12.338

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Question:
For a real symmetric matrix $\mathbf{A}$, which of the following statements is true?

a) The matrix is always diagonalizable and invertible.
b) The matrix is always invertible but not necessarily diagonalizable.
c) The matrix is always diagonalizable but not necessarily invertible.
d) The matrix is always neither diagonalizable nor invertible.

Solution:
Checking for diagonalizability of matrix A
given,

$$\mathbf{A} = \mathbf{A}^\top \tag{0.1}$$

$\therefore$ eigenvalues of $\mathbf{A}$ are real.
for distinct eigenvalues λ_i, λ_j corresponding eigenvectors are $\mathbf{x_i}$, $\mathbf{x_j}$.

$$\mathbf{A}\mathbf{x_i} = \lambda_i\mathbf{x_i} \quad \text{and} \quad \mathbf{A}\mathbf{x_j} = \lambda_j\mathbf{x_j} \tag{0.2}$$

$$\mathbf{x_j}^\top\mathbf{A}\mathbf{x_i} = \lambda_i\mathbf{x_j}^\top\mathbf{x_i} \tag{0.3}$$

$$(\mathbf{A}\mathbf{x_j})^\top\mathbf{x_i} = \lambda_i\mathbf{x_j}^\top\mathbf{x_i} \tag{0.4}$$

$$\because \quad \mathbf{A}\mathbf{x_j} = \lambda_j\mathbf{x_j} \tag{0.5}$$

$$\lambda_j\mathbf{x_j}^\top\mathbf{x_i} = \lambda_i\mathbf{x_j}^\top\mathbf{x_i} \tag{0.6}$$

$$(\lambda_j - \lambda_i)\mathbf{x_j}^\top\mathbf{x_i} = 0 \tag{0.7}$$

$$\lambda_i \neq \lambda_j \tag{0.8}$$

$$\mathbf{x_j}^\top\mathbf{x_i} = 0 \tag{0.9}$$

$\therefore$ eigenvectors are orthogonal
$\therefore$ We can construct an orthogonal matrix with these eigenvectors

$$Q = [\mathbf{x_1}\ \mathbf{x_2}\ \mathbf{x_3}\ \dots\ \mathbf{x_n}] \tag{0.10}$$

$$Q^\top Q = I \tag{0.11}$$

$$A = QMQ^\top \tag{0.12}$$

Where $\mathbf{M}$ is diagonal matrix with it's entries as eigen values of matrix A
The eigenvalues are placed in the same order as their corresponding eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$

$$M = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix} \tag{0.13}$$

where $\lambda_1,\ \lambda_2,\ \cdots\ \lambda_n$ are eigen values of $\mathbf{A}$
$\therefore$ $\mathbf{A}$ is always diagonalizable.

Checking for invertibility of Matrix $\mathbf{A}$:

$$\mathbf{A} = QMQ^\top \tag{0.14}$$

$$|\mathbf{A}| = |Q||M||Q^\top| \tag{0.15}$$

$$|\mathbf{A}| = M_1 M_2 \cdots M_n \tag{0.16}$$

where $M_1, M_2, \cdots M_n$ are diagonal entries of Matrix M.
$\mathbf{A}$ is invertible only when

$$\det(\mathbf{A}) \neq 0 \tag{0.17}$$

that is $M_1, M_2, M_3 \cdots M_n \neq 0$
that is none of its eigenvalues are zero

if $\lambda_i = 0$
then A is non-invertible

$\therefore$ a real symmetric matrix may or may not be invertible.
$\therefore$ Option c is correct.

Examples of a real symmetric matrix $\mathbf{A}$:
1.

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \tag{0.18}$$

$$\mathbf{A} = \mathbf{A}^\top \tag{0.19}$$

$\mathbf{A}$ is real symmetric and diagonalizable but not invertible as $det(\mathbf{A}) = 0$
Eigen values of $\mathbf{A}$ are 0 and 2. Eigen vectors of $\mathbf{A}$ are:

$$q1 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \tag{0.20}$$

$$q2 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \tag{0.21}$$

$$\mathbf{A} = \mathbf{QMQ}^{\top} \tag{0.22}$$

$$where, \mathbf{Q} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \tag{0.23}$$

$$\mathbf{M} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \tag{0.24}$$

2.

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{0.25}$$

$$\mathbf{A} = \mathbf{A}^{\top} \tag{0.26}$$

A is real symmetric, diagonizable and invertible matrix
Eigen values of **A** are 3 and 1. Eigen vectors of **A** are:

$$q1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \tag{0.27}$$

$$q2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \tag{0.28}$$

$$\mathbf{A} = \mathbf{QMQ}^{\top} \tag{0.29}$$

$$where, \mathbf{Q} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \tag{0.30}$$

$$\mathbf{M} = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \tag{0.31}$$

det$\mathbf{A}$≠0

$$\mathbf{A}^{-}1 = \begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix} \tag{0.32}$$