

# Contents

<b>Chapter 1: Topological Manifolds</b>	<b>4</b>
Topological Manifolds	5
Smooth Manifolds	6
<b>Chapter 2: Smooth Maps</b>	<b>10</b>
Smooth Maps	11
Diffeomorphisms	12
Partitions of Unity	13
<b>Chapter 3: Tangent Spaces</b>	<b>14</b>
Algebra of Germs on $C^\infty(N)$	15
Tangent spaces of manifolds	15
Tangent spaces of $\mathbb{R}^n$	15
Differential of a smooth map $F \in C^\infty(N, M)$	16
Differential of a smooth map $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$	17
Change of Coordinates Matrix	18
<b>Chapter 4: Submersions, Immersions and Embeddings</b>	<b>21</b>
Matrices	22
Estimates in vector calculus	22
Inverse Function Theorem (Rudin)	23
Inverse Function Theorem on Manifolds	26
More on immersions and embeddings	29
Regular values and level sets	30
Submanifolds	31
Measure zero	32
Lemmas for Sard's Theorem	33
Sard's Theorem	34
Whitney Embedding Theorem	39
Whitney Approximation Theorems	39
Tubular Neighbourhood Theorem	39
Transversality	39
<b>Chapter 9: Integral Curves and Flows</b>	<b>40</b>
Time invariant flows	41
Lie Derivatives of Vector Fields	41
Flowouts from Boundary	43

Time varying flows . . . . .	43
<b>Chapter 12: Tensors</b>	<b>44</b>
Tensor Products . . . . .	45
Tensor Fields on Manifolds . . . . .	46
Technicalities for Tensor Fields . . . . .	47
Pullbacks of Tensor Fields . . . . .	47
Lie Derivatives of Tensor Fields . . . . .	48
<b>Chapter 13: Riemannian Metrics</b>	<b>50</b>
<b>Chapter 14: Differential Forms</b>	<b>51</b>
Multi-linear algebra . . . . .	52
Differential Forms on Manifolds . . . . .	54
Exterior Derivatives . . . . .	58
<b>Chapter 22: Symplectic Manifold</b>	<b>60</b>
Symplectic Tensors . . . . .	61
Matrices and bilinear forms . . . . .	61
Orthogonality . . . . .	63
Riesz Representation Theorems . . . . .	64
Isometries . . . . .	65
Hyperbolic spaces, nonsingular completions . . . . .	65
Canonical Forms . . . . .	65
Symplectic Manifolds . . . . .	65
Darboux's Theorem . . . . .	65
<b>Chapter Hofer book</b>	<b>66</b>
<b>Chapter 1: Multilinear maps</b>	<b>69</b>
Introduction . . . . .	70
Bilinear maps . . . . .	70
Notation . . . . .	71
$k$ -linear maps . . . . .	72
<b>Chapter 2: Differentiation</b>	<b>75</b>
The derivative . . . . .	76
<b>Chapter 4: Higher order derivatives</b>	<b>80</b>
Introduction . . . . .	81
The second derivative . . . . .	81
The $p$ -th derivatives . . . . .	85
<b>Chapter A: Review of Topology</b>	<b>89</b>
Set Operations . . . . .	90
Topological Spaces . . . . .	91
Basis of a Topology . . . . .	91
Product Topology . . . . .	93

Continuity . . . . .	94
Quotient Topology . . . . .	94
Product Topology . . . . .	94
Connectedness . . . . .	95
Interiors and closures . . . . .	96
Neighbourhoods . . . . .	97
Adherent points . . . . .	97
Dense and nowhere dense subsets . . . . .	98
Urysohn's Lemma . . . . .	99
Compactness . . . . .	103
Properties of Compact Spaces . . . . .	105
Locally Compact Hausdorff Spaces . . . . .	108
<b>Chapter B: Abstract Algebra</b>	<b>110</b>
Groups . . . . .	111
Homomorphisms . . . . .	114
Subgroups . . . . .	116
Cyclic groups . . . . .	117
Cosets . . . . .	118
Normal Subgroups . . . . .	118
Isomorphism Theorems . . . . .	118
<b>Chapter C: Algebraic Topology</b>	<b>119</b>
Homotopy . . . . .	120
Path and PathClass Products . . . . .	122
Fundamental Group . . . . .	124
Homomorphisms . . . . .	124
Simply connected space . . . . .	125
Covering maps . . . . .	125

# Chapter 1: Topological Manifolds

## Topological Manifolds

The study of differential geometry begins with tens of pages of definitions.

### Definition 1.1: Topological Manifold

Let  $M$  be a topological space.  $M$  is a topological manifold of dimension  $m$  if it is Hausdorff, second-countable, and locally homeomorphic to  $\mathbb{R}^n$ .

### Definition 1.2: Local homeomorphism

$M$  locally homeomorphic to  $\mathbb{R}^n$  if every point  $x \in M$  an open set  $U$ , equipped with a homeomorphism which sends points in  $U$  into an open subset of  $\mathbb{R}^n$ .

$$\phi : U \rightarrow \phi(U)$$

The tuple  $(U, \phi)$  is called a coordinate chart.

### Definition 1.3: More on coordinate charts

- A coordinate chart  $(U, \phi)$  is centered at  $p \in M$  if  $p \in U$  and  $\phi(p) = 0 \in \mathbb{R}^n$ .
- We call  $U$  the coordinate domain, and
- we call  $\phi$  the coordinate map.
- If the choice of  $(U, \phi)$  is unambiguous, then the local coordinates of  $p$  are simply the coordinates of  $\phi(p)$  in  $\mathbb{R}^n$ , and
- we sometimes also denote  $\phi(U)$  by  $\hat{U}$  if it is unambiguous to do so.
- If  $\hat{U}$  is an open ball/cube, then  $U$  is called a coordinate ball/cube.

The central theme of point-set topology (or even metric topology) is that of passing a topological argument to the basis or to a neighbourhood. Manifolds in particular have a nice basis.

### Proposition 1.1: Basis of precompact coordinate balls

Every topological manifold has a countable basis of precompact coordinate balls.

### Proposition 1.2: Additional facts about topological manifolds

If  $M$  is a topological manifold,

- $M$  is locally compact. (Lee, Proposition 1.12)
- $M$  is paracompact, and every open cover has a refinement that is another countably locally finite open cover whose elements are chosen from an arbitrary (but fixed) basis of  $M$ . (Lee, Theorem 1.15)

- $M$  is locally-path connected.
- $M$  is connected iff it is path-connected.
- $M$  is metrizable. (Munkres Chapter 6)

## Smooth Manifolds

We wish to perform calculus on manifolds.

### Definition 2.1: Smooth function $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$

Let  $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ , replacing  $\mathbb{R}^n$  and  $\mathbb{R}^m$  with open subsets if necessary.  $F$  is smooth if its (scalar-valued) component functions has continuous partial derivatives of all orders. The set of smooth functions from  $\mathbb{R}^n$  to  $\mathbb{R}^m$  is sometimes denoted by  $C^\infty(\mathbb{R}^n, \mathbb{R}^m)$ . If  $m = 1$ , we sometimes write  $C^\infty(\mathbb{R}^n)$ , similar to a test function on the Schwartz Space.

### Definition 2.2: Transition map from $\phi$ to $\psi$

Let  $(U, \phi)$  and  $(V, \psi)$  be coordinate charts on  $M$ . The composite function (whenever  $U \cap V \neq \emptyset$ )

$$\psi \circ \phi^{-1} : \phi(U \cap V) \rightarrow \psi(U \cap V)$$

is called the transition map. Notice  $\psi \circ \phi^{-1}$  is by definition a homeomorphism.

### Definition 2.3: Smoothly compatible

Two coordinate charts on  $M$ ,  $(U, \phi)$  and  $(V, \psi)$  are called smoothly compatible if either their domains are disjoint, or their transition map is a diffeomorphism on  $\mathbb{R}^m$ .

### Definition 2.4: Smooth atlas

An atlas  $\mathcal{A}$  of  $M$  is a collection of charts  $\{(U_\alpha, \phi_\alpha)\}$  whose collection of coordinate domains  $\{U_\alpha\}$  for an open cover of  $M$ .

It is called a smooth atlas if any two charts in the atlas are pairwise smoothly compatible.

### Definition 2.5: Smooth manifold

A smooth atlas  $\mathcal{A}$  on  $M$  is maximal if it is not contained (properly) in any other smooth atlas as a subset. In other words, if  $(U', \phi')$  is a chart on  $M$  that is smoothly compatible with all elements in  $\mathcal{A}$ , then  $(U', \phi') \in \mathcal{A}$  already.

This smooth atlas is often very large, it includes all translations of charts, dilations, and composition with diffeomorphisms in  $\mathbb{R}^m$ , restrictions onto open subsets, etc. A maximal smooth atlas is sometimes called a complete atlas, or a smooth manifold structure.

A smooth manifold is the tuple  $(M, \mathcal{A})$ , where  $\mathcal{A}$  is some smooth atlas. It can happen if  $M$  is originally a topological manifold with a huge number of charts, some of which are smoothly compatible with others, that  $\mathcal{A}$  is a strict subset, and both  $\mathcal{A}_1$  and  $\mathcal{A}_2$  are maximal smooth atlases on  $M$ , but  $\mathcal{A}_1 \neq \mathcal{A}_2$ . We often omit  $\mathcal{A}$  and write  $M$  if the smooth atlas is understood or not of importance.

### Definition 2.6: Smooth coordinate terminologies

Let  $(M, \mathcal{A})$  be a smooth manifold.

- Any coordinate chart  $(U, \phi) \in \mathcal{A}$  is called a smooth chart, similar to def. 1.3
- We call  $U$  the *smooth coordinate domain* or *smooth coordinate neighbourhood* of any  $p \in U$ , and
- we call  $\phi$  the *smooth coordinate map*.
- The terms *smooth coordinate ball* and *smooth coordinate cube* are used similarly.
- A set  $B \subseteq M$  is a *regular coordinate ball* if its image is a smooth coordinate ball centered at the origin; and the closure of this ball in  $\mathbb{R}^m$  is a subset of the image of another smooth coordinate ball, centered at the origin.

### Definition 2.7: Standard smooth structure on $\mathbb{R}^n$

The maximal smooth atlas containing  $(\mathbb{R}^n, \text{id}_{\mathbb{R}^n})$  is called the *standard smooth structure on  $\mathbb{R}^n$* .

Manifolds with boundary are not as important as regular manifolds for now, but they are worth mentioning.

### Definition 2.8: Closed n-dimensional upper half-plane $\mathbb{H}^n \subseteq \mathbb{R}^n$

We define the following symbols for the upper half plane.

- $\mathbb{H}^n = \{x \in \mathbb{R}^n, x^n \geq 0\}$ ,
- $\text{Int } \mathbb{H}^n = \{x \in \mathbb{R}^n, x^n > 0\}$ ,
- $\partial \mathbb{H}^n = \{x \in \mathbb{R}^n, x^n = 0\}$

### Definition 2.9: Manifolds with boundary

A topological space  $M$  is called a manifold with boundary if it is Hausdorff, second-countable, and locally homeomorphic to an open subset of  $\mathbb{H}^n$  (endowed with the subspace topology from  $\mathbb{R}^n$ ).

A chart  $(U, \phi)$  is an *interior chart* if its coordinate image is disjoint from the 'boundary' of the upper-half plane. This means  $\phi(U) \cap \partial \mathbb{H}^n = \emptyset$ . Similarly,  $(V, \psi)$  is a *boundary chart* if its range contains a point in  $\partial \mathbb{H}^n$ ; so  $\psi(V) \cap \partial \mathbb{H}^n \neq \emptyset$ .

Similar to def. 1.3 and def. 2.6, we use the terms *coordinate half-ball*, *coordinate half-cube*, *regular coordinate half-ball*.

Let  $p \in M$ , it is called an *interior point of  $M$*  (not to be confused with the topological interior) if it is in the domain of some interior chart, and  $p$  is called a *boundary point of  $M$*  if there exists a boundary chart that sends  $p$  into  $\partial\mathbb{H}^n$ . The set of interior points and boundary points of  $M$  will be denoted by  $\text{Int } M$  and  $\partial M$ .



**Example 2.1: Sphere as a topological manifold**

The  $n$ -sphere as a topological manifold. Define

$$S^n = \left\{ x \in \mathbb{R}^{n+1}, |x| = 1 \right\}$$

We claim that  $\{U_i^\pm\}_{i=1}^{n+1}$  form an open cover, where

$$U_i^+ = \left\{ x \in S^n, x^i > 0 \right\} \quad U_i^- = \left\{ x \in S^n, x^i < 0 \right\}$$

Each  $U_i^\pm$  is the inverse image of  $\pi_i^{-1}((0, +\infty)) \cap S^n$  or  $\pi_i^{-1}((0, -\infty)) \cap S^n$ , hence open. For every  $x \in S^n$ , there exists at least some  $1 \leq j \leq n+1$  that makes the  $j$ -th coordinate of  $x$ ,  $x^j \neq 0$ . So

$$S^n = \bigcup_i U_i^\pm$$

Denote the unit ball  $\left\{ x \in \mathbb{R}^n, |x| < 1 \right\}$  in  $\mathbb{R}^n$  by  $\mathbb{B}^n$ .

## Chapter 2: Smooth Maps

## Smooth Maps

### Definition 1.1: Smooth functions $C^\infty(M, \mathbb{R}^k)$

Let  $F : M \rightarrow \mathbb{R}^k$  be a vector-valued function on a smooth manifold  $M$ . We say  $F$  is a smooth function if for every  $p \in M$ , there exists a smooth chart  $p \in (U, \phi)$  such that the *coordinate representation of  $F$  at  $p$ , with respect to  $(U, \phi)$*  is a smooth function from  $\mathbb{R}^m$  to  $\mathbb{R}^k$ , denoted by  $\hat{F}$  (in the sense of Definition 2.1).

$$\hat{F} = F \circ \phi^{-1} : \phi(U) \rightarrow \mathbb{R}^k \in C^\infty(\phi(U), \mathbb{R}^k)$$

if  $k = 1$ , then we denote the space of *test functions* on  $M$  by  $C^\infty(M) = C^\infty(M, \mathbb{R})$

### Definition 1.2: Smooth maps between manifolds $C^\infty(N, M)$

Let  $F : N \rightarrow M$  be a map between smooth manifolds  $N$  and  $M$  (note we switched the order).  $F$  is a smooth map if at every  $p \in M$ , there exists

- a chart in the smooth atlas of  $N$  (the domain),  $p \in (U, \phi)$ ,
- another chart in the smooth atlas of  $M$  (the range),  $F(U) \subseteq (V, \psi)$ ,
- such that, the *coordinate representation of  $F$  at  $p$  with respect to  $(U, \phi)$ , and  $(V, \psi)$*  is a smooth function from  $\mathbb{R}^n$  to  $\mathbb{R}^m$ , also denoted by  $\hat{F}$ .

$$\hat{F} = \psi \circ F \circ \phi^{-1} : \phi(U) \rightarrow \psi(V) \in C^\infty(\mathbb{R}^n, \mathbb{R}^m) \quad (1)$$

The following propositions summarizes common operations on smooth maps, a few sources of them.

### Proposition 1.1: Smooth maps are continuous

If  $F : N \rightarrow M$  is a smooth map, then  $F$  is continuous with respect to the topologies on  $N$  and  $M$ .

*Proof.* Let  $p \in N$  be fixed, because  $F$  is smooth this induces two smooth charts, one in the domain and another in the range; as in def. 1.2.  $F(p)$  is a point in  $M$ . From eq. (1),  $\hat{F}|_{\phi(U)}$  is a smooth hence continuous function. Since  $\phi : U \mapsto \phi(U)$  and  $\psi : V \mapsto \psi(V)$  are homeomorphisms,

$$F|_U = \underbrace{\psi^{-1}}_{\text{continuous}} \circ \underbrace{\hat{F}|_{\phi(U)}}_{\text{smooth}} \circ \underbrace{\phi}_{\text{continuous}} \quad \text{is continuous on } U$$

Let the point  $p$  range through all the points in  $N$ , so  $F$  is continuous at every  $p$ , hence on  $N$ . ■

### Proposition 1.2: Characterizations of Smooth Maps in $C^\infty(N, M)$

Let  $N$  and  $M$  be smooth manifolds, and  $F : N \rightarrow M$ .  $F$  is a smooth map iff

- For every  $p \in N$ , there exists smooth charts  $p \in (U, \phi)$  and  $F(p) \in (V, \psi)$  such that  $U \cap F^{-1}(V)$  is an open set in  $N$ , and the composite map (the coordinate representation)

$$\psi \circ F \circ \phi^{-1}|_{(U \cap F^{-1}(V))} : \phi(U \cap F^{-1}(V)) \rightarrow \psi(V) \quad \text{is smooth}$$

- $F$  is continuous and there exist smooth atlases  $\{(U_\alpha, \phi_\alpha)\} \subseteq \mathcal{A}_N$ , and  $\{(V_\beta, \psi_\beta)\} \subseteq \mathcal{A}_M$  such that the coordinate representation

$$\psi_\beta \circ F \circ \phi_\alpha^{-1} : \phi(U_\alpha \cap F^{-1}(V_\beta)) \rightarrow \psi(V_\beta) \quad \text{is smooth}$$

whenever it makes sense.

- the restriction of  $F$  onto any arbitrary open set  $U$ ,  $F|_U : U \mapsto M$  is smooth (in the sense of open submanifold).

*Proof.* By Proposition 1.1, and the fact that complete atlases are closed under restrictions onto open sets, it is clear that the original definition implies the two. The first definition also clearly implies the original definition, as we can restrict

$$(U, \phi) \mapsto \left( U \cap F^{-1}(V), \phi \Big|_{U \cap F^{-1}(V)} \right)$$

since  $U \cap F^{-1}(V)$  is open in the domain manifold.

The second definition implies the original one as well, since the smooth atlases are taken from the maximal atlas, we can pass the argument to any smoothly-compatible chart. Atlases must cover both the domain and the range, and coordinate transitions between smoothly compatible charts are diffeomorphisms. If  $F$  is smooth on a subcollection of those charts, meaning

$$\psi_\beta \circ F \circ \phi_\alpha^{-1} \in C^\infty(\phi_\alpha(U_\alpha) \cap F^{-1}(V_\beta), \psi_\beta(V_\beta))$$

it is smooth with respect to every pair of (smooth) charts in the two atlases  $\mathcal{A}_N$ ,  $\mathcal{A}_M$ , as a composition of smooth maps:

$$\psi \circ \underbrace{\psi^{-1} \circ \psi_\beta}_{\text{smooth}} \circ F \circ \underbrace{\phi_\alpha^{-1} \circ \phi}_{\text{smooth}} \circ \phi^{-1}$$

where we can restrict  $\phi_\alpha \mapsto \phi_\alpha|_{(U_\alpha \cap F^{-1}(V_\beta))}$  by continuity of  $F$ .

I will prove the third and last equivalence later. ■

### Proposition 1.3: Sources of smooth maps

Let  $N, M, P$  be smooth manifolds, then

- Every constant map is smooth,
- The identity map  $\text{id}_M : M \rightarrow M$  is smooth,
- The inclusion map  $\iota : W \rightarrow M$  is smooth, where  $W$  is an open submanifold of  $M$ .
- The composition of smooth maps is again a smooth map: if  $F \in C^\infty(N, M)$  and  $G \in C^\infty(M, P)$ , then  $(G \circ F) \in C^\infty(N, P)$

## Diffeomorphisms

**Definition 2.1: Diffeomorphism between Manifolds  $\mathcal{D}(N, M)$** 

Let  $N$  and  $M$  be smooth manifolds,  $F : N \rightarrow M$  is a diffeomorphism if it is a smooth bijective map with a smooth inverse. We denote the space of diffeomorphisms from  $N$  to  $M$  by  $\mathcal{D}(N, M)$ .

**Proposition 2.1: Properties of Manifold Diffeomorphisms**

Let  $N$ ,  $M$  and  $P$  be smooth manifolds, then

- The composition of diffeomorphisms is again a diffeomorphism, that is, if  $F \in \mathcal{D}(N, M)$  and  $G \in \mathcal{D}(M, P)$ , then  $(G \circ F) \in \mathcal{D}(N, P)$ .
- The open-manifold restriction of a diffeomorphism onto its image is again a diffeomorphism,
- Every diffeomorphism is a homeomorphism and an open map.

*Proof.* Trivial. ■

**Partitions of Unity**

See Folland Chapters 4 and 8. Including Urysohn's Lemma, Tietze's Extension Theorem, the usual construction of  $C_c^\infty$  bump functions.

## Chapter 3: Tangent Spaces

## Algebra of Germs on $C^\infty(N)$

The tangent space is a powerful concept that acts almost like the dual in distribution theory.

### Definition 1.1: Algebra of Germs at $p$ : $C_p^\infty(N)$

Let  $N$  be a smooth manifold and  $p \in N$ . We define an equivalence relation on the space of test functions on  $N$ ,  $C^\infty(N)$ . If  $f, g \in C^\infty(N)$ , we write  $f \sim g$  if  $f = g$  for some open neighbourhood about  $p$ . We denote this equivalence class by  $C_p^\infty(N)$ , and it is clear  $C^\infty(N)$  is closed under pointwise multiplication by the product rule, and form an algebra; so  $C_p^\infty(N)$  is an algebra too.

## Tangent spaces of manifolds

### Definition 2.1: Vector space of derivations at $p$ : $T_p N$

Let  $\nu : C_p^\infty(N) \rightarrow \mathbb{R}$  be a linear functional on the vector space of germs at  $p$ . It is called a derivation at  $p$  if it satisfies the product rule, if  $f, g \in C_p^\infty(N)$ ,

$$\nu(fg) = g(p)\nu(f) + f(p)\nu(g)$$

then we say

- $\nu$  is a tangent vector at  $p$ ,
- $\nu \in T_p N$ ,
- $\nu$  is an element of the *tangent space* at  $p$ .
- $\nu$  is a derivation on  $N$  at  $p$ .

### Proposition 2.1: Properties of derivations at $p$

Let  $N$  be a smooth manifold and  $p \in N$ .

- If  $f \in C_p^\infty$  is constant in some neighbourhood of  $p$ , then  $\nu(f) = 0$  for every  $\nu \in T_p N$ ,
- If  $f(p) = g(p) = 0$ , then  $\nu(fg) = 0$  for tangent vector  $\nu$  at  $p$ .

## Tangent spaces of $\mathbb{R}^n$

### Proposition 3.1: Basis of $T_p \mathbb{R}^n$

Let  $\mathbb{R}^n$  be equipped with the standard smooth structure as in Definition 2.7. The vector space of derivations at  $p \in \mathbb{R}^n$  are spanned by the  $n$  partial derivatives at  $p$

$$\left. \frac{\partial}{\partial x^j} \right|_p : f \mapsto \left. \frac{\partial}{\partial x^j} f(x) \right|_p, \quad 1 \leq j \leq n, \quad f \in C^\infty(\mathbb{R}^n)$$

Moreover, the  $n$  vectors form a basis, and  $\dim T_p \mathbb{R}^n = n$ .

**Definition 3.1: Standard Basis of  $T_p\mathbb{R}^n$**

The standard basis for the tangent space at  $p \in \mathbb{R}^n$  is the  $n$  partial derivatives at  $p$ .

$$T_p\mathbb{R}^n = \left\{ \frac{\partial}{\partial x^1} \Big|_p, \dots, \frac{\partial}{\partial x^n} \Big|_p \right\} \quad (2)$$

**Definition 3.2: Coordinate functions  $x^j$  on  $\mathbb{R}^n$**

Let  $x^1, \dots, x^n$  be the standard coordinates on  $\mathbb{R}^n$ . If  $1 \leq j \leq n$ ,  $x^j$  is a function (also represented by the same symbol as the standard coordinate) from  $\mathbb{R}^n$  to  $\mathbb{R}$ , which maps each  $p = (p^1, \dots, p^j, \dots, p^n)$  to its  $j$ -th coordinate  $p \mapsto p^j$ .

This map is linear, hence smooth, has the matrix representation of  $\mathcal{M}\{x^j\} = (\delta_{jk})_{1,k} \in \mathbb{R}^{1 \times n}$  where  $\delta_{jk}$  denotes the discrete mass at  $j$ .

Furthermore, the coordinate functions behave like the dual basis for the derivations  $\partial/\partial x^j|_p$  at  $p$

$$\frac{\partial}{\partial x^j} \Big|_p (x^k) = \delta_{jk}$$

If  $\nu \in T_p\mathbb{R}^n$ , and has the basis representation

$$\nu = \sum_j \nu^j \frac{\partial}{\partial x^j} \Big|_p = \nu^j \frac{\partial}{\partial x^j} \Big|_p$$

then

$$\nu = \sum_j \nu(x^j) \frac{\partial}{\partial x^j} \Big|_p = \nu(x^j) \frac{\partial}{\partial x^j} \Big|_p \quad (3)$$

**Differential of a smooth map  $F \in C^\infty(N, M)$**

**Definition 4.1: Differential of a smooth map  $dF_p : T_pN \rightarrow T_{F(p)}M$**

Let  $F \in C^\infty(N, M)$ ,  $p \in N$  and  $\nu \in T_pN$  be a tangent vector at  $p$ . The differential of a smooth map is a linear map that sends tangent vectors in  $T_pN$  to tangent vectors in  $T_{F(p)}M$ . If  $f \in C^\infty(M)$  is a test function on  $M$ , then  $f \circ F$  is a test function on  $N$ , and

$$dF_p(\nu)(f) = \nu \left( \underbrace{f \circ F}_{C^\infty(N)} \right)$$

We state the following without proof, as the proof is tedious. It simply involves unboxing the definition of the differential  $dF_p : T_pN \rightarrow T_pM$ , and projecting onto the coordinates.

**Proposition 4.1: Properties of the differential**



Let  $N$ ,  $M$  and  $P$  be smooth manifolds, if  $F \in C^\infty(N, M)$ ,  $G \in C^\infty(M, P)$  then

- $dF_p$  is a linear map between  $T_p$  and  $T_{F(p)}N$ ,
- $d(G \circ F)_p = dG_{F(p)} \circ dF_p$
- $d(\text{id}_N)_p = \text{id}_{T_p N}$ ,
- if  $F \in \mathcal{D}(N, M)$ , then  $dF_p$  is a linear isomorphism between  $T_p N$  and  $T_{F(p)}M$ , and

$$(dF_p)^{-1} = d(F^{-1})_{F(p)}$$

**Proposition 4.2: Matrix representation of the differential of  $F : N \rightarrow M$**

Let  $F \in C^\infty(N, M)$ , and  $p \in N$  induces two charts  $p \in (U, \phi)$  and  $F(p) \in (V, \psi)$ . The matrix representation of the differential at  $p$ ,  $dF_p : T_p N \rightarrow T_{F(p)}M$  is nothing but the Jacobian matrix of size  $m \times n$  of the coordinate representation at  $p$ .

$$\mathcal{M}\{dF_p\} = \begin{bmatrix} \frac{\partial \hat{F}^1}{\partial x^1} \Big|_{\phi(p)} & \frac{\partial \hat{F}^1}{\partial x^2} \Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^1}{\partial x^n} \Big|_{\phi(p)} \\ \frac{\partial \hat{F}^2}{\partial x^1} \Big|_{\phi(p)} & \frac{\partial \hat{F}^2}{\partial x^2} \Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^2}{\partial x^n} \Big|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \hat{F}^m}{\partial x^1} \Big|_{\phi(p)} & \frac{\partial \hat{F}^m}{\partial x^2} \Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial \hat{F}^m}{\partial x^n} \Big|_{\phi(p)} \end{bmatrix} \quad (4)$$

Alternately, if we write  $\hat{p} = \phi(p)$  as the  $\mathbb{R}^n$  coordinates at  $p$ , then

$$\mathcal{M}\{dF_p\} = \begin{bmatrix} \frac{\partial \hat{F}^1}{\partial x^1} \Big|_{\hat{p}} & \frac{\partial \hat{F}^1}{\partial x^2} \Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^1}{\partial x^n} \Big|_{\hat{p}} \\ \frac{\partial \hat{F}^2}{\partial x^1} \Big|_{\hat{p}} & \frac{\partial \hat{F}^2}{\partial x^2} \Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^2}{\partial x^n} \Big|_{\hat{p}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial \hat{F}^m}{\partial x^1} \Big|_{\hat{p}} & \frac{\partial \hat{F}^m}{\partial x^2} \Big|_{\hat{p}} & \cdots & \cdots & \frac{\partial \hat{F}^m}{\partial x^n} \Big|_{\hat{p}} \end{bmatrix} \quad (5)$$

**Differential of a smooth map  $F \in C^\infty(\mathbb{R}^n, \mathbb{R}^m)$**

An important application of this is the following. We begin with the  $\mathbb{R}^m \rightarrow \mathbb{R}^n$  case. We will see that if  $p$  and  $F(p)$  are represented by another pair of coordinate charts (smoothly compatible with the previous pair), then the rank of  $dF_p$  does not change. So the rank of the differential is an invariant of the choice of coordinate chart.

**Definition 5.1: Matrix representation of the differential of  $F : \mathbb{R}^m \rightarrow \mathbb{R}^n$**

Let  $F \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$ , and  $p \in \mathbb{R}^m$  induces two charts  $p \in (U, \text{id}_{\mathbb{R}^m})$  and  $F(p) \in (V, \text{id}_{\mathbb{R}^n})$ , where  $U \subseteq \mathbb{R}^m$  and  $V \subseteq \mathbb{R}^n$ . The matrix representation of the differential at  $p$ ,  $dF_p : T_p\mathbb{R}^m \rightarrow T_{F(p)}\mathbb{R}^n$  is nothing but the Jacobian matrix of  $F$  at  $p$ .

$$\mathcal{M}\{dF_p\} = DF(p) = \begin{bmatrix} \left. \frac{\partial F^1}{\partial x^1} \right|_p & \left. \frac{\partial F^1}{\partial x^2} \right|_p & \cdots & \cdots & \left. \frac{\partial F^1}{\partial x^m} \right|_p \\ \left. \frac{\partial F^2}{\partial x^1} \right|_p & \left. \frac{\partial F^2}{\partial x^2} \right|_p & \cdots & \cdots & \left. \frac{\partial F^2}{\partial x^m} \right|_p \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \left. \frac{\partial F^n}{\partial x^1} \right|_p & \left. \frac{\partial F^n}{\partial x^2} \right|_p & \cdots & \cdots & \left. \frac{\partial F^n}{\partial x^m} \right|_p \end{bmatrix} \quad (6)$$

**Change of Coordinates Matrix**

We will go through the section on the Change of Coordinates, and how different coordinate charts change the representation of a derivation at  $p \in M$ , where  $M$  is some smooth manifold.

**Definition 6.1: Standard basis of  $T_pN$**

From prop. 2.1, since  $\phi^{-1}$  is a diffeomorphism,  $d(\phi^{-1}|_{\phi(p)}) : T_{\phi(p)}\mathbb{R}^n \rightarrow T_pN$  is a linear isomorphism. Hence  $T_pN$  is a  $n$ -dimensional vector space, and the standard basis vectors of  $T_p\mathbb{R}^n$  are denoted by

$$\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right\} \quad (7)$$

where each basis vector  $\left. \frac{\partial}{\partial x^j} \right|_p \triangleq d(\phi^{-1}|_{\phi(p)})$  is the push-forward derivation (through  $\phi^{-1}$ ) of the  $j$ -th standard basis vector in  $T_{\phi(p)}N$ .

**Proposition 6.1: Differential of  $\psi \circ \phi^{-1} : M \rightarrow M$**

Let  $M$  be a smooth manifold, and fix  $p \in M$ . If  $\nu \in T_pM$  is given with respect to the bases

$$\left\{ \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^m} \right|_p \right\} \quad \text{and} \quad \left\{ \left. \frac{\partial}{\partial y^1} \right|_p, \dots, \left. \frac{\partial}{\partial y^m} \right|_p \right\}$$

Defined by

$$\left. \frac{\partial}{\partial x^j} \right|_p \triangleq d\left(\phi^{-1}\Big|_{\phi(p)}\right)\left(\left. \frac{\partial}{\partial x^j} \right|_{\phi(p)}\right) \quad \text{and} \quad \left. \frac{\partial}{\partial y^j} \right|_p \triangleq d\left(\psi^{-1}\Big|_{\psi(p)}\right)\left(\left. \frac{\partial}{\partial y^j} \right|_{\psi(p)}\right)$$

and we write  $\nu$  in terms of the first basis

$$\nu = \nu^j \frac{\partial}{\partial x^j} \Big|_p = \sum_{j=1}^m \nu^j \frac{\partial}{\partial x^j} \Big|_p$$

and the second basis

$$\nu = \nu^j \frac{\partial y^k}{\partial x^j} \Big|_{\phi(p)} \frac{\partial}{\partial y^k} \Big|_p = \sum_{k=1}^m \sum_{j=1}^m \nu^j \frac{\partial y^k}{\partial x^j} \Big|_{\phi(p)} \frac{\partial}{\partial y^k} \Big|_p$$

If  $f \in C^\infty(M)$ , then

$$\nu(f) = \nu^j \frac{\partial}{\partial x^j} \Big|_p f = \nu^j \frac{\partial y^k}{\partial x^j} \Big|_{\phi(p)} \frac{\partial}{\partial y^k} \Big|_p f$$

*Proof.* Recall  $\frac{\partial}{\partial x^j} \Big|_p f \triangleq \frac{\partial}{\partial x^j} \Big|_{\phi(p)} f \circ \phi^{-1}$ , similarly for  $\frac{\partial}{\partial y^j} \Big|_p f$ . Deriving  $f$  and  $p$  and by vector space operations on  $T_p M$ , the first basis expansion gives

$$\nu^j \frac{\partial}{\partial x^j} \Big|_p f = \nu^j \frac{\partial}{\partial x^j} \Big|_{\phi(p)} f \circ \phi^{-1} \quad (8)$$

and the second expression reads

$$\nu^j \frac{\partial y^k}{\partial x^j} \Big|_{\phi(p)} \frac{\partial}{\partial y^k} \Big|_p f = \nu^j \frac{\partial y^k}{\partial x^j} \Big|_{\phi(p)} \frac{\partial}{\partial y^k} \Big|_{\psi(p)} f \circ \psi^{-1} \quad (9)$$

Since  $f \circ \phi^{-1} \in C^\infty(\mathbb{R}^m, \mathbb{R})$ , we see the expressions are indeed equal. By the chain rule, if

$$\psi \circ \phi^{-1}(x^1, \dots, x^m) = (y^1, \dots, y^m)$$

then

$$D(\psi \circ \phi^{-1})(\phi(p)) = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} \Big|_{\phi(p)} & \frac{\partial y^1}{\partial x^2} \Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial y^1}{\partial x^m} \Big|_{\phi(p)} \\ \frac{\partial y^2}{\partial x^1} \Big|_{\phi(p)} & \frac{\partial y^2}{\partial x^2} \Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial y^2}{\partial x^m} \Big|_{\phi(p)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y^m}{\partial x^1} \Big|_{\phi(p)} & \frac{\partial y^m}{\partial x^2} \Big|_{\phi(p)} & \cdots & \cdots & \frac{\partial y^m}{\partial x^m} \Big|_{\phi(p)} \end{bmatrix}$$

It follows from Proposition 3.6d) that the matrix  $D(\psi \circ \phi^{-1})|_{\phi(p)}$  is invertible, as  $\psi \circ \phi^{-1}$  is a diffeomorphism. ■

**Proposition 6.2: Rank of a  $dF_p$  is invariant under coordinate change**

Let  $F$  be a smooth map between  $M$  and  $N$ , at every  $p \in M$ ,  $\text{rank } dF_p$  is an invariant over (smoothly compatible) pairs of charts in  $M$  and  $N$ .

