Physics Letters A 380 (2016) 1953–1958

Contents lists available at ScienceDirect

Physics Letters A

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Patterns and singular features of extreme fluctuational paths of a periodically driven system

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ARTICLE INFO

Article history:
Received 22 February 2016
Received in revised form 3 April 2016
Accepted 4 April 2016
Available online 8 April 2016
Communicated by C.R. Doering

Keywords:
Large fluctuations
Optimal path
Prehistory probability distribution
Caustics

ABSTRACT

Large fluctuations of an overdamped periodically driven oscillating system are investigated theoretically and numerically in the limit of weak noise. Optimal paths fluctuating to certain point are given by statistical analysis using the concept of prehistory probability distribution. The validity of statistical results is verified by solutions of boundary value problem. Optimal paths are found to change topologically when terminating points lie at opposite side of a switching line. Patterns of extreme paths are plotted through a proper parameterization of Lagrangian manifold having complicated structures. Several extreme paths to the same point are obtained by multiple solutions of boundary value solutions. Actions along various extreme paths are calculated and associated analysis is performed in relation to the singular features of the patterns.

1. Introduction

In the past two decades, a great deal of mathematical and experimental effort has been devoted to the study of large fluctuations in nonequilibrium systems [1–4], using Hamiltonian formalism [5] or equivalent path integral formulations [6–8]. The method is based on the concept of optimal paths along which the system fluctuates to a remote state with overwhelming probability. Through a proper formulation of a statistical distribution function, i.e., prehistory probability distribution [1,9], experimental approaches by means of analogue electronic circuits [10,11] were proposed to prove the physical existence of such paths. The mathematical foundation of these concepts involves a set of dynamical equations providing extremums of certain action functional in the path integral formulation of the stochastic process. The optimal path gives rise to the absolute minimum to the action functional which characterizes the difficulty of the arrival at a given point along different paths. The quantity of the action serves as the exponential rate of a stationary probability density in approximated WKB form in the weak noise limit [12]. As noise intensity tends to zero, these probabilities become exponentially small but the rate of falling off is path dependent. It follows that for a given noise intensity the probability for the system moving along the optimal path is exponentially larger than ones for motions along other paths.

We remark that extreme paths provide local extremums to the action functional, rather than global minimum. Using Hamiltonian formalism, one can obtain an auxiliary Hamiltonian system, whose trajectories emanating from a stable state trace out unstable Lagrangian manifold. Even if Hamiltonian trajectories do not intersect in Lagrangian manifold, the manifold may have complicated structures leading to intersections of their projections which in fact are extreme paths. Consequently several extreme paths may arrive at the same point and patterns of extreme paths for nonequilibrium systems may have singular features, such as cusps and caustics [13–15]. They stem from complicated topological structures of Lagrangian manifold, a schematic plot of which is given below in Fig. 1 for illustration.

As can be seen, each cusp gives rise to a pair of folds in Lagrangian manifold and the projections of folds into the coordinate space are caustics. When a Hamiltonian trajectory in Lagrangian manifold goes over a fold, the corresponding extreme path in coordinate space is reflected by one of the caustics. Limited work has been done in the study of topology of Lagrangian manifold and its singularities. Smelyanskiy, Dykman and Maier [16] investigated the pattern of extreme paths in the vicinity of an unstable focus of a periodically oscillating system. They found the Lagrangian manifold has a novel structure with folds spiraling into the focus. D.G. Luchinsky [15] considered a noise driven exit in a two dimensional bistable system lacking detailed balance and found a bifurcation of optimal exit path as parameters of system are changed through analog and digital stochastic simulation. M.I. Dykman et al. [14] investigated the distribution of large fluctuational

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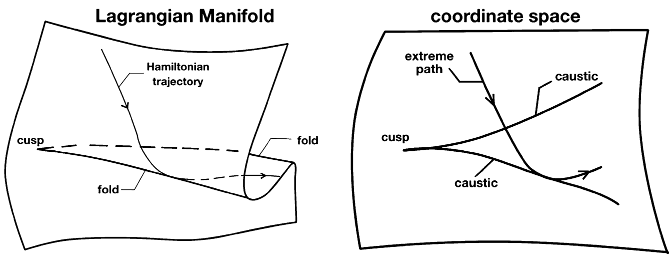
http://dx.doi.org/10.1016/j.physleta.2016.04.004



