# MODELING

## AND

# SIMULATION

***LECTURE 5***

**Dr. Hamdy Heniedy**

*hheniedy@mans.edu.eg*

# CHAPTER 5
# Monte Carlo Simulation

- We define Monte Carlo simulation as ***a scheme* (مخطط) *employing random numbers, that is, U (0, 1) random variates, which is used for solving certain stochastic or deterministic problem.*** Thus, a stochastic discrete-event simulation is included in this definition. The name "***Monte Carlo***" simulation or method originated during World War II, when this approach was applied to problems related to the development of the ***atomic bomb and other weapons***.

## *5.1 Introduction to Monte Carlo Simulation*

### *5.1.1 Introduction*

- **Monte Carlo technique deals with taking random samples from distribution of a variable in order to supply a series of values for use in the model.**

- This requires ***a method of generating or obtaining random variates from the uniform distribution*** on the interval [0, 1] this distribution is denoted by ***U(0, 1)*** and random variates generated from the ***U(0, 1)*** distribution will be called random numbers.

# Introduction Monte Carlo Simulation

- The probability of any random number in that domain [0, 1] is equal. Random values can be conveniently and efficiently generated from a desired probability distribution by ***the inverse transform method*** for use in executing simulation models. Other methods of generating random numbers like ***linear congruential and mixed generators*** are explained in ***chapter 3.***
- **Monte Carlo methods are stochastic and are typically simple to implement.**
- **It is often used to solve *stochastic problems* to mimic randomness of physical behaviour.**
- **Monte Carlo can also be used to solve *deterministic problems* by using probability for estimation.**

# Introduction Monte Carlo Simulation

## *5.1.2 Requirements of Stochastic Simulation*

- Knowledge of relevant probability distributions depend on the **theoretical or empirical information about physical system being simulated.**

- **Supply of random numbers for making random choices.** By simulating large number of trials, probability distribution of overall results can be approximated with accuracy attained depending on number of trials. ***As the number of iterations increases we come to actual value.***

# Introduction Monte Carlo Simulation

## Example 5.1

In a queueing system, the arrival rate and service rate in minutes with their probabilities are given as:

| Arrival rate | Probability | | Service rate | Probability |
|---|---|---|---|---|
| 3 | 15 % | | 4 | 30 % |
| 4 | 30 % | | 5 | 40 % |
| 5 | 20 % | | 6 | 30 % |
| 6 | 35 % | | | |

As a conclusion from the given data, ***they are not following any typical distribution.***

To get help from random numbers in predicting the values of number of arrival and service rate, we find the ***cumulative frequency*** and ***distribute the uniformly distributed two digit random numbers (from 00 to 99) according to frequency as shown in the following table***:

# Introduction Monte Carlo Simulation

| Arrival rate | Frequency | Cumulative frequency | Two digit random numbers |
|---|---|---|---|
| 3 | 15 | 15 | 00 – 14 |
| 4 | 30 | 45 | 15 – 44 |
| 5 | 20 | 65 | 45 – 64 |
| 6 | 35 | 100 | 65 -99 |

| Service rate | Frequency | Cumulative frequency | Two digit random numbers |
|---|---|---|---|
| 4 | 30 | 30 | 00 – 29 |
| 5 | 40 | 70 | 30 – 69 |
| 6 | 30 | 100 | 70 - 99 |

Generating a stream of random numbers (***from 00 to 99***): (***78, 47, 13, 87, 89, 05 …***) and pick the first number **78** which corresponds to arrival rate of **6 minutes**, and similarly for the service rate. We can use ***the random function software*** to generate uniformly distributed random numbers from the ***excel sheet*** as shown in the following example.

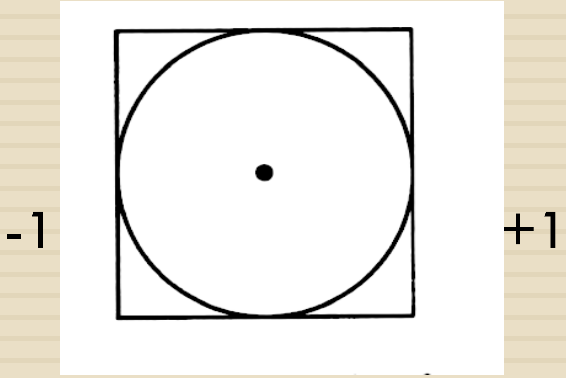
# Introduction Monte Carlo Simulation

## *Example 5.2*

Calculate the deterministic value of **π** by using random numbers.

***Solution***

Assume a circle of ***radius=1*** and its center is at the origin, then its area is **π**. The circle is surrounded by a square, the length of its ***side=2***.



If we select a point ***(from -1 to 1***) at random, it ***may be located inside the circle or outside the circle but still inside the square***. If we repeat this selection for ***10 times, we may get 7 times the point is located inside the circle*** and ***3 times outside the circle*** but ***still inside the square***.

# Introduction Monte Carlo Simulation

This means that: ***Probability of a point located inside the circle= 0.7***

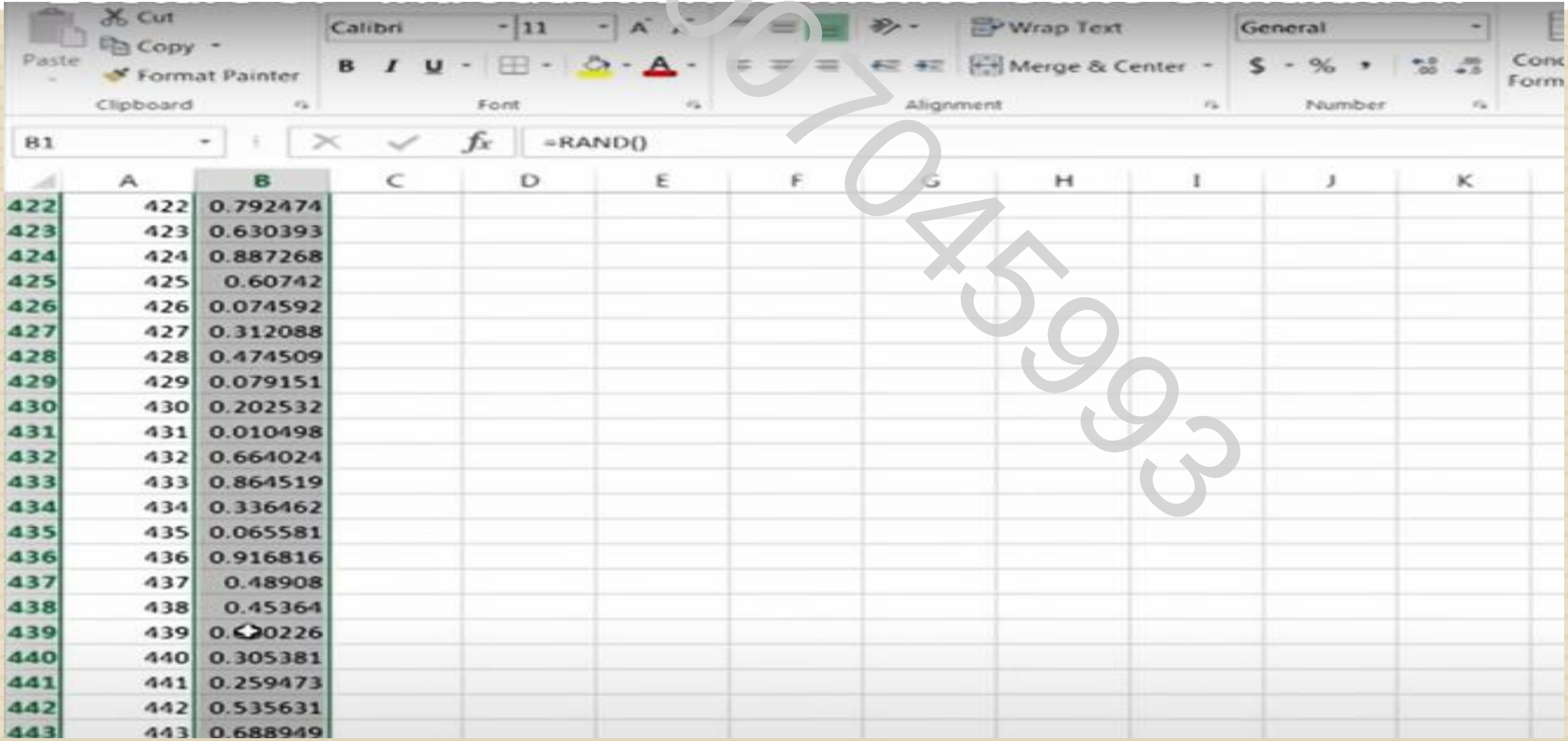
***Probability of a point located outside the circle = 0.3***

Therefore: ***The area of the circle = 0.7 the area of the square***

***The area of the circle = 0.7 x (2 x 2) = 2.8***

- ***Actually the area of the circle = π ≈ 3.14,*** so the result of ***2.8*** is an approximation and ***as the number of iterations increases*** it will ***go closer to the actual value.***
- We can do that on the ***Excel sheet*** by generating a stream of random numbers using ***rand ( )*** function. Let us first have the numbering (1, 2… to 500) in column A, then generate random numbers between [0, 1] by the command ***rand ( )*** in column B.

