Chapter Two

Algebra Review

Of all the chapters in this book, this is the one most safely skipped. Most of this chapter is taken up by a review of arithmetic and algebra, which should be familiar to most readers. If you feel comfortable with this material, skip it. If it is only vaguely familiar, don't. The third section briefly discusses the utility of computational aids for performing calculations and checking work.

2.1 BASIC PROPERTIES OF ARITHMETIC

There are several properties of arithmetic that one uses when simplifying equations. These arise from the real numbers or integers for which the variables stand. In other words, because the variables we use in political science generally take values in $\mathbb{R}$ or $\mathbb{Z}$, these five properties generally apply. This will be true nearly throughout the book; however, in Part IV we will see that matrix variables can fail to commute under multiplication, for example, and do not always possess multiplicative inverses. But for variables that stand for real numbers or integers, these properties will always hold. Most of these are expressed in terms of addition and multiplication, but the first three properties apply to subtraction and division, respectively, as well, except for division by zero.

The **associative properties** state that $(a+b)+c = a+(b+c)$ and $(a\times b)\times c = a \times (b \times c)$. In words, the properties indicate that the grouping of terms does not affect the outcome of the operation.

The **commutative properties** state that $a + b = b + a$ and $a \times b = b \times a$. In words, the properties claim that the order of addition and multiplication is irrelevant.

The **distributive property** states that $a(b + c) = ab + ac$. In words, the property says that multiplication distributes over addition (and subtraction).

The **identity properties** state that there exists a zero such that $x + 0 = x$ and that there exists a one such that $x \times 1 = x$. In other words, there exist values that leave x unchanged under addition and multiplication (and subtraction and division, respectively, as well).

The **inverse property** states that there exists a $-x$ such that $(-x) + x = 0$. In other words, there exist values that when added to any x produce the identity under addition. We might also consider an inverse under multiplication, x^{-1}, such that $(x^{-1}) \times x = 1$. The existence of this inverse is a property of the real numbers (and the rational numbers), but not the integers, so one must be careful. For example, if $x = 2$, then $x^{-1} = 0.5$ in the real numbers, but

no integer multiplied by two equals one. Whether or not an inverse exists will depend, therefore, on the set of values the variable can take.

It is useful to recall at this stage that division by zero is undefined. The expression $x/0 = \infty$ is true for any $x \neq 0$, but is completely undefined for $x = 0$. Other useful facts include that $x = 1x = x^1 = 1x^1$, and that $x^0 = 1$. Recall also that multiplication by a variable with a negative value changes the sign of the product: $-1 \times x = -x$. The product of two terms with negative signs is positive: $(-x) \cdot (-y) = xy$.

2.1.1 Order of Operations

The order of operations is also important and can trip people up. In arithmetic and algebra the order of operations is parentheses, exponents, multiplication, division, addition, subtraction. A common mnemonic device people use to memorize order of operations is PEMDAS, or <u>P</u>lease <u>E</u>xcuse <u>M</u>y <u>D</u>ear <u>A</u>unt <u>S</u>ally.

2.1.2 Ratios, Proportions, and Percentages

Ratios, proportions, and percentages sometimes give people trouble, so let's briefly review those. The **ratio** of two quantities is one divided by the other $\frac{x}{y}$ is the ratio of x to y. Ratios are also written as $x : y$. Keep in mind that one can only take the ratio of two variables measured at a ratio level of measurement (i.e., there is a constant scale between values, and a meaningful zero). Though a ratio may be negative, we typically consider ratio variables that range from 0 to ∞. To get this, we take the absolute value of the ratio, denoted $\left|\frac{x}{y}\right|$. All this does is turn any negative number positive. As an example, international relations scholars are often interested in the ratio of military power between two countries (e.g., Organski and Kugler, 1981).

The **proportion** of two variables, on the other hand, is the amount one variable represents of the sum of itself and a second variable: $\left|\frac{x}{x+y}\right|$. A proportion ranges from a minimum of 0 to a maximum of 1. Students of budgetary politics are often interested in the proportion of expenditures that is spent in a given category (e.g., health and welfare, pork barrel politics, defense spending; see Ames, 1990).

