WTS 7.1.**

*Proof.*



**Theorem 1.8**

**WTS 8.1.**

*Proof.*





**Theorem 1.9**

**WTS 9.1.**

*Proof.*



**Theorem 1.10****WTS 10.1.***Proof.*

**Theorem 1.11**

**WTS 11.1** (Caratheodory's Theorem). *If  $\mu^*$  is an outer measure on  $\mathbf{X}$ , the collection  $\mathcal{M}$  of  $\mu^*$ -measurable sets is a  $\sigma$ -algebra, and the restriction of  $\mu^*$  to  $\mathcal{M}$  is a complete measure.*

*Proof.* We quote the definition for a set  $A \subseteq X$  to be  $\mu^*$  measurable. For any  $E \subseteq X$ ,

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \setminus A) \quad (3)$$

- Show  $\mathcal{M}$  is an algebra.
- $\mu^*$  is finitely additive on  $\mathcal{M}$ .
- $\mathcal{M}$  is closed under countable disjoint (this makes  $\mathcal{M}$  a sigma algebra, since it is an algebra that is closed under countable disjoint unions)

**Lemma 11.1** *The family of  $\mu^*$ -measurable sets is an algebra.*

*Proof of Lemma 11.1.* Clearly  $\mathcal{M}$  is closed under complements. To show that it is a  $\sigma$ -algebra, and if  $A, B \in \mathcal{M}$ , then  $\left\{ \underbrace{E \cap A}_1, \underbrace{E \setminus A}_2 \right\} \subseteq \mathbb{P}(\mathbf{X})$ . And because  $B$  is  $\mu^*$ -measurable,

$$\mu^*(E) = \underbrace{\mu^*(E \cap A \cap B) + \mu^*(E \cap A \setminus B)}_1 + \underbrace{\mu^*(E \cap B \setminus A) + \mu^*(E \setminus (A \cup B))}_2$$

By subadditivity of  $\mu^*$ ,  $A \cup B = A \cap B + A \setminus B + B \setminus A$  with  $+$  denoting the disjoint union, hence

$$\mu^*(E \cap (A \cap B)) \leq \mu^*(E \cap (A \cap B)) + \mu^*(E \cap (A \setminus B)) + \mu^*(E \cap (B \setminus A))$$

and

$$\mu^*(E) \geq \mu^*(E \cap (A \cup B)) + \mu^*(E \setminus (A \cup B))$$

■

**Lemma 11.2**  *$\mu^*$  is finitely additive on  $\mathcal{M}$ , the family of  $\mu^*$ -measurable sets.*

*Proof of Lemma 11.2.* Let  $A, B$  be disjoint  $\mu^*$ -measurable sets. It suffices to show  $\mu^*(A \cup B) \geq \mu^*(A) + \mu^*(B)$ , as the reverse estimate follows from subadditivity. From Lemma 11.1,  $A \cup B \in \mathcal{M}$ , so

$$\begin{aligned}\mu^*(A \cup B) &= \mu^*(A \cup B \cap A) + \mu^*(A \cup B \setminus A) \\ &= \mu^*(A \cup \emptyset) + \mu^*(A \setminus A \cup B \setminus A) \\ &= \mu^*(A) + \mu^*(B)\end{aligned}$$

■

**Corollary 11.1** *If  $\{A_j\}_{j \geq N} \subseteq \mathcal{M}$  is a finite disjoint family, then*

$$\mu^*\left(\bigcup A_{j \leq N}\right) = \sum \mu^*(A_{j \leq N})$$

**Lemma 11.3** *Let  $\{A_j\}_{j \geq 1}$  be a countable disjoint sequence in  $\mathcal{M}$ , and denote  $B_n = \bigcup A_{j \leq n} \in \mathcal{M}$  by Lemma 11.1. For every  $E \subseteq X$ ,*

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j \leq n})$$

*Proof of Lemma 11.3.* We will proceed by induction. If  $n = 1$  then we have equality, suppose the result holds for  $n \in \mathbb{N}^+$ , and  $A_{n+1} \in \mathcal{M}$  so

$$\begin{aligned}\mu^*(E \cap B_{n+1}) &= \mu^*(E \cap B_{n+1} \cap A_{n+1}) = \mu^*(E \cap B_{n+1} \setminus A_{n+1}) \\ &= \mu^*(E \cap A_{n+1}) + \mu^*(E \cap B_n) \\ &= \sum_{j \leq n+1} \mu^*(E \cap A_j)\end{aligned}$$

as  $A_j \cap A_{n+1} = \emptyset \iff A_j \setminus A_{n+1} = A_j$ , and  $B_n \cap A_n = A_n \iff A_n \subseteq B_n$ . ■

To show  $\mathcal{M}$  is a sigma-algebra, fix any disjoint sequence  $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$ , and denote  $B_n$  as in lemma 11.3. Define  $B = \bigcup A_{j \geq 1} \supseteq B_n$  and for every  $n \geq 1$ ,

we have

$$\begin{aligned}
\mu^*(E) &= \mu^*(E \cap B_n) + \mu^*(E \setminus B_n) \\
&= \sum \mu^*(E \cap A_{j \leq n}) + \mu^*(E \setminus B_n) \\
&\geq \sum \mu^*(E \cap A_{j \leq n}) + \mu^*(E \setminus B) \quad \text{since } B_n \subseteq B \iff B^c \subseteq B_n^c \\
&\geq \sup_n \left[ \sum \mu^*(E \cap A_{j \leq n}) \right] + \mu^*(E \setminus B)
\end{aligned}$$

Let  $J \subseteq \mathbb{N}^+$  be a finite non-empty set. And  $\sup J \in \mathbb{N}^+$ ,  $\sup J < +\infty$ . By the Archimedean Property we can find a large  $N \in \mathbb{N}^+$ , with  $N > J$  so that

$$\sum_{j \in J} \mu^*(E \cap A_j) \leq \sum_{j \leq N} \mu^*(E \cap A_j)$$

Applying the estimate  $\sup_n \left[ \sum \mu^*(E \cap A_{j \leq n}) \right] + \mu^*(E \setminus B) \leq \mu^*(E)$  reads

$$\left[ \sum_{j \in J} \mu^*(E \cap A_j) \right] + \mu^*(E \setminus B) \leq \mu^*(E)$$

Now by Chapter 0, the infinite sum

$$\sum_{j \geq 1} \mu^*(E \cap A_j) = \sup \left\{ \sum_{j \in J} \mu^*(E \cap A_j), J \subseteq \mathbb{N}^+, 0 < |J| < +\infty \right\}$$

and  $\bigcup A_{j \geq 1} = B$  is  $\mu^*$ -measurable. Since  $\mu^*(\emptyset) = 0$ , and  $\mu^*$  is countably additive on  $\mathcal{M}$ , (perhaps by replacing  $E$  with the union over all disjoint sets),  $\mu^*$  is a measure on  $\mathcal{M}$ . To show  $\mu^*$  is a complete measure, fix  $A \in \mathcal{M}$  where  $\mu^*(A) = 0$ . Then any  $B \subseteq A$  is also null, and for  $E \subseteq X$ ,

$$\mu^*(E) \geq \underbrace{\mu^*(E \cap B)}_0 + \mu^*(E \setminus B) \implies B \in \mathcal{M}$$

■

**Theorem 1.12**

**WTS 12.1.**

*Proof.*



**Theorem 1.13**

**WTS 13.1.**

*Proof.*



**Theorem 1.14**

**WTS 14.1.**

*Proof.*





**Theorem 1.15****WTS 15.1.**

*Proof.* If  $\{E_j\}_{j \geq 1} \subseteq \mathcal{A}$  such that each  $E_j = FDU(I_{ji})$  over finitely many  $i$ , and suppose  $E_j$  are disjoint, and that  $DU(E_j) \in \mathcal{A}$ . So that  $DU(E_j) = FDU(I_\alpha)$  for some finite collection of half-intervals  $\{I_\alpha\}$ .

We will first prove the simpler case. Suppose we have already proven:

$$\{E_j\}_{j \geq 1} \subseteq \mathcal{A}, DU(E_j) = I_\alpha \in \mathcal{A} \implies \mu_0\left(DU(E_j)\right) = \sum \mu_0(E_j) = \mu_0(I_\alpha) \quad (4)$$

but each  $E_j$  is a FDU of  $I_{ji}$ , and for every  $j \geq 1$ ,  $E_j \cap I_\alpha \in \mathcal{A}$  (closure under intersections, because the family of FDU of h-intervals is an algebra).

Thus we have a disjoint sequence whose union is one h-interval. In symbols:

$$DU(E_j) = FDU(I_\alpha) \implies I_\alpha = DU(E_j \cap I_\alpha)$$

$$\forall j \geq 1, E_j \cap I_\alpha \in \mathcal{A} \implies$$

$$\begin{aligned} \mu_0(FDU(I_\alpha)) &= \sum_{\alpha < +\infty} \mu_0(I_\alpha) \\ &= \sum_{\alpha < +\infty} \sum_{j \geq 1} \mu_0(E_j \cap I_\alpha) \\ &= \sum_{j \geq 1} \sum_{\alpha < +\infty} \mu_0(E_j \cap I_\alpha) \\ &= \sum_{j \geq 1} \mu_0(E_j) \end{aligned}$$

It is permissible to swap the two summations because we are using the supremum definition for a sum of non-negative terms. And we applied finite-additivity (see earlier), to conclude that  $\sum_{j \geq 1} \sum_{\alpha} \mu_0(E_j \cap I_\alpha) = \sum_{j \geq 1} \mu_0(E_j)$ . ■

Define

- $\mathcal{H}_1 = \left\{ (a, b], -\infty \leq a < b < +\infty \right\},$
- $\mathcal{H}_2 = \left\{ (a, +\infty), a \in \mathbb{R} \cup \{-\infty\} \right\},$
- $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2 + \{\emptyset\}$ . Where  $+$  denotes the disjoint union.
- $DU$ : disjoint union,  $FDU$ : finite disjoint union.

Steps:

1. Show that  $\mathcal{H}$  is an elementary family.
2. Show that if  $I_\alpha \in \mathcal{H}_1$ , then for every  $I_\beta \in \mathcal{H}_1 \cup \mathcal{H}_2$ ,  $I_\alpha \cap I_\beta \in \mathcal{H}_1$ . We write this as

$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1, I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_1$$

3. Show that if  $I_\alpha \in \mathcal{H}_2$ , then

$$I_\alpha \cap \mathcal{H}_1 = \mathcal{H}_1, I_\alpha \cap \mathcal{H}_2 = \mathcal{H}_2$$

4. Show that  $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$  is well defined. (modify the proof in Folland to check for  $a = -\infty$  with

$$\overline{F} : \overline{\mathbb{R}} \rightarrow \overline{\mathbb{R}}, \quad \begin{cases} \overline{F}|_{\mathbb{R}} &= F \\ \overline{F}(+\infty) &= \sup_x F(x), \\ \overline{F}(-\infty) &= \inf_x F(x) \end{cases}$$

5. Show that  $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$  is well defined for  $b < +\infty$ . If  $E = (a, b] \in \mathcal{A}$ , then  $E$  is an FDU of  $\mathcal{H}_1$ , and  $\mathcal{H}_2$ . So we write

$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2) = FDU(\mathcal{H}_1)$$

since  $E$  is bounded above, the  $\mathcal{H}_2$  part of the FDU must be null. Now fix  $E = FDU_{\mathcal{H}_1}(I_j) = FDU_{\mathcal{H}_1}(I_2)$ . And follow the proof in Folland to see the 'well-definedness' of  $\mu_0$  if  $E \in \mathcal{H}_1$ .

6. Next, suppose  $E \in \mathcal{H}_2$  and

$$E = FDU(\mathcal{H}_1) + FDU(\mathcal{H}_2)$$

Clearly  $FDU(\mathcal{H}_2) \neq \emptyset$ , since  $E$  is unbounded above, and  $FDU(\mathcal{H}_2)$  consists of exactly one element, so we write

$$E = FDU(\mathcal{H}_1) + (z, +\infty)$$

7. Show that  $\mu_0((a, b]) = \overline{F}(b) - \overline{F}(a)$  is well defined. Hint: use the fact that if  $E \in \mathcal{A}$ , such that  $E = FDU(E, \mathcal{H}_1) + FDU(E, \mathcal{H}_2)$ , then  $FDU(E, \mathcal{H}_2)$  contains at most one element (after throwing away empty sets), then use this to deduce  $E \cap I_\alpha$  has a  $FDU(E \cap I_\alpha, \mathcal{H}_2)$  of exactly one  $\mathcal{H}_2$  interval, where  $I_\alpha$  participates in  $FDU(E, \mathcal{H}_2)$ , if  $E$  is unbounded above. Then take  $E \setminus I_\alpha = FDU(E \setminus I_\alpha, \mathcal{H}_1) = FDU(E, \mathcal{H}_1)$ .
8. Now show that  $\mu_0$  is well-defined on all  $E \in \mathcal{A}$ .
9. Continue the proof for Folland until you reach the unbounded intervals, then modify the 'right continuity argument' to add an extra  $\mathcal{H}_2$  interval. Let  $I = \mathcal{H}_1 + \mathcal{H}_2 = I_\alpha + I_\beta$ , meaning  $I$  can be represented by at most one  $\mathcal{H}_1$  and  $\mathcal{H}_2$  interval. If  $(I_k) \subseteq \mathcal{H}_1 \cup \mathcal{H}_2$ , then  $\{I_k \cap I_\alpha\} \subseteq \mathcal{H}_1$ , and continue the proof as usual.

**Theorem 1.16****WTS 16.1.***Proof.*

**Theorem 1.17****WTS 17.1.***Proof.*

**Theorem 1.18****WTS 18.1.***Proof.*

## Exercises

### Exercise 1.1

WTS 19.1.

*Proof.*



**Exercise 1.2****WTS 20.1.***Proof.*



**Exercise 1.3****WTS 21.1.***Proof.*

**Exercise 1.4**

**WTS 22.1.** *An algebra  $\mathcal{A}$  is a  $\sigma$ -algebra  $\iff$  it is closed under countable increasing unions.*

*Proof.*  $\Leftarrow$  is trivial. And it suffices to show that  $\mathcal{A}$  is closed under countable disjoint unions. Indeed, if  $\{E_j\}_{j \geq 1} \subseteq \mathcal{A}$  is a countable disjoint sequence of sets, write

$$F_n = \bigcup E_{j \leq n}$$

Clearly,  $F_j$  is increasing, and denote  $F = \bigcup E_{j \geq 1}$ , which is a member of  $\mathcal{A}$ . We claim that

$$\bigcup F_{n \geq 1} = \bigcup E_{j \geq 1}$$

Fix any  $x \in \bigcup E_{j \geq 1}$ , then  $x$  belongs in some  $E_j \subseteq F_j$ , and  $\supseteq$  is proven. Also, if  $x \in \bigcup F_{n \geq 1}$ , then there exists some  $F_n$  for which  $x$  is a member of. For this particular  $F_n$ , means that  $x \in E_j$  where  $j \leq n$  and  $x \in \bigcup E_{j \geq 1}$ .  $\blacksquare$

**Exercise 1.5**

**WTS 23.1.** Let  $\mathcal{M}(\mathcal{E})$  be the  $\sigma$ -algebra generated by  $\mathcal{E} \subseteq X$ , and

$$\mathcal{N} = \left\{ \mathcal{M}(\mathcal{F}), \mathcal{F} \subseteq \mathcal{E}, \mathcal{F} \text{ is countable} \right\}$$

Show that  $\mathcal{M}(\mathcal{E}) = \mathcal{N}$ .

*Proof.* The outline of the proof is as follows,

1. Prove that  $\mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$ ,
2. Show that  $\mathcal{N}$  is a  $\sigma$ -algebra,
3. Show that  $\mathcal{N}$  contains  $\mathcal{E}$  as a subset, and hence  $\mathcal{M}(\mathcal{E}) \subseteq \mathcal{N}$ .

First, for any  $\mathcal{F} \subseteq \mathcal{E}$ , where  $\mathcal{F}$  is countable, it follows from Lemma 1.1 that  $\mathcal{M}(\mathcal{F}) \subseteq \mathcal{M}\mathcal{E}$ . Taking the union over all of such  $\mathcal{F}$ , we get  $\bigcup \mathcal{M}(\mathcal{F}) = \mathcal{N} \subseteq \mathcal{M}(\mathcal{E})$ .

To show that  $\mathcal{N}$  is a  $\sigma$ -algebra, fix any  $A \in \mathcal{N}$ , and  $A$  belongs to  $\mathcal{M}(\mathcal{F})$ , therefore  $A^c \in \mathcal{M}(\mathcal{F}) \subseteq \mathcal{N}$ . To show closure under countable unions, fix a sequence  $\{E_j\} \subseteq \mathcal{N}$ , then each of these  $E_j$  belongs to a corresponding  $\mathcal{M}(\mathcal{F}_j)$ , for  $j \in \{1, 2, \dots\}$ . Now define

$$\overline{\mathcal{F}} = \bigcup \mathcal{F}_{j \geq 1} \subseteq \mathcal{E}$$

and  $\overline{\mathcal{F}}$  is obviously countable. Hence for every  $j \geq 1$ ,  $\mathcal{M}(\mathcal{F}_j) \subseteq \mathcal{M}(\overline{\mathcal{F}})$  and taking the union yields

$$\bigcup \mathcal{M}(\mathcal{F}_{j \geq 1}) \subseteq \mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$$

It is also clear that our sequence  $\{E_j\}$  is contained in  $\mathcal{M}(\overline{\mathcal{F}})$ , and  $E = \bigcup E_j$  belongs to  $\mathcal{M}(\overline{\mathcal{F}}) \subseteq \mathcal{N}$  as an element. Therefore  $\mathcal{N}$  is a  $\sigma$ -algebra.

Let  $\alpha \in A$  index the family of sets in  $\mathcal{E}$ , (so that  $E_\alpha \in \mathcal{E}$ ) and the singleton set of a set  $\{E_\alpha\}$  is a countable subset of  $\mathcal{E}$ . For every  $\alpha \in A$ , we have

$$E_\alpha \in \mathcal{M}(\{E_\alpha\}) \subseteq \mathcal{N} \implies \mathcal{E} \subseteq \mathcal{N}$$

And one final application of Lemma 1.1 finishes the proof. ■

**Exercise 1.6**

**WTS 24.1.**

*Proof.*



**Exercise 1.7**

**WTS 25.1.** If  $\mu_1, \dots, \mu_n$  are measures on  $(X, \mathcal{M})$ , and  $a_1, \dots, a_n \in [0, +\infty)$ , then  $\mu = \sum_1^n \mu_j$  is a measure on  $(X, \mathcal{M})$ .

*Proof.* If  $\{E_j\}$  is a disjoint sequence in  $\mathcal{M}$ , and denote  $E = \bigcup (E_j)$ . If for each  $k \leq n$ ,  $\mu_k(E) < +\infty$ ,

$$\mu_k(E) = \sum \mu_k(E_j) \implies a_k \mu_k(E) = \sum a_k \mu_k(E_j)$$

Then,

$$\mu(E) = \sum_{k \leq n} a_k \mu_k(E) = \sum_{k \leq n} \sum_{j \geq 1} a_k \mu_k(E_j) = \sum_{j \geq 1} \sum_{k \leq n} a_k \mu_k(E_j) = \sum_{j \geq 1} \mu(E_j)$$

If there exists some  $\mu_k$  such that  $\mu_k(E) = +\infty$ , then

$$\mu(E) = \sum_{k \leq n} \sum_{j \geq 1} a_k \mu_k(E_j)$$

Now if there exists some  $\mu_{k'}$  with  $\mu_{k'}(E) = +\infty$ , then  $\mu(E) = \sum_{k \leq n} \mu_k(E) = +\infty$ , and

$$\sum_{j \geq 1} \mu(E_j) = \sup_N \sum_{j \leq N} \sum_{k \leq n} a_k \mu_k(E_j) \geq \mu_{k'}(E)$$

Therefore  $\mu(E) = \sum_{j \geq 1} \mu(E_j)$ , and  $\mu$  is a measure. ■

**Exercise 1.8**

**WTS 26.1.** *If  $(X, \mathcal{M}, \mu)$  is a measure space, and  $\{E_j\} \subseteq \mathcal{M}$ , then  $\mu(\liminf E_j) \leq \liminf \mu(E_j)$ . Also,  $\mu(\limsup E_j) \geq \limsup \mu(E_j)$  provided that  $\mu(\bigcup E_{j \geq 1}) < +\infty$*

*Proof.* If  $\{E_j\}_{j \geq 1}$  is a sequence in  $\mathcal{M}$ , and define  $F_m = \bigcap_{j \geq m} E_j$

$$\liminf E_j = \bigcup_{m \geq 1} \bigcap_{j \geq m} E_j = \bigcup_{m \geq 1} F_m$$

Also, for every  $m \geq 1$ ,  $F_m \subseteq F_{m+1}$ , and  $F_m$  is an increasing sequence, because

$$[m, +\infty) \supseteq [m+1, +\infty) \implies F_m \subseteq F_{m+1}$$

Using continuity above, and writing  $F = \bigcup F_{m \geq 1} = \liminf E_j$ , we have

$$\begin{aligned} \mu(\liminf E_j) &= \mu(F) \\ &= \liminf \mu(F_m) \\ &\leq \liminf \mu(E_m) \end{aligned}$$

The second part of the proof is similar, if  $G_m = \bigcup_{j \geq m} E_j$ , then

$$\limsup E_j = \bigcap_{m \geq 1} \bigcup_{j \geq m} E_j = \bigcap_{m \geq 1} G_m$$

Similarly,  $G_m$  is a decreasing sequence, and since  $\mu(\bigcup E_{j \geq 1}) = \mu(G_1)$  is finite, we can use continuity from above in the same manner, and the proof is complete. ■

**Exercise 1.9****WTS 27.1.***Proof.*

**Exercise 1.10****WTS 28.1.***Proof.*



**Exercise 1.11****WTS 29.1.***Proof.*

## Exercise 1.12

**WTS 30.1.** Let  $(X, \mathcal{M}, \mu)$  be a finite measure space,

- If  $E, F \in \mathcal{M}$ , and  $\mu(E \Delta F) = 0$ , then  $\mu(E) = \mu(F)$ ,
- Say that  $E \sim F$  if  $\mu(E \Delta F) = 0$ , then  $\sim$  is an equivalence relation on  $\mathcal{M}$ ,
- For every  $E, F \in \mathcal{M}$ , define  $\rho(E, F) = \mu(E \Delta F)$ . Show that  $\rho$  defines a metric on the space of  $\mathcal{M}/\sim$  equivalence classes.

*Proof of Part A.* Use the fact that  $\mu(F) = \mu(E \cap F) + \mu(F \cap E^c)$ , and by monotonicity,

$$\mu(F \cap E^c) \leq \mu(E \Delta F) = 0$$

And  $\mu(F) = \mu(E \cap F) = \mu(E)$ , the last equality follows after a simple modification. ■

*Proof of Part B.* Suppose that  $\mu(E \Delta F) = \mu(F \Delta G) = 0$ , then

- $\mu(E \cap F^c) = \mu(F \cap E^c) \leq \mu(E \Delta F) = 0$  by monotonicity,
- Similarly, we have  $\mu(F \cap G^c) = \mu(G \cap F^c) = 0$ , and
- By subadditivity,
  - $\mu(E \cap G^c) = \mu(E \cap F^c \cap G^c) + \mu(E \cap F \cap G^c) \leq 0$ , and  $\mu(E \cap G^c) = 0$ ,  
and
  - $\mu(G \cap E^c) = 0$
- Therefore  $\mu(E \Delta G) = \mu(E \cap G^c) + \mu(G \cap E^c) = 0$

It is clear that the relation is reflexive, since  $E \Delta E = \emptyset$ , and symmetry is trivial. ■

*Proof of Part C.* Since  $\rho(E, F) = \rho(F, E)$ , and  $\rho(E, F) \geq 0$  for every  $E, F \in \mathcal{M}$ , and  $\rho(E, F) = 0 \iff E \sim F$ . We only have to prove the Triangle Inequality. Notice that

$$\begin{aligned} \mu(E \setminus F) &= \mu(E \cap F^c \cap G) + \mu(E \cap F^c \cap G^c) \\ &\leq \mu(F^c \cap G) + \mu(E \cap F^c) \end{aligned}$$

and in the same fashion,

$$\mu(F \setminus E) \leq \mu(F \cap G^c) + \mu(E^c \cap F)$$

Combining the two inequalities, and applying additivity finishes the proof. ■

**Exercise 1.13****WTS 31.1.** *Every  $\sigma$ -finite measure is semi-finite*

*Proof.* Suppose  $\mu$  is  $\sigma$ -finite then there exists an increasing sequence of sets  $E_j \nearrow X$  with  $\mu(E_j) < +\infty$ . Now for every  $W \in \mathcal{M}$ , if  $\mu(W) = +\infty$  then  $\mu(W) = \lim_{j \rightarrow \infty} \mu(E_j \cap W) = +\infty$ . Since this real-valued limit converges to its supremum  $+\infty$ , there exists a non-null subset  $E_j \cap W$  of positive and finite measure. ■

**Exercise 1.14**

**WTS 32.1.** *If  $\mu$  is a semi-finite measure, and if  $\mu(E) = +\infty$ , for every  $C > 0$ , there exists an  $F \subseteq E$  with  $0 < \mu(F) < +\infty$ .*

*Proof.* Suppose by contradiction that there exists a  $C > 0$  so for every  $F \subseteq E$ , if  $F$  is of finite measure, then  $0 \leq \mu(F) \leq C$ . Let  $s = \sup\{\mu(F), F \subseteq E, 0 < \mu(F) < +\infty\}$ , and for any  $n^{-1} > 0$ , this induces a  $F_n$  with measure

$$\mu(F_n) > s - n^{-1}$$

and take  $A_n = \bigcup_{j \leq n} F_j$ . A simple induction will show that  $\mu(A_n) \leq \sum_{j \leq n} \mu(F_j) < +\infty$ , therefore  $\mu(A_n) \leq s$  for every  $n \geq 1$ . By continuity from below

$$\lim_{n \rightarrow \infty} \mu(A_n) = \mu\left(\bigcup_{j \geq 1} F_j\right) \leq s$$

Next, by monotonicity, denoting the union over  $A_n$  by  $A$ , for every  $n^{-1} > 0$

$$s - n^{-1} \leq \mu(A_n) \leq \mu(A) \leq s \implies \mu(A) = s$$

Now,  $E \setminus A$  is a set of infinite measure, and by semi-finiteness. Find a set  $B \subseteq E \setminus A$  with strictly positive measure, so that

$$\mu(A \cup B) = \mu(A) + \mu(B) > s$$

And this finishes the proof. ■

### Exercise 1.15

**WTS 33.1.** Given a measure  $\mu$  on  $(X, \mathcal{M})$ , and define  $\mu_0 = \sup\{\mu(F), F \subseteq E, \mu(F) < +\infty\}$ . Show  $\mu_0$  is semi-finite. Then, show that if  $\mu$  is semi-finite,  $\mu = \mu_0$ . Lastly, there exists a measure  $\nu$  on  $(X, \mathcal{M})$ , with  $\mu = \nu + \mu_0$ , where  $\nu$  only assumes the values 0 or  $+\infty$ .

*Proof.* First, a small Lemma. We claim that  $\mu_0 = \mu$  on finite sets. Let  $E \in \mathcal{M}$ , and  $\mu(E) < +\infty$ , since

$$\mu(E) \in \{\mu(F), F \subseteq E, \mu(F) < +\infty\} \implies \mu(E) \leq \mu_0(E)$$

Next, for every  $W \subseteq E$ ,  $\mu(W) \leq \mu(E)$ , so  $\mu_0(E) \leq \mu(E)$ . This proves the equality.

If  $E$  is any measurable subset of  $X$ , and suppose also  $\mu_0(E) = +\infty$ , one can easily find subsets of  $E$ ,  $\{E_n\}_{n \geq 1}$  with

$$n \geq \mu(E_n) < +\infty$$

But  $E_n$  is a subset of finite measure, so  $0 < \mu(E_n) = \mu_0(E_n) < +\infty$ . This proves the semi-finiteness of  $\mu_0$ .

Next, suppose  $\mu$  is semi-finite, and fix any measurable set  $E$ . If  $E$  is of finite measure, then  $\mu(E) = \mu_0(E)$ , and if  $\mu(E) = +\infty$ , apply Exercise 14, so there exists a sequence of subsets of finite measure  $E_n \subseteq E$  for every  $n \geq 1$ , with  $\mu(E_n) \rightarrow \mu(E)$ . Therefore  $\mu_0(E) = \mu(E)$ .

For the last part of the proof, let  $\mu$  be an arbitrary measure. And let  $E \in \mathcal{M}$ . If  $\mu(E) < +\infty$ , then  $\nu(E) = 0$  would suffice (this proves the first property of the measure). If  $\mu(E) = +\infty$ , and if  $\mu(E)$  is not semi-finite, then set  $\nu(E) = +\infty$ . So that  $\mu_0(E) + \nu(E) = 0 + \infty = \infty = \mu(E)$ . The additivity of  $\nu$  is immediate, since  $\nu$  can only assume two values. This finishes the proof. ■

**Exercise 1.16****WTS 34.1.***Proof.*

### Exercise 1.17

**WTS 35.1.** Let  $\{A_j\}_{j \geq 1}$  be a countable disjoint sequence in  $\mathcal{M}$ , and denote  $B_n = \bigcup A_{j \leq n} \in \mathcal{M}$ . For every  $E \subseteq X$ ,

$$\mu^*(E \cap B_n) = \sum \mu^*(E \cap A_{j \leq n})$$

*Proof.* Proven in Theorem 1.11 as a Lemma. ■



**Exercise 1.18**

**WTS 36.1.** Let  $\mathcal{A} \subseteq \mathbb{P}(\mathbf{X})$  be an algebra.  $\mathcal{A}_\sigma$  the collection of countable unions of sets in  $\mathcal{A}$ , and  $\mathcal{A}_{\sigma\delta}$  the collection of countable intersection of sets in  $\mathcal{A}_\sigma$ . Let  $\mu_0$  be a premeasure on  $\mathcal{A}$ , and  $\mu^*$  be the induced outer-measure.