(a) (b)

Fig. 1. (a) The Lagrangian manifold is traced out by trajectories of auxiliary Hamiltonian system. The Lagrangian manifold has folds and cusps and a trajectory on it is plotted by a curve with arrows. (b) Extreme paths are projections onto the coordinate space. The caustics are projections of folds in (a) and a trajectory going over the fold is described by an extreme path reflected by the caustic.

paths and found critical broadening of the distribution of paths coming to a cusp point. In their further work [17], more visualized results were given.

In this paper, we continue predecessors' work and investigate the problem in more detail. Both the optimal path and patterns of extreme paths are obtained and analyzed. Boundary value problems of the auxiliary Hamiltonian system are solved to give deeper insight into the behaviors of extreme paths and properties of general singularities. The paper is organized as follows. In Section 2, the problem is formulated and the corresponding deterministic system is discussed. In Section 3 the Hamiltonian approach is used to give the auxiliary dynamical system and the pattern of extreme paths is given through parameterizing the Lagrangian manifold. Optimal paths to given points are investigated in Section 4 and singular features of the pattern of extreme paths are studied in Section 5.

2. Formulation

We investigate an overdamped system driven by a periodic force $K(q;t)$ and white noise $\xi(t)$, the equation of which is

$$\dot{q} = K(q;t) + \xi(t), \qquad K(q;t) = K(q;t+T)$$
$$\langle \xi(t)\xi(t') \rangle = D\delta(t-t') \tag{1}$$

The model (1) has a wide application in many physical systems and a great deal of effort in its study has been devoted [18–20]. We consider a simple example of model (1) with

$$\dot{q} = -U'(q) + A\cos(\omega t) + \xi(t) \tag{2}$$
$$U(q) = -\frac{1}{2}q^2 + \frac{1}{4}q^4 \tag{3}$$

where A, ω are parameters, neither of which need to be small. Only the noise intensity D will be assumed small.

It can be seen from Fig. 2 that the potential (3) has two wells, so that the system (2) without noise is bistable with two stable periodic states, moving around the original stable equilibriums $q_s = \pm 1$ respectively. In what follows we study the fluctuations in the domain of attraction of one attractor, the position of which is a periodic function of time,

$$\dot{q}^{(0)} = K\left(q^{(0)};t\right), \qquad q^{(0)}\left(t + 2\pi\omega^{-1}\right) = q^{(0)}(t) \tag{4}$$

3. Hamiltonian formalism and patterns of extreme paths

3.1. Hamiltonian approach of fluctuations

If the noise intensity D is small, the system spends most of the time fluctuating about the attractor, only occasionally far away from it (of scale $\gg \sqrt{D}$). Furthermore, escape from the domain of attraction could occur. When it does occur, escape follows a unique optimal trajectory with overwhelming probability, seemingly in an almost deterministic way. To determine the optimal trajectory, we must turn to investigate the asymptotic solution of the corresponding Fokker–Planck equation as $D \to 0$. In the limit of weak noise intensity D one can seek an approximate solution in an eikonal or WKB form:

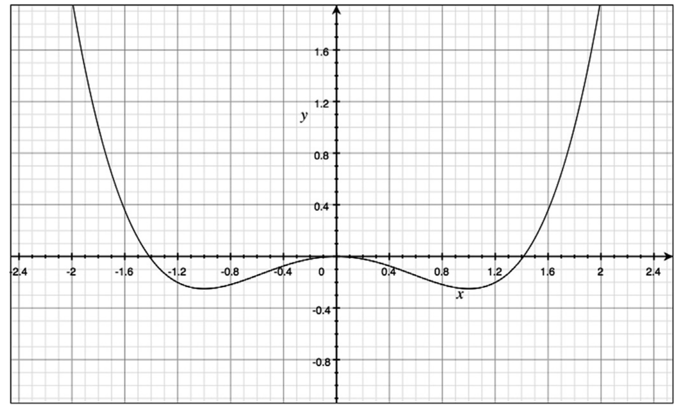


Fig. 2. The potential given by $U(q)$ in Eqn. (3), from which one can see two wells corresponding to two attractors of system (2).

$$P(q) \sim C(q)\exp\left[-S(q)/D\right] \tag{5}$$

with $C(q)$ a prefactor not investigated in this paper and $S(q)$ the "activation energy" of fluctuations to the vicinity of the point q in the state space [21]. $S(q)$ is also called quasipotential or nonequilibrium potential [22].