*Proof.* Let  $p \in (U_1, \phi_1) \cap (U_2, \phi_2)$ , and  $F(p) \in (V_1, \psi_1) \cap (V_2, \psi_2)$ . Where all charts are smoothly compatible if it makes sense to talk about it. Both  $\phi_2 \circ \phi_1^{-1}$  and  $\psi_2 \circ \psi_1^{-1}$  are diffeomorphisms, and the change of basis matrices  $D(\phi_2 \circ \phi_1^{-1})|_{\phi_1(p)}$  and  $D(\psi_2 \circ \psi_1^{-1})|_{\psi_1(F(p))}$  are invertible by Proposition 3.6d) again, so the ranks  $dF_p$  with respect to any of the two charts are equal.

$$\underbrace{D(\psi_2 \circ \psi_1^{-1})|_{\psi_1(F(p))}}_{\text{invertible}} \left( \mathcal{M}\{dF_p\} \right) \underbrace{D(\phi_2 \circ \phi_1^{-1})|_{\phi_1(p)}}_{\text{invertible}}$$

■

## Chapter 4: Submersions, Immersions and Embeddings

## Matrices

The following is of utmost importance. It states that that rank of a matrix, square or otherwise, is an 'open condition'.

### Example 1.1: Lee Example 1.28 (Matrices of Full Rank)

Let  $A \in \mathcal{M}(m \times n, \mathbb{R})$  be the set of  $m \times n$  matrices with real entries.  $A$  has rank  $m$  iff there exists some  $m \times m$  sub-matrix of  $A$ , denoted by  $S$  st  $S$  is invertible. We wish to show the set of rank- $m$  matrices is invertible. Indeed, let

$$F : \mathcal{M}(m \times n, \mathbb{R}) \rightarrow \mathbb{R}, \Delta_{m \times m}(A) = \sum_{\substack{S \text{ is a } m \times m \\ \text{sub-matrix of } A}} |\det\{S\}|$$

Since  $S \mapsto \det\{S\}$  is continuous in the entries of  $S$ , hence continuous in the entries of  $A$ ,  $\Delta_{m \times m}$  is continuous.

So the set  $\left\{A \in \mathcal{M}(m \times n, \mathbb{R}), \text{rank } A = m\right\} = F^{-1}(\mathbb{R} \setminus \{0\})$  is open.

## Estimates in vector calculus

Before proving the inverse function theorem, we will need several Lemmas

### Proposition 2.1: Rudin Theorem 9.7

If  $A$  and  $B$  are in  $L(\mathbf{X}, \mathbf{Y})$ , then

$$\|BA\| \leq \|B\|\|A\|$$

*Proof.* Let  $\|x\| = 1$ , and

$$\|B(Ax)\| \leq \|B\|\|Ax\| \leq \|B\|\|A\|\|x\|$$

this holds for every  $\|x\| = 1$ , hence

$$\|BA\| \leq \|B\|\|A\|$$

■

### Proposition 2.2: Rudin Theorem 9.19

Let  $f$  map a convex open set  $U \subseteq \mathbb{R}^n$  into  $\mathbb{R}^m$ , if  $f$  is differentiable (pointwise) in  $U$ , and there exists some  $M$  st its derivative is bounded (in the operator norm)

$$\|Df(x)\| \leq M \quad x \in U$$

then, for every pair of elements  $x_1, x_2$  in  $U$ ,

$$\|f(x_1) - f(x_2)\| \leq M\|x_1 - x_2\|$$

*Proof.* This proof 'passes the argument' to the scalar-valued version, in short: if  $x_1$  and  $x_2$  are in  $U$ . Define

$$c(t) = (1 - t)x_1 + tx_2$$

as the convex combination of  $x_1$  and  $x_2$ . The takeaway intuition here is that it suffices to check on the line joining the two points', to obtain an estimate for  $\|f(x_1) - f(x_2)\|$ . Indeed, define

$$g(t) = f(c(t)) \text{ is a curve } g : \mathbb{R} \rightarrow \mathbb{R}^m$$

Using Theorem 5.19, of which we will state below

**Proposition 2.3: Rudin Theorem 5.19**

Let  $g : [0, 1] \rightarrow \mathbb{R}^m$ , and  $g$  be differentiable on  $(0, 1)$ , then there exists some  $x \in (0, 1)$  with

$$\|g(1) - g(0)\| \leq \|g'(x)\|$$

*Proof.* Read from Rudin Theorem 5.19. ■

Since  $Dg(t) = Df(c(t)) \circ Dc(t)$  by the Chain Rule, and  $Dc(t) = b - a$  by inspection,

$$\|Dg(t)\| = \|Df(c(t)) \circ Dc(t)\| \leq \|Df\| \|Dc\| = \|Df\| \|b - a\|$$

This holds for every  $t \in [0, 1]$ . Applying Theorem 5.19 gives

$$\underbrace{\|g(1) - g(0)\|}_{\text{curve endpoints}} \leq \|Df\| \|b - a\|$$

Replacing  $\|g(1) - g(0)\| = \|f(x_1) - f(x_2)\|$  and  $\|Df\| \leq M$  we get

$$\|f(x_1) - f(x_2)\| \leq M \|x_1 - x_2\|$$
■

## Inverse Function Theorem (Rudin)

**Proposition 3.1: Rudin Theorem 9.24**

Suppose  $f \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ , and  $Df(a)$  is invertible for some  $a \in \mathbb{R}^n$ , and define  $b = f(a)$ . Then,

- (a) there exist open sets  $U$  and  $V$  in  $\mathbb{R}^n$  such that  $a \in U$ ,  $b \in V$ , and  $f$  is one-to-one on  $U$ , and  $f(U) = V$ .
- (b) if  $g$  is the inverse of  $f$  (which exists, by Part a), defined in  $V$  by  $g(f(x)) = x$  for every  $x \in U$  then  $g \in C^1(\mathbb{R}^n, \mathbb{R}^n)$

*Proof of Part A.* We define  $Df(a) = A \in \mathbb{R}^{n \times n}$ , so  $A$  is invertible, and  $\|A^{-1}\| \neq 0$ , where  $\|\cdot\|$  denotes the operator norm. Recall all norms on finite-dimensional vector spaces are equivalent, this will be useful later.

Choose  $\lambda > 0$  st

$$\lambda = \|A^{-1}\|^{-1} 2^{-1} \tag{10}$$

By continuity of  $Df(x)$  at the point  $a$ , let  $\lambda > 0$ , this induces a  $B(\delta, a)$  with  $x \in B(\delta, a)$  means

$$\underbrace{\|Df(x) - Df(a)\|}_{\text{operator norm}} < \lambda \tag{11}$$

as  $Df : \mathbb{R}^n \rightarrow L(\mathbb{R}^n, \mathbb{R}^n)$  takes a point in  $\mathbb{R}^n$  and returns a linear map., with  $L(\mathbb{R}^n, \mathbb{R}^n)$  endowed with the usual vector space structure. Fix  $y \in \mathbb{R}^n$ , and define

$$\phi(x) = x + \underbrace{A^{-1}(y - f(x))}_{\text{offset}}$$

this is now a function solely in  $x$ , and  $\phi(x) = x \iff f(x) = y$  is clear, but such a fixed point is not necessarily unique. We claim that it is unique in  $B(\delta, a)$ . We will use the contractive mapping principle.

Differentiating  $\phi(x)$  reads

$$D\phi(x) = \underbrace{I}_{I=A^{-1}A} - A^{-1}Df(x) = A^{-1}(A - Df(x))$$

Proposition 2.1 tells us the norm of a product is bounded above by the product of the norms. Using eqs. (10) and (11), if  $x \in U$  we have

$$\|D\phi(x)\| = \|A^{-1}(A - Df(x))\| \leq \|A^{-1}\| \|A - Df(x)\| \leq 2^{-1}$$

The total derivative of  $\phi$  is uniformly bounded in  $U$ , applying Proposition 2.2 tells us that  $\phi$  is a contractive mapping

$$\|D\phi(x)\| \leq 2^{-1} \implies \|\phi(x_1) - \phi(x_2)\| \leq 2^{-1}\|x_1 - x_2\|$$

for  $x_1, x_2$  in  $U$ .

To show  $f|U$  is indeed a bijection, fix  $y \in f(U)$  so  $y = f(x)$  for some  $x \in U$ , and there can only be one fixed point stemming from  $\phi|U$ , with  $\phi(z) = z + A^{-1}(y - f(z))$  being the 'fixed point detector'. Write  $(f|U)^{-1}(y) = \lim\{(\phi|U)(x_n)\}_n$  and every point in  $f(U)$  has a unique inverse.

For the last part of the proof, we wish to show  $V = f(U)$  is open. Let  $y_0 \in V$  and we can 'hone into' the inverse of  $y_0$  using the same construction as earlier. So  $f(x_0) = y_0$  for some unique  $x_0 \in U$ .

If  $x_0$  is in  $U$ , it induces an open ball (see fig. 1) st

$$x_0 \in B(r, x_0) \subseteq \overline{B(r, x_0)} \subseteq U, \quad r > 0$$

We claim the open ball  $B(\lambda r, y_0) \subseteq V$ . Indeed, suppose  $y \in \mathbb{R}^n$  with

$$d(y, y_0) < \lambda r$$

If  $\phi$  is the 'fixed-point detector' with respect to  $y$  (the point we are trying to prove that is in  $f(U)$ ), in fact: we will prove  $y \in f(\overline{B(r, x_0)}) \subseteq f(U)$ .

$$\underbrace{\phi(x_0) - x_0}_{\text{removing the offset from } \phi(x_0)} = A^{-1}(y - f(x_0)) = A^{-1}(y - y_0)$$

using the operator norm on  $A^{-1}(y - y_0)$  reads

$$\|\phi(x_0) - x_0\| = \|A^{-1}(y - y_0)\| \leq \|A^{-1}\| \|y - y_0\| \leq \|A^{-1}\| \lambda r = r 2^{-1}$$



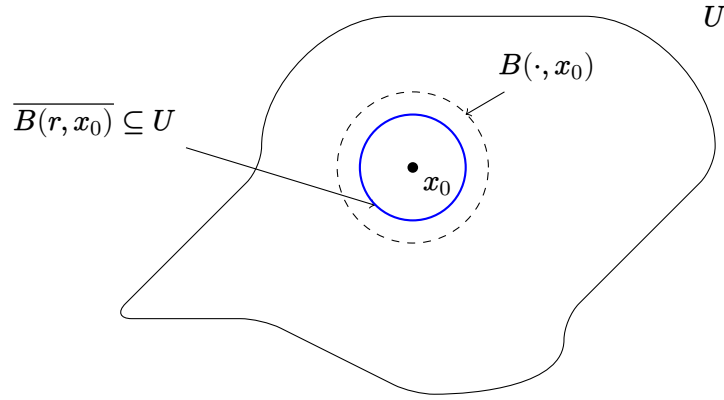


Figure 1: Every point  $x_0$  in an open set  $U$  admits an open ball that hides in  $U$

We will drag  $y$  into the image of the closed ball as follows: suppose  $x$  is another point that lies in the closed ball,  $\phi$  is contractive on  $\overline{B} \subseteq U$  regardless of the point  $y$  that induces  $\phi$ . But  $\overline{B}$  is closed, hence it is complete. So the Cauchy sequence (from the contractive mapping theorem) produces exactly one point in  $\overline{B}$ . It remains to show that if we start our sequence at some point  $x \in \overline{B}$ , then  $\phi(x) \in \overline{B}$  as well, and a simple induction will produce our contractive sequence.

To this, fix  $x \in \overline{B}$ , and

$$\begin{aligned} |\phi(x) - x_0| &\leq |\phi(x) - \phi(x_0)| + |\phi(x_0) - x_0| \\ &\leq \underbrace{2^{-1}|x - x_0|}_{\text{contraction on } \overline{B} \subseteq U} + \underbrace{r2^{-1}}_{\text{earlier}} \\ &= r \end{aligned}$$

therefore  $\phi$  contracts to a fixed point  $x^* \in \overline{B}$ , and  $f(x^*) = y$ . So  $y \in f(\overline{B}) \subseteq f(U)$  as desired.  $\blacksquare$

*Proof of Part B.* The proof is quite long, and we will only focus on the important bits. Rudin uses the technique of approximating smooth functions using first-order terms. He writes

$$\begin{cases} f(x) &= y \\ f(x+h) &= y+k \end{cases} \implies k = f(x+h) - f(x)$$

Furthermore, if  $x \in U$ , then the derivative  $Df(x)$  is invertible, this is from Theorem 9.8, obtains an estimate on the open ball in  $GL(n, \mathbb{R})$ . Roughly speaking, this open ball 'drags' other matrices into  $GL(n, \mathbb{R})$ . If  $A$  is invertible, and  $B$  is a conformable matrix with  $A$ , then

$$\underbrace{\|B - A\|}_{\text{distance between } A, B} \|A^{-1}\| < 1 \implies B \in GL(n, \mathbb{R})$$

If  $x \in B(\delta, a)$ , then Equation (11) reads

$$\|Df(x) - A\| < \lambda \implies \|Df(x) - A\| \|A^{-1}\| < 2^{-1} < 1$$

so  $Df(x)$  is invertible with inverse  $T$ .

And we estimate the deviation  $|k|^{-1} \leq \lambda|h|^{-1}$  by using the contraction inequality with  $y$  as the basepoint for  $\phi$ . Skipping a few lines ahead (to the confusing part), we see that

$$|h| \leq |h - A^{-1}k| + |A^{-1}k| \leq 2^{-1}|h| + |A^{-1}k|$$

subtracting over, and multiplying across gives an upper bound on  $|k|^{-1}$

$$2^{-1}|h| \leq |A^{-1}k| \implies 2^{-1}|h| \leq \|A^{-1}\||k| \implies |k|^{-1} \leq \underbrace{\frac{2}{\|A^{-1}\|}}_{\lambda} |h|^{-1}$$

Notice  $2\lambda\|A^{-1}\| = 1$ , so  $2/\|A^{-1}\| = \lambda$ . Finally, we 'factor out'  $-T$  on the line just before the difference quotient.

$$\begin{aligned} \overbrace{g(y+k) - g(y) - Tk}^{\text{numerator in difference quotient}} &= h - Tk \\ &= -T \left( \underbrace{f(x+h) - f(x)}_{=k} - \underbrace{Df(x)h}_{=T^{-1}h} \right) \end{aligned}$$

We see that  $T = Dg(y)$ , indeed:

$$\begin{aligned} \frac{|g(y+k) - g(y) - Tk|}{|k|} &\leq \frac{\|T\|}{\lambda} \frac{|f(x+h) - f(x) - Df(x)h|}{|h|} \\ &\lesssim \frac{|f(x+h) - f(x) - Df(x)h|}{|h|} \\ &= \underbrace{o(h)}_{|h| \lesssim |k|} = o(k) \rightarrow 0 \end{aligned}$$

Finally,  $Df|U : U \rightarrow GL(n, \mathbb{R})$  is a continuous mapping. By Theorem 9.8,  $(Df|U)^{-1} : U \rightarrow GL(n, \mathbb{R})$  is continuous as well. Therefore  $g \in C^1(U, U)$ , and  $f|U$  is a  $C^1$ -diffeomorphism. ■

### Remark 3.1

The inverse function theorem is extremely powerful. If a  $f$  is a  $C^1$  map from and into  $\mathbb{R}^n$ , and the total differential of  $f$  is full rank (hence invertible, as it is square) at some point  $a \in \mathbb{R}^n$ , the theorem states three things:

- For points  $x$  within a small enough neighbourhood  $a$ , the total differential  $Df(x)$  is invertible,
- On this same neighbourhood (denoted by  $U$ ),  $f(U)$  is a bijection,
- the inverse of  $f$  is a  $C^1$  map. This makes  $f|U$  a  $C^1$ -diffeomorphism

## Inverse Function Theorem on Manifolds

Let  $F$  be a smooth map between two smooth manifolds  $M$  and  $N$ , with dimensions  $m$  and  $n$  respectively.

**Definition 4.1: Rank of a map**

The rank of  $F$  at  $p \in M$  is the rank of the linear map:

$$dF_p : T_p M \rightarrow T_{F(p)} N$$

**Definition 4.2: Constant rank maps**

A smooth map  $F \in C^\infty(M, N)$  has constant rank if its differential  $dF_p : T_p M \rightarrow T_{F(p)} N$  has the same rank at every point  $p \in M$ .

There are three types of constant rank maps that are of interest.

**Definition 4.3: Smooth submersion**

$F$  is a smooth submersion if  $dF_p$  is a surjection onto  $T_{F(p)} N$  at  $p$ -everywhere. That is,  $\text{rank } dF_p = \dim T_{F(p)} N = \dim N$

**Definition 4.4: Smooth immersion**

$F$  is a smooth immersion if  $dF_p$  is an injection onto  $T_{F(p)} N$  at  $p$ -everywhere. That is,  $\text{rank } dF_p = \dim T_p M = \dim M$

**Definition 4.5: Smooth embedding**

$F$  is a smooth embedding if it is a smooth immersion, and it is a topological homeomorphism onto its range  $F(M) \subseteq N$ .

**Definition 4.6: Topological embedding**

$F \in C(X, Y)$ , where  $X$  and  $Y$  are topological spaces is a *topological embedding* if  $F$  is a homeomorphism onto its range.

Inclusion maps are topological embeddings, as the restriction onto the range of the map is the identity.

If  $X$  is compact and  $F$  is injective, then  $F$  is a topological embedding. If  $F$  is injective and an open map, then  $F$  is a topological embedding as well.

**Definition 4.7: Topological submersion**

**Definition 4.8: Local diffeomorphism**

$F$  is a local diffeomorphism if every  $p \in M$  in its domain induces a neighbourhood  $U \subseteq M$ ,  $F(U)$  is open in  $N$  with  $F|U : U \rightarrow F(U)$  is a diffeomorphism (in the sense of two open sub-manifolds).

**Proposition 4.1: Rank as an open condition**

Suppose  $F : M \rightarrow N$  is a smooth map, and  $p \in M$ . If  $dF_p$  is a surjection (resp. injection), pointwise at  $p$ , there exists a neighbourhood  $U$  of  $p$  where  $F|U$  is a smooth submersion (resp. immersion)

*Proof.* Trivial. See Example 1.1. ■

Proposition 4.1 roughly states that, if the differential of  $F$  at some point  $p$  is injective or surjective, then there exists a neighbourhood  $U$  about  $p$  such that  $dF|U(p)$  is an injection or surjection. The continuity of the map  $dF|U(p) \mapsto \Delta_{m \times m}(dF|U(p))$ , induces a neighbourhood in the vector space of matrices about the differential  $dF|U(p)$ . This vector space is endowed with any of the equivalent norms on  $\mathcal{M}(m \times n, \mathbb{R})$ , which is equivalent to the entrywise 2-norm. Since all partials of the form  $\left. \frac{\partial \hat{F}^k}{\partial x^j} \right|_p$  are continuous, we take the intersection over all  $n \times m$  partials such that  $dF|U(p)$  is an injection or surjection. Finally, send this neighbourhood about  $\hat{p}$  through to  $p$  by using the continuity of  $\phi$ .

**Proposition 4.2: Inverse Function Theorem on Manifolds**

Let  $M$  and  $N$  be smooth manifolds, and  $F : M \rightarrow N$  be a smooth map. Suppose the differential of  $F$  is invertible at some point  $p \in M$ , then there exists connected neighbourhoods  $U_0$  of  $p$ , and  $V_0$  of  $F(p)$  such that  $F|U_0 : U_0 \rightarrow V_0$  is a diffeomorphism.

*Proof.* Trivial. See the regular inverse function theorem Proposition 3.1 on Euclidean space, and pass the argument back to the manifolds using coordinate charts. ■

**Proposition 4.3: Rank Theorem for Manifolds**

Let  $F : M \rightarrow N$  be a smooth map with constant rank  $r$ , then at every  $p \in M$ , there exists smooth charts  $p \in (U, \phi)$  and  $F(U) \subseteq (V, \psi)$ , where the coordinate representation of  $F$  takes the form

$$\hat{F}(x) = \begin{bmatrix} \text{id}_{r \times r} & 0_{r \times m-r} \\ 0_{n-r \times r} & 0_{n-r \times m-r} \end{bmatrix} x, \quad \text{or equivalently} \quad (12)$$

$$\hat{F}(x^1, \dots, x^r, x^{r+1}, \dots, x^m) = (x^1, \dots, x^r) \quad (13)$$

*Proof.* Tedious. However, some techniques are worth remembering:

- Passing the argument to the Euclidean case as usual,
- We are free to shrink the sizes of open cubes and balls, and exploit local compactness,
- Suppose we are given a matrix of size  $m \times n$ , which has rank  $r$ , then we can attach a sub-matrix to make it square and invertible, then rehearse the usual arguments with the Inverse Function Theorems Propositions 3.1 and 4.2 to obtain a neighbourhood small enough that preserves the rank of the square matrix. Then pass the argument back to the smaller sub-matrix.

The last bullet point is worth elaborating, suppose we are given a rectangular matrix, where  $A$  is square and invertible. Take  $z = (x, y)^T$  with dimensions that make the formulas below make sense.

$$Mz = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \Rightarrow \begin{bmatrix} M \\ I \end{bmatrix} (z, y)^T = \begin{bmatrix} A & B \\ 0 & I \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix} \text{ is square and invertible}$$

by Proposition 4.1, we see that there exists a neighbourhood about the square matrix  $\begin{bmatrix} M \\ I \end{bmatrix}$  such that it remains invertible, hence a neighbourhood about  $M$  that makes  $A$  invertible (as a sub-matrix), so the rank of  $M$  is preserved. ■

### Corollary 4.1: Rank Theorem for Manifolds - Special Cases

Let  $F : M \rightarrow N$  be a smooth map with constant rank. If  $F$  is a smooth immersion, then Equation (12) takes the form:

$$\hat{F}(x) = \begin{bmatrix} \text{id}_{m \times m} \\ 0_{n-m \times m} \end{bmatrix} x, \quad \text{or equivalently} \quad (14)$$

$$\hat{F}(x^1, \dots, x^m) = (x^1, \dots, x^m, 0, \dots, 0) \quad (15)$$

If  $F$  is a smooth submersion,

$$\hat{F}(x) = \begin{bmatrix} \text{id}_{n \times n} & 0_{n \times m-n} \end{bmatrix} x, \quad \text{or equivalently} \quad (16)$$

$$\hat{F}(x^1, \dots, x^n, x^{n+1}, \dots, x^m) = (x^1, \dots, x^n) \quad (17)$$

## More on immersions and embeddings

### Proposition 5.1: Characterization of smooth immersions

$F$  is a smooth immersion iff every point  $p \in M$  has a neighbourhood  $U \subseteq M$  where  $F|U : U \rightarrow N$  is a smooth embedding.

*Proof.* We will prove it for when  $M$  and  $N$  are smooth manifolds, see Lee for the full proof with boundary. It involves extending the argument by composing  $F$  with an inclusion map. From Lemma 3.11 (Lee), if  $a \in \partial \mathbb{H}^n$ , then the differential of the inclusion map  $\iota : \mathbb{H}^n \rightarrow \mathbb{R}^n$  is a linear isomorphism between tangent spaces.

$$d\iota_a : T_a \mathbb{H}^n \rightarrow T_a \mathbb{R}^n, \quad \underbrace{T_a \mathbb{H}^n \cong T_a \mathbb{R}^n}_{\text{isomorphic}}$$

If for every  $p \in M$ , there exists a neighbourhood  $U$  of  $p$  with  $F|U : U \rightarrow N$  a smooth embedding, then  $dF|U_p$  has rank  $m$ , so  $dF_p$  has rank  $m$ , and the differential is injective pointwise everywhere. Conversely, if  $dF_p$  is a smooth immersion, the Rank Theorem (Proposition 4.3) tells us there exists connected neighbourhoods of  $p$  and  $F(p)$ , where  $F$  has coordinate representation in Equation (14) with respect to an appropriate choice of coordinate charts centered at  $p$ , so  $\hat{F}(\hat{p}) = 0 \in \mathbb{R}^n$ . Let  $\hat{p} \in \hat{U}$  and  $\hat{F}(\hat{p}) \in \hat{V}$ , the proof then devolves into a linear-map problem.  $\hat{F}$  given by the expression in Equation (14) is clearly

injective. Therefore it is bijective onto its range, its inverse is nothing but the map that removes the extra zeroes at the end. Therefore  $F|U$  is a smooth embedding. ■

**Definition 5.1: Section of  $\pi : M \rightarrow N$**

If  $\pi : M \rightarrow N$  is a continuous map, a *section of  $\pi$*  is a continuous right inverse for  $\pi$ , i.e  $\sigma : N \rightarrow M$ ,  $\sigma \in C(N, M)$ ,  $\pi \circ \sigma = \text{id}_N$ .

A *local section for  $\pi$*  is a continuous function  $\sigma$  from an open set  $U \subseteq V$  into  $M$  with  $\pi \circ \sigma = \text{id}_U$ .

**Proposition 5.2: Characterization of smooth submersion**

Let  $\pi : M \rightarrow N$  be smooth, then  $\pi$  is a smooth submersion iff every point of  $M$  is in the image of a smooth local section of  $\pi$ .

*Proof.* Suppose  $\pi$  is a smooth submersion, and fix  $p \in M$ , by the Rank Theorem Proposition 4.3, and Equation (16),  $\pi$  has the coordinate representation

$$\hat{\pi}(x^1, \dots, x^n, x^{n+1}, x^m) = (x^1, \dots, x^n)$$

between two open sets  $U \subseteq M$  and  $V \subseteq N$ , (it really does not matter). Now, define

$$\sigma : V \rightarrow M, (x^1, \dots, x^n) \mapsto \underbrace{(x^1, \dots, x^n, 0, \dots, 0)}_{\mathbb{R}^m} \in U$$

the charts by assumption are centered, and  $\pi \circ \sigma$  is clearly smooth (check coordinatewise), so  $\sigma$  reaches  $p$ . Conversely, recall if the composition of maps  $(g \circ f)$  is a surjection, then  $g$  is a surjection. Now, fix  $p \in M$ , this induces an open set  $V$  containing  $\pi(p)$ , and a smooth local section  $\sigma_V : V \rightarrow M$ . By Proposition 4.1, *the differential of a composition is equal to the composition of the differentials*

$$\text{id}_{T_q N} = d(\text{id}_U) = d(\pi)_{\sigma(q)} \circ d(\sigma)_q$$

so  $d(\pi)_{\sigma(q)} = d(\pi)_p$  is a surjection and the proof is complete. ■

**Regular values and level sets**

If  $F : X \rightarrow Y$ , and  $c \in Y$ , we call the  $F^{-1}(\{y\})$  a *level set at  $c$* , and  $c$  the *level value*. We often write  $F^{-1}(y)$  in place of  $F^{-1}(\{y\})$ . If  $Y = \mathbb{R}^k$ , then  $F^{-1}(0)$  is the *zero set of  $F$* .

**Definition 6.1: Critical point of  $F \in C^\infty(N, M)$**

$p \in M$  is a *critical point* of  $F$  if the differential  $dF_p : T_p N \rightarrow T_{F(p)} M$  fails to be surjective at  $p$ , otherwise  $p$  is a *regular point* of  $F$ .

**Definition 6.2: Critical value of  $f \in C^\infty(N, \mathbb{R})$**

Let  $f$  be a test function on  $N$ ,  $c \in \mathbb{R}$  is a critical value if there exists a  $p \in f^{-1}(c)$  where  $df_p$  is not surjective.

Otherwise  $c$  is called a regular value. Caveat: if  $c$  is not in the image of  $f$ , we call  $c$  a regular value as

well. It is clear  $c$  is a regular value iff  $c \notin f(M)$  or for every  $p \in f^{-1}(c)$ ,  $df_p$  is a surjection.

Notice  $df_p : T_p N \rightarrow T_{f(p)} \mathbb{R}$  is not surjective  $\iff$  all partials of the coordinate representation of  $f$  vanish. Since the matrix representation of  $df_p$  has the form

$$\mathcal{M}\{df_p\} = \begin{bmatrix} \frac{\partial f}{\partial x^1} \Big|_p & \cdots & \frac{\partial f}{\partial x^m} \Big|_p \end{bmatrix}$$

$$\text{rank } \mathcal{M}\{df_p\} \neq 1 \iff \frac{\partial f}{\partial x^j} \Big|_p = 0 \text{ for } 1 \leq j \leq m.$$

### Definition 6.3: Regular level set of $f \in C^\infty(N, \mathbb{R})$

If  $c$  is a regular value of  $f$ , then  $f^{-1}(c)$  is called a regular level set.

Define  $g = f - c$ , the partials at  $p \in N$  for both  $f$  and  $g$  agree, so the matrix representations of  $df_p$  and  $dg_p$  are identical (whereas their ranges might not be),

$$\begin{aligned} \mathcal{M}\{dg_p\} &= \begin{bmatrix} \frac{\partial g}{\partial x^1} \Big|_p & \cdots & \frac{\partial g}{\partial x^m} \Big|_p \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f-c}{\partial x^1} \Big|_p & \cdots & \frac{\partial f-c}{\partial x^m} \Big|_p \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial f}{\partial x^1} \Big|_p & \cdots & \frac{\partial f}{\partial x^m} \Big|_p \end{bmatrix} \\ &= \mathcal{M}\{df_p\} \end{aligned}$$

## Submanifolds

### Definition 7.1: Embedded submanifold of $M$

A subset  $S$  of  $M$  is called an *embedded submanifold* of  $M$ , when:

- equipped with the subspace topology from  $M$ ,
- it has a smooth structure that makes it a smooth manifold, and
- the inclusion map  $\iota : S \hookrightarrow M$  is a smooth embedding.

### Definition 7.2: Immersed submanifold of $M$

A subset  $S$  of  $M$  is called an *immersed submanifold* of  $M$ , when:

- equipped with an *arbitrary topology*,
- it has a smooth structure that makes it a smooth manifold (with or without boundary), and
- the inclusion map  $\iota : S \hookrightarrow M$  is a smooth immersion.

**Definition 7.3: Properly embedded submanifold**

is an embedded submanifold, and the inclusion map  $\iota : S \rightarrow M$  is a *proper* map.

**Definition 7.4: Weakly embedded submanifold**

is an immersed submanifold of  $M$ , whenever  $F : N \rightarrow M$  is a smooth map and the range of  $F$  lies in  $S$ , then  $F : N \rightarrow S$  as a map between two manifolds.

**Measure zero**

**Definition 8.1: Measure zero**

A subset  $E \subseteq \mathbb{R}^n$  has *measure zero* if  $\mu(E) = 0$

$$\mu(E) = \inf \left\{ \sum_{j \geq 1} m(U_j), E \subseteq \bigcup U_j, U_j \in \mathcal{M} \right\}$$

where  $\mathcal{M}$  denotes the Borel  $\sigma$ -algebra on  $\mathbb{R}^n$ . Each  $U_j$  can be taken to be open/closed balls or cubes, as they are measurable.

We sometimes refer to a measure zero set as *null*. If  $M$  is a smooth manifold of dimension  $m$ , then  $E \subseteq M$  has *measure zero* if for every chart  $(U, \varphi) \in \mathcal{A}_M$ ,

$$\varphi(U \cap E) \subseteq \mathbb{R}^m \quad \text{has measure zero in } \mathbb{R}^m$$

**Proposition 8.1: Subadditivity and Monotonicity of null sets**

The countable union of null sets is again null. Any measurable subset of a null set is again null. This holds for  $\mathbb{R}^n$  and the smooth manifold case.

*Proof.* See any textbook on the theory of integration. ■

**Proposition 8.2: Slice criterion for measure zero**

Suppose  $A \subseteq \mathbb{R}^n$  is compact (not necessary), and for every  $c \in \mathbb{R}$ , define the  $c$ -section of  $A$

$$A_c = \left\{ x \in \mathbb{R}^{n-1}, (c, x) \in A \right\}$$

If  $A_c$  has measure zero in  $\mathbb{R}^{n-1}$  for every  $c \in \mathbb{R}$ , then  $A$  has measure zero

*Proof.* By Tonelli's Theorem, since the Lebesgue measure on  $\mathbb{B}_{\mathbb{R}^n}$  is a Radon measure, it is finite on compact sets. Hence

$$\mu(A) = \int f d\mu_n = \int \chi_A d\mu_1 \times \cdots \times \mu_n$$

where  $f(c) = \int \chi_{A_c} d\mu_1 \times \cdots \times \mu_{n-1}$  is 0 for all  $c$ , hence  $\int f d\mu_n$  integrates to zero. ■



**Proposition 8.3: Equivalent condition for manifold measure zero**

Let  $E \subseteq M$ , if  $\{(U_\alpha, \varphi_\alpha)\}$  is a collection of charts that cover  $A$ , and  $\varphi_\alpha(E \cap U_\alpha)$  has measure zero for every  $\alpha$ , then  $E$  has measure zero.

**Proposition 8.4: Smooth maps send null sets to null sets**

If  $F \in C^\infty(M, N)$ , and  $A \subseteq M$  has measure zero, so does  $F(A)$ . In particular, this holds for  $F \in C^\infty(\mathbb{R}^m, \mathbb{R}^n)$  with the standard smooth structures.

**Lemmas for Sard's Theorem**

**Lemma 9.1: Lemmas on regular/critical points**

Let  $M$  and  $N$  be smooth manifolds, and  $F \in C^\infty(M, N)$ . Then,

1. The set of regular points of  $F$  are open in  $M$ .
2. The set of critical points of  $F$  are closed in  $M$ .
3. The set of critical values of  $F$  are precisely the direct image of its critical points. In symbols:

$$\text{Critical Values} = F(\text{critical points}) \quad (18)$$

**Lemma 9.2: Local characterization of critical points**

Let  $p \in M$ , and if  $p \in (U, \varphi)$ ,  $F(U) \subseteq (V, \psi)$ . Then the rank of  $dF_p$  is independent of the choice of coordinate charts used, and it is equal to the rank of  $d\hat{F}_{\hat{p}}$ . Where  $\hat{F}$  is the coordinate representation of  $F$  with respect to the charts  $(U, \varphi)$  and  $(V, \psi)$ .

$$\hat{F} : \varphi(U) \rightarrow \psi(V) \quad \hat{F} \triangleq \psi \circ F \circ \varphi^{-1}$$

Moreover, if  $p$  a critical point of  $F$ , it is also a critical point of  $F|_{U'} : U' \rightarrow V'$ . Where  $U'$  is an open subset of  $U$ , and  $V'$  is an open subset of  $V$  that contains  $F(U')$  entirely.

**Lemma 9.3: LCH Lemma for Sard's Theorem**

Let  $\mathbf{X}$  be a LCH space. Suppose  $x \in \mathbf{X}$ , and  $U$  is an open set containing  $x$ . There exists an open precompact neighbourhood about  $x$  that *hides in*  $U$ . If  $V$  denotes this neighbourhood, then

$$x \in V \subseteq \bar{V} \subseteq U$$

**Lemma 9.4: Covering Lemmas for Sard's Theorem**

The following holds for an arbitrary  $m$ .

1. If  $U \subseteq \mathbb{R}^m$  is an open set, it can be covered by
  - countably many open balls,
  - countably many open cubes,
  - countably many closed balls,
  - countably many closed cubes,
 all of the above are subsets of  $U$ .
2. If  $E \subseteq \mathbb{R}^m$  is a closed cube with side length  $R$ , then it can be covered by  $K^m$  many closed cubes with equal side lengths  $R' = RK^{-1}$ .