# Introduction Monte Carlo Simulation

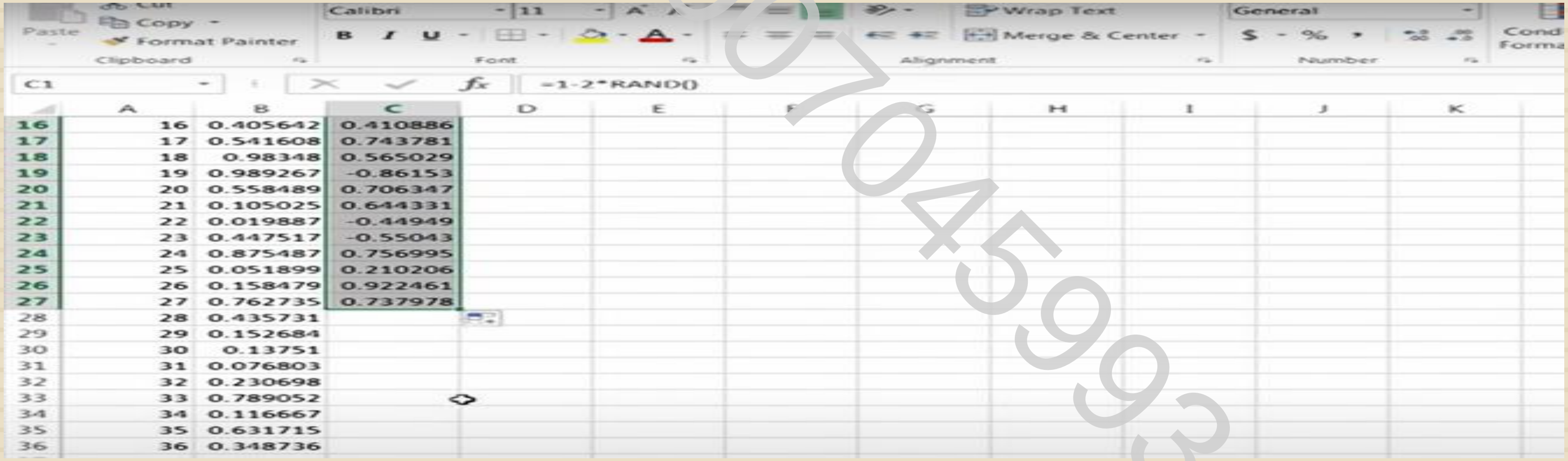


B1 =RAND()

|  | A | B |
|---|---|---|
| 422 | 422 | 0.792474 |
| 423 | 423 | 0.630393 |
| 424 | 424 | 0.887268 |
| 425 | 425 | 0.60742 |
| 426 | 426 | 0.074592 |
| 427 | 427 | 0.312088 |
| 428 | 428 | 0.474509 |
| 429 | 429 | 0.079151 |
| 430 | 430 | 0.202532 |
| 431 | 431 | 0.010498 |
| 432 | 432 | 0.664024 |
| 433 | 433 | 0.864519 |
| 434 | 434 | 0.336462 |
| 435 | 435 | 0.065581 |
| 436 | 436 | 0.916816 |
| 437 | 437 | 0.48908 |
| 438 | 438 | 0.45364 |
| 439 | 439 | 0.[illegible]0226 |
| 440 | 440 | 0.305381 |
| 441 | 441 | 0.259473 |
| 442 | 442 | 0.535631 |
| 443 | 443 | 0.688949 |

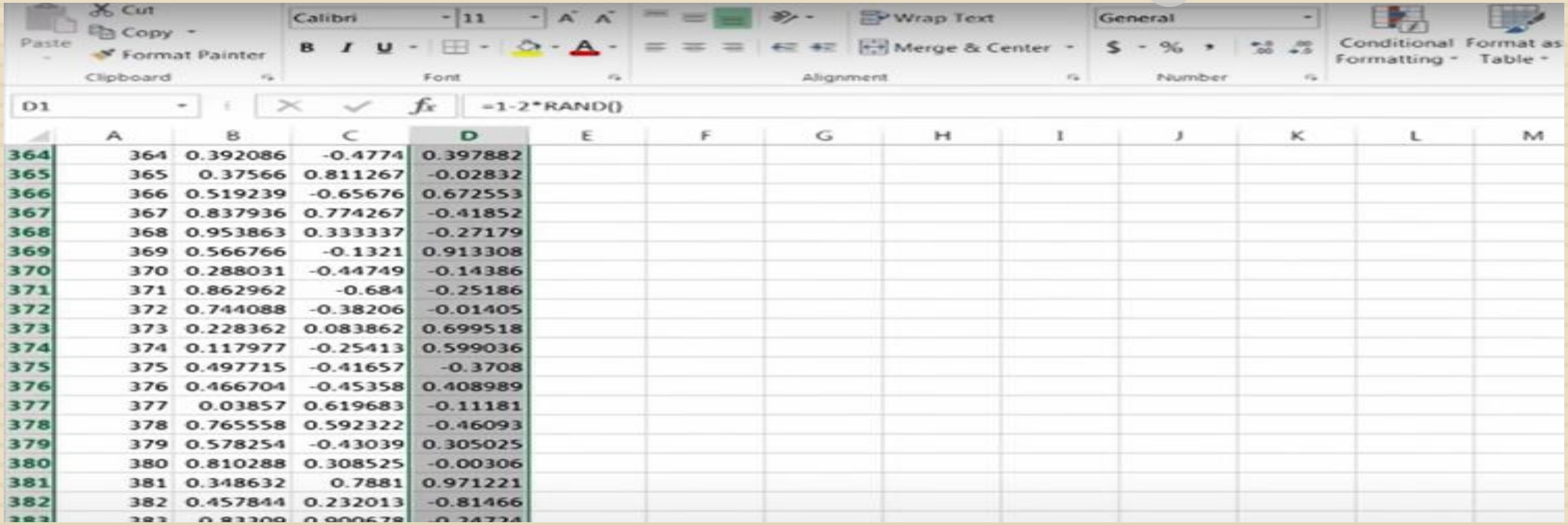
For the circle example, the random numbers should be in between ***[-1, 1]***, so we use the function as: ***1-2*rand ( )*** and the result will be in column C, then generating a similar stream in column D:

# Introduction Monte Carlo Simulation



C1 =1-2*RAND()

| | A | B | C |
|---|---|---|---|
| 16 | 16 | 0.405642 | 0.410886 |
| 17 | 17 | 0.541608 | 0.743781 |
| 18 | 18 | 0.98348 | 0.565029 |
| 19 | 19 | 0.989267 | -0.86153 |
| 20 | 20 | 0.558489 | 0.706347 |
| 21 | 21 | 0.105025 | 0.644331 |
| 22 | 22 | 0.019887 | -0.44949 |
| 23 | 23 | 0.447517 | -0.55043 |
| 24 | 24 | 0.875487 | 0.756995 |
| 25 | 25 | 0.051899 | 0.210206 |
| 26 | 26 | 0.158479 | 0.922461 |
| 27 | 27 | 0.762735 | 0.737978 |
| 28 | 28 | 0.435731 | |
| 29 | 29 | 0.152684 | |
| 30 | 30 | 0.13751 | |
| 31 | 31 | 0.076803 | |
| 32 | 32 | 0.230698 | |
| 33 | 33 | 0.789052 | |
| 34 | 34 | 0.116667 | |
| 35 | 35 | 0.631715 | |
| 36 | 36 | 0.348736 | |

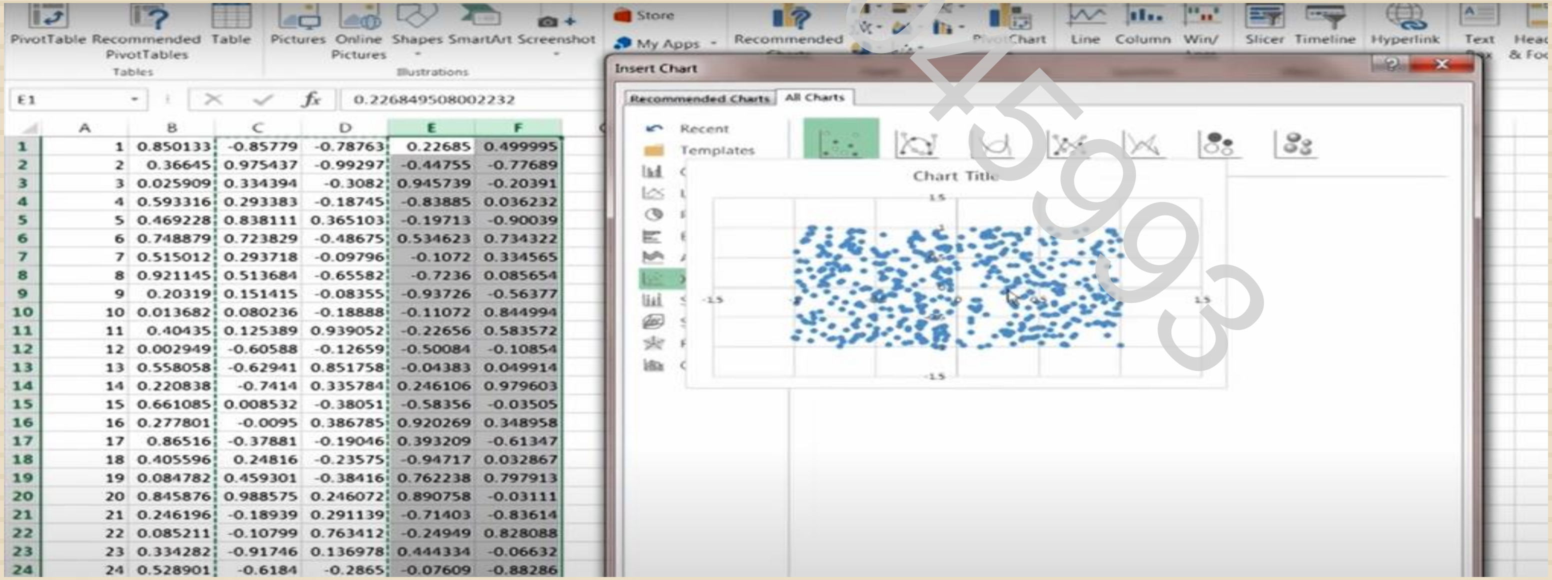


D1 =1-2*RAND()

