# MODELING

# AND

# SIMULATION

***LECTURE 5***



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# CHAPTER 5
# Monte Carlo Simulation



- We define Monte Carlo simulation as ***a scheme*** **(مخطط)** ***employing random numbers, that is, U (0, 1) random variates, which is used for solving certain stochastic or deterministic problem.*** Thus, a stochastic discrete-event simulation is included in this definition. The name "***Monte Carlo***" simulation or method originated during World War II, when this approach was applied to problems related to the development of the ***atomic bomb and other weapons***.

## 5.1 Introduction to Monte Carlo Simulation

### 5.1.1 Introduction

- **Monte Carlo technique deals with taking random samples from distribution of a variable in order to supply a series of values for use in the model.**
- This requires ***a method of generating or obtaining random variates from the uniform distribution*** on the interval **[0, 1]** this distribution is denoted by ***U(0, 1)*** and random variates generated from the ***U(0, 1)*** distribution will be called random numbers.

# Introduction Monte Carlo Simulation



- The probability of any random number in that domain [0, 1] is equal. Random values can be conveniently and efficiently generated from a desired probability distribution by ***the inverse transform method*** for use in executing simulation models. Other methods of generating random numbers like ***linear congruential and mixed generators*** are explained in ***chapter 3.***
- **Monte Carlo methods are stochastic and are typically simple to implement.**
- **It is often used to solve *stochastic problems* to mimic randomness of physical behaviour.**
- **Monte Carlo can also be used to solve *deterministic problems* by using probability for estimation.**

# Introduction Monte Carlo Simulation



## *5.1.2 Requirements of Stochastic Simulation*

- Knowledge of relevant probability distributions depend on the **theoretical or empirical information about physical system being simulated.**

- **Supply of random numbers for making random choices.** By simulating large number of trials, probability distribution of overall results can be approximated with accuracy attained depending on number of trials. ***As the number of iterations increases we come to actual value.***

# Introduction Monte Carlo Simulation



## *Example 5.1*

In a queueing system, the arrival rate and service rate in minutes with their probabilities are given as:

| Arrival rate | Probability | | Service rate | Probability |
|---|---|---|---|---|
| 3 | 15 % | | 4 | 30 % |
| 4 | 30 % | | 5 | 40 % |
| 5 | 20 % | | 6 | 30 % |
| 6 | 35 % | | | |

As a conclusion from the given data, ***they are not following any typical distribution.***

To get help from random numbers in predicting the values of number of arrival and service rate, we find the ***cumulative frequency*** and ***distribute the uniformly distributed two digit random numbers (from 00 to 99) according to frequency as shown in the following table***:

# Introduction Monte Carlo Simulation



| Arrival rate | Frequency | Cumulative frequency | Two digit random numbers |
|---|---|---|---|
| 3 | 15 | 15 | 00 – 14 |
| 4 | 30 | 45 | 15 – 44 |
| 5 | 20 | 65 | 45 – 64 |
| 6 | 35 | 100 | 65 -99 |

| Service rate | Frequency | Cumulative frequency | Two digit random numbers |
|---|---|---|---|
| 4 | 30 | 30 | 00 – 29 |
| 5 | 40 | 70 | 30 – 69 |
| 6 | 30 | 100 | 70 - 99 |

Generating a stream of random numbers (***from 00 to 99***): (***78, 47, 13, 87, 89, 05 …***) and pick the first number **78** which corresponds to arrival rate of **6 minutes**, and similarly for the service rate. We can use ***the random function software*** to generate uniformly distributed random numbers from the ***excel sheet*** as shown in the following example.

# Introduction Monte Carlo Simulation

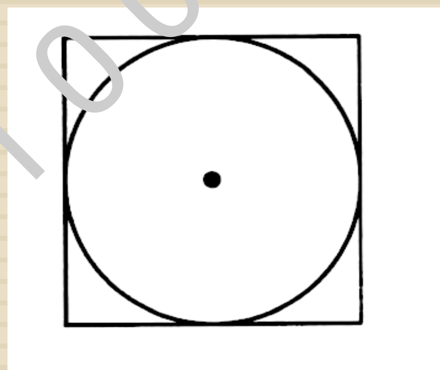


## *Example 5.2*

Calculate the deterministic value of **π** by using random numbers.

***Solution***

Assume a circle of ***radius=1*** and its center is at the origin, then its area is **π**. The circle is surrounded by a square, the length of its ***side=2***.




If we select a point ***(from -1 to 1***) at random, it ***may be located inside the circle or outside the circle but still inside the square***. If we repeat this selection for ***10 times, we may get 7 times the point is located inside the circle*** and ***3 times outside the circle*** but ***still inside the square.***

# Introduction Monte Carlo Simulation



This means that: ***Probability of a point located inside the circle= 0.7***

***Probability of a point located outside the circle = 0.3***

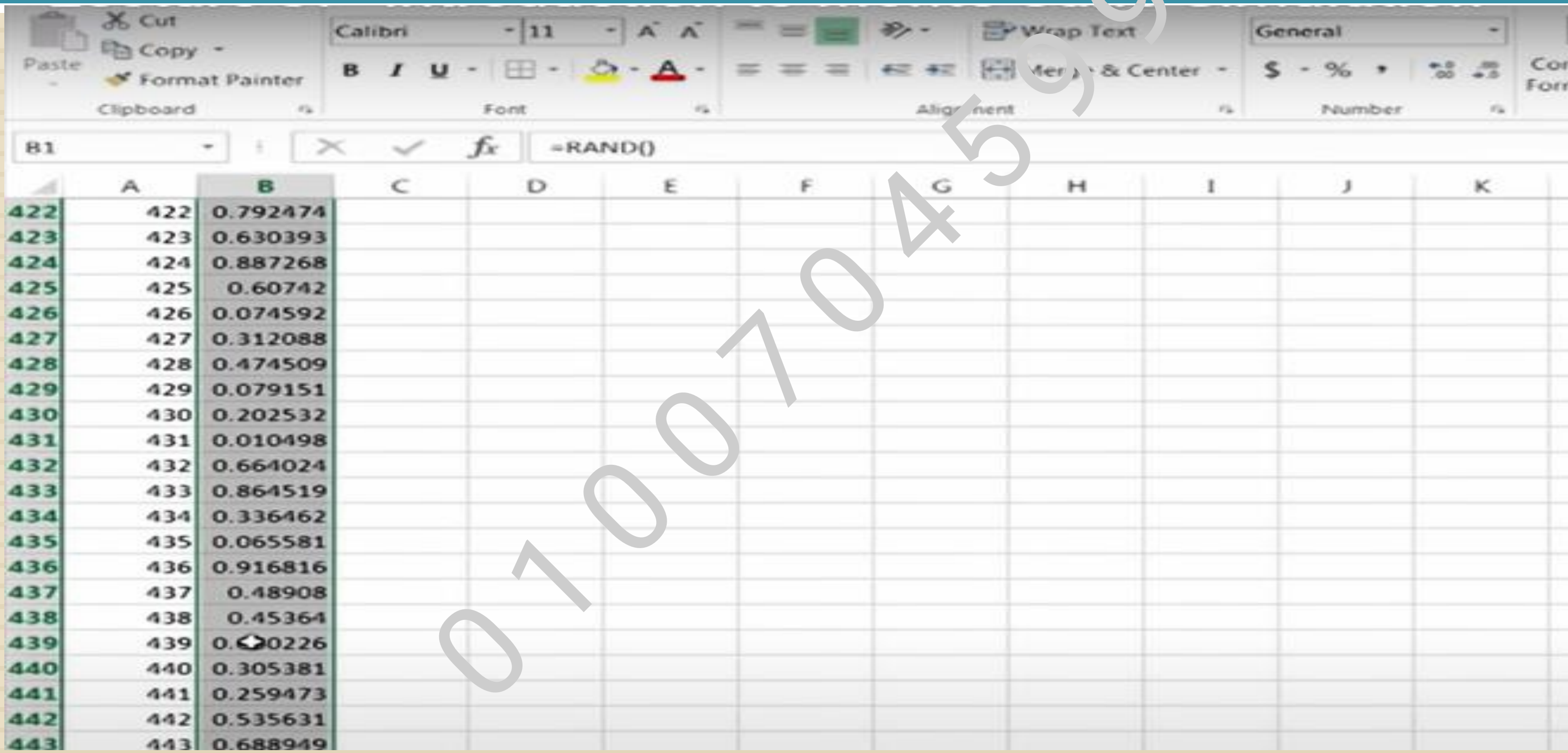
Therefore: ***The area of the circle = 0.7 the area of the square***