**Lemma 9.5: Diffeomorphism invariances for Sard's Theorem**

Let  $M, N, P, Q$  be smooth manifolds. Let  $F \in C^\infty(M, N)$ , and  $G \in \mathcal{D}(M, M)$ ,

1. If  $p \in M$ , define  $H = F \circ G$ , then
  - $\text{rank } dF_p = \text{rank } dH_p$ ,
  - $p$  is a critical point of  $F$  iff it is a critical point of  $H$ ,
  - $E \subseteq M$  has measure zero iff  $G(E)$  has measure zero,
  - topological properties are preserved (applies for homeomorphisms  $G$ ).  $E$  is compact, open, closed, connected or path-connected iff  $G(E)$  is.
2. Let  $H \in \mathcal{D}(Q, M)$ , if  $U$  is an open subset of  $Q$ , then  $H|_U : U \rightarrow H(U)$  is again a diffeomorphism, where  $U$  and  $H(U)$  are interpreted as open-submanifolds of  $Q$  and  $M$ .

**Sard's Theorem**

**Proposition 10.1: Sard's Theorem**

Let  $M$  and  $N$  be smooth manifolds with dimensions  $m$  and  $n$ , then the set of critical values of  $F$  has measure-zero in  $N$ .

The proof for Sard's Theorem is extremely long. We will use induction on the dimension of  $M$ . The base case will be divided into two sub-cases, one where  $n = 0$  and  $n \geq 1$ .

**Step 1: Base case:  $m = 0$**

Let  $G \in C^\infty(P, N)$  where  $P$  and  $N$  are smooth manifolds with dimensions  $p = 0$  and  $n \geq 1$ . The set of critical values of  $G$  have measure zero in  $N$ .

As we will soon see, the induction hypothesis allows us to use several dimension reduction techniques to prove the induction step. These techniques will depend on the rank theorem.

**Step 2: Induction hypothesis**

Let  $G \in C^\infty(P, N)$  where  $P$  and  $N$  are smooth manifolds with dimensions  $0 \leq p \leq m - 1$  and  $n \geq 1$ . The set of critical values of  $G$  have measure zero in  $N$ .

The induction step is extremely fussy. First, we dissect the domain and range of  $M$  and  $N$  into countably many charts  $\{(U_{jk}, \varphi_{jk}), (V_{jk}, \psi_{jk})\}_{j,k \geq 1}$  where

$$F(U_{jk}) \subseteq V_{jk} \quad (19)$$

that also cover  $M$  and  $N$ .

By Lemma 9.2, if  $p$  is a critical point of  $F$  and  $p \in U_{jk}$ , then  $p$  is a critical point of

$$F_{jk} : U_{jk} \rightarrow V_{jk} \quad (20)$$

**Step 3: Construction of charts**

We construct a countable sequence of  $\{(U_{jk}, \varphi_{jk}), (V_{jk}, \psi_{jk})\}_{j,k \geq 1}$  charts that satisfy Equations (19) and (20).

*Proof.* Let  $\{(U_\alpha, \varphi_\alpha)\}$  be the maximal smooth atlas on  $M$ . Each  $U_\alpha$  is open in  $M$ , and  $M$  is second-countable. It follows that there exists a countable open cover (indexed by  $j \geq 1$ ) that covers  $M$ , and a countable smooth atlas  $\{(U_j, \varphi_j)\}$ . Similarly for  $\{(V_k, \psi_k)\}$ .

Let us write  $U_{jk} \triangleq U_j \cap F^{-1}(V_k)$ . Note  $U_{jk}$  is open, since smoothness implies continuity (Proposition 2.4). And

$$(V_{jk}, \psi_{jk}) \triangleq (V_k, \psi_k) \quad F_{jk} = F|_{U_{jk}}$$

so  $F_{jk}(U_{jk}) \subseteq V_{jk}$ . Finally, define  $\varphi_{jk} = \varphi_j|_{U_{jk}}$ . ■

Using Proposition 8.4, smooth maps preserve measure-zero sets. If the critical values of some  $F_{jk}$  have measure zero in  $N_{jk}$ , then so does  $\iota(\text{critical points of } F_{jk})$ , where is the smooth  $\iota : N_{jk} \hookrightarrow N$  is the inclusion map (which is a smooth embedding by Definition 7.1). Thus,

**Step 4: Pass the measure-zero condition to each chart**

Prove the following

$F(\text{critical points of } F)$  has measure zero

$$\iff \text{For } j, k \geq 1, \quad F_{jk}(\text{critical points of } F_{jk}) \text{ null in } V_{jk} \subseteq N$$

$$\iff \text{For } j, k \geq 1, \quad \hat{F}_{jk}(\text{critical points of } \hat{F}_{jk}) \text{ null in } \psi_{jk}(V_{jk}) \subseteq \mathbb{R}^n$$

where  $\hat{F}_{jk}$  denotes the coordinate representation of  $F$  with respect to the charts  $(U_{jk}, \varphi_{jk})$  and  $(V_{jk}, \psi_{jk})$ .

$$F_{jk} : \varphi_{jk}(U_{jk}) \rightarrow \psi_{jk}(V_{jk}), \quad \hat{F}_{jk} \triangleq \psi_{jk} \circ F_{jk} \circ \varphi_{jk} \quad (21)$$

*Proof.* The first equivalence follows from Lemma 9.2, and the observation that the image of  $F$  commutes with unions. The second follows from Proposition 8.4 and lem. 9.5. ■

To simplify the problem even further, we assume  $U_{jk}$  and  $V_{jk}$  are open subsets of  $\mathbb{R}^m$  and  $\mathbb{R}^n$ , as the core of the 'measure-zero' argument is invariant under diffeomorphisms of the coordinate charts (see Lemma 9.5). Let  $j, k$  be fixed, and relabel

$$((U_{jk}, \varphi_{jk}), (V_{jk}, \psi_{jk})) \mapsto ((U, \varphi), (V, \psi)) \quad \text{and} \quad F_{jk} \mapsto F$$

similarly for  $\hat{F}_{jk} \mapsto \hat{F}$ .

**Step 5: Pass measure-zero condition to  $\mathbb{R}^m$  and  $\mathbb{R}^n$**

Let  $U$  and  $V$  be open sub-manifolds of  $M$  and  $N$ , and  $\varphi$  and  $\psi$  be global (smooth) charts.

$$F(\text{critical points of } F) = (\psi^{-1} \circ \hat{F} \circ \varphi)(\text{critical points of } F)$$

and

$$\text{critical points of } F = \varphi^{-1}(\text{critical points of } \hat{F})$$

*Proof.* The first equation follows from the definition of  $\hat{F}$ , after unboxing the domains and ranges. ■

We again relabel:

$$U \mapsto \varphi(U), \quad V \mapsto \psi(V), \quad F \mapsto \hat{F}$$

and  $M \mapsto \mathbb{R}^m$ ,  $N \mapsto \mathbb{R}^n$ , and critical points of  $F$ . Now  $F : U \rightarrow V$  is a smooth map between open subsets of  $\mathbb{R}^m$  and  $\mathbb{R}^n$ . Let us state the goal of the induction.

**Step 6: Simplified induction goal**

Let  $U \subseteq \mathbb{R}^m$  and  $V \subseteq \mathbb{R}^n$  be open subsets.  $F \in C^\infty(U, \mathbb{R}^n)$  is a smooth map. Where  $\mathbb{R}^m$  and  $\mathbb{R}^n$  are considered to be smooth manifolds with the standard smooth structure. Prove that the set of critical values of  $F$  have measure zero.

We define some standard notation for partials.

- $\mathbb{N}_0 \triangleq \{0, 1, \dots\}$ ,
- $\alpha$  is a  *$m$ -multi-index* or a *multi-index of order  $m$*  if  $\alpha = (\alpha_1, \dots, \alpha_m)$  is a  $m$ -tuple with entries in  $\mathbb{N}_0$ . The set of  $m$ -multi-indices are denoted by  $\mathbb{N}_0^m$ .
- The order of a multi-index  $\alpha$ , denoted by  $|\alpha|$  is its  $l^1$  norm. In symbols:

$$|\alpha| = \sum |\alpha_j|$$

- If  $g \in C^\infty(\mathbb{R}^m, \mathbb{R})$ , and  $\alpha \in \mathbb{N}_0^m$ , we define

$$\partial^\alpha g \triangleq \left( \frac{\partial}{\partial x^1} \right)^{\alpha_1} \cdot \dots \cdot \left( \frac{\partial}{\partial x^m} \right)^{\alpha_m}$$

$$\text{so that } \partial^\alpha \triangleq \prod_{j=1}^m \left( \frac{\partial}{\partial x^j} \right)^{\alpha_j} = \prod_{j=1}^m \partial_j^{\alpha_j}.$$

Let  $C \subseteq U$  denote the set of critical points of  $F$ , we decompose  $F(C)$  into the following

$$F(C) = F(C \setminus C_1) + \bigcup_{j=1}^{k-1} F(C_j \setminus C_{j+1}) + F(C_k) \quad (22)$$

and show each of the three sets in the union have measure zero. (the direct image commutes with unions). Where

$$C_k \triangleq \left\{ x \in C, \forall i \leq n, \alpha \in \mathbb{N}_0^m, |\alpha| \leq k, \partial^\alpha F^i(x) = 0 \right\}$$

and  $F^i$  are the *component functions* of  $F$ , or  $F(x) = (F^1(x), \dots, F^n(x))$ .

#### Step 7: Properties of $C, C_k$

$C, C_k$  is closed for every  $k \geq 1$ , and  $U \supseteq C \supseteq C_1 \supseteq C_2 \supseteq \dots$ .

*Proof.*  $C$  is closed by Lemma 9.1. The function

$$h(x) \triangleq \sum_{\substack{\alpha \in \mathbb{N}_0^m \\ |\alpha| \leq k \\ i=1, \dots, n}} |\partial^\alpha F^i(x)|$$

is a continuous map from  $U \rightarrow \mathbb{R}$ . and  $h^{-1}(0) \cap C$  is closed by continuity.  $C \supseteq C_1$  follows by definition of  $C_1$ , and  $C_k \supseteq C_{k+1}$  follows from comparing  $h$  for the two values of  $k$ . ■

#### Step 8: Case I: $C \setminus C_1$ has measure zero.

Let  $a \in C \setminus C_1$ , noting that  $U \setminus C_1$  is an open set containing  $a$ . Since  $a \notin C_1$ , there exists a first order non-vanishing partial for a component function of  $F$ . Suppose

$$\frac{\partial}{\partial x^j} F^i(a) \neq 0 \quad \text{with respect to the standard charts}$$

We wish to swap the columns and rows of the coordinate charts so that  $\frac{\partial}{\partial x^i} F^{-1}(a) \neq 0$ . More concretely, the transposition of the  $i$ th and the 1st row is a permutation; so is the transposition of the  $j$ th and the 1st column. By invariance, we are free to relabel

$$F \mapsto P_{i,1} \circ F \circ P_{j,1}$$

Next, we use a convenient change of coordinates, to arrive at the form described below.

#### Step 9: Change of Coordinates

Perform a change of coordinates, such that if  $(u, v^2, \dots, v^m) = (u, v)$ , where  $v \in \mathbb{R}^{n-1}$ , then

$$F(u, v) = (u, F^2(u, v), \dots, F^n(u, v)) \quad (23)$$

and the differential is given by

$$dF_{(u,v)} = \begin{bmatrix} 1 & 0 \\ \frac{\partial F^i}{\partial u} \Big|_{(u,v)} & \frac{\partial F^i}{\partial v^j} \Big|_{(u,v)} \end{bmatrix} \quad \begin{matrix} i=2, \dots, n \\ j=2, \dots, m \end{matrix} \quad (24)$$

where  $dF_{(u,v)}$  is not surjective for each  $(u,v) \in C \setminus C_1 \subseteq C$ . This change of coordinates is valid on some open, precompact set containing  $a$ , denoted by  $V_a$ . More precisely

$$a \in V_a \subseteq \bar{V}_a \subseteq B \subseteq U \setminus C_1 \quad (25)$$

*Proof.* If  $(x^1, \dots, x^m)$  are the standard coordinates of  $\mathbb{R}^m$ , the map

$$T : (x^1, \dots, x^m) \mapsto (F^1(a), x^2, \dots, x^m) \quad \text{is a diffeomorphism}$$

on some open, precompact neighbourhood  $V_a$  of  $a$ .

$$a \in V_a \subseteq \bar{V}_a \subseteq U$$

The Jacobian of  $T$  is given by

$$\text{Jacobian}(T)(x) = \begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \cdots \\ * & Id_{m-1} \end{bmatrix}$$

and it is invertible at  $a$ . The continuity of  $\frac{\partial F^1}{\partial x^1}$  at  $a$  gives us an open ball  $B \subseteq U \setminus C_1$  containing  $a$ , on which  $\frac{\partial F^1}{\partial x^1} \neq 0$ . Let  $V_a \subseteq \bar{V}_a \subseteq B$  be an open, precompact neighbourhood that hides in this ball as in Lemma 9.3.

Next, we absorb the change of coordinates into  $F$ , so

$$F \mapsto (F \circ T^{-1})|_B$$

Verification of the Equation (23) is straightforward, as

$$T^{-1}(u, v) = ((F^1)^{-1}(u, v), v^2, \dots, v^m)$$

$$\begin{aligned} (F \circ T^{-1})(u, v) &= \left( F^1[(F^1)^{-1}(u, v)], \dots, \dots \right) \\ &= (u, F^2(u, v), \dots, F^n(u, v)) \end{aligned}$$

Computing the Jacobian matrix gives Equation (24), and  $V_a$  clearly satisfies Equation (25). ■

Let  $a$  range through all points in  $C \setminus C_1$ , then  $\{C \setminus C_1 \cap V_a\}_{a \in C \setminus C_1}$  is an open cover  $\text{Rel } C \setminus C_1$ , so there exists a countable subcover. We will prove something slightly stronger, we prove that  $F(C \setminus C_1 \cap \bar{V}_a)$  has measure zero for each  $a$  in this countable collection. Proposition 8.1 will then prove  $F(C \setminus C_1)$  is null.

Notice  $C \setminus C_1 \cap \bar{V}_a$  is a closed subset of  $\bar{V}_a$ , hence compact. Continuous functions send compact sets to compact sets, so  $F(C \setminus C_1 \cap \bar{V}_a)$  is compact in  $\mathbb{R}^n$ . We wish to use the induction hypothesis and 'lower the dimension' of  $F$ , or at the very least — pass the argument to a lower dimension.

**Step 10:  $F(C \setminus C_1 \cap \bar{V}_a)$  has measure zero.**

In this step, we show every point in  $C \setminus C_1$ , admits a neighbourhood which is null. So

$$F(C \setminus C_1 \cap V_a) \subseteq F(C \setminus C_1 \cap \bar{V}_a)$$

$V_a$  is open  $\text{Rel } C \setminus C_1$ , and  $a$  ranges through all the points in  $C \setminus C_1$ .  $C \setminus C_1$  is second countable in the subspace topology, so  $\{V_a\}$  admits a countable subcover, and we can apply subadditivity.

*Proof.* Let  $a$  an element in the countable subcollection described above. Let  $c \in \mathbb{R}^1$  be arbitrary, if

$$\{c\} \times \mathbb{R}^{n-1} \cap F(C \setminus C_1 \cap \bar{V}_a) = \emptyset$$

then the  $c$ -section  $E_c$  of  $F(C \setminus C_1 \cap \bar{V}_a)$ ,

$$E_c \triangleq \{q \in \mathbb{R}^{n-1}, (c, q) \in F(C \setminus C_1 \cap \bar{V}_a)\}$$

is empty, so  $E_c$  is null. Next, if  $E_c$  is not empty, we will pass the argument to some  $F_c : \underbrace{B_c}_{\subseteq \mathbb{R}^{n-1}} \rightarrow \mathbb{R}^{n-1}$

and show  $E_c$  is *exactly* (double inclusion) the set of critical points of  $F_c$ . Indeed, define

$$B_c = \left\{ v, (c, v) \in \bar{V}_a \right\} \quad \text{and} \quad F(c, v) = (c, F_c(v))$$

so  $F_c(v) = (F^2(c, v), \dots, F^n(c, v))$  for  $v \in B_c$ . The Jacobian (differential) of  $F_c$  resembles that of  $F$ . With  $c$  fixed,

$$(dF_c)_v = \left[ v^j_{(u,v)} \right]_{\substack{i=2,\dots,n \\ j=2,\dots,m}}$$

Compare the previous equation with Equation (24), what we have done essentially is *removing the first column and row* of  $dF_{(c,v)}$ . By elementary linear algebraic arguments about canonical forms,  $\text{rank } dF_{(c,v)} \neq n \iff \text{rank}(dF_c)_v \neq n-1$ . So  $(c, v) \in C \setminus C_1 \cap \bar{V}_a$  if and only if  $v$  is a critical point of  $F_c$ . If  $(c, x)$  is a critical value in  $C \cap \bar{V}_a$ , then  $DF$  cannot have full rank so

$$\left[ \frac{\partial F^i}{\partial v^j} \right] (x) \text{ as a function of } x \in \mathbb{R}^{n-1} \text{ cannot have full rank.}$$

Now we wish to use the  $c$ -section Lemma in Proposition 8.2 in conjunction with our induction hypothesis.  $C \setminus C_1 \cap \bar{V}_a$  is a closed subset of compact  $\bar{V}_a$ , hence  $F(C \setminus C_1 \cap \bar{V}_a)$  is again compact. The  $c$ -section

$$\left[ F(C \cap \bar{V}_a) \right]_c = \left\{ x \in \mathbb{R}^{n-1}, (c, x) \in F(C \cap \bar{V}_a) \right\}$$

so  $(c, x)$  is a critical value of  $F|_{\bar{V}_a}$ . Another key idea is the following:

- Define  $F_c(v) = (F^2(c, v), \dots, F^n(c, v))$ ,
- so  $(DF_c)|_v = \left[ \frac{\partial F^i}{\partial v^j} \right]$  from slicing the bottom right corner of the matrix in Equation (24).
- $\left[ F(C \cap \bar{V}_a) \right]_c = \left\{ x \in \mathbb{R}^{n-1}, (c, x) \in F(C \cap \bar{V}_a) \right\}$

■

This completes Case I.

## Whitney Embedding Theorem

## Whitney Approximation Theorems

## Tubular Neighbourhood Theorem

## Transversality

## Chapter 9: Integral Curves and Flows



## Time invariant flows

### Lie Derivatives of Vector Fields

#### Definition 2.1: Lie Derivatives of Vector Fields

Let  $M$  be a smooth manifold and  $V, W$  be smooth vector fields on  $M$ , and  $\theta$  be the flow of  $V$ . The *Lie derivative of  $W$  with respect to  $V$*  is a rough vector field  $\mathcal{L}_V W$

$$(\mathcal{L}_V W)_p \triangleq \left. \frac{d}{dt} \right|_{t=0} d(\theta_t)_{\theta_t(p)}(W_{\theta_t(p)}) \quad (26)$$

The above uses the Frechet derivative on the (normed) vector space  $T_p M$ , and the result is an element in  $T_p M$ . Equivalently,

$$(\mathcal{L}_V W)_p = \lim_{t \rightarrow 0} \frac{d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) - W_p}{t} \quad (27)$$

#### Remark 2.1

If we use the regular the 'velocity of curves' definition, we will see the result is an element in  $T_{W_p}(T_p M)$ , which is not what we want. However, Proposition 3.13 allows us to identify, through the canonical linear isomorphism  $T_p M \cong T_{W_p}(T_p M)$  for a fixed  $W_p \in T_p M$ .

#### Proposition 2.1: Existence of $\mathcal{L}_V W$

Let  $M$  be a smooth manifold and  $V, W \in \mathfrak{X}(M)$ ,  $(\mathcal{L}_V W)_p$  exists for every  $p \in M$ , and  $\mathcal{L}_V W$  defines a smooth vector field.

*Proof.* Let  $\theta$  denote the flow of  $V$ , and  $p \in M$  be arbitrary. Let  $(x^i)$  be the local coordinates in some neighbourhood  $U$  about  $p$ . Since  $(0, p)$  is a point in the flow domain of  $V$ , there exists an open interval containing 0  $J_0$ , and an open neighbourhood  $U_0$  containing  $p$ , such that  $U_0 \subseteq U$ .

Since every single point  $\theta_t(x)$  for  $(t, x) \in J_0 \times U_0$  share the same local coordinates as  $p$ , the matrix representation of the differential  $d(\theta_{-t})_{\theta_t(x)} : T_{\theta_t(x)} M \rightarrow T_x M$  is precisely the Jacobian of the coordinate representation of  $\theta$ .

$$d(\theta_{-t})_{\theta_t(x)} = \left( \frac{\partial \theta^i}{\partial x^j} \right) (-t, \theta(t, x))$$

is a matrix of smooth functions  $\frac{\partial \theta^i}{\partial x^j}$  evaluated at  $(-t, \theta(t, x)) \in J_0 \times U_0$ . Thinking of  $T_p M$  as a normed vector space, it has basis

$$\left( \left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p \right)$$

To show that  $(\mathcal{L}_V W)_p$  exists, let  $X : J_0 \rightarrow T_p M$  be a map (not a curve) starting at  $W_p$ , given by  $X(t) = d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})$ , computing its Frechet derivative with respect to the above basis,

$$\left. \frac{d}{dt} \right|_{t=0} X(t) = \left. \frac{d}{dt} \right|_{t=0} \underbrace{\frac{\partial \theta^i}{\partial x^j}}_{\text{smooth matrix}} \underbrace{W^j(\theta(t, p))}_{W \text{ vector}} \underbrace{\left. \frac{\partial}{\partial x^i} \right|_p}_{\text{basis}} \quad (28)$$

$X(t)$  clearly has smooth coefficients in  $J_0 \times U_0$ , so the Frechet derivative exists. Further, the coefficients in Equation (28) vary smoothly with  $t$  and  $p$  in  $J_0 \times U_0$ . So the rough vector field  $(\mathcal{L}_V W)$  is smooth around any  $p$  in  $M$ , and the proof is complete. ■

**Remark 2.2**

The above holds for manifold with boundary as well, we require the vector field  $V$  to be tangent to  $\partial M$ , for the flow to be defined on the boundary. Further, we can view this Lie derivative as the Frechet limit of the *pullback* through the diffeomorphism  $\theta_{-t} : M_{-t} \rightarrow M_t$ . This point of view will be useful later on, when we look at Lie derivatives of (covariant) tensor fields.

**Proposition 2.2: Computing Lie derivatives at  $t_0$**

Let  $M$  be a smooth manifold,  $V, W \in \mathfrak{X}(M)$ , and the flow of  $V$  by  $\theta$  with domain  $\mathcal{D}$ . If  $(t_0, p) \in \mathcal{D}$ ,

$$\left. \frac{d}{dt} \right|_{t=t_0} d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)}) = d(\theta_{-t_0})_{\theta_{t_0}(p)}((\mathcal{L}_V W)_{\theta_{t_0}(p)})$$

*Proof.* Let  $J_0 \times U_0$  is the product of an open interval containing 0 and an open neighbourhood of  $p$ , and  $X : J_0 \rightarrow T_p M$ , where

$$X(t) \triangleq d(\theta_{-t})_{\theta_t(p)}(W_{\theta_t(p)})$$

This is a smooth curve, and we can shrink  $J_0$  and  $U_0$  respectively so that  $\theta$  maps  $J_0 \times U_0$  into the same chart  $(U, (x^i))$ . Writing  $X$  in local coordinates, noting that  $X$  maps into the *vector space*  $T_p M$ . Equation (28) tells us  $X$  is indeed a smooth curve in  $T_p M$ . The composition of smooth maps is again smooth, define  $\tau : J_0 \rightarrow \tau(J_0)$ , with  $t \mapsto t_0 + t$

$$(X \circ \tau)(t) = X(t + t_0) \quad \text{is smooth on } \tau(J_0)$$

and

$$(X \circ \tau)(t) = d(\theta_{-t_0})_{\theta_{t_0}(p)} \circ d(\theta_{-t})_{\theta_{t_0}(p)}(W_{\theta_{t_0}(p)})$$

$$\begin{aligned} X'(t_0) &= \left. \frac{d}{dt} \right|_{t=t_0} X(t) \\ &= \left. \frac{d}{dt} \right|_{t=0} X(t + t_0) \\ &= \left. \frac{d}{dt} \right|_{t=0} d(\theta_{-t_0}) \circ d(\theta_{-t})_{\theta_{t_0}(p)}(W_{\theta_{t_0}(p)}) \\ &= d(\theta_{-t_0}) \left( \left. \frac{d}{dt} \right|_{t=0} W_{\theta_{t_0}(p)} \right) \\ &= d(\theta_{-t_0})(\mathcal{L}_V W)_{\theta_{t_0}(p)} \end{aligned}$$

One way to justify the second last equality is to view the whole computation using Frechet derivatives. ■

## Flowouts from Boundary

This will be a draft of Problem 9.11. We wish to modify the proof for Theorem 9.20 (Flowout Theorem). We need to modify the following aspects

- $M$  is now a smooth manifold with boundary,
- The original embedded submanifold of the flow  $\mathcal{D}$ , is  $\mathcal{O} = (\mathbb{R} \times S) \cap \mathcal{D}$ . We need to change it to

$$\mathcal{O} = ([0, +\infty) \times \partial M) \cap \mathcal{D}$$

where  $\mathcal{D}$  is the flow of the smooth vector field  $N$ , which is everywhere inward pointing on the boundary of  $M$ .

- The induced smooth embedding  $\Phi : P_\delta \rightarrow M$  (and thus diffeomorphism, since  $\partial M$  has codimension 1), is defined on

$$P_\delta = \left\{ (t, p), p \in \partial M, 0 \leq t < \delta(p) \right\}$$

where  $\delta : \partial M \rightarrow \mathbb{R}^+$  is a smooth, strictly positive function.

- For each  $p \in \partial M$ , the map  $t \mapsto \Phi(t, p)$  is an integral curve of  $N$  starting at  $p$ .

## Time varying flows

## Chapter 12: Tensors

## Tensor Products

### Definition 1.1: Covariant Tensor

Let  $V$  be a finite-dimensional vector space. Covariant tensors are multi-linear maps  $\alpha : V \times \cdots \times V \rightarrow \mathbb{R}$ . If  $\alpha$  sends  $k$ -copies of  $V$  into  $\mathbb{R}$ , it is called a  $k$ -tensor, or a *covariant tensor of rank  $k$* . It is clear that  $k$ -tensors form a vector space over  $\mathbb{R}$  for any  $k \geq 0$ . The convention is that 0-tensors are real numbers as they have 0 arguments in  $V$ .

$$T^k(V^*) = \left\{ \alpha : \underbrace{V \times \cdots \times V}_{k \text{ copies}} \rightarrow \mathbb{R}, \alpha \text{ is multi-linear.} \right\}$$

### Definition 1.2: Contravariant Tensor

Contravariant tensors are multi-linear maps  $\alpha : V^* \times \cdots \times V^* \rightarrow \mathbb{R}$ , similar to covariant tensors. Denoted by

$$T^k(V) = \left\{ \alpha : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \rightarrow \mathbb{R}, \alpha \text{ is multi-linear.} \right\}$$

### Definition 1.3: Mixed Tensors

A  $(k, l)$ -mixed tensor, or just  $(k, l)$ -tensor is a multi-linear map that takes  $k$  copies of  $V^*$ , and  $l$  copies of  $V$  into  $\mathbb{R}$ . In symbols,

$$T^{(k,l)}(V) = \left\{ \alpha : \underbrace{V^* \times \cdots \times V^*}_{k \text{ copies}} \times \underbrace{V \times \cdots \times V}_{l \text{ copies}} \rightarrow \mathbb{R}, \alpha \text{ is multi-linear.} \right\}$$

The space of  $k$ -covariant,  $k$ -contravariant, and  $(k, l)$ -tensors form a vector space over  $\mathbb{R}$ . We will rarely use contra-variant tensors, so we will use the word *tensor* to refer to covariant tensors only.

### Definition 1.4: Tensor Product

Let  $F$  and  $G$  be  $k$  and  $l$ -tensors on  $V$  respectively, we define a bi-linear map,

$$F \otimes G : \underbrace{V \times \cdots \times V}_{k+l \text{ copies}} \rightarrow \mathbb{R}$$

by

$$(F \otimes G)(\underline{v}_k, \underline{w}_l) = \underbrace{F(\underline{v}_k)G(\underline{w}_l)}_{\substack{\text{scalar} \\ \text{multiplication} \\ \text{in } \mathbb{R}}}$$

Tensor product is clearly associative, and commutative (although we will not use this fact).

**Proposition 1.1: Basis of  $T^k(V^*)$** 

Let  $V$  be a  $n$ -dimensional vector space with basis  $(E_i)$ , and dual basis  $(\varepsilon^i)$ .  $T^k(V^*)$  has basis

$$\mathbb{B} = \left\{ \bigotimes \varepsilon^{i_k}, i_k \in \{n\} \right\} = \left\{ \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}, 1 \leq i_j \leq n, 1 \leq j \leq k \right\}$$

This means, if  $A$  is a  $k$ -tensor, and for every multi-index  $I = (i_k)$ , define

$$A_I = A_{i_1, \dots, i_k} = A_{i_k} = A(E_{i_k}) = A(E_{i_1}, \dots, E_{i_k}) = A(E_I)$$

are precisely the *basis coefficients* of  $F$  with respect the basis  $\mathbb{B}$ ,

$$F = F_{i_1, \dots, i_k} \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k} \quad (29)$$

**Tensor Fields on Manifolds**

Similar to the tangent bundle  $TM = \coprod_{p \in M} T_p M = \bigcup_{p \in M} \{p\} \times T_p M$ , is endowed with a unique smooth structure that makes the canonical projection  $\pi : TM \rightarrow M$ ,  $\pi(p, v) = p$  a smooth embedding. We can use the algebraic machinery we have built up from Chapter 10 to construct abstract tensor bundles on  $M$ . For completeness, we will state the definitions for covariant and mixed bundles as well.

**Definition 2.1: Tensor Bundles**

Let  $M$  be a smooth manifold with or without boundary, define

- Covariant  $k$ -bundle over  $M$  by

$$T^k T^* M = \coprod_{p \in M} T^k(T_p^* M)$$

- Contravariant  $k$ -bundle over  $M$  by

$$T^k TM = \coprod_{p \in M} T^k(T_p M)$$

- $(k, l)$ -mixed bundle over  $M$  by

$$T^{(k, l)} TM = \coprod_{p \in M} T^{(k, l)}(T_p M)$$

**Definition 2.2: Tensor Field on  $M$** 

A  $k$ -covariant tensor field, or just a  $k$ -tensor field over a smooth manifold (with or without boundary)  $M$ , is a smooth section of the vector bundle as defined in Definition 2.1. As with vector fields  $\mathfrak{X}(M)$ , they form a  $R$ -module over  $C^\infty(M)$  by pointwise multiplication.

We denote the space of  $k$ -tensor fields over  $M$  by  $\mathcal{T}^k(M)$ . Moreover, if  $A \in \mathcal{T}^k(M)$ , we can write  $A$  in

local coordinates  $(x^i)$  by

$$A = A_{i_1, \dots, i_k} dx^{i_1} \otimes \dots \otimes dx^{i_k}$$

with summation convention in effect, since  $(dx^1, \dots, dx^n)$ , when evaluated pointwise in local coordinates, form a dual basis of  $T_p^*M$ . Where each  $dx^i$  is the Chapter 11 differential of the  $i$ -th coordinate function.

## Technicalities for Tensor Fields

We will skip the technicalities for now. Recall  $k$ -tensors on a vector space act on a  $k$ -tuple of vectors, the same is true for  $k$ -tensor fields over a manifold  $M$ . If  $A \in \mathcal{T}^k(M)$ , and  $(X_1, \dots, X_k) \subseteq \mathfrak{X}(M)$ , then

$$A(X_k) = A(X_1, \dots, X_k) \quad \text{such that} \quad A(X_k)(p) = A_p(X_k|_p) = A_p(X_1|_p, \dots, X_k|_p) \in \mathbb{R}$$

We should expect, that if  $A$  is a rough  $k$ -tensor field over  $M$ , it is smooth if and only if for every  $k$ -tuple of smooth vector fields, the resulting function (as defined pointwise above), is  $C^\infty(M)$ . See Lee Proposition 12.19 for more details.

## Pullbacks of Tensor Fields

**Definition 4.1: Pointwise pullback**  $dF_p^* : T^k(T_{F(p)}N) \rightarrow T^k(T_pM)$

Let  $M$  and  $N$  be smooth manifolds with or without boundary. Let  $F : M \rightarrow N$  be a smooth map. For every  $p \in M$ , if  $\alpha$  is a  $k$ -tensor on  $T_{F(p)}N$ , (so that  $\alpha \in T^k(T_{F(p)}^*N)$ ), the *pointwise pullback* of  $\alpha$  through  $dF_p$  is a  $k$ -tensor on  $T_pM$ . Denoted by  $dF_p^*(\alpha)$ , if  $(v_k)$  are tangent vectors in  $T_pM$ , then

$$dF_p^*(\alpha)(v_k) = \alpha(dF_p(v_k)) \quad (30)$$

**Definition 4.2: Tensor Pullback**  $F^* : T^kN \rightarrow T^kM$

Let  $F : M \rightarrow N$  be a smooth map, between smooth manifolds with or without boundary, if  $A$  is a  $k$ -tensor field on  $N$ , we define

$$(F^*A)_p(v_k) = dF_p^*(A_{F(p)}) = A_{F(p)}(dF_p(v_k)) \quad (31)$$

the result is a  $k$ -tensor field (which is smooth, by the next Proposition).

**Proposition 4.1: Proposition 12.25, Properties of Tensor Pullbacks**

Let  $F : M \rightarrow N$  and  $G : N \rightarrow P$  be smooth maps between smooth manifold with or without boundary, and  $A$  and  $B$  are  $k$ -tensor fields on  $N$ , then

(i) For every  $f \in C^\infty(N)$ ,

$$F^*(fB) = \underbrace{(f \circ F)}_{C^\infty(M)} F^*B$$

(ii)  $F^*$  commutes with tensor products,

$$F^*(A \otimes B) = F^*(A) \otimes F^*(B)$$

(iii)  $F^*$  is linear over  $\mathbb{R}$ ,

$$F^*(aA + bB) = aF^*(A) + bF^*(B)$$

(iv)  $F^*(B)$  is smooth,

(v) The tensor pullback satisfies the following co-functorial properties

- The (tensor) pullback of the composition is the (tensor) pullback of the *pre*-composition,

$$(G \circ F)^* = F^* \circ G^*$$

- The tensor pullback of the identity map on  $M$  is the identity map on tensor fields over  $M$ ,

$$\text{id}_M^* = \text{id}_{\mathcal{T}^k(M)}$$

#### Proposition 4.2: Tensor Field Pullback in Coordinates

Let  $M$  and  $N$  be smooth manifolds with or without boundary, and  $F : M \rightarrow N$  be a smooth map.  $A$  is a smooth covariant  $k$ -tensor field on  $N$ , and  $p \in M$  induces local coordinates  $(U, (x^i))$  about  $p$ ,  $(V, (y^j))$  about  $F(p)$ . If  $A$  has the following coordinate representation about  $F(p)$ ,

$$A = \underbrace{A_{i_1, \dots, i_k}}_{\substack{\text{smooth} \\ \text{function in } N}} dy^{i_1} \otimes \dots \otimes dy^{i_k} = A_{i_{\underline{k}}} \left( \bigotimes dy^{i_{\underline{k}}} \right) \quad (32)$$

then  $F^*A$  is given by

$$F_p^*A = (A_{i_1, \dots, i_k} \circ F) d(y^{i_1} \circ F) \otimes \dots \otimes d(y^{i_k} \circ F) \quad (33)$$

$$= (A_{i_1, \dots, i_k} \circ F) d(F^{i_1}) \otimes \dots \otimes d(F^{i_k}) \quad (34)$$

$$= (A_{i_{\underline{k}}} \circ F) \left( \bigotimes d(F^{i_{\underline{k}}}) \right) \quad (35)$$

in local coordinates.