- (a) For any  $E \subseteq \mathbf{X}$ , and  $\varepsilon > 0$ , there exists  $A \in \mathcal{A}_\sigma$  with  $E \subseteq A$  and  $\mu^*(A) \leq \mu^*(E) + \varepsilon$ .
- (b) If  $\mu^*(E) < +\infty$ , then  $E$  is  $\mu^*$ -measurable  $\iff$  there exists  $B \in \mathcal{A}_{\sigma\delta}$  with  $E \subseteq B$  and  $\mu^*(B \setminus E) = 0$ .
- (c) If  $\mu_0$  is  $\sigma$ -finite, the restriction  $\mu^*(E) < +\infty$  in (b) is superfluous.

*Proof of Part A.* Let  $E \subseteq \mathbf{X}$  and  $\varepsilon > 0$ , then by definition of  $\mu^*$ ,

$$\mu^*(E) + \varepsilon \geq \sum \mu_0(A_j) = \sum \mu^*(A_j) \geq \mu^*(A)$$

by subadditivity and  $A = \bigcup A_j$ . ■

*Proof of Part B.* Suppose  $E$  is outer-measurable and of finite outermeasure, then by part A we have a sequence of  $A_n \in \mathcal{A}_\sigma$  with

$$\mu^*(E) + n^{-1} \geq \mu^*(A_n) \implies \mu^*(E) = \mu^*(A)$$

if we define  $A = \bigcap A_n \supseteq E$ . Using the  $\mu^*$ -measurability of  $E$ , we get

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E) < +\infty \implies \mu^*(A \setminus E) = 0$$

Conversely, if  $\mu^*(A \setminus E) = 0$ , for any  $V \subseteq \mathbf{X}$ , with  $\mu^*(V) < +\infty$ , we have

$$\begin{aligned} \mu^*(V) &= \mu^*(V \cap A) + \mu^*(V \setminus A) \\ &\geq \mu^*(V \cap E) + \mu^*(V \setminus A) + \mu^*(V \cap [A \setminus E]) \\ &\geq \mu^*(V \cap E) + \mu^*(V \setminus E) \end{aligned}$$
■

*Proof of Part C.* Suppose  $\mu_0$  is  $\sigma$ -finite, then  $E \in \mathcal{M}^*$  induces a sequence  $E_j \nearrow E$ , where each  $E_j$  is of finite measure. By part b) we obtain  $\{A_j\} \subseteq \mathcal{A}_{\sigma\delta}$  with

$$\mu^*(A_j \setminus E_j) = 0$$

Now define  $B = \bigcup A_j$ , so that  $B \in \mathcal{A}_{\sigma\delta}$ . Observe  $\bigcup (A_j \setminus E_j) = B \setminus E_1 \supseteq B \setminus E$  (verify these). And  $\mu^*(B \setminus E) \leq \sum \mu^*(A_j \setminus E_j) = 0$  by subadditivity. Since  $B \supseteq E$ , and  $B \in \mathcal{A}_{\sigma\delta}$ , this proves  $\implies$ .

Conversely, suppose  $E \subseteq X$  and there exists a  $B \in \mathcal{A}_{\sigma\delta}$  with  $E \subseteq B$ ,  $\mu^*(B \setminus E) = 0$ . Let  $\{X_j\} \nearrow X$  as a sequence of sets of finite measure. Then,

$$(X_j \cap B) \setminus (X_j \cap E) = X_j \cap (B \setminus E) \subseteq B \setminus E$$

$X_j \cap B \in \mathcal{A}_{\sigma\delta}$ , and  $X_j \cap B \supseteq (X_j \cap E)$ . Each  $E_j = X_j \cap E$  is  $\mu^*$  measurable by monotonicity, so is their countable union. ■

**Exercise 1.19**

**WTS 37.1.** Let  $\mu^*$  be an outer measure on  $\mathbf{X}$  induced from a finite pre-measure  $\mu_0$ . If  $E \subseteq \mathbf{X}$ , define the inner measure of  $E$  to be  $\mu_*(E) = \mu_0(\mathbf{X}) - \mu^*(E^c)$ . Then  $E$  is  $\mu^*$ -measurable iff  $\mu^*(E) = \mu_*(E)$ .

*Proof.* Suppose  $E \subseteq \mathbf{X}$  is  $\mu^*$ -measurable. Then

$$\mu^*(\mathbf{X}) = \mu^*(\mathbf{X} \cap E) + \mu^*(\mathbf{X} \setminus E) = \mu_0(\mathbf{X})$$

Rearranging gives the result, since all quantities are finite.

If  $\mu^*(E) = \mu_*(E)$ , then  $\mu^*(E^c) = \mu_*(E^c)$ , since the definition of  $\mu_*$  is symmetric. Let  $B \in \mathcal{A}_{\sigma\delta}$ , with  $\mu^*(B) = \mu^*(E)$ ,  $E \subseteq B$ . We can always find such a  $B$  by taking the intersection over all  $B_n \in \mathcal{A}_\sigma$ ,

$$\mu^*(E) + n^{-1} \geq \sum_j \mu^*(B(j, n)) \geq \mu^*\left(\bigcup_j B(j, n) = B_n\right)$$

Notice  $E \subseteq B \iff E^c \supseteq B^c \iff E^c \cap B^c = B^c$ . Since  $B$  is  $\mu^*$ -measurable, we have

$$\begin{aligned} \mu^*(E^c \cap B) + \mu^*(E^c \setminus B) &= \mu^*(E^c) \\ &= \mu^*(\mathbf{X}) - \mu^*(E) \\ \mu^*(B \setminus E) + \mu^*(B^c) &= \mu^*(\mathbf{X}) - \mu^*(E) \\ &= \mu^*(B) + \mu^*(B^c) - \mu^*(E) \\ \mu^*(B \setminus E) &= \mu^*(B) - \mu^*(E) \\ &= 0 \end{aligned}$$

■

**Exercise 1.20****WTS 38.1.***Proof.*

**Exercise 1.21**

**WTS 39.1.** Let  $\mu^*$  be an outermeasure induced from a premeasure, and  $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$ , where  $\mathcal{M}^*$  denotes the family of  $\mu^*$ -measurable sets. Show that  $\bar{\mu}$  is saturated. That is,  $\widetilde{\mathcal{M}^*} = \mathcal{M}^*$

*Proof.* Suppose  $E$  is locally measurable (with respect to  $\bar{\mu} = \mu^*|_{\mathcal{M}^*}$ ). Fix  $V \subseteq \mathbf{X}$ , with  $\mu^*(V) < +\infty$ . It suffices to show  $\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$ .

By 18a), find a  $V' \in \mathcal{A}_{\sigma\delta}$ , with  $V \subseteq V'$ , and  $\mu^*(V') = \mu^*(V) < +\infty$ . so that  $E \cap V'$  is  $\mu^*$ -measurable.

$$\mu^*(V) = \mu^*(V \cap E \cap V') + \mu^*(V \setminus (V \cap (V' \cap E)))$$

therefore

$$\mu^*(V) = \mu^*(V \cap E) + \mu^*(V \setminus E)$$

■

## Exercise 1.22

**WTS 40.1.**

*Proof.* To show  $\bar{\mu}$  is complete, Fix  $U \subseteq F$ , where  $F \in \mathcal{M}^*$ , with  $\bar{\mu}(F) = 0$ . Let  $F' \in \mathcal{A}_{\sigma\delta}$ , with  $F' \supseteq F$ , and

$$\mu^*(F') = \mu^*(F) \geq \mu^*(F' \setminus U)$$

Since  $F' \supseteq U$ , applying Exercise 18b gives  $\overline{\mathcal{M}^*} \subseteq \mathcal{M}^*$ . For the other direction,

■

**Exercise 1.23****WTS 41.1.***Proof.*

**Exercise 1.24**

**WTS 42.1.** If  $\mu$  is a finite measure on  $(X, \mathcal{M})$ , and let  $\mu^*$  be the outer measure. Suppose that  $E \subseteq X$  satisfies  $\mu^*(E) = \mu^*(X)$  (but  $E \notin \mathcal{M}$  necessarily). Show that

- (a) For any  $A, B \in \mathcal{M}$ , and  $A \cap E = B \cap E$ , then  $\mu(A) = \mu(B)$ .
- (b) Let  $\mathcal{M}_E = \{A \cap E, A \in \mathcal{M}\}$ , and define  $\nu$  on  $\mathcal{M}$  with  $\nu(A \cap E) = \mu(A)$ . Then  $\mathcal{M}_E$  is a  $\sigma$ -algebra, and  $\nu$  is a measure on  $\mathcal{M}_E$ .

*Proof of Part A.*

$$\mu^*(E) = \mu^*(X) \implies \mu^*(X \setminus E) = 0$$

This is a simple consequence of the  $\mu^*$ -measurability of  $X$ , since  $X \in \mathcal{M}$ , and the  $\mu$  is a pre-measure on  $\mathcal{M}$ , and by monotonicity,

$$\begin{cases} A \cap (X \setminus E) \subseteq (X \setminus E) \\ B \cap (X \setminus E) \subseteq (X \setminus E) \end{cases} \implies \begin{cases} \mu^*(A \cap (X \setminus E)) = 0 \\ \mu^*(B \cap (X \setminus E)) = 0 \end{cases}$$

Write  $A \cap X = (A \cap E) \cup (A \cap X \setminus E)$ , and by subadditivity of  $\mu^*$ ,

$$\begin{aligned} \mu(A) &= \mu^*(A \cap X) \\ &\leq \mu^*(A \cap E) + \mu^*(X \setminus E) \\ &= \mu^*(B \cap E) \\ &\leq \mu^*(B \cap X) \\ &= \mu(B) \end{aligned}$$

Therefore  $\mu(A) \leq \mu(B)$ , and  $\mu(B) \leq \mu(A)$  is trivial. ■

*Proof of Part B.* We want to show  $\mathcal{M}_E$  is a  $\sigma$ -algebra.

- Closure under complements,

$$\forall A \cap E \in \mathcal{M}_E, A \in \mathcal{M} \implies (E \setminus A^c) \in \mathcal{M}_E$$

Therefore  $(E \setminus A^c) \cap E \in \mathcal{M}_E$ . Note that the question mentions that  $\mathcal{M}_E$  is a  $\sigma$ -algebra on  $E$ , therefore we take complements relative to  $E$ .



- Closure under countable unions. Fix any countable sequence  $\{A_j \cap E\} \subseteq \mathcal{M}_E$  where  $\{A_j\} \subseteq \mathcal{M}$ . It is obvious that  $A = \cup A_j \in \mathcal{M}$ , therefore  $\cup(A_j \cap E) = E \cap A \in \mathcal{M}_E$  as well.

Since  $\nu(\emptyset) = \mu(\emptyset \cap E) = 0$ , and for countable additivity, fix any disjoint sequence  $\{A_j \cap E\}_{j \geq 1} \subseteq \mathcal{M}_E$ , where  $\{A_j\}_{j \geq 1} \subseteq \mathcal{M}$ , and let  $A = \cup A_{j \geq 1}$

$$\begin{aligned} \nu(A \cap E) &= \mu(A) \\ &= \sum \mu(A_{j \geq 1}) \\ &= \sum \nu(A_{j \geq 1} \cap E) \end{aligned}$$

■

## Chapter 2

**Theorem 2.1**

**WTS 1.1.**

*Proof.*



## Chapter 3

## Notes on Chapter 3

**WTS 0.2.** *Prove two things,*

1.  $\limsup_{r \rightarrow R} \phi(r) = \lim_{\varepsilon \rightarrow 0} \sup_{0 < |r-R| < \varepsilon} \phi(r) = \inf_{\varepsilon > 0} \sup_{0 < |r-R| < \varepsilon} \phi(r),$
2.  $\lim_{r \rightarrow R} \phi(r) = c \iff \limsup_{r \rightarrow R} |\phi(r) - c| = 0$

*Proof.* ■

**WTS 0.3.** *If  $U \subseteq B(1, 0) = \{|x| < 1\}$ , and  $U \in \mathbb{B}$ , and if  $m(U) > 0$ , then the family of sets*

$$E_r = \left\{ x + ry, y \in U \right\}$$

*shrinks nicely to  $x \in \mathbb{R}^n$ .*

*Proof.* Let  $r > 0$  be fixed then  $\forall z \in E_r \ni z = x + ry$ . Hence,

$$\begin{aligned} d(x, z) &= d(x, x + ry) \\ &= |r|d(0, y) < |r| \end{aligned}$$

by translation invariance. ■

**Theorem 3.1****WTS 1.1.**

*Proof.* Let  $\nu$  be a signed measure, and fix any increasing sequence  $E_j \nearrow E = \bigcup E_{j \geq 1}$  of sets. This induces a disjoint sequence in  $\{F_n\}$ . Define  $F_1 = E_1$ , and if  $n \geq 2$ ,

$$F_n = E_n \setminus \bigcup E_{j \leq n-1}$$

and from this, the finite It is clear that  $\bigcup F_{n \geq 1} = E$ , and let us assume  $\nu(E)$  is of finite measure.

By countable additivity, and the absolute convergence of the series  $\sum_{j \leq n} \nu(F_j)$

$$\begin{aligned} \nu\left(\bigcup E_{j \geq 1}\right) &= \sum_{j \geq 1} \nu(F_j) \\ &= \lim_n \sum_{j \leq n} \nu(F_j) \\ &= \lim \nu(E_n) \end{aligned}$$

■

**Theorem 3.2**

**WTS 2.1.**

*Proof.*





**Theorem 3.3**

**WTS 3.1.**

*Proof.*



**Theorem 3.4**

**WTS 4.1.**

*Proof.*



**Theorem 3.5**

**WTS 5.1.**

*Proof.*



**Theorem 3.6**

**WTS 6.1.**

*Proof.*



**Theorem 3.7**

**WTS 7.1.**

*Proof.*



**Theorem 3.8**

**WTS 8.1.**

*Proof.*



**Theorem 3.9**

**WTS 9.1.**

*Proof.*



**Theorem 3.10**

**WTS 10.1.**

*Proof.*





**Theorem 3.11**

**WTS 11.1.**

*Proof.*



**Theorem 3.12****WTS 12.1.***Proof.*

**Theorem 3.13****WTS 13.1.***Proof.*

**Theorem 3.14**

**WTS 14.1.**

*Proof.*



**Theorem 3.15****WTS 15.1.***Proof.*

**Theorem 3.16****WTS 16.1.***Proof.*

**Theorem 3.17**

**WTS 17.1.** *Let the maximal function of any measurable  $f \in \mathbb{B}_{\mathbb{R}^n}$  be denoted by  $Hf(x)$ , more precisely,*

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) dy$$

where  $A_r|f|$  is the average of  $|f|$  on a ball with radius  $r > 0$  centered at  $x \in \mathbb{R}^n$ . In symbols,

$$A_r|f| = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y) dy$$

The maximal theorem makes two claims:

1.  $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$ , and  $Hf$  is measurable for every  $f \in L^1_{loc}$ .
2. There exists a  $C > 0$ , for every  $f \in L^1$

$$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha} \|f\|_1$$

for every  $\alpha > 0$ .

*Proof.* Let  $\alpha > 0$  and fix  $z \in (Hf)^{-1}((\alpha, +\infty))$ , so  $Hf(z) > \alpha$  and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with  $Hf(z) - \alpha > 0$ , we get some  $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let  $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$ , it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some  $r_0 > 0$ . Since  $A_r|f|$  (a function indexed by  $r > 0$ ) is continuous in  $x \in \mathbb{R}^n$ ,  $(A_r|f|)^{-1}((\alpha, +\infty))$  is open, and  $Hf$  is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{ Hf > \alpha \right\} = \bigcup_{r>0} \{A_r |f| > \alpha\}$$

Let  $x \in E_\alpha$ , this induces a  $r_x > 0$  where  $x \in \left\{ A_{r_x} |f| > \alpha \right\}$ . Rearranging gives

$$\left( \frac{1}{\alpha} \int_{B(r,x)} |f| dz \right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each  $x \in E_\alpha \mapsto r_x > 0 \mapsto A_{r_x} |f|$ ,
- If  $U = \bigcup_{x \in E_\alpha} B(r_x, x)$ , then  $E_\alpha \subseteq U$ ,
- Choose  $c < m(E_\alpha) \leq m(U)$  (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by

$$x_1, \dots, x_N \in E_\alpha$$

so that  $\bigsqcup_{j \leq N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$ , and  $c < 3^n \sum_{j \leq k} m(B_j)$

- Define  $B_j = B(r_{x_j}, x_j)$  for all  $j \leq k$ , and

$$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$

by finite additivity,

$$c 3^{-n} < \sum_{j \leq k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \leq k} \int_{B_j} |f| dz$$

and finally

$$c < \frac{3^n}{\alpha} \sum_{j \leq k} \int_{B_j} |f| dz \leq \frac{3^n}{\alpha} \|f\|_1$$



- By inner regularity, of  $m$  on  $\mathbb{B}$ , since

$$m(E_\alpha) = \sup \left\{ m(K), K \in \mathcal{J}_{\mathbb{R}^n}, K \subseteq E_\alpha \right\}$$

for any  $K \in \mathcal{J}_{\mathbb{R}^n}$ ,  $K \subseteq E_\alpha$ , we have  $m(K) < +\infty$ ,  $m(K) \leq m(E_\alpha)$  and

$$m(K) = c < \frac{3^n}{\alpha} \|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha} \|f\|_1$$

**Remark 17.1** *We used the properties of a Radon Measure here, without relying on the phrase ‘sending  $c \rightarrow E_\alpha$ ’, which would require us to deal with two cases  $m(E_\alpha) < +\infty$  and  $m(E_\alpha) = +\infty$ .*

■

**Theorem 3.18****WTS 18.1.***Proof.*

**Theorem 3.19**

**WTS 19.1.**

*Proof.*



**Theorem 3.20****WTS 20.1.***Proof.*

**Theorem 3.21**

**WTS 21.1.** *The Lebesgue Differentiation Theorem. Suppose  $f \in L^1_{loc}$ , and for every  $x \in \mathcal{L}_f$ , (so that  $x \in \mathbb{R}^n$  a.e). We have*

1.  $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| dy = 0,$
2.  $\lim_{r \rightarrow 0} \frac{1}{m(E_r)} \int_{E_r} f(y) dy = f(x),$

For every family  $\{E_r\}_{r>0}$  that shrinks nicely to  $x \in \mathbb{R}^{n'}$ .

*Proof.* Since the family  $\{E_r\}_{r>0}$  shrinks nicely, we have

$$m(E_r) \gtrsim m(B(r, x)) \implies m(E_r) > \alpha \cdot m(B(r, x))$$

for some  $\alpha > 0$ , independent on  $r$ . Rearranging gives

$$m^{-1}(E_r) < \alpha^{-1} m^{-1}(B(r, x))$$

And monotonicity of the integral

$$\int_{E_r} |f(y) - f(x)| dy \leq \int_{B(r, x)} |f(y) - f(x)| dy$$

Combining the last two results, for every  $\varepsilon > 0$ , if  $0 < r < \varepsilon$ , then

$$m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

Taking the supremum on both sides,

$$\sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \leq \sup_{0 < r < \varepsilon} m^{-1}B(r, x) \int_{B(r, x)} |f(y) - f(x)| dy$$

and sending  $\varepsilon \rightarrow 0$ , proves the first claim. The second claim is immediate upon applying the  $L^1$  inequality.

Fix any  $\varepsilon > 0$ , and

$$\begin{aligned}
 \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} f(y) dy = f(x) &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} f(y) dy - f(x) \right| \\
 &\iff \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} \left| m^{-1}(E_r) \int_{E_r} [f(y) - f(x)] dy \right| \\
 &\leq \lim_{\varepsilon \rightarrow 0} \sup_{0 < r < \varepsilon} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
 &= \lim_{r \rightarrow 0} m^{-1}(E_r) \int_{E_r} |f(y) - f(x)| dy \\
 &= 0
 \end{aligned}$$

■

**Theorem 3.22****WTS 22.1.***Proof.*

**Theorem 3.23**

**WTS 23.1.**

*Proof.*





**Theorem 3.24**

**WTS 24.1.**

*Proof.*



**Theorem 3.25**

**WTS 25.1.**

*Proof.*



**Theorem 3.26**

**WTS 26.1.**

*Proof.*



**Theorem 3.27****WTS 27.1.***Proof.*

**Theorem 3.28****WTS 28.1.***Proof.*

**Theorem 3.29****WTS 29.1.***Proof.*

**Theorem 3.30****WTS 30.1.***Proof.*

**Theorem 3.31**

**WTS 31.1.**

*Proof.*





**Theorem 3.32**

**WTS 32.1.**

*Proof.*



**Theorem 3.33**

**WTS 33.1.**

*Proof.*



**Theorem 3.34**

**WTS 34.1.**

*Proof.*



**Theorem 3.35**

**WTS 35.1.**

*Proof.*



**Theorem 3.36****WTS 36.1.***Proof.*

## Chapter 4

## Notes on Chapter 4

### General Topology Proofs

**Definition 0.1**  $A^\circ$  is defined to be the largest open subset of  $A$ ,

$$A^\circ = \bigcup_{U \text{ open, } U \subseteq A} U$$

**Corollary 0.1** The union of subsets of  $A$  is again a subset of  $A$ , therefore Corollary 0.1 implies  $A^\circ \subseteq A$  for any  $A \subseteq X$ .

**Definition 0.2** and  $\bar{A}$  is the smallest closed superset of  $A$ ,

$$\bar{A} = \bigcap_{K \text{ closed, } A \subseteq K} K$$

**WTS 0.2.** The complement of the closure is the interior of the complement, or equivalently:  $(\bar{A})^c = A^{\circ c}$

*Proof.* Taking complements, and the substitution  $U = K^c$  reads

$$\begin{aligned} (\bar{A})^c &= \left( \bigcap_{K \text{ closed, } A \subseteq K} K \right)^c \\ &= \bigcup_{K \text{ closed, } K^c \subseteq A^c} K^c \\ &= \bigcup_{U \text{ open, } U \subseteq A^c} U \\ &= A^{\circ c} \end{aligned}$$

■

**Remark 0.1** Personally, I remember this as pushing the complement inside and flipping the bar to a  $c$ !

**Definition 0.3** A neighbourhood of  $x \in \mathbf{X}$  is a set  $U \subseteq \mathbf{X}$  where  $x \in U^\circ$ . The set of neighbourhoods for a point  $x \in \mathbf{X}$  will sometimes be denoted by  $\mathcal{N}(x)$ .

**WTS 0.3.** If  $W = \left\{x \in \mathbf{X}, \text{ there exists a neighbourhood } U \text{ of } x, U \subseteq A\right\}$ , then  $W = A^\circ$ .

*Proof.* If  $x \in A^\circ$ , then  $A$  is a neighbourhood of  $x$ , and  $A \subseteq A$ , so  $x \in W$ . Conversely, if  $x$  is a member of  $W$ , it has a neighbourhood  $U \subseteq A$  (not necessarily open). By monotonicity of the interior,

$$x \in U^\circ \subseteq A^\circ$$

and  $x \in A^\circ$ . ■

It is easy to see that  $A$  is open  $\iff A^\circ = A \iff A$  is a neighbourhood of itself.

- The first equivalence follows from:

$$E \subseteq \mathbf{X} \implies E^\circ \subseteq E$$

and if  $A$  is an open set, it is an open subset of itself, by Corollary 0.1  $A \subseteq A^\circ$ . If  $A^\circ = A$ , then it suffices to show that  $A^\circ$  is open. Which it is, since it is the arbitrary union of open sets.

- To prove the second equivalence: suppose  $A^\circ = A$ , then each  $x \in A$  has a neighbourhood contained (as a subset) in  $A$ , namely  $A$  itself. (This statement is hard to parse, the reader is encouraged to really work through this and be honest).

$$x \in A^\circ \subseteq A \implies A \subseteq A^\circ$$

so  $A$  is a neighbourhood of itself. Conversely, if  $A \subseteq A^\circ$ , then  $A = A^\circ$ , since the reverse inclusion follows immediately from Corollary 0.1.



We will now discuss the closure of a set.

**WTS 0.4.** *Let  $A \subseteq X$ , if  $W = \left\{ x \in X, \text{ every neighbourhood } U \text{ of } x, U \cap A \neq \emptyset \right\}$ , then  $\overline{A} = W$*

*Proof.* Suppose  $x \notin W$ , then there exists a neighbourhood  $U$  of  $x$  where

$$U \cap A = \emptyset \iff U \subseteq A^c$$

this is exactly the definition of the interior of  $A^c$ , so  $x \in A^{c\circ}$  and recall (from WTS 0.2) that  $(\overline{A})^c = A^{c\circ}$ , so  $x \notin \overline{A}$ . For the reverse inclusion, read the proof backwards, by flipping  $\forall \rightarrow \exists$  within the set, and we see that

$$W^c = A^{c\circ} = (\overline{A})^c$$

■

## Urysohn's Lemma Notes

Notes on the construction of the countable 'onion' sequence within a normal space  $\mathbf{X}$ .

If  $\mathbf{X}$  is a normal space, and  $A$  and  $B$  are disjoint closed subsets, then we can easily find an open  $U$  with

$$A \subseteq U \subseteq \overline{U} \subseteq B^c \quad (5)$$

We say that  $U$  hides in  $B^c$  if the closure of  $U$  is contained in  $B^c$ . Define  $\Delta_n = \left\{ k2^{-n}, 1 < k < 2^n \right\}$ , so that  $\Delta_n \subseteq (0, 1)$  for all  $n \geq 1$ . Notice

$$\Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

and the even indices for  $\Delta_{n+1}$  are contained in  $\Delta_n$ . Suppose  $\Delta_n$  is well defined, it suffices to choose the odd indices for  $\Delta_{n+1}$ . If  $r = j2^{-(n+1)}$ , where  $j$  is odd, then  $r$  sits in between precisely two elements in  $\Delta_n \cup \{0, 1\}$ . If  $r$  sits between an endpoint, then define  $\overline{U}_0 = A$ , and  $B^c = U_1$ . And denote the closest left and neighbours by  $s, t$  respectively. If  $s < r < t$ , it is clear that  $\overline{U}_s$  and  $U_t^c$  are disjoint closed sets.

Use the 'normal space' construction to obtain an superset of  $\overline{U}_s$  that hides in  $U_t$ , denote this open set by  $U_r$ , and similar to Equation (5)

$$\overline{U}_s \subseteq U_r \subseteq \overline{U}_r \subseteq U_t$$

Now that the construction of this sequence is complete, we wish to prove Urysohn's Lemma. Let  $A$  and  $B$  be disjoint closed sets. And define

$$f(x) = \inf \left\{ r \in \Delta \cup \{1\}, x \in U_r \right\}$$

where  $U_1 = \mathbf{X}$ . So that  $0 \leq f(x) \leq 1$  is immediate. If  $x \in A$ , then  $x$  is in all  $U_r$ , and by density of  $\Delta \subseteq (0, 1)$ , we have  $f(x) = 0$ . Conversely, if  $x \in B$  then  $x \notin U_r$  for all  $r \in \Delta$ , if  $E$  denotes the indices in  $\Delta$  where  $x \in U_s$  when  $s \in E$ ,

$$(-\infty, r) \subseteq E^c \iff E \subseteq [r, +\infty) \iff \inf(E) \geq r \quad (6)$$

Send  $r \rightarrow 1$  and  $f(x) = 1$ . Thus  $f|_A = 0$  and  $f|_B = 1$ .

To show continuity, it suffices to show that the inverse images of the open half  $\left\{ (x > \alpha), (x < \alpha) \right\}_{\alpha \in \mathbb{R}}$  lines are indeed open in  $\mathbf{X}$ . Let  $\alpha$  be fixed. And if  $x \in \{f < \alpha\}$ , we can 'wiggle' the infimum towards the right (towards  $\alpha$ ), and using density of  $\Delta$  within  $(0, 1)$ , there exists a  $r \in E$  that satisfies  $f(x) < r < \alpha$ . This is equivalent to

$$x \in \bigcup_{r < \alpha} U_r$$

If there exists an  $r < \alpha$  st  $x$  belongs to  $U_r$  as an element, then  $f(x) \leq r < \alpha$ .

If  $f(x) > \alpha$ , then  $(-\infty, \alpha) \subseteq E^c$ , by Equation (6). Suppose  $\alpha < 1$ , otherwise  $\{f > \alpha\} = \emptyset$ . Wiggle  $f(x)$  to the left and obtain an  $r \in \Delta$ ,  $\alpha < r < f(x)$  with  $x \notin U_r$ . By density again, take any  $s < r$  by a small amount (st  $s > \alpha$ ,  $s \in \Delta$ ), and

$$\overline{U}_s \subseteq U_r \iff U_r^c \subseteq \overline{U}_s$$

so that  $x \in \overline{U}_s^c$  for some  $s > \alpha$ . This is equivalent to

$$x \in \bigcup_{s > \alpha} \overline{U}_s^c$$

Conversely, if  $x \notin \overline{U}_s^c$  for some  $s > \alpha$ , since  $\{U_r\}$  (thus  $\{\overline{U}_r\}$ ) is increasing, and  $x \notin U_r$  for every  $r \leq s$ . Hence,

$$(-\infty, s] \subseteq E^c \iff E \subseteq (s, +\infty) \iff f(x) \geq s > \alpha$$

**Theorem 4.1**

**WTS 1.1.** Suppose that  $A$  is a subset of  $X$ , let  $\text{acc } A$  be the set of accumulation points of  $A$ , then

$$\overline{A} = A \cup \text{acc } (A) \quad (7)$$

and  $A$  is closed if and only if  $\text{acc } (A) \subseteq A$ .

