Linear equations, inequalities, sets and functions, quadratics, and logarithms

1 Simplify expressions

Simplify the following expressions as much as possible:

a. $(-x^4y^2)^2$

1. Distribute exponents over products.

$$(-1)^2x^{(2\times4)}y^{(2\times2)}$$

2. Multiply 2 and 2 together.

$$(-1)^2x^{(2\times4)}y^4$$

3. Multiply 2 and 4 together.

$$(-1)^2x^8y^4$$

4. Evaluate $(-1)^2$.

$$x^8y^4$$

b. $9(3^0)$

1. Any nonzero number to the zero power is 1.

$$9(1)$$

2. Anything times 1 is the same value.

$$9$$

c. $(2a^2)(4a^4)$

1. Combine products of like terms.

$$2a^2\times4a^4=2\times4a^{(2+4)}$$

2. Evaluate $2+4$.

$$2\times4a^6$$

3. Multiply 2 and 4 together.

$$8a^6$$

d. $\dfrac{x^4}{x^3}$

1. For all exponents, $\dfrac{a^n}{a^m} = a^{(n-m)}$.

$$x^{(4-3)}$$

2. Evaluate $4 - 3$.

$$x$$

e. $(-2)^{7-4}$

1. Subtract 4 from 7.

$$(-2)^3$$

2. In order to evaluate 2^3 express 2^3 as 2×2^2.

$$-2 \times 2^2$$

3. Evaluate 2^2.

$$-2 \times 4$$

4. Multiply -2 and 4 together.

$$-8$$

f. $\left(\dfrac{1}{27b^3}\right)^{1/3}$

1. Separate component terms.

$$\frac{1}{27}^{1/3} \times \frac{1}{b^3}^{1/3}$$

2. Evaluate cube roots.

$$\frac{1}{3} \times \frac{1}{b}$$

3. Combine terms.

$$\frac{1}{3b}$$

g. $y^7 y^6 y^5 y^4$

1. Combine products of like terms.

$$y^{(7+6+5+4)}$$

2. Evaluate $7+6+5+4$.

$$y^{22}$$

h. $\dfrac{2a/7b}{11b/5a}$

1. Write as a single fraction by multiplying the numerator by the reciprocal of the denominator.

$$\frac{2a}{7b} \times \frac{5a}{11b}$$

2. Product property of exponents: $x^a \times x^b = x^{(a+b)}$

$$\frac{5a \times 2a}{7b \times 11b} = \frac{5 \times 2a^{1+1}}{7 \times 11b^{1+1}}$$

3. Evaluate $1+1$.

$$\frac{5 \times 2a^2}{7 \times 11b^2}$$

4. Multiple scalars together.

$$\frac{10a^2}{77b^2}$$

i. $(z^2)^4$

1. Nested exponents rule: $(x^a)^b = x^{ab}$

$$z^{2\times 4}$$

2. Evaluate 2×4

$$z^8$$

2 Simplify a (more complex) expression

Simplify the following expression:

$$(a+b)^2 + (a-b)^2 + 2(a+b)(a-b) - 3a^2$$

1. Expand $(a+b)^2$ with FOIL.

$$a^2 + 2ab + b^2 + (a-b)^2 + 2(a+b)(a-b) - 3a^2$$

2. Expand $(a-b)^2$ with FOIL.

$$a^2+2ab+b^2+a^2-2ab+b^2+2(a+b)(a-b)-3a^2$$

3. Multiply $a+b$ and $a-b$ together using FOIL.

$$a^2+2ab+b^2+a^2-2ab+b^2+2(a^2-b^2)-3a^2$$

4. Distribute 2 over a^2-b^2.

$$a^2+2ab+b^2+a^2-2ab+b^2+2a^2-2b^2-3a^2$$

5. Group like terms.

$$(a^2+a^2+2a^2-3a^2)+(b^2+b^2-2b^2)+(2ab-2ab)$$

6. Combine like terms.

$$a^2+(b^2+b^2-2b^2)+(2ab-2ab)$$

7. Look for the difference of two identical terms.

$$a^2$$

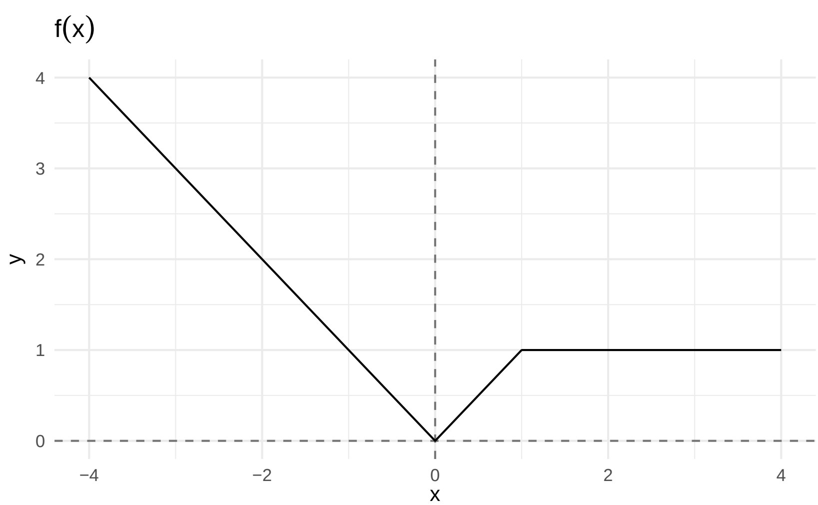
3 Graph sketching

Let the functions $f(x)$ and $g(x)$ be defined for all $x \in \Re$ by

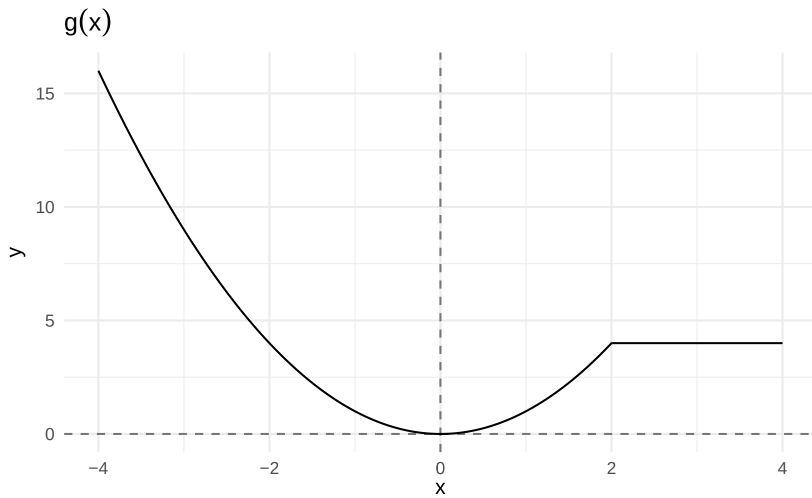
$$f(x)=\begin{cases} |x| & \text{if } x<1 \\ 1 & \text{if } x \geq 1 \end{cases}, \quad g(x)=\begin{cases} x^2 & \text{if } x<2 \\ 4 & \text{if } x \geq 2 \end{cases}$$

Sketch the graphs of:

1. $y=f(x)$



2. $y = g(x)$



3. $y = f(g(x))$

To sketch the composite function, we first evaluate $g(x)$ for different values of x, and then evaluate $f(g(x))$ for different outputs of $g(x)$.

- For $x \geq 2$, $g(x)$ is a constant value:

$$\begin{aligned} x &\geq 2 \\ g(x) &= 4 \\ f(g(x)) &= f(4) = 1 \end{aligned}$$

- For $x < 2$, $g(x)$ is not constant: $g(x) = x^2$. $f(x)$ evaluates differently depending on its input, so we have two cases based on the output of $g(x)$:
  - if $g(x) < 1$, $f(g(x)) = |g(x)| = |x^2| = x^2$. This is the case when:

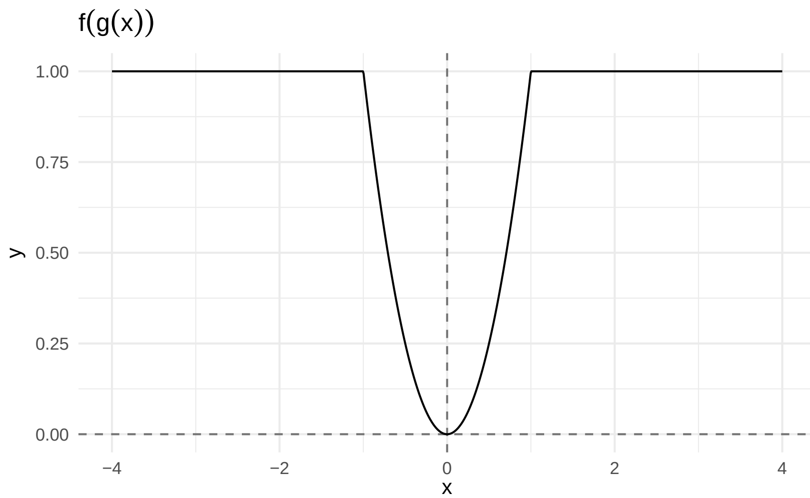
$$\begin{aligned} g(x) &< 1 \\ x^2 < 1 \text{ and } x &< 2 \\ -1 < x &< 1 \end{aligned}$$

  - if $g(x) \geq 1$, $f(g(x)) = 1$. This is the case when:

$$\begin{aligned} g(x) &\geq 1 \\ x^2 \geq 1 \text{ and } x &< 2 \\ x \leq -1 \text{ or } 1 \leq x &< 2 \end{aligned}$$

- Therefore, $f(g(x))$ has the following values:

$$f(g(x)) = \begin{cases} 1 & \text{if } x \leq -1 \\ x^2 & \text{if } -1 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



4. $y = g(f(x))$

   To sketch the composite function, we first evaluate $f(x)$ for different values of x, and then evaluate $g(f(x))$ for different outputs of $f(x)$.

   - For $x \geq 1$, $f(x)$ is a constant value:

   $$\begin{aligned} x &\geq 1 \\ f(x) &= 1 \\ g(f(x)) = f(1) &= 1^2 = 1 \end{aligned}$$

   - For $x < 1$, $f(x)$ is not constant: $f(x) = |x|$. $g(x)$ evaluates differently depending on its input, so we have two cases based on the output of $f(x)$:

     – if $f(x) < 2$, $g(f(x)) = f(x)^2 = |x|^2 = x^2$. This is the case when:

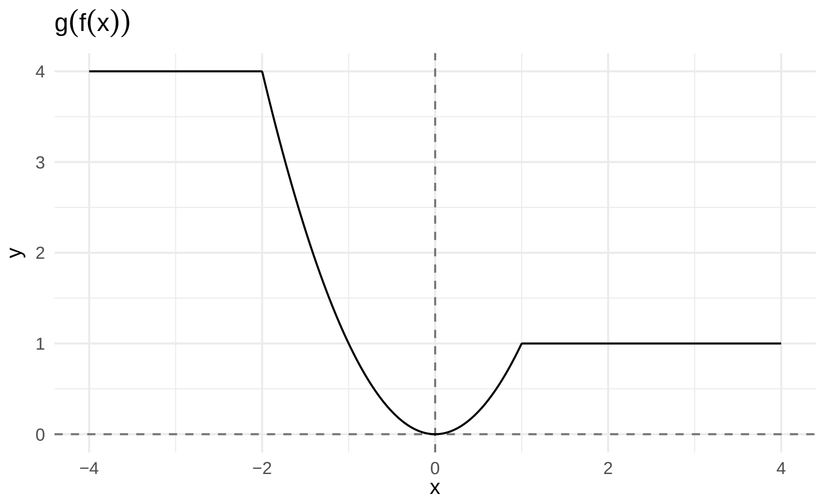
   $$\begin{aligned} f(x) &< 2 \\ |x| < 2 \text{ and } x &< 1 \\ -2 < x &< 1 \end{aligned}$$

     – if $f(x) \geq 2$, $g(f(x)) = 4$. This is the case when:

   $$\begin{aligned} f(x) &\geq 2 \\ |x| \geq 2 \text{ and } x &< 1 \\ x &\leq -2 \end{aligned}$$

   - Therefore, $g(f(x))$ has the following values:

   $$g(f(x)) = \begin{cases} 4 & \text{if } x \leq -2 \\ x^2 & \text{if } -2 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$



4 Root finding

Find the roots (solutions) to the following quadratic equations.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

a. $9x^2 - 3x - 12 = 0$

- Factor the left-hand side.

$$3(x+1)(3x-4) = 0$$

- Divide both sides by 3 to simplify the equation.

$$(x+1)(3x-4) = 0$$

- Find the roots of each term in the product separately by solving for x.