Substituting Eqn. (5) into the Fokker–Planck equation and keeping only the terms of lowest order in D, we obtain the Hamilton–Jacobi equation for $S(q)$:

$$H(q,p) \equiv K(q)p + \frac{1}{2}p^2 = 0, \quad p \equiv \frac{\partial S}{\partial q} \tag{6}$$

To solve the Hamilton–Jacobi equation (6) one can employ the method of characteristics, arriving at the following equations:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = K(q) + p$$
$$\frac{dp}{dt} = -\frac{\partial H}{\partial q} = -\frac{\partial K}{\partial q} \tag{7}$$

Note that Eqn. (7) leads to an auxiliary Hamiltonian dynamical system, with the Wenzel–Freidlin Hamiltonian $H(q,p)$. From this point of view $S(q)$ can be interpreted as the classical action at zero energy [16].

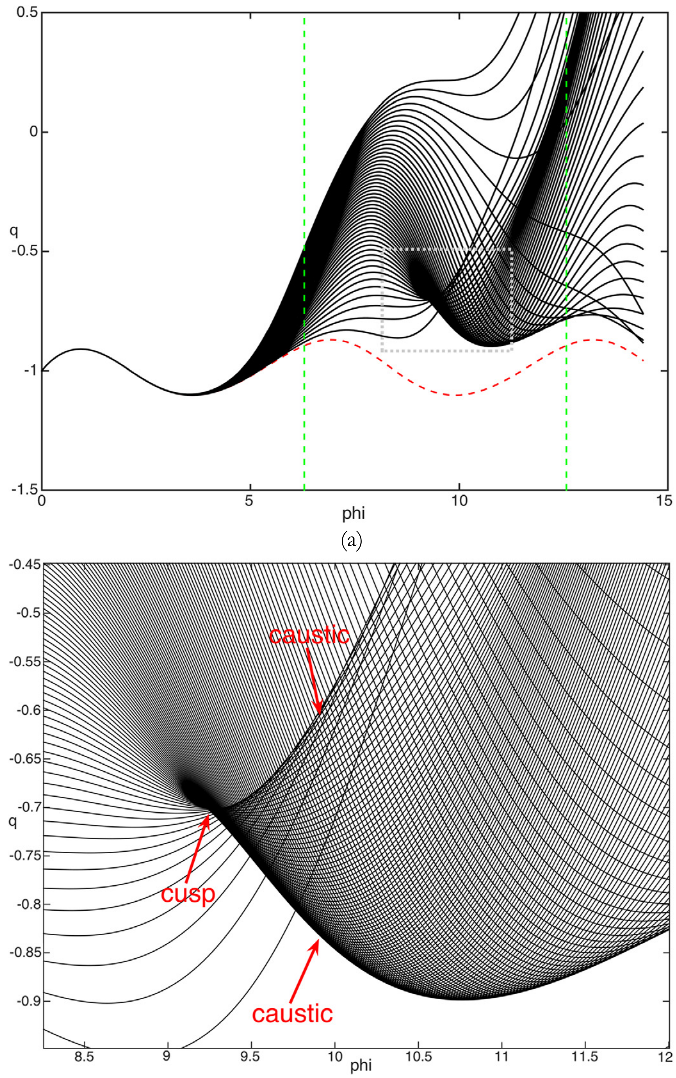
Solutions of Eqn. (7) describe trajectories yielding extreme values of the cost functional [23] of the form:

$$S\left[q(t)\right] = \frac{1}{2}\int_{t_0}^{t_f} \xi^2(t)dt \tag{8}$$

with $q(t)$ being certain trajectory driven by corresponding realization of $\xi(t)$ and satisfying $q(t_0) = q_0$ and $q(t_f) = q_f$. Since using (2) the cost functional (8) can be transformed into an action functional:

$$S\left[q(t)\right] = \int_{t_0}^{t_f} dt L(q,\dot{q}), \qquad L(q,\dot{q}) = \frac{1}{2}(\dot{q} - K)^2 \tag{9}$$

