# Chapter 1: Topological Manifolds

The $n$-sphere as a topological manifold. Define

$$S^n = \left\{ x \in \mathbb{R}^{n+1},\ |x| = 1 \right\}$$

We claim that $\{U_i^{\pm}\}_{i=1}^{n+1}$ form an open cover, where

$$U_i^+ = \left\{ x \in S^n, x^i > 0 \right\} \quad U_i^- = \left\{ x \in S^n, x^i < 0 \right\}$$

Each $U_i^{\pm}$ is the inverse image of $\pi_i^{-1}((0,+\infty)) \cap S^n$ or $\pi_i^{-1}((0,-\infty)) \cap S^n$, hence open. For every $x \in S^n$, there exists at least some $1 \leq j \leq n+1$ that makes the $j$-th coordinate of $x$, $x^j \neq 0$. So

$$S^n = \bigcup_i U_i^{\pm}$$

Denote the unit ball $\left\{ x \in \mathbb{R}^n,\ |x| < 1 \right\}$ in $\mathbb{R}^n$ by $\mathbb{B}^n$.

# Chapter 3: Tangent Spaces

**Proposition 0.1.** *Let $M$ be a smooth manifold, and fix $p \in M$. If $\nu \in T_pM$ is given with respect to the bases*

$$\left\{ \left.\frac{\partial}{\partial x^1}\right|_p, \dots, \left.\frac{\partial}{\partial x^m}\right|_p \right\} \quad \textit{and} \quad \left\{ \left.\frac{\partial}{\partial y^1}\right|_p, \dots, \left.\frac{\partial}{\partial y^m}\right|_p \right\}$$

*Defined by*

$$\left.\frac{\partial}{\partial x^j}\right|_p \triangleq d\left( \left.\phi^{-1}\right|_{\phi(p)} \right) \left( \left.\frac{\partial}{\partial x^j}\right|_{\phi(p)} \right) \quad \textit{and} \quad \left.\frac{\partial}{\partial y^j}\right|_p \triangleq d\left( \left.\psi^{-1}\right|_{\psi(p)} \right) \left( \left.\frac{\partial}{\partial y^j}\right|_{\psi(p)} \right)$$

*and we write $\nu$ in terms of the first basis*

$$\nu = \nu^j \left.\frac{\partial}{\partial x^j}\right|_p = \sum_{j=1}^{m} \nu^j \left.\frac{\partial}{\partial x^j}\right|_p$$

*and the second basis*

$$\nu = \nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_p = \sum_{k=1}^{m} \sum_{j=1}^{m} \nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_p$$

*If $f \in C^\infty(M)$, then*

$$\nu(f) = \nu^j \left.\frac{\partial}{\partial x^j}\right|_p f = \nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_p f$$

*Proof.* Recall $\left.\frac{\partial}{\partial x^j}\right|_p f \triangleq \left.\frac{\partial}{\partial x^j}\right|_{\phi(p)} f \circ \phi^{-1}$, similarly for $\left.\frac{\partial}{\partial y^j}\right|_p f$. Deriving $f$ and $p$ and by vector space operations on $T_pM$, the first basis expansion gives

$$\nu^j \left.\frac{\partial}{\partial x^j}\right|_p f = \nu^j \left.\frac{\partial}{\partial x^j}\right|_{\phi(p)} f \circ \phi^{-1} \tag{1}$$

and the second expression reads

$$\nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_p f = \nu^j \left.\frac{\partial y^k}{\partial x^j}\right|_{\phi(p)} \left.\frac{\partial}{\partial y^k}\right|_{\psi(p)} f \circ \psi^{-1} \tag{2}$$

Since $f \circ \phi^{-1} \in C^\infty(\mathbb{R}^m, \mathbb{R})$, we see the expressions are indeed equal. By the chain rule, if

$$\psi \circ \phi^{-1}(x^1, \dots x^m) = (y^1, \dots y^m)$$

then

$$D(\psi \circ \phi^{-1})(\phi(p)) = \begin{bmatrix} \left.\frac{\partial y^1}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^1}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^1}{\partial x^m}\right|_{\phi(p)} \\ \left.\frac{\partial y^2}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^2}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^2}{\partial x^m}\right|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left.\frac{\partial y^m}{\partial x^1}\right|_{\phi(p)} & \left.\frac{\partial y^m}{\partial x^2}\right|_{\phi(p)} & \cdots & \cdots & \left.\frac{\partial y^m}{\partial x^m}\right|_{\phi(p)} \end{bmatrix}$$

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