*Proof.* Suppose that  $x \notin \overline{A}$ , then  $x \in (\overline{A})^c = A^{\circ}$ , then  $A^c \in \mathcal{N}_B(x)$ . But this means that  $x \notin \text{acc } (A)$ , since there exists a neighbourhood of  $x$  (in the form of  $A^c$ ), such that

$$A \cap A^c \setminus \{x\} = A \cap A^c = \emptyset$$

Also,  $A \subseteq \overline{A} \implies (\overline{A})^c \subseteq A^c$  which means that

$$x \notin \overline{A} \implies x \notin A$$

Since  $x \notin \overline{A} \implies x \notin A$  and  $x \notin \text{acc } (A)$ ,

$$(\overline{A})^c \subseteq A^c \cap \text{acc } (A)^c = (A \cup \text{acc } (A))^c$$

Now, if  $x \notin \text{acc } (A) \cup A$ , then  $x \notin \text{acc } (A)$ , therefore there exists some  $U \in \mathcal{N}_B(x)$  such that

$$A \cap U \setminus \{x\} = A \cap U = \emptyset$$

Where for the second last equality we used the fact that  $x \notin A \implies A \setminus \{x\} = A$ , and taking complements gives us

$$U \subseteq A^c$$

And since  $U \in \mathcal{N}_B(x)$ , then  $x \in U^{\circ} \subseteq A^{\circ}$  (since  $U^{\circ}$  is an open subset of  $A^c$ ). then

$$x \in A^{\circ} = (\overline{A})^c \implies x \notin (\overline{A})^c$$

Therefore  $(A \cup \text{acc } (A))^c \subseteq (\overline{A})^c$ . ■

**Theorem 4.2**

**WTS 2.1.** *If  $\mathcal{T}_X$  is a topology on  $X$  and  $\mathcal{E} \subseteq \mathcal{T}_X$  then  $\mathcal{E}$  is a base for  $\mathcal{T}_X$  if and only if for every*

$$\forall U \in \mathcal{T}_X, U \neq \emptyset, \implies U = \bigcup_{V \in B} V$$

Where  $B$  is a subset of  $\mathcal{E}$ .

*Proof.* Suppose that  $\mathcal{E}$  is a base, then fix any non-empty  $U \in \mathcal{T}_X$ , then for every  $x \in U$ , there exists a neighbourhood base for this  $x$  and a member  $V \in \mathcal{E}$  such that  $x \in V_x \subseteq U$ . Take the union over all  $V_x$  and

$$U \subseteq \bigcup_{x \in U} V_x$$

But each  $V_x \subseteq U$ , so  $U = \bigcup_{x \in U} V_x$ , where  $\{V_x\} \subseteq \mathcal{E}$ .

Conversely, if every non-empty  $U$  is a union of members in  $\mathcal{E}$  then fix any  $x \in X$ , we claim that we have a neighbourhood base in

$$\{V \in \mathcal{E}, x \in V\}$$

The reason is as follows

- $x$  belongs to every  $E \in \{V \in \mathcal{E}, x \in V\}$  and
- For every open  $U$ , if  $x \in U$  then there exists a union of members of  $\mathcal{E}$  such that  $U = \bigcup E_\alpha$ , then  $x \in U \iff \exists E_\alpha \in \{V \in \mathcal{E}, x \in V\}$  and
- Using this particular  $E_\alpha \in \mathcal{E}$  that we just found,  $x \in E_\alpha \subseteq U$ , and we are done.

■

**Theorem 4.3**

**WTS 3.1.** For every  $\mathcal{E} \subseteq \mathbb{P}(X)$ ,  $\mathcal{E}$  is base for a topology on  $X$  if and only if

- (a) each  $x \in X$  is contained in some  $V \in \mathcal{E}$ , and
- (b) if  $U, V \in \mathcal{E}$ , and  $x \in U \cap V$ , then there must exist some  $W \in \mathcal{E}$  with  $x \in W \subseteq U \cap V$ .

*Proof.* Suppose that  $\mathcal{E}$  is a base, then we get a), and b) follows since for every  $U, V \in \mathcal{E} \subseteq \mathcal{T}_X$ , and by closure over finite intersections,  $U \cap V \in \mathcal{T}_X$  implies that there exists some  $W \in \mathcal{E}$  with

$$x \in W \subseteq U \cap V$$

Now, suppose both a) and b) hold, then we claim that this  $\mathcal{E} \subseteq \mathbb{P}(X)$  induces a topology on  $X$

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, \text{ with } x \in V \subseteq U\}$$

Intuitively speaking, this means that  $\mathcal{T}$  is just fine (and not too fine) to satisfy the conditions for  $\mathcal{E} \subseteq \mathcal{T}$  to be a base of  $\mathcal{T}$ .

We first show that  $\mathcal{T}$  is a topology.

- $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$ , the first is trivial and the second is from a)
- Closure under unions: fix  $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T}$ , and  $U = \bigcup U_\alpha$ , and for every  $x \in U$  there exists some  $V_\alpha \in \mathcal{E}$  such that  $x \in V_\alpha \subseteq U_\alpha \subseteq U$ , therefore  $U \in \mathcal{T}$ .
- Closure under finite intersections, fix any  $U_1, U_2$  as elements in  $\mathcal{T}$ , then suppose that they are not disjoint (if they are disjoint then their intersection is the empty set, which is also contained in  $\mathcal{T}$ ). If  $U_1 \cap U_2 \neq \emptyset$ , then for every  $x \in U_1 \cap U_2$  induces two sets  $V_1, V_2 \in \mathcal{E}$  with  $x \in V_1 \subseteq U_1$  and  $x \in V_2 \subseteq U_2$ , taking their intersection and applying b) gives us some  $V \subseteq V_1 \cap V_2$  with  $V \in \mathcal{E}$  therefore  $x \in V \subseteq U_1 \cap U_2$ , and  $\mathcal{T}$  is closed under finite intersections.

Now to show that  $\mathcal{E}$  is a base for  $\mathcal{T}$ ,  $\mathcal{E} \subseteq \mathcal{T}$  is obvious since every  $V \in \mathcal{E}$  satisfies the properties laid out by  $\mathcal{T}$  by simply choosing  $V$  again for any

$x \in V$ . Now fix any member  $U \in \mathcal{T}$ , then for every  $x \in U$ , there exists some  $V \in \mathcal{E}$  with

$$x \in V \subseteq U$$

(This is an immediate consequence of how we defined  $\mathcal{T}$ ). And we can conclude that  $\mathcal{E}$  is a base for this induced topology  $\mathcal{T}$ . ■

**Theorem 4.4**

**WTS 4.1.** *If  $\mathcal{E} \subseteq \mathbb{P}(X)$ , the topology  $\mathcal{T}(\mathcal{E})$  generated by  $\mathcal{E}$  consists of  $\emptyset, X$  and all unions of finite intersections of  $\mathcal{E}$ , in symbols*

$$\mathcal{T}(\mathcal{E}) = \{\emptyset, X\} \cup \left\{ \bigcup W_\alpha, W_\alpha = \bigcap E_{j \leq n}, E_j \in \mathcal{E} \right\}$$

*Proof.* Denote the set

$$W = \{X\} \cup \left\{ \bigcap V_{j \leq n}, V_j \in \mathcal{E} \right\}$$

We claim this set  $W$  satisfies Theorem 4.3. Since 4.3a) is satisfied with  $X \in W$ . 4.3b) follows since the right member in  $W$  is closed under intersections.

And if we are taking an element from each member,  $E_1 \in \{\emptyset, X\}$  and  $E_2$  is an element in the right member, then it is trivial to verify that their intersection is always contained within  $W$ . Therefore  $W$  induces a topology by Theorem 4.2, and we call this topology  $\mathcal{T}$  — and for the sake of completeness

$$\mathcal{T} = \{U \subseteq X, \forall x \in U, \exists V \in \mathcal{E}, x \in V \subseteq U\}$$

We so claim that if we define  $\overline{W}$  as the union of all members  $w \in W$ , together with the empty set, is equal to the set  $\mathcal{T}$ .

$$\overline{W} = \left\{ \bigcup_{w \in W} w \right\} \cup \{\emptyset\}$$

- We want to show  $\mathcal{T} \subseteq \overline{W}$ , since  $W$  is a base for the topology  $\mathcal{T}$ , every (non-empty)  $U \in \mathcal{T}$  is the union of members in  $W$  (Theorem 4.2), and there exists some  $B \subseteq W$  with

$$U = \bigcup E_{\alpha \in B} \in \overline{W}$$

Now if  $U$  is the empty set then it is trivially contained within  $\overline{W}$ .

- Next, we show that  $\overline{W} \subseteq \mathcal{T}$ , fix any element  $E \in \overline{W}$ , if  $E = \emptyset$  then there is nothing to prove since  $\mathcal{T}$  is a topology. Now for every  $x \in E$ ,

$$x \in E = \bigcup_{w \in W} w \implies x \in w$$

Therefore  $E \in \mathcal{T}$  by definition. This proves that  $\mathcal{T} = \overline{W}$ .



Now that  $\overline{W}$  is a topology, that contains  $\mathcal{E}$  as a subset, and by definition of  $\mathcal{T}(\mathcal{E})$

$$\mathcal{T}(\mathcal{E}) = \bigcap \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Tells us

$$\mathcal{T}(\mathcal{E}) \subseteq \overline{W}, \quad \text{since } \overline{W} \in \{A, \text{ is a topology, and } \mathcal{E} \subseteq A\}$$

Conversely, fix any member  $E \in \overline{W}$ , if  $E = \emptyset$  then  $E \in \mathcal{T}(\mathcal{E})$ , if not, then there exists some subset  $B \subseteq W$  such that

$$E = \bigcup_{w \in B} w = \bigcup_{w \in B} \bigcap_{j \leq n} V_{j \leq n}^w V_j \in \mathcal{E} \cup \{X\}$$

Since  $\mathcal{T}(\mathcal{E})$  is closed under finite intersections and unions, and it contains  $\mathcal{E}$  as a subset,  $\overline{W} = \mathcal{T}(\mathcal{E})$  and we are done. ■

**Theorem 4.5**

**WTS 5.1.** *Every second countable space is separable. (Countable dense subset).*

*Proof.* What we wish to prove is that if a space  $X$  has a countable base, then it has a countable dense subset. Denote this base of  $X$  by  $\mathcal{E}$  as usual, then we claim that

$$W = \{x_u, U \in \mathcal{E}\}$$

Is a dense subset in  $X$ . Note that  $(\overline{W})^c = W^{\circ} \in \mathcal{T}_X$ . If  $W^{\circ} = \emptyset$  then we simply take complements and we get  $\overline{W} = X$ . So suppose that  $W^{\circ}$  is non-empty, then for each  $x \in W^{\circ}$  (by definition of a base), it should induce some  $V_x \in \mathcal{E}$  with

$$x \in V_x \subseteq W^{\circ}$$

But clearly, for every element in  $\mathcal{E}$ , the second estimate can never be satisfied, since for every  $U \in \mathcal{E}$ ,  $x_U \notin W^{\circ}$  for this particular set  $W^{\circ}$ . Therefore  $W^{\circ}$  must be empty, and this completes the proof. ■

**Theorem 4.6**

**WTS 6.1.** *If  $X$  is first countable, then for every  $A \subseteq X$ ,  $x \in \overline{A} \iff$  there exists some sequence  $\{x_j\}_{j \geq 1} \subseteq A$  such that  $x_j \rightarrow x$ .*

*Proof.* Suppose that  $X$  is first countable, and  $A \subseteq X$ , and fix any element  $x \in \overline{A}$ . Since  $X$  is first countable, there is a sequence of descending neighbourhoods of  $\{U_j\}_{j \geq 1}$  of  $x$  such that

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_j \supseteq U_{j+1}$$

If  $x \in A$ , take  $x_n = x$  for all  $n \geq 1$ . If  $x \in \text{acc}(A)$ , then take  $x_n \in U_n \cap A \setminus \{x\} = U_n \cap A$ , which is not empty. Then it remains to show that this sequence converges to  $x$ . Fix any neighbourhood  $U \in \mathcal{N}_B(x)$  then there exists some  $N$ , for every  $n \geq N$

$$x \in U^o \implies \exists N \in \mathbb{N}^+, x \in U_N \subseteq U^o$$

Then every  $x_n \in A \cap U_N \subseteq A \cap U^o \subseteq U^o$ . And this establishes  $\implies$ .

Now suppose that  $x \notin \overline{A}$ , so that  $x \notin A$  and  $x \notin \text{acc}(A)$ , then fix any sequence  $\{x_j\} \subseteq A$ . We wish to show that  $x_j \not\rightarrow x$ .

Since  $x \notin \text{acc}(A)$ , there exists some  $V \in \mathcal{N}_B(X)$  with

$$A \cap V \setminus \{x\} = \emptyset \implies V \subseteq A^c$$

Since  $\{x_j\}_{j \geq 1} \subseteq A \implies x_j \notin A^c$  for every  $j \geq 1$ , then choose  $V$  as the neighbourhood around  $x$ , and  $x_j \not\rightarrow x$  for any arbitrary sequence  $x_j$  in  $A$ . ■

**Remark 6.1** *To truly understand what is going on one should recall that all metric space spaces are first countable.*

**Theorem 4.7**

**WTS 7.1.**  $X$  is a  $T_1$  space  $\iff \{x\}$  is closed for every  $x \in X$ .

*Proof.* If  $X$  is  $T_1$  and  $x \in X$ , then for every  $y \neq x$  there exists some open  $U_y$  that contains  $y$  but not  $x$ . Following Folland's argument closely, every  $y \neq x$  is in  $\cup U_{y \neq x}$ . Hence  $\{x\}^c \subseteq \cup U_{y \neq x}$ . To show the converse, for every  $z \in \cup U_{y \neq x}$  that is open, there exists a  $y \neq x$  such that  $z \in U_y$ . But every  $U_y$  does not contain  $x$  as an element, so  $z \neq x$  implies that  $z \notin \{x\}$ . And  $z \in \{x\}^c$ . Hence  $\cup U_{y \neq x} = \{x\}^c$ .

Now conversely if every  $x \in X$  satisfies the fact that  $\{x\}^c$  is open, then  $\{x\}^c$  is an open set that contains every  $y \neq x$ . Now fix some  $y \neq x$ , since  $\{y\}$  is also closed, we have  $X \cap \{x\}^c$  is an open set that contains  $x$  but not  $y$ . Also,  $\{x\}^c$  is an open set that contains  $y$  but not  $x$ . And therefore  $X$  is  $T_1$ . ■

**Theorem 4.8**

**WTS 8.1.** *The map  $f : X \rightarrow Y$  is continuous if and only if at  $f$  is continuous at every  $x \in X$ .*

*Proof.* Suppose that  $f$  is continuous, then fix any  $f(x) \in Y$  and any of its neighbourhood  $V \in \mathcal{N}_B(f(x))$ ,

$$f(x) \in V^o \implies f^{-1}(V^o) \in \mathcal{N}_B(x)$$

But by continuity,  $f^{-1}(V^o)$  is an open set that contains  $x$ , with

$$f\left(f^{-1}(V^o)\right) \subseteq V^o$$

Therefore  $f$  is continuous at  $x$ . Now suppose that  $f$  is continuous at every  $x \in X$ , then for every open subset  $V \subseteq Y$ , and for every point  $f(x) \in V = V^o$  means that  $V \in \mathcal{N}_B(f(x))$  for all such points  $f(x)$ . By continuity, for every  $x$  in  $f^{-1}(V)$ , implies that  $f^{-1}(V)$  is a neighbourhood of all of its elements, therefore  $f^{-1}(V) \subseteq (f^{-1}(V))^o$ , and  $f^{-1}(V)$  is open. ■

**Theorem 4.9**

**WTS 9.1.** *If  $\mathcal{E}_Y$  generates the topology on  $Y$ , and  $f$  is a mapping from  $X \rightarrow Y$ , then  $f : X \rightarrow Y$  is continuous if and only if  $f^{-1}(V) \in \mathcal{T}_X$  for every  $V \in \mathcal{E}_Y$ .*

*Proof.* The inverse image commutes with intersections, complements, and unions. To prove  $\Leftarrow$ , use Theorem 4.4, since every  $U \in \mathcal{T}_Y$  can be represented the union of finite intersections of elements  $\mathcal{E}_Y$ , and use the fact that  $\mathcal{T}_X$  is closed under arbitrary unions and finite intersections.

To show  $\Rightarrow$ , since  $\mathcal{E}_Y \subseteq \mathcal{T}_Y$ , if  $f^{-1}$  is open for every  $U \in \mathcal{T}_Y$ , then it is open for every  $U \in \mathcal{E}_Y$  as well. ■

**Theorem 4.10**

**WTS 10.1.** *If  $X_\alpha$  is Hausdorff for each  $\alpha \in A$ , then  $X = \prod_{\alpha \in A} X_\alpha$  is Hausdorff.*

*Proof.* If two elements in  $X$ ,  $x \neq y$  then there exists some  $\alpha \in A$  such that  $\pi_\alpha(x) \neq \pi_\alpha(y) \in X_\alpha$ , but this  $X_\alpha$  is Hausdorff, then there exists two open, disjoint sets  $V_x, V_y \subseteq X_\alpha$  such that

- $x \in \pi_\alpha^{-1}(V_x)$ , and  $y \in \pi_\alpha^{-1}(V_y)$
- $\pi_\alpha^{-1}(V_x) \cap \pi_\alpha^{-1}(V_y) = \pi_\alpha^{-1}(V_x \cap V_y) = \emptyset$
- $\pi_\alpha^{-1}(V_x), \pi_\alpha^{-1}(V_y) \in \mathcal{T}_X$

Where for the last bullet point we used the fact that the product topology makes all the projection maps continuous. This proves that  $X$  is Hausdorff. ■

**Theorem 4.11**

**WTS 11.1.** *If  $X_\alpha$  and  $Y$  are topological spaces, and  $X = \prod_{\alpha \in A} X_\alpha$ , and  $f : Y \rightarrow X$  is a mapping. Then  $f$  is continuous if and only if  $\pi_\alpha \circ f$  is continuous for each  $\alpha \in A$ .*

*Proof.* If  $\pi_\alpha \circ f$  is continuous at each  $\alpha$ , this means that

$$\forall \alpha \in A, \forall E_\alpha \in \mathcal{T}_\alpha, f^{-1}(\pi_\alpha^{-1}(E_\alpha)) \in \mathcal{T}_Y$$

But it is exactly sets of the form  $\pi_\alpha^{-1}(E_\alpha)$  which generate the weak topology for  $\mathcal{T}_X$ . Therefore  $f$  is continuous.

Now, suppose that  $f$  is continuous, by definition of the weak topology (as it is generated by the set of inverse projections), for every  $\alpha \in A$ ,  $\pi_\alpha^{-1}(E_\alpha) \in \mathcal{T}_X$  and by continuity of  $f$ , its inverse image is open in  $Y$  as well. ■

**Remark 11.1** *The take-away intuition here is that if the range space is generated by some  $\mathcal{E}$ , then a function is continuous if and only if all inverse images of sets in  $\mathcal{E}$  are open in the domain space. Furthermore, if the range space is endowed with the product topology (which is generated by sets of the form  $\pi_\alpha^{-1}(E_\alpha)$ , where  $E_\alpha \in \mathcal{T}_\alpha$ ), then it suffices to check all inverse images of those. And this is equivalent to checking that  $\pi_\alpha(\cdot) \circ f$  is continuous at each  $\alpha$ .*



**Theorem 4.12**

**WTS 12.1.** *If  $X$  is a topological space, and  $A$  is any non-empty set,  $\{f_n\} \subseteq X^A$  is a sequence, then  $f_n \rightarrow f$  with respect to the product topology if and only if  $f_n \rightarrow f$  pointwise.*

*Proof.* Suppose that  $f_n \rightarrow f$  pointwise. Since the product topology  $\mathcal{T}_X$  is generated from sets of the form

$$\pi_\alpha^{-1}(E_\alpha), \quad E_\alpha \in \mathcal{T}_\alpha$$

And by Theorem 4.4,  $\mathcal{T}_X$  consists of  $\emptyset, X$  and unions of finite intersections of  $\pi_\alpha^{-1}(E_\alpha)$ . We claim that for every  $f \in X^A$ , the following is a valid neighbourhood base for  $f$

$$\left\{ \bigcap_{j \leq n} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j} \cap \mathcal{N}_B(\pi_{\alpha_j}(f)) \right\}$$

A couple things to note

- Each  $E_{\alpha_j}$  is open in  $X_{\alpha_j}$ , so that its inverse image is also open (in  $X$ ). Since any neighbourhood base has to be a subset of  $\mathcal{T}_X$ .
- Only finitely many intersections are involved, so each element in the above set is open in  $X$ .
- Each  $E_{\alpha_j}$  is a neighbourhood of  $\pi_{\alpha_j}(f)$ , meaning  $f \in E_{\alpha_j}^\circ = E_{\alpha_j}$ .
- Last and perhaps most importantly for intuition, fix any non-empty open set  $U \in \mathcal{T}_X$  then by Theorem 4.4 (or my reading of it),  $U$  can be written as the union of sets like

$$\bigcap_{j \leq m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}$$

Then applying Theorem 4.2, the family of finite intersections of  $\pi_\alpha^{-1}(E_\alpha)$  is a base for  $\mathcal{T}_X$ . Then,

$$N_{base}(f) = \left\{ V = \bigcap_{j \leq m} \pi_{\alpha_j}^{-1}(E_{\alpha_j}), \quad E_{\alpha_j} \in \mathcal{T}_{\alpha_j}, \quad f \in V \right\}$$

Has to be a neighbourhood base for any  $f \in X$ .

Now to show that  $f_n \rightarrow f$  in the product topology, fix any neighbourhood  $U \in \mathcal{N}_B(f)$ , then  $f \in U^o$ , and by definition of a neighbourhood base, there exists some  $E \in N_{base}(f)$  such that  $f \in E \subseteq U^o$ , but this  $E$  is just the finite intersection of  $\pi_{\alpha_j}^{-1}(E_{\alpha_j})$ , then at every  $\alpha_j$

- Let  $N_j$  be an integer such that for every  $n \geq N_j$ ,  $\pi_{\alpha_j}(f_n) \in E_{\alpha_j}$
- Set  $N = \sum_{j \leq m} N_j \geq N_j$  for every  $j \leq m$ .

Then for every  $n \geq N$ ,  $f_n \in E \subseteq U^o \subseteq U$  for any arbitrary neighbourhood  $U$  of  $f$ . So  $f_n \rightarrow f$  in the product topology.

Conversely, suppose that  $f_n \rightarrow f$  in the product topology, then fix any  $\alpha \in A$ , and for every neighbourhood  $E_\alpha$  of  $\pi_\alpha(f)$ ,  $\pi_\alpha^{-1}(E_\alpha)$  is a neighbourhood of  $f$ . Hence for every  $\alpha \in A$ , and for every neighbourhood  $E_\alpha$  of  $\pi_\alpha(f)$ ,  $\pi_\alpha(f_n)$  is eventually in  $E_\alpha$ . This completes the proof. ■

**Theorem 4.13**

**WTS 13.1.** *If  $X$  is a topological space then  $BC(X)$  is a closed subspace of  $B(X)$  in the uniform metric, and  $BC(X)$  is complete.*

*Proof.* Suppose that  $\{f_n\} \subseteq BC(X)$  converges to some  $f$ . There are a couple things that we need to show prior to tackling the main proof.

- (a)  $B(X)$  endowed with the uniform norm of an  $f \in B(X)$

$$\|f\|_u = \sup\{|f(x)|, x \in X\}$$

Is indeed a normed vector space.

- (b)  $B(X)$  with its norm (and induced metric), is a complete metric space.  
So that our  $\{f_n\} \rightarrow f$  at worst, converges to  $f \in B(X)$ .

To show that  $B(X)$  is a normed vector space, for any  $k \in \mathbb{C}$ ,  $f_1, f_2 \in B(X)$ , then at every  $x \in X$

$$|f_1(x) + kf_2(x)| \leq |f_1(x)| + |k| \cdot |f_2(x)| \leq \|f_1\|_u + |k|\|f_2\|_u$$

And to show absolute homogeneity, note that  $\sup |kA| = |k| \cdot \sup A$  for any non-empty bounded above set of reals  $A$ . This proves (a).

To show (b), fix any Cauchy sequence (with respect to the uniform metric), then for every  $\varepsilon > 0$ , there exists an  $N$  so large that for every  $n, m \geq N$  we have

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_u < \varepsilon$$

This shows that  $\{f_n(x)\}_{n \geq 1} \subseteq \mathbb{C}$  is a Cauchy, and it makes sense to call its limit  $f(x) = \lim f_n(x)$ . To show that for this  $f$ ,

- $f_n \rightarrow f$  uniformly, and
- $f \in B(X)$

Fix an  $\varepsilon > 0$ , and there exists an  $N$  so large that for every  $m, n \geq N$  implies that

$$\|f_n(x) - f_m(x)\|_u < \varepsilon$$

Since  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$ , this means that

$$\lim_{n \rightarrow \infty} |f_n(x) - f_m(x)| = |f(x) - f_m(x)| \leq \varepsilon$$

In the above we replaced the strict inequality with an inequality since the sequence may converge to its supremum. Since this holds for any  $x \in X$ , we have

$$\|f_m - f\|_u \leq \varepsilon$$

One can easily replace all the  $\varepsilon$  with  $\varepsilon/2$  to obtain strict inequalities, to finish the proof, simply send  $m \rightarrow \infty$  (since  $f_m \rightarrow f$  pointwise everywhere, the uniform norm goes to zero as well). This proves both bullet points.

Now we will prove Theorem 4.13, for any sequence  $\{f_n\} \subseteq \text{BC}(X)$ , if it does converge to some  $f$  uniformly, then we claim that  $f \in \text{BC}(X)$ . Note that  $f \in B(X)$ , so it suffices for us to show that  $f$  is continuous at every point  $x \in X$ .

Fix any ball with radius  $\varepsilon > 0$  at  $f(x) \in \mathbb{C}$ , and since

- $\varepsilon/3 > 0$  induces some  $N$  such that for every  $n \geq N$ , at every point  $x \in X$

$$|f_n(x) - f(x)| \leq \|f_n - f\|_u < \varepsilon/3$$

- Another  $\varepsilon/3$  ball around  $f_n(x)$  (using the same point  $x \in X$ ), such that its inverse image is an open set  $U \in \mathcal{T}_X$ , because  $f_n \in \text{BC}(X)$

$$f_n^{-1}(V_{\varepsilon/3} f_n(x)) = U \in \mathcal{T}_X$$

- The last  $\varepsilon/3$  comes from the fact that  $y \in U \subseteq X$  so it satisfies

$$|f_n(y) - f(y)| \leq \|f_n - f\|_i < \varepsilon/3$$

Combining these three,

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f(x) - f_n(x)| + |f_n(x) - f_n(y)| < \varepsilon$$

So there exists some open set  $U \in \mathcal{T}_X$  (and hence neighbourhood of every  $x$ ), for every open ball of radius  $\varepsilon > 0$ , around every  $f(x) \in \mathbb{C}$ , such that

$$f(U) \subseteq B \in \mathcal{T}_{\mathbb{C}}$$

Since the open balls are a neighbourhood base at every point in  $\mathbb{C}$ , and  $f$  is continuous at every point  $x \in X$ , we must conclude that  $f \in \text{BC}(X)$ . ■

**Theorem 4.14**

**WTS 14.1.** Suppose that  $A$  and  $B$  are disjoint closed subsets of the normal space  $X$ , and let  $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$  be the set of dyadic rationals in  $(0, 1)$ . There is a family  $\{U_r : r \in \Delta\}$  of open sets such that

1.  $A \subseteq U_r \subseteq B^c$  for every  $r \in \Delta$ ,
2.  $\overline{U_r} \subseteq U_s$  for  $r < s$ , and
3. For every  $r < s$ ,  $\overline{U_r} \subseteq U_s$

*Proof.* The goal of this proof is to show that for every  $r \in \Delta$ , there exists an open  $U_r$  that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some  $N \geq 1$  that all previous  $U_r$  in previous layers have been constructed properly, meaning if  $r = k/2^n$ , then for every  $1 \leq n \leq N - 1$ , we have

$$r = \frac{k}{2^n}, 1 \leq n \leq N - 1, 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each  $U_r$ ,

- $A \subseteq U_r \subseteq B^c$  and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer  $N \geq 1$ , we only have to construct the  $U_{k/2^N}$  for every odd  $k$ , this is because if  $k$  is an even number, then  $k = 2j$  and  $r = 2j/2^N = j/2^{N-1}$  and for this particular  $U_r$  is already constructed. So for every odd  $k = 2j + 1$ , the sets of the form  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  are already defined, and satisfy

$$A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every  $k - 1 \neq 0$  and  $k + 1 \neq 1$ . (We will consider these cases later). We claim that for every pair of open sets,  $E_1, E_2 \in \mathcal{T}_X$ , then there exists some open set  $G \in \mathcal{T}_X$  such that if  $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$  where  $H$  is defined as the set

$$H = \{(E_1, E_2) \in (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \emptyset\}$$

Then there exists some  $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$  such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any  $(E_1, E_2) \in H$ , then this pair induces a pair of disjoint sets  $\overline{E_1}$  and  $E_2^c$  since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \emptyset$$

And by normality, there exists disjoint open sets  $G_1, G_2$  such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \emptyset \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since  $G_2^c$  is a closed set that contains  $G_1$  as a subset,  $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of  $G_2$  (so we may discard the notion of  $G_2$  in our minds). Let us now replace  $G$  with  $G_1$  then it is an easy task to verify that  $G = G_1 = \mathcal{J}(E_1, E_2)$  has the required properties.