This has the form of a Lagrangian L for a classical mechanical system. As $D \to 0$, these path integrals (9) can be evaluated by means



(b)

Fig. 3. (a) Patterns of extreme paths emanating from the stable periodic orbit in the vicinity of $q_s = -1$, plotted by a red dashed curve. The two dashed green lines represent a complete period in φ. (b) A local enlargement of the pattern in the grey dotted box in (a), showing a cusp and caustics with arrows. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of steepest descents and the paths dominating the integrals are the ones giving rise to $\delta S/\delta x = 0$. This results in an Euler–Poisson equation for extreme paths, which is a $2n$th-order nonlinear partial differential equation. Through applying appropriate transform [5], it can be converted into $2n$ first-order ordinary differential equations (7). In other words, the action $S(q)$ mentioned above is just given by the variational problem of (9). Combining (2) and formulas (9)–(11), we have the Wenzel–Freidlin Hamiltonian for (2)

$$H(q, p; \varphi) = \frac{1}{2}p^2 + p\big(q - q^3 + A\cos(\varphi)\big), \quad \varphi = \omega t + \varphi_0 \tag{10}$$

with $\varphi_0 = 0$, and the auxiliary Hamiltonian dynamics

$$\begin{aligned} \dot{q} &= q - q^3 + A\cos(\varphi) + p \\ \dot{p} &= -p\big(1 - 3q^2\big) \end{aligned} \tag{11}$$

We have introduced a new variable φ into the Hamiltonian (10) such that $H(q, p; \varphi)$ is periodic in φ. From the point of view of dynamical system, different values of φ determine different Poincaré cross-sections of system (11), meaning that the paths $(q(t), p(t))$ fluctuating to a given point $(q_f, \varphi_f + 2\pi)$ are the same paths as those to the point (q_f, φ_f). In addition, the frequency of the harmonic driving is set to $\omega = 1$ and the amplitude to $A = 0.25$ in this paper.

3.2. Parameterizations of extreme paths

The fluctuational dynamics is determined by the patterns of extreme paths emanating from the stable periodic orbit. Next we consider the behavior of these paths. It follows from Eqn. (11) that in the thee-dimensional phase space (q, p, φ), a stable periodic state in the (q, φ) space corresponds to a closed orbit on the hyperplane $p = 0$ if we identify the section $\varphi = 0$ with the section $\varphi = 2\pi$. The Hamiltonian trajectories providing solutions to Eqn. (11) emanating from this closed orbit trace out a two-dimensional Lagrangian unstable manifold. The projections of these trajectories onto the (q, φ) plane form the extreme paths.

Selecting initial coordinates in a sufficiently small vicinity of $q_s = -1$, the paths can be represented by a one-parameter set $\{(q(t; \delta q), p(t; \delta q))\}$. The pattern of extreme paths in (q, φ) plane is given in Fig. 3. The red dashed curve is the stable periodic orbit in the neighborhood of $q_s = -1$, from which large fluctuations emanate. The evolutions of extreme paths for more than two periods are plotted, with two green dashed lines representing one period of 2π. A local amplification of the grey dotted box in Fig. 3(a) is given in Fig. 3(b), from which one can clearly see that the pattern shows singular features: a cusp and a pair of caustics, marked by red arrows. The cause of these singularities has been discussed in Section 1 and in what follows we turn to investigate their effect on fluctuations.

4. Statistical analysis and the optimal path

In general, there will be several extreme trajectories of (11) starting from initial state q_0 to final state q_f, thus the action $S(q_f)$ is multivalued. Each extreme path corresponds to one realization of noise. Taking all such realizations into consideration there exists one giving the least action. This least action $S_{min}(q_f)$ is the actual value that appears in the WKB approximation (5) and the corresponding $q(t)$ is the optimal trajectory $q_{opt}(t)$. On the other hand, the effect of singularities on fluctuations virtually relies on the effect on optimal paths, since only the optimal one is physically observable in reality.