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned} \qquad \begin{aligned} 3x &= 4 \\ x &= \frac{4}{3} \end{aligned}$$

b. $x^2 - 2x - 16 = 0$

1. Complete the square

$$\begin{aligned} x^2 - 2x - 16 &= 0 \\ x^2 - 2x &= 16 \\ x^2 - 2x + 1 &= 17 \\ (x-1)^2 &= 17 \\ x - 1 &= \pm\sqrt{17} \\ x &= 1 \pm \sqrt{17} \end{aligned}$$

2. Quadratic formula
   - Using the quadratic formula, solve for x

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - (4 \times 1 \times 16)}}{2 \times 1}$$
$$x = \frac{2 \pm \sqrt{4 + 64}}{2}$$
$$x = \frac{2 \pm \sqrt{68}}{2}$$

   - Simplify the radical

$$x = \frac{2 \pm \sqrt{2^2 \times 17}}{2}$$
$$x = \frac{2 \pm 2\sqrt{17}}{2}$$

   - Factor the greatest common divisor

$$x = 1 \pm \sqrt{17}$$

c. $6x^2 - 6x - 6 = 0$

- Divide both sides by 6 to simplify the equation.

$$x^2 - x - 1 = 0$$

- Using the quadratic formula, solve for x

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - (4 \times 1 \times -1)}}{2 \times 1}$$
$$x = \frac{1 \pm \sqrt{1 - 4(-1)}}{2}$$
$$x = \frac{1 \pm \sqrt{1 + 4}}{2}$$
$$x = \frac{1 \pm \sqrt{5}}{2}$$

5 Systems of linear equations

Solve the following systems of equations for their unknown values. If there is no solution, indicate as such.

a. Two unknowns

$$3x - 2y = 18$$
$$5x + 10y = -10$$

Via substitution:

1. Solve for x in the first equation

$$
\begin{aligned}
3x - 2y &= 18 \\
3x &= 18 + 2y \\
x &= \frac{2}{3}y + 6
\end{aligned}
$$

1. Substitute $x = \frac{2}{3}y + 6$ into the second equation and solve for y

$$
\begin{aligned}
5\left(\frac{2}{3}y + 6\right) + 10y &= -10 \\
\frac{10}{3}y + 30 + 10y &= -10 \\
\frac{40}{3}y + 30 &= -10 \\
\frac{40}{3}y &= -40 \\
y &= -3
\end{aligned}
$$

1. Substitute y back into the first equation

$$
\begin{aligned}
3x - 2(-3) &= 18 \\
3x + 6 &= 18 \\
3x &= 12 \\
x &= 4
\end{aligned}
$$

$$x = 4, y = -3$$

b. Three unknowns

$$
\begin{aligned}
5x - 2y + 3z &= 20 \\
2x - 4y - 3z &= -9 \\
x + 6y - 8z &= 21
\end{aligned}
$$

1. Subtract 2/5 × (equation 1) from equation 2:

$$
\begin{aligned}
5x - 2y + 3z &= 20 \\
0x - \frac{16y}{5} - \frac{21z}{5} &= -17 \\
x + 6y - 8z &= 21
\end{aligned}
$$

2. Multiply equation 2 by -5:

$$
\begin{aligned}
5x - 2y + 3z &= 20 \\
0x + 16y + 21z &= 85 \\
x + 6y - 8z &= 21
\end{aligned}
$$

3. Subtract 1/5 × (equation 1) from equation 3:

$$
\begin{aligned}
5x - 2y + 3z &= 20 \\
0x + 16y + 21z &= 85 \\
0x + \frac{32y}{5} - \frac{43z}{5} &= 17
\end{aligned}
$$

4. Multiply equation 3 by 5:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 16y + 21z &= 85 \\ 0x + 32y - 43z &= 85 \end{aligned}$$

5. Swap equation 2 with equation 3:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 32y - 43z &= 85 \\ 0x + 16y + 21z &= 85 \end{aligned}$$

6. Subtract $1/2 \times$ (equation 2) from equation 3:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 32y - 43z &= 85 \\ 0x + 0y + \frac{85z}{2} &= \frac{85}{2} \end{aligned}$$

7. Multiply equation 3 by 2/85:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 32y - 43z &= 85 \\ 0x + 0y + z &= 1 \end{aligned}$$

8. Add $43 \times$ (equation 3) to equation 2:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + 32y + 0z &= 128 \\ 0x + 0y + z &= 1 \end{aligned}$$