| | A | B | C | D |
|---|---|---|---|---|
| 364 | 364 | 0.392086 | -0.4774 | 0.397882 |
| 365 | 365 | 0.37566 | 0.811267 | -0.02832 |
| 366 | 366 | 0.519239 | -0.65676 | 0.672553 |
| 367 | 367 | 0.837936 | 0.774267 | -0.41852 |
| 368 | 368 | 0.953863 | 0.333337 | -0.27179 |
| 369 | 369 | 0.566766 | -0.1321 | 0.913308 |
| 370 | 370 | 0.288031 | -0.44749 | -0.14386 |
| 371 | 371 | 0.862962 | -0.684 | -0.25186 |
| 372 | 372 | 0.744088 | -0.38206 | -0.01405 |
| 373 | 373 | 0.228362 | 0.083862 | 0.699518 |
| 374 | 374 | 0.117977 | -0.25413 | 0.599036 |
| 375 | 375 | 0.497715 | -0.41657 | -0.3708 |
| 376 | 376 | 0.466704 | -0.45358 | 0.408989 |
| 377 | 377 | 0.03857 | 0.619683 | -0.11181 |
| 378 | 378 | 0.765558 | 0.592322 | -0.46093 |
| 379 | 379 | 0.578254 | -0.43039 | 0.305025 |
| 380 | 380 | 0.810288 | 0.308525 | -0.00306 |
| 381 | 381 | 0.348632 | 0.7881 | 0.971221 |
| 382 | 382 | 0.457844 | 0.232013 | -0.81466 |
| 383 | 383 | 0.83209 | 0.900678 | -0.24724 |

# Introduction Monte Carlo Simulation

- If we plot these points on the scatter (from ***Insert*** menu on the excel sheet choose ***scatter***) to notice that their uniform distribution:



E1 fx 0.226849508002232

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | 1 | 0.850133 | -0.85779 | -0.78763 | 0.22685 | 0.499995 |
| 2 | 2 | 0.36645 | 0.975437 | -0.99297 | -0.44755 | -0.77689 |
| 3 | 3 | 0.025909 | 0.334394 | -0.3082 | 0.945739 | -0.20391 |
| 4 | 4 | 0.593316 | 0.293383 | -0.18745 | -0.83885 | 0.036232 |
| 5 | 5 | 0.469228 | 0.838111 | 0.365103 | -0.19713 | -0.90039 |
| 6 | 6 | 0.748879 | 0.723829 | -0.48675 | 0.534623 | 0.734322 |
| 7 | 7 | 0.515012 | 0.293718 | -0.09796 | -0.1072 | 0.334565 |
| 8 | 8 | 0.921145 | 0.513684 | -0.65582 | -0.7236 | 0.085654 |
| 9 | 9 | 0.20319 | 0.151415 | -0.08355 | -0.93726 | -0.56377 |
| 10 | 10 | 0.013682 | 0.080236 | -0.18888 | -0.11072 | 0.844994 |
| 11 | 11 | 0.40435 | 0.125389 | 0.939052 | -0.22656 | 0.583572 |
| 12 | 12 | 0.002949 | -0.60588 | -0.12659 | -0.50084 | -0.10854 |
| 13 | 13 | 0.558058 | -0.62941 | 0.851758 | -0.04383 | 0.049914 |
| 14 | 14 | 0.220838 | -0.7414 | 0.335784 | 0.246106 | 0.979603 |
| 15 | 15 | 0.661085 | 0.008532 | -0.38051 | -0.58356 | -0.03505 |
| 16 | 16 | 0.277801 | -0.0095 | 0.386785 | 0.920269 | 0.348958 |
| 17 | 17 | 0.86516 | -0.37881 | -0.19046 | 0.393209 | -0.61347 |
| 18 | 18 | 0.405596 | 0.24816 | -0.23575 | -0.94717 | 0.032867 |
| 19 | 19 | 0.084782 | 0.459301 | -0.38416 | 0.762238 | 0.797913 |
| 20 | 20 | 0.845876 | 0.988575 | 0.246072 | 0.890758 | -0.03111 |
| 21 | 21 | 0.246196 | -0.18939 | 0.291139 | -0.71403 | -0.83614 |
| 22 | 22 | 0.085211 | -0.10799 | 0.763412 | -0.24949 | 0.828088 |
| 23 | 23 | 0.334282 | -0.91746 | 0.136978 | 0.444334 | -0.06632 |
| 24 | 24 | 0.528901 | -0.6184 | -0.2865 | -0.07609 | -0.88286 |

To draw these random numbers in a circle of radius = 1, the circle equation is $x^2 + y^2 = 1$ with the center in the origin. On the Excel sheet in column G we have the condition: **IF** ***(Column E)*** $^{2}$ **+ *(column F)*** $^{2}$ **< 1, *then* 1, *else* 0**

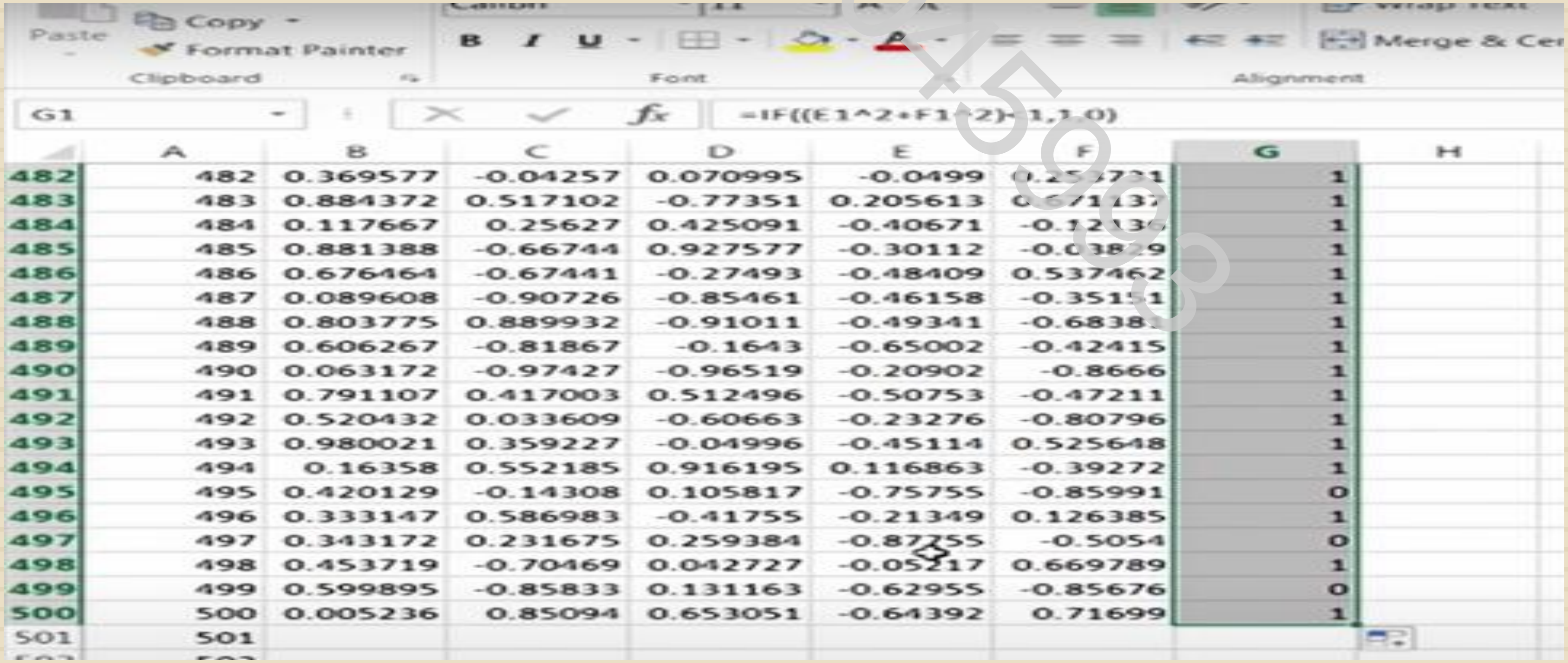
$$= if\ ((E1^\wedge 2 + F1^\wedge 2) < 1, 1, 0)$$

Gives the points ***in the quarter of the circle***.

