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On the Stochastic Dynamical Behaviors of a Nonlinear Oscillator Under Combined Real Noise and Harmonic Excitations

The exit problem and global stability of a nonlinear oscillator excited by an ergodic real noise and harmonic excitations are examined. The real noise is assumed to be a scalar stochastic function of an n-dimensional Ornstein–Uhlenbeck vector process which is the output of a linear filter system. Due to the existence of t-dependent excitation, two two-dimensional Fokker–Planck–Kolmogorov (FPK) equations governing the van der Pol variables process and the amplitude-phase process, respectively, are obtained and discussed through a perturbation method and the spectrum representations of the FPK operator and its adjoint operator of the linear filter system, while the detailed balance condition and the strong mixing condition are removed. Based on these FPK equations, the global properties of one-dimensional nonlinear oscillators with external or (and) internal periodic excitations under external or (and) internal real noises can be examined. Finally, a Duffing oscillator excited by a parametric real noise and parametric harmonic excitations is presented as an example, and the mean first-passage time (MFPT) about the oscillator's exit behavior between limit cycles is obtained under both wide-band noise and narrow-band noise excitations. The analytical result is verified by digital simulation. [DOI: 10.1115/1.4034735]

1 Introduction

Real noise is always observed in real life, in contrast, white noise is just an idealized model and is the most commonly used model in researches because of its mathematical convenience. Although based upon the method of stochastic averaging, the effect of a white noise can substitute for the one of a real noise, if its bandwidth is wide enough, however, it is regretful that many real noises process finite or narrow bandwidths, for which the effects are significantly different from the ones of white noises and some special phenomena in the nonlinear or even linear oscillators were observed [1–3]. Up to now, there are already a lot of analytical and numerical results reported to investigate the effects of the real noise on the linear and nonlinear oscillators.

By way of the asymptotic analysis and the singular perturbation method, Schuss and coworkers did analytical works to study the exit problem of autonomous nonlinear systems that were excited by colored noise [4]. For a two-coupled autonomous linear oscillator driven by real noise, Namachchivaya and Van Roessel investigated its stochastic stability by applying the asymptotic method and the method of stochastic averaging and obtained the moment Lyapunov exponent, the maximal Lyapunov exponent, and rotation numbers analytically [5–7]. The similar asymptotic analysis was also used by Roy to develop a method of stochastic averaging for a nonlinear oscillator which was autonomous and was excited by colored Gaussian process [8]. However, for a multidimensional noise system, the condition of detailed balance is a necessary condition for the method of stochastic averaging method.

In the series work of Liu and Liew, they used this method to study an autonomous codimension two-bifurcation system which was on a three-dimensional central manifold and was subjected to a stochastic parametric excitation of a real noise, and an autonomous von der Pol–Duffing oscillator driven by a parametric real noise to obtain the maximal Lyapunov exponent and the moment Lyapunov exponent analytically by linearizing these two systems, respectively [9–12]. In Refs. [11] and [12], the results were obtained when the detailed balance and strong mixing condition were both removed, and they also derived a standard FPK equation governing the amplitude process of an autonomous nonlinear stochastic system to analyze its almost-sure stability and stability in probability by the same method. In Ref. [13], a non-Gaussian colored noise system was considered, and the moment stability of a linearized system was investigated by the same method.

In the most engineering models, there always exist harmonic excitations due to the rotation parts in mechanical structure, then in some works, the effects of both harmonic excitations and noises were investigated. In this case, a difficulty arises that an explicit parameter t exists in the expressions, and then, the dimension of the model of the autonomous system is increased. Since even a small noise can induce the phenomenon of large derivation in dynamical systems, the global properties of a nonlinear system with multisteady states were studied widely in recent decades. And because the large derivation is involved, linearization is not available in global analysis.