The **percentage** one variable represents of a total is the proportion represented over the range from 0 to 100. In other words, the percentage is a linear transformation of the proportion $\left|\frac{x}{x+y}\right| \times 100\%$. Many people find a percentage representation more intuitive than a proportion representation, but they provide the same information.

You will also encounter the **percentage change** in a variable, which is calculated as $\frac{(x_{t+1}-x_t)}{x_t}$, where the subscript t indicates the first observation and the subscript $t+1$ indicates the second observation. For instance, according to the Center for Defense Information's *Almanac*, the United States spent \$75.4 billion for military personnel wages in 2001 and an estimated \$80.3 billion in

2002. The expenditures in 2002 represented a 6.5% increase over 2001 expenditures: $\frac{(80.3-75.4)}{75.4} \simeq 6.5\%$. Note that the percentage change can range from $-\infty$ to ∞.

2.1.3 Why Should I Care?

You care about these properties because you need to know them to follow along. People who use mathematics to communicate their ideas, whether in formal theory or statistics, assume that you can do the operations allowed by these properties. They often "skip steps" when writing down manipulations and expect you to do them in your head. If you cannot do them, you will get lost.

2.2 ALGEBRA REVIEW

This section reviews the most common algebraic manipulations you will encounter. Most of you will be familiar with these; the trick is trying to minimize errors, which are easy to make. We note some common errors to avoid.

2.2.1 Fractions

Many students find fractions the most frustrating part of algebra. People generally find whole numbers more intuitive than fractions, and that makes calculations with fractions more difficult to perform. As such, whenever possible it is best to convert fractions to whole numbers. Recall that the number on the top of a fraction is the **numerator** and the number on the bottom of a fraction is the **denominator**

$$\frac{Numerator}{Denominator}.$$

Thus, one can convert to a whole number whenever the denominator divides evenly into the numerator.

Many people find mixed numbers such as $2\frac{3}{4}$ even more frustrating. To convert these mixed numbers to fractions, follow these two steps. First, multiply the denominator of the fraction by the whole number (i.e., multiply 4×2, which equals 8). Second, take this product and add it to the numerator and place that sum over the original denominator (add 8 to 3, which equals 11, and place that over 4 for the final fraction $\frac{11}{4}$). These two quantities are equivalent.

Two common algebraic manipulations relating to fractions that often trouble students are cancellations and adding fractions.

2.2.1.1 Cancellations

The reason we want to reduce fractions is to make them easier to use (if the fraction can be converted to a whole number, this is ideal). For example, $\frac{10x}{2}$ can be reduced to $5x$. One that you might encounter in game theory could look like this: $\frac{7+3x}{2x}$.

One of the most common mistakes made is to cancel the xs and simplify $\frac{7+3x}{2x}$ to $\frac{10}{2}$, and then simplify this quantity to 5. However, $7+3x \neq 10x$, so $\frac{7+3x}{2x} \neq 5$.

In this example $\frac{7+3x}{2x}$ can be simplified to $\frac{7}{2x} + \frac{3x}{2x}$. The fraction $\frac{7}{2x}$ is in its simplest form. The fraction $\frac{3x}{2x}$ can be simplified to $\frac{3}{2}$, as long as $x \neq 0$.[1]

Therefore, $\frac{7+3x}{2x} = \frac{7}{2x} + \frac{3}{2}$.

2.2.1.2 Adding Fractions

Adding or subtracting fractions can be a bit frustrating as they do not follow the same rules as whole numbers. More specifically, you can only add the numerators of two or more fractions when the denominators of each fraction are the same (i.e., you cannot add fractions with different denominators). You can add $\frac{4}{\beta} + \frac{\alpha}{\beta}$, which equals $\frac{4+\alpha}{\beta}$. When two fractions have different denominators, such as $\frac{\beta}{4} + \frac{\alpha}{2}$, one must transform one or both of the denominators to make addition possible: the numerators of all fractions can be added once their denominators are made equal.

To pursue the above example, $\frac{\beta}{4} + \frac{\alpha}{2}$, if we multiply $\frac{\alpha}{2}$ by $\frac{2}{2}$ (which equals one; you can always multiply by things equal to one, or add things equal to zero because of the identity property), it becomes $\frac{2\alpha}{4}$. Since the two fractions now have the same denominator, we can add their numerators:[2] $\frac{\beta}{4} + \frac{2\alpha}{4} = \frac{2\alpha+\beta}{4}$.