# Introduction Monte Carlo Simulation

the condition: ***(Column E)*** $^{2}$ ***+ (column F)*** $^{2}$ ***≤ 1, then 1, else 1***

***Gives the points in the quarter of the square.***



G1 =IF((E1^2+F1^2)<1,1,0)

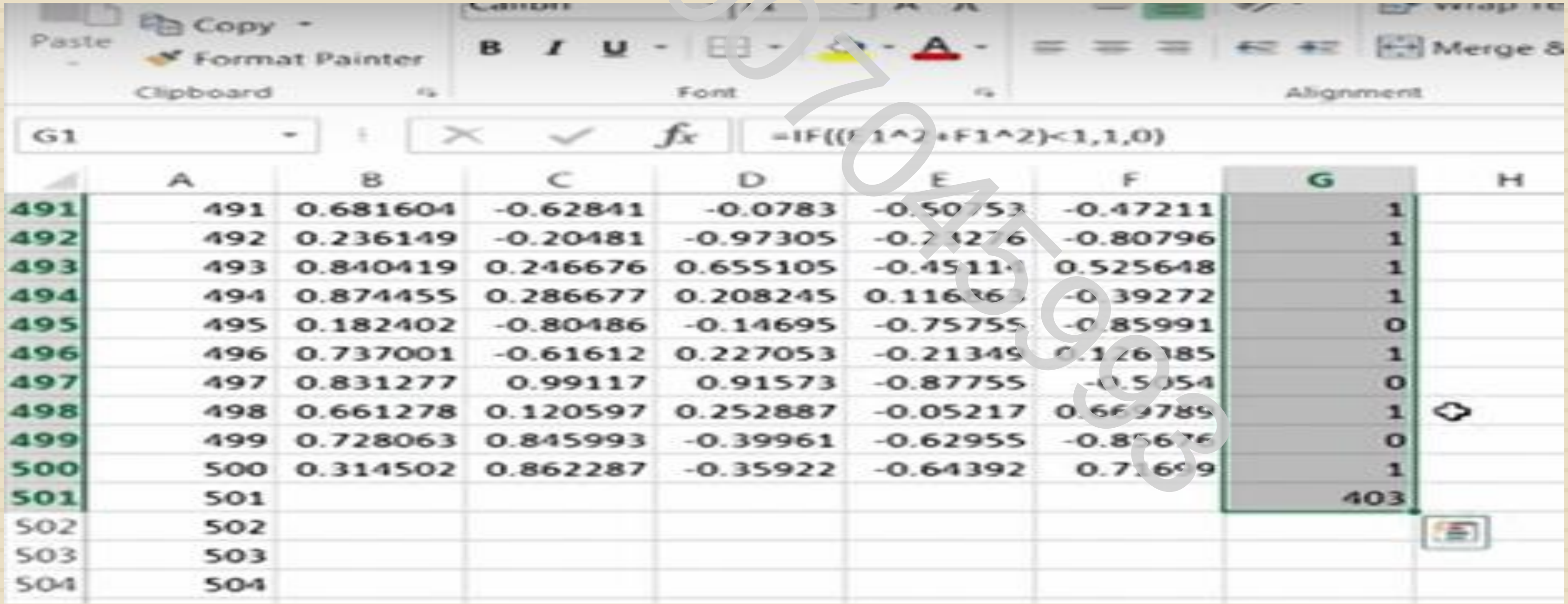
| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 482 | 482 | 0.369577 | -0.04257 | 0.070995 | -0.0499 | 0.253731 | 1 | |
| 483 | 483 | 0.884372 | 0.517102 | -0.77351 | 0.205613 | 0.671137 | 1 | |
| 484 | 484 | 0.117667 | 0.25627 | 0.425091 | -0.40671 | -0.12136 | 1 | |
| 485 | 485 | 0.881388 | -0.66744 | 0.927577 | -0.30112 | -0.03829 | 1 | |
| 486 | 486 | 0.676464 | -0.67441 | -0.27493 | -0.48409 | 0.537462 | 1 | |
| 487 | 487 | 0.089608 | -0.90726 | -0.85461 | -0.46158 | -0.35151 | 1 | |
| 488 | 488 | 0.803775 | 0.889932 | -0.91011 | -0.49341 | -0.68381 | 1 | |
| 489 | 489 | 0.606267 | -0.81867 | -0.1643 | -0.65002 | -0.42415 | 1 | |
| 490 | 490 | 0.063172 | -0.97427 | -0.96519 | -0.20902 | -0.8666 | 1 | |
| 491 | 491 | 0.791107 | 0.417003 | 0.512496 | -0.50753 | -0.47211 | 1 | |
| 492 | 492 | 0.520432 | 0.033609 | -0.60663 | -0.23276 | -0.80796 | 1 | |
| 493 | 493 | 0.980021 | 0.359227 | -0.04996 | -0.45114 | 0.525648 | 1 | |
| 494 | 494 | 0.16358 | 0.552185 | 0.916195 | 0.116863 | -0.39272 | 1 | |
| 495 | 495 | 0.420129 | -0.14308 | 0.105817 | -0.75755 | -0.85991 | 0 | |
| 496 | 496 | 0.333147 | 0.586983 | -0.41755 | -0.21349 | 0.126385 | 1 | |
| 497 | 497 | 0.343172 | 0.231675 | 0.259384 | -0.87755 | -0.5054 | 0 | |
| 498 | 498 | 0.453719 | -0.70469 | 0.042727 | -0.05217 | 0.669789 | 1 | |
| 499 | 499 | 0.599895 | -0.85833 | 0.131163 | -0.62955 | -0.85676 | 0 | |
| 500 | 500 | 0.005236 | 0.85094 | 0.653051 | -0.64392 | 0.71699 | 1 | |
| 501 | 501 | | | | | | | |

***For example:***

$$(0.87)^2 + (0.505)^2 = 0.7569 + 0.255 = 1.0119 > 1 \rightarrow 0 \text{ i.e. outside the circle}$$

$$(0.05)^2 + (0.66)^2 = 0.0025 + 0.4356 = 0.4381 < 1 \rightarrow 1 \text{ i.e. inside the circle}$$

# Introduction Monte Carlo Simulation



G1 =IF((E1^2+F1^2)<1,1,0)

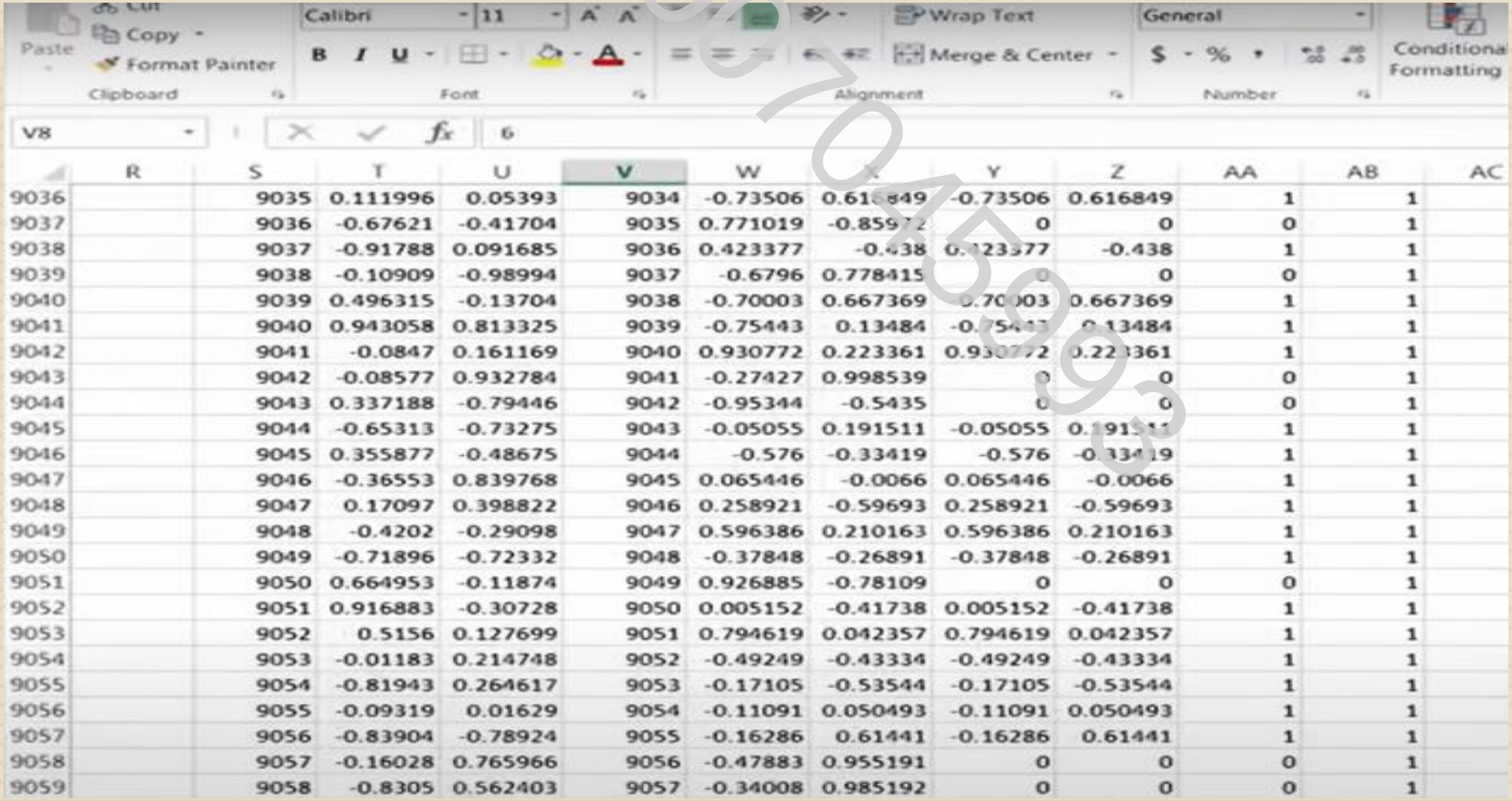
| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 491 | 491 | 0.681604 | -0.62841 | -0.0783 | -0.50753 | -0.47211 | 1 | |
| 492 | 492 | 0.236149 | -0.20481 | -0.97305 | -0.24276 | -0.80796 | 1 | |
| 493 | 493 | 0.840419 | 0.246676 | 0.655105 | -0.45114 | 0.525648 | 1 | |
| 494 | 494 | 0.874455 | 0.286677 | 0.208245 | 0.116863 | -0.39272 | 1 | |
| 495 | 495 | 0.182402 | -0.80486 | -0.14695 | -0.75755 | -0.85991 | 0 | |
| 496 | 496 | 0.737001 | -0.61612 | 0.227053 | -0.21349 | 0.126385 | 1 | |
| 497 | 497 | 0.831277 | 0.99117 | 0.91573 | -0.87755 | -0.5054 | 0 | |
| 498 | 498 | 0.661278 | 0.120597 | 0.252887 | -0.05217 | 0.669789 | 1 | |
| 499 | 499 | 0.728063 | 0.845993 | -0.39961 | -0.62955 | -0.85676 | 0 | |
| 500 | 500 | 0.314502 | 0.862287 | -0.35922 | -0.64392 | 0.71699 | 1 | |
| 501 | 501 | | | | | | 403 | |
| 502 | 502 | | | | | | | |
| 503 | 503 | | | | | | | |
| 504 | 504 | | | | | | | |