Now define for every odd  $k$ , since  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (we note in passing that  $\mathcal{J}$  is not a function as the set  $G$  may not be unique).

$$U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$$

Then, if  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  is 'well constructed' we have

$$A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore  $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$  sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U_{(k-1)/2^N}} \subseteq U_{k/2^N}$  and
- $\overline{U_{k/2^N}} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed'  $U_{k/2^N}$  for every  $k - 1 \neq 0$  and  $k + 1 \neq 1$ . Now let us deal with the remaining pathological cases.

If  $k - 1$  so happens to be 0, then no  $r \in \Delta$  satisfies  $r = 0/2^N$ , and we substitute

$$\overline{U}_0 = A, \quad \text{or alternatively, } U_0 = A^c$$

Then  $U_0 \in \mathcal{T}_X$ ,  $\overline{U}_0 = A \subseteq B^c$ . It is at this point that we must mention that  $0, 1 \notin \Delta$ , so  $U_0$  and  $U_1$  do not have to obey the rules we have laid out for  $U_{r \in \Delta}$ .

Now if  $k + 1$  is equal to  $2^N$  (this makes  $r = (k + 1)/2^N = 1$ ) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every  $0 \leq m \leq 2^N - 1$ ,  $U_{m/2^N}$  must satisfy

$$\overline{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (even for when  $N = 1$ , since  $A = \overline{U}_0 \subseteq U_1 = B^c$ ) and a corresponding  $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$  such that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\overline{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when  $N = 1$ . We would only have to construct for  $k = 1$ , since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■

**Theorem 4.15**

**WTS 15.1.** *Urysohn's Lemma.* Let  $X$  be a normal space, if  $A$  and  $B$  are disjoint closed subsets of  $X$ , then there exists a  $f \in C(X, [0, 1])$  such that  $f = 0$  on  $A$  and  $f = 1$  on  $B$ .

*Proof.* Let  $r \in \Delta$  be as in Lemma 4.14, and set  $U_r$  accordingly except for  $U_1 = X$ . Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write  $W = \{k : x \in U_k\}$ , Then for every  $x \in A$  we have  $f(x) = 0$ , since by the construction of the 'onion' function in Lemma 4.14, for each  $r \in \Delta \cap (0, 1)$ ,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since  $r > 0$  is arbitrary, and  $0 \in W$ , we can use a classic  $\varepsilon$  argument. If  $f(x) > 0$  then there exists some  $0 < r < f(x)$  by density of the dyadic rationals on the line, if  $f(x) < 0$  then this implies that there exists some  $f(x) < r < 0$  such that  $x \in U_r$ , but no  $r \in \Delta$  can be negative, hence  $f(x) = 0$ .

Now, for every  $x \in B$ , since  $A$  and  $B$  are disjoint, and  $A \subseteq U_r \subseteq B^c$ , then for every  $x \in B$  means that  $x$  is not a member of any  $U_r$ , but we set  $U_1 = X$ . Since none of the  $r \in (0, 1)$  is a member of the set we are taking the infimum, and  $x \in U_1 = X$ . The  $\varepsilon$  argument follows: suppose for every  $\varepsilon > 0$ ,  $(1 - \varepsilon) \notin W$ , and  $1 \in W$ , then  $f(x) = 1$ .

Since  $x \in U_1 = X$ , for every  $x \in X$ ,  $f(x) \leq 1$ , and  $f(x)$  cannot be negative as  $r > 0$  for every  $r \in \Delta$ . So  $0 \leq f(x) \leq 1$ . Now we have to show that this  $f(x)$  is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in  $X$ . So  $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$  and  $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$ .

Suppose that  $f(x) < \alpha$ , so  $\inf W < \alpha$ , and using the density of  $\Delta$ , there exists an  $r$ ,  $f(x) < r < \alpha$  such that  $x \in U_r$  such that  $x \in \bigcup_{r < \alpha} U_r$ . So  $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r < \alpha} U_r$ .

Fix an element  $x \in \bigcup_{r < \alpha} U_r$ , this induces an  $r$  such that  $\inf W \leq r < \alpha$  therefore  $f(x) < \alpha$ , and  $\bigcup_{r < \alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$ .



For the second case, suppose that  $f(x) > \alpha$ , then  $\inf W > \alpha$ , and there exists an  $r$  (by density) such that  $\inf W > r > \alpha$  such that for every  $k \in W$ ,  $k \neq r$ . Therefore  $x \notin U_r$ , but by density again, and using the property of the onion function: for every  $s < r$  we get  $\overline{U_s} \subseteq U_r$ , taking complements (which reverses the estimate) — we have  $x \notin \overline{U_s}$ , but  $(\overline{U_s})^c$  is open in  $X$ . It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq (\overline{U_s})^c \subseteq \bigcup_{s > \alpha} (\overline{U_s})^c$$

So  $f^{-1}((\alpha, +\infty))$  is a subset of  $\bigcup_{s > \alpha} (\overline{U_s})^c$ . To show the reverse, fix an element  $x$  in the union, then this induces some  $x \in (\overline{U_s})^c \subseteq (U_s)^c$ . Then for this  $s > \alpha$ ,  $(-\infty, s)$  contains no elements of  $W$ . This is because for every  $p < s$  implies that  $(U_s)^c \subseteq (U_p)^c$ , so  $p \notin W$ . Our chosen  $s$  is a lower bound for  $W$ , and  $\alpha < s \leq \inf W = f(x)$ .

Since all of the inverse images from the generating set of  $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$  are open in  $X$ , using Theorem 4.9 finishes the proof. ■

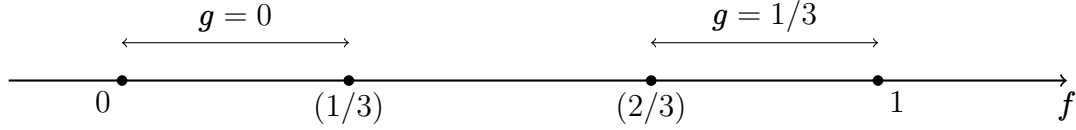


Figure 1: Lemma 16.1 for Theorem 4.16: Separate the range of  $f \in C(A, [0, 1])$  into three parts. Subtract an additional  $g$  that reduces the error even further.

### Theorem 4.16

**WTS 16.1.** *The Tietze's Extension Theorem. Let  $X$  be a normal space, and for any closed subset  $A \subseteq X$ , and  $f \in C(A, [a, b])$ , there exists an  $F \in C(X, [a, b])$  which extends  $f$ .*

*Proof.* We begin with an important lemma that will serve as a 'black box' for the induction.

**Lemma 16.1** *For every  $f \in C(A, [0, 1])$ , there exists a  $g \in C(X, [0, 1/3])$  such that*

$$0 \leq f - g \leq 2/3 \quad \text{pointwise on } A \quad (8)$$

*Proof.* Since  $f$  is continuous,  $B = f^{-1}([0, 1/3])$ , and  $C = f^{-1}([2/3, 1])$  are closed, disjoint subsets. Applying Urysohn's Lemma (Theorem 4.15) we get a continuous function  $g \in C(X, [0, 1])$  such that  $g|_B = 0$  and  $g|_C = 1$ . Rescale  $g$  by a factor of  $1/3$ , and  $g \in C(X, [0, 1/3])$ .

To show that Equation (8) holds, suppose  $x \in B$ , then  $f(x) \in [0, 1/3]$  and  $g(x) = 0 \implies 0 \leq f - g \leq 1/3 \leq 2/3$ . Now suppose that  $x \in C$ , then  $f(x) \in [2/3, 1]$  and  $g(x) = 1/3$  (recall that we relabelled  $g$ ). So we have  $0 \leq 1/3 \leq f - g \leq 2/3$ . Lastly, for the case where  $x \notin (B \cup C)$ , then  $f(x) \in (1/3, 2/3)$ , and  $g(x) \in [0, 1/3]$  implies that

$$\begin{aligned} 1/3 < f(x) < 2/3 & \implies 1/3 \leq f(x) \leq 2/3 \\ 0 \leq g(x) \leq 1/3 & \implies -1/3 \leq -g(x) \leq 0 \end{aligned}$$

Therefore  $0 \leq f(x) - g(x) \leq 2/3$ . See Figure 1. ■

We can assume that  $f \in C(A, [0, 1])$ , since we can relabel  $f = (f - a)/(b - a)$ . The main part of this proof consists of constructing a sequence of  $\{g_n\} \subseteq C(X, \mathbb{R})$  where  $0 \leq g_n \leq (2/3)^n(1/2)$ , and  $0 \leq f - \sum_{j \leq n} g_j \leq (2/3)^n$  on  $A$ . Let us begin with the base case with  $n = 1$ . We can apply Lemma 16.1 to get  $g_1 \in C(X, [0, 1/3])$

$$0 \leq f - g_1 \leq (2/3)^1$$

Now let us suppose that  $\{g_j\}_{j \leq n}$  has been chosen, we will find our  $g_{n+1}$  by noting that

$$0 \leq f(x) - \sum_{j \leq n} g_j(x) \leq (2/3)^n$$

Here is where my proof deviates from that of Folland's, we multiply both sides by  $(2/3)^{-n}$  and we obtain a new function in  $C(A, [0, 1])$ .

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) \left( \frac{3}{2} \right)^n \leq 1$$

Applying Lemma 16.1, we get a function  $h \in C(X, [0, 1/3])$  that reduces the error between  $f$  and the partial sums of  $g_{j \leq n-1}$ . For every  $x \in A$

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) \left( \frac{3}{2} \right)^n - h \leq 2/3$$

Multiplying across gives

$$0 \leq \left( f(x) - \sum_{j \leq n} g_j(x) \right) - h \left( \frac{2}{3} \right)^n \leq \left( \frac{2}{3} \right)^{n+1}$$

Set  $g_{n+1} = h \left( \frac{2}{3} \right)^n$  and  $g_{n+1} \in C(X, [0, 2^n/3^{n+1}])$ . Furthermore, the sum of all  $g_j$  pointwise converges uniformly, as

$$\sum_{j \geq 1} \|g_j\|_u \leq \sum_{j \geq 1} \left( \frac{2}{3} \right)^j \cdot \frac{1}{2} < +\infty$$

Denote the pointwise sum  $F = \sum g_j$ , then this  $F \in BC(X)$  (by Theorem 4.13 and 5.1). And

$$\left\| f - \sum_{j \leq n} g_j \right\|_u \leq \left( \frac{2}{3} \right)^n \rightarrow 0$$

So  $F = f$  on  $A$ , now if we want to obtain our  $F$  on  $[a, b]$  we simply relabel  $F = F(b - a) + a$ . This finishes the proof.  $\blacksquare$

**Theorem 4.17**

**WTS 17.1.** *If  $X$  is a normal space, and  $A$  is a closed subspace of  $X$ , and  $f \in C(A)$ , then there exists an  $F \in C(X)$  such that  $F$  extends  $f$ .*

*Proof.* First we suppose that  $f$  is real valued, so  $f \in C(X, \mathbb{R})$ . And define a  $g \in C(A, (-1, +1)) \subseteq C(A, [-1, +1])$ , using

$$g = \frac{f}{1 + |f|}$$

Since  $g$  satisfies the assumption of Theorem 4.16 (note that we do not require  $g$  to be injective), there exists a  $G \in C(X, [-1, +1])$  such that  $G|_A = g$ . Since the set  $\{-1, +1\}$  is closed in  $\mathbb{R}$ ,  $G^{-1}(\{-1, +1\})$  is closed as well. Since  $G^{-1}((-1, +1)) \subseteq A$ , this makes  $A$  and  $B = G^{-1}(\{-1, +1\})$  disjoint closed sets in  $X$ .

By Urysohn's Lemma, there exists a continuous function  $h \in C(X, [0, 1])$  such that  $h|_B = 0$  and  $h|_A = 1$ , so that the product  $|hG| < 1$  for all  $x \in X$ . We can think of this  $h$  as a continuous indicator function that filters out the parts we do not want, namely  $G^{-1}\{-1, +1\}$ . Now define  $F$  in the following manner, since division is permissible

$$F = \frac{hG}{1 - |hG|}$$

We will show that  $F|_A = g/(1 - |g|) = f$  indeed. Since  $|g| = \frac{|f|}{1+|f|}$ , and  $g(1 + |f|) = f$  implies that  $g/(1 - |g|) = f$ , because  $g \in C(A, (-1, +1))$ . This completes the proof for any  $f \in \mathbb{R}$  if  $f \in C(A)$ , then

1.  $\operatorname{Re}(f) = f_1 \in C(A, \mathbb{R})$
2.  $\operatorname{Im}(f) = f_2 \in C(A, \mathbb{R})$

And by our previous argumentation, there exists two functions in  $C(X, \mathbb{R})$  that extends  $f_1$  and  $f_2$ , and  $F_1 + iF_2 = f$  on  $A$  and  $F_1 + iF_2 \in C(X)$ , and the proof is complete. ■

**Theorem 4.18**

**WTS 18.1.** *If  $X$  is a topological space, and  $E \subseteq X$  and  $x \in X$ , then  $x \in \text{acc } E \iff$  there exists a net in  $E \setminus \{x\}$  that converges to  $x$ , and  $x \in \overline{E} \iff$  there exists a net in  $E$  that converges to  $x$ .*

*Proof.* Suppose that  $x \in \text{acc } E$ , then for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $E \cap U \setminus \{x\} \neq \emptyset$ , then choose  $\mathcal{N}(x)$  as the set of neighbourhoods directed by reverse inclusion (and this makes  $(\mathcal{N}(x), \supseteq)$  a directed set), and we will define the net as follows.

Map each  $U \in \mathcal{N}(x)$  to some  $x_U \in E \cap U \setminus \{x\}$ , then this net converges to  $x$ . Suppose that we fix a neighbourhood,  $V \in \mathcal{N}(x)$ , then for every  $U \supseteq V$  we have  $x_U \in U \subseteq V$ . So  $\langle x_U \rangle$  is eventually in  $V$ .

Conversely, if  $\langle x_\alpha \rangle \subseteq E \setminus \{x\}$ , and  $x_\alpha \rightarrow x$ , then every  $U \in \mathcal{N}(x)$  there exists a  $x_\alpha \in E \cap U \setminus \{x\}$  that makes

$$E \cap U \neq \emptyset \quad \forall U \in \mathcal{N}(x)$$

Hence  $x \in \text{acc } E$ .

Now for the second part of the Theorem, suppose that  $x \in \overline{E}$ , if  $x \notin E$  then  $E = E \setminus \{x\}$  and  $x \in \text{acc } E$ , so there exists a net in  $E \setminus \{x\} \subseteq E$  such that  $x_\alpha \rightarrow x$ . If  $x \in E$  then simply choose  $\langle x_\alpha \rangle = x$  for every  $\alpha \in A$ .

Now, suppose that there is a net that converges to  $x$ , and this net  $\langle x_\alpha \rangle \subseteq E$ , if  $x \in E$  then there is nothing to prove, since  $E \subseteq \overline{E}$ , so suppose that  $x \notin E$ , then there exists a net in  $E \setminus \{x\} = E$  such that

$$x_\alpha \rightarrow x \implies x \in \text{acc } E \subseteq \overline{E}$$

■

**Theorem 4.19**

**WTS 19.1.** *Let  $X$  and  $Y$  be topological spaces, then every  $f : X \rightarrow Y$  is continuous at a point  $x \in X \iff$  every net  $\langle x_\alpha \rangle$  that converges to  $x$  implies that  $\langle f(x_\alpha) \rangle$  converges to  $f(x)$ .*

*Proof.* If  $f$  is continuous at a point  $x \in X$ , then  $V \in \mathcal{N}(f(x)) \implies f^{-1}(V) \in \mathcal{N}(x)$ , then for every net  $\langle x_\alpha \rangle$  that converges to this  $x$ , there exists an  $\alpha_0$  such that for every  $\alpha \gtrsim \alpha_0$  implies that  $x_\alpha \in f^{-1}(V)$ . Hence

$$f(x_\alpha) \in f(f^{-1}(V)) \subseteq V$$

And this is equivalent to saying that for every  $V \in \mathcal{N}(f(x))$ ,  $\langle f(x_\alpha) \rangle$  is eventually in  $V$ , and this proves convergence.

Now suppose that  $f$  is not continuous at some  $x$ , then there exists a  $V \in \mathcal{N}(f(x))$  such that  $f^{-1}(V) \notin \mathcal{N}(x)$ , so

$$x \notin (f^{-1}(V))^o \implies x \in (f^{-1}(V))^{oc} = \overline{f^{-1}(V^c)}$$

Where for the last equality we pulled the complement inside the inverse image. Then by Theorem 4.18, our  $x \in \overline{f^{-1}(V^c)}$  induces a net  $\langle x_\alpha \rangle \subseteq f^{-1}(V^c)$  that converges to  $x$ . But every element in the net is contained within  $f^{-1}(V^c)$ , and for every  $\alpha \in A$

$$f(x_\alpha) \in f(f^{-1}(V^c)) \subseteq V^c$$

gives  $f(x_\alpha) \notin V$ , but  $V$  is a neighbourhood of  $f(x)$ , hence there exists some  $x_\alpha \rightarrow x$  and  $f(x_\alpha) \not\rightarrow f(x)$ . ■

**Theorem 4.20**

**WTS 20.1.** *If  $\langle x_\alpha \rangle$  is a net in  $X$ , and  $x \in X$  is a cluster point of  $\langle x_\alpha \rangle \iff$  there exists a subnet of  $\langle x_\alpha \rangle$  that converges to  $x$ .*

*Proof.* Suppose that  $\langle y_\beta \rangle_{\beta \in B}$  is a subnet of  $\langle x_\alpha \rangle$  that converges to  $x$ , then for every neighbourhood  $U \in \mathcal{N}(x)$ , there exists a  $\beta_1$  such that for every  $\beta \gtrsim \beta_1$  we get  $y_\beta = x_{\alpha_\beta} \in U$ .

Furthermore, let us fix a  $\alpha_0 \in A$  to attempt to show that  $\langle x_\alpha \rangle$  is frequently in  $U$ , then by the subnet property of  $\langle y_\beta \rangle$ , there exists some  $\beta_2 \in B$  such that for every  $\beta \gtrsim \beta_2$ ,  $\alpha_\beta \gtrsim \alpha_0$ . (Intuitively this property means that the directed set of  $B$  'grows' as much as the directed set of  $A$ , so we can always find elements that are greater than any fixed  $\alpha_0$ .)

Since  $\langle y_\beta \rangle$  is a net, we there exists some  $\beta \in B$  such that  $\beta \gtrsim \beta_1$  and  $\beta \gtrsim \beta_2$ , we then apply the  $\beta \mapsto \alpha_\beta$  map and we obtain some  $\alpha = \alpha_\beta$  that satisfies:

- $\alpha = \alpha_\beta \gtrsim \alpha_0$
- $x_\alpha = x_{\alpha_\beta} \in U$

Where for the second property we used the fact that  $\beta \gtrsim \beta_1$  so that  $y_\beta$  falls into  $U$ .

Conversely, suppose that  $x$  is a cluster point of  $\langle x_\alpha \rangle$ , then by definition

$$\forall U \in \mathcal{N}(x), \forall \alpha_0 \in A, \exists \alpha \gtrsim \alpha_0, x_\alpha \in U$$

Denote the directed neighbourhoods of  $x$  by  $\mathcal{N}(x)$ , and construct our directed set  $B$  for our subnet as follows, define

$$B = \mathcal{N}(x) \times A$$

Where for every  $(U, \gamma) \in B$  we can map it to some  $\alpha_{(U, \gamma)} \in A$ , if we choose some  $\alpha_{(U, \gamma)} \gtrsim \gamma$  and  $\alpha_{(U, \gamma)} \in U$ .

To show that  $B$  is a directed set, we say that  $(U, \gamma) \gtrsim (U', \gamma')$  if and only if  $U \subseteq U'$  and  $\gamma \gtrsim \gamma'$ . And to show that  $\langle y_\beta \rangle = \langle x_{\alpha_{(U, \gamma)}} \rangle$  is indeed a subnet of  $\langle x_\alpha \rangle$ , fix any  $\alpha_0 \in A$ , then simply take any neighbourhood  $U$  of  $x$  (we always

have  $X \in \mathcal{N}(x)$  — and therefore  $(U, \alpha_0) \in B$ .

Now for every  $(U', \alpha'_0) \gtrsim (U, \alpha_0)$  implies that  $\alpha'_0 \gtrsim \alpha_0$ , therefore we have

$$\alpha_{(U', \alpha'_0)} \gtrsim \alpha'_0 \gtrsim \alpha_0$$

And this satisfies the subnet property. Now to show that  $\langle y_\beta \rangle$  indeed converges to  $x$ , fix any  $V \in \mathcal{N}(x)$ , then with any  $\alpha_0 \in A$ , and for every  $(V', \alpha'_0) \gtrsim (V, \alpha_0) \in B$ , we have

$$x_{\alpha_{(V', \alpha'_0)}} \in V' \subseteq V$$

So  $\langle x_{\alpha_{(U, \gamma)}} \rangle$  converges to  $x$ . ■



**Theorem 4.21**

**WTS 21.1.** *A topological space  $X$  is compact  $\iff$  every family of closed sets,  $\{F_\alpha\}_{\alpha \in A}$  that has the finite intersection property, implies that*

$$\bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

*Proof.* We first examine the assertion, Theorem 4.21 proposes for any family of closed sets  $\{F_\alpha\}_{\alpha \in A}$ , and for every finite subset  $B \subseteq A$  then,

$$\bigcap_{\alpha \in B} F_\alpha \neq \emptyset \implies \bigcap_{\alpha \in A} F_\alpha \neq \emptyset$$

Taking the contrapositive (which is logically equivalent), we get

$$\bigcap_{\alpha \in A} F_\alpha = \emptyset \implies \text{there exists a finite } B \subseteq A, \bigcap_{\alpha \in B} F_\alpha = \emptyset$$

Applying DeMorgan's theorem, and since every  $\{F_\alpha\}_{\alpha \in A}$  induces a family of open sets (and vice versa), where  $U_\alpha = F_\alpha^c$ , so for any family of open sets  $\{U_\alpha\}_{\alpha \in A}$  we have

$$\bigcup_{\alpha \in A} U_\alpha = X \implies \text{there exists a finite } B \subseteq A, \bigcup_{\alpha \in B} U_\alpha = X$$

Which is equivalent to saying that  $X$  is compact. ■

**Theorem 4.22**

**WTS 22.1.** *A closed subset of a compact space  $X$  is compact.*

*Proof.* Suppose  $F \subseteq X$  and  $F$  is open, then fix an open cover for  $F$ , so

$$F \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since  $F^c$  is an open set, we can obtain a valid open cover for  $X$ , then we pick out a finite subcover, for some finite  $B \subseteq A$

$$X = F \cup F^c \subseteq F^c \cup \left( \bigcup_{\alpha \in B} U_\alpha \right)$$

Taking the intersection with  $F$  on both sides yields

$$\begin{aligned} F &= X \cap F \subseteq (F^c \cap F) \cup \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \\ F &= \left( F \cap \left( \bigcup_{\alpha \in B} U_\alpha \right) \right) \iff \\ F &\subseteq \bigcup_{\alpha \in B} U_\alpha \end{aligned}$$

Therefore every open cover of  $F$  has a finite subcover, and  $F$  is compact. ■

**Theorem 4.23**

**WTS 23.1.** *If  $F$  is a compact subset of a Hausdorff space  $X$ , and  $x \notin F$ , there are disjoint open sets  $U, V$  such that  $x \in U$  and  $F \subseteq V$ .*

*Proof.* Since  $x \in F^c$ , for every  $y \in F$ ,  $x \neq y$  induces two sets  $U_y, V_y$  (because  $X$  is  $T_2$ ).

- $U_y \cap V_y = \emptyset$
- $x \in U_y$
- $y \in V_y$

But  $\{V_y\}_{y \in F}$  is an open cover for the compact set  $F$ , then there exists a finite subcollection  $H \subseteq F$  such that

$$F \subseteq \bigcup_{y \in H} V_y$$

Since  $H$  is finite,  $U = \bigcap_{y \in H} U_y$  is an open set that contains  $x$ , also define  $V = \bigcup_{y \in H} V_y$ . If for every  $y \in H$ ,  $U_y \cap V_y = \emptyset$ , then  $U \cap V_y = U \cap V = \emptyset$ . This completes the proof. ■

**Remark 23.1** *Every metric space  $(X, d)$  is first countable, and  $T_2$  (it is actually  $T_4$ , but that will require some effort to prove, see Exercise 3). The first claim is easily verified if we fix any element  $x \in X$  and we notice that  $W_x = \{V_r(x), r \in \mathbb{Q}^+\}$  is a countable neighbourhood base for every  $x$ . To show that  $(X, d)$  is  $T_2$ , for every pair of elements  $x \neq y$ , we can take  $r = d(x, y)/2$  and there exists disjoint open sets  $V_r(x)$  and  $V_r(y)$  such that  $x \in V_r(x)$  and  $y \in V_r(y)$ .*

**Theorem 4.24**

**WTS 24.1.** *Every compact subset of a Hausdorff ( $T_2$ ) space is closed.*

*Proof.* If  $F$  is compact, then for every  $x \in F^c$ , by Theorem 4.23, there exists two disjoint open sets such that  $x \in U$  and  $F \subseteq V$ , but

$$U \cap V = \emptyset \implies U \cap F = \emptyset \implies U \subseteq F^c$$

But since  $x \in F^c$  is arbitrary, and  $U$  is an open subset of  $F^c$ ,

$$x \in U \subseteq F^{co} \implies F^c \subseteq F^{co}$$

Which shows that  $F^c$  is open and  $F$  is closed. ■

**Theorem 4.25****WTS 25.1.** *Every compact Hausdorff ( $T_2$ ) space is normal ( $T_4$ ).*

*Proof.* Fix  $A, B$  which are disjoint closed subsets of  $X$ , by Theorem 4.22, we know that these two sets are compact. Hence for every  $y \in B$  there exists two disjoint open sets  $U, V_y$  (by Theorem 4.23)

$A \subseteq U_y$  and  $y \in V_y$ . But the family  $\{V_y\}_{y \in B}$  is a valid open cover for the compact set  $B$ , hence there exists a finite subcollection  $H \subseteq B$  such that

$$B \subseteq \bigcup_{y \in H} V_y, \quad U_y \cap V_y = \emptyset$$

The second equality holds for every  $y \in H$  so that  $U_y \cap (\bigcup_{y \in H} V_y) = \emptyset$ . Define  $U = \bigcap_{y \in H} U_y$  and  $V = \bigcup_{y \in H} V_y$ , where both of these are disjoint open sets that contain  $A$  and  $B$  as subsets, since for each  $y \in H$ ,  $A \subseteq U_y$  hence the intersection of all  $U_y$  also contains  $A$  as a subset. Therefore  $X$  is normal. ■

**Theorem 4.26**

**WTS 26.1.** *If  $X$  is compact, and  $f : X \rightarrow Y$  is continuous, then  $f(X)$  is compact.*

A small lemma.

**Lemma 26.1** *For every  $\{E_j\} \subseteq X$ ,  $f(\cup E_j) = \cup f(E_j)$ .*

The proof is trivial.

*Proof.* If  $\{V_{\alpha \in A}\}$  is an open cover for  $f(X)$ , then

$$X \subseteq f^{-1}(f(X)) = f^{-1}\left(\bigcup_{\alpha \in A} V_{\alpha}\right) = \bigcup_{\alpha \in A} f^{-1}(V_{\alpha}) \subseteq X$$

Since  $f$  is continuous, we have an open cover in the form of  $\{f^{-1}(V_{\alpha})\}$  for  $X$ , then there exists a finite subset  $B \subset A$  such that

$$X \subseteq \bigcup_{\alpha \in B} f^{-1}(V_{\alpha})$$

Then we wish to show that for this  $B \subseteq A$ ,  $\{V_{\alpha \in B}\}$  is a finite open cover for  $f(X)$ . Fix any element  $y \in f(X)$ , then this induces a  $x \in X$  such that  $y = f(x)$ , but because  $\{f^{-1}(V_{\alpha \in B})\}$  is an open cover for  $X$ , there exists some  $\alpha \in B$  such that  $x \in f^{-1}(V_{\alpha})$ , hence by definition of the inverse image

$$f(x) \in V_{\alpha} \implies f(X) \subseteq \bigcup_{\alpha \in B} V_{\alpha}$$

Therefore  $f(X)$  is compact and this completes the proof. ■

**Theorem 4.27**

**WTS 27.1.** *If  $X$  is compact, then  $C(X) = BC(X)$ .*

*Proof.* Notice that  $BC(X) \subseteq C(X)$ , so we only have to show the reverse estimate. Fix any  $f \in C(X)$ , since  $X$  is compact, by Theorem 4.26 we know that  $f(X)$  is also compact. Since  $\mathbb{C} = \mathbb{R}^2$  is a complete metric space,  $f(X)$  is bounded and  $f \in BC(X)$ . ■

**Theorem 4.28**

**WTS 28.1.** *If  $X$  is compact, and if  $Y$  is Hausdorff, then any continuous bijection  $f : X \rightarrow Y$  is a homeomorphism.*

*Proof.* If  $E \subset X$  is closed, then since  $X$  is compact,  $E$  is compact as well. By continuity of  $f$ ,  $f(E)$  is a compact set in  $Y$ , but compact subsets of  $Y$  are closed, so  $f$  is continuous.