Roy studied the noise-induced transition in several different kinds of nonlinear systems with multisteady states under periodic excitations and Gaussian white noise by using averaging method and perturbation method developed by Schuss and coworkers [14–17]. In their theory, stationary probability density function is estimated by the use of WKB approximation and ray method. By using stochastic averaging method, Zhu and his group studied the first failure passage of nonlinear system under harmonic and Gaussian white noise and solved the backward Kolmogorov equation and generalized Pontryagin equation by the use of finite difference scheme [18–21]. They also studied the first failure passage of nonlinear system under harmonic and wide-band noise by using the same methods [22,23]. And even strongly nonlinear oscillators are under consideration [24]. But because of the limit

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of stochastic averaging method, only wide-band noise was considered in these papers.

In order to analyze the global properties of general nonlinear oscillators under real noise, a nonlinear term which is t-dependent and a parametric real noise excitation are considered in this paper. Based on the theory of Schuss' work [4] to study exit problem, the evaluation of stationary probability density function $p(\mathbf{x})$ is under consideration at first. Then, MFPT measuring the mean time cost during the process that the oscillator starting from one state to another is expressed in terms of $p(\mathbf{x})$ by using singular perturbation method [4,8] in the limit of small noise

$$T_{\mathrm{MFPT}} = \frac{\int_D p(\mathbf{x})\mathrm{d}\mathbf{x}}{\int_{\partial D} \mathbf{Pc}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x})\mathrm{d}s} \tag{1}$$

where $\mathbf{x}$ is the state vector, D is the domain of interest in the attracting region of one attractor to study the exit behavior to exit this domain, $\mathbf{Pc}$ is the stationary probability current density vector, $\mathbf{n}$ is the outer normal vector of the boundary ∂D of the domain D, and s is the coordinate along this boundary.

To determine the stationary probability density function of a nonlinear oscillator under real noise, two FPK equations based on different variables are obtained and discussed in Sec. 4 by the use of perturbation method and spectrum representations of the FPK operator and its adjoint operator of the linear filter system, while the detailed balance condition and the strong mixing condition are removed. And, these equations can be degenerated to the one that is derived from the stochastic averaging method in the case of Gaussian white noise. Due to the existence of t-dependent excitations, one more dimension is extended and the standard one-dimensional FPK equation governing the amplitude process is hardly obtained from the analysis in Refs. [8], [11], and [12]. With the estimation of the probability density function of a general nonlinear oscillator under real noise, the global properties of one-dimensional nonlinear oscillators with external or (and) internal periodic excitations under external or (and) internal real noises can be examined. In this paper, a simple example is given to illustrate this point by calculating the MFPT of a nonlinear oscillator.

Based on the FPK equation governing the von der Pol variables, the global stability of the subharmonic solutions in a Duffing nonlinear oscillator which is excited by parametric excitations is examined by the use of singular perturbation and ray method. MFPT about its exit behavior is obtained under both wide-band and narrow-band real noise excitation. Both results are verified by the digital simulation.

2 Real Noise and Spectrum Analysis

In this paper, the real noise is assumed as a scalar function of a n-dimensional vector process which is an output of a linear filter system. A general linear filter system is governed by the following It ô stochastic differential equation:

$$\dot{\mathbf{u}}(t) = \mathbf{A}\mathbf{u}(t) + \mathbf{W}.(t) \tag{2}$$

where $\mathbf{A} = [a_{ij}]$, $i,j = 1,\dots,n$ is a n-dimensional square matrix, $\mathbf{W}.(t)$ is an n-dimensional zero-mean Gaussian white noise with $\mathrm{E}\left(\mathbf{W}.(t+\tau)\,\mathbf{W}.(t)\right) = \mathbf{V}\delta(\tau)$, and $\mathbf{V} = [v_{ij}]$, $i,j = 1,\dots,n$ is a symmetric, non-negative defined constant matrix. Thus, $\mathbf{u} = (u_1,\dots,u_n)^{\mathrm{T}}$ is an n-dimensional Ornstein–Uhlenbeck vector process and also is a stationary Gaussian diffusion process. According to Refs. [11] and [12], matrices $\mathbf{A}$ and $\mathbf{V}$ satisfy the following two conditions:

(a) the eigenvalues of $\mathbf{A}$, $\alpha_1,\dots,\alpha_n$, satisfy $\alpha_i \neq \alpha_j$, $i \neq j$, and $\mathrm{Re}(\alpha_i) < 0$, $i = 1,\dots,n$, to ensure that the process is asymptotically stable, and the corresponding eigenvectors are denoted by $\mathbf{e}_1,\dots,\mathbf{e_n}$.