### Lie Derivatives of Tensor Fields

#### Definition 5.1: Lie Derivative of Tensor Fields

Let  $M$  be a smooth manifold and  $V \in \mathfrak{X}(M)$ , and let  $\theta$  be the flow of  $V$ . If  $A \in \mathcal{T}^k(M)$  is a smooth  $k$ -tensor field, we define the *Lie derivative of  $A$  with respect to  $V$* , denoted by  $\mathcal{L}_V A$ ,

$$(\mathcal{L}_V A)_p \triangleq \left. \frac{d}{dt} \right|_{t=0} (\theta_t^* A)_p = \lim_{t \rightarrow 0} \frac{d(\theta_t)_p^* (A_{\theta_t(p)}) - A_p}{t} \quad (36)$$



Similar to Definition 2.1, we view  $T^k(T_p^*M)$  as a vector space, and  $\frac{d}{dt}\big|_{t=0}(\theta_t^*A)_p$  is the Frechet Derivative on  $T^k(T_p^*M)$ .

**Proposition 5.1: Existence of  $\mathcal{L}_V A$**

Let  $M$  be a smooth manifold, and  $V \in \mathfrak{X}(M)$  with integral flow  $\theta$  defined on  $\mathcal{D} \subseteq \mathbb{R} \times M$ . If  $A$  is a smooth covariant  $k$ -tensor field on  $M$ , then  $(\mathcal{L}_V A)_p$  exists for every  $p \in M$ , and defines a smooth  $k$ -tensor field.

*Proof.* Fix  $p \in M$ , following Proposition 2.1 closely, since  $(0, p) \in \mathcal{D}$  induces an open interval  $J_0$  containing 0 and an open neighbourhood  $U_0$  containing  $p$ . We write  $A$  in local coordinates for  $(t, x) \in J_0 \times U_0$ . Shrinking  $U_0$  if necessary, we it suffices to assume  $\theta$  maps  $J_0 \times U_0$  into the same chart  $(U, (x^i))$ .

If  $(t, x) \in J_0 \times U_0$ ,  $A_{\theta_t(x)} \in T^k(T_{\theta_t(x)}M)$  has coordinate representation

$$A_{\theta_t(x)} = \underbrace{A_{i_1, \dots, i_k}(\theta(t, x))}_{\text{smooth function}} \underbrace{dx_{\theta_t(x)}^{i_1} \otimes \dots \otimes dx_{\theta_t(x)}^{i_k}}_{\text{basis vector}}$$

By Equation (33), the pullback  $d(\theta_t^*)_p(A_{\theta_t(p)})$  precomposes  $A_{i_k}$  with  $\theta_t(p)$ . Since  $\theta(t, x) \in (U, (x^i))$ ,  $\theta^j(t, x)$  is precisely the  $j$ th coordinate function for all  $(t, x) \in J_0 \times U_0$ .

$$d(\theta_t^*)_p(A_{\theta_t(p)}) = (A_{i_k} \circ \theta^{(p)})(t) \left( \bigotimes dx_p^{i_k} \right) \in T^k(T_p^*M)$$

$(A_{i_k} \circ \theta^{(p)})(t)$  is clearly smooth in both arguments. So  $(\mathcal{L}_V A)_p$  exists, and is smooth about every  $p \in M$ . ■

**Proposition 5.2: Computing Lie derivatives at  $t_0$  (Tensor Fields)**

Let  $M$  be a smooth manifold,  $V \in \mathfrak{X}(M)$ , and the flow of  $V$  by  $\theta$  with domain  $\mathcal{D}$ . Let  $A$  be a smooth covariant  $k$ -tensor field on  $M$ . If  $(t_0, p) \in \mathcal{D}$ ,

$$\frac{d}{dt}\bigg|_{t=t_0} d(\theta_{-t})_{\theta_t(p)}(A_{\theta_t(p)}) = d(\theta_{-t_0})\big((\mathcal{L}_V A)_{\theta_{t_0}(p)}\big)$$

*Proof.* The proof is the same as in Proposition 2.2. ■

# Chapter 13: Riemannian Metrics

## Chapter 14: Differential Forms

## Multi-linear algebra

### Definition 1.1: Wedge Product

Let  $\omega \in \Lambda^k(V^*)$ ,  $\eta \in \Lambda^l(V^*)$  the wedge product of  $\omega$  and  $\eta$  is defined

$$\omega \wedge \eta \triangleq \frac{(k+l)!}{k!l!} \text{Alt}(\omega \otimes \eta) \quad (37)$$

with  $\text{Alt}(\omega \otimes \eta) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma)^\sigma (\omega \otimes \eta)$ , and if  $(v_1 \dots v_k, v_{k+1}, \dots, v_{k+l})$  are vectors in  $V$ , then

$$\text{Alt}(\omega \otimes \eta)(v_1 \dots v_k, v_{k+1}, \dots, v_{k+l}) \triangleq \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \eta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)}) \quad (38)$$

To simplify the expressions, we will use the following shorthand,

$$(v_1, \dots, v_k) = (\underline{v_k}) \quad (39)$$

$$(v_{k+1}, \dots, v_{k+l}) = (\underline{v_{k+1, k+l}}), \text{ alternatively} \quad (40)$$

$$(v_{k+1}, \dots, v_{k+l}) = (\underline{v_{k+l}}) \quad (41)$$

and Equation (38) becomes

$$\text{Alt}(\omega \otimes \eta)(\underline{v_{k+l}}) = \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \omega(\underline{v_{\sigma(k)}}) \eta(\underline{v_{\sigma(k), \sigma(k+l)}}) \quad (42)$$

### Definition 1.2: Elementary $k$ -covectors

Let  $V$  be a finite dimensional vector space with basis  $(E_i)$ , and if  $(\varepsilon^i)$  is a dual basis for  $V^*$ , and if  $I$  is a  $k$ -index, where  $I = (\underline{i_k})$ , then

$$\varepsilon^I(\underline{v_k}) = \det \left( \begin{bmatrix} \varepsilon^{i_1}(v_1) & \dots & \varepsilon^{i_1}(v_k) \\ \vdots & \ddots & \vdots \\ \varepsilon^{i_k}(v_1) & \dots & \varepsilon^{i_k}(v_k) \end{bmatrix} \right) = \det \left( \begin{bmatrix} v_1^{i_1} & \dots & v_k^{i_1} \\ \vdots & \ddots & \vdots \\ v_1^{i_k} & \dots & v_k^{i_k} \end{bmatrix} \right) = \det \left( (\varepsilon^{i_j} v_l)_{j,l} \right)$$

where  $\varepsilon^{i_j}(v_l)$  is the projection of  $v_l$  onto the  $i_j$  basis vector  $E_{i_j}$ . The space of all  $\varepsilon^I$  where  $I$  is a  $k$ -index are called *elementary  $k$ -covectors*. It is clear that  $\varepsilon^I$  are alternating  $k$ -covectors.

### Definition 1.3: Concatenating multi-indices

Let  $I$  and  $J$  be multi-indices,  $I = (\underline{i_k})$  and  $J = (\underline{j_l})$ , then

$$IJ = (\underline{i_k, j_l})$$

is a  $k + l$  multi-index obtained by concatenating  $I$  and  $J$ .

**Proposition 1.1: Proposition 14.8**

Let  $V$  be a  $n$ -dimensional vector space with basis  $(E_i)$ , and if  $(\varepsilon^i)$  is a dual basis for  $V^*$ , then  $\Lambda^k(V^*)$  has the basis:

$$\bar{\mathcal{E}} = \left\{ \varepsilon^I, I \text{ is an increasing } k\text{-index} \right\}$$

In particular, this means the vector space (algebra) of alternating  $k$ -covectors have dimension  $\binom{n}{k}$ . If  $k = n$ , then it is spanned by

$$\varepsilon^{(1, \dots, n)} = \varepsilon^{(\underline{n})}$$

and all alternating  $n$ -covectors are of the form  $A\varepsilon^{(\underline{n})}$ , where  $A \in \mathbb{R}$ . This roughly means there is, up to a scalar multiple of  $\varepsilon^{(\underline{n})}$ , only one (oriented) way of measuring volume in a basis-independent, and dimension-independent manner.

**Proposition 1.2: Lemma 14.10**

Let  $I$  and  $J$  be multi-indices,  $I = (\underline{i_k})$  and  $J = (\underline{j_l})$ , then

$$\varepsilon^I \wedge \varepsilon^J = \varepsilon^{IJ} \quad (43)$$

**Proposition 1.3: Lemma 14.11, Properties of the Wedge Product**

The wedge product satisfies the following properties

- It is bi-linear over  $\mathbb{R}$ ,
- It is associative,
- It is anti-commutative, for  $\omega \in \Lambda^k(V^*)$  and  $\eta \in \Lambda^l(V^*)$ , then

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$

- Formula for elementary covectors, let  $\varepsilon^{i_k}$  be covectors (covectors are assumed to have rank one), then

$$\varepsilon^I \triangleq \varepsilon^{i_1} \wedge \dots \wedge \varepsilon^{i_k}$$

is an alternating  $k$ -covector.

- Determinant Law

$$\omega^1 \wedge \dots \wedge \omega^k(v_{\underline{k}}) = \det\left(\omega^j(v_i)\right)_{ji}$$

where  $(A_i^j)_{ji}$  denotes the matrix with entries  $A_i^j$  in the  $j$ th row and  $i$ th column.

One of the better ways of introducing the wedge product is through the 'determinant' convention, and working backwards. That way, interior multiplication will come naturally.

**Definition 1.4: Interior multiplication**

Let  $V$  be a finite dimensional vector space. If  $x \in V$  is a vector, it induces a linear map  $i_x : \Lambda^k(V^*) \rightarrow \Lambda^{k-1}(V^*)$  such that for every  $\omega \in \Lambda^k(V^*)$ , and  $\underline{v_{k-1}} \in V$ ,

$$i_x(\omega)(\underline{v_{k-1}}) \triangleq \omega(x, \underline{v_{k-1}})$$

by fixing  $x$  in the first argument of  $\omega$ . The result is an alternating  $k-1$  covector, and it is clearly linear in  $\omega$  over  $\mathbb{R}$ . We also write

$$i_x(\omega) = x \lrcorner \omega$$

**Definition 1.5: Graded algebra**

An algebra  $A$  is said to be *graded* if it has a direct sum decomposition  $A = \bigoplus_{k \in \mathbb{Z}} A^k$  such that the algebra-product  $(\cdot, \cdot) : A \times A \rightarrow A$  satisfies

$$(a, b) \in A^{k+l} \quad \text{for every } a \in A^k, b \in A^l$$

A graded algebra is said to be *anti-commutative*, if the algebra product satisfies

$$(a, b) = (-1)^{kl}(b, a)$$

We see that  $\bigoplus_{k \in \mathbb{Z}} \Lambda^k(V^*)$  is a graded algebra, if we define  $\Lambda^k(V^*) = \{0\}$  for  $k \leq -1$ , or  $k \geq n+1$ .

**Proposition 1.4: Lemma 14.13, Properties of interior multiplication**

Let  $V$  be a finite dimensional vector space and fix  $x \in V$ ,

- (i)  $i_x \circ i_x \equiv 0$ , where we interpret  $i_v$  as a linear map on the entire graded algebra.
- (ii)  $i_x$  satisfies some kind of product rule. For every  $\omega \in \Lambda^k(V^*)$  and  $\eta \in \Lambda^l(V^*)$ ,

$$i_x(\omega \wedge \eta) = (i_x \omega) \wedge \eta + (-1)^k \omega \wedge (i_x \eta)$$

**Differential Forms on Manifolds**

**Definition 2.1: Differential  $k$ -form on  $M$**

Let  $M$  be a smooth manifold with or without boundary, and  $p \in M$  be fixed,  $\Lambda^k(T_p^*M)$  is a linear subspace of  $T^k(T_p^*M)$  of dimension  $\binom{n}{k}$ , and

$$\Lambda^k T^*M \triangleq \coprod_{p \in M} \Lambda^k(T_p^*M)$$

is a smooth vector bundle of rank  $\binom{n}{k}$  over  $M$ .

A differential  $k$ -form on  $M$  is a  $k$ -tensor field (or a smooth global section of the  $k$ -tensor bundle  $T^k T^* M$ ) that is *pointwise alternating*, or equivalently: it is a smooth global section of the vector bundle  $\Lambda^k T^* M$ . The space of differential  $k$ -forms on  $M$  is denoted by

$$\Omega^k(M) = \Gamma(\Lambda^k T^* M)$$

**Definition 2.2: Graded algebra of all  $k$ -forms**

$\Omega^k(M)$  is a vector space over  $\mathbb{R}$ , and a left  $C^\infty(M)$  module. If we stick with scalars over  $\mathbb{R}$ , and define the wedge product of two tensor fields  $\omega, \eta \in \Omega^k(M)$  (recall these are smooth global sections of some vector bundle),

$$(\omega \wedge \eta)_p \triangleq \omega_p \wedge \eta_p$$

We can inherit all the properties of the pointwise wedge product (see Proposition 1.3), namely

- Bi-linearity over  $\mathbb{R}$ ,
- Associativity,
- Anti-commutativity

With this, the direct sum of all  $\Omega^k(M)$  forms an associative, anticommutative graded algebra over  $\mathbb{R}$ , with the tensor wedge product as the algebra product.

$$\Omega^*(M) = \bigoplus_{k \geq 0} \Omega^k(M)$$

**Definition 2.3:  $\Omega^k(M)$  in local coordinates**

Let  $\omega \in \Omega^k(M)$ , and  $(x^i)$  be the local coordinates on some open subset  $U$  of  $M$ . If

$$\left( \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right) \text{ is a coordinate frame on } TU \triangleq \pi^{-1}(U) \quad (44)$$

where  $\pi : TM \rightarrow M$  is the canonical projection of the tangent bundle. Then,

$$(dx^1, \dots, dx^n) \text{ is the dual co-frame to the frame in Equation (44).}$$

by Proposition 1.1, we can write  $\omega$  uniquely as a linear combination (over  $C^\infty(M)$ ) of  $k$ -tensors which are *increasing elementary  $k$ -covectors* pointwise.

$$\omega = \sum_I' \omega_I dx^{i_1} \wedge \dots \wedge dx^{i_k} = \sum_I' \omega_I dx^I$$

where  $\sum_I'$  a sum over all increasing  $k$ -indices, and  $dx^I$  is the shorthand for the wedge product above,

and  $\omega_I$  is a number obtained by evaluating  $\omega$  at the following

$$\omega_I \triangleq \omega\left(\frac{\partial}{\partial x^{i_1}}, \dots, \frac{\partial}{\partial x^{i_k}}\right), \quad \text{where } I = (i_1, \dots, i_k)$$

Since differential  $k$ -forms are (smooth)  $k$ -tensor fields, the pullback  $F^* : \Omega^k(N) \rightarrow \Omega^k(M)$  is well defined, in particular,

**Proposition 2.1: Lemma 14.16, Properties of  $k$ -form pullback**

Let  $F : M \rightarrow N$  be a smooth map, then

- The  $k$ -form pullback is linear over  $\mathbb{R}$ ,
- The  $k$ -form pullback commutes with wedge products, or

$$F^*(\omega \wedge \eta) = F^*\omega \wedge F^*\eta$$

- In any smooth chart, if  $(y^i)$  are the local coordinates of  $N$ . For any  $k$ -form on  $N$   $\omega = \sum' \omega_I dy^{i_1} \wedge \dots \wedge dy^{i_k} = \sum' \omega_I dy^I$ ,

$$F^*\left(\sum' \omega_I dy^{i_1} \wedge \dots \wedge dy^{i_k}\right) = \sum' (\omega_I \circ F) d(y^{i_1} \circ F) \wedge \dots \wedge d(y^{i_k} \circ F)$$

*Proof.* Linearity of the  $k$ -form pullback over  $\mathbb{R}$  is an immediate consequence of the linearity of the tensor field pullback. To prove the second claim, notice

$$F^*(A \otimes B) = F^*(A) \otimes F^*(B)$$

and

$$\begin{aligned} F^*(\text{Alt}(A \otimes B)) &= F^*\left(\frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma(A \otimes B)\right) \\ &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) F^*(\sigma(A \otimes B)) \end{aligned}$$

The tensor pullback commutes with permutations, let  $C$  be a  $k$ -tensor field on  $N$ , and  $\sigma \in S_k$ , then

$$(\sigma F^*(C))_p(v_k) = (\sigma C)(dF_p(v_k)) = C(dF_p(v_{\sigma(k)})) = F^*(\sigma C)_p(v_k)$$

Hence,

$$\begin{aligned} F^*(\text{Alt}(A \otimes B)) &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma[F^*(A \otimes B)] \\ &= \frac{1}{(k+l)!} \sum_{\sigma \in S_{k+l}} (\text{sgn } \sigma) \sigma[F^*(A) \otimes F^*(B)] \\ &= \text{Alt}(F^*(A) \otimes F^*(B)) \end{aligned}$$

Since the wedge product differs by the alternating product by a constant,  $F^*$  commutes with the wedge product. The general case follows upon induction, and associativity of the wedge product.



For the third claim, use linearity, and the pullback of the wedge product is the wedge product of the pullback, noting that each  $y^{ij}$  is a smooth function on  $N$ , so  $F^*(dy^{ij}) = d(y^{ij} \circ F)$  by Proposition 11.25b in Lee, we state this Proposition for completeness:

Let  $F : M \rightarrow N$  be a smooth map, and  $u$  is a smooth function on  $N$ , then

$$F^* du = d(u \circ F)$$

■

**Proposition 2.2: Corollary 14.21, Tensor Pullback of Top-Degree Forms**

Let  $F : M \rightarrow N$  be a smooth map between manifolds with or without boundary of the same rank  $n$ . If  $(x^i)$  and  $(y^j)$  are local coordinates on open subsets  $U$  and  $V$  of  $M$  and  $N$  respectively, for every  $u \in C(V, \mathbb{R})$ ,

$$F^*(udy^1 \wedge \cdots \wedge dy^n) = (u \circ F)(\det DF) dx^1 \wedge \cdots \wedge dx^n$$

on an open set  $U \cap F^{-1}(V)$ , where  $DF$  is the Jacobian matrix of the coordinate representation of  $F$ .

*Proof.* The proof uses the fact that both manifolds are of dimension  $n$ , and any  $n$ -form on  $U$  is necessarily of the form

$$g dx^1 \wedge \cdots \wedge dx^n \tag{45}$$

where  $g$  is some continuous, real valued function on  $U$ . Solving for  $g$  pointwise yields the result. Suppose  $p \in U \cap F^{-1}(V)$ , and evaluating eq. (45) with the coordinate derivations  $\left(\frac{\partial}{\partial x^i}\right)_p$  reads

$$\begin{aligned} \left(g dx^1 \wedge \cdots \wedge dx^n\right)_p \left(\frac{\partial}{\partial x^i}\right)_p &= g(p)(dx^1 \wedge \cdots \wedge dx^n)_p \left(\frac{\partial}{\partial x^i}\right)_p && \text{(pw. eval. of tensor field)} \\ &= g(p) dx^1_p \wedge \cdots \wedge dx^n_p \left(\frac{\partial}{\partial x^i}\right)_p && \text{(wedge pw. is pw. wedge)} \\ &= g(p) \det \left( dx^i_p \left( \frac{\partial}{\partial x^j} \right) \right)_{ij} && \text{(det. law from Proposition 1.3)} \\ &= g(p) \det \left( \delta^i_j \right)_{ij} \\ &= g(p) \end{aligned}$$

Let  $F^i = y^i \circ F$  be the  $i$ th coordinate component of  $F$  (which is a smooth function on  $M$ ). So that  $dF^i$  and  $\frac{\partial}{\partial x^j}$  form a covector-vector pair if evaluated pointwise. Apply the determinant law from Proposition 1.3 for every  $p \in U$  reads

$$(dF^1_p \wedge \cdots \wedge dF^n_p) \left( \frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p \right) = \det \left( \left[ dF^i_p \left( \frac{\partial}{\partial x^j}\bigg|_p \right) \right]_{ij} \right) \tag{46}$$

and

$$dF^1 \wedge \cdots \wedge dF^n \left( \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right) = \det \left( dF^i \left( \frac{\partial}{\partial x^j} \right) \right)$$

Using the tensor pullback formula from Chapter 12, and by Lemma 14.16,

$$\begin{aligned} F^*(udy^1 \wedge \cdots \wedge dy^n) &= (u \circ F)d(y^1 \circ F) \wedge \cdots \wedge d(y^n \circ F) \\ &= (u \circ F)dF^1 \wedge \cdots \wedge dF^n \end{aligned}$$

Evaluating the last expression using the coordinate frame on  $\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$  at some  $p \in U \cap F^{-1}(V) \subseteq U$ ,

$$\begin{aligned} F^*(udy^1 \wedge \cdots \wedge dy^n)_p \left( \frac{\partial}{\partial x^n} \Big|_p \right) &= \underbrace{(u \circ F)(p)}_{\text{scalar}} (dF^1 \wedge \cdots \wedge dF^n)_p \left( \frac{\partial}{\partial x^n} \Big|_p \right) \quad (\text{pw. eval. of tensor field}) \\ &= (u \circ F)(p) (dF_p^1 \wedge \cdots \wedge dF_p^n) \left( \frac{\partial}{\partial x^n} \Big|_p \right) \quad (\text{wedge pw. is pw. wedge}) \\ &= (u \circ F)(p) \det \left( dF^i \left( \frac{\partial}{\partial x^j} \right) \right)_p \quad (\text{Equation (46)}) \end{aligned}$$

Equating the two expressions we see that

$$g(p) = (u \circ F)(p) \det \left( dF^i \left( \frac{\partial}{\partial x^j} \right) \right)_p = (u \circ F)(p) \det(DF)_p$$

■

## Exterior Derivatives

### Definition 3.1: Exterior Derivative on Euclidean Space

Let  $U$  be an open subset of  $\mathbb{R}^n$  or  $\mathbb{H}^n$ , and  $(x^i)$  be the local coordinates in  $U$ . If  $\omega \in \Omega^k(U)$ ,

$$\omega = \sum_J' \omega_J dx^J$$

and we define the *exterior derivative* of  $\omega$  by

$$d\omega = d \left( \sum_J' \omega_J dx^J \right) = \sum_J' d\omega_J \wedge dx^J$$

where  $d\omega_J$  is the Chapter 11 differential of the smooth function  $\omega_J$ , and  $d\omega_J$  is a covector field on  $U$ . If  $p \in u$  and  $v \in T_p U$ ,

$$d(\omega_J)_p(v) \triangleq v(\omega_J)$$

Since  $\Omega^1(U) = \mathfrak{X}(U)$ , the wedge product  $d\omega_J \wedge dx^J = d\omega_J \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$  is alternating.

With this,  $d : \Omega^k(\mathbb{R}^n) \rightarrow \Omega^{k+1}(\mathbb{R}^n)$  (resp.  $\mathbb{H}^n$ ) is a linear map over  $\mathbb{R}$ . We can rewrite  $d\omega_J$  in terms of the coframe on  $U$ , suppose

$$d\omega_J = \frac{\partial \omega_J}{\partial x^i} dx^i \implies d\omega_J \wedge dx^J = \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^J = \sum_{i=1}^n \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^J$$

Summing over all increasing  $k$ -indices  $J$ , we get

$$d\omega = d\left(\sum_J' \omega_J dx^J\right) = \sum_J' \sum_{i=1}^n \frac{\partial \omega_J}{\partial x^i} dx^i \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_k}$$

## Chapter 22: Symplectic Manifold

## Symplectic Tensors

**Definition 1.1: Bilinear forms**

Let  $V$  be a vector space, a *bilinear form*  $\omega : V \times V \rightarrow \mathbb{R}$  is a 2-tensor on  $V$ .

**Definition 1.2: Characterization of bilinear forms**

Let  $\omega$  be a bilinear form on  $V$ , it is

- *symmetric* if

$$\omega(x, y) = \omega(y, x)$$

- *skew-symmetric* or *anti-symmetric* if

$$\omega(x, y) = (-1)\omega(y, x)$$

- *alternating* if

$$\omega(x, x) = 0$$

If  $V$  is a vector space over the field  $F$  and  $\text{char}(F) \neq 2$ , then the last two conditions are equivalent. Moreover,

- $V$  is called an *orthogonal geometry* if  $\omega$  is symmetric.
- $V$  is called a *symplectic geometry* if  $\omega$  is alternating.

**Definition 1.3: Metric vector space**

A vector space (not necessarily finite dimensional) is called a *metric vector space* if it is a orthogonal or symplectic geometry.

## Matrices and bilinear forms

**Definition 2.1: Matrix of bilinear form**

If  $B = (b_1, \dots, b_n)$  is an ordered basis for  $V$ , we define the *matrix representation of  $\omega$*  by

$$\mathcal{M}(\omega) = (a_{ij}) = (\omega(b_i, b_j))$$

**Proposition 2.1: Matrix induces a bilinear form**

Let  $A = (a_{ij})$  be a matrix on  $V$  with respect to some basis  $B = (b_n)$  it is clear that  $A$  induces a bilinear form, on  $V$  through  $A(x, y) = [x]_B^T A [y]_B$ , where  $[\cdot]_B$  denotes the canonical isomorphism  $V \cong \mathbb{R}^n$  with respect to the basis  $B$ .

$$[x]_B^T A [y]_B = \begin{bmatrix} x^1 & \dots & x^n \end{bmatrix} A \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}$$

for  $x = x^i b_i$  and  $y = y^j b_j$ .

Moreover,

$$A[x]_B = \begin{bmatrix} A(b_1, x) \\ \vdots \\ A(b_n, x) \end{bmatrix}$$

is a *column* vector  
whose entries are given  
by applying  $x$  on the  
second coordinate

and

$$[x]_B^T A = \begin{bmatrix} A(x, b_1) & \dots & A(x, b_n) \end{bmatrix}$$

is a *row* vector whose  
entries are given by ap-  
plying  $x$  on the first co-  
ordinate

Let  $A_B$  be the matrix representation of  $\omega$  with respect to the  $B$ , if  $C$  is another basis on  $V$ , then how do we compute  $A_C$ ? The answer is simple, recall for any vector  $x \in V$ ,  $x = x_B^i b_i$  and  $x = x_C^j c_j$ , then

$$[x]_B = M_{C,B} [x]_C \quad \text{for some matrix of an automorphism } M_{C,B}$$

$$\omega(x, y) = [x]_B^T A_B [y]_B = ([x]_C^T M_{C,B}^T) A_B (M_{C,B} [y]_C) = [x]_C^T A_C [y]_C, \text{ then}$$

$$M_{C,B}^T A_B M_{C,B} = A_C \tag{47}$$

We can describe this relation between the two matrices  $A_B$  and  $A_C$  by the following

**Definition 2.2: Congruent matrices**

Two matrices  $M$  and  $N$  are said to be *congruent*, if there exists an invertible matrix  $P$  for which

$$P^T M P = N$$

Congruence is an equivalence relation on the space of matrices, and the equivalence classes over congruence are called *congruence classes*.

**Proposition 2.2: Characterization of matrices using congruence**

Let  $A_1$  and  $A_2$  be matrix representations of two bilinear forms with respect to the basis  $B$ .

$$A_1 = (A_1(b_i, b_j))_{ij} \quad A_2 = (A_2(b_i, b_j))_{ij}$$

They induce the same bilinear form if and only if they are congruent.

**Definition 2.3: Alternate matrices**

Let  $M$  be a matrix with  $F$ -coefficients, it is *alternate* if it is skew symmetric and is *hollow*; meaning it

has 0s on the main diagonal. If  $F = \mathbb{R}$  or  $\text{char}(F) \neq 2$ , then alternate matrices are and are precisely the skew-symmetric matrices.

## Orthogonality

For this section,  $(V, \omega)$  will denote a metric vector space, not necessarily finite-dimensional unless we are using matrix representations.

### Definition 3.1: Orthogonal complements

A vector  $x \in V$  is orthogonal to another vector  $y \in V$ , written  $x \perp y$ , if  $\omega(x, y) = 0$ .

If  $V$  is an orthogonal or symplectic geometry then  $\perp$  is a symmetric relation. If  $E$  is a subset of  $V$ , we denote the *orthogonal complement of  $E$*  by

$$E^\perp \triangleq \{v \in V, v \perp E\}$$

### Definition 3.2: Characterization of metric vector spaces

- A nonzero vector  $x \in V$  is *isotropic*, or *null* if  $\omega(x, x) = 0$
- $V$  is *isotropic* if it contains at least one isotropic vector.
- $V$  is *anisotropic* or *nonisotropic* if for every  $x \in V$ ,  $\omega(x, x) = 0 \implies x = 0$ ,
- $V$  is *totally isotropic* (that is, symplectic if  $\text{char}(F) \neq 2$ ) if  $\omega(x, x) = 0$  for every vector  $x \in V$ .

The first bullet point above is about vectors in  $V$ , while the others are properties of  $V$ .

- A vector  $x \in V$  is called *degenerate* if  $x \perp V$ , that is,

$$\forall y \in V, \omega(x, y) = 0$$

- The *radical* of  $V$ , denoted by  $\text{rad}(V)$  is the set of all degenerate vectors in  $V$ ,

$$\text{rad}(V) \triangleq V^\perp$$

- $V$  is *singular* or *degenerate* if  $\text{rad}(V) \neq \{0\}$ ,
- $V$  is *non-singular* or *non-degenerate* if  $\text{rad}(V) = \{0\}$ ,
- $V$  is *totally singular*, if  $\text{rad}(V) = V$ .

To summarize,

- $V$  is isotropic if there exists a non-zero isotropic vector, meaning  $\omega(x, x) = 0$ , for some  $x \neq 0$ ,
- $V$  is degenerate if there exists a degenerate vector,  $x \perp V$ .

**Proposition 3.1: Matrix invariants under congruence**

Non-singularity, symmetry, and skew-symmetry are invariants under congruence.

*Proof.* ■

**Proposition 3.2: Characterization of non-degeneracy**

$V$  is non-degenerate if and only if every matrix representation  $A$  of  $\omega$  is non-singular.

*Proof.* Suppose  $V$  is non-degenerate, then let  $B = (b_n)$  be a basis for  $V$ , if  $A$  is the matrix representation of  $\omega$  with respect to  $B$ , let  $x$  be a non-zero vector in  $V$ , so  $x \notin \text{rad}(V)$

$$b_i^T A[x]_B = \omega(b_i, x) \neq 0 \implies A[x]_B \neq 0$$

so  $A$  is non-singular. If  $A'$  is another matrix representation with respect to another basis  $C$ , by Equation (47)  $A'$  is non-singular as well.

Conversely, if every matrix representation of  $\omega$  is non-singular, let  $x$  be a non-zero vector in  $V$ , then  $A[x]_B \neq 0$  is a non-zero vector so there exists some basis component  $(A[x]_B)^j$  that is non zero, and

$$[b_j]_B^T A[x]_B = \omega(b_j, x) \neq 0$$

therefore  $V$  is non-degenerate. ■

**Proposition 3.3: Characterisation of bilinear forms from matrix representations**

Let  $\omega$  be a bilinear form on  $V$ , if  $\mathcal{M}(\omega)$  the induced matrix representation relative to any basis. Assume  $V$  is a vector space over  $\mathbb{R}$ , then

- it is symmetric iff  $\mathcal{M}(\omega)$  symmetric as a matrix,
- it is skew-symmetric, iff alternating iff  $\mathcal{M}(\omega)$  is skew-symmetric as a matrix.

**Corollary 3.1: Characterisation of non-singular symplectic form**

Let  $(V, \omega)$  be a finite dimensional vector space over  $\mathbb{R}$ , equipped with a bilinear form  $\omega$ .  $(V, \omega)$  is a non-singular symplectic vector space iff the matrix representation of  $\omega$  with respect to every basis is non-singular and skew-symmetric.

## Riesz Representation Theorems

**Proposition 4.1**

Let  $(V, \omega)$  be a nonsingular metric vector space, the map  $x \mapsto x \lrcorner \omega \in V^*$  defined by

$$x \lrcorner \omega = \omega(x, \cdot), \quad \text{and} \quad (x \lrcorner \omega)(y) = \omega(x, y), \quad \forall y \in V$$



is a linear isomorphism from  $V$  to  $V^*$ .

## Isometries

### Definition 5.1: Isometry between MVS

Let  $(V, \omega)$  and  $(W, \eta)$  be metric vector spaces. An *isometry*  $\tau \in L(V, W)$  is a linear isomorphism that preserves the bilinear form.

$$\omega(u, v) = \eta(\tau u, \tau v)$$

### Definition 5.2: Orthogonal, symplectic groups

Let  $V$  be a nonsingular metric vector space. If  $V$  is an orthogonal (resp. symplectic) geometry, the set of all isometries on  $V$  is called the *orthogonal* (resp. *symplectic*) *group on  $V$* . It is a group under composition, and is denoted by  $\mathcal{O}(V)$  (resp.  $\text{Sp}(V)$ ).

## Hyperbolic spaces, nonsingular completions

### Canonical Forms

### Symplectic Manifolds

### Darboux's Theorem

#### Proposition 9.1: Lie Derivatives of Tensor Fields (along time-varying vector fields)

Let  $M$  be a smooth manifold. Suppose  $V : J \times M \rightarrow TM$  is a smooth time-varying vector field on  $M$ . Denote the time-varying flow of  $V$  by  $\psi : \mathcal{E} \rightarrow M$ . Let  $A \in \mathcal{T}^k(M)$  be a smooth time-invariant covariant  $k$ -tensor field on  $M$ . For every  $(t_1, t_0, p) \in \mathcal{E}$ ,

$$\left. \frac{d}{dt} \right|_{t=t_1} (\psi_{t,t_0}^* A)_p = (\psi_{t_1,t_0}^* (\mathcal{L}V_{t_1} A))_p \quad (48)$$

## Chapter Hofer book

**Definition 0.1: Symplectic vector space**

Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$ . It is a *symplectic vector space* if it admits a non-singular, antisymmetric bilinear form  $\omega : V \times V \rightarrow \mathbb{R}$ .

$$\omega(u, v) = -\omega(v, u)$$

for  $u, v \in V$ . By the previous section on Riesz Representation, the linear map

$$\hat{\omega} : V \rightarrow V^*, \quad v \mapsto \omega(v, \cdot)$$

is a linear isomorphism of  $V$  onto its dual vector  $V^*$ .

We define the *standard symplectic vector space*  $(\mathbb{R}^{2n}, \omega_0)$ , where  $n \in \mathbb{N}^+$ , where

$$\omega_0(u, v) = \langle Ju, v \rangle \quad J \triangleq \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$$

where  $\langle \cdot, \cdot \rangle$  denotes the standard inner product on  $\mathbb{R}^{2n}$ .

$$\omega_0(u, v) = \langle Ju, v \rangle = \langle u, J^T v \rangle = u^T J^T v \quad (49)$$

$J^T = -J$  by Corollary 3.1.

We will mainly deal with non-singular symplectic forms because of Riesz isomorphism.