We used the fact that the inverse of  $f^{-1}$  is  $f$ , since it suffices to check that every inverse image of a closed set is also closed,  $f^{-1}$  is continuous. And by definition of a homeomorphism ( $f$  has to be bijective and both  $f$  and  $f^{-1}$  have to be continuous),  $f$  is a homeomorphism. ■



**Theorem 4.29**

**WTS 29.1.** *If  $X$  is any topological space, the following are equivalent.*

- (a)  $X$  is compact.
- (b) Every net has a cluster point.
- (c) Every net in  $X$  has a convergent subnet.

*Proof.* By Theorem 4.20, every net in  $X$  has a cluster point  $\iff$  there exists a subnet that converges to this cluster point, so these two points are equivalent.

Suppose a) holds, then  $X$  is compact, and fix an arbitrary net  $\langle x_\alpha \rangle$  in  $X$ . and define the 'tail' of the net

$$E_\alpha := \{x_\beta, \beta \succeq \alpha\}$$

We wish to show that the arbitrary intersection of  $\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \emptyset$ . Where  $\overline{E}_\alpha$  is closed, so it suffices to check that every finite  $B \subseteq A$ , the intersection over  $\overline{E}_\alpha$  is non-empty.

Suppose we are given a finite  $B \subseteq A$ , then fix any two elements  $\alpha$  and  $\beta \in B$ , by the definition of a net there exists a  $\gamma \in A$  such that  $\gamma \succeq \alpha$  and  $\gamma \succeq \beta$ , and

$$\emptyset \neq E_\alpha \cap E_\beta \implies \overline{E}_\alpha \cap \overline{E}_\beta \neq \emptyset$$

Therefore for any finite collection of  $\{\overline{E}_{\alpha \in B}\}$ , then

$$\bigcap_{\alpha \in A} \overline{E}_\alpha \neq \emptyset$$

Now fix an element  $x \in \bigcap_{\alpha \in A} \overline{E}_\alpha$ . Then for every  $\alpha \in A$ ,  $x \in \overline{E}_\alpha$ , and for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $U \cap E_\alpha \neq \emptyset$ . This is because if  $x \in E_\alpha$ , then  $U \cap E_\alpha$  contains at least  $\{x\}$ , if  $x \in \text{acc } E_\alpha$ , then by definition of an accumulation point,  $U \cap E_\alpha \setminus \{x\} \neq \emptyset$ , so the intersection is non empty.

Now let us turn our attention to how we defined the 'tail' of the net,  $E_\alpha$ , if for every  $\alpha \in A$ ,  $x \in E_\alpha$  if and only if there exists some  $\gamma \succeq \alpha$ ,  $x_\gamma \in U \cap E_\alpha$ ,

this is equivalent to saying that  $x$  is a cluster point of  $\langle x_\alpha \rangle$ . So  $a) \implies b)$ .

Now let us suppose that  $X$  is not compact, then there exists an open cover  $\{U_\alpha \in A\}$  of  $X$  that has no finite subcover. Let  $\mathbb{B}$  be the collection of all finite subsets of  $A$ , directed by set inclusion (we will show that this set is indeed a directed set at another time, for now it is a needless distraction).

Now for every  $B \in \mathbb{B}$ , find some  $x_B \in (\bigcup_{\alpha \in B} U_\alpha)^c$ . So we have a net in  $X$ . Now we will show that no  $x \in X$  can be a cluster point of this net. Suppose not, then take a neighbourhood  $U_\beta$  with  $\beta \in A$  such that  $U_\beta$  belongs to the open cover we first discussed. Then for any  $B \in \mathbb{B}$  such that  $B \gtrsim \{\beta\}$  (meaning that  $\{\beta\} \subseteq B$ , where  $B$  is a finite set), then

$$x_B \in \left( \bigcup_{\alpha \in B} U_\alpha \right)^c \implies x_B \notin \left( \bigcup_{\alpha \in \{\beta\}} U_\alpha \right) \implies x_B \in U_\beta^c$$

Hence no point in  $X$  can be a cluster point for this net, and the proof is complete. ■

**Theorem 4.30**

**WTS 30.1.** *If  $X$  is a LCH space, and for every  $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$ , there exists a compact  $N \subseteq U$  where  $N \in \mathcal{N}_B(x)$ .*

*Proof.* For every  $U \in \mathcal{N}_B(x) \cap \mathcal{T}_X$ , we can find an  $E$  open subset of  $U$  that has a compact closure, since every  $x \in X$  induces some compact  $F \in \mathcal{N}_B(x)$ , therefore

$$E := U \cap F^o \implies \overline{E} \subseteq F$$

Since closed subsets of compact sets are compact (by Theorem 4.22),  $\overline{E}$  is compact. More is true, since  $E$  is open,

$$x \in U \cap F^o \implies x \in E^o \implies E \in \mathcal{N}_B(x)$$

Now it suffices to show that there exists some compact  $N \subseteq E \subseteq U$  such that  $N \in \mathcal{N}_B(x)$ . Since  $\overline{E}$  is compact, the closed subset  $\partial E = \overline{E} \cap \overline{E}^c$  of  $\overline{E}$  is also compact.

Since  $\partial E \cap E^o = \emptyset$ ,  $x \in E^o = E$  means that  $x \notin \partial E$ . Applying Theorem 4.23 to the compact set  $\partial E$  and  $x \notin \partial E$  gives us two disjoint open sets  $V'$  and  $W'$ . We list their properties

1.  $V', W' \in \mathcal{T}_X$
2.  $x \in V'$
3.  $\partial E \subseteq W'$
4.  $V' \cap W' = \emptyset$

The two disjoint pairs induce another pair of open sets relative to  $\overline{E}$ , recall the definition of the topology relative to  $\overline{E}$ ,

$$\mathcal{T}_{\overline{E}} = \{A \cap \overline{E} : A \in \mathcal{T}_X\}$$

We now agree to define

- $V = V' \cap \overline{E}$
- $W = W' \cap \overline{E}$

Then evidently  $V, W \in \mathcal{T}_{\overline{E}}$  and

1.  $x \in V' \cap \overline{E} \implies x \in V$
2.  $\partial E \subseteq \overline{E} \implies \partial E \subseteq W$
3.  $V' \cap W' = \emptyset \implies V \cap W = \emptyset$

Furthermore,

$$\partial E \subseteq W \implies W^c \subseteq (\partial E)^c = E^o \cup E^{co}$$

Taking the intersection over  $\overline{E}$  gives us

$$\overline{E} \setminus W \subseteq \overline{E} \cap (E^o \cup E^{co})$$

Note that  $E^{co} = (\overline{E})^c$ , since  $(E^c)^{oc} = \overline{(E^{cc})} = \overline{E}$  therefore  $\overline{E} \cap E^{oc} = \emptyset$ , hence

$$\overline{E} \setminus W \subseteq \overline{E} \cap E^o = E^o$$

Using the fact from 3,  $V \subseteq W^c$  and  $V \subseteq \overline{E}$  and  $V \subseteq W^c$  implies that  $V \subseteq \overline{E} \setminus W$ . Compiling everything, we have

$$V \subseteq \overline{E} \setminus W \subseteq E$$

Note that the set  $\overline{E} \setminus W$  is closed in  $\mathcal{T}_X$  (and hence closed in  $\overline{E}$ ) by closure over intersections,  $\overline{V}$  is therefore a closed subset of  $\overline{E} \setminus W$ , and  $\overline{V}$  is compact. Also

$$\overline{V} \subseteq \overline{E} \setminus W \subseteq E$$

To check that  $\overline{V} \in \mathcal{N}_B(x)$ , note that

$$x \in V^o \subseteq (\overline{V})^o \implies \overline{V} \in \mathcal{N}_B(x)$$

The subset relation  $V^o \subseteq \overline{V}^o$  comes from the fact that  $V^o$  is an open subset of  $\overline{V}$ , and hence is contained in  $(\overline{V})^o$  as a subset. Now let us define  $N = \overline{V}$ , and  $N$  satisfies the assertions in the Theorem, since

- $N \in \mathcal{N}_B(x)$
- $N$  is compact
- $N \subseteq E \subseteq U$

And this completes the proof. ■

**Remark 30.1** *Intuitively speaking, this means that if  $X$  is any LCH space, then for every open neighbourhood  $U \in \mathcal{N}_B(x)$ , there exists a compact  $E \in \mathcal{N}_B(x)$  such that  $x \in E \subseteq U^o$ . This property is indeed a very strong one as it allows us to have effectively 'infinite' descending compact neighbourhoods of  $x$ .*

**Theorem 4.31**

**WTS 31.1.**  *$X$  is a LCH space, and  $K \subseteq U \subseteq X$  where  $K$  is compact, and  $U$  is open, then there exists some precompact, open  $V$  with*

$$K \subseteq V \subseteq \bar{V} \subseteq U$$

*Proof.* For every  $x \in K$ , we can apply Proposition 4.30, since  $x \in K \subseteq U$ , this induces some compact  $F_x \subseteq U$  where  $F_x \in \mathcal{N}_B(x)$ . Then we can obtain an open cover of  $U$  in the form of  $\{F_x^o\}_{x \in K}$ . By compactness of  $K$ , there exists a finite  $B \subseteq K$  such that

$$K \subseteq \bigcup_{x \in B} F_x^o$$

Let  $V = \bigcup_{x \in B} F_x^o$ , then clearly  $V$  is open, and  $K \subseteq V$ . Since each  $F_x$  is closed (compact sets are closed in any Hausdorff Space), we have

$$V \subseteq \bigcup_{x \in B} F_x \implies \bar{V} \subseteq \bigcup_{x \in B} F_x$$

Since  $\bigcup_{x \in B} F_x$  is a finite union of compact sets, we claim that it is also compact. Consider two compact sets  $E_1$  and  $E_2$ , then if  $\{U_\alpha\}_{\alpha \in A}$  is any open cover of  $E_1 \cup E_2$ , it must be an open cover for  $E_1$  and  $E_2$  as well, because

$$E_1, E_2 \subseteq E_1 \cup E_2 \subseteq \bigcup_{\alpha \in A} U_\alpha$$

Since  $E_1$  and  $E_2$  are both compact sets, they each induce two finite subsets of  $B_1, B_2$  of  $A$  whose union  $B = B_1 \cup B_2$  is also compact. Therefore

$$E_1 \cup E_2 \subseteq \bigcup_{\alpha \in B} U_\alpha$$

Then a simple proof by induction will show that if  $\{E_{j \leq n}\}$  is a family of compact sets, then  $E = \bigcup E_{j \leq n}$  is also compact.

Returning to the main part of the proof,  $\bigcup_{x \in B} F_x$  is a compact set, therefore  $\bar{V}$  is also compact. Moreover

$$\forall x \in K, F_x \subseteq U \implies \bar{V} \subseteq \bigcup_{x \in B} F_x \subseteq U$$

Combining, we have

- $K \subseteq V \subseteq \overline{V}$ ,
- $V$  is open and  $\overline{V}$  is compact, and
- $\overline{V} \subseteq U$

This completes the proof. ■

**Theorem 4.32**

**WTS 32.1.** *Urysohn's Lemma, Locally Compact Version.* For any LCH space  $X$ , and if  $K \subseteq U \subseteq X$  where  $K$  is compact and  $U$  is open, then there exists some  $f \in C(X, [0, 1])$  with

- $f = 1$  on  $K$
- $f = 0$  outside some compact  $\bar{V} \subseteq U$

*Proof.* Let  $V$  be as in Theorem 4.31, for our fixed  $K \subseteq U \subseteq X$ , there exists a pre-compact, open  $V$  that satisfies

$$K \subseteq V \subseteq \bar{V} \subseteq X$$

It follows that this  $(\bar{V}, \mathcal{T}_{\bar{V}})$  is a normal space by Theorem 4.25 (compact Hausdorff spaces are normal), and by Urysohn's Lemma (Theorem 4.15) on normal spaces, since we can easily find two disjoint closed subsets of  $\bar{V}$  in the form of

- $K \subseteq V^\circ = V \subseteq \bar{V}$  (compact sets in Hausdorff spaces are closed)
- $\partial V = \bar{V} \cap \bar{V}^c$  (closed sets in compact spaces are compact)
- $K \subseteq V^\circ$  implies that  $K \cap \partial V = K \cap (\bar{V} \setminus V^\circ) = \emptyset$

Then there exists a continuous  $f|_{\bar{V}} \in C(\bar{V}, [0, 1])$  that evaluates to

- $f|_{\bar{V}} = 1$  on closed  $K$
- $f|_{\bar{V}} = 0$  on closed  $\partial V$

Now let us extend  $f|_{\bar{V}}$  to  $f$  by defining

$$f|_{(\bar{V})^c} = 0$$

We will show that this extension of  $f$  is indeed continuous. Indeed, for every closed set  $E \subseteq [0, 1]$  that does not contain 0, we have:

$$0 \notin E \implies \{0\} \cap E = \emptyset \implies f^{-1}(\{0\}) \cap f^{-1}(E) = \emptyset$$

But  $(\bar{V})^c \subseteq f^{-1}(\{0\})$  therefore

$$(\bar{V})^c \cap f^{-1}(\{0\}) \cap f^{-1}(E) = (\bar{V})^c \cap f^{-1}(E) = \emptyset \implies f^{-1}(E) \subseteq \bar{V}$$

We can write

$$f^{-1}(E) = f|_{\bar{V}}^{-1}(E)$$

But we know that  $f|_{\bar{V}}$  is continuous, so  $f|_{\bar{V}}^{-1}(E)$  must be closed (with respect to  $\bar{V}$ ), and therefore is closed wrt  $X$ , since  $\bar{V}$  is closed wrt  $X$ .

For the case where  $0 \in E$ , note that

$$f^{-1}(E) = (f^{-1}(E) \cap \bar{V}) \cup (f^{-1}(E) \cap (\bar{V})^c) = (f|_{\bar{V}})^{-1}(E) \cup (f|_{\bar{V}^c})^{-1}(E)$$

The above equalities are messy in print. They are but a simple consequence of disjoint decomposition of the pre-images, since

$$\bar{V} \cap f^{-1}(E) = \{x \in \bar{V} : f(x) \in E\} = f|_{\bar{V}}^{-1}(E)$$

Back to our main discussion, recall that for every  $x \in \partial V$

$$f(x) = 0 \in f^{-1}(\{0\}) \subseteq f^{-1}|_{\bar{V}}(E)$$

Therefore  $\partial V \subseteq f^{-1}|_{\bar{V}}(E)$ , and  $(\bar{V})^c = f^{-1}|_{(\bar{V})^c}(E)$  gives us (since  $V^c$  is closed),

$$\begin{aligned} f^{-1}(E) &= f^{-1}|_{\bar{V}}(E) \cup \partial V \cup (\bar{V})^c \\ &= f^{-1}|_{\bar{V}}(E) \cup \overline{(V^c)} \cup (\bar{V})^c \\ &= f^{-1}|_{\bar{V}}(E) \cup (V^c \cup V^{\text{co}}) \\ &= f^{-1}|_{\bar{V}}(E) \cup V^c \end{aligned}$$

Since  $f^{-1}|_{\bar{V}}(E)$  and  $V^c$  are closed subsets of  $X$ , then  $f^{-1}(E)$  is also closed, and  $f \in C(X, [0, 1])$ . ■



**Theorem 4.33**

**WTS 33.1.** *Every LCH space is completely regular (or  $T_{3.5}$ ).*

*Proof.* Recall that a space  $X$  is completely regular if it is  $T_1$  and every closed subset  $A$  and every  $x \notin A$  there exists some

$$f \in C(X, [0, 1]), \quad f(x) = 1, \quad f|_A = 0$$

Fix a closed set  $A \subseteq X$ , then for every  $x \in A^c$ , there exists a compact  $E_x \in \mathcal{N}_B(x)$  with  $E_x \subseteq A^c$  (by Theorem 4.30).

Note that  $E_x \subseteq A^c$  where  $E_x$  is compact and  $A^c$  is closed, then an application of Theorem 4.31 tell us that there exists an  $f \in C(X, [0, 1])$  such that for every  $x \in E_x$ ,  $f(x) = 1$  and for points  $y \notin A^c$  (which means that  $y \in A$ ),  $f(y) = 0$ . Therefore  $X$  is completely regular. ■

**Theorem 4.34**

**WTS 34.1.**

*Proof.*



**Theorem 4.35**

**WTS 35.1.** *If  $X$  is a LCH space, we claim that*

$$\overline{C_c(X)} = C_0(X)$$

*Proof.* We begin by proving several things that are mentioned before this Theorem, namely

$$C_c(X) \subseteq C_0(X) \subseteq BC(X)$$

Fix an  $f \in C_c(X)$ , and for every  $\varepsilon > 0$ ,

$$x \in |f|^{-1}([\varepsilon, +\infty)) \implies |f(x)| \geq \varepsilon > 0$$

Therefore  $|f|^{-1}([\varepsilon, +\infty))$  is a closed subset of  $\text{supp}(f)$ , since  $(-\infty, \varepsilon)$  is open in  $\mathbb{R}$ , then  $[\varepsilon, +\infty)$  is a closed set. And by continuity of  $|\cdot| \circ f$  (a composition of two continuous functions),  $|f|^{-1}([\varepsilon, +\infty))$  is closed. Using the fact that closed subsets of compact  $\text{supp}(f)$  are also compact, we get  $f \in C_0(X)$ .

Next, we show that  $C_0(X) \subseteq BC(X)$ . Fix any element  $f$  of  $C_0(X)$  with an arbitrary  $\varepsilon > 0$ , then  $E_\varepsilon = \{x \in X : |f(x)| \geq \varepsilon\}$  is compact. The continuity of  $f$  guarantees that the direct image of a compact set is another compact set (Theorem 4.26)

$$|f|(E_\varepsilon) \text{ is a compact subset of } \mathbb{R}$$

And therefore for every  $x \in E_\varepsilon \implies |f(x)| \in |f|(E_\varepsilon)$ , then by Heine-Borel, there exists some  $M \geq 0$  such that  $|f(x)| \leq M$ . If  $x \notin E_\varepsilon$ , then by definition of  $E_\varepsilon$ , implies that  $|f(x)| < \varepsilon$ . Then  $|f(x)| \leq M + \varepsilon$  for every  $x \in X$ . Hence  $f \in BC(X)$ .

Here I wish to offer an alternate proof for  $C_0(X) \subseteq BC(X)$ , we begin by constructing an open cover for  $\text{supp}(f)$  such that

$$\{U_n\}_{n>0} = \{x \in X : |f(x)| < n\}$$

Then there exists a finite subcollection of  $\{U_n\}_{n \in B}$  where  $B$  is a finite set, then define  $M = 1 + \sum_{n \in B} n$  and for every  $x \in \text{supp}(f)$  we have  $|f(x)| < n$  and since  $n > 0$  this holds for every  $x \in X$  too. Therefore  $f \in BC(X)$ .

For the main proof of Theorem 4.35, since  $BC(X)$  is endowed with the uniform metric, it is also first countable, and therefore by Theorem 4.6, it suffices to show that every sequence  $\{f_n\}_{n \geq 1} \subseteq C_c(X)$  converges in  $C_0(X)$ . And every element  $f \in C_0(X)$  has a convergence sequence in  $C_c(X)$ .

Fix a convergent sequence  $\{f_n\}_{n \geq 1} \subseteq C_c(X)$  that converges uniformly to some  $f \in BC(X)$  (since  $BC(X)$  is a closed subset of  $C(X)$  with respect to the uniform norm), then for every  $\varepsilon > 0$ , there exists some  $n \geq 1$  with

$$\|f_n - f\|_u < \varepsilon$$

We aim to show that  $(\text{supp}(f_n))^c \subseteq |f|^{-1}((-\infty, \varepsilon))$ , so fix any  $x \notin \text{supp}(f_n)$ , then

$$|f(x) - f_n(x)| = |f(x)| \leq \|f - f_n\|_u < \varepsilon$$

This establishes the estimate, and taking complements

$$|f|^{-1}([\varepsilon, +\infty)) \subseteq \text{supp}(f_n)$$

Therefore for any arbitrary  $\varepsilon > 0$ ,  $\{x \in X, |f(x)| \geq \varepsilon\}$  is compact, and  $\overline{C_c(X)} \subseteq C_0(X)$ . Conversely, fix any  $f \in C_0(X)$ , and for every  $n \geq 1$ , define

$$K_n = \{x \in X, |f(x)| \geq 1/n\}$$

Using Urysohn's Lemma for our LCH space  $X$ , there exists some  $g_n$  that has a compact support, and  $g_n(x) = 1$  for every  $x \in K_n$ . We then write  $f_n = g_n \cdot f \in C_c(X)$ . We wish to show that  $f_n \rightarrow f$  uniformly. Notice that for any fixed  $n \geq 1$ , if  $x \in K_n$  then

$$f_n(x) = f(x) \implies |f_n - f|(x) = 0$$

If  $x \notin K_n$ ,  $|f(x)| < 1/n$  (recall what  $K_n$  does), and  $f_n = g_n \cdot f \in [0, 1]$  by definition of  $g_n$  from Theorem 4.32, hence

$$|f_n(x) - f(x)| = |f(x)| \cdot |1 - g_n| \leq |f(x)| < 1/n$$

Taking the supremum over  $x \in X$ , we have

$$\|f_n - f\|_u < 1/n \rightarrow 0$$

As we send  $n$  to  $+\infty$ , and  $f_n \rightarrow f$  uniformly. This completes the proof. ■

**Theorem 4.36**

**WTS 36.1.**

*Proof.*



**Theorem 4.37**

**WTS 37.1.** *If  $X$  is an LCH space and  $E \subseteq X$ .  $E$  is closed if and only if  $E \cap K$  is closed for every compact  $K \subseteq X$ .*

*Proof.* Suppose that  $E$  is closed, then  $E \cap K$  is closed, since compact subsets of Hausdorff spaces are closed, and  $E \cap K \subseteq K$  tells us that  $E \cap K$  is indeed compact.

Now suppose that  $E$  is not closed, by Theorem 4.1,  $E \neq \overline{E}$ , so pick some  $x \in (\overline{E} \setminus E) = \text{acc}(E) \cap E^c$ , since  $X$  is locally compact, let  $K_x$  be a compact neighbourhood of  $x$ , then for every neighbourhood  $U \in \mathcal{N}_B(x)$ , we have

$$x \in U^\circ, x \in K_x^\circ \implies x \in (U^\circ \cap K_x^\circ) \subseteq (U \cap K_x)^\circ$$

Since  $(U^\circ \cap K_x^\circ)$  is an open subset of  $(U \cap K_x)$ , then  $(U \cap K_x) \in \mathcal{N}_B(x)$ , and recall that  $x \in \text{acc}(E)$ , therefore

$$(U \cap K_x) \cap E \setminus \{x\} = U \cap (K_x \cap E) \neq \emptyset$$

But  $x \notin E \implies x \notin E \cap K_x$ . So  $x$  is an accumulation point of  $E \cap K_x$  that is not in  $E \cap K_x$ . Therefore there exists some  $E \cap K_x$  (with  $K_x$  compact) that is not closed. ■

**Theorem 4.38**

**WTS 38.1.**

*Proof.*



**Theorem 4.39**

**WTS 39.1.**

*Proof.*





**Theorem 4.40****WTS 40.1.***Proof.*

**Theorem 4.41**

**WTS 41.1.**

*Proof.*



## Exercises

### Exercise 4.1

**WTS 42.1.** *If  $\text{card } \mathbf{X} \geq 2$ , there is a topology on  $\mathbf{X}$  that is  $T_0$  but not  $T_1$ .*

*Proof.* Let  $\mathcal{T}_{\mathbf{X}} = \{\emptyset\} \cup \{\{x\} \cup B, B \subseteq \mathbf{X}\}$ , where  $x \in \mathbf{X}$  is any point in  $\mathbf{X}$ . Suppose  $U_1$ , and  $U_2$  are open sets in  $\mathcal{T}_{\mathbf{X}}$ , if either is empty then their intersection must be contained in  $\mathcal{T}_{\mathbf{X}}$ . Otherwise  $U_1 = \{x\} \cup B_1$ , and  $U_2 = \{x\} \cup B_2$ , where  $B_1$  and  $B_2$  are subsets of  $\mathbf{X}$ .

$$U_1 \cap U_2 = \{x\}(B_1 \cap B_2) \in \mathcal{T}_{\mathbf{X}}$$

Notice also  $\{\emptyset, \mathbf{X}\} \subseteq \mathcal{T}_{\mathbf{X}}$ . Fix an arbitrary family of open sets  $\{U_{\alpha}\}_{\alpha \in A}$ , in similar fashion we have  $\bigcup U_{\alpha} = \{x\} \cup \left(\bigcup B_{\alpha \in A}\right)$  so their union is contained in  $\mathcal{T}_{\mathbf{X}}$  as well.

This topology is  $T_0$ . Fix  $y \neq z$  in  $\mathbf{X}$ , if either  $y$  or  $z$  is  $x$ , then choosing  $\{x\}$  does the job. So assume  $x \neq y \neq z \neq x$ , and  $\{y\} \cup \{x\}$  is an open set that does not contain  $z$ . This topology cannot be  $T_1$ , as  $x$  sticks onto every open set, so there are no open sets which separate  $x$  from the other points in  $\mathbf{X}$ . ■

## Exercise 4.2

**WTS 43.1.** *If  $\mathbf{X}$  is an infinite set, the cofinite topology on  $\mathbf{X}$  is  $T_1$  but not  $T_2$ , and is first countable iff  $\mathbf{X}$  is countable.*

*Proof.* We will first verify that the cofinite topology  $\mathcal{T}_{\mathbf{X}}$  is a topology.

$$\mathcal{T}_{\mathbf{X}} = \left\{ U, \quad U^c \text{ is finite.} \right\} \cup \{\emptyset\}$$

So that  $\{\emptyset, \mathbf{X}\} \subseteq \mathcal{T}_{\mathbf{X}}$ . Let  $U_1$  and  $U_2$  be a pair of open sets, assuming if neither of them are empty, then  $U_2^c$  and  $U_1^c$  are finite sets, so that  $U_1^c \cup U_2^c$  is finite as well. Use DeMorgan to see that  $U_1 \cap U_2 \in \mathcal{T}_{\mathbf{X}}$ .

If  $\{U_{\alpha}\}_{\alpha \in A}$  is an arbitrary collection of open sets, then

$$\bigcap_{\alpha \in A} U_{\alpha}^c \subseteq U_{\beta}^c$$

where  $\beta \in A$  is arbitrary, so  $U_{\beta}^c$  is finite. And the union  $\bigcup U_{\alpha}$  is contained in  $\mathcal{T}_{\mathbf{X}}$ .

To show that  $\mathcal{T}_{\mathbf{X}}$  is  $T_1$ , every singleton set is closed. To show that  $\mathcal{T}_{\mathbf{X}}$  is not  $T_2$ , fix  $x \neq y$ . If  $B_x$  and  $B_y$  are open sets that contain  $x$  and  $y$  respectively. If  $B_x$  and  $B_y$  disjoint, then

$$B_x \subseteq B_y^c$$

Which means  $B_x$  is an open, finite subset. But the only open and finite subset of  $\mathbf{X}$  is the empty set. This contradicts  $x \in B_x$ .

If  $\mathbf{X}$  is countable, we will find a neighbourhood base  $\mathcal{N}_B(x)$  for any  $x \in \mathbf{X}$  as follows:

- We can index  $\mathbf{X}$  using  $\mathbb{N}^+ \cup \{0\}$ , so without loss of generality, let  $x_0 = x$ , and
- Define  $U_1 = \{x_1\}^c$ , and  $U_n = \bigcap_{j=1}^n \{x_j\}^c$  are open sets that contain  $x$ . Equivalently,

$$U_n = \left\{ x_j, j \geq n+1 \right\} \cup \{x_0\}$$

- If  $V$  is an open set that contains  $x_0$ , then  $V^c$  is finite, let  $M \in \mathbb{N}^+$  be the largest index of  $x_j \notin V$  (the negation of this is that if  $j \geq M + 1$ , then  $x_j \in V$ ) then  $U_{M+1} \subseteq V$  as needed, and  $\mathbf{X}$  is first countable.

Conversely, if  $\mathbf{X}$  is first countable, we can find a descending sequence of neighbourhoods which form a neighbourhood base,  $\{U_j\}_{j \geq 1} \subseteq \mathcal{T}_{\mathbf{X}}$ . And each  $U_j^c$  is finite, so  $\bigcup U_j^c$  is countable. Assume for contradiction that  $\mathbf{X}$  is uncountable, then

$$\bigcup U_j^c = \left( \bigcap U_j \right)^c$$

is countable, hence the intersection  $\bigcap U_j$  must be uncountable (hence infinite). Pick  $y \neq x$ , where  $y$  belongs in the intersection of all neighbourhoods  $U_j$ . This contradicts the fact that  $\{U_j\}$  is a neighbourhood base, as  $x$  is an element in the open set  $\{y\}^c$  therefore there must be a  $U_k$

$$x \in U_k \subseteq \{y\}^c$$

But  $y \in U_k$  for each  $U_k$  and the proof is complete. ■

### Exercise 4.3

**WTS 44.1.** *Every metric space is normal. (If  $A, B$  are disjoint closed sets in the metric space, consider the set of points  $x$  where  $d(x, A) < d(x, B)$  or  $d(x, A) > d(x, B)$ ).*

*Proof.* First, we show that if  $A$  is closed, then  $d(x, A) = 0 \iff x \in A$ . If  $x \in A$ , then  $d(x, A) \leq d(x, x) = 0$ . if  $x \notin A$ , then there exists a ball of radius  $\varepsilon > 0$  where  $B(\varepsilon, x) \cap A = \emptyset$ . Hence,  $\varepsilon$  is a lower bound for the set  $\{d(x, y), y \in A\}$ , taking the infimum over this set we see that  $d(x, A) \geq \varepsilon > 0$ .