***The sum of column G gives 403*** which means ***403 ones out of 500***, i.e. ***1/4 of the area = 403/500 = 0.806 ≈ π/4*** which means ***π ≈ 3.224***

***While the actual value of π is 3.14.***

If we increase the number of points ***more than 500 the result will be closer to the actual value.***

# Introduction Monte Carlo Simulation



| | R | S | T | U | V | W | X | Y | Z | AA | AB | AC |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 9036 | | 9035 | 0.111996 | 0.05393 | 9034 | -0.73506 | 0.616849 | -0.73506 | 0.616849 | 1 | 1 | |
| 9037 | | 9036 | -0.67621 | -0.41704 | 9035 | 0.771019 | -0.85972 | 0 | 0 | 0 | 1 | |
| 9038 | | 9037 | -0.91788 | 0.091685 | 9036 | 0.423377 | -0.438 | 0.423377 | -0.438 | 1 | 1 | |
| 9039 | | 9038 | -0.10909 | -0.98994 | 9037 | -0.6796 | 0.778415 | 0 | 0 | 0 | 1 | |
| 9040 | | 9039 | 0.496315 | -0.13704 | 9038 | -0.70003 | 0.667369 | -0.70003 | 0.667369 | 1 | 1 | |
| 9041 | | 9040 | 0.943058 | 0.813325 | 9039 | -0.75443 | 0.13484 | -0.75443 | 0.13484 | 1 | 1 | |
| 9042 | | 9041 | -0.0847 | 0.161169 | 9040 | 0.930772 | 0.223361 | 0.930772 | 0.223361 | 1 | 1 | |
| 9043 | | 9042 | -0.08577 | 0.932784 | 9041 | -0.27427 | 0.998539 | 0 | 0 | 0 | 1 | |
| 9044 | | 9043 | 0.337188 | -0.79446 | 9042 | -0.95344 | -0.5435 | 0 | 0 | 0 | 1 | |
| 9045 | | 9044 | -0.65313 | -0.73275 | 9043 | -0.05055 | 0.191511 | -0.05055 | 0.191511 | 1 | 1 | |
| 9046 | | 9045 | 0.355877 | -0.48675 | 9044 | -0.576 | -0.33419 | -0.576 | -0.33419 | 1 | 1 | |
| 9047 | | 9046 | -0.36553 | 0.839768 | 9045 | 0.065446 | -0.0066 | 0.065446 | -0.0066 | 1 | 1 | |
| 9048 | | 9047 | 0.17097 | 0.398822 | 9046 | 0.258921 | -0.59693 | 0.258921 | -0.59693 | 1 | 1 | |
| 9049 | | 9048 | -0.4202 | -0.29098 | 9047 | 0.596386 | 0.210163 | 0.596386 | 0.210163 | 1 | 1 | |
| 9050 | | 9049 | -0.71896 | -0.72332 | 9048 | -0.37848 | -0.26891 | -0.37848 | -0.26891 | 1 | 1 | |
| 9051 | | 9050 | 0.664953 | -0.11874 | 9049 | 0.926885 | -0.78109 | 0 | 0 | 0 | 1 | |
| 9052 | | 9051 | 0.916883 | -0.30728 | 9050 | 0.005152 | -0.41738 | 0.005152 | -0.41738 | 1 | 1 | |
| 9053 | | 9052 | 0.5156 | 0.127699 | 9051 | 0.794619 | 0.042357 | 0.794619 | 0.042357 | 1 | 1 | |
| 9054 | | 9053 | -0.01183 | 0.214748 | 9052 | -0.49249 | -0.43334 | -0.49249 | -0.43334 | 1 | 1 | |
| 9055 | | 9054 | -0.81943 | 0.264617 | 9053 | -0.17105 | -0.53544 | -0.17105 | -0.53544 | 1 | 1 | |
| 9056 | | 9055 | -0.09319 | 0.01629 | 9054 | -0.11091 | 0.050493 | -0.11091 | 0.050493 | 1 | 1 | |
| 9057 | | 9056 | -0.83904 | -0.78924 | 9055 | -0.16286 | 0.61441 | -0.16286 | 0.61441 | 1 | 1 | |
| 9058 | | 9057 | -0.16028 | 0.765966 | 9056 | -0.47883 | 0.955191 | 0 | 0 | 0 | 1 | |
| 9059 | | 9058 | -0.8305 | 0.562403 | 9057 | -0.34008 | 0.985192 | 0 | 0 | 0 | 1 | |

***For a stream of 10000 random numbers we get:***

***7857 ones out of 10000, i.e. the area = (7857/10000) x 4 = 3.1428***

# *Inventory Control Simulation*

## *5.2 Inventory Control Simulation using Monte Carlo Technique*

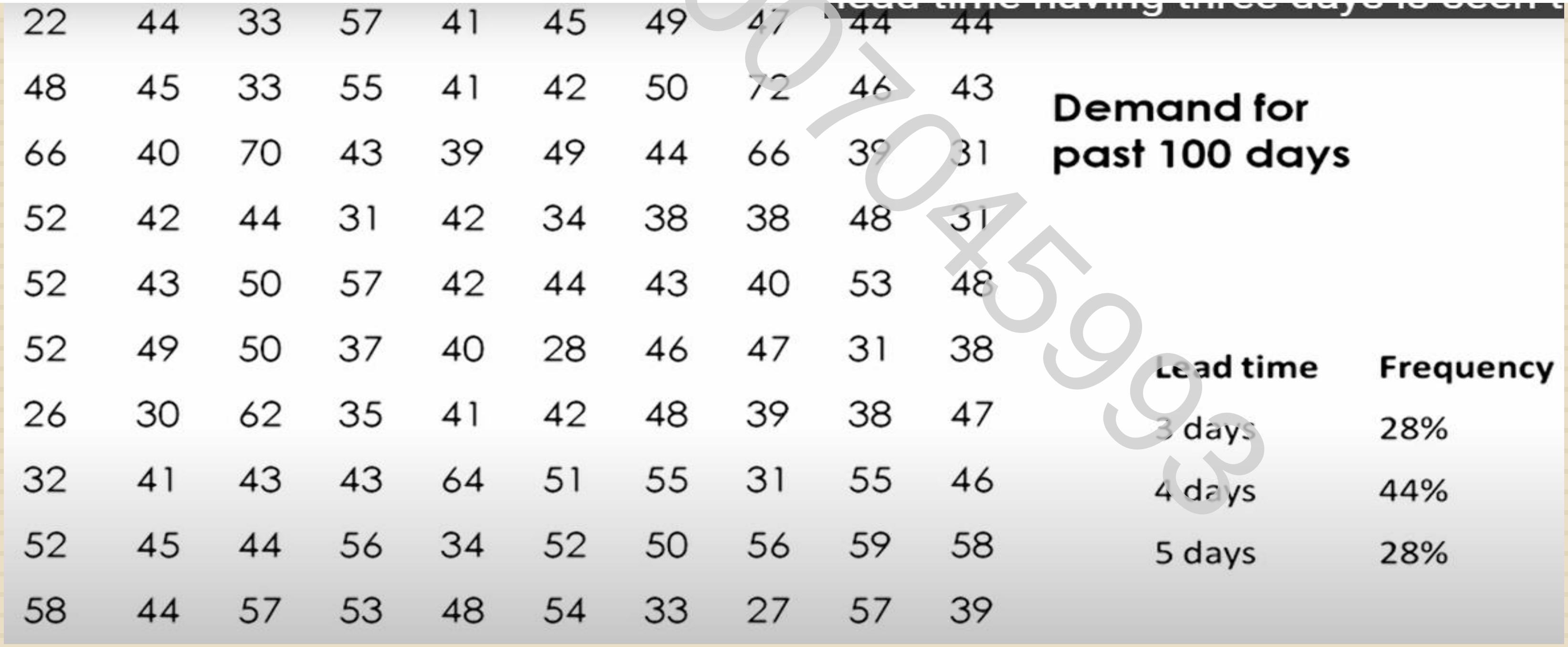
- For inventory control problem we are interested in finding ***when to reorder**, **what is the reorder level**, **what is the daily demand**, **what is the stock in hand**, **what is the total cost of holding material**, **and what is the shortage cost***. Those issues are to be modeled trough this technique.