(b) $(\mathbf{A},\tilde{\mathbf{V}})$ is a controllable pair, i.e., $\mathrm{rank}\left(\tilde{\mathbf{V}},\mathbf{A}\tilde{\mathbf{V}},\dots,\mathbf{A^{n-1}}\tilde{V}\right) = n$, and $\mathbf{V} = \tilde{\mathbf{V}}\tilde{\mathbf{V}}^{\mathrm{T}}$.

The covariance matrix of this process is denoted as $\mathbf{K_u} = \mathrm{E}(\mathbf{u}(t)\mathbf{u}(t))$, and it is the solution of the steady-state variance equation

$$\mathbf{A}\mathbf{K_u} + \mathbf{K_u}\mathbf{A}^{\mathrm{T}} + \mathbf{V} = 0 \tag{3}$$

According to the theory of stochastic process, the FPK operator for diffusion process $\mathbf{u}(t)$ and its adjoint operator, i.e., the backward Kolmogorov operator, are given by

$$\begin{aligned}
&\mathcal{L}_u(\cdot) = -\partial_{u_i}(a_{ij}u_j\cdot) + \frac{1}{2}v_{ij}\partial^2_{u_iu_j}(\cdot),\\
&\mathcal{L}^*_u(\cdot) = a_{ij}u_j\partial_{u_i}(\cdot) + \frac{1}{2}v_{ij}\partial^2_{u_iu_j}(\cdot), \quad \partial_{u_i} = \frac{\partial}{\partial u_i}, \quad \partial^2_{u_iu_i} = \frac{\partial^2}{\partial u_i \partial u_j}
\end{aligned} \tag{4}$$

where the repeated indexes indicate the summation convention in this paper. And, the eigenvalue problems of these two operators are described as

$$\mathcal{L}_u(\psi_\lambda(\mathbf{u})) = \lambda\psi_\lambda(\mathbf{u}), \quad \mathcal{L}^*_u\left(\psi^*_{\lambda'}(\mathbf{u})\right) = \lambda'\psi^*_{\lambda'}(\mathbf{u}) \tag{5}$$

According to the theory of spectral analysis of FPK operator, $\mathcal{L}_u$ and $\mathcal{L}^*_u$ have the same eigenvalues, i.e., $\lambda = \lambda'$. In Refs. [12] and [11], the eigenvalues and eigenfunctions are obtained without the detailed balance condition, respectively.

(a) $\lambda_{\mathbf{m}} = m_i\alpha_i$, $i = 1,\dots,n$, where $\mathbf{m} = (m_1,\dots,m_n)^{\mathrm{T}}$, $m_1 + m_2 + \cdots + m_n = m$, and m_i are all non-negative integers. Since $\alpha_i < 0$, all $\lambda_{\mathbf{m}}$ are non-negative too.

(b) The corresponding eigenfunctions of $\mathcal{L}_u$ and $\mathcal{L}^*_u$ can be obtained by the use of multivariate Hermite polynomials without the detained balance condition [11,12]. They can be expressed as below

$$\begin{aligned}
&\psi^*_{\mathbf{m}}(\mathbf{u}) = (-1)^m \exp\left[\frac{\mathbf{v}^{\mathrm{T}}\mathbf{C}v}{2}\right]\partial^m_{w_1^{m_1}w_2^{m_2}\cdots w_n^{m_n}}\exp\left[-\frac{\mathbf{v}^{\mathrm{T}}\mathbf{C}\mathbf{v}}{2}\right],\\
&\psi_{\mathbf{m}}(\mathbf{u}) = \psi_0(\mathbf{u})\prod_{i=1}^{n}\left(\mathbf{U}_i^{\mathrm{T}}\mathbf{u}\right)^{m_i}
\end{aligned} \tag{6}$$