**Definition 0.2: Symplectic linear map**

Let  $(V, \omega)$  be a symplectic vector space. A linear map  $F \in \text{Hom}(V)$  is *symplectic* if it preserves symplectic form  $\omega$ . For every  $u \in V$ ,

$$\langle u, v \rangle = \langle Au, Av \rangle \triangleq A^* \omega(u, v)$$

where  $A^* : \mathcal{T}^*(V) \rightarrow \mathcal{T}^*(V)$  denotes the tensor pullback by precomposing any tensor  $S$  by  $A$

$$\forall S \in \mathcal{T}^k(V), \quad A^* S(v_{\underline{k}}) \triangleq S(Av_{\underline{k}})$$

The set of linear symplectic maps on a  $2n$ -dimensional vector space form a group under composition. It is a Lie Group denoted by  $\text{Sp}(n)$ .

**Proposition 0.2: Symplectic Maps are Area-preserving**

Let  $(\mathbb{R}^{2n}, \omega_0)$  denote the standard symplectic space. If  $\varphi \in \text{Sp}(n)$ , then  $\det \varphi = 1$ .

*Proof.* See page 4. ■

$$(\Lambda_{s-|\alpha|} \partial^\alpha f)^\wedge = (1 + |\zeta|^2)^{s/2-|\alpha|/2} \cdot (\partial^\alpha f)^\wedge \quad (50)$$

$$= (1 + |\zeta|^2)^{s/2-|\alpha|/2} \cdot (2\pi i \zeta)^\alpha \cdot \hat{f} \quad (51)$$

$$= (2\pi i)^{|\alpha|} (1 + |\zeta|^2)^{(s-|\alpha|)/2} \cdot |\zeta|^{|\alpha|} \cdot \hat{f} \quad (52)$$

$$\leq |\alpha| (1 + |\zeta|^2)^{s/2} \hat{f} \quad (53)$$

# Chapter 1: Multilinear maps

## Introduction

A *Banach space* is a normed vector space that is Cauchy-complete under the usual metric induced by its norm.

If  $E$  and  $F$  are Banach spaces over  $\mathbb{R}$ . We will denote the norms on  $E$ , and  $F$  by single lines, so

$$|x| = \|x\|_E \quad \text{and} \quad |y| = \|y\|_F \quad \forall x \in E, y \in F$$

$\mathcal{L}(E, F)$  will denote the space of linear maps between  $E$  and  $F$ . In the category of Banach spaces, the space of morphisms are called *toplinear morphisms* - or *CLMs* (*continuous linear maps*); which we will denote by  $L(E, F)$  for toplinear morphisms between  $E$  and  $F$ .

We use  $\|\cdot\|_{L(E, F)}$  or  $\|\cdot\|$  to denote the operator norm, depending on how much emphasis we wish to place on  $L(E, F)$ . Recall,

$$\begin{aligned} \|\varphi\|_{L(E, F)} &= \inf \left\{ A \geq 0, |\varphi(x)| \leq A|x| \forall x \in E \right\} \\ &= \sup \left\{ |\varphi(x)|, x \in E, |x| = 1 \right\} \end{aligned}$$

By the open mapping theorem: any continuous surjective linear map is an open map. Hence invertible elements in  $L(E, F)$  are naturally called *toplinear isomorphisms*. If  $\varphi \in L(E, F)$  such that  $\varphi$  preserves the norm between the Banach Spaces, that is for every  $x \in E$ ,  $|x| = |\varphi(x)|$  then we call  $\varphi$  an *isometry*, or a *Banach space isomorphism*. If  $E_1$  and  $E_2$  are Banach spaces, we will use the usual *product norm*  $(x_1, x_2) \mapsto \max(|x_1|, |x_2|)$ .

## Bilinear maps

### Definition 2.1: Bilinear map

A map  $\varphi : E_1 \times E_2 \rightarrow F$ , where  $F$  is also a Banach space, is said to be *bilinear* if

$$\varphi(x, \cdot) : E_2 \rightarrow F \quad \text{and} \quad \varphi(\cdot, y) : E_1 \rightarrow F$$

are linear for every  $x \in E_1$  and  $y \in E_2$ .

### Proposition 2.1: Continuity of a bilinear map

Let  $E_1, E_2, F$  be Banach spaces, a bilinear map  $m : E_1 \times E_2 \rightarrow F$  is continuous if and only if there exists a  $C \geq 0$ , where

$$|m(x, y)| \leq C|x||y| \tag{54}$$

*Proof.* Suppose such a  $C$  exists, fix a convergent sequence  $(x_n, y_n) \rightarrow (x, y)$  in  $E_1 \times E_2 = E$ . Because the projection maps are continuous, this means  $x_n \rightarrow x$  and  $y_n \rightarrow y$ . Using inspiration from the proof where  $x_n y_n \rightarrow xy$ , where

$$x_n(y_n - y) + (x_n - x)y = x_n y_n - xy \quad x, y, x_n, y_n \in \mathbb{R}$$

Using the inspiration, and replacing multiplication in  $\mathbb{R}$  with the bilinear map  $m$ , we have:

$$\begin{aligned} m(x_n, y_n - y) + m(x_n - x, y) &= m(x_n, y_n) - m(x, y) \\ |m(x_n, y_n) - m(x, y)| &\leq C[|x_n| \cdot |y_n - y| + |x_n - x| \cdot |y|] \rightarrow 0 \end{aligned}$$

Conversely, if  $m$  is continuous, then it is continuous at the origin  $(0,0) = 0$ . There exists a  $\delta$  where  $|(x,y)| \leq \delta$  implies  $|m(x,y)| \leq 1$ . Now, if  $x, y \neq 0$  are elements in  $E$ , we normalize so that  $(x,y)$  has length  $\delta$

$$|(x|x|^{-1}\delta, y|y|^{-1}\delta)| = \delta|(x|x|^{-1}, y|y|^{-1})| = \delta$$

So that  $|m(x|x|^{-1}\delta, y|y|^{-1}\delta)| \leq 1$ , using bilinearity of  $m$ :

$$|m(x, y)| \leq \delta^{-2}|x| \cdot |y|$$

Setting  $\delta^{-2} = C$  finishes the proof (notice if either  $x$  or  $y$  is 0, then  $m$  is trivially 0 and the inequality holds). ■

**Proposition 2.2:**  $L(E_1, E_2; F)$  is isomorphic to  $L(E_1, L(E_2, F))$

For each bilinear map  $\omega \in L(E_1, E_2; F)$ , there exists a unique map  $\varphi_\omega \in L(E_1, L(E_2, F))$  such that  $|\omega| = |\varphi_\omega|$ ; such that for every  $(x, y) \in E_1 \times E_2$ ,  $\omega(x, y) = \varphi(x)(y)$ .

*Proof.* Let  $\varphi_\omega : E_1 \rightarrow L(E_2, F)$  be the unique map such that  $\varphi_\omega(x)(y) = \omega(x, y)$ . Proposition 2.1 shows that  $\varphi_\omega(x)$  is a continuous linear map into  $F$  at each  $x$ , and  $|\varphi_\omega(x)| \leq |\omega||x|$ . This holds for an arbitrary  $x$ , and  $\varphi_\omega(\cdot)$  is clearly linear, hence  $|\varphi_\omega| \leq |\omega|$ . Reversing the roles of  $\omega$  and  $\varphi$  shows proves the other estimate.

The rule as outlined above is linear in  $\omega$ ; and it is not hard to see  $\varphi : L(E_1, E_2; F) \rightarrow L(E_1, L(E_2, F))$  is an injection. By the open mapping theorem, the proposition is proven if  $\varphi$  is a surjection. Fix  $\theta \in L(E_1, L(E_2, F))$ , define a map  $\omega : E_1 \times E_2 \rightarrow F$  such that  $\omega(x, \cdot) = \theta(x)(\cdot)$ . So that  $\omega$  is linear in its second argument. To show  $\omega$  is linear in its first: fix a linear combination  $A = \sum ax$  in  $E_1$ , and  $y \in E_2$ .

$$\omega(A, y) = \theta(\sum ax)(y) = \sum a\theta(x)(y) = \sum a\omega(x, y)$$

Continuity follows from Equation (54), and  $\varphi_\omega = \theta$  as needed. ■

## Notation

We will use the following notation to simplify computations with multilinear maps. Let  $E$  and  $F$  be sets, and  $v_1, \dots, v_k \in E$ .  $f : E \rightarrow F$ .

- Listing individual elements:  $v_{\underline{k}}$  means  $v_1, \dots, v_k$  as separate elements.
- Creating a  $k$ -list:  $(v_{\underline{k}}) = (v_1, \dots, v_k) \in \prod E_{j \leq k}$  if  $v_i \in E_i$  for  $i = \underline{k}$ .
- Double indices:  $(v_{\underline{n_k}}) = (v_{n_{\underline{k}}}) = (v_{n_1}, \dots, v_{n_k})$ , and

$$(v_{\underline{n_k}}) \neq (v_{n_{(1, \dots, k)}})$$

- Closest bracket convention:

$$(v_{(\underline{n_k})}) = (v_{(n_1, \dots, n_k)}) \quad \text{and} \quad (v_{n_{(\underline{k})}}) = (v_{n_{(1, \dots, k)}})$$

- Underlining 0 means it is iterated 0 times:

$$(v_{\underline{0}}, a, b, c) = (a, b, c)$$

- Skipping an index:

$$(v_{\underline{i-1}}, v_{i+\underline{k-i}}) = (v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_k)$$

for  $i = \underline{k}$ .

- Applying  $f$  to a particular index:

$$(v_{\underline{i-1}}, f(v_i), v_{i+\underline{k-i}}) = (v_1, \dots, v_{i-1}, f(v_i), v_{i+1}, \dots, v_k)$$

Of course, if  $i = 1$ , then the above expression reads  $(f(v_1), v_2, \dots, v_k)$  by the  $\underline{0}$  interpretation.

- In any list using this 'underline' notation, we can find the size of a list by summing over all the underlined terms, and the number of terms with no underline.
- If  $\wedge : E \times E \rightarrow F$  is any associative binary operation,

$$\bigcirc(\underline{v_k}) = v_1 \wedge \dots \wedge v_k$$

### Remark 3.1: Preview of exterior calculus

We can write the formula for the determinant of a  $\mathbb{R}^{k \times k}$  matrix in this notation. Suppose  $a_i \in \mathbb{R}$ , and  $b_i \in \mathbb{R}^{k-1}$  for  $i = \underline{k}$ .

$$M = \begin{bmatrix} a_1 & \cdots & a_k \\ | & & | \\ b_1 & \cdots & b_k \\ | & & | \end{bmatrix}$$

The determinant of  $M$  is a linear combination of determinants of  $k-1$ -sized matrices, given in terms of the columns of  $b$

$$\det(M) = \sum_{i=\underline{k}} (-1)^{i-1} a_i \det(b_{\underline{i-1}}, b_{i+\underline{k-i}})$$

## $k$ -linear maps

### Definition 4.1: $k$ -linear maps

Let  $E_{\underline{k}}, F$  be Banach spaces. A map  $\varphi : \prod E_{\underline{k}}$  is  $k$ -linear if for every  $i = \underline{k}$ ,  $v_i \in E_i$ ,

$$\varphi(\cdot, \cdot, v_i, \cdot, \cdot) : \bigcirc(E_{\underline{i-1}}, E_{i+\underline{k-i}}) \rightarrow F \quad \text{is } (k-1)\text{-linear}$$



The following theorem should give confidence to the notation we have adopted to use.

**Proposition 4.1**

Let  $E_{\underline{k}}$  and  $F$  be Banach spaces, a  $k$ -linear map  $\varphi : \prod E_{\underline{k}} \rightarrow F$  is continuous iff there exists a  $C > 0$ , such that for every  $x_i \in E_i$ ,  $i = \underline{k}$

$$|\varphi(x_{\underline{k}})| \leq C \prod |x_{\underline{k}}|$$

*Proof.* Suppose  $\varphi$  is continuous, then it is continuous at the origin. Picking  $\varepsilon = 1$  induces a  $\delta > 0$  such that for  $|(x_{\underline{k}})| \leq \delta$ ,  $|\varphi(x_{\underline{k}})| \leq 1$ . The usual trick of normalizing an arbitrary vector  $(x_{\underline{k}}) \in \prod E_{\underline{k}}$  does the job:

$$|\varphi(x_{\underline{k}} \cdot |x_{\underline{k}}|^{-1} \cdot \delta)| \leq 1 \implies |\varphi(x_{\underline{k}})| \leq \delta^{-k} \prod |x_{\underline{k}}|$$

Conversely, fix a sequence (indexed by  $n$ , in  $k$  elements in the product space  $\prod E_{\underline{k}}$ ), so

$$(x_{\underline{n}}^k) \rightarrow (x_{\underline{k}}^k) \quad \text{as } n \rightarrow +\infty \quad (55)$$

To proceed any further, we need to prove an important equation that decomposes a difference in  $\varphi$ .

$$\varphi(b_{\underline{k}}^k) - \varphi(a_{\underline{k}}^k) = \sum_{i=\underline{k}} \varphi(b_{\underline{k}}^{i-1}, \Delta_i, a_{\underline{k}}^{i+k-i}) \quad (56)$$

where  $(b_{\underline{k}}^k)$  and  $(a_{\underline{k}}^k)$  are elements in  $\prod E_{\underline{k}}$ , and  $\Delta_i = b^i - a^i$  for  $i = \underline{k}$ . The proof is in the following note, which is in more detail than usual - to help the reader ease into the new notation.

**Note 4.1**

We proceed by induction, and eq. (56) follows by setting  $m = k$  in

$$\varphi(a_{\underline{k}}^k) = \varphi(b_{\underline{k}}^m, a_{\underline{k}}^{m+k-m}) - \sum_{i=\underline{m}} \varphi(b_{\underline{k}}^{i-1}, \Delta_i, a_{\underline{k}}^{i+k-i}) \quad (57)$$

Base case: set  $m = 1$ , by definition of  $k$ -linearity (def. 4.1) of  $\varphi$ . Since  $a^1 = b^1 - \Delta_1$ ,

$$\varphi(a_{\underline{k}}^k) = \varphi(b^1 - \Delta_1, a^{1+k-1}) = \varphi(b^1, a^{1+k-1}) - \varphi(\Delta_1, a^{1+k-1})$$

Induction hypothesis: suppose eq. (57) holds for a fixed  $m$ . Since  $a^{m+1} = b^{m+1} - \Delta_{m+1}$ ,

$$\begin{aligned} \varphi(a_{\underline{k}}^k) &= \varphi(b_{\underline{k}}^m, a_{\underline{k}}^{m+k-m}) - \sum_{i=\underline{m}} \varphi(b_{\underline{k}}^{i-1}, \Delta_i, a_{\underline{k}}^{i+k-i}) \\ &= \varphi(b_{\underline{k}}^m, a_{\underline{k}}^{m+1}, a_{\underline{k}}^{(m+1)+k-(m+1)}) - \sum_{i=\underline{m}} \varphi(b_{\underline{k}}^{i-1}, \Delta_i, a_{\underline{k}}^{i+k-i}) \\ &= \varphi(b_{\underline{k}}^{m+1}, a_{\underline{k}}^{(m+1)+k-(m+1)}) - \varphi(b_{\underline{k}}^{m+1}, \Delta_{m+1}, a_{\underline{k}}^{(m+1)+k-(m+1)}) - \sum_{i=\underline{m}} \varphi(b_{\underline{k}}^{i-1}, \Delta_i, a_{\underline{k}}^{i+k-i}) \end{aligned}$$

and this proves eq. (56)

We substitute  $a^i = x^i$ , and  $b^i = x_n^i$  for  $i = \underline{k}$ , and eq. (56) becomes eq. (58)

$$\varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) = \sum_{i=\underline{k}} \varphi(x_n^{i-1}, x_n^i - x^i, x^{i+k-i}) \quad (58)$$

Then the triangle inequality reads

$$\begin{aligned} \left| \varphi(x_n^{\underline{k}}) - \varphi(x^{\underline{k}}) \right| &\leq \sum_{i=\underline{k}} \left| \varphi(x_n^{i-1}, x_n^i - x^i, x^{i+k-i}) \right| \\ &\leq \sum_{i=\underline{k}} |\varphi| \cdot \bigoplus \left( x_n^{i-1}, \Delta_i, x^{i+k-i} \right) \\ &\leq \sum_{i=\underline{k}} |\varphi| \cdot \left| x_n^i - x^i \right| \bigoplus \left( x_n^{i-1}, x^{i+k-i} \right) \\ &\lesssim_n |\varphi| \sup_{i=\underline{k}} |x_n^i - x^i| \rightarrow 0 \end{aligned}$$

where we identify the product  $\bigoplus(v^{\underline{k}})$  with the product of their norms  $\bigoplus(|v^{\underline{k}}|)$ . ■

#### Remark 4.1

The  $k$ -linear variant of prop. 2.2 holds. We will use but not prove this fact.

#### Remark 4.2

We denote the space of  $k$ -linear maps from  $E$  into  $F$  by  $L(E_{\underline{k}}; F) = L(E^{\underline{k}}, F) = L^k(E, F)$ . *Tensors* on  $E$  are  $k$ -linear maps from the product space of  $E$  into  $\mathbb{R}$ , by replacing  $F$  with  $\mathbb{R}$ .

## Chapter 2: Differentiation

## The derivative

### Definition 1.1: Open sets and neighbourhoods

If  $U$  is an open subset of a topological space  $X$ , we denote this by  $U \subseteq X$ . If  $U$  is a *neighbourhood* of a point  $p \in X$ , we write  $p \in U$ .

We do not require neighbourhoods to be open sets; rather, we say  $U$  is a neighbourhood of  $p$  when the interior of  $U$  contains  $p$ .

### Definition 1.2: Little $o$

A real-valued function in a real variable defined for all  $t$  sufficiently small is said to be  $o(t)$  if  $\lim_{t \rightarrow 0} o(t)/t = 0$ . A map  $\psi : U \rightarrow F$  where  $U \subseteq E$  contains 0 in  $E$ , is said to be  $o(h)$  if  $|\psi(h)|/|h| \rightarrow 0$  as  $h \rightarrow 0$  in  $E$ .

### Definition 1.3: Differentiability

Let  $f : E \rightarrow F$  be a map, replacing  $E$  and  $F$  by their open subsets if necessary. We say  $f$  is *differentiable* at  $x \in E$  when there exists a **continuous linear map on  $E$** :  $\lambda \in L(E, F)$  such that

$$f(x + h) = f(x) + \lambda h + o(h) \quad \text{for sufficiently small } h \quad (59)$$

The role  $o(h)$  plays here is a map from  $U \rightarrow F$ , where  $U$  is some neighbourhood of 0.

### Proposition 1.1: Basic properties of the derivative

If  $f$  is differentiable at  $x$ , then the  $\lambda$  in eq. (59) is unique. We write  $f'(x) = Df(x) = \lambda$  as in ?? . Furthermore, if  $f'(x)$  and  $g'(x)$  exist, then  $(f + g)'(x) = f'(x) + g'(x)$  as linear maps, similar for scalar multiplication.

*Proof.* Suppose  $\lambda_i \in L(E, F)$  are both derivatives of  $f$  at  $x$ . Then,

$$\begin{cases} f(x + h) = f(x) + \lambda_1(h) + o(h) \\ f(x + h) = f(x) + \lambda_2(h) + o(h) \end{cases}$$

And  $(\lambda_1 - \lambda_2)(h) = o(h) = \varphi(h) \cdot |h|$ , where  $\varphi(h) \rightarrow 0$  as  $h \rightarrow 0$ . Using the operator norm, we see that

$$\|\lambda_1 - \lambda_2\|_{L(E, F)} \leq |\varphi(h)| \rightarrow 0$$

This proves uniqueness. Suppose  $f$  and  $g$  are differentiable at  $x$ , denote  $\lambda_f = f'(x)$  (resp.  $g'(x)$ ). The definition of def. 1.3 reads

$$\begin{aligned} f(x + h) + g(x + h) &= (f(x) + g(x)) + (\lambda_f(h) + \lambda_g(h)) + o(h) + o(h) \\ (f + g)(x + h) &= (f + g)(x) + (\lambda_f + \lambda_g)(h) + o(h) \end{aligned} \quad (60)$$

since eq. (60) satisfies eq. (59), the proof is complete. ■

**Proposition 1.2: Chain rule**

Let  $E, F, G$  be Banach spaces. If  $f \in C^1(E, F)$ ,  $g \in C^1(F, G)$ , for every  $x \in E$ ,

$$(g \circ f)'(x) = g'(f(x)) \circ f'(x) \quad (61)$$

*Proof.* Since  $f$  is differentiable at  $x$ ,  $f(x + h) = f(x) + f'(x)(h) + o_1(h)$ , (resp. for  $g$ ,  $o_2(h)$ ). Set  $k(h) = f(x + h) - f(x)$ , and

$$\begin{aligned} g(f(x + h)) &= g(f(x)) + g'(f(x))(k(h)) + o_2(k(h)) \\ &= g(f(x)) + g'(f(x))(f'(x)(h) + o_1(h)) + o_2(k(h)) \\ (g \circ f)(x + h) &= (g \circ f)(x) + g'(f(x)) \circ f'(x)(h) + g'(f(x))(o_1(h)) + o_2(k(h)) \\ (g \circ f)(x + h) &= (g \circ f)(x) + g'(f(x)) \circ f'(x)(h) + o(h) \end{aligned}$$

because  $|A(o_1(h))| \leq |A||o_1(h)|$  for all  $A \in L(E, F)$ ; and  $o(k(h)) = o(h)$  for every continuous  $k : E \rightarrow F$  such that  $k(h) \rightarrow 0$  as  $h \rightarrow 0$ . ■

**Proposition 1.3: Derivatives of CLMs**

**Proposition 1.4: Product rule in  $k$  variables**

Let  $m : \prod F_{\underline{k}} \rightarrow G$  be a  $k$ -linear map between Banach spaces  $F_{\underline{k}}$  and  $G$ . Suppose  $f_i \in C^1(E, F_i)$  with  $i = \underline{k}$ , writing

$$m(f_{\underline{k}})(x) = m(f_{\underline{k}}(x)) \quad (62)$$

then  $m(f_{\underline{k}})$  is in  $C^1(E, G)$  and for every  $y \in E$ ,

$$Dm(f_{\underline{k}})(x)(y) = \sum_{i=\underline{k}} m(f_{\underline{i}-1}(x), Df_i(x)(y), f_{i+\underline{k}-i}(x)) \quad (63)$$

*Proof.* Let  $x$  be fixed. Equation (63) is proven if we show eq. (64)

$$m(f_{\underline{k}})(x + h) = m(f_{\underline{k}})(x) + \left( \sum_{i=\underline{k}} m(f_{\underline{i}-1}(x), Df_i(x)(h), f_{i+\underline{k}-i}(x)) \right) + o(h) \quad (64)$$

and for sufficiently small  $h$  we have

$$f_i(x + h) - f_i(x) = Df_i(x)(h) + o(h^i) \quad (65)$$

We will use the difference formula in eq. (57), with the following substitutions

$$f_i(x + h) = b^i \quad f_i(x) = a^i \quad (66)$$

$$Df_i(x)(h) = c^i \quad o(h^i) = \varepsilon^i \quad (67)$$

$$f_i(x + h) - f_i(x) = c^i + \varepsilon^i \quad \Delta^i = o(h^i) + c^i \quad (68)$$

With these substitutions, the equation we want to prove (eq. (63)) becomes eq. (69)

$$m(b^{\underline{k}}) - m(a^{\underline{k}}) = \left( \sum_{i=\underline{k}} m(a^{i-1}, c^i, a^{i+k-i}) \right) + o(h) \quad (69)$$

Starting from eq. (57),

$$m(b^{\underline{k}}) - m(a^{\underline{k}}) = \sum_{i=\underline{k}} m(b^{i-1}, \Delta^i, a^{i+k-i})$$

We can expand each term, if  $i = \underline{k}$ ,

$$m(b^{i-1}, \Delta^i, a^{i+k-i}) = m(b^{i-1}, c^i, a^{i+k-i}) + m(b^{i-1}, o(h^i), a^{i+k-i}) \quad (70)$$

Let us study the first term in eq. (70), and with  $i$  held fixed, define

$$m_i(z^{i-1}) = m(z^{i-1}, c_i, a^{i+k-i}) \quad (71)$$

Expanding the first term within eq. (70), and because  $m_i$  as defined in eq. (71) is  $i-1$ -linear (because it is a  $k$ -linear map with  $k - (i-1)$  variables held constant); we use eq. (57) again.

$$m_i(b^{i-1}) = \left( \sum_{j=\underline{k}} m_i(b^j, \Delta^j, a^{j+(i-1)-j}) \right) + m_i(a^{i-1}) \quad (72)$$

Unboxing the last term in eq. (72) using the definition of  $m_i$  reads

$$m(b^{i-1}, \Delta^i, a^{i+k-i}) = m(a^{i-1}, c^i, a^{i+k-i}) + \sum_{j=i-1} m_i(b^j, \Delta^j, a^{j+(i-1)-j}) \quad (73)$$

We wish to remove all of the  $b^i$ 's. Since  $\Delta^i = c^i + \varepsilon^i$  (eq. (68)), we have

$$\begin{aligned} m(b^{\underline{k}}) - m(a^{\underline{k}}) &= \sum_{i=\underline{k}} m(b^{i-1}, c^i, a^{i+k-i}) + m(b^{i-1}, \varepsilon^i, a^{i+k-i}) \\ &= \left( \sum_{i=\underline{k}} m_i(b^{i-1}) \right) + \sum_{i=\underline{k}} m(b^{i-1}, \varepsilon^i, a^{i+k-i}) \\ &= \left( \sum_{i=\underline{k}} m_i(a^{i-1}) + \sum_{j=i-1} m_i(b^{j-1}, \Delta^j, a^{j+(i-1)-j}) \right) + \sum_{i=\underline{k}} m(b^{i-1}, \varepsilon^i, a^{i+k-i}) \\ &= \left( \sum_{i=\underline{k}} m_i(a^{i-1}) \right) + \sum_{\substack{i=\underline{k} \\ j=i-1}} m_i(b^{j-1}, \Delta^j, a^{j+(i-1)-j}) + \sum_{i=\underline{k}} m(b^{i-1}, \varepsilon^i, a^{i+k-i}) \end{aligned} \quad (74)$$

The last term within eq. (74) is  $o(h)$ , since it is a linear combination of  $o(h^i)$ 's.

$$\left| \sum_{i=\underline{k}} m(b^{i-1}, \varepsilon^i, a^{i+k-i}) \right| \lesssim_{m,a,b} |o(h)| \quad (75)$$

Each summand in the second last term in eq. (74) is  $o(h)$  as well, as

$$\begin{aligned}
 |m_i(b^{j-1}, \Delta^j, a^{j+(i-1)-j})| &\leq |m_i| \left( \prod (b^{j-1}, \Delta^j, a^{j+(i-1)-j}) \right) \\
 &\leq |m| \cdot \left( \prod (c^i, a^{i+k-i}) \right) \left( \prod (b^{j-1}, \Delta^j, a^{j+(i-1)-j}) \right) \\
 &\lesssim_{m,a,b} \sup_{\substack{i=k \\ j=i-1}} |c^i| \cdot |\Delta^j| \\
 &\lesssim_{m,a,b} \sup_{\substack{i=k \\ j=i-1}} |Df_i(x)(h)| \cdot |f_j(x+h) - f_j(x)| \\
 &\lesssim_{m,a,b} |Df_i(x)||h| \sup_{\substack{i=k \\ j=i-1}} |\Delta^j| \\
 &\lesssim_{m,a,b} |o(h)|
 \end{aligned} \tag{76}$$

for the second last estimate we used  $\Delta^j \rightarrow 0$ . Therefore the second term in eq. (74) is  $o(h)$ , and eq. (64) is proven. Therefore  $m(f_k)$  is differentiable at  $x$ . Continuity of  $Dm(f_k)$  follows from the fact that

$$Dm(f_k)(x) = \sum_{i=k} m(f_{i-1}(x), Df_i(x)(\cdot), f_{i+k-i}(x)) \tag{77}$$

and each of the summands eq. (77) can be broken down as the product of the compositions shown in eqs. (78) and (79)

$$x \mapsto (f_{i-1}(x), f_{i+k-i}(x)) \mapsto m(f_{i-1}(x), \cdot, f_{i+k-i}(x)) \tag{78}$$

$$x \mapsto Df_i(x)(\cdot) \tag{79}$$

which are continuous from  $E$  to  $L(E, F)$ . ■

## Chapter 4: Higher order derivatives



## Introduction

We start with the definition of  $C^p(E, F)$ . Let  $E$  and  $F$  be Banach Spaces, if  $p \geq 1$  is an integer, we define the class  $C^p$  to be the set of maps which are  $p$  times differentiable, and  $D^p f \in C(E, X)$ , where

$$X = L(E, L(E, L(E, \dots F))) p \text{ times } \xLeftrightarrow{\mathcal{L}} L(E^p, F)$$

Sometimes we replace  $E$  with an open subset  $U \subseteq E$  if necessary, and we write  $f \in C(U, F)$  if  $D^p \in C(U, X)$ . Note, even if  $f \in C^1(U, F)$ ,  $Df$  is still a map from  $U$  into  $L(E, F)$ .

We will prove two major results in this section.

- The structure of the derivative  $D^p f$ , in particular, if  $f \in C^p(E, F)$ , then  $D^p f(x)$  is a *symmetric multilinear map* in  $p$  arguments.
- Taylor's Theorem

## The second derivative

### Proposition 2.1: Product rule in 2 variables

Let  $E_1$ ,  $E_2$  and  $F$  be Banach spaces, if  $\omega : E_1 \times E_2 \rightarrow F$  is bilinear and continuous, then  $\omega$  is differentiable, and for every  $(x_1, x_2) \in E_1 \times E_2$ ,  $(v_1, v_2) \in E_1 \times E_2$ ,

$$D\omega(x_1, x_2)(v_1, v_2) = \omega(x_1, v_2) + \omega(v_1, x_2)$$

Furthermore,  $D^2\omega(x, y) = D\omega \in L(E^2, F)$ , and  $D^3\omega = 0$ .

*Proof.* By the definition of  $\omega$ , using the familiar interpolation method

$$\omega(x_1 + h_2, x_2 + h_2) = \omega(x_1, x_2) + \omega(x_1, h_2) + \omega(h_1, x_2) + \omega(h_1, h_2)$$

by continuity of  $\omega$ , the last term (which we wish to make  $o(h)$ ):

$$|\omega(h_1, h_2)| \leq \|\omega\| \cdot |(h_1, h_2)|^2$$

so that  $\omega(h_1, h_2) = o(h)$ , and  $D\omega(x_1, x_2)$  exists and is continuous, and is given by the *linear map*  $\omega(x_1, \cdot) + \omega(\cdot, x_2)$ . The rest of the proof follows, if it is not immediately obvious then read the following note.

### Note 2.1

Write  $E = E_1 \times E_2$  for convenience. The linear map  $A = D\omega(x_1, x_2)$  takes arguments  $E$  into  $F$ , consider the projections  $\pi_1$  and  $\pi_2$ , and  $v \in E_1 \times E_2$ , then

$$A(v) = \omega(x_1, \pi_1 v) + \omega(\pi_2 v, x_2)$$

We can view  $A(x) = D\omega(x_1, x_2) \in L(E, F)$ . It is clear that  $A$  is linear in  $x$ , if we fix  $v \in E$ ,

$$A(x + y, v) = \omega(\pi_1(x + y), \pi_2 v) + \omega(\pi_1 v, \pi_2(x + y)) = A(x, v) + A(y, v)$$

and similarly for scalar multiplication. Hence  $DA(x) = A \in L(E, L(E, F))$  and  $D^2A(x) = D^3\omega = 0$ .

Our next result is the following, which states that if  $f : U \rightarrow F$  where  $U \overset{\circ}{\subseteq} E$ , and  $Df, DDf = D^2f$  exists and are continuous maps from  $U$  into  $L(E, F)$  and  $L(E, L(E, F))$  respectively, then  $D^2f(x)$  is a *symmetric bilinear map*. The proof is non-trivial, and relies on computing the 'Lie Bracket':

$$D^2f(x)(v, w) - D^2f(x)(w, v)$$

Which we will prove is equal to 0 for every  $x \in U$ , and  $v, w \in E$ .

**Note 2.2: Notation for open subsets**

The symbol ' $\overset{\circ}{\subseteq}$ ' means  $U$  is an open subset of  $E$ .

**Proposition 2.2: Second derivative is symmetric**

Let  $f \in C^2(U, F)$ , where  $U \overset{\circ}{\subseteq} E$  with the possibility that  $U = E$ . For every point  $x \in U$ , the *second derivative*  $D^2f(x)$  is bilinear and symmetric.

*Proof.* Fix  $x \in U \ni B(r) + x \overset{\circ}{\subseteq} U$ . We restrict our attention to vectors  $v, w \in E$  where  $|v|, |w| < r2^{-1}$  for now, so that the

$$\{x, x + w, x + v, x + v + w\} \subseteq U$$

We will denote the following quantity by  $\Delta$

$$\Delta = f(x + w + v) - f(x + w) - f(x + v) + f(x)$$

By rearranging terms, we see that  $\Delta$  can be approximated in two ways:

- Postponing the discussion about the the domain of  $y$ , set  $g(y) = f(y + v) - f(y)$  is  $C^2$ , and

$$\Delta = g(x + w) - g(x) \tag{80}$$

- Again, for  $y$  sufficiently close to  $x$ , define  $h(y) = f(y + w) - f(y)$ , and

$$\Delta = h(x + v) - h(x) \tag{81}$$

- To find the domain for  $y$ , an easy argument using the Triangle inequality gives us  $g, h \in C^2(B(r2^{-1}) + x, F)$ ,
- Leaving the computations of  $h$  as an exercise, we compute  $Dg$ , recall the shift map  $y \mapsto y + v$  commutes with  $D$ , and

$$Dg(y) = D(\tau_{-v}f)(y) - Df(y) = Df(y + v) - Df(y) \tag{82}$$

Using MVT twice, once on Equation (80) (the line segment  $x + tw$ ,  $0 \leq t \leq 1$  is contained in the domain of  $g$ ), and another time on Equation (82) (with  $y = x + tw$  in the integrand). We obtain:

$$\begin{aligned}\Delta &= g(x + w) - g(x) \\ &= \int_0^1 Dg(x + tw) \cdot w dt \\ &= \int_0^1 \int_0^1 D^2 f(x + tw + sv) \cdot v ds dt \cdot w \\ &= \int_0^1 \int_0^1 D^2 f(x + tw + sv) ds dt \cdot v \cdot w\end{aligned}$$

We can rewrite the application of  $v$  then  $w$  by  $\cdot(v, w)$ , and using the approximation  $D^2 f(x + tw + sv) \cdot (v, w) = D^2 f(x) \cdot (v, w) + \delta_1(tw, sv)$ . Integrating over  $s, t$  gives

$$\Delta = D^2 f(x) \cdot (v, w) + \int_0^1 \int_0^1 \delta_1(tw, sv) ds dt$$

### Note 2.3

The error term  $\delta_1$  in the integrand is given by

$$\delta_1(tw, sv) = D^2 f(x + tw + sv)(v, w) - D^2 f(x)(v, w)$$

for  $v, w$  sufficiently small and  $0 \leq s, t \leq 1$ .

A similar argument for  $h$  shows that  $\Delta = D^2 f(x) \cdot (w, v) + \int_0^1 \int_0^1 \delta_2(tw, sv) ds dt$ . Combining the two together, the following holds for all  $v, w$  sufficiently small:

$$D^2 f(x) \cdot (v, w) - D^2 f(x) \cdot (w, v) = \int_0^1 \int_0^1 \delta_1(tw, sv) ds dt - \int_0^1 \int_0^1 \delta_2(tw, sv) ds dt \quad (83)$$

To show the right hand side is 0, we will need the following note.

### Note 2.4

We wish to show the RHS of Equation (83) is 0. We begin by controlling the RHS and show that it is super-bilinear; meaning it shrinks after than the product  $|v||w|$ . Then, we will prove a lemma which will show the only bilinear map that satisfies this property is the 0 map.

