Theorem 3.17

WTS. *Let the maximal function of any measurable* $f \in \mathbb{B}_{\mathbb{R}^n}$ *be denoted by* $Hf(x)$, *more precisely,*

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

where $A_r|f|$ *is the average of* $|f|$ *on a ball with radius* $r > 0$ *centered at* $x \in \mathbb{R}^n$. *In symbols,*

$$A_r|f| = \frac{1}{m(B(r,x))} \int_{B(r,x)} f(y)dy$$

The maximal theorem makes two claims:

1. $(Hf)^{-1}((\alpha, +\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, *and* Hf *is measurable for every* $f \in L^1_{loc}$.
2. *There exists a* $C > 0$, *for every* $f \in L^1$

   $$m(\{Hf(x) > \alpha\}) \leq \frac{C}{\alpha}\|f\|_1$$

   for every $\alpha > 0$.

Proof. Let $\alpha > 0$ and fix $z \in (Hf)^{-1}((\alpha, +\infty))$, so $Hf(z) > \alpha$ and

$$\sup_{r>0} A_r|f|(z) > \alpha$$

and with $Hf(z) - \alpha > 0$, we get some $r_0 > 0$

$$Hf(z) - (Hf(z) - \alpha) = \alpha < A_{r_0}|f|(z) \implies z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$$

Next, let $z \in \bigcup_{r>0} (A_r|f|)^{-1}((\alpha, +\infty))$, it is clear that

$$Hf(z) \geq A_{r_0}|f|(z) > \alpha$$

for some $r_0 > 0$. Since $A_r|f|$ (a function indexed by $r > 0$) is continuous in $x \in \mathbb{R}^n$, $(A_r|f|)^{-1}((\alpha, +\infty))$ is open, and Hf is measurable.

The second claim is slightly more intricate than the first. Define

$$E_\alpha = \left\{ Hf > \alpha \right\} = \bigcup_{r>0} \{A_r|f| > \alpha\}$$

Let $x \in E_\alpha$, this induces a $r_x > 0$ where $x \in \left\{ A_{r_x}|f| > \alpha \right\}$. Rearranging gives

$$\left(\frac{1}{\alpha} \int\limits_{B(r,x)} |f| dz \right) < m(B(r,x))$$

We wish to apply Theorem 3.15 to this family of open balls. Notice

- Each $x \in E_\alpha \looparrowright r_x > 0 \looparrowright A_{r_x}|f|$,
- If $U = \bigcup\limits_{x \in E_\alpha} B(r_x, x)$, then $E_\alpha \subseteq U$,
- Choose $c < m(E_\alpha) \le m(U)$ (by monotonicity) arbitrarily,
- By Theorem 3.15, there exists a finite disjoint subcollection of points indexed by
$$x_1, \ldots, x_N \in E_\alpha$$
so that $\bigsqcup_{j \le N} B(r_{x_j}, x_j) = U \supseteq E_\alpha$, and $c < 3^n \sum_{j \le k} m(B_j)$
- Define $B_j = B(r_{x_j}, x_j)$ for all $j \le k$, and
$$m(B_j) < \frac{1}{\alpha} \cdot \int_{B_j} |f| dz$$
by finite additivity,
$$c3^{-n} < \sum_{j \le k} m(B_j) < \frac{1}{\alpha} \cdot \sum_{j \le k} \int_{B_j} |f| dz$$
and finally
$$c < \frac{3^n}{\alpha} \sum_{j \le k} \int_{B_j} |f| dz \le \frac{3^n}{\alpha} \|f\|_1$$

- By inner regularity, of m on $\mathbb{B}$, since

$$m(E_\alpha) = \sup\left\{m(K),\ K \in ♪_{\mathbb{R}^n},\ K \subseteq E_\alpha\right\}$$

for any $K \in ♪_{\mathbb{R}^n}$, $K \subseteq E_\alpha$, we have $m(K) < +\infty$, $m(K) \leq m(E_\alpha)$ and

$$m(K) = c < \frac{3^n}{\alpha}\|f\|_1 \implies m(E_\alpha) \leq \frac{3^n}{\alpha}\|f\|_1$$

Remark. *We used the properties of a Radon Measure here, without relying on the phrase 'sending $c \to E_\alpha$', which would require us to deal with two cases $m(E_\alpha) < +\infty$ and $m(E_\alpha) = +\infty$.*

$\square$