where $\mathbf{v} = \mathbf{U}^{-1}\mathbf{u}, \mathbf{C} = \mathbf{U}^{\mathrm{T}}\mathbf{K_u}^{-1}\mathbf{U}, \mathbf{w} = \mathbf{U}^{\mathrm{T}}\mathbf{K_u}^{-1}\mathbf{u}, \mathbf{U} = (\mathbf{e}_1,\dots,\mathbf{e}_n)$, and $\mathbf{U}_i^{\mathrm{T}}$ is the ith row vector of $\mathbf{U}^{\mathrm{T}}$.

It is noted that $\lambda_0 = 0$, the corresponding eigenfunction $\psi^*_0(\mathbf{u}) = 1$ and $\psi_0(\mathbf{u})$ is the stationary transition probability density function $p_s(\mathbf{u})$, which satisfies $\int_{Rn}\psi_0(\mathbf{u})\mathrm{d}\mathbf{u} = \int_{Rn}p_s(\mathbf{u})\mathrm{d}\mathbf{u} = 1$.

Next, a new function $\tilde{\psi}_{\mathbf{m}}$ is introduced

$$\tilde{\psi}_{\mathbf{m}}(\mathbf{u}) = \psi_{\mathbf{m}}(\mathbf{u})/\psi_0(\mathbf{u}) \tag{7}$$

and the expectation of $\tilde{\psi}_{\mathbf{m}1}\psi^*_{\mathbf{m}2}$ is estimated

$$\begin{aligned}
E\left(\tilde{\psi}_{\mathbf{m}1}\psi^*_{\mathbf{m}2}\right) &= \int_{Rn}\tilde{\psi}_{\mathbf{m}1}\psi^*_{\mathbf{m}2}\psi_0(\mathbf{u})\mathrm{d}\mathbf{u} = \delta_{\mathbf{m}1,\mathbf{m}2}\\
&= \begin{cases} 1, & \mathbf{m1} = \mathbf{m2}\\ 0, & \mathbf{m1} \neq \mathbf{m2}\end{cases}
\end{aligned} \tag{8}$$

Based on the spectral analysis of linear filter systems, the transition probability density of process $\mathbf{u}(t)$ is given by

$$p(\mathbf{u},t|\mathbf{u}_0,0) = \psi_0(\mathbf{u})\sum_{m_1=0,\dots,m_n=0}^{\infty}\tilde{\psi}_{\mathbf{m}}(\mathbf{u})\psi^*_{\mathbf{m}}(\mathbf{u}_0)\exp(\lambda_{\mathbf{m}}t), \quad t \geq 0 \tag{9}$$

Then, the covariance and spectral functions of a scalar stochastic function for $f(\mathbf{u})$ which satisfies that $E(f(\mathbf{u})) = 0$, $E(f^2(\mathbf{u})) = 0 < +\infty$ are obtained easily with the relationship $\psi_0(\mathbf{u}) = p_s(\mathbf{u})$ and Eq. (9)

$$R_f(\tau) = \int_{Rn} d\mathbf{u} \int_{Rn} d\mathbf{u}_0 \left[f(\mathbf{u}_0) f(\mathbf{u}) p(\mathbf{u}, \tau | \mathbf{u}_0, 0) p_s(\mathbf{u}_0) \right]$$
$$= \sum_{m_1=0,\ldots,m_n=0}^{\infty} E\left(f(\mathbf{u}_0)\psi^*_{\mathbf{m}}(\mathbf{u}_0)\right) E\left(f(\mathbf{u})\tilde{\psi}_{\mathbf{m}}(\mathbf{u})\right) \exp(\lambda_{\mathbf{m}}\tau) \tag{10}$$