- For  $j = 1, 2$ , relabel  $\delta = \delta_j$  for convenience. We can use the  $L^1$  inequality, to obtain the estimate

$$\left| \int_0^1 \int_0^1 \delta(tw, sv) ds dt \right| \leq \int_0^1 \int_0^1 |\delta(tw, sv)| ds dt \quad (84)$$

- $\delta(tw, sv)$  is controlled by  $|D^2 f(x + tw + sv) - D^2 f(x)| |v| |w|$ . Take  $y = tw + sv$ , then  $|y| \leq |tw| + |sv|$ . Hence,

$$|\delta_j| \leq |D^2 f(x + tw + sv) - D^2 f(x)| |v| |w| \quad (85)$$

- Let  $A$  denote the span of  $w, v$  for scalars  $s, t \in [0, 1]$ . In symbols,

$$A = \left\{ tw + sv, s, t \in [0, 1] \right\}$$

$A$  is clearly compact, and the continuity of  $D^2f$  means

$$R(v, w, \delta) = \sup_{y \in A} \left| D^2f(x + y) - D^2f(x) \right| \text{ is finite, and } \lim_{(v, w) \rightarrow 0} R(v, w, \delta) = 0 \quad (86)$$

See the remark after this proof for a generalized version of this 'compact linear combination' argument.

- Relabel  $R(v, w)$  to be the maximum across  $R(v, w, \delta_1)$  and  $R(v, w, \delta_2)$ .
- Combining Equations (84) to (86), we obtain the following bound on Equation (83)

$$\begin{aligned} \left| D^2f(x) \cdot (v, w) - D^2f(x) \cdot (w, v) \right| &\leq \left| \iint \delta_1(tw, sv) ds dt - \iint \delta_2(tw, sv) ds dt \right| \\ &\leq \iint |\delta_1| ds dt + \iint |\delta_2| ds dt \\ &\leq |v||w|R(v, w) \end{aligned} \quad (87)$$

The following Lemma gives a useful criterion to check when a multilinear map is identically 0.

### Lemma 2.1

Let  $E$  be a Banach space, and  $k \geq 1$  be an integer. If  $\lambda \in L(E^k, F)$  and there exists another map  $\theta : E^k \rightarrow F$  (defined perhaps on an open neighbourhood of the origin), such that

$$|\lambda(\underline{u}_k)| \leq |\theta(\underline{u}_k)| \cdot \prod |\underline{u}_k|$$

for all  $(\underline{u}_k)$  sufficiently small. And  $\lim_{(\underline{u}_k) \rightarrow 0} \theta(\underline{u}_k) = 0$ , then,  $\lambda = 0$ .

*Proof.* Fix arbitrary  $(\underline{u}_k) \in E^k$ , for  $s > 0$  sufficiently small, the left hand side of the equation reads

$$|s|^k |\lambda(\underline{u}_k)| \leq |\theta(s\underline{u}_k)| \cdot |s|^k \prod |\underline{u}_k|$$

The rest of the argument is Archimedean: divide by  $|s|^k$  and send  $s \rightarrow 0$  (while paying attention to the term with  $\theta$ ): perhaps after relabelling  $v_s = s\underline{u}_k$  for sufficiently small  $s$ , then  $|\theta(v_s)| \rightarrow 0$  as  $s \rightarrow 0$ . ■

■

### Remark 2.1

Generalization of the "compact linear combination" argument used above. Let  $(t_k) \subseteq \mathbb{C}^k$  or  $\mathbb{R}^k$ , and vectors  $v_k \in E$ . Suppose further  $(t_k) \subseteq A$  is compact in  $\mathbb{C}^k$  or  $\mathbb{R}^k$ . It is clear that if  $y = t_i v^i \in E$ ,

where the summation convention is in effect. Then,

$$|y| \lesssim_A |(v^k)|_{E^k}$$

Now, fix a continuous function  $f \in C(E, F)$ , we can approximate the maximum error over all such  $y$

$$\sup_{y \in B} |f(x + y) - f(x)| < \varepsilon \quad \forall |y| \lesssim_A |(v^k)| < \delta$$

where

$$B = \left\{ \sum t_i v^i, (t_k) \subseteq A, (v^k) \in E^k \right\}$$

### The $p$ -th derivatives

If  $f$  is  $p$  times differentiable, and  $f, Df, D^2f, \dots, D^p f$  are all continuous, then we say  $f \in C^p(E, F)$  (replacing  $E$  with an open subset of  $E$  if necessary). A *symmetric,  $k$ -linear* map between vector spaces  $V, W$  is a map  $A \in \mathcal{L}(V^k, W)$  such that for every  $k$ -permutation  $\theta \in S_k$ ,

$$A(v_k) = A(v_{\theta(k)})$$

#### Note 3.1

- We say a map  $F$  is *between* the spaces  $X$  and  $Y$  if  $F : X \rightarrow Y$ .
- $\mathcal{L}(V^K, W)$  denotes the space of  $k$ -linear maps from  $V$  to  $W$  that are not necessarily continuous.

#### Proposition 3.1

If  $f \in C^p(E, F)$ , then  $D^p f(x)$  is symmetric for every  $x \in E$ . (Replace  $E$  with an open set if necessary).

*Proof.* The main proof proceeds as follows. We will use induction on  $p$ , with  $p = 2$  serving as the base case. Our induction hypothesis is that for every  $f \in C^{p-1}(E, F)$ , for every permutation  $\beta \in S_{p-1}$ , at every point  $x \in E$ , for every possible choice of  $p - 1$  vectors  $(v_2, \dots, v_p) = (v_{1+\underline{p-1}})$ ,

$$D^{p-1} f(x)(v_{1+\underline{p-1}}) = D^{p-1} f(x)(v_{1+\beta(\underline{p-1})})$$

To prove the assertion for  $p$ , it suffices to show  $D^p f(x)(v_p)$  is invariant under transpositions of indices; since the transpositions generate  $S_p$ . Furthermore, the transpositions in  $S_p$  are generated by

- the transposition  $(1, 2, \dots) \mapsto (2, 1, \dots)$  where the omitted indices are held fixed, and
- the transpositions which leave the first index fixed:

$$(1, 1 + \underline{p-1}) \mapsto (1, 1 + \beta(\underline{p-1}))$$

where  $\beta \in S_{p-1}$

so it suffices to prove invariance under those two types of transpositions. Let  $g = D^{p-2}f$ , so  $g \in C^2(E, L(E^{p-2}, F))$ . Because the application of vectors (currying) on a multilinear map  $A \in L(E^p, F)$  is associative, illustrated as follows:

$$(A \cdot v_1) \cdot v_2 = A \cdot (v_1, v_2) = A(v_1, v_2, \cdot) \in L(E^{p-2}, F)$$

Then, let  $\lambda : L(E^{p-2}, F) \rightarrow F$  be the evaluation map at  $(v_3, \dots, v_p) = (v_{2+\underline{p-2}})$ . Using the base case on  $D^{p-2}f = g \in C^2(E, L(E^{p-2}, F))$ ,

$$(D^2g)(x)(v_1, v_2) = (D^2g)(x)(v_2, v_1) \implies \lambda((D^2g)(x)(v_1, v_2)) = \lambda((D^2g)(x)(v_2, v_1))$$

But  $\lambda$  is the map that *applies* the rest of the vectors, and

$$(D^2g)(x)(v_1, v_2) \cdot (v_{2+\underline{p-2}}) = (D^2g)(x)(v_2, v_1) \cdot (v_{2+\underline{p-2}}) \quad (88)$$

Since  $D$  commutes with continuous linear maps (and  $\lambda$  is continuous because  $(v_{2+\underline{p-2}})$  is fixed),

$$\lambda(D^2(D^{p-2}f)) = D(\lambda(D(D^{p-2}f))) = D(D\lambda \circ D^{p-2}f) = D^2(\lambda \circ D^{p-2}f) \quad (89)$$

Substituting Equation (88) for the rightmost hand side of Equation (89) gives the result.

### Note 3.2

There are no magic 'identifications' being made here. To be perfectly clear, for each  $x \in E$ ,  $g(x)$  is an element in  $L(E^{p-2}, F)$ , and  $(D^2g)(x) \in L(E^2, L(E^{p-2}, F))$ . Evaluating  $g$  at a point  $x$  gives a bilinear map that takes values in the Banach space  $L(E^{p-2}, F)$ .

For the second case, beginning from the induction hypothesis. If  $\theta$  is a  $p$ -permutation that leaves the first coordinate unchanged, then there exists a unique  $p-1$ -permutation  $\beta \in S_{p-1}$  such that

$$\begin{aligned} (\theta(\underline{p})) &= (1, \theta(1 + \underline{p-1})) \\ &= (1, 1 + \beta(\underline{p-1})) \end{aligned} \quad (90)$$

Using a similar argument as the first case, set  $g = D^{p-1}f$  and  $\lambda, \lambda' \in L(E^{p-1}, F)$  to be the evaluation maps of  $(v_1, v_{1+\underline{p-1}}) = (v_{\underline{p}})$  and  $(v_1, v_{1+\beta(\underline{p-1})})$  respectively. Rehearsing the same proof as before:

$$\begin{aligned} (D^p f)(x)(v_{\underline{p}}) &= D(\lambda D^{p-1} f)(x)(v_1) && \text{Equation (89)} \\ &= D(\lambda' D^{p-1} f)(x)(v_1) && \text{ind. hyp.} \\ &= (D^p f)(x)(v_{\theta(\underline{p})}) && \text{Equation (89)} \end{aligned}$$

This proves the induction step, and the proof is complete. ■

Before stating and proving Taylor's Theorem, an important remark on the 'postcomposition' of linear maps. Summarized in the following note.

### Note 3.3

Let  $f \in C^p(E, F)$ , and  $\lambda \in L^p(F, G)$ .  $\lambda$  induces a map between  $L(E^p, F)$  and  $L(E^p, G)$  by post-composing any multi-linear map  $A \in L(E^p, F)$  by  $\lambda$ . Denoting this map by  $\lambda_*$ ,

$$\lambda_* : L(E^p, F) \rightarrow L(E^p, G)$$

It is clear  $\lambda_*$  is linear and continuous. And its action on  $A$ , evaluated at  $(v_{\underline{p}}) \in E^p$  is given by

$$\lambda_*(A) \in L(E^p, G) \quad (\lambda_*(A))(v_{\underline{p}}) = \lambda(A(v_{\underline{p}})) = (\lambda \circ A)(v_{\underline{p}})$$

Now, recall that for  $p = 1$

$$[D(\lambda \circ f)](x) = \lambda[(Df)(x)]$$

To simplify the notation, we want to 'move' the evaluation  $x$  outside of the brackets, and somehow write  $x \mapsto \lambda[(Df)(x)]$  as one map between  $E$  and  $L(E, G)$ . We further *identify*  $\lambda$  as this map, so that

$$[D(\lambda \circ f)](x) = \lambda = (\lambda \circ Df)(x)$$

Dropping the  $x$  from the expression, for  $p \geq 2$  *assuming a similar formula holds*, then we write  $[D^p(\lambda \circ f)] = \lambda_* \circ D^p f$ . We make a final identification, of  $\lambda = \lambda_*$  (thereby conflating the two different maps, the first is a map from  $E$  to  $F$ , the second is a map from  $L(E^p, F)$  into  $L(E^p, G)$ ).

### Proposition 3.2

If  $p \geq 2$ ,  $f \in C^p(E, F)$ ,  $\lambda \in L(F, G)$ , then

$$D^p(\lambda \circ f) = \lambda \circ D^p f$$

Where we have identified  $\lambda$  as the same map that acts on  $L(E^p, F)$  to produce another map in  $L(E^p, G)$ , and suppressed the point  $x$ .

*Proof.* Use induction on  $p$ . ■

$$\ln(e^a) = e^{\ln(a)} = a$$



# Chapter A: Review of Topology

## Set Operations

This section is meant for reference.

### Proposition 1.1: Direct and Inverse Images of Maps

Let  $f : \mathbf{X} \rightarrow \mathbf{Y}$ , where  $\mathbf{X}$  and  $\mathbf{Y}$  are sets. If  $A \subseteq \mathbf{X}$ ,  $B \subseteq \mathbf{Y}$ , and  $\{E_\alpha\}$  is an indexed collection of subsets of  $\mathbf{X}$ ,  $\{G_\beta\}$  is an indexed collection of subsets of  $\mathbf{Y}$ , then

Direct images

$$f\left(\bigcap E_\alpha\right) \subseteq \bigcap f(E_\alpha) \quad \text{equality if injective} \quad (91)$$

$$f\left(\bigcup E_\alpha\right) = \bigcup f(E_\alpha) \quad (92)$$

Estimates

$$f\left(f^{-1}(B)\right) \subseteq B \quad \text{equality if surjective} \quad (93)$$

$$A \subseteq f^{-1}(f(A)) \quad \text{equality if injective} \quad (94)$$

Inverse images

$$f^{-1}\left(\bigcup G_\beta\right) = \bigcup f^{-1}(G_\beta) \quad (95)$$

$$f^{-1}\left(\bigcap G_\beta\right) = \bigcap f^{-1}(G_\beta) \quad (96)$$

$$f^{-1}(B^c) = \left(f^{-1}(B)\right)^c \quad (97)$$

### Proposition 1.2: Composition of Maps

Let  $h = g \circ f$ , we assume this composition is well defined.

- If  $h$  is a surjection, then  $g$  is a surjection,
- If  $h$  is an injection, then  $f$  is an injection.

*Proof.* Take the contrapositive. ■

### Proposition 1.3: Left and Right inverses

Let  $F : \mathbf{X} \rightarrow \mathbf{Y}$ ,

- $F$  is surjective if and only if there exists right inverse  $G : \mathbf{Y} \rightarrow \mathbf{X}$ ,

$$F \circ G = \text{id}_{\mathbf{Y}}$$

if  $A \subseteq \mathbf{X}$ ,

$$G^{-1}(A) \subseteq F(A)$$

- $F$  is injective if and only if there exists a left inverse  $H : F(\mathbf{X}) \rightarrow \mathbf{X}$

$$H \circ F = \text{id}_{\mathbf{X}}$$

and if  $B \subseteq Y$ ,

$$F^{-1}(B) \subseteq H(B)$$

## Topological Spaces

This section will roughly follow Munkres text on General Topology, in particular we hope to cover Chapters 2, 3, 4 and 9. The rest of the Chapters should be covered proper by the subsequent section.

### Definition 2.1: Topology

Let  $X$  be a non-empty set. A topology  $\mathcal{T}$  on  $X$ , sometimes denoted by  $\mathcal{T}_X$  is a family of subsets of  $X$ ,

- $\{\emptyset, X\} \subseteq \mathcal{T}$ ,
- If  $U_1$  and  $U_2$  are elements of  $\mathcal{T}$ , so is their intersection.
- If  $\{U_\alpha\}$  is an arbitrary family of sets in  $\mathcal{T}$ , their union is also contained in  $\mathcal{T}$  as an element.

We call the elements of  $\mathcal{T}$  open sets. The complements of elements in  $\mathcal{T}$  are closed sets.

## Basis of a Topology

### Definition 3.1: Basis of a topology

A basis  $\mathbb{B}$  is a family of subsets of  $X$ , that satisfies:

- Every  $x \in X$  belongs (as an element) in some  $V \in \mathbb{B}$ .
- If  $B_1$  and  $B_2$  are basis elements, such that their intersection is non-empty. Then every  $x \in B_1 \cap B_2$  induces a  $B_3 \in \mathbb{B}$  with

$$x \in B_3 \subseteq B_1 \cap B_2$$

This roughly means a basis is 'finitely' fine at every point in  $x$ .

If  $\mathbb{B}$  is a basis, it 'generates' a topology  $\mathcal{T}$  through

$$\mathcal{T} = \left\{ U \subseteq X, \forall x \in U, x \in B \subseteq U \text{ for some } B \in \mathbb{B} \right\} \quad (98)$$

Notice this is equivalent to  $\mathcal{T}$  is the collection of all unions of basis elements in  $\mathbb{B}$ .

### Proposition 3.1

Let  $\mathbb{B}$  be a basis as defined in Definition 3.1, then  $\mathcal{T}$  as defined in Equation (98) is a valid topology on  $X$ . And every member of  $\mathcal{T}$  is and is precisely the union of elements in  $\mathbb{B}$ .

*Proof.* Every point in  $X$  belongs in some basis element, so  $X \in \mathcal{T}$ , so does  $\emptyset$ . Next, if  $U_1$  and  $U_2$  are in  $\mathcal{T}$ , then

$$\begin{cases} x \in U_1 \rightarrow x \in B_1 \subseteq U_1 \\ x \in U_2 \rightarrow x \in B_2 \subseteq U_2 \end{cases} \implies x \in B_3 \subseteq B_1 \cap B_2 \subseteq U_1 \cap U_2$$

for some  $B_3 \in \mathbb{B}$ , so  $\mathcal{T}$  is closed under finite intersections (perhaps after a standard induction argument).

If  $\{U_\alpha\} \subseteq \mathcal{T}$ , and  $x$  belongs in the union of all  $U_\alpha$ , then  $x \in B_\alpha \subseteq U_\alpha$ , which is a subset of the entire union. So the union over  $U_\alpha$  is again contained in  $\mathcal{T}$ , and  $\mathcal{T}$  is a topology on  $\mathbf{X}$ .

It is worth noting that  $\mathbb{B} \subseteq \mathcal{T}$ . Finally, if  $U \in \mathcal{T}$ ,

$$U = \bigcup_{x \in U} B_x$$

where  $B_x$  is the basis element taken to satisfy  $x \in B_x \subseteq U$ . Every point in  $U$  is included in some  $B_x$ , and hence is included in the union. For the reverse inclusion, notice the union of subsets of  $U$  is again a subset of  $U$ .

Now, if  $E \subseteq \mathbf{X}$  is the union of basis elements in  $\mathbb{B}$ , if  $E$  is non-empty, then every point  $x \in E$  belongs in some  $B_x$ . Recycling the previous argument, and we see that  $E$  is open in  $\mathcal{T}$ . If  $E$  is empty, we define the 'union' of no sets as the empty set. So  $\mathcal{T}$  is precisely the collection of all unions of basis elements  $\mathbb{B}$ . ■

We are now in a position to compare the relative 'fineness' of topologies.

**Definition 3.2: Fineness of topologies**

If  $\mathcal{T}'$  and  $\mathcal{T}$  are both topologies on some non-empty set  $\mathbf{X}$ . We say  $\mathcal{T}'$  is finer than  $\mathcal{T}$ , or  $\mathcal{T}$  is coarser than  $\mathcal{T}'$  if

$$\mathcal{T}' \supseteq \mathcal{T}$$

**Proposition 3.2**

If  $\mathbb{B}$  and  $\mathbb{B}'$  are bases for  $\mathcal{T}'$  and  $\mathcal{T}$ , the following are equivalent:

- $\mathcal{T}'$  is finer than  $\mathcal{T}$ ,
- If  $B$  is an arbitrary basis element in  $\mathbb{B}$ , then every point  $x \in B$  induces a basis element in  $\mathbb{B}'$  with

$$x \in B' \subseteq B$$

*Proof.* Suppose  $\mathcal{T}'$  is finer than  $\mathcal{T}$ . Notice  $\mathbb{B} \subseteq \mathcal{T}'$  as well. By Equation (98), each  $x \in B$  induces a  $B' \in \mathbb{B}'$

$$x \in B' \subseteq B$$

Conversely, fix any open set  $U \in \mathcal{T}$ , and for each  $x \in U$ ,

$$x \in B' \subseteq B \subseteq U$$

Applying Definition 3.1 tells us  $U$  is open in  $\mathcal{T}'$ . ■

The last of the big three 'generating' definitions for topologies will be the sub-basis. It simply means the first condition (but not necessarily the second, is satisfied in Definition 3.1

**Definition 3.3: Sub-basis of a topology**

A sub-basis  $\mathcal{S} \in \mathbb{P}(\mathbf{X})$  is a family of subsets of  $\mathbf{X}$  that satisfies one property. Any point  $x$  in  $\mathbf{X}$  belongs to at least one member of  $\mathcal{S}$ .

A sub-basis can be upgraded to a basis by collecting all of its finite intersections.

**Proposition 3.3**

Let  $\mathcal{S}$  be a sub-basis of  $\mathbf{X}$ , then the collection of all finite intersections of  $\mathcal{S}$  forms a basis  $\mathbb{B}$  of  $\mathbf{X}$ .

*Proof.* Every point in  $\mathbf{X}$  lies in some element of  $\mathcal{S}$ , hence in some element of  $\mathbb{B}$ . The second basis property is immediate, since  $\mathbb{B}$  is closed under finite intersections. ■

## Product Topology

We will start with products of a finite collection of topological spaces.

**Definition 4.1: Finite Product of Topological Spaces**

Let  $(\mathbf{X}, \mathcal{T}_{\mathbf{X}})$  and  $(\mathbf{Y}, \mathcal{T}_{\mathbf{Y}})$  be topological spaces. The product topology (denoted by  $\mathcal{T}_{\mathbf{X} \times \mathbf{Y}}$ ) on  $\mathbf{X} \times \mathbf{Y}$  is defined as the topology generated by the basis

$$\mathbb{B}_{\mathbf{X} \times \mathbf{Y}} = \left\{ U \times V, (U, V) \in \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}} \right\} \quad (99)$$

Since bases are easier to describe than topologies, we have the following statement concerning the basis of the product topology.

**Proposition 4.1**

If  $\mathbb{B}_{\mathbf{X}}$  and  $\mathbb{B}_{\mathbf{Y}}$  are bases for  $\mathcal{T}_{\mathbf{X}}$  and  $\mathcal{T}_{\mathbf{Y}}$ , then the product topology (as described in Definition 4.1) is also generated by

$$\mathcal{M} = \left\{ U \times V, (U, V) \in \mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \right\} \quad (100)$$

*Proof.* We will introduce (and use) the technique of 'double inclusion' by proving that the topologies generated are both finer than the other. Let us denote the topology generated by  $\mathcal{M}$  in Equation (100) by  $\mathcal{T}_{\mathcal{M}}$ .

Since  $\mathbb{B}_{\mathbf{X}} \times \mathbb{B}_{\mathbf{Y}} \subseteq \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}$ , if  $U \times V \in \mathcal{M}$  as in Equation (100), then we can pick the same 'open rectangle' again. We trivially have

$$x \in \underbrace{U \times V}_{\text{member of } \mathcal{T}_{\mathbf{X}} \times \mathcal{T}_{\mathbf{Y}}} \subseteq U \times V$$

and by Proposition 3.2,  $\mathcal{T}_{\mathbf{X} \times \mathbf{Y}}$  is finer than  $\mathcal{T}_{\mathcal{M}}$ .

Fix any set  $U \times V \in \mathbb{B}_{\mathbf{X} \times \mathbf{Y}}$ , and if  $(p, q) \in U \times V$ , each coordinate induces basis elements from  $\mathbb{B}_{\mathbf{X}}$  and  $\mathbb{B}_{\mathbf{Y}}$ , more precisely:

$$\begin{cases} p \in U \implies p \in \text{Basis element of } \mathbb{B}_{\mathbf{X}} \subseteq U \\ q \in V \implies q \in \text{Basis element of } \mathbb{B}_{\mathbf{Y}} \subseteq V \end{cases} \implies (p, q) \in \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{X}}} \times \underbrace{\quad}_{\text{in } \mathbb{B}_{\mathbf{Y}}} \subseteq U \times V$$

by Proposition 3.2,  $\mathcal{T}_M$  is finer than  $\mathcal{T}_{X \times Y}$  and  $\mathcal{T}_{X \times Y} = \mathcal{T}_M$ . ■

## Continuity

### Definition 5.1: Continuous maps $C(X, Y)$

Let  $f$  be a map from  $X$  to  $Y$ . It is called *continuous* if  $f^{-1}(U)$  is open in  $X$  for every open set  $U$  in  $Y$ . We denote the set of continuous functions from  $X$  to  $Y$  by  $C(X, Y)$ .

### Proposition 5.1: Continuity preserving operations

The composition of continuous functions is again continuous, and the product of continuous functions is again continuous.

*Proof.* Suppose  $f \in C(X, Y)$  and  $g \in C(Y, Z)$ . Fix an open set  $U \subseteq Z$ . Then  $g^{-1}(U)$  is open in  $Y$ , hence

$$f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U) \quad \text{is open in } X$$

Next, let  $\{f_\alpha\}_{\alpha \in A}$  be a collection of continuous functions, where each  $f_\alpha \in C(X_\alpha, Y_\alpha)$ . Let us write

$$X \triangleq \prod X_\alpha \quad \text{and} \quad Y \triangleq \prod Y_\alpha$$

and the projection maps:

$$\pi_\alpha^X : X \rightarrow X_\alpha, \quad \text{For every } x \in X, \pi_\alpha^X(x) = x(\alpha) \in X_\alpha$$

similarly for  $\pi_\alpha^Y : Y \rightarrow Y_\alpha$ . The product function  $F = \prod f_\alpha$ , is defined through its behaviour 'on each coordinate'

$$\pi_\alpha^Y \circ F = f_\alpha \circ \pi_\alpha^X \tag{101}$$

A function  $F : X \rightarrow \prod Y_\alpha$  is continuous iff  $\pi_\alpha^Y \circ F$  is continuous for each  $\alpha \in A$ . By Equation (101), it is clear that each  $\pi_\alpha^Y \circ F$  is continuous, since the right member is the composition of two continuous functions, which is again continuous by the first part of this proof, therefore  $F$  is continuous. ■

### Definition 5.2: Open/Closed Maps

Let  $f : X \rightarrow Y$  be a map (not necessarily continuous), it is called *open* (resp. *closed*) if for every open (resp. closed) set  $E \subseteq X$ ,  $f(E)$  is open (resp. closed).

Clearly, the composition of open (resp. closed) maps is again open (resp. closed).

## Quotient Topology

## Product Topology

The Cartesian Product of an arbitrary family of topological spaces, if equipped with the product topology, preserves a lot of the structure. If  $\{X_\alpha\}_{\alpha \in A}$  is a family of topological spaces which are \_\_\_\_\_, then  $\prod X_\alpha$  is \_\_\_\_\_. Replace \_\_\_\_\_ with:

1. Hausdorff, (Folland)
2. Regular,
3. Connected, (Munkres chp23, exercise 10)
4. First countable, if  $A$  is countable,
5. Second countable, if  $A$  is countable,
6. Compact (Tynchonoff's Theorem, Folland)

**Proposition 7.1: Product of Closed sets again Closed**

The product of closed sets is again closed. More concretely, if  $\{E_\alpha\}_{\alpha \in A}$  is a family of sets such that  $E_\alpha \subseteq X_\alpha$ , then

$$\prod \overline{E_\alpha} = \overline{\prod E_\alpha}$$

**Connectedness**

**Definition 8.1: Connectedness**

A topological space  $X$  is connected if  $U$  and  $V$  are disjoint open subsets whose union is  $X$ , then at least one of  $U$  or  $V$  is empty.

See Folland Exercise 4.10 for more properties.

**Definition 8.2: Path-connectedness**

A topological space  $X$  is path-connected if for any two pair of points  $x, y \in X$ . There exists a continuous function  $f : [a, b] \rightarrow X$ , with  $f(a) = x$  and  $f(b) = y$ .

**Definition 8.3: Connected component**

The connected components of  $X$  is the family of equivalence classes on  $X$ , where  $x \sim y$  if there is a connected subspace of  $X$  that contains both of them.

**Proposition 8.1**

Continuous functions map connected spaces to connected spaces (in the subspace topology).

*Proof.* Let  $X$  and  $Y$  be topological spaces and  $f : X \rightarrow Y$  be continuous. If  $f(X)$  is disconnected, then we can find  $U$  and  $V$ , open and disjoint in  $\mathcal{T}_{f(X)}$  such that

$$U \cup V = f(X) \implies f^{-1}(U) \cup f^{-1}(V) = X$$

where  $f^{-1}(f(X)) = X$ . Both  $f^{-1}(U)$  and  $f^{-1}(V)$  are open, non-empty, and are pairwise disjoint. So  $X$  is separated. ■

**Proposition 8.2**

Let  $(X_\alpha, \mathcal{T}_\alpha)$  be a family of connected topological spaces indexed by  $\alpha \in A$ . Then  $\prod_{\alpha \in A} X_\alpha$  is disconnected in the product topology.

*Proof.* We will attempt the contrapositive. Suppose  $\prod_{\alpha \in A} X_\alpha$  is disconnected, then ■

**Interiors and closures**

**Definition 9.1: Interior of a set**

$A^\circ$  is defined to be the largest open subset of  $A$ ,

$$A^\circ = \bigcup_{\substack{U \text{ open,} \\ U \subseteq A}} U$$

**Corollary 9.1**

The union of subsets of  $A$  is again a subset of  $A$ , therefore Corollary 9.1 implies  $A^\circ \subseteq A$  for any  $A \subseteq X$ .

**Definition 9.2: Closure of a set**

and  $\bar{A}$  is the smallest closed superset of  $A$ ,

$$\bar{A} = \bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K$$

**Proposition 9.1**

The complement of the closure is the interior of the complement, or equivalently:  $(\bar{A})^c = A^{co}$

*Proof.* Taking complements, and the substitution  $U = K^c$  reads

$$\begin{aligned} (\bar{A})^c &= \left[ \bigcap_{\substack{K \text{ closed,} \\ A \subseteq K}} K \right]^c \\ &= \bigcup_{\substack{K \text{ closed,} \\ K^c \subseteq A^c}} K^c \\ &= \bigcap_{\substack{U \text{ open,} \\ U \subseteq A^c}} U \\ &= A^{co} \end{aligned}$$

■



**Remark 9.1**

Personally, I remember this as pushing the complement inside and flipping the bar to a c!

## Neighbourhoods

The concept of a neighbourhood allows us to characterize the interior of a set 'locally'.

**Definition 10.1: Neighbourhood (not necessarily open)**

A neighbourhood of  $x \in \mathbf{X}$  is a set  $U \subseteq \mathbf{X}$  where  $x \in U^\circ$ . The set of neighbourhoods for a point  $x \in \mathbf{X}$  will sometimes be denoted by  $\mathcal{N}(x)$ .

**Proposition 10.1: Characterization of the interior**

If  $W = \left\{ x \in \mathbf{X}, \text{ there exists a neighbourhood } U \text{ of } x, U \subseteq A \right\}$ , then  $W = A^\circ$ .

*Proof.* If  $x \in A^\circ$ , then  $A$  is a neighbourhood of  $x$ , and  $A \subseteq A$ , so  $x \in W$ . Conversely, if  $x$  is a member of  $W$ , it has a neighbourhood  $U \subseteq A$  (not necessarily open). By monotonicity of the interior,

$$x \in U^\circ \subseteq A^\circ$$

and  $x \in A^\circ$ . ■

It is easy to see that  $A$  is open  $\iff A^\circ = A \iff A$  is a neighbourhood of itself.

- The first equivalence follows from:

$$E \subseteq \mathbf{X} \implies E^\circ \subseteq E$$

and if  $A$  is an open set, it is an open subset of itself, by Corollary 9.1  $A \subseteq A^\circ$ . If  $A^\circ = A$ , then it suffices to show that  $A^\circ$  is open. Which it is, since it is the arbitrary union of open sets.

- To prove the second equivalence: suppose  $A^\circ = A$ , then each  $x \in A$  has a neighbourhood contained (as a subset) in  $A$ , namely  $A$  itself. (This statement is hard to parse, the reader is encouraged to really work through this and be honest).

$$x \in A^\circ \subseteq A \implies A \subseteq A^\circ$$

so  $A$  is a neighbourhood of itself. Conversely, if  $A \subseteq A^\circ$ , then  $A = A^\circ$ , since the reverse inclusion follows immediately from Corollary 9.1.

## Adherent points

Similar to the neighbourhood, the concept of an adherent point of a set allows us to speak of the closure in more concrete terms. The following definition is key in understanding the relationship between the closure, interior, and the boundary.

**Definition 11.1: Adherent point of a set**

Let  $A \subseteq X$ ,  $x \in X$  is an adherent point of  $A$  if every neighbourhood  $U$  of  $x$  intersects  $A$ . In symbols,

$$U \cap A \neq \emptyset, \quad \forall U \in \mathcal{N}(x)$$

**Proposition 11.1: Characterization of the closure**

Let  $A \subseteq X$ , and let  $W$  be the set of adherent points of  $A$ , then  $\overline{A} = W$

*Proof.* Suppose  $x \notin W$ , then there exists a neighbourhood  $U$  of  $x$  where

$$U \cap A = \emptyset \iff U \subseteq A^c$$

this is exactly the definition of the interior of  $A^c$ , so  $x \in A^{co}$  and recall (from Proposition 9.1) that  $(\overline{A})^c = A^{co}$ , so  $x \notin \overline{A}$ . For the reverse inclusion, read the proof backwards, by flipping  $\forall \rightarrow \exists$  within the set, and we see that

$$W^c = A^{co} = (\overline{A})^c$$

■

## Dense and nowhere dense subsets

**Definition 12.1: Dense subset**

A subset of a topological space  $E \subseteq X$  is dense if  $\overline{E} = X$ .

**Definition 12.2: Nowhere dense subset**

A subset of a topological space  $E \subseteq X$  is nowhere dense if  $\overline{E}^o = \emptyset$ .

This means  $E$  is dense in none of the (non-trivial) open subspaces of  $X$ .

**Proposition 12.1**

$E$  is dense in  $X$  iff for every non-empty, open set  $U \subseteq X$ ,  $U \cap E \neq \emptyset$ .

*Proof of Proposition 12.1.* Suppose  $E$  is dense, then  $\overline{E} = X$ . Every point of  $X$  is an adherent point of  $E$ . Let  $U \subseteq X$  be a non-empty open set. If  $x \in U$  then  $U$  is a neighbourhood of  $x$ , thus  $U$  intersects  $E$ . Conversely, suppose every non-empty open set  $U$  intersects  $E$ . Fix any point  $x \in X$ , and any neighbourhood  $U$  of  $x$ .  $U$  has a non-empty interior (because it must contain  $x$ ). But  $U^o$  is a non-empty open set, therefore  $\emptyset \neq U^o \cap E \subseteq U \cap E$  ■

**Proposition 12.2**

Let  $f : X \rightarrow X$  be a homeomorphism.  $E$  is nowhere dense iff  $f(E)$  is nowhere dense.

*Proof.* Since  $f^{-1}$  is a homeomorphism, suppose  $\overline{f^{-1}(E)}^o \neq \emptyset$ , there exists a non-empty, open subset  $U \subseteq \mathbf{X}$  with

$$\overline{f^{-1}(E)} \cap U = U$$

The direct image yields

$$f\left(\overline{f^{-1}(E)} \cap U\right) = f(U)$$

since  $f$  is a bijection (injectivity is necessary here), it commutes with intersections.

$$f(\overline{f^{-1}(E)}) \cap f(U) = f\left(\overline{f^{-1}(E)} \cap U\right) = f(U) \quad (102)$$

and  $f$  is continuous, so  $f(\overline{A}) \subseteq \overline{f(A)}$  for any  $A \subseteq \mathbf{X}$ . For the reverse inclusion,  $f$  is a closed map, so  $f(\overline{A})$  is a closed superset of  $f(A)$  so

$$f(\overline{A}) = \overline{f(A)}$$

Take  $A = f^{-1}(E)$ , and  $f(\overline{f^{-1}(E)}) = \overline{f(f^{-1}(E))} = \overline{E}$ . From eq. (102), we see that

$$\overline{E} \cap f(U) = f(U)$$

$f(U)$  is a non-empty open subset of  $\mathbf{X}$ , since  $f$  is an open map, so  $E$  is not no-where dense. The reverse implication can be proven by replacing  $f$  with  $f^{-1}$ . ■

## Urysohn's Lemma

### Proposition 13.1: Folland Theorem 4.14

Suppose that  $A$  and  $B$  are disjoint closed subsets of the normal space  $X$ , and let  $\Delta = \{k2^{-n} : n \geq 1 \text{ and } 0 < k < 2^n\}$  be the set of dyadic rationals in  $(0, 1)$ . There is a family  $\{U_r : r \in \Delta\}$  of open sets such that

1.  $A \subseteq U_r \subseteq B^c$  for every  $r \in \Delta$ ,
2.  $\overline{U_r} \subseteq U_s$  for  $r < s$ , and
3. For every  $r < s$ ,  $\overline{U_r} \subseteq U_s$

*Proof.* The goal of this proof is to show that for every  $r \in \Delta$ , there exists a open  $U_r$  that satisfies the above. As usual for these types of proofs we will proceed by induction. We can divide the problem by 'layers' (as I will hereinafter explain).