***The area of the circle = 0.7 x (2 x 2) = 2.8***

- ***Actually the area of the circle = π ≈ 3.14,*** so the result of ***2.8*** is an approximation and ***as the number of iterations increases*** it will ***go closer to the actual value.***
- We can do that on the ***Excel sheet*** by generating a stream of random numbers using ***rand ( )*** function. Let us first have the numbering (1, 2… to 500) in column A, then generate random numbers between [0, 1] by the command ***rand ( )*** in column B.

# Introduction Monte Carlo Simulation



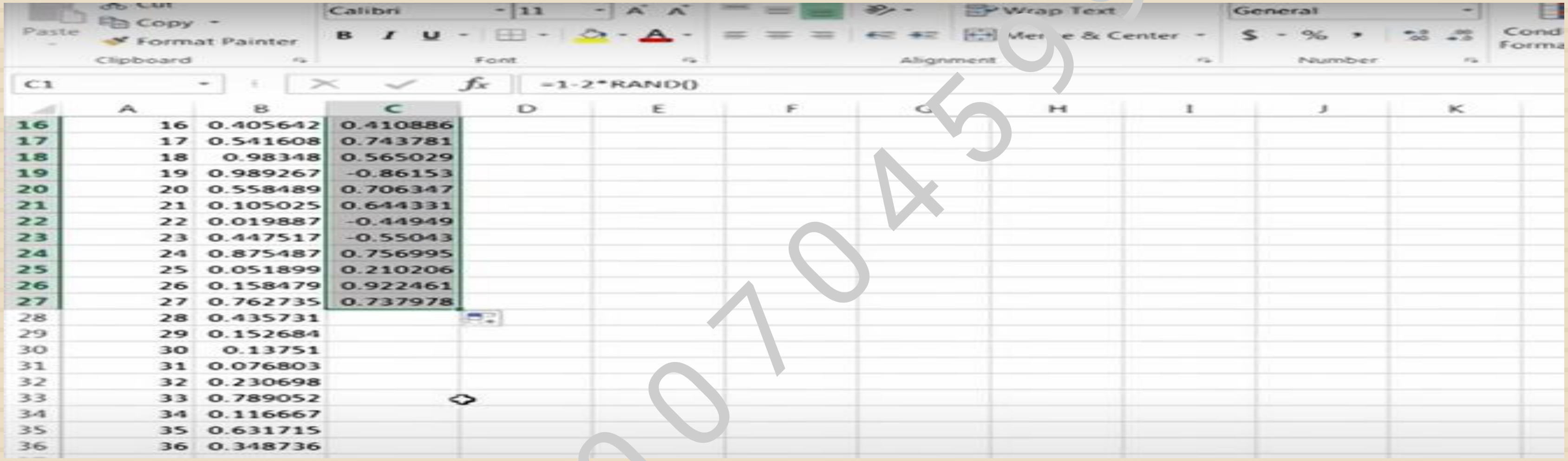




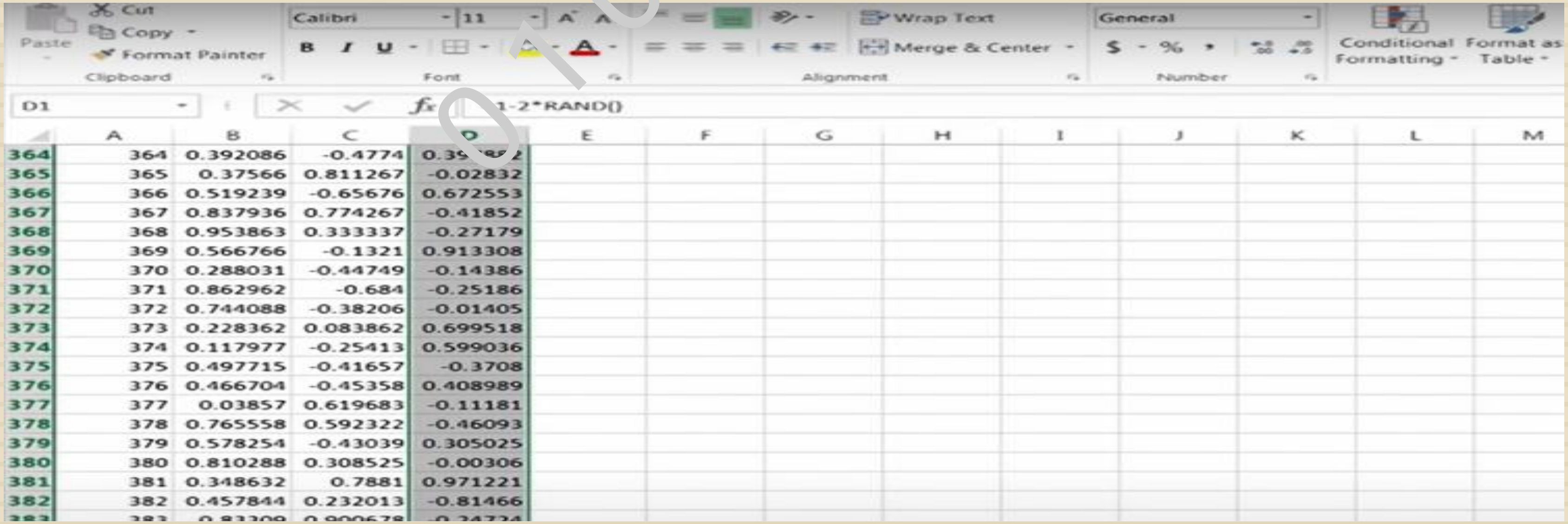

For the circle example, the random numbers should be in between ***[-1, 1]***, so we use the function as: ***1-2*rand ( )*** and the result will be in column C, then generating a similar stream in column D:

# Introduction Monte Carlo Simulation






| | A | B | C |
|---|---|---|---|
| 16 | 16 | 0.405642 | 0.4108886 |
| 17 | 17 | 0.541608 | 0.7743781 |
| 18 | 18 | 0.98348 | 0.565029 |
| 19 | 19 | 0.989267 | -0.86153 |
| 20 | 20 | 0.558489 | 0.706347 |
| 21 | 21 | 0.105025 | 0.644331 |
| 22 | 22 | 0.019887 | -0.44949 |
| 23 | 23 | 0.447517 | -0.55043 |
| 24 | 24 | 0.875487 | 0.756995 |
| 25 | 25 | 0.051899 | 0.210206 |
| 26 | 26 | 0.158479 | 0.922461 |
| 27 | 27 | 0.762735 | 0.737978 |
| 28 | 28 | 0.435731 | |
| 29 | 29 | 0.152684 | |
| 30 | 30 | 0.13751 | |
| 31 | 31 | 0.076803 | |
| 32 | 32 | 0.230698 | |
| 33 | 33 | 0.789052 | |
| 34 | 34 | 0.116667 | |
| 35 | 35 | 0.631715 | |
| 36 | 36 | 0.348736 | |

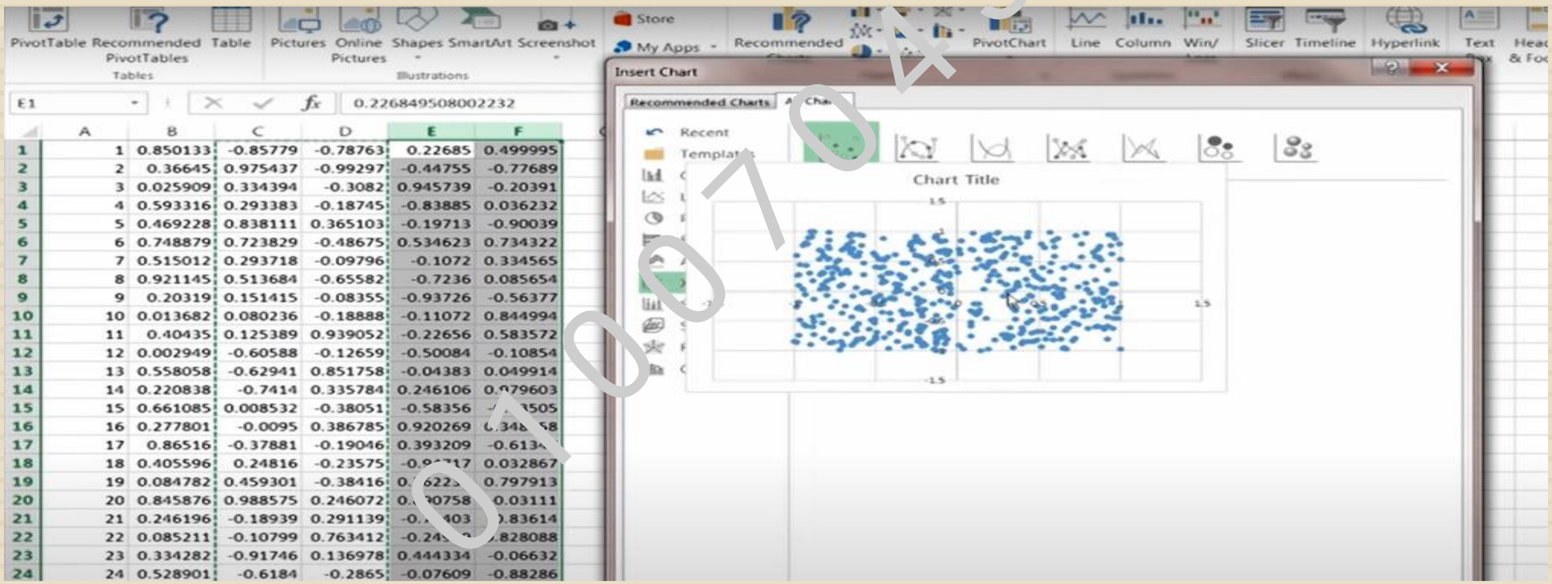



| | A | B | C | D |
|---|---|---|---|---|
| 364 | 364 | 0.392086 | -0.4774 | 0.39[illegible]2 |
| 365 | 365 | 0.37566 | 0.811267 | -0.02832 |
| 366 | 366 | 0.519239 | -0.65676 | 0.672553 |
| 367 | 367 | 0.837936 | 0.774267 | -0.41852 |
| 368 | 368 | 0.953863 | 0.333337 | -0.27179 |
| 369 | 369 | 0.566766 | -0.1321 | 0.913308 |
| 370 | 370 | 0.288031 | -0.44749 | -0.14386 |
| 371 | 371 | 0.862962 | -0.684 | -0.25186 |
| 372 | 372 | 0.744088 | -0.38206 | -0.01405 |
| 373 | 373 | 0.228362 | 0.083862 | 0.699518 |
| 374 | 374 | 0.117977 | -0.25413 | 0.599036 |
| 375 | 375 | 0.497715 | -0.41657 | -0.3708 |
| 376 | 376 | 0.466704 | -0.45358 | 0.408989 |
| 377 | 377 | 0.03857 | 0.619683 | -0.11181 |
| 378 | 378 | 0.765558 | 0.592322 | -0.46093 |
| 379 | 379 | 0.578254 | -0.43039 | 0.305025 |
| 380 | 380 | 0.810288 | 0.308525 | -0.00306 |
| 381 | 381 | 0.348632 | 0.7881 | 0.971221 |
| 382 | 382 | 0.457844 | 0.232013 | -0.81466 |

# Introduction Monte Carlo Simulation



- If we plot these points on the scatter (from ***Insert*** menu on the excel sheet choose ***scatter***) to notice that their uniform distribution:




To draw these random numbers in a circle of radius = 1, the circle equation is $x^2 + y^2 = 1$ with the center in the origin. On the Excel sheet in column G we have the condition: **IF** ***(Column E)*** $^2$ **+ *(column F)*** $^2$ **< *1, then 1, else 0***

$$= if\ ((E1^\wedge 2 + F1^\wedge 2) < 1, 1, 0)$$

Gives the points ***in the quarter of the circle***.