To obtain the optimal path we have to resort to some approximate methods, one of which is the concept of prehistory probability distribution $P_h(\boldsymbol{q}, t \mid \boldsymbol{q_i}, t_i, \boldsymbol{q_f}, t_f)$, first proposed by Dykman and McClintock [1]. It turns out to be an appropriate statistical method to describe the distribution of fluctuational paths, both theoretically and experimentally [10]. By definition, $P_h(\boldsymbol{q}, t \mid \boldsymbol{q_i}, t_i, \boldsymbol{q_f}, t_f)$ gives the probability that the system is at q at time t, provided that its position is at $\boldsymbol{q}_i$ at initial time t_i and at $\boldsymbol{q}_f$ at final time t_f. Due to the fact that optimal paths are ones along which the system moves during escape with overwhelming probability, the prehistory probability distribution peaks sharply around the optimal path as $D \to 0$. In this method we first selected an initial point ($q_0 = -1$, $\varphi_0 = 0$) and integrate the stochastic differential equations (2) numerically, with the noise intensity $D = 3 \times 10^{-3}$. The evolution of the system $q(t)$ and the random force $\xi(t)$ is tracked continuously until the system drops into a small neighborhood of the given end point (q_f, φ_f). Then this particular path and noise realization are both conserved. Repeating this process thousands of times one obtains an ensemble of trajectories starting at point (q_0, φ_0) and terminating at point (q_f, φ_f). By sample averaging the prehistory probability distribution can be constructed for the time interval of observation and it contains all information

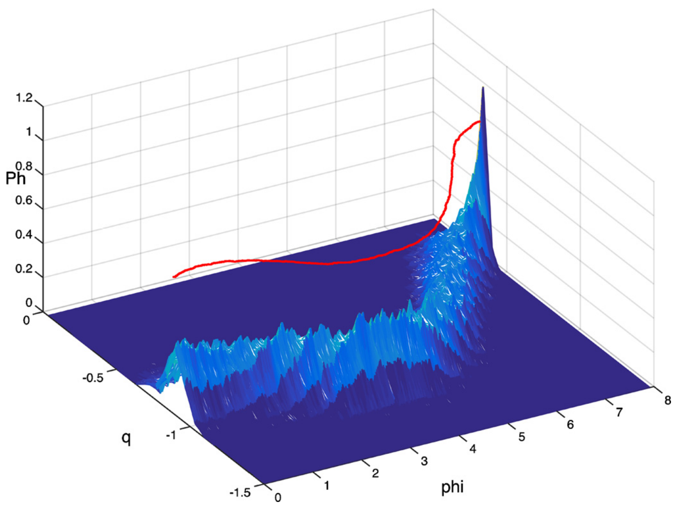


Fig. 4. Prehistory probability distribution of fluctuational paths to a given point, showing obvious ridges along which the optimal path lies. The red curve, corresponding to the optimal path, is plotted above. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

with respect to the evolution process of the fluctuation. In view of the fact that most of the time would be spent on fluctuating about the attractor, the initial time t_0 is often treated as $-\infty$ and the exact value of q_0 makes no difference. Thus they can be omitted for simplicity in the expression of P_h.

As an example, we consider the case $(q_f, \varphi_f) = (-0.3671, 7.92)$, in which the terminating point locates outside the region surrounded by caustics. With thousands of samples fluctuating to the given point obtained in the way above, we present the prehistory probability distribution of paths fluctuating to (q_f, φ_f) in Fig. 4. It's apparent to see that the distribution P_h has sharp peaks along certain trajectory indicated by a red curve in Fig. 4. By definition it is actually the optimal path fluctuating to (q_f, φ_f).

The optimal path is also plotted in Fig. 5(a) in a blue curve, in comparison with extreme paths. Since the terminating point locates outside the region surrounded by caustics, there exists a unique extreme path emanating from the stable periodic orbit and fluctuating to (q_f, φ_f) in this case, as is obviously shown in Fig. 5(a). This particular extreme path, indicated by a black curve, is the optimal one.