Let us suppose that for some  $N \geq 1$  that all previous  $U_r$  in previous layers have been constructed properly, meaning if  $r = k/2^n$ , then for every  $1 \leq n \leq N - 1$ , we have

$$r = \frac{k}{2^n}, 1 \leq n \leq N - 1, 1 \leq k \leq 2^{n-1}$$

And by 'constructed properly', we mean that for each  $U_r$ ,

- $A \subseteq U_r \subseteq B^c$  and
- $U_r \in \mathcal{T}_X$

Then for this fixed layer  $N \geq 1$ , we only have to construct the  $U_{k/2^N}$  for every odd  $k$ , this is because if  $k$  is an even number, then  $k = 2j$  and  $r = 2j/2^N = j/2^{N-1}$  and for this particular  $U_r$  is already constructed. So for every odd  $k = 2j + 1$ , the sets of the form  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  are already defined, and satisfy

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

For every  $k - 1 \neq 0$  and  $k + 1 \neq 1$ . (We will consider these cases later). We claim that for every pair of open sets,  $E_1, E_2 \in \mathcal{T}_X$ , then there exists some open set  $G \in \mathcal{T}_X$  such that if  $(E_1, E_2) \in H \subseteq (\mathcal{T}_X \times \mathcal{T}_X)$  where  $H$  is defined as the set

$$H = \left\{ (E_1, E_2) \subseteq (\mathcal{T}_X \times \mathcal{T}_X) : \overline{E_1} \cap E_2^c = \emptyset \right\}$$

Then there exists some  $G = \mathcal{J}(E_1, E_2) \in \mathcal{T}_X$  such that

$$E_1 \subseteq \overline{E_1} \subseteq G \subseteq \overline{G} \subseteq E_2$$

Now consider any any  $(E_1, E_2) \in H$ , then this pair induces a pair of disjoint sets  $\overline{E_1}$  and  $E_2^c$  since

$$\overline{E_1} \subseteq E_2 \implies \overline{E_1} \cap E_2^c = \emptyset$$

And by normality, there exists disjoint open sets  $G_1, G_2$  such that

- $\overline{E_1} \subseteq G_1 \in \mathcal{T}_X$
- $E_2^c \subseteq G_2 \in \mathcal{T}_X$
- $G_1 \cap G_2 = \emptyset \implies G_1 \subseteq G_2^c \subseteq E_2$
- Since  $G_2^c$  is a closed set that contains  $G_1$  as a subset,  $\overline{G_1} \subseteq G_2^c \subseteq E_2$

It is at this point that we will make no further mention of  $G_2$  (so we may discard the notion of  $G_2$  in our minds). Let us now replace  $G$  with  $G_1$  then it is an easy task to verify that  $G = G_1 = \mathcal{J}(E_1, E_2)$  has the required properties.

Now define for every odd  $k$ , since  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (we note in passing that  $\mathcal{J}$  is not a function as the set  $G$  may not be unique).

$$U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$$

Then, if  $U_{(k-1)/2^N}$  and  $U_{(k+1)/2^N}$  is 'well constructed' we have

$$A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$$

Therefore  $U_{k/2^N} = \mathcal{J}(U_{(k-1)/2^N}, U_{(k+1)/2^N})$  sits 'right inbetween' the two sets so that

- $A \subseteq \overline{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$  and
- $\overline{U}_{k/2^N} \subseteq U_{(k+1)/2^N} \subseteq B^c$

Combining the above two estimates will give us a 'well constructed'  $U_{k/2^N}$  for every  $k-1 \neq 0$  and  $k+1 \neq 1$ . Now let us deal with the remaining pathological cases.

If  $k-1$  so happens to be 0, then no  $r \in \Delta$  satisfies  $r = 0/2^N$ , and we substitute

$$\bar{U}_0 = A, \quad \text{or alternatively, } U_0 = A^c$$

Then  $U_0 \in \mathcal{T}_X$ ,  $\bar{U}_0 = A \subseteq B^c$ . It is at this point that we must mention that  $0, 1 \notin \Delta$ , so  $U_0$  and  $U_1$  do not have to obey the rules we have laid out for  $U_{r \in \Delta}$ .

Now if  $k+1$  is equal to  $2^N$  (this makes  $r = (k+1)/2^N = 1$ ) we define

$$U_1 = B^c \in \mathcal{T}_X$$

With this, for every  $0 \leq m \leq 2^N - 1$ ,  $U_{m/2^N}$  must satisfy

$$\bar{U}_{m/2^N} \subseteq B^c = U_1$$

And the pair  $(U_{(k-1)/2^N}, U_{(k+1)/2^N}) \in H$  (even for when  $N = 1$ , since  $A = \bar{U}_0 \subseteq U_1 = B^c$ ) and a corresponding  $U_{k/2^N} = \mathcal{J}(\cdot, \cdot)$  such that

- $A \subseteq \bar{U}_{(k-1)/2^N} \subseteq U_{k/2^N}$
- $\bar{U}_{(k+1)/2^N} \subseteq B^c$

Now as a final step, we complete the base case for when  $N = 1$ . We would only have to construct for  $k = 1$ , since

$$U_{1/2} = \mathcal{J}(U_0, U_1) = \mathcal{J}(A, B^c)$$

Apply the induction step, and the proof is complete, at long last. ■

**Proposition 13.2: Folland Theorem 4.15: Urysohn's Lemma**

Urysohn's Lemma. Let  $X$  be a normal space, if  $A$  and  $B$  are disjoint closed subsets of  $X$ , then there exists a  $f \in C(X, [0, 1])$  such that  $f = 0$  on  $A$  and  $f = 1$  on  $B$ .

*Proof.* Let  $r \in \Delta$  be as in Lemma 4.14, and set  $U_r$  accordingly except for  $U_1 = X$ . Define

$$f(x) = \inf\{k : x \in U_k\}$$

Let us also write  $W = \{k : x \in U_k\}$ , Then for every  $x \in A$  we have  $f(x) = 0$ , since by the construction of the 'union' function in Lemma 4.14, for each  $r \in \Delta \cap (0, 1)$ ,

$$x \in A \subseteq U_r \implies f(x) \leq r$$

Since  $r > 0$  is arbitrary, and  $0 \in W$ , we can use a classic  $\varepsilon$  argument. If  $f(x) > 0$  then there exists some  $0 < r < f(x)$  by density of the dyadic rationals on the line, if  $f(x) < 0$  then this implies that there exists some  $f(x) < r < 0$  such that  $x \in U_r$ , but no  $r \in \Delta$  can be negative, hence  $f(x) = 0$ .

Now, for every  $x \in B$ , since  $A$  and  $B$  are disjoint, and  $A \subseteq U_r \subseteq B^c$ , then for every  $x \in B$  means that  $x$  is not a member of any  $U_r$ , but we set  $U_1 = X$ . Since none of the  $r \in (0, 1)$  is a member of the set we

are taking the infimum, and  $x \in U_1 = X$ . The  $\varepsilon$  argument follows: suppose for every  $\varepsilon > 0$ ,  $(1 - \varepsilon) \notin W$ , and  $1 \in W$ , then  $f(x) = 1$ .

Since  $x \in U_1 = X$ , for every  $x \in X$ ,  $f(x) \leq 1$ , and  $f(x)$  cannot be negative as  $r > 0$  for every  $r \in \Delta$ . So  $0 \leq f(x) \leq 1$ . Now we have to show that this  $f(x)$  is continuous. The remainder of the proof is divided into two parts. We would like to show that the inverse images of the half lines are open in  $X$ . So  $f^{-1}((-\infty, \alpha)) \in \mathcal{T}$  and  $f^{-1}((\alpha, +\infty)) \in \mathcal{T}$ .

Suppose that  $f(x) < \alpha$ , so  $\inf W < \alpha$ , and using the density of  $\Delta$ , there exists an  $r$ ,  $f(x) < r < \alpha$  such that  $x \in U_r$  such that  $x \in \bigcup_{r < \alpha} U_r$ . So  $f^{-1}((-\infty, \alpha)) \subseteq \bigcup_{r < \alpha} U_r$ .

Fix an element  $x \in \bigcup_{r < \alpha} U_r$ , this induces an  $r$  such that  $\inf W \leq r < \alpha$  therefore  $f(x) < \alpha$ , and  $\bigcup_{r < \alpha} U_r \subseteq f^{-1}((-\infty, \alpha))$ .

For the second case, suppose that  $f(x) > \alpha$ , then  $\inf W > \alpha$ , and there exists an  $r$  (by density) such that  $\inf W > r > \alpha$  such that for every  $k \in W$ ,  $k \neq r$ . Therefore  $x \notin U_r$ , but by density again, and using the property of the onion function: for every  $s < r$  we get  $\overline{U_s} \subseteq U_r$ , taking complements (which reverses the estimate) — we have  $x \notin \overline{U_s}$ , but  $(\overline{U_s})^c$  is open in  $X$ . It immediately follows that

$$x \in f^{-1}((\alpha, +\infty)) \implies x \in (U_r)^c \subseteq (\overline{U_s})^c \subseteq \bigcup_{s > \alpha} (\overline{U_s})^c$$

So  $f^{-1}((\alpha, +\infty))$  is a subset of  $\bigcup_{s > \alpha} (\overline{U_s})^c$ . To show the reverse, fix an element  $x$  in the union, then this induces some  $x \in (\overline{U_s})^c \subseteq (U_s)^c$ . Then for this  $s > \alpha$ ,  $(-\infty, s)$  contains no elements of  $W$ . This is because for every  $p < s$  implies that  $(U_s)^c \subseteq (U_p)^c$ , so  $p \notin W$ . Our chosen  $s$  is a lower bound for  $W$ , and  $\alpha < s \leq \inf W = f(x)$ .

Since all of the inverse images from the generating set of  $(\mathbb{R}, \mathcal{T}_{\mathbb{R}})$  are open in  $X$ , using Theorem 4.9 finishes the proof. ■

Notes on the construction of the countable 'onion' sequence within a normal space  $\mathbf{X}$ .

If  $\mathbf{X}$  is a normal space, and  $A$  and  $B$  are disjoint closed subsets, then we can easily find an open  $U$  with

$$A \subseteq U \subseteq \overline{U} \subseteq B^c \tag{103}$$

We say that  $U$  hides in  $B^c$  if the closure of  $U$  is contained in  $B^c$ . Define  $\Delta_n = \left\{ k2^{-n}, 1 < k < 2^n \right\}$ , so that  $\Delta_n \subseteq (0, 1)$  for all  $n \geq 1$ . Notice

$$\Delta_1 \supseteq \cdots \supseteq \Delta_n \supseteq \Delta_{n+1}$$

and the even indices for  $\Delta_{n+1}$  are contained in  $\Delta_n$ . Suppose  $\Delta_n$  is well defined, it suffices to choose the odd indices for  $\Delta_{n+1}$ . If  $r = j2^{-(n+1)}$ , where  $j$  is odd, then  $r$  sits in between precisely two elements in  $\Delta_n \cup \{0, 1\}$ . If  $r$  sits between an endpoint, then define  $\overline{U_0} = A$ , and  $B^c = U_1$ . And denote the closest left and neighbours by  $s, t$  respectively. If  $s < r < t$ , it is clear that  $\overline{U_s}$  and  $U_t^c$  are disjoint closed sets.

Use the 'normal space' construction to obtain an superset of  $\overline{U_s}$  that hides in  $U_t$ , denote this open set by  $U_r$ , and similar to Equation (103)

$$\overline{U_s} \subseteq U_r \subseteq \overline{U_r} \subseteq U_t$$

Now that the construction of this sequence is complete, we wish to prove Urysohn's Lemma. Let  $A$  and  $B$  be disjoint closed sets. And define

$$f(x) = \inf \left\{ r \in \Delta \cup \{1\}, x \in U_r \right\}$$

where  $U_1 = \mathbf{X}$ . So that  $0 \leq f(x) \leq 1$  is immediate. If  $x \in A$ , then  $x$  is in all  $U_r$ , and by density of  $\Delta \subseteq (0, 1)$ , we have  $f(x) = 0$ . Conversely, if  $x \in B$  then  $x \notin U_r$  for all  $r \in \Delta$ , if  $E$  denotes the indices in  $\Delta$  where  $x \in U_s$  when  $s \in E$ ,

$$(-\infty, r) \subseteq E^c \iff E \subseteq [r, +\infty) \iff \inf(E) \geq r \quad (104)$$

Send  $r \rightarrow 1$  and  $f(x) = 1$ . Thus  $f|_A = 0$  and  $f|_B = 1$ .

To show continuity, it suffices to show that the inverse images of the open half  $\left\{ (x > \alpha), (x < \alpha) \right\}_{\alpha \in \mathbb{R}}$  lines are indeed open in  $\mathbf{X}$ . Let  $\alpha$  be fixed. And if  $x \in \{f < \alpha\}$ , we can 'wiggle' the infimum towards the right (towards  $\alpha$ ), and using density of  $\Delta$  within  $(0, 1)$ , there exists a  $r \in E$  that satisfies  $f(x) < r < \alpha$ . This is equivalent to

$$x \in \bigcup_{r < \alpha} U_r$$

If there exists an  $r < \alpha$  st  $x$  belongs to  $U_r$  as an element, then  $f(x) \leq r < \alpha$ .

If  $f(x) > \alpha$ , then  $(-\infty, \alpha) \subseteq E^c$ , by Equation (104). Suppose  $\alpha < 1$ , otherwise  $\{f > \alpha\} = \emptyset$ . Wiggle  $f(x)$  to the left and obtain an  $r \in \Delta$ ,  $\alpha < r < f(x)$  with  $x \notin U_r$ . By density again, take any  $s < r$  by a small amount (st  $s > \alpha$ ,  $s \in \Delta$ ), and

$$\overline{U}_s \subseteq U_r \iff U_r^c \subseteq \overline{U}_s$$

so that  $x \in \overline{U}_s^c$  for some  $s > \alpha$ . This is equivalent to

$$x \in \bigcup_{s > \alpha} \overline{U}_s^c$$

Conversely, if  $x \notin \overline{U}_s^c$  for some  $s > \alpha$ , since  $\{U_r\}$  (thus  $\{\overline{U}_r\}$ ) is increasing, and  $x \notin U_r$  for every  $r \leq s$ . Hence,

$$(-\infty, s] \subseteq E^c \iff E \subseteq (s, +\infty) \iff f(x) \geq s > \alpha$$

## Compactness

Compactness is one of the most important concepts in topology and analysis.

### Definition 14.1: Compact topological space

A topological space  $\mathbf{X}$  is compact if every open covering  $\{U_\alpha\}$  contains a finite subcover. That is, if  $\{U_\alpha\}$  is an arbitrary collection of open sets, then

$$\mathbf{X} = \bigcup_{\alpha \in A} U_\alpha \implies \bigcup_{j \leq n} U_{\alpha_j}$$

**Definition 14.2: Compact set**

$E \subseteq \mathbf{X}$  is compact if it is compact in the subspace topology.

**Definition 14.3: Precompact set**

$E \subseteq \mathbf{X}$  is precompact if its closure is compact (as a subset).

**Definition 14.4: Paracompact space**

A topological space  $\mathbf{X}$  is paracompact if every open covering of  $\mathbf{X}$  has a locally finite open refinement that covers  $\mathbf{X}$ .

**Definition 14.5: Locally finite collection of sets**

Let  $\mathcal{A}$  be a collection of subsets of  $\mathbf{X}$ . It is called locally finite, if at every point  $p \in \mathbf{X}$ , we can find a neighbourhood  $U$  of  $p$  (not necessarily open), that intersects only finitely many members of  $\mathcal{A}$ . In symbols,

$$U \cap E = \emptyset \quad \text{for all but finitely many } E \in \mathcal{A}$$

We do not require  $\mathcal{A}$  to be a cover of  $\mathbf{X}$ , nor do we require  $\mathcal{A}$  to be a collection of open sets.

**Definition 14.6: Countably locally finite**

A collection  $\mathbb{B}$  is countably locally finite if it is the countable union of locally finite families.

$$\mathbb{B} = \bigcup_{\mathbb{N}}^{\text{countable union}} \mathbb{B}_n, \quad \text{where each } \mathbb{B}_n \text{ is a locally finite collection}$$

**Definition 14.7: Refinement**

If  $\mathcal{A}$  is a collection of sets,  $\mathbb{B}$  is a refinement of  $\mathcal{A}$  if every element  $B \in \mathbb{B}$ , induces an element  $A \in \mathcal{A}$ , such that  $B \subseteq A$ .

**Remark 14.1: Intuition for refinements**

If  $\mathbb{B}$  is a refinement of  $\mathcal{A}$ , we can use the 'absolute continuity' muscle. For each element in  $\mathbb{B}$  is dominated by some element (through subset inclusion) in  $\mathcal{A}$ . Recall, if  $\nu$  and  $\mu$  are non-negative measures, then  $\nu \ll \mu$  if for every measurable set  $E \in \mathcal{M}$ ,  $\mu(E) = 0 \implies \nu(E) = 0$ .

A refinement of a family of sets is another family of sets, whose elements are dominated by some other element in the un-refined family. *Refining families makes them 'smaller', cover less area.*



**Proposition 14.1**

Compact Hausdorff spaces are normal, compact subsets of Hausdorff spaces are closed, and closed subsets of compact sets are again compact.

**Properties of Compact Spaces**

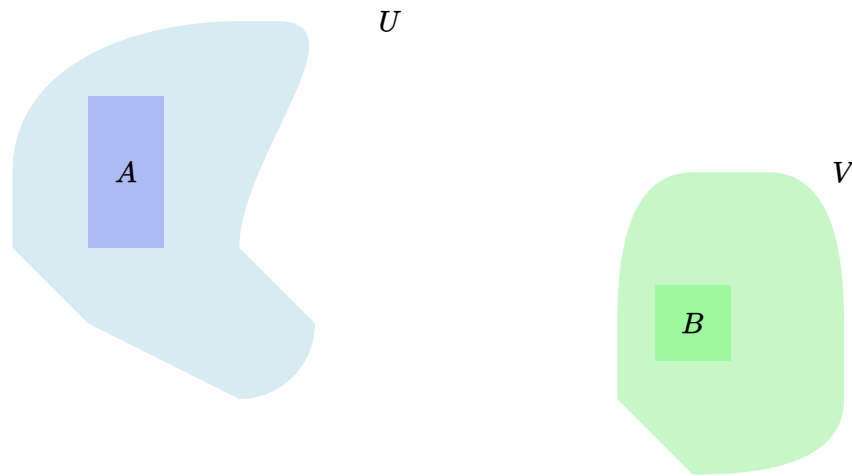


Figure 2: Closed sets  $A$  and  $B$  within open sets  $U$  and  $V$ , respectively.

**Proposition 15.1**

Let  $\mathbf{X}$  and  $\mathbf{Y}$  be topological spaces.

- (a) If  $F \in C(\mathbf{X}, \mathbf{Y})$ , and  $\mathbf{X}$  is compact, then  $F(\mathbf{X})$  is compact.
- (b) If  $\mathbf{X}$  is compact and  $F \in C(\mathbf{X}, \mathbb{R})$ , then  $F(\mathbf{X})$  is bounded, and  $F$  attains its supremum and infimum on  $\mathbf{X}$ .
- (c) A finite union of compact subspaces of  $\mathbf{X}$  is again compact.
- (d) If  $\mathbf{X}$  is Hausdorff, and  $A, B$  are disjoint, compact subspaces of  $\mathbf{X}$ , there exists open  $U$  and  $V$ , (see fig. 2).
- (e) Every closed subset of a compact space is compact.
- (f) Every compact subset of a Hausdorff space is closed.
- (g) Every compact subset of a metric space is bounded.
- (h) Every finite product of compact spaces is compact.
- (i) Every quotient of a compact space is compact.

*Proof of Proposition 15.1 Part A.* Let  $f \in C(\mathbf{X}, \mathbf{Y})$  with  $\mathbf{X}$  compact. Fix an open cover of  $f(\mathbf{X})$  in the

relative topology,

$$\{U_\alpha \cap f(\mathbf{X})\}_{\alpha \in A} \text{ covers } \mathbf{X}, U_\alpha \text{ open in } \mathbf{Y}$$

So that  $\bigcup f^{-1}(U_\alpha) = \bigcup f^{-1}(U_\alpha \cap f(\mathbf{X})) = \mathbf{X}$ . Since  $\{f^{-1}(U_\alpha)\}_{\alpha \in A}$  is an open cover for  $\mathbf{X}$ , this induces a finite subcollection of indices  $\{\alpha_1, \dots, \alpha_n\}$  with

$$\bigcup_{j=1}^n f^{-1}(U_{\alpha_j}) = \bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))$$

The direct image commutes with unions, therefore

$$f(\mathbf{X}) = f\left(\bigcup_{j=1}^n f^{-1}(U_{\alpha_j} \cap f(\mathbf{X}))\right) = \bigcup_{j=1}^n f\left(f^{-1}(U_{\alpha_j})\right) = \bigcup_{j=1}^n U_{\alpha_j}$$

■

*Proof of Proposition 15.1 Part B.* Let  $\mathbf{X}$  be compact, and  $f \in C(\mathbf{X}, \mathbb{R})$ , so that  $f(\mathbf{X}) \subseteq \mathbb{R}$  is compact. Compact subsets are closed and bounded in  $\mathbb{R}$ , let  $A = \sup f(\mathbf{X})$  and  $B = \inf f(\mathbf{X})$ . Both  $A$  and  $B$  are accumulation points of  $f(\mathbf{X})$ , so  $A = f(x)$  and  $B = f(y)$  for some  $x, y$  in  $\mathbf{X}$ . ■

*Proof of Proposition 15.1 Part C.* Let  $\mathbf{X}$  be a topological space, and  $K_1, \dots, K_n$  be compact subspaces. Denote  $K = \bigcup_{j=1}^n K_j$ . Let  $\{U_\alpha \cap K\}_{\alpha \in A}$  be an open cover for  $K$ , where  $U_\alpha$  is open in  $\mathbf{X}$ . We can pass the argument to each individual  $K_j$  as follows. Let  $1 \leq j \leq n$ , then  $\{U_\alpha \cap K_j\}_{\alpha \in A}$  is an open cover for  $K_j$ , so there exists a finite subcollection of indices  $I_j \subseteq A$ , (a finite subset of  $A$ ) whose open sets cover  $K_j$ . Repeat this process for each  $j$  and

$$I = \bigcup_{j=1}^n I_j \text{ is a finite subset of } A$$

with  $K_j \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K_j) \subseteq \bigcup_{\alpha \in I_j} (U_\alpha \cap K)$ . Taking the union over all  $K_j$  reads

$$K = \bigcup_{j=1}^n K_j \subseteq \bigcup_{j=1}^n \bigcup_{\alpha \in I_j} (U_\alpha \cap K) = \bigcup_{\alpha \in I} U_\alpha \cap K$$

■

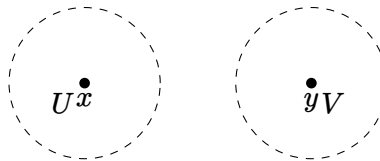


Figure 3: In a Hausdorff space, any two distinct points  $x$  and  $y$  can be separated by disjoint open neighbourhoods  $U$  and  $V$ .

*Proof of Proposition 15.1 Part D.* Let  $\mathbf{X}$  be Hausdorff. We first prove that compact subspaces of  $\mathbf{X}$  are closed. Indeed, if  $K$  is compact in  $\mathbf{X}$ , fix any  $x \in K^c$ . Let  $y$  range through the elements of  $K$ , then  $x \neq y$  induces a pair of disjoint open sets  $U_y$  and  $V_y$ , such that

- $x \in U_y$
- $y \in V_y$
- $U_y \cap V_y = \emptyset$
- See fig. 3

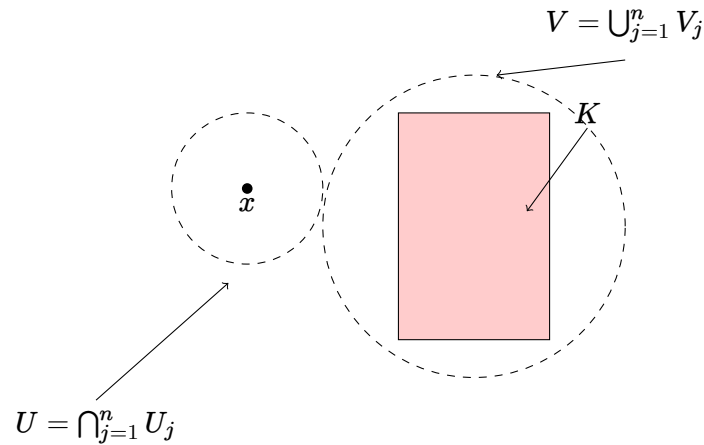


Figure 4: Compact sets are closed in Hausdorff spaces

Let  $V_y$  range through all possible  $y \in K$ , So that  $\{V_y\}_{y \in K}$  is an open cover. There exists a finite subcollection of 'anchor points' of  $K$ ,  $y_1, \dots, y_n$  that corresponds with  $\{V_{y_j}\}_{j=1}^n$ . A finite intersection of open sets is again open, so

$$U = \bigcap_{j=1}^n U_{y_j} \text{ is open}$$

Define  $V = \bigcup_{j=1}^n V_{y_j}$ , so  $V \subseteq K$  and  $U \cap V = \emptyset$  and  $x \in U \subseteq K^c$  (see fig. 4). Therefore  $K$  is closed.

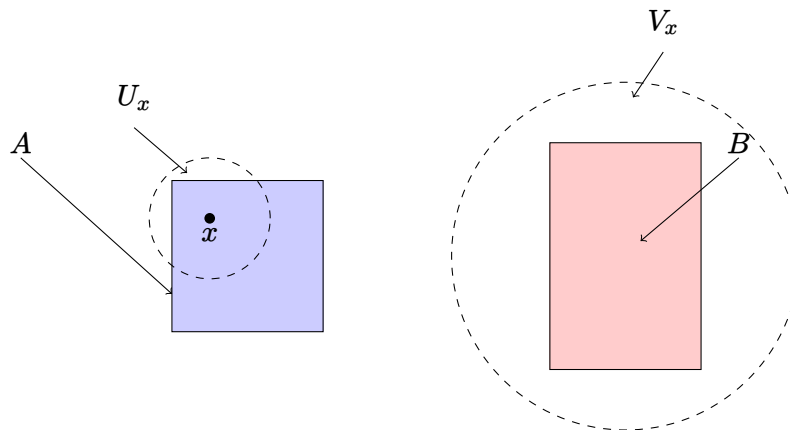


Figure 5: Closed sets  $A$  and  $B$ , point  $x$  in  $A$ , and disjoint neighbourhoods  $U$  around  $x$  and  $V$  around  $B$ .

Finally, if  $A$  and  $B$  are disjoint compact sets, then each  $x \in A \subseteq B^c$  induces neighbourhoods  $x \in U_x$ , and  $B \subseteq V_x$  (see fig. 5), let  $x$  range through all the elements of  $A$ . By compactness of  $A$ , this produces a finite subcover, and

$$U = \bigcup_{j=1}^n U_{x_j} \quad V = \bigcap_{j=1}^n V_{x_j}$$

are disjoint open sets that contain  $A$  and  $B$  respectively. ■

*Proof of Proposition 15.1 Part E.* Let  $K \subseteq \mathbf{X}$  be a closed set of a compact space. Let  $\{U_\alpha \cap K\}$  be an open cover for  $K$ , where each  $U_\alpha$  is open in  $\mathbf{X}$ . We can append an extra set  $K^c$  which is open in  $\mathbf{X}$ . The collection

$$W = \{U_\alpha\} \cup \{K^c\} \text{ covers } \mathbf{X}$$

so there exists a finite subcollection of  $W_1, \dots, W_n$  that cover  $\mathbf{X}$  (since  $\mathbf{X}$  is compact by itself). Remove  $K^c$  from this finite subcollection if it exists, and take the intersection with  $K$  for each element  $W_j$ , and

$$\{W_1 \cap K, \dots, W_n \cap K\} = \{U_1 \cap K, \dots, U_n \cap K\} \text{ covers } K$$

so  $K$  is compact. ■

*Proof of Proposition 15.1 Part F.* Proven in Part D. ■

*Proof of Proposition 15.1 Part G.* let  $K \subseteq \mathbf{X}$  be a compact subset of the metric space  $(\mathbf{X}, d)$ . Compact subsets of  $\mathbf{X}$  are totally bounded, and hence bounded. ■

*Proof of Proposition 15.1 Part H.* See Tychonoff's Theorem in Folland Chapter 4. ■

*Proof of Proposition 15.1 Part I.* Let  $\mathbf{X}$  and  $\mathbf{Y}$  be topological spaces and  $\pi : \mathbf{X} \rightarrow \mathbf{Y}$  be a quotient map. So that  $\mathbf{Y}$  is endowed with the quotient topology. So that  $\pi$  is a surjective continuous map. and  $\pi(\mathbf{X}) = \mathbf{Y}$ . Apply Part A, and we see that  $\mathbf{Y}$  is compact. ■

## Locally Compact Hausdorff Spaces

Compactness is an intrinsic topological property (in the subspace topology). We see from Proposition 4.25 that compact Hausdorff spaces are normal, which gives a sufficient condition for us to approximate and extend any continuous function; and allows us to extend certain 'local' properties to 'global' properties.

If given a Hausdorff space, not necessarily compact, the natural question is to ask 1) whether a topological space has 'enough' compact subsets to work with, and 2) whether we can embed a given topological space in a larger one to force it to be compact.

### Definition 16.1: LCH space

Let  $\mathbf{X}$  be a Hausdorff space. We call  $\mathbf{X}$  a LCH space if every point  $p \in \mathbf{X}$  admits a compact neighbourhood. That is, a compact set  $K$  whose interior contains  $p$ .

We note in passing that the above definition differs slightly from the usual 'local' definitions.

**Definition 16.2: Locally connected**

Let  $\mathbf{X}$  be a topological space, it is locally connected if for every  $x \in \mathbf{X}$ , and open neighbourhood  $U$  containing  $x$ , there exists a connected, open neighbourhood  $V$  of  $x$  such that  $x \in V \subseteq U$ .

**Definition 16.3: Locally path-connected**

Let  $\mathbf{X}$  be a topological space, it is locally connected if for every  $x \in \mathbf{X}$ , and open neighbourhood  $U$  containing  $x$ , there exists a path-connected, open neighbourhood  $V$  of  $x$  such that  $x \in V \subseteq U$ .

**Definition 16.4: Local homeomorphism**

$\mathbf{X}$  locally homeomorphic to  $\mathbb{R}^n$  if every point  $x \in \mathbf{X}$  belongs to a coordinate chart  $(U, \phi)$ , where  $U$  is an open neighbourhood of  $x$  and  $\phi$  is a homeomorphism from  $U \rightarrow \phi(U) \subseteq \mathbb{R}^n$ .

**Definition 16.5: Local diffeomorphism**

Let  $M$  be a smooth manifold and  $F \in C^\infty(M, N)$ .  $F$  is a local diffeomorphism if every  $p \in M$  in its domain induces a neighbourhood  $U \subseteq M$  with  $F|_U : U \rightarrow F(U)$  is a diffeomorphism (in the sense of two open sub-manifolds).

# Chapter B: Abstract Algebra

## Groups

### Definition 1.1: Semigroups, Monoids

A non-empty set  $G$  equipped with an associative binary operation  $G \times G \rightarrow G$  is called a semigroup. For every  $a, b, c \in G$ , we have

$$a(bc) = (ab)c \quad (105)$$

A *monoid* is a semigroup  $G$  which contains a *two-sided identity* element  $e \in G$  such that  $ae = ea$  for all  $a \in G$ . (not necessarily unique)

Monoids admit unique two-sided identities.

### Lemma 1.1: Monoids: unique identity

Let  $e$  and  $i$  be two-sided identities for a monoid  $G$ , then

*Proof.*

$$e = ei = i$$

■

### Definition 1.2: Group

A semigroup  $G$  is a group if every element  $a \in G$  admits a two-sided inverse  $a^{-1}$ . (not necessarily unique)

$$aa^{-1} = a^{-1}a = e$$

### Proposition 1.1: Properties of Groups (Hungerford: Theorem 1.2)

Let  $G$  be a group with identity  $e$ , which is unique by lem. 1.1. Then

(i)  $c \in G$  and  $cc = c$  implies  $c = e$ .

(ii) Left/Right cancellation:

$$\begin{cases} ab = ac \implies b = c \\ ba = ca \implies b = c \end{cases}$$

(iii) If  $a \in G$ , its two-sided inverse is unique.

(iv) Let  $a \in G$ , then the inverse of its two-sided inverse (uniqueness guaranteed by iii), is  $a$  itself; or  $(a^{-1})^{-1} = a$ .

(v) If  $a, b \in G$ , then the following equations in  $x, y$  admit unique solutions

$$\begin{cases} ax = b \\ ya = b \end{cases}$$

*Proof of Proposition 1.1.*

Proof of Part (i):

$$cc = c \implies (cc)c^{-1} = cc^{-1} \implies c(cc^{-1}) = e \implies ce = c = e$$

Proof of Part (ii): First claim:

$$\begin{aligned} ab = ac &\implies a^{-1}(ab) = a^{-1}(ac) \\ &\implies (a^{-1}a)b = (a^{-1}a)c \implies eb = ec \implies b = c \end{aligned}$$

Second claim is the same, just cancel from the right using  $aa^{-1} = e$  and associativity.

Proof of Part (iii): Suppose  $b$  and  $c$  are two-sided inverse for  $a$ , it follows from Part ii that

$$ab = ac \implies b = c = a^{-1}$$

Proof of Part (iv): From Part iii, the two-sided inverses of group elements exist and are unique, and  $a^{-1}a = aa^{-1}$  so  $a$  is an inverse for  $a^{-1}$ , and it is the only inverse.

Proof of Part (v): First equation: write  $ax = b = a(a^{-1}b)$ , left-cancelling reads  $x = a^{-1}b$ , uniqueness follows from Part ii. Second equation is similar. ■

### Lemma 1.2: Group: equality lemma

For any pair of elements  $a, b \in G$ ,  $a = b \iff ab^{-1} = e$ .

*Proof.* ( $\implies$ ):  $a = b \implies ab^{-1} = bb^{-1} = e$ . ( $\impliedby$ ):  $ab^{-1} = e \implies a(b^{-1}b) = eb \implies a = eb = b$ . ■

### Proposition 1.2: Semigroup: upgrade to group I (Hungerford Proposition 1.3)

Let  $G$  be a semigroup,  $G$  is also a group iff both of the conditions below hold

- Existence of a left-identity: there exists  $e \in G$  for every  $a \in G$ ,  $ea = a$ .
- Existence of left-inverses: for every  $a \in G$ , there exists a  $a^{-1} \in G$  with  $a^{-1}a = e$ , where  $e$  is any left-identity element.