Unlike addition, multiplication does *not* require a common base, and one does multiply both numerator and denominator: $\frac{2}{3} \times \frac{1}{4} = \frac{2}{12} = \frac{1}{6}$.

Another common mistake people make when adding fractions is to assume that all aspects of fractions follow the same rules of addition. For example, they assert that $\frac{1}{\Delta+\Theta}$ is equal to $\frac{1}{\Delta} + \frac{1}{\Theta}$. It is not. To see why this is so, let's add real numbers to the expression. If we substitute 2 for Δ and 1 for Θ and sum the denominator, we get $\frac{1}{2+1}$, which is equal to $\frac{1}{3}$. If we split the fraction improperly, including the numerator over both parts of the denominator as above, we will conclude that $\frac{1}{2+1} = \frac{1}{2} + \frac{1}{1}$, which equals $1\frac{1}{2}$, or 1.5,[3] not $\frac{1}{3}$.

2.2.2 Factoring

Factoring involves rearranging the terms in an equation to make further manipulation possible or to reveal something of interest. The goal is to make the expression simpler. One uses the properties described above rather extensively when factoring.

A standard algebraic manipulation involves combining like terms in an expression. For example, to simplify $\delta + \delta^2 + 4\delta - 6\delta^2 + 18\delta^3$, we combine all like terms. In this case we combine all the δ terms that have the same exponent, which gives us $18\delta^3 - 5\delta^2 + 5\delta$.

[1] Remember, anything divided by itself is one, and anything multiplied by one equals itself.

[2] Note that we do **not** take the sum of the denominators. One only adds the numerators.

[3] If you have forgotten how to convert fractions into decimals, the solution is to do the division implied by the fraction (you can use a calculator if you wish): $\frac{1}{2} = 0.5$.

Another standard factoring manipulation involves separating a common term from unlike ones. We first establish what we want to pull out of the equation, then apply the distributive property of multiplication in reverse. For example, we might want to pull x out of the following: $3x + 4x^2 = x(3 + 4x)$. Another example is $6x^2 - 12x + 2x^3 = 2x(3x - 6 + x^2)$.

A more complex example is $12y^3 - 12 + y^4 - y$.

We can factor 12 out of the first two terms in the expression and y out of the next two terms.

The expression is then $12(y^3 - 1) + y(y^3 - 1)$, which can be regrouped as $(12 + y)(y^3 - 1)$.

2.2.2.1 *Factoring Quadratic Polynomials*

Quadratic polynomials are composed of a constant and a variable that is both squared and raised to the power of one: $x^2 - 2x + 3$, or $7 - 12x + 6x^2$.[4] Quadratic polynomials can be factored into the product of two terms: $(x \pm ?) \times (x \pm ?)$, where you need to determine whether the sign is + or −, and then replace the question marks with the proper values.

Hopefully, it is apparent that one can multiply many products of two sums or two differences to get a quadratic polynomial; this is the reverse of factoring.[5]

2.2.2.2 *Factoring and Fractions*

We can also reduce fractions by factoring. Consider the fraction $\frac{x^2-1}{x-1}$. We can factor the numerator $x^2 - 1 = (x + 1)(x - 1)$. We can thus rewrite the fraction as follows

$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1}.$$

The term $x - 1$ is in both the numerator and the denominator and thus (as long as $x \neq 1$) cancels out, leaving $x + 1$. Thus, $\frac{x^2-1}{x-1} = x + 1$ for $x \neq 1$.

This factoring need not be accomplished in one step. Consider the expression

$$\frac{3\lambda^4 + 3\lambda^3 - 6\lambda^2}{6\lambda^2 + 12\lambda}.$$

First, we can factor out the common factor from both the numerator and denominator. All of the terms in the numerator are multiples of $3\lambda^2$ and both of the terms in the denominator are multiples of 6λ. This yields

[4] Another way of putting this is that a quadratic polynomial is a second-order polynomial in a single variable x. We discuss polynomial functions in the next chapter. Finally, given that the Latin prefix *quadri* is associated with four, you may be wondering why *quadratic* is used to describe equations with a term raised to the power of two. The reason is that the Latin term *quadratum* means "square." So an equation with a variable that is squared is a quadratic equation (Weisstein, N.d.).

