Linear algebra

1 Basic matrix arithmetic

If

$$\mathbf{a} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

find:

a. $\mathbf{a} + \mathbf{b}$

b. $-4\mathbf{b}$

c. $3\mathbf{a} - 4\mathbf{b}$

2 More complex matrix arithmetic

Suppose

$$\mathbf{x} = \begin{bmatrix} 3 \\ 2q \\ 6 \end{bmatrix} \quad \text{and} \quad \mathbf{y} = \begin{bmatrix} p+2 \\ -5 \\ 3r \end{bmatrix}$$

.

If $\mathbf{x} = 2\mathbf{y}$, find p, q, r.

3 Check for linear dependence

Which of the following sets of vectors are linearly dependent?

In each part, you can denote each vector as $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively.

a. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

c. $\begin{bmatrix} 13 \\ 7 \\ 9 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 5 \\ 8 \end{bmatrix}$

d. $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

4 Vector length

Find the length of the following vectors:

a. $(3, 4)$

b. $(0, -3)$

c. $(1, 1, 1)$

d. $(1, 2, 3)$

e. $(1, 2, 3, 4)$

f. $(3, 0, 0, 0, 0)$

5 Law of cosines

The **law of cosines** states:

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}$$

where θ is the angle from $\mathbf{w}$ to $\mathbf{v}$ measured in radians. Of importance, arccos() is the inverse of cos():

$$\theta = \arccos\left(\frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \, \|\mathbf{w}\|}\right)$$

For each of the following pairs of vectors, calculate the angle between them. Report your answers in both radians and degrees. To convert between radians and degrees:

$$\text{Degrees} = \text{Radians} \times \frac{180^o}{\pi}$$

a. $\mathbf{v} = (1, 0), \quad \mathbf{w} = (2, 2)$

b. $\mathbf{v} = (4, 1), \quad \mathbf{w} = (2, -8)$

c. $\mathbf{v} = (1, 1, 0), \quad \mathbf{w} = (1, 2, 1)$

6 Matrix algebra

Using the matrices below, calculate the following. Some may not be defined; if that is the case, say so.

$$\mathbf{A} = \begin{bmatrix} 3 \\ -2 \\ 9 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 8 \\ 0 \\ -1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 7 & -1 & 5 \\ 0 & 2 & -4 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 3 & 1 \\ 3 & 4 \\ 3 & -7 \end{bmatrix} \quad \mathbf{E} = \begin{bmatrix} 5 & 2 & 3 \\ 1 & 0 & -4 \\ -2 & 1 & -6 \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} 4 & 1 & -5 \\ 0 & 7 & 7 \\ 2 & -3 & 0 \end{bmatrix} \quad \mathbf{G} = \begin{bmatrix} 2 & -8 & -5 \\ -3 & 7 & -4 \\ 1 & 0 & 3 \\ 1 & 2 & 6 \end{bmatrix} \quad \mathbf{K} = \begin{bmatrix} 9 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{L} = \begin{bmatrix} 5 & 0 & 3 & 1 \end{bmatrix} \quad \mathbf{M} = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$$

a. $\mathbf{A} + \mathbf{B}$

b. $-\mathbf{G}$

c. $\mathbf{D}'$

d. $\mathbf{C} + \mathbf{D}$

e. $\mathbf{A}'\mathbf{B}$

f. $\mathbf{BC}$

g. $\mathbf{FB}$

h. $\mathbf{E} - 5\mathbf{I_3}$

i. $\mathbf{M}^2$

7 Matrix inversion

Invert each of the following matricies by hand (you can use a calculator or computer to check your solution, but be sure to show your work). Verify you have the correct inverse by calculating $\mathbf{X}\mathbf{X}^{-1} = \mathbf{I}$. Not all of the matricies may be invertible - if not, show why.

a. $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

b. $\begin{bmatrix} 2 & 1 \\ -4 & -2 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{bmatrix}$

8 One-hot encoding for categorical variables

Ordinary least squares regression is a common method for obtaining regression parameters relating a set of explanatory variables with a continuous outcome of interest. The vector $\hat{\mathbf{b}}$ that contains the intercept and the regression slope is calculated by the equation:

$$\hat{\mathbf{b}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}$$

If an explanatory variable is nominal (i.e. ordering does not matter) with more than two classes (e.g. $\{\text{White}, \text{Black}, \text{Asian}, \text{Mixed}, \text{Other}\}$), the variable must be modified to include in the regression model. A common technique known as **one-hot encoding** converts the column into a series of $n-1$ binary (0/1) columns where each column represents a single class and n is the total number of unique classes in the original column. Explain why this method converts the column into $n-1$ columns, rather than n columns, in terms of linear algebra. **Reminder: X contains both the one-hot encoded columns as well as a column of 1s representing the intercept.**

9 Solve the system of equations

Solve the following systems of equations for x, y, z, either via matrix inversion or substitution:

a. System #1

$$
\begin{aligned}
x + y + 2z &= 2 \\
3x - 2y + z &= 1 \\
y - z &= 3
\end{aligned}
$$

b. System #2

$$
\begin{aligned}
x - y + 2z &= 2 \\
4x + y - 2z &= 10 \\
x + 3y + z &= 0
\end{aligned}
$$

10 Multiplying by 0

When it comes to real numbers, we know that if $xy = 0$, then either $x = 0$ or $y = 0$ or both. One might believe that a similar idea applies to matricies, but one would be wrong. Prove that if the matrix product $\mathbf{AB} = \mathbf{0}$ (by which we mean a matrix of appropriate dimensionality made up entirely of zeroes), then it is not necessarily true that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$. Hint: in order to prove that something is not always true, simply identify one example where $\mathbf{AB} = \mathbf{0},\ \mathbf{A}, \mathbf{B} \neq \mathbf{0}$.