## *Example 5.3*

**Suppose you have certain *demand for the past 100 days* and the *lead time* in which the ordered quantity has arrived are given in the following table.**

- Notice that for **the lead time we may have a standard frequency distribution** rather than the given frequencies to find the random numbers corresponding to a particular lead time.
- **We need to know *the demand in the coming days* and the *possible lead time*, and *when to order*.**

# Inventory Control Simulation



lead time having three days is seen t

Demand for past 100 days

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 22 | 44 | 33 | 57 | 41 | 45 | 49 | 47 | 44 | 44 |
| 48 | 45 | 33 | 55 | 41 | 42 | 50 | 72 | 46 | 43 |
| 66 | 40 | 70 | 43 | 39 | 49 | 44 | 66 | 39 | 31 |
| 52 | 42 | 44 | 31 | 42 | 34 | 38 | 38 | 48 | 31 |
| 52 | 43 | 50 | 57 | 42 | 44 | 43 | 40 | 53 | 48 |
| 52 | 49 | 50 | 37 | 40 | 28 | 46 | 47 | 31 | 38 |
| 26 | 30 | 62 | 35 | 41 | 42 | 48 | 39 | 38 | 47 |
| 32 | 41 | 43 | 43 | 64 | 51 | 55 | 31 | 55 | 46 |
| 52 | 45 | 44 | 56 | 34 | 52 | 50 | 56 | 59 | 58 |
| 58 | 44 | 57 | 53 | 48 | 54 | 33 | 27 | 57 | 39 |

| Lead time | Frequency |
|---|---|
| 3 days | 28% |
| 4 days | 44% |
| 5 days | 28% |

Suppose that in the inventory control we have the following costs:

**$C_0$ … is the cost of placing an order = *100 pound/order***

**$C_i$ … is the cost of holding an item in stock = *0.1 pound/unit/day***

**$C_s$ … is the cost of shortage = *5 pounds/unit***

# Inventory Control Simulation

*(when you have shortage you have to bring items from local market at higher price)*

**We will calculate what will be the reordered level and other quantities by statistical calculations. We have:** ***Mean demand $\mu_D$***

$\mu_D = \Sigma$ *all demand in (n = 100) days / 100 =* ***45.23 unit/day***

***Standard deviation of the demand*** $\sigma_D = \sqrt{[(\Sigma Di^2) - [(\Sigma Di)^{2} / n]]} / (n-1)$

$\sigma_D = \sqrt{[214077 - [(4523)^{2} / 100]\,]} / 99 = 9.8$

***Mean lead time*** $\mu_L = [(28x3) + (44x4) + (28x5)]/100 = 400/4 = 4$ ***days***

***Standard deviation of the lead time*** $\sigma_L$

$\sigma_L = \sqrt{\{[(28x3)^2 + (44x4)^2 + (28x5)^2] - [(400)^2 / 100]\}} / 99 = 0.75$

All these calculations can be done from the available data.

# *Inventory Control Simulation*

- Now we find the ***histogram of the demand*** which ranges from ***20 to 70***, we calculate the ***frequency, cumulative frequency***, and assign ***two digit*** random numbers (from ***00 to 99***) for each interval of the demand according to its frequency as shown in the following table. ***When we predict the demand we will take the average value of the demand in each interval.***

| Demand interval | Average demand | frequency | Cumulative frequency | Random numbers |
|---|---|---|---|---|
| 20 – 24 | 22 | 1 | 1 | 00 |
| 25 – 29 | 27 | 3 | 4 | 01 – 03 |
| 30 – 34 | 32 | 12 | 16 | 04 – 15 |
| 35 – 39 | 37 | 10 | 26 | 16 – 25 |
| 40 – 44 | 42 | 26 | 52 | 26 – 51 |
| 45 – 49 | 47 | 17 | 69 | 52 – 68 |
| 50 – 54 | 52 | 13 | 82 | 69 – 81 |
| 55 – 59 | 57 | 12 | 94 | 82 – 93 |
| 60 – 64 | 62 | 2 | 96 | 94 – 95 |
| 65 – 69 | 67 | 2 | 98 | 96 – 97 |
| 70 – 74 | 72 | 2 | 100 | 98 – 99 |

# Inventory Control Simulation

***Similarly*** we can have the calculations for ***the lead time:***

| Lead time | frequency | Cumulative frequency | Random numbers |
|---|---|---|---|
| 3 | 28 | 28 | 00 – 27 |
| 4 | 44 | 72 | 28 – 71 |
| 5 | 28 | 100 | 72 – 99 |

***economic order quantity (EOQ) is given as:***

$$EOQ = \sqrt{2\,\mu_D\,C_0\,/\,Ci} = \sqrt{(2 \times 45.23 \times 100)\,/\,0.1} \approx 300 \text{ units}$$

Whenever the stock quantity reaches below certain limit (the ***reorder level***) we have to order this quantity (***300 units***) because we are going for fixed quantity system.

***The reorder level is given as:***

$$ROL = \mu_D\,\mu_L + K\,[\sqrt{\mu_L\,\sigma D^2 + (\mu D^2\,\sigma L^2)}]$$

# *Inventory Control Simulation*

Where **K** is a constant that determines the ***probability of shortage*** of stock, e.g. for ***K = 1.65*** the chance of shortage of stock is ***5 %*** (i.e. ***95%*** sure that there will not be any shortage). If ***K = 2.33*** there will be only ***1 %*** chance of shortage.

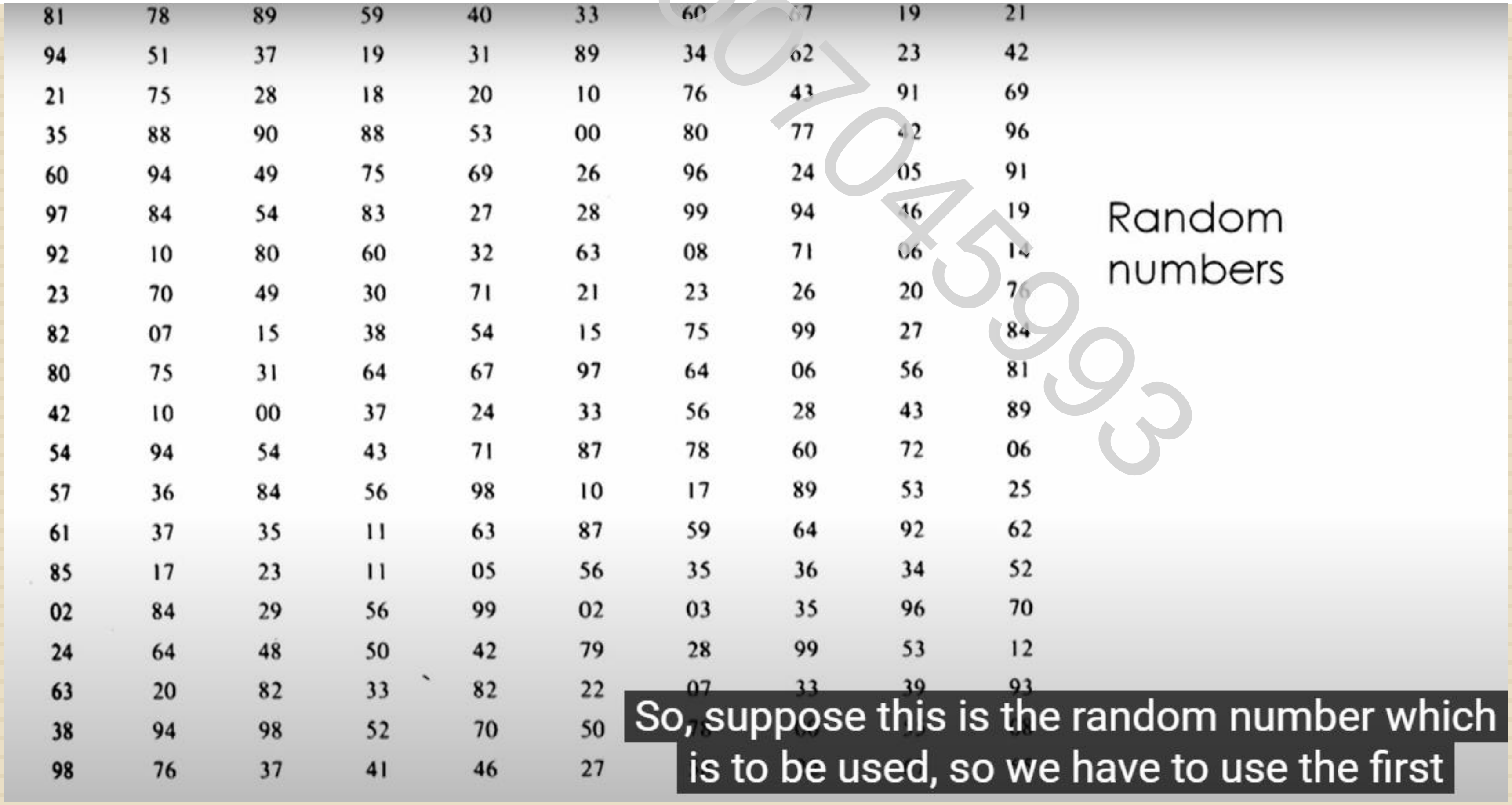
For ***K = 1.65:***

$$ROL = (45.23x4) + 1.65x\sqrt{4\,(95.98)^2 + (45.23)^2\,x0.5657} = 245$$

***Now we start the simulation with EOQ = 300 and ROL = 245 units***

- Assuming the ***initial stock = 300 units*** on ***Friday*** evening, so we will ***start simulation from Monday*** onwards.
- The random numbers can be generated from ***(00 to 99)*** if we put in the excel ***100*** into the ***rand ( )*** function. We can also generate random numbers by using other generators.