# Introduction Monte Carlo Simulation



the condition: ***(Column E)*** $^{2}$ **+ *(column F)*** $^{2} \leq$ ***1, then 1, else 1***

***Gives the points in the quarter of the square.***

Paste | Copy | Format Painter | Clipboard | B I U | Font | Merge & Cen | Alignment

G1 | $f_x$ | =IF((E1^2+F1^2)<1,1,0)

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 482 | 482 | 0.369577 | -0.04257 | 0.070995 | -0.0499 | 0.253731 | 1 | |
| 483 | 483 | 0.884372 | 0.517102 | -0.77351 | 0.205613 | 0.671137 | 1 | |
| 484 | 484 | 0.117667 | 0.25627 | -0.425091 | -0.40671 | -0.12136 | 1 | |
| 485 | 485 | 0.881388 | -0.66744 | 0.927577 | -0.30112 | -0.03829 | 1 | |
| 486 | 486 | 0.676464 | -0.67441 | -0.77493 | -0.48409 | 0.537462 | 1 | |
| 487 | 487 | 0.089608 | -0.90726 | -0.85461 | -0.46158 | -0.35151 | 1 | |
| 488 | 488 | 0.803775 | 0.889932 | -0.91011 | -0.49341 | -0.68381 | 1 | |
| 489 | 489 | 0.606267 | -0.81867 | -0.1643 | -0.65002 | -0.42415 | 1 | |
| 490 | 490 | 0.063172 | -0.97427 | -0.96519 | -0.20902 | -0.8666 | 1 | |
| 491 | 491 | 0.791107 | 0.417003 | 0.512496 | -0.50753 | -0.47211 | 1 | |
| 492 | 492 | 0.520432 | 0.033609 | -0.60663 | -0.23276 | -0.80796 | 1 | |
| 493 | 493 | 0.980021 | 0.359227 | -0.04996 | -0.45114 | 0.525648 | 1 | |
| 494 | 494 | 0.16358 | 0.552185 | 0.916195 | 0.116863 | -0.39272 | 1 | |
| 495 | 495 | 0.420129 | -0.14308 | 0.105817 | -0.75755 | -0.85991 | 0 | |
| 496 | 496 | 0.333147 | 0.586983 | -0.41755 | -0.21349 | 0.126385 | 1 | |
| 497 | 497 | 0.343172 | 0.231675 | 0.259384 | -0.87755 | -0.5054 | 0 | |
| 498 | 498 | 0.453719 | -0.70469 | 0.042727 | -0.05217 | 0.669789 | 1 | |
| 499 | 499 | 0.599895 | -0.85833 | 0.131163 | -0.62955 | -0.85676 | 0 | |
| 500 | 500 | 0.005236 | 0.85094 | 0.653051 | -0.64392 | 0.71699 | 1 | |
| 501 | 501 | | | | | | | |

***For example:***

$(0.87)^2 + (0.505)^2 = 0.7569 + 0.255 = 1.0119 > 1 \rightarrow 0$ ***i.e. outside the circle***

$(0.05)^2 + (0.66)^2 = 0.0025 + 0.4356 = 0.4381 < 1 \rightarrow 1$ ***i.e. inside the circle***

# Introduction Monte Carlo Simulation





G1 $f_x$ =IF((E1^2+F1^2)<1,1,0)

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 491 | 491 | 0.681604 | -0.62841 | -0.0783 | [illegible] | -0.47211 | 1 | |
| 492 | 492 | 0.236149 | -0.20481 | [illegible] | -0.23276 | -0.80796 | 1 | |
| 493 | 493 | 0.840419 | 0.246676 | [illegible] | -0.45114 | 0.525648 | 1 | |
| 494 | 494 | 0.874455 | 0.286677 | [illegible] | 0.116863 | -0.39272 | 1 | |
| 495 | 495 | 0.182402 | -0.80486 | [illegible] | -0.75755 | -0.85991 | 0 | |
| 496 | 496 | 0.737001 | -0.61612 | [illegible] | -0.21349 | 0.126385 | 1 | |
| 497 | 497 | 0.831277 | 0.99117 | [illegible] | -0.87755 | -0.5054 | 0 | |
| 498 | 498 | 0.661278 | 0.120597 | [illegible] | -0.05217 | 0.669789 | 1 | |
| 499 | 499 | 0.728063 | [illegible] | [illegible] | -0.62955 | -0.85676 | 0 | |
| 500 | 500 | 0.314502 | [illegible] | -0.35922 | -0.64392 | 0.71699 | 1 | |
| 501 | 501 | | | | | | 403 | |
| 502 | 502 | | | | | | | |
| 503 | 503 | | | | | | | |
| 504 | 504 | | | | | | | |

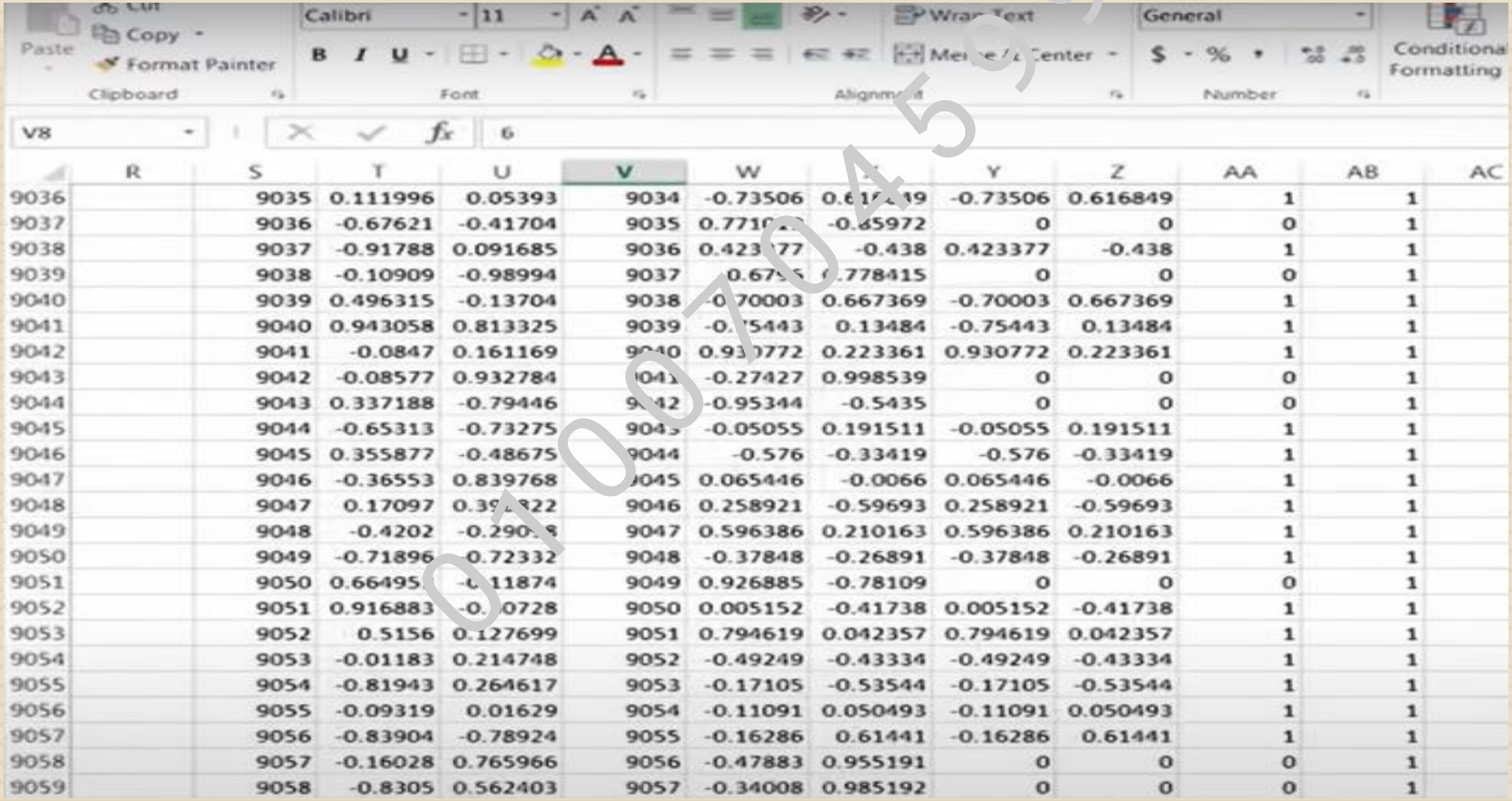
***The sum of column G gives 403*** which means ***403 ones out of 500***, i.e. ***1/4 of the area = 403/500 = 0.806 ≈ π/4*** which means ***π ≈ 3.224***

