Functions of several variables and optimization with several variables

1 Find first partial derivatives

Find all of the first partial derivatives of each function.

a. $f(x, y) = 3x - 2y^4$

b. $f(x, y) = x^5 + 3x^3y^2 + 3xy^4$

c. $g(x, y) = xe^{3y}$

d. $k(x, y) = \frac{x-y}{x+y}$

e. $f(x, y, z) = \log(x + 2y + 3z)$

f. $h(x, y, z) = x^2e^{yz}$

2 Find the gradient

Find the gradient ∇f of the following functions and evaluate them at the given points.

a. $f(x, y) = \sqrt{x^2 + y^2}, \quad (x, y) = (3, 4)$

b. $f(x, y, z) = (x + z)e^{x-y}, \quad (x, y, z) = (1, 1, 1)$

3 Find the Hessian

Find the Hessian H for the following functions.

a. $g(x, y) = x^4 - 3x^2y^3$

b. $f(x, y, z) = xyz - x^2$

4 Find the critical points

Find the local minimum values, local maximum values, and saddle point(s) of the function. Remember the process we discussed in class: Calculate the gradient, set it equal to zero to solve the system of equations, calculate the Hessian, and assess the Hessian at critical values. Be sure to show your work on each of these steps.

a. $f(x, y) = x^4 + y^4 - 4xy + 2$

b. $k(x, y) = (1 + xy)(x + y)$

5 Least squares regression

Suppose we were interested in learning about how years of schooling affect the probability that a person turns out to vote. To simplify things, let's say we just have one observation of each variable. Let Y be our single observation of the dependent variable (whether or not a person turns out to vote) and X be our single observation of the independent variable, (the number of years of education that same person has). We believe that the process used to generate our data takes the following form:

$$Y = \beta X + \epsilon$$

where ϵ is an error term. We include this error term because we think random occurrences in the world will mean our model produces estimates that are slightly wrong sometimes, but we believe that on average, this model accurately relates X to Y. We observe the values of X and Y, but what about β? How do we know the value of β that best approximates this relationship, i.e., what's the slope of this line?

There are different criteria we could use, but a popular choice is the method of least squares. In this process, we solve for the value of β that minimizes the sum of squared errors, ϵ^2, in our data. Using the tools of minimization we've been practicing, find the value of β that minimizes this quantity. (Hint: In this case there is only one observation, so the sum of squared errors is equal to the single error squared.)

6 Least squares regression, refined

Following on the previous exercise, suppose we showed a colleague our model of voter turnout and she complained. "What a lame model", our colleague said, "You definitely have to include an intercept term." So in this problem we'll follow our colleague's advice and do just that.

Let Y be our single observation of the dependent variable (whether or not a person turned out to vote) and X_1 be our single observation of an independent variable, *education*, the number of years of schooling for this individual. Now though, we're also going to include an intercept term, β_0 in our model along with β_1 a coefficient that's associated with X_1.

This produces the following model for which we want to find the values of both β_0 and β_1 that minimize the sum of square errors.

$$Y = \beta_0 + \beta_1 X_1 + \epsilon$$

where ϵ is an error term.

Use the method of least squares to solve for the values of β_0, β_1 that minimizes the sum of squared errors in the our data. Using the tools of multivariate minimization we've been practicing, find the values of β_0 and β_1 that minimize this quantity.