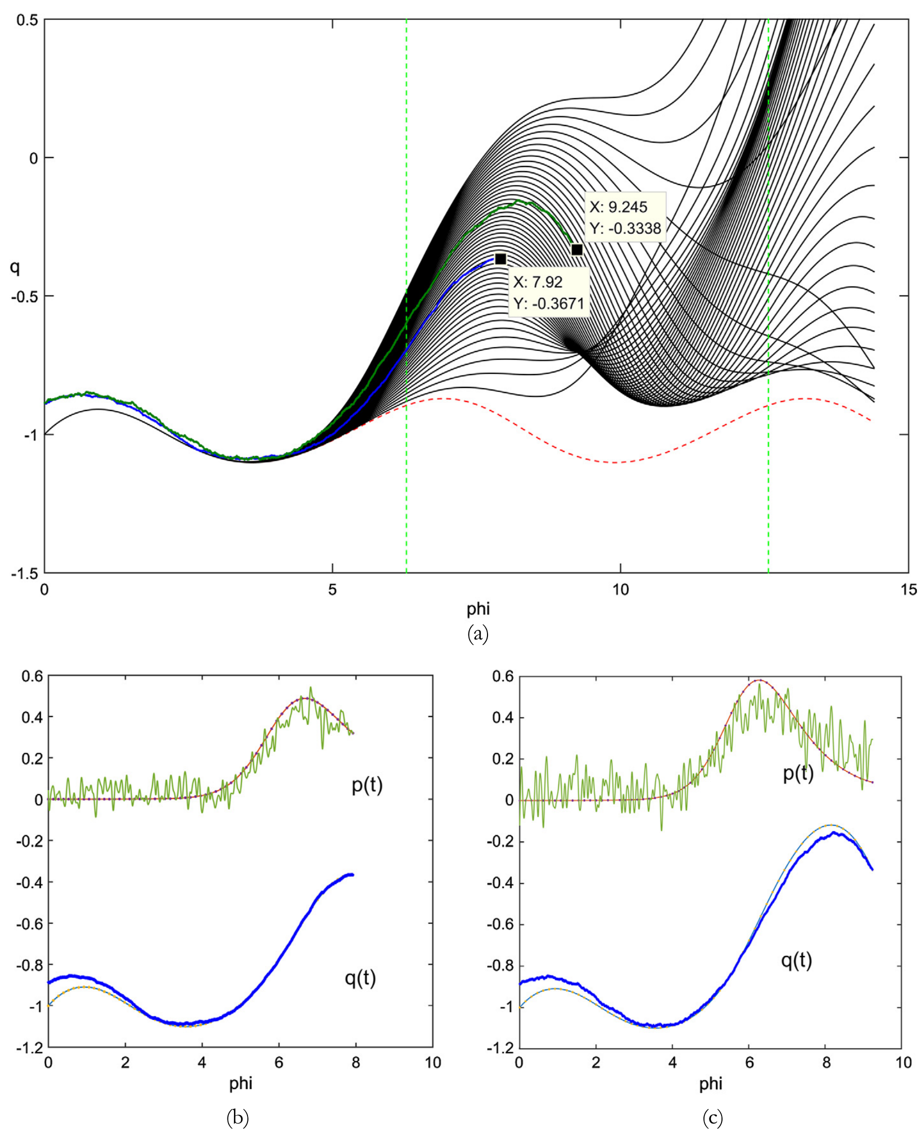


Fig. 5. (a) Patterns of extreme paths in solid black lines and the blue curve represents the optimal path to a given point indicated in the figure. (b) Solutions of boundary value problem are compared with statistical results, showing good agreement. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