Fix some  $x \in \Phi_A$  where  $\Phi_A = \left\{y \in X, d(y, A) < d(y, B)\right\}$ . We wish to find an open ball about  $x$  that is contained in  $\Phi_A$ . The Triangle Inequality works for this definition of distance as well, as

$$f(a) \leq g(a), \forall a \in A \implies \inf_{a \in A} f(a) \leq \inf_{a \in A} g(a) \quad (9)$$

If  $a \in A$ , then  $d(z, A) \leq d(x, z) + d(x, A)$ , using Equation (9) yields

$$\begin{cases} d(z, A) \leq d(x, A) + d(x, z) \\ d(x, B) - d(x, z) \leq d(z, B) \end{cases}$$

where  $z \in B(\varepsilon, x)$  so  $d(x, z) \lesssim \varepsilon$ . The second estimate above is found by 'flipping an upper bound to become a lower bound'. We can choose  $d(x, z)$  sufficiently small that

$$d(x, A) + d(x, z) < d(x, B) - d(x, z)$$

in order to 'pipe' the two inequalities, so

$$2d(x, z) < d(x, B) - d(x, A) \quad (10)$$

Take  $\varepsilon = [d(x, B) - d(x, A)]3^{-1}$ , then  $z \in B(\varepsilon, x)$  implies  $d(x, z) < \varepsilon$ , and Equation (10) holds. See Figure 2 for details. ■

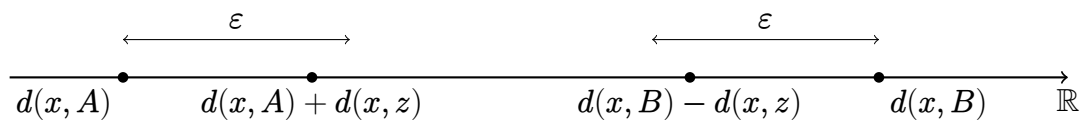


Figure 2: Exercise 4.3: Finding a  $\varepsilon$  small enough that fits within  $d(x, A) < d(x, B)$

### Exercise 4.4

**WTS 45.1.**

*Proof.*

■

## Exercise 4.5

**WTS 46.1.** *Every separable metric space is second countable.*

*Proof.* We wish to show that if  $\mathbf{X}$  is a metric space, then

$$\text{second countable} \iff \text{separable}$$

Suppose  $\mathbf{X}$  is separable, where  $A$  is a countable dense subset in  $\mathbf{X}$ , and  $x \in \mathbf{X}$ . Let  $U$  be an open set that contains  $x$ , so  $B(\varepsilon, x) \subseteq U$  for some  $\varepsilon > 0$ .  $B(\varepsilon/2, x)$  is a non-empty open set, therefore contains some  $y \in A$  (this follows from the definition of density). If we choose  $r \in \mathbb{Q}$  wisely,

$$d(x, y) < r < \varepsilon/2$$

So that  $x \in B(r, y)$ , and if  $z \in B(r, y)$ , then

$$d(x, z) \leq d(x, y) + d(z, y) < r + r < \varepsilon$$

So  $x \in B(r, y) \subseteq U$ . But  $\{B(r, y), r \in \mathbb{Q}^+, y \in A\}$  is countable. Therefore  $\mathbf{X}$  is second countable.

Conversely, Proposition 4.5 gives us the  $\Leftarrow$  direction. But we will repeat anyway, if  $\mathbf{X}$  is second-countable with  $\mathcal{E}$  as a countable base, then take

$$W = \left\{ x_\alpha \in U, \quad U \in \mathcal{E} \right\}$$

by picking a point from each set, we claim  $W$  is dense in  $\mathbf{X}$ , so  $\overline{W} = \mathbf{X}$ . If not, then  $\overline{W}^c \neq \emptyset$ , and

$$\overline{W}^c = (W^c)^o \neq \emptyset$$

Pick a point  $x \in W^c$ , which is an open set containing  $x$ . But the way we chose  $W$  does not allow for any open set  $U \in \mathcal{E}$  with  $x \in U \subseteq W^c$ , since

By picking one point from each of the base sets, and flipping the complement. We can ensure that no  $U \in \mathcal{E}$  from the countable base is fully contained as a subset of  $W^c$ , as there is always one point from each  $U \in \mathcal{E}$  that cannot be found in  $W^c$

■



**Exercise 4.6**

**WTS 47.1.**

*Proof.*



**Exercise 4.7**

**WTS 48.1.** *If  $\mathbf{X}$  is a topological space, a point  $x \in \mathbf{X}$  is called a cluster point of the sequence  $\{x_j\}$  if for every neighbourhood  $U \in \mathcal{N}(x)$ ,  $x_j \in U$  for infinitely many  $j$ . If  $\mathbf{X}$  is first countable,  $x$  is a cluster point of  $\{x_j\}$  iff some subsequence of  $\{x_j\}$  converges to  $x$ .*

*Proof.* Suppose  $\{x_n\}$  has a cluster point in  $z \in \mathbf{X}$ . Fix a descending sequence of neighbourhoods  $U_k \subseteq \mathcal{N}(z)$ , where

$$U_1 \supseteq U_2 \supseteq \cdots \supseteq U_k$$

Define  $n_k = \text{least} \left\{ j \in \mathbb{N}^+, j > n_{k-1}, x_j \in U_k \right\}$  with  $n_0 = 0$ , so that for every  $m \geq k$ ,  $x_{n_m} \in U_k$  eventually. And  $\{x_{n_j}\}_{j \geq 1}$  is a subsequence which converges to  $z$ . This proves ( $\implies$ ).

Conversely (this part does not require that  $\mathbf{X}$  be first countable, if  $\{x_{n_k}\}_{k \geq 1}$  is a subsequence that converges to  $z \in \mathbf{X}$ . Every neighbourhood of  $z$  must intersect all but infinitely many  $x_{n_k}$ , therefore  $z$  is a cluster point of  $\{x_n\}$ . ■

**Exercise 4.8**

**WTS 49.1.** *If  $\mathbf{X}$  is an infinite set with the cofinite topology and  $\{x_j\}$  is a sequence of distinct points in  $\mathbf{X}$ , then  $x_j \rightarrow x$  for every  $x \in \mathbf{X}$ .*

*Proof.* The intuition here is that the cofinite topology does not distinguish between points, so it acts as a type of jelly that hides the points.

Let  $x \in \mathbf{X}$  be arbitrary, if  $U \in \mathcal{N}(x)$  then  $U^o \in \mathcal{N}(x)$ , so that  $\{y_j\}_{j \leq k}$  are the  $k$  points that are required to extend  $U^o$  to  $\mathbf{X}$ . (All but finitely many points are in any open set of  $\mathbf{X}$ ).

There exists a large  $N \in \mathbb{N}^+$  so that for every  $n \geq N$ ,

$$x_j \notin \{y_j\}_{j \leq k} \implies x_j \in U^o$$

eventually. And  $x_j \rightarrow x$ . ■

**Exercise 4.9****WTS 50.1.***Proof.*

**Exercise 4.10**

**WTS 51.1.** *A topological space  $\mathbf{X}$  is called disconnected if there exists non-empty, disjoint open sets  $U, V$  and  $U \cup V = \mathbf{X}$ ; otherwise  $\mathbf{X}$  is connected. When we speak of connected or disconnected subsets of  $\mathbf{X}$ , we refer to the relative topology on them*

- (a)  *$\mathbf{X}$  is connected iff  $\emptyset$  and  $\mathbf{X}$  are the only two clopen sets.*
- (b) *If  $\{E_\alpha\}_{\alpha \in A}$  is a collection of connected subsets of  $\mathbf{X}$ , and  $\bigcap E_\alpha$  is non-empty, then  $\bigcup E_\alpha$  is connected.*
- (c) *If  $A \subseteq \mathbf{X}$  is connected, then  $\overline{A}$  is connected,*
- (d) *Every point in  $x \in \mathbf{X}$  contained in a unique maximal connected subset of  $\mathbf{X}$ , and this subset is closed. It is called the connected component of  $x$ .*

*Proof.* The proof is rather long, so we will ■

**Exercise 4.14**

**WTS 52.1.** If  $\mathbf{X}$  and  $\mathbf{Y}$  are topological spaces,  $f : \mathbf{X} \rightarrow \mathbf{Y}$  is continuous iff  $\overline{f(A)} \subseteq f(\overline{A})$  for every  $A \subseteq \mathbf{X}$  iff  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$  for all  $B \subseteq \mathbf{Y}$ .

*Proof. First Equivalence:* If  $f$  is continuous, fix any  $A \subseteq \mathbf{X}$ , and  $z \in \overline{A}$ , by WTS 0.4 (I will spare you the flipping by including):

$$\overline{A} = \{x \in \mathbf{X}, \forall U \in \mathcal{N}(x), U \cap A \neq \emptyset\}$$

Let  $U \in \mathcal{N}(f(z))$ , so that  $f^{-1}(U^\circ)$  is an open set containing  $z$  and  $f^{-1}(U^\circ) \in \mathcal{N}(z)$ , so

$$f^{-1}(U^\circ) \cap A \neq \emptyset \implies U^\circ \cap f(A) \subseteq U \cap f(A)$$

so  $f(\overline{A}) \subseteq \overline{f(A)}$ . Conversely, suppose  $f(\overline{A}) \subseteq \overline{f(A)}$  holds for every  $A \subseteq \mathbf{X}$ . The following is a sequence of symbolic manipulations that I found but have zero intuitive understanding about. First take the inverse image

$$\overline{A} \subseteq f^{-1}\left(\overline{f(A)}\right) \subseteq f^{-1}\left(f(\overline{A})\right)$$

Next, let  $F$  be a closed set in  $\mathbf{Y}$ , and make the substitution  $A = f^{-1}(F)$ , hence

$$\overline{f^{-1}(F)} \subseteq f^{-1}\left(\overline{f(f^{-1}(F))}\right) \subseteq f^{-1}(\overline{F}) = f^{-1}(F)$$

for the second inclusion we used the monotonicity of the closure, and since  $\overline{f^{-1}(F)} = f^{-1}(F)$ , we are done.

*Second Equivalence:* Suppose  $f \in C(\mathbf{X}, \mathbf{Y})$ , then  $\overline{B} \subseteq \mathbf{Y}$  is a closed set, so  $f^{-1}(\overline{B})$  is closed in  $\mathbf{X}$ . By monotonicity of the inverse image,

$$f^{-1}(B) \subseteq f^{-1}(\overline{B}) \implies \overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$$

Conversely, if  $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$  for any  $B \subseteq \mathbf{Y}$ , take any closed  $B \subseteq \mathbf{Y}$ , and

$$\overline{f^{-1}(B)} \subseteq f^{-1}(B) \subseteq \overline{f^{-1}(B)}$$

so  $f^{-1}(B)$  is closed, and  $f$  is in  $C(\mathbf{X}, \mathbf{Y})$ . ■

**Exercise 4.16****WTS 53.1.***Proof.*

- (a) Let  $x \in \{f \neq g\}$ , then there exists disjoint open subsets of  $\mathbf{Y}$ ,  $f(x) \in U$  and  $g(x) \in V$ ,  $U \cap V = \emptyset$ , but  $f^{-1}(U) \cap g^{-1}(V)$  is an open set in  $\mathbf{X}$  that contains  $x$ . Therefore  $\{f \neq g\}$  is open in  $\mathbf{X}$ .
- (b) Suppose  $\{f = g\} = E$  is dense in  $\mathbf{X}$ . Let  $x \in E$ , induces two disjoint open sets exactly like in part a. This is an open set that contains  $x$ , and  $y \in f^{-1}(U) \cap g^{-1}(V) \cap E$ . Since  $y \in E$ , it follows that  $f(y) = g(y)$ , and

$$\begin{cases} y \in f^{-1}(U) \implies f(y) \in U \\ y \in g^{-1}(V) \implies g(y) \in V \end{cases}$$

■

**Exercise 4.17**



## Chapter 5

**Theorem 5.1**

**WTS 1.1.**

*Proof.*



**Theorem 5.2**

**WTS 2.1.**

*Proof.*



**Theorem 5.3**

**WTS 3.1.**

*Proof.*



**Theorem 5.4**

**WTS 4.1.**

*Proof.*



**Theorem 5.5**

**WTS 5.1.**

*Proof.*



**Theorem 5.6**

**WTS 6.1.**

*Proof.*



**Theorem 5.7**

**WTS 7.1.**

*Proof.*





**Theorem 5.8**

**WTS 8.1.**

*Proof.*



## Chapter 6

## Theorem 6.1

**WTS 1.1.** *For every  $a, b \geq 0$ , and  $0 < \lambda < 1$ , then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1 - \lambda)b$$

**Theorem 6.2**

**WTS 2.1.**

*Proof.*



**Theorem 6.3**

**WTS 3.1.**

*Proof.*



**Theorem 6.4**

**WTS 4.1.**

*Proof.*



**Theorem 6.5**

**WTS 5.1.**

*Proof.*



**Theorem 6.6****WTS 6.1.**



**Theorem 6.7**

**WTS 7.1.**

**Theorem 6.8****WTS 8.1.**

**Theorem 6.9**

**WTS 9.1.**

*Proof.*



**Theorem 6.10**

**WTS 10.1.**

*Proof.*



**Theorem 6.11**

**WTS 11.1.**

*Proof.*



**Theorem 6.12****WTS 12.1.***Proof.*

**Theorem 6.13**

WTS 13.1.

**Theorem 6.14**

**WTS 14.1.**



**Theorem 6.15****WTS 15.1.**

*Proof.* First suppose that  $(X, \mathcal{M}, \mu)$  is finite measure space. If  $\mu(X) < +\infty$ , then for every  $E \in \mathcal{M}$ , by monotonicity  $E \subseteq X$  yields  $\mu(E) \leq \mu(X) < +\infty$ . Next, for any  $p < +\infty$ ,  $\|\chi_E\|_p^p < +\infty$  and  $\|\chi_E\|_{+\infty} \leq 1 < +\infty$ . So all indicator functions are in  $L^p$ .

It follows that every simple function is also in  $L^p$ , since it is a finite linear combination of indicators. We now define  $\nu(E) = \phi(\chi_E)$ , we wish to show that  $\nu : \mathcal{M} \rightarrow \mathbb{C}$  is a complex measure which is absolutely continuous with respect to  $\mu$ .

To show  $\sigma$ -additivity, fix any disjoint sequence  $\{E_j\}_{j \geq 1} \subseteq \mathcal{M}$ . Where we also note that  $\mu(E) = \mu(\cup E_j) < +\infty$ . Now suppose that  $p < +\infty$ , then the following converges in the  $p$ -norm

$$\chi_E = \sum_{j \geq 1} \chi_{E_j}$$

We divert our attention to the following,

$$E \setminus \left( \bigcup_{j \leq n} E_j \right) = \left( \bigcup_{j \geq 1} E_j \right) \setminus \left( \bigcup_{j \leq n} E_j \right) = \bigcup_{j \geq n+1} E_j$$

and define  $F_{n+1}$  as the rightmost member above. Then  $\{F_{n \geq 1}\}$  is a decreasing sequence of sets. All sets are of finite measure, hence  $\mu(E) - \mu(\cup_{j \leq n} E_j) = \mu(F_{n+1}) \rightarrow 0$ .

Now, for any fixed  $n \geq 1$ ,

$$\left| \chi_E - \sum \chi_{E_{j \leq n}} \right| = \left| \sum \chi_{E_{j \geq n+1}} \right|$$

the above holds pointwise almost everywhere. Since the above function evaluates either to 0 or to 1, taking the  $p$ th power does not change pointwise, and

$$\left| \sum \chi_{E_{j \geq n+1}} \right|^p = \left| \sum \chi_{E_{j \geq n+1}} \right| = \sum \chi_{E_{j \geq n+1}}$$

Convergence in  $p$ -norm is given by

$$\left\| \chi_E - \sum \chi_{E_{j \leq n}} \right\| = \left\| \sum \chi_{E_{j \geq n+1}} \right\| = \mu(F_{n+1})^{1/p}$$

Applying continuity, and linearity to our  $\phi \in L^{p*}$

$$\begin{aligned} \nu(E) &= \phi(\chi_E) \\ &= \phi\left(\lim_{n \rightarrow \infty} \sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \phi\left(\sum \chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \sum \phi\left(\chi_{E_{j \leq n}}\right) \\ &= \lim_{n \rightarrow \infty} \sum \nu(E_{j \leq n}) \end{aligned}$$

To show absolute convergence, recall that for any  $\phi(\chi_{E_j}) \in \mathbb{C}$ , define  $\beta_j = \overline{\text{sgn}(\|\phi(\chi_{E_j})\|)}$  then multiplication yields

$$\|\phi(\chi_{E_j})\| = \beta_j \phi(\chi_{E_j}) = \phi(\beta_j \chi_{E_j})$$

Then, the following series converges in the  $p$ -norm.

$$\left\| \sum_{j \geq 1} \beta_j \chi_{E_j} - \sum_{j \leq n} \beta_j \chi_{E_j} \right\|_p = \left\| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right\|_p$$

And because  $\left| \sum_{j \geq n+1} \beta_j \chi_{E_j} \right|$  is pointwise equal to  $\left| \sum_{j \geq n+1} \chi_{E_j} \right|$ , since  $|\beta_j| = 1$  for every  $j \geq 1$ . We can reuse the same continuity and linearity argument. We also note that  $\sum_{j \geq 1} \beta_j \chi_{E_j} \in L^p$  since its  $p$ -norm is equal to  $\mu(E)^{1/p}$ .

$$\begin{aligned}
\sum_{j \geq 1} |\nu(E_j)| &= \sup_{n \geq 1} \sum_{j \leq n} \|\nu(E_{j \leq n})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \|\phi(\chi_{E_j})\| \\
&= \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \phi(\chi_{E_j}) \\
&= \lim_{n \rightarrow \infty} \phi \left( \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&= \phi \left( \lim_{n \rightarrow \infty} \sum_{j \leq n} \beta_j \chi_{E_j} \right) \\
&\leq \|\phi\| \left\| \sum_{j \geq 1} \beta_j \chi_{E_j} \right\|_p \\
&< +\infty
\end{aligned}$$

Assuming the above estimate holds, then we only need  $\nu(E) = \phi(\chi_E) = \mu(E) = 0$  ( $\nu$  is now a measure and  $\nu \ll \mu$ ). As the indicator of a null set is equal to the zero element in  $L^p$ . Then by Radon-Nikodym we can have some  $g \in L^1(\mu)$  such that

$$d\nu = g d\mu$$

We wish to satisfy the hypothesis of Theorem 6.14 for our function  $g$ . For every  $\chi_E$  measurable,  $\|\chi_E g\|_1 \leq \|g\|_1 < +\infty$ , by monotonicity of the integral in  $L^+$ . So any simple function,  $\alpha = \sum a_j \cdot \chi_{E_j}$  means that  $\alpha g$  is in  $L^1(\mu)$ , and

$$\phi(\alpha) = \int \alpha g d\mu$$

If  $\|\alpha\|_p = 1$ , then

$$\left| \int \alpha g \right| = |\phi(\alpha)| \leq \|\phi\| \cdot \|\alpha\|_p = \|\phi\| < +\infty$$

Then

$$M_q(g) = \sup \left\{ \left| \int \alpha \cdot g \right|, \|\alpha\|_p = 1, \text{ and } \alpha \text{ is simple, and vanishes out-} \right\} < \infty$$

side a set of finite measure.

Since  $S_g = \{x \in X, g(x) \neq 0\}$  is  $\sigma$ -finite, an application of Theorem 6.14 tells us that  $g \in L^q$ , and  $M_q(g) = \|g\|_q \leq \|\phi\| < +\infty$ . Now that we know  $g$  is in  $L^q$  we can use the density of  $\alpha$  in  $L^p$  to show, for every single  $f \in L^p$

$$\phi(f) = \int f g d\mu$$

Conjure a sequence of  $\alpha$ 's, and call them  $\{f_n\} \rightarrow f$  p.w.a.e, then each  $f_n \cdot g \in L^1$ . An application of the DCT and continuity gives us

$$\phi(\lim f_n) = \lim \phi(f_n) = \lim \int f_n g d\mu = \int f g d\mu = \phi(f)$$

This completes the proof for when  $\mu$  is finite.

Let us upgrade our  $\mu$  into a  $\sigma$ -finite measure. Then there exists an increasing sequence  $\{E_n\} \nearrow X$  such that each  $E_n$  is of finite measure. Define

$$P_n = \{L^p, \forall f, |f| = |f| \cdot \chi_{E_n}\}$$

So every function in  $P_n$  vanishes outside a set of finite measure and is also in  $L^p$ . And  $Q_n$  is defined in a similar manner. Now, fix our  $\phi \in L^{p*}$ , and for each  $f \in P_n$ , there exists a corresponding  $g_n \in Q_n$ . Then  $p \in [1, +\infty)$  tells us that  $q \in (1, +\infty]$ , and the assumptions for Theorem 6.13 all hold. Therefore for each  $g_n \in Q_n$ , there is a corresponding bounded linear operator  $\phi_{g_n} \in (P_n)^*$  such that

$$\phi(f) = \phi|_{P_n}(f) = \int f g_n d\mu = \phi_{g_n}(f)$$

The remainder of the proof consists of taking the sequence of  $g_n$  towards some  $g \in L^q$ . We claim that this limit makes sense. As for any  $n < m$ , such that  $E_n \subseteq E_m$  then  $g_n = g_m$  on  $E_n$  pointwise. The proof is simple since each the restriction of our  $\phi \in L^{p*}$  onto  $E_n$  and  $E_m$  spawns two functions  $g_n$  and  $g_m \in L^1$ . To verify, take any subset  $Z \subseteq E_n$  then

$$\phi|_{P_n}(\chi_Z) = \int \chi_Z \cdot g_n = \int \chi_Z \cdot g_m = \phi|_{Q_n}(\chi_Z)$$

So  $g_n = g_m$  pointwise a.e on  $E_n$ . Now we define  $g$  measurable such that  $g|_{E_n} = g_n$  for every  $n$ . And

$$\begin{aligned}
|g_n| &= \chi_{E_n} \cdot |g_m| \implies \\
|g_n| &\leq |g_{n+1}| \implies \\
\|g_n\|_q &\leq \|g_{n+1}\|_q = \|\phi_{g_{n+1}}\|_{q^*} \leq \|\phi\|_{q^*} < +\infty
\end{aligned}$$

Where the second last estimate is from on the monotonicity of the supremum on subsets with  $(P_n \subseteq P_{n+1})$ . If  $q = +\infty$  then  $g \in L^\infty$  is trivial, but for any  $q < +\infty$ . We wish to show that  $g \in L^q$ . Since  $|g_n| \leq |g|$  pointwise for every  $n$ , and for each  $x \in X$ , there exists a  $N$ , where  $n \geq N$  implies  $|g(x)| = |g_n(x)|$ , so  $|g(x)|$  is an upperbound that is actually attained by the sequence  $|g_n(x)|$ . So,  $|g(x)| = \sup_{n \geq 1} \{|g_n(x)|\}$ .

Using the Monotone Convergence Theorem on  $|g_n|$ ,

$$\begin{aligned}
\int \lim_{n \rightarrow \infty} |g_n|^q d\mu &= \int \sup_{n \geq 1} |g_n|^q d\mu \\
&= \int |g|^q d\mu \\
&= \lim \int |g_n|^q d\mu
\end{aligned}$$

Which yields  $\|g\|_q^q = \lim \|g_n\|_q^q = \sup \|g_n\|_q^q \leq \|\phi\|_q^q < +\infty$ . It follows that  $g \in L^q$ .

Finally, we will show that  $\phi(f) = \int fg$  for every  $f \in L^p$ . Redefine  $f_n = f \cdot \chi_{E_n} \in P_n$  for every  $n \geq 1$ . We claim that  $f_n \rightarrow f$  in the  $p$ -norm.

$$\begin{aligned}
|f_n - f| &\leq |f_n| + |f| \\
&\leq |f| + |f| \\
&\leq 2|f|
\end{aligned}$$

And  $|f_n - f|^p \leq 2^p \cdot |f|^p \in L^+ \cap L^1$ . Now it is permissible to apply the Dominated Theorem, and we will do so.

$$\begin{aligned}
\lim \int |f_n - f|^p &= \int \lim |f_n - f|^p \\
\lim \|f_n - f\|_p^p &= \|\lim(|f_n - f|)\|_p^p \\
&= 0
\end{aligned}$$

And we have  $\phi(f) = \phi(\lim f_n) = \lim \phi(f_n)$

$$\begin{aligned}
 \phi(f) &= \lim \phi|_{P_n}(f_n) \\
 &= \lim \int f_n \cdot g_n \\
 &= \lim \int f \cdot g \cdot \chi_{E_n} \\
 &= \int \lim (fg \cdot \chi_{E_n}) \\
 &= \int fg
 \end{aligned}$$

Where we used the DCT again in the second last equality. The justification is a simple consequence of  $fg\chi_{E_n} \rightarrow fg$  pointwise and Holder's Inequality. This completes the proof for when  $\mu$  is of  $\sigma$ -finite measure, and  $p \in [1, +\infty)$ .

Suppose now  $\mu$  is arbitrary, and  $p \in (1, +\infty)$ , then  $q < +\infty$ . Now let us agree to define, for every  $\sigma$ -finite  $E \subseteq X$

$$P_E = \{L^p, |f| = |f| \cdot \chi_E\}$$

Where  $Q_E$  does not hold any surprises. Then for each  $E$  we have a  $\phi|_E$  which induces a  $g_E$  that vanishes outside  $E$ . We are ready for the final part of the proof.

First, if  $E \subseteq F$  and both  $E$  and  $F$  are  $\sigma$ -finite, then  $\|g_E\|_q \leq \|g_F\|_q$ . This is a simple consequence of monotonicity in  $L^+$  if we take  $|g_E|^q \leq |g_F|^q$ .

Second, we define

$$W = \{\|g_E\|_q, E \text{ is } \sigma\text{-finite, and } \phi|_{P_E} \text{ induces } g_E\}$$

Let  $M$  be the supremum of  $W$ , then there exists a sequence of  $\sigma$ -finite sets,  $\{E_n\}$  where  $\|g_{E_n}\|_q \rightarrow M \leq \|\phi\|_{p^*}$ . Take a set  $F = \cup E_{n \geq 1}$ , which is also  $\sigma$ -finite, so that  $\|g_F\|_q = M$ . Now assume there exists another  $\sigma$ -finite superset of  $F$ , let us call it  $A$ . Then

$$\int |g_F|^q + \int |g_{A \setminus F}|^q = \int |g_A|^q \leq M^q = \|g_F\|_q^q$$

Everything is finite here so there is no need for caution, subtracting we have  $g_{A \setminus F} = 0$  pointwise a.e. For any  $f \in L^p$ , the spots where  $f$  does not vanish is  $\sigma$ -finite. This comes from  $\int |f|^p < +\infty$ . So it suffices to integrate over this  $\sigma$ -finite set. But we already know, even if this set  $A$  contains  $F$  as a subset,  $\int f g_F = \int f g_A$ .

We now define  $g = g_F$ , and the proof is complete. As for every  $\phi \in L^{p*}$ , there exists a  $g \in L^q$  such that the evaluation of any  $f \in L^p$  is given by integrating  $f$  with  $g$ . ■

## Chapter 7



**Theorem 7.1**

**WTS 1.1.** *If  $I$  is a linear functional on  $C_c(X)$ , then for every compact  $K \subseteq X$ , there exists some  $C_K \geq 0$  with*

$$|I(f)| \leq C_K \cdot \|f\|_u$$

*Proof.* Since  $\text{supp}(f)$  is compact, by Urysohn's Lemma (Theorem 4.32), there exists a  $\phi \in C_c(X, [0, 1])$  such that  $\phi = 1$  on  $K$  and vanishes outside some compact  $\bar{V} \subseteq X$ . Then at every  $x$ ,

$$-\|f\|_u \leq f(x) \leq +\|f\|_u$$

Implies that

$$(-\|f\|_u)\phi \leq f(x) \leq (+\|f\|_u)\phi$$

So that  $f + \|f\|_u\phi \geq 0$  and  $+\|f\|_u - f \geq 0$ , and by linearity,

$$(-\|f\|_u)I(\phi) \leq I(f) \leq (+\|f\|_u)I(\phi)$$

Therefore  $|I(f)| \leq I(\phi)\|f\|_u$ , and taking  $C_K = I(\phi)$  will suffice. ■

## Theorem 7.2

**WTS 2.1.** *The Riesz-Markov-Kakutani Representation Theorem. If (for every)  $I$  is a positive linear functional on  $C_c(X)$ , then there exists a unique Radon measure  $\mu$  on  $X$ , such that*

$$I(f) = \int f d\mu$$

for every  $f \in C_c(X)$ .  $\mu$  also satisfies, for every open  $U$ , and every compact  $K \subseteq X$

$$\mu(U) = \sup \{I(f), f \in C_c(X), f \prec U\} \quad (11)$$

$$\mu(K) = \inf \{I(f), f \in C_c(X), f \geq \chi_K\} \quad (12)$$

For the sake of completeness, we place the definitions for a Radon measure. Let  $X$  be a LCH space, and  $\mathbb{B}$  be its usual  $\sigma$ -algebra, a measure  $\nu$  is a Radon measure iff

- (i)  $\nu(K) < +\infty$  for every compact  $K$ .
- (ii)  $\nu$  is outer-regular on all Borel sets  $E$ ,

$$\nu(E) = \inf \{\nu(U), U \supseteq E, U \in \mathcal{T}\}$$

Intuition: approximation by open supersets.