***While the actual value of π is 3.14.***

If we increase the number of points ***more than 500 the result will be closer to the actual value.***

# Introduction Monte Carlo Simulation






| | R | S | T | U | V | W | X | Y | Z | AA | AB | AC |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9036 | | 9035 | 0.111996 | 0.05393 | 9034 | -0.73506 | [illegible] | -0.73506 | 0.616849 | 1 | 1 | |
| 9037 | | 9036 | -0.67621 | -0.41704 | 9035 | [illegible] | -0.85972 | 0 | 0 | 0 | 1 | |
| 9038 | | 9037 | -0.91788 | 0.091685 | 9036 | [illegible] | -0.438 | 0.423377 | -0.438 | 1 | 1 | |
| 9039 | | 9038 | -0.10909 | -0.98994 | 9037 | [illegible] | 0.778415 | 0 | 0 | 0 | 1 | |
| 9040 | | 9039 | 0.496315 | -0.13704 | 9038 | -0.70003 | 0.667369 | -0.70003 | 0.667369 | 1 | 1 | |
| 9041 | | 9040 | 0.943058 | 0.813325 | 9039 | -0.75443 | 0.13484 | -0.75443 | 0.13484 | 1 | 1 | |
| 9042 | | 9041 | -0.0847 | 0.161169 | 9040 | 0.930772 | 0.223361 | 0.930772 | 0.223361 | 1 | 1 | |
| 9043 | | 9042 | -0.08577 | 0.932784 | 9041 | -0.27427 | 0.998539 | 0 | 0 | 0 | 1 | |
| 9044 | | 9043 | 0.337188 | -0.79446 | 9042 | -0.95344 | -0.5435 | 0 | 0 | 0 | 1 | |
| 9045 | | 9044 | -0.65313 | -0.73275 | 9043 | -0.05055 | 0.191511 | -0.05055 | 0.191511 | 1 | 1 | |
| 9046 | | 9045 | 0.355877 | -0.48675 | 9044 | -0.576 | -0.33419 | -0.576 | -0.33419 | 1 | 1 | |
| 9047 | | 9046 | -0.36553 | 0.839768 | 9045 | 0.065446 | -0.0066 | 0.065446 | -0.0066 | 1 | 1 | |
| 9048 | | 9047 | 0.17097 | [illegible] | 9046 | 0.258921 | -0.59693 | 0.258921 | -0.59693 | 1 | 1 | |
| 9049 | | 9048 | -0.4202 | [illegible] | 9047 | 0.596386 | 0.210163 | 0.596386 | 0.210163 | 1 | 1 | |
| 9050 | | 9049 | -0.71896 | -0.72332 | 9048 | -0.37848 | -0.26891 | -0.37848 | -0.26891 | 1 | 1 | |
| 9051 | | 9050 | [illegible] | -0.11874 | 9049 | 0.926885 | -0.78109 | 0 | 0 | 0 | 1 | |
| 9052 | | 9051 | 0.916883 | [illegible] | 9050 | 0.005152 | -0.41738 | 0.005152 | -0.41738 | 1 | 1 | |
| 9053 | | 9052 | 0.5156 | 0.127699 | 9051 | 0.794619 | 0.042357 | 0.794619 | 0.042357 | 1 | 1 | |
| 9054 | | 9053 | -0.01183 | 0.214748 | 9052 | -0.49249 | -0.43334 | -0.49249 | -0.43334 | 1 | 1 | |
| 9055 | | 9054 | -0.81943 | 0.264617 | 9053 | -0.17105 | -0.53544 | -0.17105 | -0.53544 | 1 | 1 | |
| 9056 | | 9055 | -0.09319 | 0.01629 | 9054 | -0.11091 | 0.050493 | -0.11091 | 0.050493 | 1 | 1 | |
| 9057 | | 9056 | -0.83904 | -0.78924 | 9055 | -0.16286 | 0.61441 | -0.16286 | 0.61441 | 1 | 1 | |
| 9058 | | 9057 | -0.16028 | 0.765966 | 9056 | -0.47883 | 0.955191 | 0 | 0 | 0 | 1 | |
| 9059 | | 9058 | -0.8305 | 0.562403 | 9057 | -0.34008 | 0.985192 | 0 | 0 | 0 | 1 | |

***For a stream of 10000 random numbers we get:***

***7857 ones out of 10000, i.e. the area = (7857/10000) x 4 = 3.1428***

## *5.2 Inventory Control Simulation using Monte Carlo Technique*

- For inventory control problem we are interested in finding ***when to reorder***, ***what is the reorder level***, ***what is the daily demand***, ***what is the stock in hand***, ***what is the total cost of holding material***, ***and what is the shortage cost***. Those issues are to be modeled trough this technique.

## *Example 5.3*

**Suppose you have certain *demand for the past 100 days* and the *lead time* in which the ordered quantity has arrived are given in the following table.**

- Notice that for **the lead time we may have a standard frequency distribution** rather than the given frequencies to find the random numbers corresponding to a particular lead time.
- **We need to know *the demand in the coming days* and the *possible lead time*, and *when to order*.**

# Inventory Control Simulation

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 22 | 44 | 33 | 57 | 41 | 45 | 49 | 47 | 44 | 44 |
| 48 | 45 | 33 | 55 | 41 | 42 | 50 | 72 | 46 | 43 |
| 66 | 40 | 70 | 43 | 39 | 49 | 44 | 66 | 39 | 31 |
| 52 | 42 | 44 | 31 | 42 | 34 | 38 | 38 | 48 | 31 |
| 52 | 43 | 50 | 57 | 42 | 44 | 43 | 40 | 53 | 48 |
| 52 | 49 | 50 | 37 | 40 | 28 | 46 | 47 | 31 | 38 |
| 26 | 30 | 62 | 35 | 41 | 42 | 43 | 39 | 38 | 47 |
| 32 | 41 | 43 | 43 | 64 | 51 | 55 | 31 | 55 | 46 |
| 52 | 45 | 44 | 56 | 34 | 52 | 50 | 56 | 59 | 58 |
| 58 | 44 | 57 | 53 | 46 | 54 | 33 | 27 | 57 | 39 |