# Inventory Control Simulation



| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 81 | 78 | 89 | 59 | 40 | 33 | 60 | 67 | 19 | 21 |
| 94 | 51 | 37 | 19 | 31 | 89 | 34 | 62 | 23 | 42 |
| 21 | 75 | 28 | 18 | 20 | 10 | 76 | 43 | 91 | 69 |
| 35 | 88 | 90 | 88 | 53 | 00 | 80 | 77 | 42 | 96 |
| 60 | 94 | 49 | 75 | 69 | 26 | 96 | 24 | 05 | 91 |
| 97 | 84 | 54 | 83 | 27 | 28 | 99 | 94 | 46 | 19 |
| 92 | 10 | 80 | 60 | 32 | 63 | 08 | 71 | 06 | 14 |
| 23 | 70 | 49 | 30 | 71 | 21 | 23 | 26 | 20 | 76 |
| 82 | 07 | 15 | 38 | 54 | 15 | 75 | 99 | 27 | 84 |
| 80 | 75 | 31 | 64 | 67 | 97 | 64 | 06 | 56 | 81 |
| 42 | 10 | 00 | 37 | 24 | 33 | 56 | 28 | 43 | 89 |
| 54 | 94 | 54 | 43 | 71 | 87 | 78 | 60 | 72 | 06 |
| 57 | 36 | 84 | 56 | 98 | 10 | 17 | 89 | 53 | 25 |
| 61 | 37 | 35 | 11 | 63 | 87 | 59 | 64 | 92 | 62 |
| 85 | 17 | 23 | 11 | 05 | 56 | 35 | 36 | 34 | 52 |
| 02 | 84 | 29 | 56 | 99 | 02 | 03 | 35 | 96 | 70 |
| 24 | 64 | 48 | 50 | 42 | 79 | 28 | 99 | 53 | 12 |
| 63 | 20 | 82 | 33 | 82 | 22 | 07 | 33 | 39 | 93 |
| 38 | 94 | 98 | 52 | 70 | 50 | 78 | [illegible] | [illegible] | [illegible] |
| 98 | 76 | 37 | 41 | 46 | 27 | [illegible] | [illegible] | [illegible] | [illegible] |

Random numbers

So, suppose this is the random number which is to be used, so we have to use the first

We will use random numbers ***for demand from the left side of the first row*** and ***for lead time from the right side of the last row (65, 67, 22 ..)***

# Inventory Control Simulation

(**initial stock** = 300 units, **EOQ** = 300 units, and **ROL** = 245 units)

## The simulation table for two working weeks (10 working days will be):

| Day | Initial stock | R. No. for demand | Daily demand | Shortage | Quantity received | Final stock | Quantity in hand + order | Quantity ordered | R. No. for lead time | Lead time |
|---|---|---|---|---|---|---|---|---|---|---|
| M | 300 | 81 | 52 | - | - | 248 | 248 | 0 | - | - |
| T | 248 | 78 | 52 | - | - | 196 | 196 | 300 | 65 | 4 |
| W | 196 | 89 | 57 | - | - | 139 | 439 | 0 | - | - |
| Th | 139 | 59 | 47 | - | - | 92 | 392 | 0 | - | - |
| F | 92 | 40 | 42 | - | - | 50 | 350 | 0 | - | - |
| M | 50 | 33 | 42 | - | 300 | 308 | 308 | 0 | - | - |
| T | 308 | 60 | 47 | - | - | 261 | 261 | 0 | - | - |
| W | 261 | 67 | 47 | - | - | 214 | 214 | 300 | 67 | 4 |
| Th | 214 | 19 | 37 | - | - | 177 | 477 | 0 | - | - |
| F | 177 | 21 | 37 | - | - | 140 | 440 | 0 | - | - |

***Final stock = initial stock – daily demand + quantity received***

On first Monday = ***300 – 52 = 248 units, Initial stock = previous final stock → on Tuesday = 248 and final stock = 248 – 52 = 196 → we have to reorder***

# *Inventory Control Simulation*

***Quantity in hand + order = final stock + quantity ordered***

When the ***lead time is 4 working days*** from ***Tuesday***, the quantity ordered will be ***received on next Monday***, final stock will be:

***Final stock on next Monday = (50 – 42) + 300 = 308 units***

The second ordered quantity will be received on next Tuesday.

We can calculate parameters like cost in certain period:

***Cost of holding inventory = holding cost per unit x final stock***

***Cost of ordering = number of orders x order cost***

# Monte Carlo Simulation of Queueing Problem

## 5.3 Monte Carlo Simulation of Queueing Problem

Monte Carlo simulation can be used in solving different kinds of problems like ***inventory control, reliability, queueing*** … etc. In this section we discuss ***Monte Carlo simulation of queueing problem.***

### Example 5.4

At a ***dentist clinic***, apply ***Monte Carlo simulation*** to find ***the average waiting time of customers***. Simulation ***starts at 8 a.m.*** in the morning and the ***arrival of customers is scheduled*** (in other cases ***it may be randomly depending on some probability distributions***). The ***first customer*** arrives at ***8 a.m.*** and every ***30 minutes one customer is coming for five different treatments to be carried out***. The five treatment services are ***filling, crown, cleaning, extraction, and checkup*** with different probabilities and service times.

# *Monte Carlo Simulation of Queueing Problem*

***We do not know the treatment service for which a customer is coming*** and that guess will be done ***by referring random numbers*** generated from any generator as shown in the following table:

| services | Service time | Probability of service | Cumulative probability | Random number intervals |
|---|---|---|---|---|
| Filling | 40 min. | 0.40 | 0.40 | 00 – 39 |
| Crown | 60 min. | 0.15 | 0.55 | 40 – 54 |
| Cleaning | 15 min. | 0.15 | 0.70 | 55 – 69 |
| Extraction | 45 min. | 0.10 | 0.80 | 70 – 79 |
| Checkup | 15 min. | 0.20 | 1.00 | 80 – 99 |

Then we can predict for what service a customer is coming.

Given the random numbers ***40, 82, 11, 34, 25, 66, 17, 79*** to be used for predicting the required service for ***8 patients*** by the dentist:

# Monte Carlo Simulation of Queueing Problem

| Patient | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| Arrival time | 8:00 | 8:30 | 9:00 | 9:30 | 10:00 | 10:30 | 11:00 | 11:30 |
| Random number of service | 40 | 82 | 11 | 34 | 25 | 66 | 17 | 79 |
| Category of service | crown | checkup | filling | filling | filling | cleaning | filling | extraction |
| Service time | 60 | 15 | 40 | 40 | 40 | 15 | 40 | 45 |

*Then we have to find when a customer arrives and when he leaves to calculate the average waiting time:*

| patient | Arrival time | Service time | Service starts at: | Waiting time |
|---|---|---|---|---|
| 1 | 8:00 | 60 | 8:00 | 0 minutes |
| 2 | 8:30 | 15 | 9:00 | 30 minutes |
| 3 | 9:00 | 40 | 9:15 | 15 minutes |
| 4 | 9:30 | 40 | 9:55 | 25 minutes |
| 5 | 10:00 | 40 | 10:35 | 35 minutes |
| 6 | 10:30 | 15 | 11:15 | 45 minutes |
| 7 | 11:00 | 40 | 11:30 | 30 minutes |
| 8 | 11:30 | 45 | 12:10 | 40 minutes |

# Monte Carlo Simulation of Queueing Problem

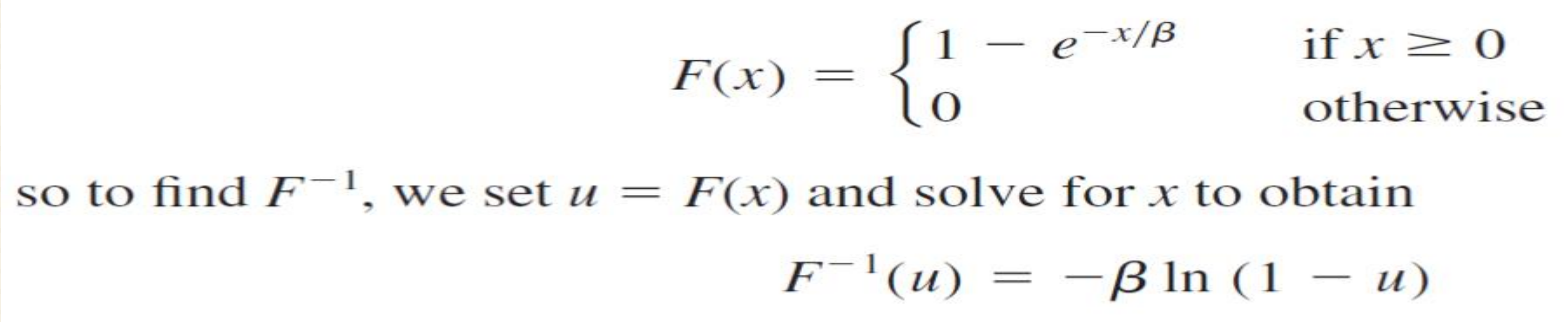
*The total waiting time = 220 minutes*

*Average waiting time = 220 / 8 = 27.5 minutes*

In many cases we have to generate the random numbers from certain distribution. ***Inter-arrival time and service time follow the exponential distribution in most cases.*** To generate random numbers using the exponential distribution we use ***the inverse transform method.***

## EXAMPLE 5.5

Let X have the exponential distribution with **mean β**, the distribution function is:



$$F(x) = \begin{cases} 1 - e^{-x/\beta} & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

so to find $F^{-1}$, we set $u = F(x)$ and solve for $x$ to obtain

$$F^{-1}(u) = -\beta \ln(1 - u)$$

# Monte Carlo Simulation of Queueing Problem

- Thus, to generate the desired random variate, we first generate a ***$U \sim U(0,1)$*** and then let **$X = -\ \beta\ \text{In}\ U$**. [It is possible in this case to use ***$U$*** instead of (***$1-U$***) since ***$(1-U)$*** and ***$U$*** have the same ***$U(0, 1)$*** distribution. [This saves a subtraction.]