[5] Note that this is true of some, *but not all*, products of two sums or two differences.

$$\frac{3\lambda^2(\lambda^2 + \lambda - 2)}{6\lambda(\lambda + 2)}.$$

Next, we factor the quadratic polynomial in the numerator to get

$$\frac{3\lambda^2(\lambda + 2)(\lambda - 1)}{6\lambda(\lambda + 2)}.$$

Then we factor out like terms. Both the numerator and denominator have $\lambda + 2$, so (as long as $\lambda \neq -2$) they cancel out, leaving

$$\frac{3\lambda^2(\lambda - 1)}{6\lambda}.$$

Finally, 3λ can be canceled (as long as $\lambda \neq 0$) from both the numerator and the denominator, leaving the expression in its simplest form

$$\frac{\lambda(\lambda - 1)}{2}.$$

2.2.3 Expansion: The FOIL Method

Sometimes we need to simplify a complex expression. At other times we need to expand a simple expression. Here is a pop quiz:

Does $(\delta + \gamma)^2 = \delta^2 + \gamma^2$?

The answer: no.

Why? The expression $(\delta + \gamma)^2 = (\delta + \gamma)(\delta + \gamma)$. This can then be expanded using the FOIL method. The expanded expression is $\delta^2 + 2\delta\gamma + \gamma^2$.

The FOIL method can be used to expand the product of two sums or differences. FOIL stands for first, outer, inner, last, and represents the products one must calculate.

F: Multiply the first terms: $(\underline{2\pi} + 7)(\underline{4} + 3\pi) = 2\pi \times 4 = 8\pi$.

O: Multiply the outer terms: $(\underline{2\pi} + 7)(4 + \underline{3\pi}) = 2\pi \times 3\pi = 6\pi^2$.

I: Multiply the inner terms: $(2\pi + \underline{7})(\underline{4} + 3\pi) = 4 \times 7 = 28$.

L: Multiply the last terms: $(2\pi + \underline{7})(4 + \underline{3\pi}) = 7 \times 3\pi = 21\pi$.

Add terms to get $8\pi + 6\pi^2 + 28 + 21\pi$.

Finally, group like terms to get $6\pi^2 + 29\pi + 28$.

To test yourself, factor the final expression and show it yields the simplified expression with which we started. This is one way to check your work for any careless mistakes.

2.2.4 Solving Equations

Solving an equation involves isolating a variable on one side (by convention, the left side of the equals sign) and all other variables and constants on the other side. One does so by performing the same calculations on both sides of the equation such that one ends up isolating the variable of interest. This often takes multiple steps and there is almost always more than one way to arrive at the solution. As an example, the equation $y = 2x$ is already solved for y. If we want to solve that equation for x, we need to do some algebra. Start with

$$y = 2x.$$

Divide both sides of the equation by 2, yielding

$$\frac{y}{2} = x.$$

Rewrite the equation:

$$x = \frac{y}{2}.$$

Note that we can go about this in a more convoluted fashion:

$$y = 2x.$$

Divide both sides by x, yielding

$$\frac{y}{x} = 2.$$

Divide both sides by y:

$$\frac{1}{x} = \frac{2}{y}.$$

Multiply both sides by x:

$$1 = x(\frac{2}{y}).$$

Multiply both sides by $\frac{y}{2}$:

$$\frac{y}{2} = x.$$

Now rewrite:

$$x = \frac{y}{2}.$$

That is hardly efficient, but the good news is that we ended up at the same place, though we would have had to be careful that neither x nor y was equal to zero when dividing by them. We also got some practice in manipulating an equation. Here are a few useful techniques for those rusty in their algebra.

1. **Focus on the variable of interest**. Work on isolating the variable you care about, and don't worry so much about what this does to the rest of the equation.

2. **Combine all like terms**. Simplifying equations is easiest when you sort out all the noise and add together like terms.

3. **Check your answer**. Substitute the value that you obtain into the original equation to make sure that your answer is correct and that you didn't make a careless mistake.

4. **Make use of identities**. Remember $\frac{a}{b} \times \frac{b}{a} = 1$ and $a - a = 0$. That means you can multiply by the first and add the second at all times, whenever convenient, without changing the equation.

5. **Operate on both sides in the same manner**. Adding the same number to each side or multiplying each side by the same number won't change the equation.