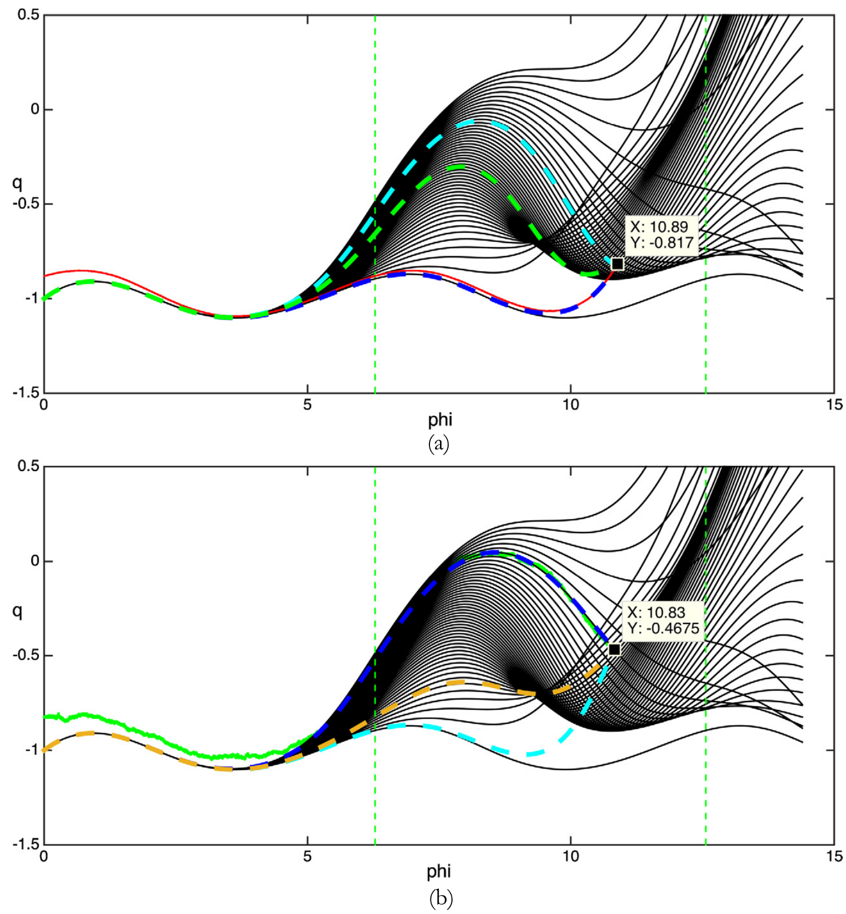


Fig. 6. Two topologically different optimal paths by statistical analysis are plotted by red and green solid lines respectively. Three extreme paths obtained by solving corresponding boundary value problems to each given point, plotted in dashed lines of three colors. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

To verify the validity of our statistical analysis, the boundary value problem of Hamiltonian system (11) is solved. The boundary conditions in this case are as follows

$$(q_0, \varphi_0) = (-1, 0), \qquad (q_f, \varphi_f) = (-0.3671, 7.92) \tag{12}$$

and the results of boundary value problem are given in Fig. 5(b), indicated by the thin curves with dots. The blue thick curve in Fig. 5(b) is the optimal path by statistical analysis and the green zigzag line above is acquired by averaging the corresponding noise realizations conserved simultaneously with escape paths and performing a low-pass filtering process. They show perfect agreement, as is demonstrated in Fig. 5(b). It can also be concluded that the optimal force $p(t)$ begins contributing to the escape only after $\varphi \approx 5$. This can be properly explained from Fig. 5(a) that prior to $\varphi \approx 5$ the system moved along the stable periodic state, costing no energy. The optimal force switches on at the moment when system leaves the attractor and overcomes its attraction.

As another illustration we investigate the fluctuation to $(q_f, \varphi_f) = (-0.3338, 9.245)$ from the same attractor and similar results are present in Fig. 5. The green curve shown in Fig. 5(a) is the optimal path obtained by the same procedure and as can be seen from Fig. 5(c), there is a good agreement between results of statistical analysis and of boundary value problem. The optimal force $p(t)$ first increases to nonzero, as we just discussed, then returns to zero. The decrease of $p(t)$ is associated with the relaxational process since when the optimal path reaches the highest position q_{max}, it should continue its motion to q_f. The motion from q_{max} to q_f is actually dominated by the attraction of the attractor and no external energy is needed. We end the section by pointing out that the nonzero and zero optimal force correspond to fluctuational and relaxational process respectively.