$$S_f(\omega) = 2\int_{Rn} \mathbf{R}_f(\tau)\cos(\omega\tau)d\tau$$
$$= -\sum_{m_1=0,\ldots,m_n=0}^{\infty} E\left(f(\mathbf{u}_0)\psi^*_{\mathbf{m}}(\mathbf{u}_0)\right) E\left(f(\mathbf{u})\tilde{\psi}_{\mathbf{m}}(\mathbf{u})\right) \frac{2\lambda_{\mathbf{m}}}{\lambda^2_{\mathbf{m}} + \omega^2} \tag{11}$$

$$\Phi_f(\omega) = 2\int_{Rn} \mathbf{R}_f(\tau)\sin(\omega\tau)d\tau$$
$$= -\sum_{m_1=0,\ldots,m_n=0}^{\infty} E\left(f(\mathbf{u}_0)\psi^*_{\mathbf{m}}(\mathbf{u}_0)\right) E\left(f(\mathbf{u})\tilde{\psi}_{\mathbf{m}}(\mathbf{u})\right) \frac{2\omega}{\lambda^2_{\mathbf{m}} + \omega^2} \tag{12}$$

With Eq. (11), the spectrum of a real noise produced by the scalar stochastic function $f(\mathbf{u})$ and Eq. (2) is determined.

3 Formulation

According to Eq. (1), the stationary probability density function of a nonlinear oscillator should be estimated before determining MFPT of the exit problem quantitatively. In this section, FPK equations and the structure of their solutions are established. Consider a common nonlinear oscillator which is excited by a real noise and a t-dependent excitation

$$\ddot{x}(t) + \varepsilon g(x(t), \dot{x}(t), t) + \omega_0^2 x(t) = \sqrt{\varepsilon} h(x(t), \dot{x}(t), t) f(\mathbf{u}) \tag{13}$$

where $\varepsilon \to 0$ indicates the weak noise, weak damping, weak nonlinearity, and weak excitation, $g(x(t), \dot{x}(t), t)$ is a nonlinear function with an explicit t variable, and $h(x(t), \dot{x}(t), t)$ is a general function. The process $\mathbf{u}$ is defined in Sec. 2, and the properties of this process are also discussed in Sec. 2.

To simplify the study of this nonlinear oscillator, the trajectories of this oscillator are assumed to be elliptic

$$x(t) = r(t)\cos(\phi(t)) = a(t)\cos(\omega_0 t) + b(t)\sin(\omega_0 t),$$
$$\dot{x}(t) = -\omega_0 r(t)\sin(\phi(t)) = -\omega_0 a(t)\sin(\omega_0 t) + \omega_0 b(t)\cos(\omega_0 t) \tag{14}$$

where $r, a, b \geq 0, \phi \in [0, \pi]$. Based on Eq. (14), the phase plane $(x, \dot{x})$ can be transformed into (r, ϕ) or $(a,\ b)$. Substituting Eq. (14) into Eq. (13), the equations of motions governing the process (r, ϕ) and (a, b) are obtained, respectively,

$$\begin{cases} \dot{r}(t) = \left[\varepsilon\sin(\phi)g(r,\phi,t) - \sqrt{\varepsilon}\sin(\phi)h(r,\phi,t)f(\mathbf{u})\right]/\omega_0 \\ \dot{\phi}(t) = \omega_0 + \left[\varepsilon\cos(\phi)g(r,\phi,t) - \sqrt{\varepsilon}\cos(\phi)h(r,\phi,t)\right)f(\mathbf{u})]/(\omega_0 r) \end{cases} \tag{15}$$

$$\begin{cases} \dot{a}(t) = \left[\varepsilon\sin(\omega_0 t)g(a,b,t) - \sqrt{\varepsilon}\sin(\omega_0 t)h(a,b,t)f(\mathbf{u})\right]/\omega_0 \\ \dot{b}(t) = \left[-\varepsilon\cos(\omega_0 t)g(a,b,t) + \sqrt{\varepsilon}\cos(\omega_0 t)h(a,b,t)f(\mathbf{u})\right]/\omega_0 \end{cases} \tag{16}$$