2.2.4.1 Solving Quadratics

Solving quadratic polynomials requires learning how to complete the square and/or knowing the quadratic equation.

Completing the Square

Many quadratic equations that you will face can be solved relatively easily by **completing the square**. The basic intuition for solving these is to isolate the variable and its square and then add a value to each side of the equation to "complete the square."

To see what we are trying to accomplish, it helps to begin with a simple example. Sometimes we are presented with a quadratic equation that factors into a squared term, i.e., $(x - n)^2 \pm c$, where c is some constant. Consider the quadratic $x^2 - 6x + 5$. We can factor this into $(x - 3)^2 - 4$ (use the FOIL method to verify). We can then rewrite this equation as $(x - 3)^2 = 4$. Finally, by taking the square root of both sides we, can solve for x:

$$(x - 3)^2 = 4 \Rightarrow$$
$$x - 3 = \pm 2 \Rightarrow$$
$$x = 5 \text{ or } x = 1.$$

Note that this quadratic equation will have two solutions in the real numbers, or zero, but not one.[6] In other words, the cardinality of the solution set for a quadratic equation will be zero or two. An example of a quadratic equation with no real solutions (i.e., no solutions in the real numbers) is $x^2 + 1 = 0$.[7]

Solving a quadratic by factoring it into a squared term $\pm$ a constant and then taking the square root is quick, but most quadratics cannot be factored in integers so easily. However, we can transform any quadratic using the following steps to "complete the square" (i.e., transform it into a squared term $\pm$ a constant) and then solve for x by taking the square roots.

[6] Quadratic equations with real coefficients will always have two solutions in complex numbers; if these solutions are complex they will come in pairs, i.e., $a \pm bi$.

[7] The solutions of this equation are $\pm i$.

1. Start with a quadratic in your variable of interest (we'll say it's x) and move the constant to the right-hand side. Divide through by the coefficient on x^2. So if you have $2x^2 - 4x - 2 = 0$, you get $x^2 - 2x = 1$.
2. Divide the coefficient on x by 2 and then square it. Add that value to both sides of the equation. So now you have $x^2 - 2x + 1 = 1 + 1$.
3. Factor the left-hand side into a "($x \pm$ some term) squared" form and simplify the right-hand side. So now you have $(x - 1)^2 = 2$.
4. Take the square root of both sides (remember that when you take the square root of a number, the solution is always $\pm$, because the square of a negative number is a positive number). So now you have $x - 1 = \pm\sqrt{2}$.
5. Solve for x. So the solutions are $x = 1 + \sqrt{2}$ and $x = 1 - \sqrt{2}$.[8]

Let's work another example. Consider the quadratic

$$x^2 + 8x + 6 = 0.$$

The first thing you might try is factoring to see if it yields the "($x \pm$ some term) squared" form. This quadratic does not, so we turn to completing the square. The first step is to isolate the squared term and variable by subtracting 6 from each side (note the coefficient on x^2 is already 1):

$$x^2 + 8x = -6.$$

Next we need to add the square of half of the value in front of x to both sides. The value in front of x is 8, so we divide 8 by 2 and then square the result $4^2 = 16$. Thus, to complete the square we need to add 16 to each side:

$$x^2 + 8x + 16 = 10.$$

We then perform step 3 and factor the left-hand side of the equation:

$$(x + 4)^2 = 10.$$

We now need to take the square root of each side, which gives us

$$x + 4 = \pm\sqrt{10}.$$

Now we can solve for x by subtracting 4 from each side. Our final answer is

$$x = -4 + \sqrt{10} \text{ and } x = -4 - \sqrt{10}.$$

[8]Irrational solutions such as $1 \pm \sqrt{2}$ will also always come in pairs.

The Quadratic Formula and Equation

Completing the square is one method for solving a quadratic equation, but it is not the only one, and you will sometimes encounter equations that are rather complicated to solve by completing the square. For example, if you are faced with $x^2 + \sqrt{15}x - 1 = 0$, you will not want to calculate half of $\sqrt{15}$, and then square it, add it to both sides, and try to factor the result. Instead, you will want to turn to the quadratic equation and formula.[9]

During your high school algebra courses you were probably required to memorize the quadratic equation and formula. The general form of a quadratic equation is[10]

$$ax^2 + bx + c = 0.$$

The general solutions to this equation are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

These solutions are called the quadratic formula. The formula can be derived from the equation by completing the square (we ask you to do this below, in an exercise). When we obtain values for x, we call these values the roots of the equation. For our purposes, this formula is used when completing the square is made difficult by fractions, decimals, or large numbers. We refer to a and b as the coefficients and c as the constant.