- (iii)  $\nu$  is inner-regular on all open sets  $U \in \mathcal{T}$ ,

$$\nu(U) = \sup \{\mu(K), K \subseteq U, K \text{ compact}\}$$

Intuition: approximation by compact subsets

The main proof is extremely long, so we will divide it into several parts. Following Folland's argumentation closely, we will prove (in order)

- (a) If  $\mu_1, \mu_2$  are Radon measures on  $X$  such that for every  $f \in C_c(X)$

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

then  $\mu_1, \mu_2$  must satisfy (11), and  $\mu_1 = \mu_2$  on  $\mathbb{B}$ .

- (b) If we define, for every open set  $U$ , define  $\mu : \mathcal{T} \rightarrow [0, +\infty]$  such that

$$\mu(U) = \sup \{I(f), f \in C_c(X), f \prec U\} \quad (13)$$

Then  $\mu$  is countably subadditive, meaning for every  $U \in \mathcal{T}$ ,  $\{U_{j \geq 1}\} \subseteq \mathcal{T}$

$$U = \bigcup U_{j \geq 1} \implies \mu(U) \leq \sum \mu(U_{j \geq 1})$$

- (c)  $\mu(\emptyset) = 0$ ,  $\{\emptyset, X\} \subseteq \mathcal{T}$ , so that by Theorem 1.10  $\mu$  induces an outer-measure  $\mu^*$

$$\mu^*(E) = \inf \left\{ \sum \mu(U_{j \geq 1}), U_j \in \mathcal{T}, E \subseteq \bigcup U_{j \geq 1} \right\} \quad (14)$$

- (d) If  $\mu^*$  is as described above, then if  $\mu$  is countably subadditive on  $\mathcal{T}$ , then

$$\mu^*(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \} \quad (15)$$

Meaning the two definitions in (14) and (15) are equal.

- (e)  $\mu^*$  and  $\mu$  agree on all open sets, and  $\mu^*|_{\mathcal{T}} = \mu$ ,  
 (f) Using again the definition in (14) and (15), we show that every open set  $U \in \mathcal{T}_X$  is  $\mu^*$ -measurable, meaning for every  $E \subseteq X$ ,

$$\mu^*(E) = \mu^*(E \cap U) + \mu^*(E \setminus U)$$

With this, since the set of all outer-measurable ( $\mu^*$ -measurable) sets,  $\mathcal{M}^*$  form a  $\sigma$ -algebra,

$$\mathcal{T} \subseteq \mathcal{M}^* \implies \mathbb{B} \subseteq \mathcal{M}^*$$

By Theorem 1.1, and define

$$\mu = \mu^*|_{\mathbb{B}} \quad (16)$$

is a Borel measure. And we note in passing that  $\mu$  is outer-regular on all  $E \in \mathbb{B}$ ,

$$\mu(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \} \quad (17)$$

(g) Using (16) for the definition of  $\mu$  on  $\mathbb{B}$ , we prove that

- $\mu$  is outer-regular on all Borel sets, and
- $\mu$  satisfies Equation (11).

(h)  $\mu$  satisfies Equation (12)

(i)  $\mu$  is finite on all compact sets.

(j)  $\mu$  is inner-regular on all open sets.

(k) For every  $f \in C_c(X, [0, 1])$ ,

$$I(f) = \int f d\mu \quad (18)$$

(l) For every  $f \in C_c(X)$ ,

$$I(f) = \int f d\mu \quad (19)$$

A small lemma needs to be made before proceeding, that concerns the 'monotonicity' of  $I$  on  $C_c X$ .

**Lemma 2.1** *Suppose that  $f, g \in C_c(X)$ , and  $f \geq g \geq 0$  for every  $X$ , then  $f - g \in C_c(X)$  and  $I(f) \geq I(g)$*

*Proof.* Suppose that  $x \in X$  where  $f(x) = 0$ , then

$$f(x) - g(x) = -g(x) \geq 0 \implies g(x) = 0 \implies f - g = 0$$

Hence

$$\begin{aligned} \{x, f(x) = 0\} &\subseteq \{x, f(x) - g(x) = 0\} \implies \{x, f(x) - g(x) \neq 0\} \subseteq \{x, f(x) \neq 0\} \\ &\implies \text{supp}(f - g) \subseteq \text{supp}(f) \end{aligned}$$

Since  $\text{supp}(f)$  is compact, and  $\text{supp}(f - g)$  is a closed subset of  $\text{supp}(f)$ , yields  $f - g \in C_c(X)$ . And if  $I$  is any positive linear functional on  $C_c(X)$ , then

$$\begin{aligned} f - g \geq 0 &\implies I(f - g) \geq 0 \\ &\implies I(f) \geq I(g) \geq 0 \end{aligned}$$

■

**Remark 2.1** If  $f \prec U$  and  $g \prec U$  for some open subset  $U \subseteq X$ , then clearly  $\text{supp}(f - g) \subseteq \text{supp}(f) \subseteq U$ , and  $1 \geq f \geq f - g \geq 0$  means that  $f - g \prec U$  as well.

### Part a

*Proof.* Suppose that  $\mu_1$  and  $\mu_2$  are Radon measures on  $X$ , and for every  $f \in C_c(X)$ ,

$$\int f d\mu_1 = I(f) = \int f d\mu_2$$

We first prove (11). Without loss of generality, by monotonicity of  $L^+$ , if  $f \prec U$  for some open  $U$ , then  $0 \leq f \leq \|f\|_u \chi_U = \chi_U$  for all  $x$  and

$$\int f d\mu_1 \leq \int \|f\|_u \chi_U d\mu_1 \leq \mu_1(U)$$

Therefore  $\mu_1(U)$  (resp.  $\mu_2(U)$ ) is an upper-bound for the set

$$\{I(f), f \in C_c(X), f \prec U\}$$

Since  $\mu_1$  is inner-regular on  $U \in \mathcal{T}$ , for every  $\varepsilon > 0$  we can find some compact  $K \subseteq U$  where

$$\mu_1(U) - \varepsilon < \mu_1(K)$$

By Urysohn's Lemma (Theorem 4.32), there exists some  $g \in C_c(X)$  with

- $g \in C_c(X, [0, 1])$ ,
- $g = 1$  on  $K \subseteq U$ ,
- $g = 0$  outside some  $\bar{V} \subseteq U$ , and
- $g \prec U$ .

Hence for every  $x \in K$ ,  $g \geq \chi_K$ . If  $x \notin K$  then  $g \geq 0 = \chi_K$ ; so  $g - \chi_K \geq 0$  for every  $x \in X$ . Since  $\chi_K \prec U$ , using Lemma 2.1, we get

$$\mu_1(K) \leq \int \chi_K d\mu_1 = I(\chi_K) \leq I(g)$$

So for every  $\varepsilon > 0$ , there exists a  $g \in C_c(X)$ , and  $g \prec U$  where

$$\mu_1(U) - \varepsilon < \mu_1(K) \leq I(g)$$

Therefore  $\mu_1(U) = \sup \{I(f), f \in C_c(X), f \prec U\}$ , and the first claim in (a) is proven. To show that  $\mu$  is indeed unique, since for every open set  $U$ , we must have  $\mu_1(U) = \mu_2(U)$ , and if  $E \in \mathbb{B}$  is any Borel set, and by outer-regularity,

$$\mu_1(E) = \inf \{\mu_1(U), U \supseteq E, U \in \mathcal{T}\} = \inf \{\mu_2(U), U \supseteq E, U \in \mathcal{T}\} = \mu_2(E)$$

Therefore this measure is unique. ■

### Part b

*Proof.* To show countable subadditivity for  $\mu$  with equation (13), fix any  $U \in \mathcal{T}$  and a sequence  $\{U_{j \geq 1}\} \subseteq \mathcal{T}$  with  $U = \bigcup U_{j \geq 1}$ . It suffices to show that the partial sum of  $\sum \mu(U_{j \leq n})$  is greater than  $I(f)$  for any  $f \in C_c(X)$ ,  $f \prec U$  (hence it is an upper bound).

Fix any  $f$ , then denote  $K = \text{supp}(f) \subseteq U$ , and since  $\{U_{j \geq 1}\}$  is an open cover for  $K$ , there exists a finite subcollection,  $B \subseteq \mathbb{N}^+$  such that

$$K \subseteq \bigcup_{j \in B} U_j$$

Using Theorem 4.41 on this finite cover of  $K$ , there exists a partition of unity in  $\{g_{j \leq n}\}$  where

- $g_j \in C_c(X, [0, 1])$ ,
- $g_j \prec U_j \subseteq U$  for every  $j \leq n$ , and
- $\sum g_j = 1$  on  $K$ ,

And notice for every  $j \leq n$ ,

$$\begin{aligned} \{f = 0\} \cup \{g_j = 0\} &\subseteq \{f \cdot g_j = 0\} \implies \{f \cdot g_j \neq 0\} \subseteq \{f \neq 0\} \cap \{g_j \neq 0\} \\ &\implies \text{supp}(f \cdot g_j) \subseteq \text{supp}(f) \cap \text{supp}(g_j) \\ &\implies \text{supp}(f \cdot g_j) \subseteq U_j \subseteq U \end{aligned}$$

Hence  $f \cdot g_j \prec U$  and  $f \cdot g_j \in C_c(X, [0, 1])$  for every  $1 \leq j \leq n$ . Moreover, if we take the sum over a finite  $n$ , we obtain  $f = \sum f \cdot g_{j \leq n}$ , this is because for every  $x \in X$ , so we have

$$\sum_{j \leq n} f(x) \cdot g_j(x) = f(x) \cdot \sum_{j \leq n} g_j(x) = f(x)$$

Then  $I(f) = I(\sum f \cdot g_j) = \sum I(f \cdot g_j)$ . And by definition of  $\mu(U_j)$ , since it is the supremum over all  $I(h_j)$ , where  $h_j \in C_c(X, [0, 1])$  and  $h_j \prec U_j$

$$I(f \cdot g_j) \leq \mu(U_j), \quad \forall j \leq n$$

Hence

$$I(f) \leq \sum_{j \leq n} \mu(U_j) \leq \sum_{j \geq 1} \mu(U_j)$$

Where for the last estimate we used the fact that  $\mu$  is non-negative, and since this holds for any  $f$ , we can conclude that  $\mu(U) \leq \sum_{j \geq 1} \mu(U_j)$ . ■

### Part c

*Proof.* By definition of a topology,  $\{\emptyset, X\} \subseteq \mathcal{T}$ , and  $\mu(\emptyset) = \sup\{I(f), f \in C_c(X), f \prec \emptyset\}$ , so  $\text{supp}(f) = \emptyset$ , and  $\{x, f(x) \neq 0\} \subseteq \emptyset$ , so the set contains one element, namely  $I(0) = 0$  by linearity. So  $\mu(\emptyset) = 0$ . The assumptions for Theorem 1.10 are satisfied and (14) is indeed an outer-measure. ■

### Part d

*Proof.* Denote the right members of (14) and (15) by  $W_1$  and  $W_2$ , we wish to show that  $\inf W_1 = \inf W_2$ . Clearly  $\inf W_1 \leq \inf W_2$ , since  $W_2 \subseteq W_1$ . Now, if  $\mu$  is countably additive, then for every  $\omega \in W_1$  induces a sequence of open sets  $\{U_{j \geq 1}\}$  such that  $E \subseteq \bigcup U_{j \geq 1}$ . Denote the union over  $\{U_{j \geq 1}\}$  by  $U$ , which is also another open set,

$$\inf W_2 \leq \mu(U) \leq \sum \mu(U_{j \geq 1}) = \omega$$

Since  $\omega$  is arbitrary, we conclude that  $\inf W_2 = \inf W_1$ , and this proves (d). ■

### Part e

*Proof.* If  $U$  and  $V$  are open subsets of  $X$ , and if  $U \subseteq V$ , then

$$\begin{aligned} U \subseteq V &\implies \{f \in C_c(X), f \prec U\} \subseteq \{f \in C_c(X), f \prec V\} \\ &\implies \{I(f), f \in C_c(X), f \prec U\} \subseteq \{I(f), f \in C_c(X), f \prec V\} \end{aligned}$$

Hence  $\mu(U) \leq \mu(V)$ . Now by equation (15),  $\mu^*(U) \leq \mu(U)$ . To show the reverse inequality, suppose by contradiction that  $\mu^*(U) < \mu(U)$ .

Since  $\mu^*(U)$  is an infimum, then for every  $\varepsilon > 0$  there exists some  $V \supseteq U$  where if we write  $\mu^*(U) + \varepsilon = \mu(U)$

$$\mu(V) < \mu^*(U) + \varepsilon = \mu(U) \implies \mu(V) < \mu(U), U \subseteq V$$

This contradicts what we have just proven, and therefore  $\mu^*(U) = \mu(U)$  for every open set  $U$ . ■

### Part f

*Proof.* We wish to show that every open set  $U$  is  $\mu^*$ -measurable. By Theorem 1.10, it suffices to show that for every  $E \subseteq X$

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U) \quad (20)$$

because the reverse inequality is given by subadditivity of  $\mu^*$ , and we can also assume that  $\mu^*(E) < +\infty$ . Let us assume that  $E$  is open, we wish to find some function  $h \in C_c(X)$ ,  $h \prec E$  with

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

The above formula is fussy, but the liberty is taken to show it beforehand to avoid any potential confusion that follows. Since  $E \cap U$  is an open subset of  $X$ , the definition of  $\mu(E \cap U) = \mu^*(E \cap U)$  in (13) tells us that every  $\varepsilon > 0$  induces some  $f \in C_c(X)$ ,  $f \prec E \cap U$  where

$$I(f) > \mu(E \cap U) - \varepsilon = \mu^*(E \cap U) - \varepsilon \quad (21)$$

Also,  $\text{supp}(f)$  is a closed set (compact subsets of Hausdorff spaces are closed), therefore  $E \setminus \text{supp}(f)$  is an open set. We make a small diversion from the current part of the proof and turn our attention to the fact that

$$\begin{aligned} \text{supp}(f) \subseteq U &\implies U^c \subseteq (\text{supp}(f))^c \\ &\implies E \setminus U \subseteq E \setminus \text{supp}(f) \end{aligned}$$

And because the outer-measure  $\mu^*$  is monotone,

$$\mu^*(U) \leq \mu^*(E \setminus \text{supp}(f)) \quad (22)$$

Now, using the definition of  $\mu(E \setminus \text{supp}(f))$  (recall that  $E \setminus \text{supp}(f)$  is an open set), for every  $\varepsilon > 0$ , there exists some  $g \in C_c(X)$ ,  $g \prec E \setminus \text{supp}(f)$  with

$$I(g) > \mu(E \setminus \text{supp}(f)) - \varepsilon = \mu^*(E \setminus \text{supp}(f)) - \varepsilon \quad (23)$$



It is at this part of the proof where we wish to define  $h = f + g$ , but first we must verify

- $f + g \in C_c(X, [0, 1])$ ,
- $f + g \prec E$

The sum of two non-negative functions is non-negative, and for every  $x \in \text{supp}(f)$ ,  $f \leq 1$ . Also

$$\begin{aligned} \text{supp}(g) \subseteq (\text{supp}(f))^c &\implies \text{supp}(f) \subseteq (\text{supp}(g))^c \\ &\implies \text{supp}(f) \subseteq \{g = 0\} \end{aligned}$$

The last implication comes from taking complements on both sides of  $\{g \neq 0\} \subseteq \text{supp}(g)$ . So  $x \in \text{supp}(f) \implies f + g \leq 1$ . Now if  $x \notin \text{supp}(f)$ , then  $f + g = g \leq 1$ . Furthermore,  $\text{supp}(f + g)$  is a closed subset of compact  $\text{supp}(f) \cup \text{supp}(g)$ . This is because  $\{f + g \neq 0\} \subseteq \{f \neq 0\} \cup \{g \neq 0\}$ , and the finite union of two compact sets is again compact.

A moment's thought should yield the fact that the last estimate should be an equality, but it is a needless distraction. Therefore  $\text{supp}(f + g)$  is compact and  $f + g \in C_c(X, [0, 1])$ .

Now both bullet points are satisfied, and we can set  $h = f + g$ . Adding equation (23) with (21) gives us

$$I(h) = I(f) + I(g) > \mu^*(E \cap U) + \mu^*(E \setminus \text{supp}(f)) - 2\varepsilon$$

Upon applying (22) to the right member of the above estimate, we have

$$I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

But this particular  $h \in C_c(X) \cap \{f \prec E\}$ , therefore

$$\mu^*(E) \geq I(h) > \mu^*(E \cap U) + \mu^*(E \setminus U) - 2\varepsilon$$

Since  $\varepsilon > 0$  is arbitrary, equation (20) holds for every open  $E$ . Now for any general  $E \subseteq X$ , fix any  $\varepsilon > 0$  and by how we defined  $\mu^*(E)$ , there exists some open  $V \supseteq E$  — recall that  $\mu^*(E)$  is the infimum over the set of  $\mu(V)$  where  $V$  is an open superset of  $E$  — hence

$$\mu^*(E) + \varepsilon > \mu(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$$

By monotonicity (twice) of the outer-measure  $\mu^*$ , we have

$$\mu^*(E) + \varepsilon > \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Let  $\varepsilon \rightarrow 0$ , and we get

$$\mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$$

Therefore every open  $U \subseteq X$  is  $\mu^*$ -measurable. So  $\mu = \mu^*|_{\mathbb{B}}$  is a Borel measure on  $X$ . ■

### Part g

*Proof.* To show outer-regularity, fix any  $E \in \mathbb{B}$ , then by definition,

$$\mu(E) = \mu^*(E) = \inf \{ \mu(U), U \supseteq E, U \in \mathcal{T} \}$$

And for every open  $U$ , (11) follows from Equation (13). ■

### Part h

*Proof.* We want to show that for every compact  $K$ , Equation (12) holds. To reduce the notational baggage that follows, we agree to define

$$\{I(f), f \in C_c(X), f \prec U\} = \{I(f), f \prec U\}$$

Similarly for  $\{I(f), f \geq \chi_K\}$ . If  $\mu(K) = 0$ , then  $\mu(K)$  is obviously a lower bound, since  $f \geq \chi_K \geq 0$  means that  $I(f) \geq 0$ , for every  $f \geq \chi_K$ . So we can suppose  $\mu(K) > 0$ .

Fix an arbitrary  $f \geq \chi_K$ , then this particular  $f$  induces an open set  $U_\alpha = \{f > 1 - \alpha\}$ , where  $\alpha > 0$ . Notice also that

$$K \subseteq \{f \geq 1\} \subseteq \{f > 1 - \alpha\} = U_\alpha$$

Since  $U_\alpha$  is an open superset of  $K$ , by Equation (17),  $\mu(K) \leq \mu(U_\alpha)$ , but  $\mu(U_\alpha)$  is simply the supremum of  $\{I(g), g \prec U_\alpha\}$ . If we wish to show that  $\mu(K) \leq \mu(U_\alpha) \leq I(f)$ , it suffices to show that  $I(f)$  is an upper-bound for  $\{I(g), g \prec U_\alpha\}$ .

Fix any  $I(g) \in \{I(g), g \prec U_\alpha\}$ , note that  $1 - \alpha \neq 0$  for any  $\alpha$  small enough, then

- $f/(1 - \alpha) > 1$  on  $U_\alpha$ ,
- $1 \geq g \geq 0$  on  $U_\alpha$ , in particular,  $f/(1 - \alpha) - g \geq 0$  on  $U_\alpha$ ,
- If  $x \notin U_\alpha$ , then  $f/(1 - \alpha) - g = f(1 - \alpha) \geq 0$ .
- Therefore  $f/(1 - \alpha) - g \geq 0$  for any  $x$ , and by Lemma 2.1,

$$I(f/(1 - \alpha)) \geq I(g) \quad \forall g \prec U_\alpha$$

Combining the above estimate with  $\mu(K) \leq \mu(U_\alpha)$  gives us

$$\mu(K) \leq \frac{1}{1 - \alpha} I(f)$$

Now write  $\varepsilon = \alpha/\mu(K) > 0$  and for every  $\varepsilon > 0$  we get

$$\mu(K) - I(f) \leq \alpha\mu(K) = \varepsilon$$

Send  $\varepsilon \rightarrow 0$  and  $\mu(K) \leq I(f)$  for every  $f \geq \chi_K$ .

To show that  $\mu(K)$  is indeed the infimum for  $\{I(f), f \geq \chi_K\}$ , notice that for every  $\varepsilon > 0$  we can obtain some open superset  $U \supseteq K$  (by outer-regularity) where  $\mu(U) < \mu(K) + \varepsilon$ . By Urysohn's Lemma, there exists some  $g \prec U$ ,  $g(x) = 1$  for every  $x \in K$ .

$$g \in \{I(f), f \prec U\} \cap \{I(f), f \geq \chi_K\}$$

Therefore  $I(g) \leq \mu(U) < \mu(K) + \varepsilon$  as desired, and Equation (12) holds. ■

### Part i

*Proof.*  $\mu(K) < +\infty$  for every compact  $K$ . Indeed, since  $I(\chi_K) \in \{I(f), f \geq \chi_K\}$ , then by Theorem 7.1, there exists a constant  $C_K \geq 0$  that bounds

$$\mu(K) \leq |I(\chi_K)| = I(\chi_K) \leq C_K \cdot \|\chi_K\| = C_K < +\infty$$

■

**Part j**

*Proof.* Fix any open set  $U$ , then for every  $\varepsilon > 0$ , there exists some  $f \prec U$  with  $\mu(U) - \varepsilon < I(f)$ . Then denote  $K = \text{supp}(f) \subseteq U$ . If we take any  $I(h) \in \{I(h), h \geq \chi_K\}$ , then  $h \geq f$  gives us  $I(h) \geq I(f)$  by Lemma 2.1. So  $I(f)$  is a lower bound of  $\{I(h), h \geq \chi_K\}$ , therefore

$$\mu(U) - \varepsilon \leq I(f) \leq \mu(K)$$

Since  $\text{supp}(f) = K \subseteq U$ , this proves inner-regularity of  $\mu$  on open sets. ■

**Part k**

*Proof.* Suppose  $f \in C_c(X, [0, 1])$ , we first show that Equation (18) holds. We divide the interval  $[0, 1]$  into  $N \geq 1$  chunks by writing

$$K_j = \{f \geq j/N\}$$

for every  $1 \geq j \geq N$ . And define  $K_0 = \text{supp}(f)$ . Each  $K_j$  is a closed subset of  $\text{supp}(f)$ , and therefore compact. More is true,

- $K_{j-1} \supseteq K_j$  for every  $1 \leq j \leq N$ .
- $x \in K_j$  iff  $f(x) \in [\frac{j}{N}, 1]$ ,
- $x \notin K_j$  iff  $f(x) \in [0, \frac{j}{N})$ , and
- $x \in (K_{j-1} \setminus K_j)$  iff  $f(x) \in [\frac{j-1}{N}, \frac{j}{N})$

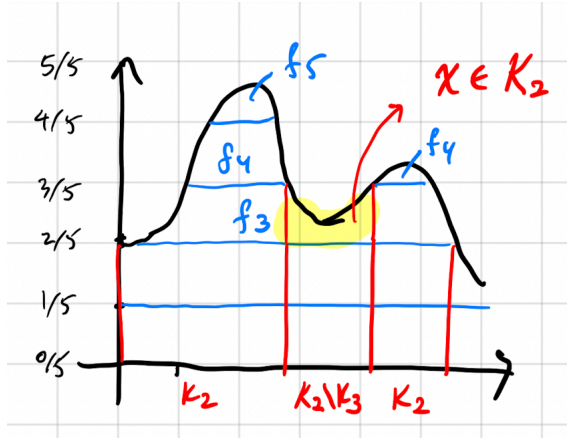
Folland constructs a finite sequence of compactly supported functions,  $\{f_j\}$ , where  $1 \leq j \leq N$  such that

- Each  $0 \leq f_j \leq 1/N$ ,
- If  $x \in (K_m \setminus K_{m+1})$  iff  $f(x) \in [\frac{m}{N}, \frac{m+1}{N})$  means that  $f_j = 1$  for all  $1 \leq j \leq m$ , and
- $f_{m+1} = f - m/N$  on  $K_m$ , such that

$$f(x) = \left(\sum f_{j \leq m}(x)\right) + \left(f(x) - \frac{m}{N}\right) = \frac{m}{N} + \left(f(x) - \frac{m}{N}\right)$$

- And for every  $m < j \leq N$ ,  $f_j = 0$ .
- If  $x \notin K_m$  iff  $f(x) \in [0, \frac{m}{N})$  then for every  $m + 1 \leq j \leq N$ ,  $f_j = 0$ .

The illustration for when  $N = 5$  below should make things clearer.



It is also trivial to verify that

- For every  $x \in K_j$ ,  $f_j = N^{-1}$ , and

$$\chi_{K_j} N^{-1} \leq f_j \quad (24)$$

Also, if  $x \notin K_j$  then  $f_j \geq 0$ , therefore  $f_j \geq \chi_{K_j} N^{-1}$  at every  $x$ .

- If  $x \notin K_{j-1}$  then  $f_j = 0 \leq \chi_{K_{j-1}} \cdot N^{-1}$ . If  $x$  is in  $K_{j-1}$  then  $f_j \leq N^{-1}$  by construction and therefore

$$f_j \leq \chi_{K_{j-1}} N^{-1} \quad (25)$$

for all  $x$ .

- $f_j \in C_c(X)$ , since  $\text{supp}(f_j) \subseteq \text{supp}(f)$ .

Combining Equations (24) with (25), and by monotonicity in  $L^+(X, \mathbb{B}, \mu)$ , since  $f_j \in L^+$

$$\int \frac{1}{N} \chi_{K_j} d\mu \leq \int f_j d\mu \leq \int \frac{1}{N} \chi_{K_{j-1}} d\mu$$

And for every  $1 \leq j \leq N$ ,

$$\frac{1}{N} \mu(K_j) \leq \int f_j d\mu \leq \frac{1}{N} \mu(K_{j-1}) \quad (26)$$

Furthermore, from Equation (24), since  $Nf_j \geq \chi_{K_j}$  then by Equation (12),

$$\mu(K_j) \leq I(Nf_j) \implies \frac{1}{N}\mu(K_j) \leq I(f_j)$$

Now for any arbitrary  $I(h) \in \{I(h), h \geq \chi_{K_{j-1}}\}$ , since

$$h \geq \chi_{K_{j-1}} \geq Nf_j \implies I(h) \geq I(Nf_j)$$

So  $NI(f_j)$  is a lower bound for  $\{I(h), h \geq \chi_{K_{j-1}}\}$  and

$$I(f_j) \leq \frac{1}{N}\mu(K_{j-1})$$

Combining the last two results, with  $I(f_j)$ , we get

$$\frac{1}{N}\mu(K_j) \leq I(f_j) \leq \frac{1}{N}\mu(K_{j-1}) \quad (27)$$

Taking the sum over  $1 \leq j \leq N$  for Equations (26) and (27). Define  $A = N^{-1} \sum_0^{N-1} \mu(K_j)$ , and  $B = N^{-1} \sum_1^N \mu(K_j)$

$$B \leq \int f d\mu \leq A$$

And also

$$B \leq I(f) \leq A$$

This is because of finite additivity of both  $I$  and the integral, and  $f = \sum f_j$  on  $K_0 = \text{supp}(f)$ . Subtracting the two equations (keeping in mind that  $\mu(K_j) < +\infty$  for any compact  $K_j$ ), we get

$$(-1)(A - B) \leq \left( \int f d\mu - I(f) \right) \leq A - B \implies \left| \int f d\mu - I(f) \right| \leq A - B$$

It is trivial to verify that

$$0 \leq A - B = N^{-1}(\mu(K_0) - \mu(K_N)) \leq N^{-1}\mu(K_0)$$

as  $K_N \subseteq K_0$ . Let  $N \rightarrow \infty$  and

$$\int f d\mu = I(f)$$

Equation (18) holds as desired. ■

**Part 1**

*Proof.* Now for any general  $f \in C_c(X)$ ,  $f$  must be bounded on the plane since  $C_c(X) \subseteq BC(X)$ , and  $|f| \leq M_0$  for some  $M_0 \geq 0$ . Since  $\text{supp}(f)$  is compact, we know that

$$\int |f| d\mu \leq \int M_0 \chi_{\text{supp}(f)} d\mu \leq M_0 \mu(\text{supp}(f)) < +\infty$$

And  $C_c(X) \subseteq L^1(\mu)$ . Furthermore,

$$\frac{1}{2}(|\text{Re } f| + |\text{Im } f|) \leq |f| \leq M_0$$

So that  $\text{Re } f$  and  $\text{Im } f$  are in  $C_c(X)$ . Without loss of generality, we may assume that  $f$  is real. Define  $f_1 = \text{Re } f^+/M_0$  and  $f_2 = \text{Re } f^-/M_0$  and it immediately follows that  $f_1, f_2 \in C_c(X, [0, 1])$ .