*Proof.* ( $\impliedby$ ) is trivial. Suppose both conditions hold, notice the proof for Proposition 1.1 Part (i) we only used left-cancellation.  $cc = c \implies e$ . To prove  $a^{-1}$  is also a right-inverse for  $a$ , we can force it as follows:

$$(aa^{-1})(aa^{-1}) = a(a^{-1}a)a^{-1} = aea^{-1} = e \implies aa^{-1} = e$$

and  $a^{-1}$  is also a right-inverse, so every element  $a \in G$  admits a two-sided inverse denoted by  $a^{-1}$ . To show  $e$  is also a right-identity for any arbitrary element  $a \in G$ ,

$$\begin{aligned} ae &= a(a^{-1}a) && \text{left inverse} \\ &= (aa^{-1})a && \text{associativity} \\ &= ea && \text{right inverse} \\ &= a && \text{left identity} \end{aligned}$$

■



**Proposition 1.3: Semigroup: upgrade to group II (Hungerford Proposition 1.4)**

Let  $G$  be a semigroup,  $G$  is a group iff for every pair of elements  $a, b \in G$ , the equations in  $x$  and  $y$

$$\begin{cases} ax = b \\ ya = b \end{cases} \quad (106)$$

have solutions (not necessarily unique).

*Proof.* If  $G$  is a group, the existence of the solutions to eq. (106) follow from Proposition 1.1. We will attempt the contrapositive. Suppose  $G$  has no left identity, for every  $e \in G$  we can always find an element  $a \in G$  such that  $ea \neq a$ , but this is precisely the (first) equation for  $a = a$  and  $b = a$ .

Now suppose  $G$  has a left identity element (not necessarily unique). Fix  $e \in G$  as any left-identity, and suppose there is an element  $a \in G$  with no left inverse, so for every  $b \in G$ ,  $ba \neq e$ . But  $b$  is precisely the solution to the (second) equation with parameters  $a = a$  and  $b = e$ . The negation of Proposition 1.2 is precisely the negation of Proposition 1.3, and the proof is complete. ■

**Proposition 1.4: Hungerford Theorem 1.5**

Let  $R/\sim$  be an equivalence relation on a group  $G$ , such that it 'preserves' the group multiplication. More precisely,

$$\begin{cases} a_1 \sim a_2 \\ b_1 \sim b_2 \end{cases} \implies a_1 b_1 \sim a_2 b_2$$

Then the set  $G/R$  of all equivalence classes of  $G$  under  $R$  is a monoid under the binary operation defined by

$$(\bar{a})(\bar{b}) = \overline{ab} \quad \text{reads: the product of two classes is the class containing the product of any pair of elements from the two classes} \quad (107)$$

where  $\bar{a}$  denotes the equivalence class containing  $a$ . If  $G$  is a group, so is  $G/R$ , if  $G$  is an abelian group, so is  $G/R$ .

*Proof.* First, notice the binary operation in Equation (107) is well defined. It is independent of the equivalence class representatives chosen, as we have restriction on  $R$  that 'forces' the operation on  $G/R$  to be well defined. Indeed, let  $\bar{a}$  and  $\bar{b}$  be elements of  $G/R$ , if  $a_1, a_2 \in \bar{a}$ , and  $b_1, b_2 \in \bar{b}$ , by definition of  $R$ :

$$a_1 \sim a_2 \quad \text{and} \quad b_1 \sim b_2$$

by Equation (107),  $a_1 b_1 \sim a_1 b_2 \implies \overline{a_1 b_1} = \overline{a_1 b_2}$ .

Associativity is proven similarly, fix  $\bar{a}, \bar{b}, \bar{c} \in G/R$ , we pass the argument to any of the representatives of the three classes, so

$$(\bar{a}\bar{b})\bar{c} \triangleq \overline{ab}\bar{c} = \overline{(ab)c} = \overline{a(bc)} \triangleq \overline{a}b\bar{c} = \bar{a}(\bar{b}\bar{c})$$

Pass the argument to the representatives, let  $e$  denote the identity element in  $G$ , it is easily shown that  $\bar{e}$  is the identity element in  $G/R$ , similarly for two-sided inverses and commutativity of the binary operation. ■

## Homomorphisms

### Definition 2.1: Homomorphism

Let  $G$  and  $H$  be semigroups,  $f : G \rightarrow H$  is a semi-group *homomorphism* if for all  $a, b \in G$ ,

$$f(ab) = f(a)f(b) \quad (108)$$

### Definition 2.2: Monomorphism

Injective homomorphism.

### Definition 2.3: Epimorphism

Surjective homomorphism.

### Definition 2.4: Isomorphism

Bijjective homomorphism.

### Definition 2.5: Endomorphism

Homomorphism for which the domain and codomain (not the range) are equal; i.e  $H = G$ .

### Definition 2.6: Automorphism

Bijjective endomorphism.

### Definition 2.7: Kernel of a homomorphism

The kernel of  $f \in \text{Hom}(G, H)$  is defined

$$\text{Ker } f = \left\{ a \in G, f(a) = e \in H \right\} \quad (109)$$

as the set of elements in  $G$  that get sent to the identity of  $H$ .

### Proposition 2.1: Hungerford Theorem 2.3

Let  $G$  and  $H$  be groups and let  $f \in \text{Hom}(G, H)$ . Denote the identity elements of  $G$  and  $H$  by  $e_G$  and  $e_H$

- (i)  $f(e_G) = e_H$ ,
- (ii)  $f(a^{-1}) = (f(a))^{-1}$  for every  $a \in G$ .

(iii)  $f$  is a monomorphism iff  $\ker f = \{e_G\}$ ,

(iv)  $f$  is an isomorphism iff there exists a homomorphism  $f^{-1} : H \rightarrow G$  that is also a two-sided inverse for  $f$ . In symbols:

$$f \circ f^{-1} = \text{id}_H \quad \text{and} \quad f^{-1} \circ f = \text{id}_G \quad (110)$$

*Proof of Proposition 2.1.*

Proof of Part (i): We will use Proposition 1.1 (i). Since  $f(e_G) = f(e_G e_G) = f(e_G)f(e_G)$  in  $H$ , we see that  $f(e_G) = e_H$  and  $e_G \in \text{Ker } f$

Proof of Part (ii): Let  $a \in G$  be arbitrary, using Part (i), we can 'pass the multiplication' between  $f(a)$  and  $f(a^{-1})$  into  $G$ ,

$$f(a)f(a^{-1}) = f(e_G) = e_H \implies f(a^{-1}) = (f(a))^{-1}$$

Proof of Part (iii): Suppose  $\ker f = e_G$ . Let  $a, b \in G$  such that  $f(a) = f(b)$ . The equality lemma Lemma 1.2 tells us  $(f(a))^{-1} = f(b)$  and  $b = a^{-1}$ , so  $a = b$  by the Lemma again;  $f$  is injective.

Conversely, suppose  $f$  is injective, Part (i) tell us  $\{e_G\} \subseteq \ker f$ . Suppose  $a \in \ker f \subseteq G$ , but  $e_G \in \ker f$ , so  $f(a) = f(e_G) = e_H$  forces  $a = e_G$ , and  $\ker f = \{e_G\}$ .

Proof of Part (iv): ( $\Leftarrow$ ) is trivial since the existence of a (functional) two-sided inverse is equivalent to bijectivity. Suppose  $f$  is an isomorphism, and define  $f^{-1}$  as its two-sided (functional) inverse, it suffices to show that  $f^{-1} \in \text{Hom}(H, G)$ . Fix  $f(a)$  and  $f(b)$  as arbitrary elements in  $H$ . We can do this because  $f$  is a bijection, so every element in  $H$  has a unique 'representative' in  $G$ .

$$f^{-1}(f(a)) f^{-1}(f(b)) = ab = f^{-1}(f(ab)) = f^{-1}(f(a)f(b))$$

■

### Definition 2.8: Subgroup $H < G$

Let  $G$  be a group. If  $H$  is a non-empty subset of  $G$  that is closed under group operations, then it is a *subgroup* of  $G$ . We write  $H < G$ .

The *trivial subgroup* of  $G$  is  $\{e\}$  and consists of one element.  $H$  is called a *proper subgroup* if  $H \neq G$  and  $H \neq \{e\}$ .

### Proposition 2.2: Hungerford Exercise 9

Let  $f \in \text{Hom}(G, H)$ ,  $A < G$  and  $B < H$ ,

(i)  $\text{Ker } f$  is a subgroup of  $G$ ,

- (ii)  $f(A)$  is a subgroup of  $H$ ,
- (iii)  $f^{-1}(B)$  is a subgroup of  $G$ .

## Subgroups

### Proposition 3.1: Subgroup criteria (Hungerford Theorem 2.5)

A non-subset  $H \subseteq G$  is a subgroup iff for any  $a, b$  in  $H$ ,  $ab^{-1} \in H$ .

*Proof.* ( $\Leftarrow$ ): Choose  $a = b$ , then  $aa^{-1} = e \in H$  acts as the two-sided identity in  $H$ , and  $ea^{-1} = a^{-1} \in H$  for every  $a \in H$ . So  $H$  is a subgroup by Proposition 1.2. ( $\Rightarrow$ ): If  $H$  is a subgroup, then Proposition 1.2 tells us  $b^{-1} \in H$  for every  $b \in H$ , hence  $ab^{-1} \in H$  for elements  $a, b \in H$ . ■

### Corollary 3.1: Hungerford Corollary 2.6

If  $G$  is a group and  $\{H_i, i \in I\}$  is a nonempty family of subgroups, then their intersection  $H \triangleq \bigcap_{i \in I} H_i$  is again a subgroup in  $G$ .

*Proof.* Let  $a, b \in H$ , then  $ab^{-1} \in H_i$  for every  $i \in I$ , hence  $ab^{-1} \in \bigcap_{i \in I} H_i = H$ , and  $H$  is a subgroup by Proposition 3.1. ■

### Definition 3.1: Subgroup generated by $A \subseteq G$

Let  $A$  be a subset of  $G$ , the *subgroup generated by  $A$*  is the smallest subgroup  $H < G$  that contains  $A$  as a subset, denoted by  $\langle A \rangle$ .

Proposition 3.1 gives us an explicit formula for  $H$ ,

$$H = \bigcap_{\substack{H_i < G \\ A \subseteq H_i}} H_i$$

If  $A$  is finite, and  $H = \langle A \rangle$  is said to be *finitely generated*. We also write

$$H = \langle a_1, \dots, a_n \rangle = \langle \{a_1, \dots, a_n\} \rangle$$

If  $A$  consists of one element,  $\{a\} = A$ , then  $\langle a \rangle = \langle \{a\} \rangle$  is called the *cyclic group generated by  $a$* .

### Proposition 3.2: Hungerford Theorem 2.8

If  $G$  is a group and  $A \subseteq G$  is a non-empty subset, the subgroup generated by  $A$  is precisely the collection of all finite products (powers included), or

$$\langle A \rangle = \left\{ a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}, a_i \in A, n_i \in \mathbb{Z} \right\}$$

If  $A = \{a\}$ , then  $\langle A \rangle = \{a^k, k \in \mathbb{Z}\}$ .

*Proof.* Let  $W = \{a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}, a_i \in A, n_i \in \mathbb{Z}\}$ . We first show  $W$  is a subgroup of  $G$ . Indeed, if  $n_1 = n_2 = \cdots = n_t = 0$ , then  $a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t} = e$ . And for any  $b \in W$ ,

$$b = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t} \implies b^{-1} = a_t^{-n_t} \cdots a_2^{-n_2} a_1^{-n_1}$$

where each  $-n_i \in \mathbb{Z}$ , so  $b^{-1} \in W$  as well.  $\langle A \rangle$  is the smallest subgroup containing  $A$ , so  $\langle A \rangle \subseteq W$ . Conversely, fix an element  $b \in W$ , so  $b$  has the form

$$b = a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t}, a_i \in A, n_i \in \mathbb{Z}$$

and a simple induction will show each  $a_i^{n_i} \in \langle A \rangle$  for  $1 \leq i \leq t$ , so  $b \in \langle A \rangle$  and  $\langle A \rangle = W$ . ■

### Definition 3.2: Lattice of subgroups

Let  $\{H_i\}_{i \in I}$  be a collection of subgroups of  $G$ , then

$$\text{glb}\{H_i\} = \bigcap_{i \in I} H_i, \quad \text{lub} = \left\langle \bigcup_{i \in I} H_i \right\rangle$$

The collection of subgroups of  $G$  is a *complete lattice*.

## Cyclic groups

The proof for the following is straight-forward, the book separates the case into  $H = \langle 0 \rangle$  and  $H \neq \langle 0 \rangle$ , then  $H$  contains a non-zero element  $h \neq 0$ , then  $|h| \in \mathbb{N}^+$  is an element in  $H$  as well, so  $H$  contains a least positive element by invoking the Well Ordering Property. For the second half of the proof, we force  $r = 0$  by the Division Algorithm.

### Definition 4.1: Order of a subgroup $H < G$

The order of a subgroup  $H$  is its cardinality  $|H|$ .

### Definition 4.2: Order of an element $a \in G$

The order of an element  $a \in G$  is the order of  $\langle a \rangle$ .

### Proposition 4.1: Hungerford Theorem 3.1

Let  $\mathbb{Z}$  be equipped with its usual addition operation. Then every subgroup  $H < \mathbb{Z}$  is cyclic, either  $H = \langle 0 \rangle$  or  $H = \langle m \rangle$ . With  $m$  being the least positive integer in  $H$ . If  $H \neq \langle 0 \rangle$ , then  $H$  is infinite.

**Proposition 4.2: Hungerford Theorem 3.2**

Every infinite cyclic group  $G = \langle a \rangle$  is isomorphic to the additive group  $\mathbb{Z}$ , and every finite cyclic group of order  $0 < m < +\infty$  is isomorphic to the additive group  $\mathbb{Z}_m$ .

*Proof of Proposition 4.2.* Let  $f : \mathbb{Z} \rightarrow G$  and  $f(k) = a^k$ . By Proposition 3.2,

$$G = \left\{ a^k, k \in \mathbb{Z} \right\}$$

$f$  is clearly surjective and  $f \in \text{Hom}(\mathbb{Z}, G)$  is an easy exercise to verify. The proof splits into two parts

- (i)  $\text{Ker } f$  is trivial: By Proposition 2.1,  $f$  is an isomorphism from  $\mathbb{Z}$  to  $G$ , and  $G \cong \mathbb{Z} \implies |G| = |\mathbb{Z}|$ .
- (ii)  $\text{Ker } f$  is not trivial: Arguing as in Proposition 4.2,  $\text{Ker } f$  is a non-trivial subgroup of  $\mathbb{Z}$ , so it contains a least positive element  $m$ , and  $\text{Ker } f = \langle m \rangle$ .  $m$  is an element of  $\text{ker } f$ , so

$$f(m) = a^m = e \implies f(jm) = a^{jm} = \prod_{l=1}^j a^m = e, j \in \mathbb{Z} \quad (111)$$

Now suppose  $r$  and  $s$  are integers with  $f(r) = f(s)$ ,

$$\begin{aligned} a^r = a^s &\iff a^{r-s} = e \\ &\iff r - s \in \text{Ker } f \\ &\iff r - s \in \langle m \rangle \\ &\iff r \equiv s \pmod{m} \\ &\iff \bar{r} = \bar{s} \end{aligned}$$

where  $\bar{r}$  denotes the  $\mathbb{Z}_m$  equivalence class of  $r$ . Let  $\beta$  be a map from  $\mathbb{Z}_m \rightarrow G$ , such that

$$\beta(\bar{k}) = f(k) = a^k$$

This is well defined, since  $\beta$  is an invariant on each equivalence class, if  $r - s$  differ by a multiple of  $m$ , then Equation (111) states that  $\beta(\bar{r}) = \beta(\bar{s})$ .  $G$  is finite, as

$$G = \langle a \rangle = \left\{ a^k, k \in \mathbb{Z}, m < k < m \right\}$$

and the kernel of  $\beta$  is trivial, it is an isomorphism and  $\mathbb{Z}_m \cong G$ .

■

Cosets

Normal Subgroups

Isomorphism Theorems

# Chapter C: Algebraic Topology

## Homotopy

This section will follow Munkres Chapters 9 and 13 closely. Possibly other chapters as well.

### Definition 1.1: Path

A *path* is a continuous function from the unit interval  $f : [0, 1] \rightarrow \mathbf{X}$ . We say  $f$  is a *path from  $x_0$  to  $x_1$*  if  $f(0) = x_0$  and  $f(1) = x_1$ .

We denote the set of paths from  $x_0$  to  $x_1$  by  $\text{Path}(x_0, x_1)$ . If  $f \in \text{Path}(x_0, x_1)$ , we sometimes denote the *reversal of  $f$*  by  $\bar{f} \in \text{Path}(x_1, x_0)$ , where  $\bar{f}(s) \triangleq f(1 - s)$ .

### Definition 1.2: Loop

A *loop* at  $x_0 \in \mathbf{X}$  is a path that begins and ends at  $x_0$ , and  $\text{Loop}(x_0) \triangleq \text{Path}(x_0, x_0)$ . The constant path (or loop) at  $x_0$  is denoted by  $e_{x_0} : [0, 1] \rightarrow \mathbf{X}$ .

$$e_{x_0}(s) = x_0, \quad \forall s \in [0, 1]$$

### Definition 1.3: Homotopy of $C(\mathbf{X}, \mathbf{Y})$

Let  $f$ , and  $g$  continuous functions from  $\mathbf{X}$  to  $\mathbf{Y}$ .  $f$  and  $g$  are homotopic, denoted by  $f \approx g$  if there exists a continuous function  $F \in C(\mathbf{X} \times I, \mathbf{Y})$  where

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x) \quad (112)$$

where  $I = [0, 1]$ .

The function  $F$  is called the *homotopy between  $f$  and  $g$* .

If  $f \simeq h$ , where  $h$  is the constant function, we say  $f$  is *nullhomotopic*.

### Definition 1.4: Path Homotopy of $\text{Path}(x_0, x_1)$

Two paths  $f_0, f_1 \in \text{Path}(x_0, x_1)$  are said to be *path homotopic*, if there exists a continuous function  $F \in C(I \times I, \mathbf{X})$ , with

- $F$  is a *homotopy between  $f_0$  and  $f_1$*  (in the sense of Definition 1.3). For every  $s \in [0, 1]$ ,

$$F(s, 0) = f_0(s) \quad \text{and} \quad F(s, 1) = f_1(s) \quad (113)$$

- $F$  leaves the endpoints fixed. For every  $t \in [0, 1]$ , then

$$F(0, t) = x_0 \quad \text{and} \quad F(1, t) = x_1 \quad (114)$$

If  $f_0$  and  $f_1$  are path-homotopic, we write  $f_0 \simeq_p f_1$ .



- The function  $F \in C(I \times I, \mathbf{X})$  is called the *path homotopy between  $f_0$  and  $f_1$* .
- If  $f \in \text{Loop}(x_0)$  is path homotopic to the constant path  $e_{x_0}$ , then  $f$  is *nullhomotopic*.
- The relation  $\simeq_p$  is defined for paths that have the same initial and final points. So it is a relation on  $\text{Path}(x_0, x_1)$ .

**Proposition 1.1: Munkres Lemma 51.1**

The relations  $\simeq$  and  $\simeq_p$  are equivalence relations on  $C(\mathbf{X}, \mathbf{Y})$  and  $\text{Path}(x_0, x_1)$  respectively.

*Proof.* ( $f \simeq f$ ): Let  $f \in C(\mathbf{X}, \mathbf{Y})$ . Define

$$F : \mathbf{X} \times I \rightarrow \mathbf{Y} \quad \text{For every } t \in [0, 1], F(x, t) = f(x)$$

$F$  is continuous, since  $F = \pi_{\mathbf{X}} \circ (f \times \text{id}_{[0,1]})$ , where  $f \times \text{id}_{[0,1]}$  is the product of two continuous functions, which is again continuous by Chapter A. Moreover,  $F(x, 0) = f(x) = F(x, 1)$ , so  $F$  is a homotopy between  $f$  and itself.

( $f \simeq g \implies g \simeq f$ ): Let  $F$  be the homotopy between  $f$  and  $g$ . Let  $G$  be the 'reversal' in the second coordinate of  $F$ , meaning

$$G(x, t) = F(x, 1 - t) \quad \text{is continuous, since } G = F \circ (\text{id}_{\mathbf{X}} \times c)$$

where  $c : I \rightarrow I$  that maps  $t \mapsto 1 - t$  is continuous, so  $\text{id}_{\mathbf{X}} \times c$  is continuous; hence  $G$  is continuous. Notice for every  $x \in \mathbf{X}$ ,

$$G(x, 0) = F(x, 1) = g(x) \quad \text{and} \quad G(x, 1) = F(x, 0) = f(x)$$

therefore  $G$  is a homotopy between  $g$  and  $f$ .

( $f \simeq g, g \simeq h \implies f \simeq h$ ): Let  $F$  be the homotopy between  $f$  and  $g$ , and  $G$  be the homotopy between  $g$  and  $h$ . Define a function  $H : \mathbf{X} \times I \rightarrow \mathbf{Y}$  that morphs  $f$  into  $g$  on  $[0, 2^{-1}]$ , then  $g$  into  $h$  on  $[2^{-1}, 1]$

$$H(x, t) = \begin{cases} F(x, 2t - \lfloor 2t \rfloor) & \text{for } 0 \leq t \leq 2^{-1} \\ G(x, 2t - \lfloor 2t \rfloor) & \text{for } 2^{-1} \leq t \leq 1 \end{cases} \quad (115)$$

where  $\lfloor \cdot \rfloor$  denotes the *floor function*.

- $H$  is well defined on the overlap  $\mathbf{X} \times 2^{-1}$ , since  $F(x, 1) = G(x, 0) = g(x)$  at every  $x \in \mathbf{X}$ .
- If  $t = 0$ , then  $H(x, 1) = F(x, 0) = f(x)$ , and  $t = 1$  gives  $H(x, 1) = G(x, 1) = h(x)$ .
- Since  $H|_{\mathbf{X} \times [0, 2^{-1}]}$  and  $H|_{\mathbf{X} \times [2^{-1}, 1]}$  are continuous functions, and they agree on the overlap,  $H$  is continuous by the pasting Lemma, and defines a homotopy between  $f$  and  $h$ .

Now consider paths  $f, g, h$  in  $\text{Path}(x_0, x_1)$ , ( $f \simeq_p f$ ) is trivial. So is symmetry of  $\simeq_p$ , as the reversal in the second coordinate (see above) of the path homotopy between  $f$  and  $g$  is path homotopy between  $g$  and  $f$ .

Suppose  $f \simeq_p g$ , and  $g \simeq_p h$ . Let  $F$ , and  $G$  be the path homotopies between  $f, g$  and  $g, h$ . Write  $H$  as in Equation (115), it is a continuous function on  $I \times I \rightarrow \mathbf{X}$ , that satisfies

$$H(s, 0) = F(s, 0) = f(s) \quad \text{and} \quad H(s, 1) = G(s, 1) = h(s) \quad \text{for every } s \in [0, 1]$$

If  $s = 0$ , it is easy to see from Equation (115) that for every  $t \in [0, 1]$ ,

$$\begin{aligned} H(0, t) &= \begin{cases} F(0, 2t - \lfloor 2t \rfloor) = x_0 & \text{for } 0 \leq t \leq 2^{-1} \\ G(0, 2t - \lfloor 2t \rfloor) = x_0 & \text{for } 2^{-1} \leq t \leq 1 \end{cases} = x_0 \quad \text{and} \\ H(1, t) &= \begin{cases} F(1, 2t - \lfloor 2t \rfloor) = x_1 & \text{for } 0 \leq t \leq 2^{-1} \\ G(1, 2t - \lfloor 2t \rfloor) = x_1 & \text{for } 2^{-1} \leq t \leq 1 \end{cases} = x_1 \end{aligned}$$

So the endpoints remain fixed throughout the deformation in  $t$ , and  $H$  is a path homotopy between  $f$  and  $h$ . This proves transitivity.  $\blacksquare$

## Path and PathClass Products

### Definition 2.1: Product of Paths $f * g$

Let  $f \in \text{Path}(x_0, x_1)$  and  $g \in \text{Path}(x_1, x_2)$ , the product of  $f$  and  $g$ , denoted by  $f * g$  is another path from  $x_0$  to  $x_2$ . For  $s \in [0, 1]$ ,

$$(f * g)(s) \triangleq \begin{cases} f(2s - \lfloor 2s \rfloor) & \text{for } 0 \leq s \leq 2^{-1} \\ g(2s - \lfloor 2s \rfloor) & \text{for } 2^{-1} \leq s \leq 1 \end{cases} \quad (116)$$

Notice the similarities between Equations (115) and (116),

### Proposition 2.1: Properties of the Path Product

Let  $f \in \text{Path}(x_0, x_1)$  and  $g \in \text{Path}(x_1, x_2)$ , let  $k \in C(\mathbf{X}, \mathbf{Y})$ , then

- (i) Invariant under left-multiplication:  $f \simeq_p g \implies k \circ f \simeq_p k \circ g$ , where  $k \circ f$  and  $k \circ g$  are elements  $\text{Paths}$  from  $k(x_0)$  to  $k(x_1)$ , and if  $F$  be a path homotopy between  $f$  and  $g$ , then  $k \circ F$  is a path homotopy between  $k \circ f$  and  $k \circ g$ .
- (ii) If we redefine  $f \in \text{Path}(x_0, x_1)$ ,  $g \in \text{Path}(x_1, x_2)$ , and  $k$  be as above, then

$$k \circ (f * g) = (k \circ f) * (k \circ g)$$

*Proof.*

Proof of Part (i): It is clear that  $k \circ f$  and  $k \circ g$  are elements of  $\text{Path}(k(x_0), k(x_1))$ , and see Part (ii) for the proof of  $k \circ f \simeq_p k \circ g$ .

Proof of Part (ii): Let  $F$  be the path homotopy between  $f$  and  $g$ . The composition  $(k \circ F)$  is in  $C(\mathbf{X} \times I, \mathbf{Y})$ . Equation (113) reads

$$\begin{aligned} (k \circ F)(s, 0) &= k(F(s, 0)) = (k \circ f)(s) \quad \text{and} \\ (k \circ F)(s, 1) &= k(F(s, 1)) = (k \circ g)(s) \quad \text{for every } s \in [0, 1] \end{aligned}$$

and Equation (114) gives

$$\begin{aligned}(k \circ F)(0, t) &= k(F(0, t)) = k(x_0) \text{ and} \\ (k \circ F)(1, t) &= k(F(1, t)) = k(x_1) \text{ for every } t \in [0, 1]\end{aligned}$$

therefore  $k \circ F$  is a path homotopy between the paths  $k \circ f$  and  $k \circ g$ . ■

**Definition 2.2: Path Homotopy class  $[f]$**

Let  $f \in \text{Path}(x_0, x_1)$ , we define the *path homotopy class of  $f$*  as

$$[f] \triangleq \left\{ g \in \text{Path}(x_0, x_1), g \simeq_p f \right\}$$

**Definition 2.3: Product of PathClasses  $[f] * [g]$**

Let  $*$  :  $\text{PathClass}(x_0, x_1) \times \text{PathClass}(x_1, x_2) \rightarrow \text{PathClass}(x_0, x_2)$  be a binary operation, where

$$[f] * [g] \triangleq [f * g] \text{ is well defined.}$$

for arbitrary  $[f] \in \text{PathClass}(x_0, x_1)$  and  $[g] \in \text{PathClass}(x_1, x_2)$ . This means it is independent of the representative chosen. More formally, if  $f \simeq_p f' \in \text{Path}(x_0, x_1)$ , and  $g \simeq_p g' \in \text{Path}(x_1, x_2)$ , then  $f * g \simeq_p f' * g'$ .

**Proposition 2.2: Properties of the PathClass product**

Let  $[f]$ ,  $[g]$  and  $[h]$  be PathClasses from and to the points  $x_0, x_1, x_2$ . Then

1. Associativity:  $([f] * [g]) * [h] = [f] * ([g] * [h])$ ,
2. Left and Right identities: if  $[f] \in \text{PathClass}(x_0, x_1)$ ,  $e_{x_0}, e_{x_1}$  denote the constant paths on  $x_0$  and  $x_1$  (the initial and final points of any  $f \in [f]$ ), then

$$[e_{x_0}] * [f] = [f] \quad \text{and} \quad [f] * [e_{x_1}] = [f]$$

3. Left and Right inverses: let  $[\bar{f}]$  be the PathClass containing the reversal of  $f$  (see Definition 1.1) for the definition, then

$$[\bar{f}] * [f] = [e_{x_1}] \quad \text{and} \quad [f] * [\bar{f}] = [e_{x_0}]$$

4. Generalized Associativity: if  $\{[f_j]\}_{j \leq n}$  is a sequence of PathClasses, such that  $[f_j] \in \text{PathClass}(x_{j-1}, x_j)$ , then

$$\prod [f_j] \triangleq [f_1] * [f_2] * \cdots * [f_n] \text{ is a well-defined object}$$

meaning we can place the brackets wherever we want.

*Proof.* We will give an outline for the proof of Generalized Associativity, the rest are trivial. Let  $\{[f_j]\}$  be defined as above. If  $\{a_j\}_{j=0}^n$ , and  $\{b_j\}_{j=0}^n$  are 'cell partitions' of the unit interval (in the sense of the

Riemann integral), meaning

$$0 = a_0 < a_1 < \cdots < a_n = 1, \quad \text{and} \quad 0 = b_0 < b_1 < \cdots < b_n = 1$$

We agree to define the following

- the lengths of each cell  $l_{a_j} \triangleq a_j - a_{j-1}$  and  $l_{b_j} \triangleq b_j - b_{j-1}$ , and
- the cells themselves are denoted by  $\text{cell}(a_j) = [a_{j-1}, a_j]$ ,  $\text{cell}(b_j) = [b_{j-1}, b_j]$ ,
- $p \in \text{Path}(0, 1)$ , where  $p$  is given explicitly by

$$p(s) = \sum_{j=1}^n \chi_{\text{cell}(a_j) \setminus \{a_{j-1}\}} \left( \frac{l_{b_j}}{l_{a_j}} (s - a_j) + b_j \right)$$

It is clear  $p$  is continuous, and for  $j = 1, \dots, n$ ,

$$p|_{\text{cell}(a_j)} \text{ is the positive linear map from } \text{cell}(a_j) \text{ to } \text{cell}(b_j)$$

Using the same line of argumentation as in the proof for associativity, we see that any two 'ways' of bracketing the expression has no impact on the path-homotopy class. ■

## Fundamental Group

### Definition 3.1: Fundamental group $\pi_1(\mathbf{X}, x_0)$

Let  $x_0 \in \mathbf{X}$ , the *fundamental group of  $\mathbf{X}$  relative to (base point)  $x_0$*  is denoted by  $\pi_1(\mathbf{X}, x_0) = \text{PathClass}(x_0, x_0)$ .

### Definition 3.2: Isomorphism induced by $\text{Path}(x_0, x_1)$

Suppose  $\alpha \in \text{Path}(x_0, x_1)$ , we define a map  $\hat{\alpha} : \pi_1(\mathbf{X}, x_0) \rightarrow \pi_1(\mathbf{X}, x_1)$ , with

$$\hat{\alpha}([f]) = [\bar{\alpha}] * [f] * [\alpha]$$

where  $\bar{\alpha}$  is the reversal of  $\alpha$ . We call  $\hat{\alpha}$  the *isomorphism induced by  $\alpha$*  (Munkres Theorem 52.1).

*Isomorphism proof.* Let  $[f]$  and  $[g]$  be elements of  $\pi_1(\mathbf{X}, x_0)$ , then

$$\begin{aligned} \hat{\alpha}([f] * [g]) &= ([\bar{\alpha}] * [f] * [\alpha]) * ([\bar{\alpha}] * [g] * [\alpha]) \\ &= [\bar{\alpha}] * ([f] * [g]) * [\alpha] \\ &= \hat{\alpha}([f]) * \hat{\alpha}([g]) \end{aligned}$$

and  $\hat{\alpha}$  is a homomorphism. We claim inverse of  $\hat{\alpha}$  is  $\hat{\bar{\alpha}}$ . Fix  $[f] \in \pi_1(\mathbf{X}, x_0)$ ,  $[g] \in \pi_1(\mathbf{X}, x_1)$ , then

$$(\hat{\bar{\alpha}} \circ \hat{\alpha})([f]) = [\alpha] * ([\bar{\alpha}] * [f] * [\alpha]) * [\bar{\alpha}] = [f]$$

so  $\hat{\bar{\alpha}}$  is the left-inverse for  $\hat{\alpha}$ . A similar argument shows it is the right inverse as well with  $(\hat{\alpha} \circ \hat{\bar{\alpha}})([g]) = [g]$ . Therefore  $\pi_1(\mathbf{X}, x_0)$  is group isomorphic to  $\pi_1(\mathbf{X}, x_1)$ . ■

## Homomorphisms

**Definition 4.1: Homomorphism induced by a continuous map**

Let  $h \in C(\mathbf{X}, \mathbf{Y})$ , and  $y_0 = h(x_0)$ , it induces a map between loops at  $x_0$  and  $y_0$ .

$$h_* : \text{Loop}(x_0) \rightarrow \text{Loop}(y_0), f \mapsto h \circ f$$

It is also a group homomorphism between fundamental groups. We use the same symbol for the two maps, relying on context to distinguish between the two.

$$h_* : \pi_1(\mathbf{X}, x_0) \rightarrow \pi_1(\mathbf{Y}, y_0), [f] \mapsto [h \circ f]$$

is well defined because of Proposition 2.2, it is a homomorphism (again by Proposition 2.2) because  $h$  'distributes' over  $*$

$$h \circ (f * g) = (h \circ f) * (h \circ g)$$

**Remark 4.1: Functorial properties of the  $h_*$**

If  $x_0 \in \mathbf{X}$ , the tuple  $(x_0, \mathbf{X})$  is an object in the category of *pointed topological spaces*, and the map  $h_*$  is a *covariant functor* from the category of pointed topological spaces to the category of groups.

Follows from Munkres Theorem 52.4, if the expressions below make sense,

$$(g \circ f)_* = g_* \circ f_* \quad \text{and} \quad h_* \circ (g \circ f)_* = (h \circ g)_* \circ f_*$$

And the identity map  $i : \mathbf{X} \rightarrow \mathbf{X}$  gets 'sent' to the identity homomorphism in  $\text{Hom}(\pi_1(\mathbf{X}, x_0), \pi_1(\mathbf{X}, x_0))$ . And if  $h$  is a homeomorphism between  $\mathbf{X}$  and  $\mathbf{Y}$ , then  $h_*$  is an isomorphism at every point.

## Simply connected space

**Definition 5.1: Simply connected space**

A topological space  $\mathbf{X}$  is *simply connected* if it is path-connected, and  $\pi_1(\mathbf{X}, x_0) = \{[e_{x_0}]\}$  for some  $x_0 \in \mathbf{X}$ . Notice this implies every fundamental group of  $\mathbf{X}$  is trivial.

**Proposition 5.1: Properties of simply connected spaces**

If  $\mathbf{X}$  is a simply connected space, then  $\text{PathClass}(x_0, x_1)$  consists of one element. That is to say, if  $f$  and  $g$  are Paths from  $x_0$  to  $x_1$ , then  $f \simeq_p g$ .

## Covering maps

**Definition 6.1: Covering maps and spaces**