What are the roots of the quadratic equation

$$1.4x^2 + 3.7x + 1.1 = 0?$$

To find the solutions, we first list the values for a, b, and c:

$$a = 1.4, b = 3.7, c = 1.1.$$

Then we plug the values for a, b, and c into the quadratic formula:

$$x = \frac{-3.7 \pm \sqrt{3.7^2 - 4 \times 1.4 \times 1.1}}{2.8}.$$

Using a calculator to solve, we find that $x = -.341$ or $x = -2.301$.

These problems can be cumbersome because it is somewhat more difficult to check them. Two pieces of advice, though, can help you minimize mistakes. First, make sure to follow order of operations (PEMDAS, remember?). Second, you can go online and use a quadratic equation solver and plug in the values for a, b, and c to verify the accuracy of your computations. We discuss this further in Section 3.

[9]It is possible to use either method—completing the square or the quadratic equation and formula—to solve a quadratic equation. But one is more likely to make errors using the latter than the former, and so many people find completing the square preferable as long as they are not faced with an unusual (e.g., radical) value in front of the x term.

[10]The equation is true when $a \neq 0$. When $a = 0$ and $b \neq 0$, it is a simple linear equation with solution $x = -\frac{c}{b}$. If both a and b are zero, the equation is false unless $c = 0$ as well.

2.2.5 Inequalities

To solve inequalities, we have to discuss a few extra properties.

First, all pairs of real numbers have exactly one of the following relations: $x = y$, $x > y$, or $x < y$.

Adding any number to each side of these relations will not change them; this includes the inequalities. That is, inequalities have the same addition and subtraction properties as equalities such that if $x > y$, then $x + a > y + a$ and $x - a > y - a$.

The properties for multiplication and division for inequalities are a bit different than for equalities. For multiplication, if a is positive and $x > y$, then $ax > ay$. If a is negative and $x > y$, then $ax < ay$. For division, if a is positive and $x > y$, then $\frac{x}{a} > \frac{y}{a}$. If a is negative and $x > y$, then $\frac{x}{a} < \frac{y}{a}$. Multiplying or dividing an inequality by zero is not allowed.

To summarize, this means that you flip the $<$ or $>$ sign when multiplying or dividing by a negative.

Example: Solve for y: $-4y > 2x + 12$. First, we want to isolate y by itself on the left side of the equation. We divide both sides by -4, which gives us $y < -\frac{x}{2} - 3$. Dividing by a -4 flips the $>$ sign to $<$. If we do not know the value of x, then we can leave it in this form.

2.2.6 Review: Avoiding Common Errors

We have included a list of some common mistakes people make when solving equations as a sort of help file for when you are struggling to find the right answer. Below this list, we've included some websites you may go to for extra examples or help.

That said, please remember that the World Wide Web is dynamic, and the links below will become dated. We found them using search engines, and you will be able to do the same.

Sign errors: Sign errors are probably the most common mistakes. Most people think of this $(-)$ sign as a negative sign. This is part of the problem. Tackling math as if it were a foreign language is the best way to approach learning the fundamentals of mathematics. This sign $(-)$ is best thought of as "the opposite of," or, in other words, "the additive inverse." (Recall that every integer and real number has an additive inverse that, when added to the number, produces zero.) When reading an equation such as $-x + y = 7$, you should say in your head, "the opposite of x added to y is 7." The reason for thinking of this sign as "the opposite of" is twofold. First, it can help you find mistakes in your work. Second, it will help you deal with situations such as $-x = 7$. You will easily interpret this as the opposite of x equals 7, so x must be the additive inverse of 7, which is -7.[11]

[11] If this is a bit confusing, remember that the number line has 0 in the middle, positive integers falling to the right of 0 and negative integers falling to the left. Any number has an opposite on the number line that is equidistant from zero. So the opposite of 8 is -8 and the opposite of -9 is 9. Thinking of negatives in these terms will also help you deal with absolute values.

