MATH 263: Section 003, Tutorial 5

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1 Second Order Linear ODE's

A **second order linear ODE** is of the form:

$$a_0(x)y" + a_1(x)y'(x) + a_2(x)y(x) = g(x)$$

In particular, it is also **homogeneous** when $g(x) = 0$.

1.1 Principle of Superposition

It can be directly shown that when y_1 and y_2 solve the general homogeneous linear ODE:

$$a_0(x)y"(x) + a_1(x)y'(x) + a_2(x)y(x) = 0$$

the linear combination $y_0(x) = c_1y_1(x) + c_2y_2(x)$ is also a solution to the differential equation.

1.2 Homogeneous Equations with Constant Coefficients

Homogeneous Equations with Constant Coefficients are of the form:

$$ay" + by' + cy = 0$$

this is solved by making the substitution $y = e^{kx}$, leading to **the characteristic equation**

$$ak^2 + bk + c = 0.$$

Problem 1.2. From Boyce and DiPrima, 10th edition (3.1, exercise 11, p.144):
Find the solution to the IVP:

$$6y" - 5y' + y = 0$$

for $y(0) = 4,\ y'(0) = 0$.

Solution: Let $y = e^{kx}$:

$$y' = ke^{kx}$$
$$y" = k^2e^{kx}$$
$$6k^2e^{kx} - 5ke^{kx} + e^{kx} = 0$$
$$6k^2 - 5k + 1 = 0$$

$$k_{1,2} = \frac{1}{2 \cdot 6}(5 \pm \sqrt{5^2 - 4 \cdot 6 \cdot 1}) = \frac{1}{12}(5 \pm 1)$$
$$k_1 = \frac{1}{2},\ k_2 = \frac{1}{3}.$$

Therefore, by superposition, the solution is:

$$y = c_1 e^{k_1 x} + c_2 e^{k_2 x} = c_1 e^{\frac{1}{2}x} + c_2 e^{\frac{1}{3}x}.$$

Now, find the solution to the IVP $y(0) = 4,\ y'(0) = 0$. To do so, we must first find the solution's derivative:

$$y' = \frac{1}{2}c_1 e^{\frac{1}{2}x} + \frac{1}{3}c_2 e^{\frac{1}{3}x}$$

Therefore,

$$y(0) = c_1 e^{\frac{1}{2}\cdot 0} + c_2 e^{\frac{1}{3}\cdot 0} = c_1 + c_2 = 4$$
$$y(0) = 4 = \frac{1}{2}c_1 e^{\frac{1}{2}\cdot 0} + \frac{1}{3}c_2 e^{\frac{1}{3}\cdot 0} = \frac{1}{2}c_1 + \frac{1}{3}c_2 = 0$$

To find the constants, solve the system of equations using any method (substitution, Gaussian elimination, inverting the matrix).

$$c_1 = -8,\ c_2 = 12,$$

$$y = 12e^{\frac{x}{3}} - 8e^{\frac{x}{2}}.$$

1.3 Repeated Roots and Reduction of Order

When $ay" + by' + cy = 0$ and $b^2 = 4ac$, the characteristic equation will have have a unique solution and only the first solution can be found:

$$y_1(x) = e^{\frac{-b}{2a}x}$$

Therefore, a method called **reduction of order** must be used to find the second solution. For any second order homogeneous linear ODE, given one solution $y_1(x) \neq 0$ that solves

$$y" + p(x)y'(x) + q(x)y(x) = 0$$

let $y(x) = v(x)y_1(x)$, find y"s derivatives and substitute them in the ODE:

$$y'(x) = v'(x)y_1(x) + v(x)y_1'(x)$$
$$y_1 v" + (2y_1' + py_1)v' + (y_1" + py_1' + qy_1)v = 0$$

$$y_1 v" + (2y_1' + py_1)v' = 0,$$

which reduces to a first order ODE when letting $\gamma(x) = v'(x)$.

For the ODE, $ay" + by' + cy = 0$ where $b^2 = 4ac$, the solution is then:

$$y(x) = c_1 e^{\frac{-b}{2a}x} + c_2 x e^{\frac{-b}{2a}x} = (c_1 + c_2 x)e^{\frac{-b}{2a}x}$$

Problem 1.3. Find the solution of:

$$y" + 2y' + y = 0$$

where $y(0) = 1,\ y'(0) = 1$. Describe the solution's long-term behaviour.

Solution: Let $y = e^{kx}$:

$$y' = ke^{kx}$$
$$y" = k^2 e^{kx}$$

Then,

$$k^2 + 2k + 1 = 0$$

$$(k+1)^2 = 0$$
$$k_{1,2} = -1.$$

Therefore, using reduction of order, the general solution is of the form:

$$y(x) = (c_1 + c_2 x)e^{-x}$$

IVP: $y(0) = 1,\ y'(0) = 1$:

$$y'(x) = -(c_1 + c_2 x)e^{-x} + c_2 e^{-x} = ([c_2 - c_1] - c_2 x)e^{-x}$$
$$y(0) = (c_1 + c_2 \cdot 0)e^{-0} = c_1 = 1$$
$$y'(0) = ([c_2 - c_1] - c_2 \cdot 0)e^{-0} = c_2 - c_1 = 1 \Rightarrow c_2 = 2$$
$$y(x) = (1 + 2x)e^{-x}.$$

Then the long-term behaviour will be

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty}(1+2x)e^{-x} = 0.$$

2 Higher Order Linear ODE's

A **linear ODE of order n** is of the form:

$$\sum_{k=0}^{n} a_k(x) y^{(k)}(x) = g(x)$$

which is **homogeneous** when $g(x) = 0$. For homogeneous ODEs, the principle of superposition still holds.

To solve constant coefficient homogeneous linear ODEs of the form

$$\sum_{k=0}^{n} a_k y^{(k)}(x) = 0,$$

we can still let $y = e^{kx} \Rightarrow y^{(n)} = k^n e^{kx}$, which gives the characteristic polynomial in k:

$$\sum_{k=0}^{n} a_k k^n = 0.$$

Then, the general solution will be the linear combination of all particular solutions. In particular, when all roots are distinct and real:

$$y(x) = \sum_{i=0}^{n} c_i e^{k_i x},$$

finding a particular solution requires n initial values $y(x_0) = y_0,\ y'(x_1) = y_1,\ \dots\ ,\ y^{(n-1)}(x_{n-1}) = y_{n-1}$.

Problem 2.1. Find the general solution of:

$$y^{(4)} - 8y" + 16y = 0.$$

Solution: Let $y = e^{kx}$, then:

$$k^4 - 8k^2 + 16 = 0$$
$$(k^2 - 4)^2 = 0$$
$$(k+2)^2(k-2)^2 = 0$$

Therefore,

$$k_{1,2} = -2,\ k_{3,4} = 2.$$

Like for second order ODE's, we can see that the general solution to the ODE is:

$$y = (c_1 + c_2 x)e^{-2x} + (c_3 + c_4 x)e^{2x}.$$