By linearity of  $I$  on  $C_c(X)$  and the integral in  $L^1(\mu)$ ,

$$I(f_1 - f_2) = I(f) = \int f d\mu = \int f_1 d\mu - \int f_2 d\mu$$

Then we may apply the above to the real and imaginary parts of a general  $f \in C_c(X)$ , and this completes the proof. ■

**Theorem 7.3**

**WTS 3.1.** *See Theorem 7.2*

*Proof.*

**Theorem 7.4**

**WTS 4.1.** *See Theorem 7.2*

*Proof.*





**Theorem 7.5**

**WTS 5.1.**

*Proof.*



**Theorem 7.6****WTS 6.1.***Proof.*

**Theorem 7.7**

**WTS 7.1.**

*Proof.*



**Theorem 7.8**

**WTS 8.1.**

*Proof.*



**Theorem 7.9**

**WTS 9.1.** *If  $\mu$  is a Radon measure on  $X$ , then  $C_c(X)$  is dense in  $L^p(\mu)$  for  $1 \leq p < +\infty$ .*

*Proof.* Theorem 6.7 tells us that the set of  $L^p$  simple functions (as Folland calls them), which are

$$\Lambda = \left\{ f, f = \sum_{j \leq n} a_j \chi_{E_j}, a_j \in \mathbb{C}, \mu(E_j) < +\infty \right\}$$

So for every  $f \in L^p$ , there exists a sequence  $\{f_n\} \subseteq \Lambda$  with  $f_n \rightarrow f$  pointwise and  $f_n \rightarrow f$  in  $L^p$ . ■

**Theorem 7.10****WTS 10.1.***Proof.*

**Theorem 7.11**

**WTS 11.1.**

*Proof.*



## Chapter 8



**Theorem 8.1**

**WTS 1.1.**

*Proof.*



**Theorem 8.2**

**WTS 2.1.**

*Proof.*



### Theorem 8.3

**WTS 3.1.** *If  $f \in C^\infty$ , then  $f \in \mathcal{S}$  if and only if  $x^\beta \partial^\alpha f$  is bounded for all multi-indices  $\alpha, \beta$*

*Proof.*



**Theorem 8.4**

**WTS 4.1.**

*Proof.*



**Theorem 8.5**

**WTS 5.1.**

*Proof.*



**Theorem 8.6**

**WTS 6.1.**

*Proof.*



**Theorem 8.7**

**WTS 7.1.**

*Proof.*



**Theorem 8.8**

**WTS 8.1.**

*Proof.*





**Theorem 8.9**

**WTS 9.1.**

*Proof.*



**Theorem 8.10****WTS 10.1.***Proof.*

**Theorem 8.11**

**WTS 11.1.**

*Proof.*



**Theorem 8.12**

**WTS 12.1.**

*Proof.*



**Theorem 8.13****WTS 13.1.***Proof.*

**Theorem 8.14**

**WTS 14.1.** Suppose  $\phi \in L^1$ , and  $\int \phi(x)dx = a$ .

- (a) If  $f \in L^p$ ,  $p \in [1, +\infty]$ , then  $f * \phi_t \rightarrow af$  in the  $L^p$  norm as  $t \rightarrow 0$ .
- (b) If  $f$  is bounded and uniformly continuous, then  $f * \phi_t \rightarrow af$  uniformly as  $t \rightarrow 0$ .
- (c) If  $f \in L^\infty$  and  $f$  is continuous on an open set  $U$ , then  $f * \phi_t \rightarrow af$  uniformly on compact subsets of  $U$  as  $t \rightarrow 0$ .

*Proof of Part A.* First, the convolution  $f * \phi_t$  is in  $L^p$  by Young's Inequality (Theorem 8.7). Furthermore,

$$f * \phi_t - af = \int_{y \in \mathbb{R}^n} f(x - y)t^{-n}\phi(t^{-1}y)dy - \int_{y \in \mathbb{R}^n} f(x)\phi(y)dy \quad (28)$$

Now apply Theorem 2.44, with  $y \mapsto y/t$ , and denote this invertible map by  $T \in GL(n, \mathbb{R})$ , so that  $|\det(T)| = t^{-n}$ , then  $y = T(y)t$  for every  $t > 0$ . It follows that

$$\begin{aligned} (f * \phi_t)(x) &= |\det(T)| \cdot \int_{y \in \mathbb{R}^n} f(x - t \cdot Ty)\phi(T(y))dy \\ &= \int_{z \in \mathbb{R}^n} f(x - tz)\phi(z)dz \\ &= \int_{z \in \mathbb{R}^n} \tau_{tz}f(x)\phi(z)dz \end{aligned} \quad (29)$$

Next,  $a = \int \phi$  so  $af = \int f(x)\phi(z)dz$ . Using Equations (28) and (29) we get

$$(f * \phi_t - af)(x) = \int_{z \in \mathbb{R}^n} (\tau_{tz}f - f)\phi(z)dz \quad (30)$$

We wish to apply Minkowski's Inequality for integrals, which states, roughly speaking:

The norm of an integral is less than the integral of the norm.

to Equation (30), and

$$\|f * \phi_t - af\|_p \leq \int_{z \in \mathbb{R}^n} \|(\tau_{tz}f - f)\phi(z)\|_p dz \quad (31)$$

The assumptions for Theorem 6.19 are satisfied by

1. Notice for every  $z \in \mathbb{R}^{n'}$ ,

$$\|(\tau_{tz}f - f)\phi(z)\|_p = \left( \int_{x \in \mathbb{R}^n} |(\tau_{tz}f(x) - f(x))\phi(z)|^p dx \right)^{1/p} \leq |\phi(z)| \left( 2\|f\|_p \right) < +\infty$$

Since  $\|\phi\|_1 < +\infty$ ,  $|\phi(z)| < +\infty$  almost everywhere.

2. Next, to show  $z \mapsto \|\phi(z)(\tau_{tz}f - f)\|_p$  is in  $L^1\mathbb{R}^n$ ,  $z$ . Reuse the last estimate in the previous bullet point, and

$$\|\phi(z)(\tau_{tz}f - f)\|_p \leq |\phi(z)| \left( 2\|f\|_p \right)$$

Taking the integral in  $L^+$  with respect to  $z$ , we get

$$\left\| \left( \|\phi(z)(\tau_{tz}f - f)\|_p \right) \right\|_1 < +\infty$$

so both assumptions are satisfied.

Therefore Equation (31) holds. Next, fix any sequence of  $t_n > 0$  with  $t_n \rightarrow 0$ . The Dominated Convergence Theorem gives, since  $|\phi(z)|\|\tau_{t_n z}f - f\|_p$  is dominated by  $|\phi(z)| \cdot 2\|f\|_p \in L^1 \cap L^+$

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{z \in \mathbb{R}^n} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz &= \int_{z \in \mathbb{R}^n} \lim_{n \rightarrow \infty} \|\tau_{t_n z}f - f\|_p |\phi(z)| dz \\ &= \int_{z \in \mathbb{R}^n} 0 dz \\ &= 0 \end{aligned}$$

The second last equality is from Lemma 8.4, as translation is continuous in the  $L^p$  norm for  $p \in [1, +\infty)$ . So almost every  $z \in \mathbb{R}^n$  (since again,  $|\phi(z)|$  can be infinite on a null set),

$$\|\tau_{t_n z}f - f\|_p \rightarrow 0 \implies \|\tau_{t_n z}f - f\|_p |\phi(z)| \rightarrow 0$$

as  $n \rightarrow +\infty$ . It follows that

$$\lim_{n \rightarrow \infty} \|f * \phi_{t_n} - af\|_p = \lim_{n \rightarrow \infty} \left\| \int_{z \in \mathbb{R}^n} [\tau_{t_n z} f(x) - f(x)] \phi(z) dz \right\|_p = 0$$

Since the sequence  $t_n \rightarrow 0$  is arbitrary, we conclude that the function  $t \mapsto \|f * \phi_t - af\|_p$  has a limit of 0 as  $t \rightarrow 0$ .  $\blacksquare$

*Proof of Part B.* Suppose  $f \in \text{UBC}(\mathbb{R}^n)$ , so that  $f$  is uniformly continuous and bounded. We wish to show  $f * \phi_t \rightarrow af$  uniformly as  $t \rightarrow 0$ . In symbols,

$$g : t \mapsto \|f * \phi_t - af\|_u, \quad g \rightarrow 0, \text{ as } t \rightarrow 0$$

The convolution between  $f$  and  $\phi_t$  makes sense at every  $x \in \mathbb{R}^n$ , as

$$\int |\tau_y f(x)| |\phi(y)| dy \leq \|f\|_u \cdot \|\phi\|_1 < +\infty$$

Taking the supremum norm on both sides of Equation (30), we get

$$\begin{aligned} \|f * \phi_t - af\|_u &= \sup_{x \in \mathbb{R}^n} \left| \int_{z \in \mathbb{R}^n} (\tau_{tz} f - f) \cdot \phi(z) dz \right| \\ &\leq \sup_{x \in \mathbb{R}^n} \int_{z \in \mathbb{R}^n} |\tau_{tz} f - f| \cdot |\phi(z)| dz \\ &\leq \int_{z \in \mathbb{R}^n} \sup_{x \in \mathbb{R}^n} |\tau_{tz} f - f| \cdot |\phi(z)| dz \\ &= \int_{z \in \mathbb{R}^n} \|\tau_{tz} f - f\|_u \cdot |\phi(z)| dz \end{aligned} \quad (32)$$

the last equality is a simple consequence of the monotonicity of the integral in  $L^+$ , indeed. For every  $x \in \mathbb{R}^n$ , the following holds pointwise for almost every  $z$

$$|\tau_{tz} f - f| \leq \|\tau_{tz} f - f\|_u \implies \sup_{x \in \mathbb{R}^n} |\tau_{tz} f - f| \leq \|\tau_{tz} f - f\|_u$$

Apply the Dominated Theorem to the right member of (32), noting that it is dominated by  $|\phi(z)| \cdot 2\|f\|_u \in L^1 \cap L^+$  as we have done for Part A of the proof. Since this holds for every sequence  $t_n \rightarrow 0$ , the proof is complete.  $\blacksquare$



*Proof of Part C.* Next, suppose that  $f \in L^\infty$ , and  $f \in C(U)$ , where  $U$  is open in  $\mathbb{R}^n$ . We claim that

$$f * \phi_t \rightarrow af$$

within the topology of uniform convergence on compact subsets of  $U$ . So that for every  $K \in \mathfrak{J}$ ,  $K \subseteq U$

$$\sup_{x \in K} |f * \phi_t - af| \rightarrow 0, \text{ as } t \rightarrow 0$$

First, a small technical Lemma.

**Lemma 14.1** *If  $\phi \in L^1(\mathbb{R}^n)$ , then for every  $\varepsilon > 0$ , there exists  $E \in \mathfrak{J}$ , with*

$$\int_{E^c} |\phi| = \|\phi \chi_{E^c}\|_1 < +\varepsilon$$

*Proof.* Assume that  $\phi \geq 0$ , if not, replace  $\phi$  by  $|\phi|$ . Since  $C_c(\mathbb{R}^n)$  is dense in  $L^1$  for every  $\varepsilon 2^{-1} > 0$  there exists some  $\psi \in C_c(\mathbb{R}^n)$  with  $\|\psi - \phi\|_1 < \varepsilon^{-1}$ , and denote  $E = \text{supp}(\psi) \in \mathfrak{J}$ , then

$$\| |\psi| - |\phi| \|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

So we can assume  $\psi \geq 0$  as well, perhaps by relabelling  $\psi$  by  $|\psi|$ . Then,

$$\|\psi - \chi_E \phi\|_1 = \|\chi_E(\psi - \phi)\|_1 \leq \|\psi - \phi\|_1 < \varepsilon 2^{-1}$$

by monotonicity in  $L^+$ . The Triangle Inequality in  $L^1$  gives

$$\|\chi_{E^c} \phi\|_1 = \|\phi - \chi_E \phi\|_1 = \|\phi(1 - \chi_E)\|_1 \leq \|\phi - \psi\|_1 + \|\psi - \chi_E \phi\|_1 < \varepsilon$$

■

Back to the main proof of Part C, fix any  $\varepsilon > 0$ , then by Lemma 14.1,  $\phi$  induces some  $E \in \mathfrak{J}$  with  $\|\chi_{E^c} \phi\|_1 < +\varepsilon$ . By Lemma 8.4,  $\chi_K f \in C_c(\mathbb{R}^n) \subseteq \text{UBC}(\mathbb{R}^n)$ . Uniform continuity of  $\chi_K f$  gives us the continuity of translations. Now for the same  $\varepsilon > 0$ , there exists  $r > 0$ , for every  $w \in \mathbb{R}^n$ ,

$$|w| < r \implies \|\tau_w \chi_K f - \chi_K f\|_u < +\varepsilon \quad (33)$$

Since  $E \in \mathfrak{J}$ , it is bounded, and let  $t$  be a small positive number such that for every  $z \in E$ ,

$$|tz| < t \cdot (1 + \sup_{z \in E} |z|) < r$$

There exists such a  $t$ , namely  $t = r2^{-1}(1 + \sup_{z \in E} |z|)^{-1}$ . And for this  $t > 0$ , it follows that for every  $z \in E$ ,

$$\sup_{x \in K} |\tau_{tz}f - f| < +\varepsilon$$

Since this holds for every  $z \in E$ , we write

$$\sup_{x \in K, z \in E} |\tau_{tz}f - f| < +\varepsilon$$

And

$$|\phi(z)| \left[ \sup_{x \in K, z \in E} |\tau_{tz}f - f| \right] < |\phi(z)|\varepsilon$$

Monotonicity in  $L^+(E, z)$  reads, for every  $x \in K$ ,

$$\int_{z \in E} |\phi(z)(\tau_{tz}f - f)| dz \leq \int_{z \in E} |\phi|\varepsilon dz = \varepsilon \|\chi_E \phi\|_1 \leq \varepsilon \|\phi\|_1$$

Since this holds for every  $x \in \mathbb{R}^n$ ,

$$\sup_{x \in K} \left\{ \int_{z \in E} |\phi(z)| \cdot |\tau_{tz}f - f| dz \right\} \leq \varepsilon \|\phi\|_1 \quad (34)$$

Next, notice for every  $t, z$ , we have

$$|\tau_{tz}f - f| \leq \|\tau_{tz}f\|_u + \|f\|_u \leq 2 \cdot \|f\|_u$$

And the following holds  $z \in E^c$  a.e,

$$|\phi(z)| \cdot |\tau_{tz}f - f| \leq |\phi(z)| \cdot 2\|f\|_u$$

Taking the integral, and applying the condition we imposed on  $E$  from Lemma (14.1), so that

$$\int_{z \in E^c} |\phi(z)| \cdot |\tau_{tz}f - f| dz \leq 2\|f\|_u \int_{z \in E^c} |\phi(z)| dz \leq 2\|f\|_u \varepsilon$$

Taking the supremum of the above estimate, so

$$\sup_{x \in K} \left\{ \int_{z \in E^c} |\phi(z)(\tau_{tz}f - f)| dz \right\} \leq 2\|f\|_u \varepsilon \quad (35)$$

Combining Equations (34) and (35). Applying the additivity of the supremum (of  $x \in K$ ), since both members are finite,

$$\sup_{x \in K} \left\{ \int_E |\phi(z)|(\tau_{tz}f - f)dz + \int_{E^c} |\phi(z)|(\tau_{tz}f - f)dz \right\} < \varepsilon(2\|f\|_u + \|\phi\|_1)$$

The left member above is equal to  $\sup_{x \in K} |f * \phi_t - af|$ . Since  $\varepsilon > 0$  is arbitrary, this completes the proof of Part C. ■

**Theorem 8.15**

**WTS 15.1.** *If  $|\phi(x)| \leq C(1 + |x|)^{-n-\varepsilon}$ , where  $\varepsilon > 0$ , and if  $f \in L^p$ , for  $p \in [1, +\infty)$ , then*

$$f * \phi_t \rightarrow af$$

*pointwise for every  $x$  in the Lebesgue set of  $f$ ,*

$$\mathcal{L}_f = \left\{ x \in \mathbb{R}^n, \lim_{r \rightarrow 0} \frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy = 0 \right\}$$

*We also claim that  $m(\mathcal{L}_f^c) = 0$ , and  $x \in \mathcal{L}_f$  at every continuous  $f(x)$ .*

The proof is long, and will be divided into several parts. Let us start with a couple of Lemmas about the Lebesgue Set of  $f$ , and several pointwise estimates that will be of use.

**Lemma 15.1** *If  $\phi : \mathbb{R}^n \rightarrow \mathbb{C}$ , and*

$$|\phi(x)| \leq C(1 + |x|)^{n-\varepsilon}, \varepsilon > 0 \quad (36)$$

*then  $\phi \in L^1$ . Furthermore,  $\phi_t \in L^1$  for every  $t > 0$ .*

*Proof of 15.1.* If  $x \neq 0$ , then

$$|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)} \leq C \cdot |x|^{-(n+\varepsilon)}$$

on some  $B^c$  as defined in Theorem 2.52, so  $\phi \in L^1(B^c)$ . Next,

$$n + \varepsilon > n > n/2 = a$$

and by monotonicity,

$$|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)} \leq C \cdot (1 + |x|)^{-(n/2)}$$

so  $\phi \in L^1(\mathbb{R}^n)$ . Next, if  $\phi \in L^1$ , then

$$|\phi_t(x)| = t^{-n} |\phi(t^{-1}x)|$$

taking the integral in  $L^+$ , and applying Theorem 2.44, with  $T : x \mapsto t^{-1}x$ , and  $\det(T) = t^{-n}$ , so that

$$\int |\phi_t|(x) dx = |\det(T)| \int |\phi| \circ T(x) dx = \int |\phi|(x) dx < +\infty$$

This completes the Lemma. ■

**Lemma 15.2** *If  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ , and if  $f \in C(\mathbb{R}^n)$ , then  $\mathcal{L}_f = \mathbb{R}^n$ .*

*Proof of 15.2.* Let  $x \notin \mathcal{L}_f$ , and there exists a sequence  $r_k \rightarrow 0$  and  $\varepsilon_0 > 0$  but

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy \geq \varepsilon_0$$

We claim that for every  $k \geq 1$ , we can find a  $y_k \in B(r_k, x) \setminus \{x\}$  with

$$|f(x) - f(y_k)| \geq \varepsilon_0$$

Indeed, suppose by contradiction that no such  $y_k$  exists, and by monotonicity,

$$\frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} |f(x) - f(y)| dy < \frac{1}{m(B(r_k, x))} \int_{y \in B(r_k, x)} \varepsilon_0 dy = \varepsilon_0$$

So choose  $y_k$  as above, and it is clear that  $y_k \rightarrow x$  as  $k \rightarrow \infty$ , but  $f(y_k) \not\rightarrow f(x)$ . Therefore  $f$  is not continuous at  $x$ . ■

**Lemma 15.3** *If  $x \in \mathcal{L}_f$ , then for every  $\delta > 0$  there exists a  $\eta > 0$ , with*

$$r \leq \eta \implies \int_{|y| < r} |f(x - y) - f(x)| dy \leq \delta \cdot r^n$$

*Proof of 15.3.* We will start with something trivial.

$$m(B(r)) = r^n m(B(1)) \tag{37}$$

where  $B(r) = \{x \in \mathbb{R}^n, |x| < r\}$ . By Theorem 2.44,

$$\begin{aligned} m(B(r)) &= \int \chi_B(x/r) dx \\ &= |\det(T)|^{-1} \int \chi_B(x) dx \\ &= r^n m(B(1)) \end{aligned}$$

where  $T : x \mapsto x/r$  and  $\det(T) = r^{-n}$ . Fix  $x \in \mathcal{L}_f$ , and take  $\varepsilon = \delta/m(B(1)) > 0$ , and by definition this induces some  $\eta > 0$ , and for every  $r \leq \eta$

$$\frac{1}{m(B(r, x))} \int_{y \in B(r, x)} |f(x) - f(y)| dy \leq \varepsilon$$

By translation invariance of  $m$ ,

$$m(B(r, x)) = m(B(r)) = r^n \cdot m(B(1))$$

and apply the map  $y \mapsto x - y$ , which is a composition a rotation by  $|-1|$  and a translation by  $x \in \mathbb{R}^n$ . By Theorems 2.44 and 2.42,

$$\int_{|y| \in B(r)} |f(x) - f(x - y)| dy = \int_{y \in B(r, x)} |f(x) - f(y)| dy < \varepsilon m(B(1)) \cdot r^n = \delta r^n$$

where we used the fact that

$$\begin{aligned} d(x - y, x) < r &\iff d(-y, 0) < r \\ &\iff d(y, 0) < r \end{aligned}$$

hence

$$\chi_{B(r, x)}(x - y) = \chi_{B(r, 0)}(y)$$

■

**Lemma 15.4** *Let  $A_j = \left\{ |y| \in [2^{-j}\eta, 2^{1-j}\eta] \right\}$ , and if Equation (36) holds for  $\phi$  then  $\phi_t$  satisfies*

$$|\phi_t| \leq C \cdot t^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \quad (38)$$

on  $A_j$  for every  $t > 0$ , where  $\alpha = t^{-1}\eta$  for some  $\eta > 0$ .

Moreover, if  $A_0 = \left\{ |y| < 2^{-K}\eta \right\}$ , where  $K \geq 0$ , then

$$|\phi_t(y)| \leq C \cdot t^{-n} \quad (39)$$

on  $A_0$

*Proof of 15.4.* Notice that

$$t^{-1}y \in [2^{-j} \cdot \eta/t, 2^{1-j} \cdot \eta/t) = [2^{-j} \cdot \alpha, 2^{1-j} \cdot \alpha)$$

And

$$1 + |t^{-1}y| \geq |t^{-1}y| \geq 2^{-j}\alpha$$

Therefore

$$C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}$$

and applying Equation (36) establishes the first claim.

The second claim follows from Equation (36),

$$|\phi_t(y)| \leq C \cdot t^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq C \cdot t^{-n}$$

■

*Main Proof of Theorem 8.15.* The outline of the proof is as follows,

1.  $|\phi| \leq C \cdot (1 + |x|)^{-(n+\varepsilon)}$  for  $\varepsilon > 0$  and
2.  $f \in L^p$  for  $p \in [1, +\infty)$ ,
3. for any  $x \in \mathcal{L}_f$ , we wish to show

$$|f * \phi_t - af|(x) \rightarrow 0, \quad \text{as } t \rightarrow 0$$

4. To prove this, we fix some  $\beta > 0$  and show that

$$|f * \phi_t - af|(x) < \beta$$

since  $\beta$  is arbitrary, the proof will be complete.

5. By Lemma 15.3, for every  $\delta > 0$  there exists a  $\eta > 0$  where  $r \leq \eta$  implies

$$\int_{|y| < r} |f(x) - f(x - y)| dy \leq \delta \cdot r^n$$

and using the  $L^1$  inequality,

$$\begin{aligned} |f * \phi_t - af|(x) &= \left| \int [f(x - y) - f(x)] \cdot \phi_t(y) dy \right| \\ &\leq \int |f(x - y) - f(x)| \cdot |\phi_t(y)| dy \\ &= \int_{|y| < \eta} |f(x - y) - f(y)| \cdot |\phi_t(y)| dy + \int_{|y| \geq \eta} |f(x - y) - f(y)| \cdot |\phi_t(y)| dy \\ &= I_1 + I_2 \end{aligned}$$

6. Let  $\delta = \beta(2A)^{-1}$ , where

$$A = 2^n \cdot C \left[ \frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$$

we make the claim that this choice of  $\delta$  will give us  $I_1 < \beta/2$

7. After choosing  $\delta > 0$ , (which induces  $\eta > 0$ ), we will show that  $I_2 < \beta/2$  (for a fixed  $\eta > 0$ ) for  $t$  sufficiently small, and applying the Triangle Inequality finishes the proof.

Let  $\eta$  be as above, and for  $t > 0$  and suppose we can find a  $K \in \mathbb{N}^+$  with

$$2^K \leq \eta/t \leq 2^{K+1} \quad (40)$$

and define  $\alpha = \eta/t$  for convenience.

Notice for any  $K \geq 1$ , the interval  $[0, 1)$  can be partitioned in the following manner

$$[0, 1) = [0, 2^{-K}) \cup \left( \bigcup_{j=1}^K [2^{-j}, 2^{1-j}) \right)$$

and let us define

$$A_j = \left\{ |y| \in [2^{-j}\eta, 2^{1-j}\eta) \right\}, \quad A_0 = \left\{ |y| \in [0, 2^{-K}\eta) \right\}$$

If no such  $K$  exists, then let  $A_j = \emptyset$  and set  $A_0 = \{|y| \in [0, \eta)\}$ . The disjoint union of all  $A_{j \geq 0}$  is the open ball  $\{|y| \in [0, \eta)\}$ . By Lemma 15.4 and Lemma 15.3 each  $j \geq 0$ ,

$$\begin{aligned} I_1 &= \sum_{j=0}^K \int_{y \in A_j} |f(x-y) - f(y)| |\phi_t(y)| dy \\ &\leq Ct^{-n} \delta (2^{-K}\eta)^n + \sum_{j=1}^K \int_{y \in A_j} |f(x-y) - f(y)| |\phi_t(y)| dy \\ &\leq Ct^{-n} \delta (2^{-K}\eta)^n + \sum_{j=1}^K Ct^{-n} (2^{-j}\alpha)^{-(n+\varepsilon)} \delta (2^{1-j}\eta)^n \end{aligned}$$



The left member reads,

$$\begin{aligned} Ct^{-n}\delta(2^{-K}\eta)^n &\leq C\delta\alpha^n 2^{-Kn} \\ &\leq C\delta 2^{n(K+1)} 2^{-Kn} \\ &= C\delta 2^n \end{aligned}$$

and termwise for the right,

$$\begin{aligned} Ct^{-n}(2^{-j}\alpha)^{-(n+\varepsilon)}\delta(2^{1-j}\eta)^n &= C\delta \cdot t^\varepsilon \cdot 2^{j\varepsilon+n}\eta^{-\varepsilon} \\ &= (C\delta 2^n \alpha^{-\varepsilon}) \cdot 2^{j\varepsilon} \end{aligned}$$

Summing over the geometric series,

$$\begin{aligned} \sum_{j=1}^K 2^{j\varepsilon} &= 2^\varepsilon \sum_{j=0}^{K-1} 2^{j\varepsilon} \\ &= \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} \end{aligned}$$

using the estimate for  $\alpha$  in Equation (40)

$$\alpha \in [2^K, 2^K + 1) \implies \alpha^{-\varepsilon} \in [2^{-\varepsilon(K+1)}, 2^{-\varepsilon K})$$

and combining the last few equations, the right member becomes

$$\begin{aligned} (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)} - 2^\varepsilon}{2^\varepsilon - 1} &\leq (C\delta 2^n) \cdot \alpha^{-\varepsilon} \frac{2^{\varepsilon(K+1)}}{2^\varepsilon - 1} \\ &\leq (C\delta 2^n) \cdot \frac{2^\varepsilon}{2^\varepsilon - 1} \end{aligned}$$

Finally,  $I_1 \leq (C\delta 2^n) \left[ \frac{2^\varepsilon}{2^\varepsilon - 1} + 1 \right]$ , and by Step 6,  $I_1 \leq \beta/2$ .

Obtaining an estimate for  $I_2$  is another laborious enterprise. Let us define  $W = \{|y| \geq \eta\}$ , and

- By Holder's Inequality,

$$I_2 \leq \|f\|_p \|\chi_W \cdot \phi_t\|_q + |f(x)| \|\chi_W \cdot \phi_t\|_1$$

where  $q$  is the conjugate exponent to  $p$ . Since  $p \in [1, +\infty)$ , it suffices to show  $\|\chi_W \cdot \phi_t\|_q \rightarrow 0$  as  $t \rightarrow 0$  for  $q \in [1, +\infty]$ .

- Suppose  $q = +\infty$ ,

$$y \in W \iff |y| \geq \eta \iff |t^{-1}y| \geq \alpha$$

$$\text{then } \|\chi_W \cdot \phi_t\|_\infty \leq Ct^{-n}(1 + |t^{-1}y|)^{-(n+\varepsilon)} \leq Ct^\varepsilon \eta^{-(n+\varepsilon)}$$

- Now suppose  $q \in [1, +\infty)$ , by polar integration and Theorems 2.51, 2.52 (brace yourselves):

$$\begin{aligned} \|\chi_W \cdot \phi_t\|_q^q &= t^{-nq} \cdot \int_{y \in W} C^q \cdot |t^{-1}y|^{-q \cdot (n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \int_{|y| \geq \eta} |y|^{-q \cdot (n+\varepsilon)} dy \\ &= C^q \cdot t^{\varepsilon q} \sigma(S^{n-1}) \int_{r \geq \eta} r^{n-1} \cdot r^{-q \cdot (n+\varepsilon)} dr \\ &= \frac{C^q t^{\varepsilon q}}{n - q \cdot (n + \varepsilon)} r^{n - q \cdot (n + \varepsilon)} \Big|_\eta^\infty \\ &= \frac{C^q t^{\varepsilon q}}{q \cdot (n + \varepsilon) - n} \eta^{n - q \cdot (n + \varepsilon)} \\ \|\chi_W \cdot \phi_t\|_q &= \left[ \frac{C}{(q \cdot (n + \varepsilon) - n)^{1/q}} \left( \eta^{n - q \cdot (n + \varepsilon)} \right)^{1/q} \right] t^\varepsilon \\ &= C_3(q) t^\varepsilon \end{aligned}$$

- Find a  $t$  sufficiently small so that

$$t^\varepsilon < \min \left\{ \beta(4C_3(1)|f(x)|)^{-1}, \beta(4C_3(q)\|f\|_p)^{-1}, \beta(4C \cdot \eta^{-(n+\varepsilon)})^{-1} \right\}$$

- Therefore  $I_2 < \beta/2$ , and the proof is complete upon sending  $\beta \rightarrow 0$ .

■

**Theorem 8.16**

**WTS 16.1.** *See Theorem 8.15*

*Proof.*



**Theorem 8.17****WTS 17.1.***Proof.*

**Theorem 8.18****WTS 18.1.***Proof.*

**Theorem 8.19**

**WTS 19.1.**

*Proof.*



**Theorem 8.20****WTS 20.1.***Proof.*