Only changing one side or term in an equation: Think of an equation as a scale or seesaw. Whatever you do to one side you must do to the other.[12] An equation must be in equilibrium. If you divide one side by 12, you must divide the other by 12. In addition, you must divide all terms on both sides by 12.

Not distributing: Always distribute across addition (and subtraction). If you have an expression such as $4x(2 + 6y + 3t)$, many people simply multiply $4x$ by 2. Each term inside a parentheses must be multiplied by what is outside the parentheses. The correct expression is $8x + 24xy + 12xt$.

Distributing with radicals and exponents: Radicals and exponents have different rules, which we discuss in depth in the next chapter. They do not follow the same rules as multiplication and addition. For example, $\sqrt{9+16}$ is not the same as $\sqrt{9} + \sqrt{16}$. Also, as discussed above, $(\alpha + \beta)^2$ is not the same as $\alpha^2 + \beta^2$.

You can find other lists of common errors at several websites. For example, Eric Schecter maintains a page of the most common math errors by undergraduates (`http://atlas.math.vanderbilt.edu/~schectex/commerrs/`). See Schecter's page for entries on "multiplying by a negative one and other sign errors," "loss of invisible parentheses," "everything is additive," and "everything is commutative."

Other common algebra mistakes include canceling terms instead of factors, misunderstanding fractions, and misunderstanding negative and fractional components. See `http://tutorial.math.lamar.edu/pdf/algebra_Cheat_Sheet.pdf`.

Beyond this, some of you may be interested in more practice, especially with algebra. One of the authors finds Huettenmueller (2010) a useful resource, but there are a number of other self-teaching guides. You can also find a number of useful resources available on the Web. We recommend `http://www.purplemath.com/`, `http://mathworld.wolfram.com/`, and `http://math.com/`. Wikipedia (`http://en.wikipedia.org/`) also has many good entries for mathematical concepts, though many of these can substantially be found elsewhere.

For more information on set theory, see Peter Suber's "A Crash Course in the Mathematics of Infinite Sets" (`http://www.earlham.edu/~peters/writing/infapp.htm`). Oregon State's "Field Guide to Functions" (`http://oregonstate.edu/instruct/mth251/cq/FieldGuide/`) is a good guide to functions (the topic of the next chapter). R.H.B. Exell's page on relations is also useful for

[12] Of course, as noted above, adding zero to one side or multiplying one side by one is acceptable, as these are identities and leave the value of the expression unchanged. This may be may be useful when working with fractions.

a more detailed introduction (`http://www.jgsee.kmutt.ac.th/exell/Logic/Logic42.htm`).

2.2.7 Why Should I Care?

algebra is the set of rules one uses to manipulate equations that have variables in place of numerical values, whereas arithmetic is the set of rules we use to manipulate equations made of numerical values. arithmetic is thus essential for making specific calculations, but algebra is needed if we want to study general concepts. You care about algebra for the same reason you care about arithmetic: people use it to communicate their ideas precisely, and they often assume you can do algebraic operations in your head. To follow along, then, you need to do the algebra. This is true in both the study of statistics and the study of formal theory. If you do not master this basic algebra, you will get lost. Solving equations and simplifying inequalities in order to find the range of solutions also proves highly useful in both game theory and statistics. Indeed, as we explain in Chapter 12, which introduces vector algebra and matrix algebra, the algebra covered here (which is called scalar algebra) is a foundation for both vector and matrix algebra.

2.3 COMPUTATIONAL AIDS

Throughout this book we assume that you will be performing all mathematical manipulations by hand, or at most using a (simple) calculator. We believe this is pedagogically important: one needs to be able to do the relevant calculations oneself in order to understand them; if one doesn't understand them, then one doesn't really know what one is saying; and if one doesn't know what one is saying, there is very little point to formalizing one's concepts with mathematics at all. So this book is intended to lead you through doing the calculations yourself. That said, it is often helpful to have access to computational aids for arithmetic, algebra, and the later topics in this book. One reason simply is as a check for your work. We all make mistakes, and it is nice to have a second pair of eyes, so to speak, to check one's work. A second reason is to increase speed once one is sure of one's understanding. As the techniques of math become more familiar to you, the boundary of your skills will expand, and you will want to devote more of your time to the harder stuff rather than simple Algebraic manipulation. Computational aids can help with this. Finally, a third reason is to help with the writeup. Some aids allow output in formats that may be easily converted to word processors or typographical languages such as LaTeX (`http://www.latex-project.org/`).