**Demand for past 100 days**

| Lead time | Frequency |
|---|---|
| 3 days | 28% |
| 4 days | 44% |
| 5 days | 28% |

Suppose that in the inventory control we have the following costs:

**$C_0$ … is the cost of placing an order = *100 pound/order***

**$C_i$ … is the cost of holding an item in stock = *0.1 pound/unit/day***

**$C_s$ … is the cost of shortage = *5 pounds/unit***

# *Inventory Control Simulation*



(*when you have shortage you have to bring items from local market at higher price*)

**We will calculate what will be the reordered level and other quantities by statistical calculations. We have: *Mean demand $\mu_D$***

***$\mu_D$ = Σ all demand in (n = 100) days / 100 = 45.23 unit/day***

***Standard deviation of the demand $\sigma_D = \sqrt{[(\Sigma Di^2) - [(\Sigma Di)^2 / n]] / (n-1)}$***

$$\sigma_D = \sqrt{[214077 - [(4523)^2 / 100]] / 99} = 9.8$$

***Mean lead time $\mu_L$ = [(28x3) + (44x4) + (28x5)]/100= 400/4 =4 days***

***Standard deviation of the lead time $\sigma_L$***

$$\sigma_L = \sqrt{\{[(28\times3)^2 + (44\times4)^2 + (28\times5)^2] - [(400)^2 / 100]\} / 99} = 0.75$$

All these calculations can be done from the available data.

# *Inventory Control Simulation*



- Now we find the ***histogram of the demand*** which ranges from ***20 to 70***, we calculate the ***frequency, cumulative frequency,*** and assign ***two digit*** random numbers (from ***00 to 99***) for each interval of the demand according to its frequency as shown in the following table. ***When we predict the demand we will take the average value of the demand in each interval.***

| Demand interval | Average demand | frequency | Cumulative frequency | Random numbers |
|---|---|---|---|---|
| 20 – 24 | 22 | 1 | 1 | 00 |
| 25 – 29 | 27 | 3 | 4 | 01 – 03 |
| 30 – 34 | 32 | 12 | 16 | 04 – 15 |
| 35 – 39 | 37 | 10 | 26 | 16 – 25 |
| 40 – 44 | 42 | 26 | 52 | 26 – 51 |
| 45 – 49 | 47 | 17 | 69 | 52 – 68 |
| 50 – 54 | 52 | 13 | 82 | 69 – 81 |
| 55 – 59 | 57 | 12 | 94 | 82 – 93 |
| 60 – 64 | 62 | 2 | 96 | 94 – 95 |
| 65 – 69 | 67 | 2 | 98 | 96 – 97 |
| 70 – 74 | 72 | 2 | 100 | 98 – 99 |

# Inventory Control Simulation



***Similarly*** we can have the calculations for ***the lead time:***

| Lead time | frequency | Cumulative frequency | Random numbers |
|---|---|---|---|
| 3 | 28 | 28 | 00 – 27 |
| 4 | 44 | 72 | 28 – 71 |
| 5 | 28 | 100 | 72 – 99 |

***economic order quantity (EOQ) is given as:***

$EOQ = \sqrt{2\,\mu_D\,C_0\,/\,Ci} = \sqrt{(2 \times 45.23 \times 100)\,/\,0.1} \approx$ **300 units**

Whenever the stock quantity reaches below certain limit (the ***reorder level***) we have to order this quantity (***300 units***) because we are going for fixed quantity system.

***The reorder level is given as:***

$$ROL = \mu_D\,\mu_L + K\,[\sqrt{\mu_L\,\sigma D^2 + (\mu D^2\,\sigma L^2)}]$$

# *Inventory Control Simulation*



Where **K** is a constant that determines the ***probability of shortage*** of stock, e.g. for ***K = 1.65*** the chance of shortage of stock is ***5 %*** (i.e. ***95%*** sure that there will not be any shortage). If ***K = 2.33*** there will be only ***1 %*** chance of shortage.

For ***K = 1.65:***

$$ROL = (45.23 \times 4) + 1.65 \times \sqrt{4\,(95.98)^2 + (45.23)^2 \times 0.5657} = 245$$

***Now we start the simulation with EOQ = 300 and ROL = 245 units***

- Assuming the ***initial stock = 300 units*** on ***Friday*** evening, so we will ***start simulation from Monday*** onwards.
- The random numbers can be generated from ***(00 to 99)*** if we put in the excel ***100*** into the ***rand ( )*** function. We can also generate random numbers by using other generators.

## Inventory Control Simulation





| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 81 | 78 | 89 | 59 | 40 | 33 | 60 | 67 | 19 | 21 |
| 94 | 51 | 37 | 19 | 31 | 89 | 34 | 62 | 23 | [illegible] |
| 21 | 75 | 28 | 18 | 20 | 10 | 76 | 43 | 91 | 69 |
| 35 | 88 | 90 | 88 | 53 | 00 | 80 | 77 | 42 | 96 |
| 60 | 94 | 49 | 75 | 69 | 26 | 96 | 24 | 05 | 91 |
| 97 | 84 | 54 | 83 | 27 | 28 | 99 | 94 | [illegible] | 19 |
| 92 | 10 | 80 | 60 | 32 | 63 | 08 | 71 | 06 | 14 |
| 23 | 70 | 49 | 30 | 71 | 21 | 23 | 26 | 20 | 76 |
| 82 | 07 | 15 | 38 | 54 | 15 | 75 | 99 | 27 | 84 |
| 80 | 75 | 31 | 64 | 67 | 97 | [illegible] | 06 | 56 | 81 |
| 42 | 10 | 00 | 37 | 24 | 33 | 56 | 28 | 43 | 89 |
| 54 | 94 | 54 | 43 | 71 | [illegible] | 78 | 60 | 72 | 06 |
| 57 | 36 | 84 | 56 | 98 | 10 | 17 | 89 | 53 | 25 |
| 61 | 37 | 35 | 11 | 63 | 87 | 59 | 64 | 92 | 62 |
| 85 | 17 | 23 | 11 | [illegible] | 56 | 35 | 36 | 34 | 52 |
| 02 | 84 | 29 | 56 | [illegible] | 02 | 03 | 35 | 96 | 70 |
| 24 | 64 | 48 | 50 | 42 | 79 | 28 | 99 | 53 | 12 |
| 63 | 20 | 82 | 33 | 82 | 22 | 07 | 33 | 39 | 93 |
| 38 | 94 | 98 | 52 | 70 | 50 | [illegible] | [illegible] | [illegible] | [illegible] |
| 98 | 76 | 37 | 41 | 46 | 27 | [illegible] | [illegible] | [illegible] | [illegible] |

So, suppose this is the random number which
is to be used, so we have to use the first

Random numbers

We will use random numbers ***for demand from the left side of the first row*** and ***for lead time from the right side of the last row (65, 67, 22 ..)***