9. Divide equation 2 by 32:

$$\begin{aligned} 5x - 2y + 3z &= 20 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 1 \end{aligned}$$

10. Add $2 \times$ (equation 2) to equation 1:

$$\begin{aligned} 5x + 0y + 3z &= 28 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 1 \end{aligned}$$

11. Subtract $3 \times$ (equation 3) from equation 1:

$$\begin{aligned} 5x + 0y + 0z &= 25 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 1 \end{aligned}$$

12. Divide equation 1 by 5:

$$\begin{aligned} x + 0y + 0z &= 5 \\ 0x + y + 0z &= 4 \\ 0x + 0y + z &= 1 \end{aligned}$$

$$x = 5, y = 4, z = 1$$

c. An animal shelter has a total of 350 animals comprised of cats, dogs, and rabbits. If the number of rabbits is 5 less than one-half the number of cats, and there are 20 more cats than dogs, how many of each animal are at the shelter?

- Let $x =$ number of cats
- Let $y =$ number of dogs
- Let $z =$ number of rabbits

This gives us the system of equations

$$\begin{aligned} x + y + z &= 350 \\ z &= \frac{1}{2}x - 5 \\ x &= 20 + y \end{aligned}$$

1. Substitute $z = \frac{1}{2}x - 5$ into the first equation

$$\begin{aligned} x + y + \frac{x}{2} - 5 &= 350 \\ \frac{3}{2}x + y - 5 &= 350 \end{aligned}$$

2. Substitute $x = y + 20$ into the first equation

$$\begin{aligned} \frac{3}{2}(y + 20) + y - 5 &= 350 \\ \frac{3}{2}y + 30 + y - 5 &= 350 \\ \frac{5}{2}y + 25 &= 350 \\ \frac{5}{2}y &= 325 \\ y &= 130 \end{aligned}$$

3. Substitute $y = 130$ into the third equation

$$\begin{aligned} x &= y + 20 \\ x &= 130 + 20 \\ x &= 150 \end{aligned}$$

4. Substitute $x = 150$ into the second equation

$$z = \frac{1}{2}x - 5$$
$$z = \frac{1}{2}(150) - 5$$
$$z = 70$$

$$x = 150, y = 130, z = 70$$

There are 150 cats, 130 dogs, and 70 rabbits.

6 Work with sets

Using the sets

$$A = \{2, 3, 7, 9, 13\}$$
$$B = \{x : 4 \leq x \leq 8 \text{ and } x \text{ is an integer}\}$$
$$C = \{x : 2 < x < 25 \text{ and } x \text{ is prime}\}$$
$$D = \{1, 4, 9, 16, 25, \ldots\}$$

identify the following:

1. $A \cup B$

   $E = \{2, 3, 4, 5, 6, 7, 8, 9, 13\}$, combine all integers between 4 and 8 inclusive with the numbers in set A.

2. $(A \cup B) \cap C$

   $F = \{3, 5, 7, 13\}$, Since C is only prime numbers greater than 2 and less than 25, we take all the prime numbers that are also included in E, but remember to drop out 2 since it is not included in C.

3. $C \cap D$

   $G = \emptyset$, there are no prime numbers in D, so nothing is shared between C and D.

7 Simplify logarithms

Express each of the following as a single logarithm:

a. $\log(x) + \log(y) - \log(z)$

- Multiplication rule of logarithms: $\log(x \times y) = \log(x) + \log(y)$
- Division rule of logarithms: $\log(\frac{x}{y}) = \log(x) - \log(y)$
- Applying the log rules, we combine logs that are added through multiplication and then combine logs that are subtracted with division.

$$\log(x) + \log(y) - \log(z)$$

$$\log(xy) - \log(z)$$

$$\log(\frac{xy}{z})$$

b. $2\log(x) + 1$

- Exponentiation rule of logarithms: $\log(x^y) = y\log(x)$
- $\log(e) = 1$

$$2\log(x) + 1$$

$$2\log(x) + \log(e)$$

$$\log(x^2) + \log(e)$$

$$\log(ex^2)$$

c. $\log(x) - 2$

- $\log(e) = 1$

$$\log(x) - 2$$

$$\log(x) - 2\log(e)$$

$$\log(x) - \log(e^2)$$

$$\log(\frac{x}{e^2})$$