There are many computational aids out there. Some are freeware, meaning you can download them from the Internet, or use them in your browser directly, with no further obligation. Some examples of these include Eigenmath (`http://eigenmath.sourceforge.net/`) and Maxima (`http://maxima.sourceforge.net/`), along with the website `http://www.wolframalpha.com/`, which allows

you to input expressions directly into your browser. There are also some useful tools at `http://www.math.com/students/tools.html`, including a function plotter. LaTeXis a free typographical language that is very good at typesetting math; we wrote this book using it, and thus were able to deliver it typeset to Princeton University Press, retaining greater control over its "look and feel" and reducing production costs. Various options exist to make LaTeXmore user-friendly, e.g., LyX (`http://www.lyx.org/`).

Other tools are potentially more powerful, but they are also more expensive. However, they can have more functionality in some areas. If you are located at a university with access to them, they can be worthwhile to try. Mathematica (`http://www.wolfram.com/mathematica/`) and Maple (`http://www.maplesoft.com/products/maple/`) work well with symbolic math, and Matlab (`http://www.mathworks.com/products/matlab/`) is well suited to matrix algebra.

2.4 EXERCISES

2.4.1 Arithmetic Rules

Complete the following equations:

1. $x^1 =$ _____.
2. $-a \times (-b)^2 =$ _____.
3. $\sum_{i=1}^{4} x_i =$ _____.
4. $\prod_{m=6}^{9} x_m =$ _____.
5. $4! =$ _____.
6. $z^4 =$ _____.
7. $\sqrt[2]{9} =$ _____.
8. $\sqrt[3]{27} =$ _____.
9. $\left(\frac{3(2-4)}{2+3}\right)^3 =$ _____.

2.4.2 Ratios, Proportions, Percentages

10. Represent the following as a ratio, a proportion, and a percentage:
    a) Latinos relative to all others: African American 98,642; Asian 62,346; Caucasian 436,756; Latino 105,342; Other 32,654.
    b) Independent registered voters relative to Republicans: Democrat 432; Independent 221; Republican 312.

c) Republican relative to Democrat from no. 2.

11. If the Latino population shrunk to 100,322 in no. 1 above, what would be the percentage change in the Latino population?

12. If the other populations remained constant, what would be the percentage change in the proportion of Latinos to all others?

13. If voter turnout in the United States in 1996 was 56% and in 2000 it was 62%, what was the percentage change in turnout from 1996 to 2000?

14. Express these two quantities as a simplified ratio: 18 and 12.

2.4.3 Algebra Practice

15. Simplify into one term the following expressions:

    a) $xz + yz$.

    b) $mn + ln - pn$.

    c) $z \times y \times x - 2 \times y \times x$.

    d) $(z + x) \times y \times \frac{1}{z}$.

16. Simplify this expression as much as possible: $\frac{2x^2+20x+50}{2x^2-50}$.

17. Simplify this expression: $\frac{5+17x+4x+7}{42x}$.

18. Add these fractions: $\frac{2g+13}{3g} + \frac{4g-5}{4g}$.

19. Factor: $-7\theta^2 + 21\theta - 14$.

20. FOIL: $(2x - 3)(5x + 7)$.

21. Factor: $q^2 - 10q + 9$.

22. Factor and reduce: $\frac{\beta-\alpha}{\alpha^2-\beta^2}$.

23. Solve: $15\delta + 45 - 6\delta = 36$.

24. Solve: $.30\Omega + .05 = .25$.

25. Solve: $11 = (y + 1)2 + (6y - 12y)\frac{7}{2y}$.

26. Solve: $-4x^2 + 64 = 8x - 32$.

27. Complete the square and solve for x: $x^2 + 14x - 14 = 0$.

28. Complete the square and solve for y: $\frac{1}{3}y^2 + \frac{2}{3}y - 16 = 0$.

29. Solve using the quadratic formula: $2x^2 + 5x - 7$.

30. Derive the quadratic formula by completing the square for the equation $ax^2 + bx + c = 0$.

31. Solve: $-\delta > \frac{\delta+4}{7}$,