## Example 5.6

Suppose in ***example 5.4*** the ***inter-arrival time is exponentially distributed*** with mean value **$\beta_1 = 2$** and ***service time is also exponentially distributed*** with mean value **$\beta_2 = 1.33$**. From the simulation table for the arrival, service, and departure times of 8 customers ***calculate the average waiting time for a customer.***

### Solution

We get the random variates representing the ***inter-arrival time $X_i$*** and ***service time $Y_i$*** from the inverse transform equations:

$$X_i = -\beta_1\ \text{In}\ U_1 \qquad \text{and} \qquad Y_i = -\beta_2\ \text{In}\ U_2$$

Where the random numbers ***$0 \leq$ (both $U_1$ and $U_2$) $\leq 1$.*** Given a stream of ***two digit*** random numbers ***$u_1, u_2$*** [from ***00*** *to* ***99***], we can get ***$U_1, U_2$*** by dividing each random number in the stream by 100, then we calculate ***$X_i, Y_i$*** from the above equations, we get the following results:

# *Monte Carlo Simulation of Queueing Problem*

***$u_1$ →20, 23, 86, 09, 92, 35, 38, 01 & $u_2$=21, 44, 27, 70, 73, 36, 59, 42***

For $u_1 = 20 \rightarrow U_1 = 20/100 = 0.20 \rightarrow X_1 = -2 \ln (0.20) = 3.22$

For $u_2 = 21 \rightarrow U_2 = 21/100 = 0.21 \rightarrow Y_1 = -1.33 \ln (0.21) = 2.08$

Similarly for all values of $u_1$ and $u_2$ as shown in the simulation table:

| $u_1$ | Inter-arrival time $X_i$ | Arrival time | $u_2$ | Service time $Y_i$ | Departure time | Waiting time |
|---|---|---|---|---|---|---|
| 20 | 3.22 | 3.22 | 21 | 2.08 | 5.30 | 0 |
| 23 | 2.94 | 6.16 | 44 | 1.09 | 7.25 | 0 |
| 86 | 0.30 | 6.46 | 27 | 1.75 | 9.0 | 0.79 |
| 09 | 4.82 | 11.28 | 70 | 0.48 | 11.76 | 0 |
| 92 | 0.17 | 11.45 | 73 | 0.42 | 12.18 | 0.31 |
| 35 | 2.10 | 13.55 | 36 | 1.36 | 14.91 | 0 |
| 38 | 1.94 | 15.49 | 59 | 0.70 | 16.19 | 0 |
| 01 | 9.21 | 24.70 | 42 | 1.16 | 25.86 | 0 |

*The average waiting time = (0.79 + 0.31) / 8 = 0.14 and we can calculate other measures of performance.*

# *Monte Carlo Simulation of Queueing Problem*

## *Example 5.7*

Simulate the operation on a ***teller window*** for ***one hour***, given the following data:

***$u_1 \rightarrow$ 25, 37, 91, 00, 61, 62, 80, 15, 23***

***$u_2 \rightarrow$ 84, 01, 59, 40, 03, 29, 50, 77, 32***

| Inter-arrival time (minutes) | 5 | 6 | 7 | 8 |
|---|---|---|---|---|
| Probability | 0.15 | 0.35 | 0.35 | 0.15 |
| Random No. | 00-14 | 15-49 | 50-84 | 85-99 |

| Service time (minutes) | 5 | 6 | 7 |
|---|---|---|---|
| Probability | 0.25 | 0.50 | 0.25 |
| Random No. | 00-24 | 25-74 | 75-99 |

# *Monte Carlo Simulation of Queueing Problem*

*The simulation table for one hour will be:*

| Random number $u_1$ | Inter-arrival time | Arrival time | Random number $u_2$ | Service time | Departure time | Waiting time | Server idle time |
|---|---|---|---|---|---|---|---|
| 25 | 6 | 6 | 84 | 7 | 13 | 0 | 6 |
| 37 | 6 | 12 | 01 | 5 | 18 | 1 | 0 |
| 91 | 8 | 20 | 59 | 6 | 26 | 0 | 2 |
| 00 | 5 | 25 | 40 | 6 | 32 | 1 | 0 |
| 61 | 7 | 32 | 03 | 5 | 37 | 0 | 0 |
| 62 | 7 | 39 | 29 | 6 | 45 | 0 | 2 |
| 80 | 7 | 46 | 50 | 6 | 52 | 0 | 1 |
| 15 | 6 | 52 | 77 | 7 | 59 | 0 | 0 |
| 23 | 6 | 58 | 32 | 6 | 65 stop | 1 | 0 |
| | | | | | | Total = 3 | Total =11 |

When departure time reaches beyond 60 minutes we have to stop.

***Average waiting time per customer = 3/9 = 1/3 minutes***

***Percentage utilization of server = [(65-11)/65] x 100 = 83 %***

# *Monte Carlo Simulation in Manufacturing*

## *5.4 Monte Carlo Simulation in Manufacturing*

We can have the generated random numbers by suitable software programs or by using different kinds of random number generators like linear congruential generator or mixed generator as shown in ***Ex. 5.8***

## *Example 5.8*

Given the units produced in a manufacturing facility and the number of vehicles available for transportation at a particular station with the corresponding probabilities as shown in the following table:

| Units produced $X_i$ | Probability $P(x = X_i)$ | Random number intervals $u_1$ | Number of vehicles available $Y_i$ | Probability $P(y = Y_i)$ | Random number intervals $u_2$ |
|---|---|---|---|---|---|
| 500 | 0.06 | 00 – 05 | 5 | 0.16 | 00 – 15 |
| 550 | 0.14 | 06 – 19 | 6 | 0.36 | 16 – 51 |
| 600 | 0.20 | 20 – 39 | 7 | 0.20 | 52 – 71 |
| 650 | 0.40 | 40 – 79 | 8 | 0.16 | 72 – 87 |
| 700 | 0.20 | 80 – 99 | 9 | 0.12 | 88 – 99 |

# CHAPTER 8
# Monte Carlo Simulation

Using ***mixed congruential generator*** of 3 parameters ***a, c,*** and ***m*** to generate two digit random numbers between ***00*** and ***99***, *given:*

***For $u_1$ → a = 21, c = 3, m = 100, seed $X_0$ = 20***

***For $u_2$ → a = 41, c = 7, m = 100, seed $X_0$ = 1, where:***

$$X_i = a\,X_{i-1} + c \bmod (m)$$

***For $u_1$: $X_0 = 20$***

***$X_1 = a\,X_0 + c \bmod (m) = (21 \times 20) + 3 \bmod (100) = 423 \bmod (100) = 23$***

***$X_2 = (21 \times 23) + 3 \bmod (100) = 486 \bmod (100) = 86$ and so on to $X_9$***

***Similarly for $u_2$: $X_0 = 1$***

***$X_1 = a\,X_0 + c \bmod (m) = (41 \times 1) + 7 \bmod (100) = 48 \bmod (100) = 48$***

***$X_2 = (41 \times 48) + 7 \bmod (100) = 1975 \bmod (100) = 75$ and so on to $X_9$***

# CHAPTER 8
# Monte Carlo Simulation

Now we can have the simulation table for ***10 days***.

| Day | R.No. for units produced | Units produced | R.No. for vehicles availability | Vehicles available |
|---|---|---|---|---|
| 1 | 20 | 600 | 01 | 5 |
| 2 | 23 | 600 | 48 | 6 |
| 3 | 86 | 700 | 75 | 8 |
| 4 | 09 | 550 | 82 | 8 |
| 5 | 92 | 700 | 69 | 7 |
| 6 | 35 | 600 | 36 | 6 |
| 7 | 38 | 600 | 83 | 8 |
| 8 | 01 | 500 | 10 | 5 |
| 9 | 24 | 600 | 17 | 6 |
| 10 | 07 | 550 | 04 | 5 |
| | | **Total = 6000** | | **Total = 64** |

Average number of units produced per day = ***6000 / 10 = 600*** units/day

Average number of available vehicles per day = ***64/10 = 6.4*** vehicle/day

## Reference Book

# Simulation Modeling and Analysis

Averill M. Law