Since t is an explicit parameter of the function $g(x(t), \dot{x}(t), t)$ in Eq. (13), a new parameter is introduced

$$\theta = \omega_0 t, \quad \dot{\theta}(t) = \omega_0 \tag{17}$$

with Eqs. (17), (15), and (16) are rewritten, respectively, as

$$\begin{cases} \dot{r}(t) = \left[\varepsilon\sin(\phi)g(r,\phi,\theta) - \sqrt{\varepsilon}\sin(\phi)h(r,\phi,\theta)f(\mathbf{u})\right]/\omega_0 \\ \dot{\phi}(t) = \omega_0 + \left[\varepsilon\cos(\phi)g(r,\phi,\theta) - \sqrt{\varepsilon}\cos(\phi)h(r,\phi,\theta)f(\mathbf{u})\right]/(\omega_0 r) \\ \dot{\theta}(t) = \omega_0 \end{cases} \tag{18}$$

$$\begin{cases} \dot{a}(t) = \left[\varepsilon\sin(\theta)g(a,b,\theta) - \sqrt{\varepsilon}\sin(\theta)h(a,b,\theta)f(\mathbf{u})\right]/\omega_0 \\ \dot{b}(t) = \left[-\varepsilon\cos(\theta)g(a,b,\theta) + \sqrt{\varepsilon}\cos(\theta)h(a,b,\theta)f(\mathbf{u})\right]/\omega_0 \\ \dot{\theta}(t) = \omega_0 \end{cases} \tag{19}$$

In the limit $\varepsilon \to 0$, the variables $r(t)$ in Eq. (18) and $a(t)$ and $b(t)$ in Eq. (19) are slow variables, and $\phi(t)$ and $\theta(t)$ are fast variables. It is noted that there are two fast variables in Eq. (18), $\phi(t)$ and $\theta(t)$, and they have the same term of order ε^0. The existence of explicit t in Eq. (13) induces the appearance of the two fast variables in Eq. (18). This makes the analysis to obtain FPK equation for Eq. (18) more complicated than the one in Refs. [8], [11], and [12]. And, a new variable can be introduced to reduce the number of fast variables

$$\vartheta(t) = \phi(t) - \omega_0 t, \quad \dot{\vartheta}(t) = \dot{\phi}(t) - \omega_0, \quad \vartheta \in [0, 2\pi] \tag{20}$$

Thus, Eq. (18) can be rewritten as

$$\begin{cases} \dot{r}(t) = \left[\varepsilon\sin(\vartheta+\theta)g(r,\vartheta,\theta) - \sqrt{\varepsilon}\sin(\vartheta+\theta)h(r,\vartheta,\theta)\right)f(\mathbf{u})]/\omega_0 \\ \dot{\vartheta}(t) = \left[\varepsilon\cos(\vartheta+\theta)g(r,\vartheta,\theta) - \sqrt{\varepsilon}\cos(\vartheta+\theta)h(r,\vartheta,\theta)f(\mathbf{u})\right]/(\omega_0 r) \\ \dot{\theta}(t) = \omega_0 \end{cases} \tag{21}$$

In Sec. 4, it is shown that Eqs. (18) and (21) lead to the same result.

There are three different processes, (r, ϕ, θ), (r, ϑ, θ), and (a, b, θ), to describe the same nonlinear oscillator (Eq. (13)). They are all non-Markovian processes, since the noise excitation $f(\mathbf{u})$ is a real noise. However, the extended processes, $(r, \phi, \theta, \mathbf{u})$, $(r, \vartheta, \theta, \mathbf{u})$, and $(a, b, \theta, \mathbf{u})$, are all Markovian, and their stationary probability density functions are denoted by $p_s(r, \phi, \theta, \mathbf{u})$, $p_s(r, \vartheta, \theta, \mathbf{u})$, and $p_s(a, b, \theta, \mathbf{u})$, respectively. They are all 2π -periodic in variable ϕ, θ, and ϑ and satisfy the following FPK equation which is expanded asymptotically:

$$\mathcal{L}p_s = \left(\mathcal{L}_0 + \sqrt{\varepsilon}\mathcal{L}_1 + \varepsilon\mathcal{L}_2\right)p_s = 0 \tag{22}$$

where $\mathcal{L}_0$, $\mathcal{L}_1$, and $\mathcal{L}_2$ are defined in three different cases:

