Math 3C03: Mathematical Physics I
Assignment 4

This assignment is due in the locker in the basement of Hamilton Hall by 3:00 PM on Friday, November 2nd.

1. (6 pts.) For $x > 0$, find the general solution of

$$xy' = \frac{3}{x}\ln(x) - 3y.$$

2. Set up the appropriate form for a particular solution of

(a) (4 pts.) $(D-1)^3(D^2-4)y = xe^x + e^{2x} + e^{-2x}$.

(b) (4 pts.) $y'' - 2y' + 5y = x^3 e^x \sin 2x$.

Here $D = \frac{d}{dx}$, for instance, $(D-3)(D+1)y = (D^2-2D-3)y = y'' - 2y' - 3y$. *Do not compute the values of the coefficients in the particular solution*, just give the correct form.

3. Let $y_1(x) = x^3$ and $y_2(x) = |x|^3$, for $-1 < x < 1$.

(a) (3 pts.) Show that these functions are linearly independent on $(-1, 1)$. Hint: Otherwise, one function is a constant multiple of the other. Show that this cannot be the case.

(b) (2 pts.) Find $W(y_1, y_2)(x)$ by considering the cases $x > 0$, $x = 0$, $x < 0$. You may use without proof that $y_2'(0) = 0$.

(c) (2 pts.) Use the previous parts, and the properties of the Wronskian, to conclude that y_1 and y_2 cannot simultaneously satisfy an equation of the form $y''+p(x)y'+q(x)y = 0$.

4. (8 pts.) Use variation of parameters to find the solution of

$$y'' + y = \frac{3}{\cos x}, \quad x \in (-\pi/4, \pi/4),$$

satisfying $y(0) = 1$, $y'(0) = 2$. Hint: $\int \tan x \; dx = -\ln(\cos x) + C$.

5. Find and classify the singular points of:

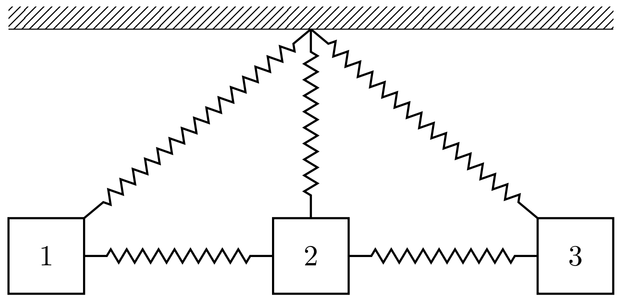
(a) (3 pts.) $x(x-1)y'' + (3x-1)y' + y = 0$.

(b) (3 pts.) $x^2y'' + (3x-1)y' + y = 0$.

6. (6 pts.) Find the general solution of

$$x^2y'' - 3xy' + 5y = 0 \quad (x > 0).$$

7. Consider the system in the figure: (no gravity)



All masses have mass m and all springs have constant k. The slanted springs have natural length $\sqrt{2}L$, while the remaining springs have natural length L and constant k. Let (x_j, y_j) be the displacement of the mass j from the equilibrium. As usual, we seek oscillations of the form $\mathbf{x}\cos(\omega t)$. We will assume that, in a suitable approximation, the potential energy is $V = \mathbf{q}^T B\mathbf{q}$, where

$$B = \frac{k}{2}\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The kinetic energy is just $T = \dot{\mathbf{q}}^T A\dot{\mathbf{q}}$, for $A = \frac{m}{2}I$.

(a) (3 pts.) Let $\lambda = \omega^2 m/k$. Find three vectors $\mathbf{x}^1$, $\mathbf{x}^2$, and $\mathbf{x}^3$, to obtain the modes of oscillation corresponding to $\lambda = 1$.

(b) (2 pts.) The mode of oscillation corresponding to the lowest frequency is given by $\mathbf{x}^4 = (1, 0, \sqrt{2}, 0, 1, 0)$. Use the symmetry of the system to find one mode of oscillation $\mathbf{x}^5$ other than $\mathbf{x}^1, \ldots, \mathbf{x}^4$.

(c) (4 pts.) Find a vector $\mathbf{x}^6$ orthogonal to $\mathbf{x}^1, \ldots, \mathbf{x}^5$ to find the last mode of oscillation. (In principle this is a 5×6 system, but it quickly reduces to 2×3.)

Total score: 50 points.