Properties of random variables and limit theorems

1 Iterated expectations

a. Given $\mathrm{E}[X|Y] = 2Y$ and $f(Y) = .5$ with $Y \in [-3, -1]$, what is $\mathrm{E}[X]$?

b. Given $\mathrm{E}[Z|H] = 15H - 10$ and $H \sim \text{Bernoulli}(.2)$, what is $\mathrm{E}[Z]$?

2 Moment generating functions

Consider a variable $Y = \log(X)$ that is distributed $N(\mu, \sigma^2)$. Derive $\mathrm{E}[X]$ and $\mathrm{Var}[X]$. (**Hint**: the moment generating function of a normal random variable is $\exp(\mu t + 0.5\sigma^2 t^2)$.)

3 Changing coordinates

Suppose $f(X) = 3x^2$ with $X \in [0, 1]$ and $Y = X^2$.

a. What is $f(Y)$?
b. What is $\mathrm{E}[Y]$?

4 Markov and Chebyshev inequalities

A statistician wants to estimate the mean height h (in meters) fo a population, based on n independent samples $X_1, \ldots, X_n$, chosen uniformly from the entire population. She uses the sample mean $M_n = \frac{X_1 + \ldots + X_n}{n}$ as the estimate of h, and a rough guess of 1.0 meters for the standard deviation of the samples X_i.

a. How large should n be so that the standard deviation of M_n is at most 1 centimeter?

b. How large should n be so that Chebyshev's inequality guarantees that the estimate is within 5 centimeters from h, with probability at least 0.99?

5 Gambling in the casino

Before starting to play roulette in a casino, you want to look for biases that you can exploit. You therefore watch 100 rounds that result in a number between 1 and 36, and count the number of rounds for which the result is odd. If the count exceeds 55, you decide that the roulette is not fair. Assuming that the roulette is fair, find an approximation for the probability that you will make the wrong decision.

Hint: Use a standard normal distribution CDF table.

6 Making some widgets

A factory produces X_n widgets on day n, where X_n are independent and identically distributed random variables, with mean 5 and variance 9.

a. Find an approximation to the probability that the total number of gadgets produced in 100 days is less than 440.

b. Find (approximately) the largest value of n such that

$$\Pr(X_1 + \ldots + X_n \geq 200 + 5n) \leq 0.05$$

c. Let N be the first day on which the total number of widgets produced exceeds 1000. Calculate an approximation to the probability that $N \geq 220$.