5. Effect of singularities and multiple extreme paths

In this section the terminating point will be chosen in the region where various extreme paths intersect and the effect of singularities on the optimal path will be investigated. Two cases are considered: Case 1. $(q_f, \varphi_f) = (-0.8170, 10.89)$ and Case 2. $(q_f, \varphi_f) = (-0.4675, 10.83)$. Results for two cases are plotted in Fig. 6 from which one can see that the two terminating points are reached along topologically different optimal paths, in red and green solid lines respectively. According to [16], there exists a continuous line separating the region surrounded by caustics. Points on the opposite side of this line have qualitatively different optimal trajectories and thus such a line is called switching line. Comparisons with solutions of boundary value problem of Hamiltonian system are drawn and omitted here.

Besides the optimal one there are several extreme paths fluctuating to the same point in the singular region, due the topological structure of Lagrangian manifold. A schematic plot is given in Fig. 1 and it is observed that the Lagrangian manifold has three sheets, leading to at least three extreme paths through the same point. The upper and lower sheets are traced out by trajectories that haven't gone over the fold. By solving the boundary value problem, we obtain three extreme paths for each case illustrated in Fig. 6. These three extreme paths correspond to Hamiltonian trajectories moving in three sheets of the Lagrangian manifold. Only one of them is physically observable. The middle extreme paths

Table 1
Actions calculated along the upper, middle, lower extreme path for both Case 1 and Case 2.

	Case 1	Case 2
Upper	0.2735	0.2860 (optimal)
Middle	0.2972	0.5656
Lower	0.1064 (optimal)	0.4850

in Fig. 6(a) and Fig. 6(b), indicated by green and yellow dashed curves, are projections of Hamiltonian trajectories that go over a fold or seem to pass through the cusp.

Three extreme paths leading to the same point q give rise to three values of the action $S(q)$ and it's known that the optimal one has the absolute minimum of the action. We present the values of actions in Table 1 to demonstrate the minimality of the optimal path. It can be seen that the lower path for Case 1 and the upper path for Case 2 have the minimum actions, which is in consistent with our results of statistical analysis. In addition, the maximum of the action is always attained by the middle path. It's known that the middle path is either reflected by one of the caustics or it seems to traverse the cusp directly, inferring that the corresponding Hamiltonian trajectory goes over the fold and moves in the middle sheet of Lagrangian manifold. As a result, we may conclude that in general cases optimal paths never drop into the middle sheet of Lagrangian manifold, neither by going over a fold nor traversing the cusp.

6. Conclusions

In this paper we investigated the pattern of extreme paths in the vicinity of a stable periodic orbit of an overdamped periodically driven dynamical system. Hamiltonian formalism was employed to give the auxiliary dynamical system describing the behaviors of extreme path. We discussed the method parameterization of the unstable Lagrangian manifold and proper initial conditions were given by linear analysis. We have shown that the pattern of extreme paths displays singular behaviors: cusps and caustics.

Using the concept of prehistory probability distribution, the optimal path fluctuating to a given point can be determined by statistical analysis. To prove the validity of our result, boundary value problems of the auxiliary Hamiltonian system were solved by setting appropriate boundary conditions. Both results show good agreement and the role of the optimal force in the process of escape was also analyzed in detail. It follows that energy is consumed only when the attraction of a stable state is overcome.

The effect of singularities on fluctuations was studied, by considering two cases of terminating points, located on both sides of the switching line. It is found that optimal paths for these two cases are topologically different and furthermore, the multiple solutions of boundary value problem were obtained, verifying the existence of several extreme paths and the multivaluedness of the action. A computation of the action along each extreme path shows that the optimal path attains the minimum and paths dropped into the middle sheet of Lagrangian manifold acquire the maximum, leading to a conclusion of behaviors of optimal paths in general cases.

Acknowledgements

This research was supported by the National Natural Science Foundation of China (Grant No. 11472126, 11232007) and the Project funded by the Priority Academic Program Development of Jiangsu Higher Education Institutions (PAPD).

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