Multivariate distributions

1 Properties of a joint PDF

Continuous random variables X and Y have the following joint probability density function (PDF):

$$f_{XY}(x, y) = \begin{cases} kx^2y^3 & \text{where } 0 < x, y < 6 \\ 0 & \text{otherwise} \end{cases}$$

Note: $0 < x, y < 6$ means that both x and y are between 0 and 6; it does not mean that x is greater than 0 and y is less than 6. This notation is not uncommon, so keep it in mind.

a. Find k.

b. Find the marginal PDF of X, $f_X(x)$.

c. Find the marginal PDF of Y, $f_Y(y)$.

d. Find $\mathrm{E}[X]$.

e. Find $\mathrm{E}[Y]$.

f. Find $\mathrm{Var}(X)$.

g. Find $\mathrm{Var}(Y)$.

h. Find $\mathrm{Cov}(X, Y)$.

i. Are X and Y independent? Why?

j. What is the PDF of X conditional on Y, $f_{X|Y}(x|y)$?

k. What is the PDF of Y conditional on X, $f_{Y|X}(y|x)$?

2 Properties of joint random variables

- $\mathrm{E}[D] = 10$
- $\mathrm{E}[F] = 4$
- $\mathrm{E}[DF] = 8$
- $\mathrm{Var}(D) = 60$
- $\mathrm{Var}(F) = 60$

a. What is $\mathrm{Cov}(D, F)$?

b. What is the correlation between D and F?

c. Suppose you multiplied F by 2 to generate a new variable, H. What is $\mathrm{Cov}(D, H)$?

d. What is $\mathrm{Cor}(D, H)$? How does this compare to your answer to Part (b) of this question?

e. Suppose instead that $\mathrm{Var}(D) = 30$. How would this change $\mathrm{Cor}(D, F)$?

3 Calculating conditional PDF

Let $f(x, y) = 15x^2y$ for $0 \leq x \leq y \leq 1$. Find $f(x|y)$.

4 Deriving a joint PDF

We start with a stick of length l. We break it at a point which is chosen according to a uniform distribution and keep the piece, of length Y, that contains the left end of the stick. We then repeat the same process on the piece that we were left with, and let X be the length of the remaining piece after breaking for the second time.

a. Find the joint PDF of Y and X

b. Find the marginal PDF of X

c. Use the PDF of X to evaluate $\mathrm{E}[X]$

d. Evaluate $\mathrm{E}[X]$, by exploiting the relation $X = Y \times \dfrac{X}{Y}$

5 Continuous Bayes' theorem

Previously, we used Bayes' theorem to link the conditional probability of discrete events A given B to the probability of B given A. There is an analogous Bayes' theorem that relates the conditional densities of random variables X and θ:

$$f(\theta \mid X) = \frac{f(X \mid \theta)f(\theta)}{\int f(X \mid \theta)f(\theta)d\theta}$$

Prove the continuous Bayes' theorem.