# Inventory Control Simulation

## (initial stock = 300 units, EOQ = 300 units, and ROL = 245 units)



**The simulation table for two working weeks (10 working days will be):**

| Day | Initial stock | R. No. for demand | Daily demand | Shortage | Quantity received | Final stock | Quantity in hand + order | Quantity ordered | R. No. for lead time | Lead time |
|---|---|---|---|---|---|---|---|---|---|---|
| M | 300 | 81 | 52 | - | - | 248 | 248 | 0 | - | - |
| T | 248 | 78 | 52 | - | - | 196 | 196 | 300 | 65 | 4 |
| W | 196 | 89 | 57 | - | - | 139 | 439 | 0 | - | - |
| Th | 139 | 59 | 47 | - | - | 92 | 392 | 0 | - | - |
| F | 92 | 40 | 42 | - | - | 50 | 350 | 0 | - | - |
| M | 50 | 33 | 42 | - | 300 | 308 | 308 | 0 | - | - |
| T | 308 | 60 | 47 | - | - | 261 | 261 | 0 | - | - |
| W | 261 | 67 | 47 | - | - | 214 | 214 | 300 | 67 | 4 |
| Th | 214 | 19 | 37 | - | - | 177 | 477 | 0 | - | - |
| F | 177 | 21 | 37 | - | - | 140 | 440 | 0 | - | - |

***Final stock = initial stock – daily demand + quantity received***

On first Monday = ***300 – 52 = 248 units, Initial stock = previous final stock → on Tuesday = 248 and final stock = 248 – 52 = 196 → we have to reorder***

# *Inventory Control Simulation*



***Quantity in hand + order = final stock + quantity ordered***

When the ***lead time is 4 working days*** from ***Tuesday***, the quantity ordered will be ***received on next Monday***, final stock will be:

***Final stock on next Monday = (50 – 42) + 300 = 308 units***

The second ordered quantity will be received on next Tuesday.

We can calculate parameters like cost in certain period:

***Cost of holding inventory = holding cost per unit x final stock***

***Cost of ordering = number of orders x order cost***

# Monte Carlo Simulation of Queueing Problem



## 5.3 Monte Carlo Simulation of Queueing Problem

Monte Carlo simulation can be used in solving different kinds of problems like ***inventory control, reliability, queueing*** … etc. In this section we discuss ***Monte Carlo simulation of queueing problem***.

### Example 5.4

At a ***dentist clinic***, apply ***Monte Carlo simulation*** to find ***the average waiting time of customers***. Simulation ***starts at 8 a.m.*** in the morning and the ***arrival of customers is scheduled*** (in other cases ***it may be randomly depending on some probability distributions***). The ***first customer*** arrives at ***8 a.m.*** and every ***30 minutes one customer is coming for five different treatments to be carried out***. The five treatment services are ***filling, crown, cleaning, extraction, and checkup*** with different probabilities and service times.

# *Monte Carlo Simulation of Queueing Problem*



***We do not know the treatment service for which a customer is coming*** and that guess will be done ***by referring random numbers*** generated from any generator as shown in the following table:

| services | Service time | Probability of service | Cumulative probability | Random number intervals |
|---|---|---|---|---|
| Filling | 40 min. | 0.40 | 0.40 | 00 – 39 |
| Crown | 60 min. | 0.15 | 0.55 | 40 – 54 |
| Cleaning | 15 min. | 0.15 | 0.70 | 55 – 69 |
| Extraction | 45 min. | 0.10 | 0.80 | 70 – 79 |
| Checkup | 15 min. | 0.20 | 1.00 | 80 – 99 |

Then we can predict for what service a customer is coming.

Given the random numbers ***40, 82, 11, 34, 25, 66, 17, 79*** to be used for predicting the required service for ***8 patients*** by the dentist:

# Monte Carlo Simulation of Queueing Problem



| Patient | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Arrival time | 8:00 | 8:30 | 9:00 | 9:30 | 10:00 | 10:30 | 11:00 | 11:30 |
| Random number of service | 40 | 82 | 11 | 34 | 25 | 66 | 17 | 79 |
| Category of service | crown | checkup | filling | filling | filling | cleaning | filling | extraction |
| Service time | 60 | 15 | 40 | 40 | 40 | 15 | 40 | 45 |

*Then we have to find when a customer arrives and when he leaves to calculate the average waiting time:*

| patient | Arrival time | Service time | Service starts at: | Waiting time |
|---|---|---|---|---|
| 1 | 8:00 | 60 | 8:00 | 0 minutes |
| 2 | 8:30 | 15 | 9:00 | 30 minutes |
| 3 | 9:00 | 40 | 9:15 | 15 minutes |
| 4 | 9:30 | 40 | 9:55 | 25 minutes |
| 5 | 10:00 | 40 | 10:35 | 35 minutes |
| 6 | 10:30 | 15 | 11:15 | 45 minutes |
| 7 | 11:00 | 40 | 11:30 | 30 minutes |
| 8 | 11:30 | 45 | 12:10 | 40 minutes |

# *Monte Carlo Simulation of Queueing Problem*



***The total waiting time = 220 minutes***

***Average waiting time = 220 / 8 = 27.5 minutes***

In many cases we have to generate the random numbers from certain distribution. ***Inter-arrival time and service time follow the exponential distribution in most cases.*** To generate random numbers using the exponential distribution we use ***the inverse transform method.***

## *EXAMPLE 5.5*

Let X have the exponential distribution with **mean β**, the distribution function is:

$$F(x) = \begin{cases} 1 - e^{-x/\beta} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

so to find $F^{-1}$, we set $u = F(x)$ and solve for $x$ to obtain

$$F^{-1}(u) = -\beta \ln (1 - u)$$

# *Monte Carlo Simulation of Queueing Problem*



- Thus, to generate the desired random variate, we first generate a ***U~U(0,1)*** and then let **X = - β ln U**. [It is possible in this case to use ***U*** instead of (***1– U)*** since ***(1 – U)*** and ***U*** have the same ***U(0, 1)*** distribution. [This saves a subtraction.]

## Example 5.6

Suppose in ***example 5.4*** the ***inter-arrival time is exponentially distributed*** with mean value $\boldsymbol{\beta_1 = 2}$ and ***service time is also exponentially distributed*** with mean value $\boldsymbol{\beta_2 = 1.33}$. From the simulation table for the arrival, service, and departure times of 8 customers ***calculate the average waiting time for a customer.***

### Solution

We get the random variates representing the ***inter-arrival time*** $X_i$ and ***service time*** $Y_i$ from the inverse transform equations:

$$X_i = -\beta_1 \ln U_1 \qquad \text{and} \qquad Y_i = -\beta_2 \ln U_2$$

Where the random numbers ***0 ≤ (both*** $U_1$ ***and*** $U_2$***) ≤ 1***. Given a stream of ***two digit*** random numbers $u_1, u_2$ [from ***00*** to ***99***], we can get $U_1, U_2$ by dividing each random number in the stream by 100, then we calculate $X_i$, $Y_i$ from the above equations, we get the following results:

# *Monte Carlo Simulation of Queueing Problem*



***$u_1 \rightarrow$20, 23, 86, 09, 92, 35, 38, 01 & $u_2$=21, 44, 27, 70, 73, 36, 59, 42***

For $u_1 = 20 \rightarrow U_1 = 20/100 = 0.20 \rightarrow X_1 = -2 \ln(0.20) = 3.22$

For $u_2 = 21 \rightarrow U_2 = 21/100 = 0.21 \rightarrow Y_1 = -1.33 \ln(0.21) = 2.08$

Similarly for all values of $u_1$ and $u_2$ as shown in the simulation table:

| $u_1$ | Inter-arrival time $X_i$ | Arrival time | $u_2$ | Service time $Y_i$ | Departure time | Waiting time |
|---|---|---|---|---|---|---|
| 20 | 3.22 | 3.22 | 21 | 2.08 | 5.30 | 0 |
| 23 | 2.94 | 6.16 | 44 | 1.09 | 7.25 | 0 |
| 86 | 0.30 | 6.46 | 27 | 1.75 | 9.0 | 0.79 |
| 09 | 4.82 | 11.28 | 70 | 0.48 | 11.76 | 0 |
| 92 | 0.17 | 11.45 | 73 | 0.42 | 12.18 | 0.31 |
| 35 | 2.10 | 13.55 | 36 | 1.36 | 14.91 | 0 |
| 38 | 1.94 | 15.49 | 59 | 0.70 | 16.19 | 0 |
| 01 | 9.21 | 24.70 | 42 | 1.16 | 25.86 | 0 |

*The average waiting time = (0.79 + 0.31) / 8 = 0.14 and we can calculate other measures of performance.*

# *Monte Carlo Simulation of Queueing Problem*



## Example 5.7

Simulate the operation on a ***teller window*** for ***one hour***, given the following data:

$u_1 \longrightarrow$ ***25, 37, 91, 00, 61, 62, 80, 15, 23***

$u_2 \longrightarrow$ ***84, 01, 59, 40, 03, 22, 50, 77, 32***

| Inter-arrival time (minutes) | 5 | 6 | 7 | 8 |
|---|---|---|---|---|
| Probability | 0.15 | 0.35 | 0.35 | 0.15 |
| Random No. | 00-14 | 15-49 | 50-84 | 85-99 |

| Service time (minutes) | 5 | 6 | 7 |
|---|---|---|---|
| Probability | 0.25 | 0.50 | 0.25 |
| Random No. | 00-24 | 25-74 | 75-99 |

# Monte Carlo Simulation of Queueing Problem



*The simulation table for one hour will be:*

| Random number $u_1$ | Inter-arrival time | Arrival time | Random number $u_2$ | Service time | Departure time | Waiting time | Server idle time |
|---|---|---|---|---|---|---|---|
| 25 | 6 | 6 | 84 | 7 | 13 | 0 | 6 |
| 37 | 6 | 12 | 01 | 5 | 18 | 1 | 0 |
| 91 | 8 | 20 | 59 | 6 | 26 | 0 | 2 |
| 00 | 5 | 25 | 40 | 6 | 32 | 1 | 0 |
| 61 | 7 | 32 | 03 | 5 | 37 | 0 | 0 |
| 62 | 7 | 39 | 29 | 6 | 45 | 0 | 2 |
| 80 | 7 | 46 | 50 | 6 | 52 | 0 | 1 |
| 15 | 6 | 52 | 77 | 7 | 59 | 0 | 0 |
| 23 | 6 | 58 | 32 | 6 | 65 stop | 1 | 0 |
| | | | | | | Total = 3 | Total =11 |

When departure time reaches beyond 60 minutes we have to stop.

***Average waiting time per customer = 3/9 = 1/3 minutes***

***Percentage utilization of server = [(65-11)/65] x 100 = 83 %***

# Monte Carlo Simulation in Manufacturing



## 5.4 Monte Carlo Simulation in Manufacturing

We can have the generated random numbers by suitable software programs or by using different kinds of random number generators like linear congruential generator or mixed generator as shown in ***Ex. 5.8***

## Example 5.8

Given the units produced in a manufacturing facility and the number of vehicles available for transportation at a particular station with the corresponding probabilities as shown in the following table:

| Units produced $X_i$ | Probability $P(x = X_i)$ | Random number intervals $u_1$ | Number of vehicles available $Y_i$ | Probability $P(y = Y_i)$ | Random number intervals $u_2$ |
|---|---|---|---|---|---|
| 500 | 0.06 | 00 – 05 | 5 | 0.16 | 00 – 15 |
| 550 | 0.14 | 06 – 19 | 6 | 0.36 | 16 – 51 |
| 600 | 0.20 | 20 – 39 | 7 | 0.20 | 52 – 71 |
| 650 | 0.40 | 40 – 79 | 8 | 0.16 | 72 – 87 |
| 700 | 0.20 | 80 – 99 | 9 | 0.12 | 88 – 99 |

Using ***mixed congruential generator*** of 3 parameters ***a***, ***c***, and ***m*** to generate two digit random numbers between ***00*** and ***99***, *given:*

***For*** $u_1 \rightarrow a = 21, c = 3, m = 100$, ***seed*** $X_0 = 20$

***For*** $u_2 \rightarrow a = 41, c = 7, m = 100$, ***seed*** $X_0 = 1$, ***where:***

$$X_i = a\, X_{i-1} + c \bmod (m)$$

***For*** $u_1$: $X_0 = 20$

$X_1 = a\, X_0 + c \bmod (m) = (21 \times 20) + 3 \bmod (100) = 423 \bmod (100) = 23$

$X_2 = (21 \times 23) + 3 \bmod (100) = 486 \bmod (100) = 86$ ***and so on to*** $X_9$

***Similarly for*** $u_2$: $X_0 = 1$

$X_1 = a\, X_0 + c \bmod (m) = (41 \times 1) + 7 \bmod (100) = 48 \bmod (100) = 48$

$X_2 = (41 \times 48) + 7 \bmod (100) = 1975 \bmod (100) = 75$ ***and so on to*** $X_9$

# CHAPTER 8
# Monte Carlo Simulation



Now we can have the simulation table for ***10 days***.

| Day | R.No. for units produced | Units produced | R.No. for vehicles availability | Vehicles available |
|---|---|---|---|---|
| 1 | 20 | 600 | 01 | 5 |
| 2 | 23 | 600 | 48 | 6 |
| 3 | 86 | 700 | 75 | 8 |
| 4 | 09 | 550 | 82 | 8 |
| 5 | 92 | 700 | 69 | 7 |
| 6 | 35 | 600 | 36 | 6 |
| 7 | 38 | 600 | 83 | 8 |
| 8 | 01 | 500 | 10 | 5 |
| 9 | 24 | 600 | 17 | 6 |
| 10 | 07 | 550 | 04 | 5 |
| | | **Total = 6000** | | **Total = 64** |

Average number of units produced per day = ***6000 / 10 = 600*** units/day

Average number of available vehicles per day = ***64/10 = 6.4*** vehicle/day

## Reference Book



# Simulation Modeling and Analysis

Averill M. Law