(a) with respect to Eq. (18)

$$\begin{cases} \mathcal{L}_0(\cdot) = -\omega_0\partial_\phi(\cdot) - \omega_0\partial_\theta(\cdot) + \mathcal{L}_{\mathbf{u}}(\cdot) \\ \mathcal{L}_1(\cdot) = \sin(\phi)f(\mathbf{u})\partial_r(h(r,\phi,\theta)\cdot)/\omega_0 \\ \qquad\qquad + f(\mathbf{u})\partial_\phi(\cos(\phi)h(r,\phi,\theta)\cdot)/(\omega_0 r) \\ \mathcal{L}_2(\cdot) = -\sin(\phi)\partial_r(g(r,\phi,\theta)\cdot)/\omega_0 - \partial_\phi(\cos(\phi)g(r,\phi,\theta)\cdot)/(\omega_0 r) \end{cases} \tag{23}$$

(b) with respect to Eq. (19)

$$\begin{cases} \mathcal{L}_0(\cdot) = -\omega_0\partial_\theta(\cdot) + \mathcal{L}_{\mathbf{u}}(\cdot) \\ \mathcal{L}_1(\cdot) = \sin(\theta)f(\mathbf{u})\partial_a(h(a,b,\theta)\cdot)/\omega_0 \\ \qquad - \cos(\theta)f(\mathbf{u})\partial_b(h(a,b,\theta)\cdot)/\omega_0 \\ \mathcal{L}_2(\cdot) = -\sin(\theta)\partial_a(g(a,b,\theta)\cdot)/\omega_0 + \cos(\theta)\partial_b(g(a,b,\theta)\cdot)/\omega_0 \end{cases} \tag{24}$$

(c) with respect to Eq. (21)

$$\begin{cases} \mathcal{L}_0(\cdot) = -\omega_0\partial_\theta(\cdot) + \mathcal{L}_{\mathbf{u}}(\cdot) \\ \mathcal{L}_1(\cdot) = \sin(\vartheta+\theta)f(\mathbf{u})\partial_r(h(r,\vartheta,\theta)\cdot)/\omega_0 \\ \qquad + f(\mathbf{u})\partial_\vartheta\big(\cos(\vartheta+\theta)h(r,\vartheta,\theta)\cdot\big)/(\omega_0 r) \\ \mathcal{L}_2(\cdot) = -\sin(\vartheta+\theta)\partial_r(g(r,\vartheta,\theta)\cdot)/\omega_0 \\ \qquad - \partial_\vartheta\big(\cos(\vartheta+\theta)g(r,\vartheta,\theta)\cdot\big)/(\omega_0 r) \end{cases} \tag{25}$$

where $\mathcal{L}_{\mathbf{u}}$ is defined in Eq. (4).

The solution of Eq. (22), p_s, is assumed to be expanded asymptotically

$$p_s = p_0 + \sqrt{\varepsilon}p_1 + \varepsilon p_2 \tag{26}$$

It is assumed that the normalization condition of p_s is satisfied by fixing the integral of p_0 equal to 1, and the integrals of p_1 and p_2 both vanish. Substituting Eq. (26) into Eq. (22) and identifying terms of order ε^0, $\sqrt{\varepsilon}$, and ε, the following equations are obtained:

$$\begin{cases} \mathcal{L}_0 p_0 = 0 \\ \mathcal{L}_0 p_1 = -\mathcal{L}_1 p_0 \\ \mathcal{L}_0 p_2 = -(\mathcal{L}_1 p_1 + \mathcal{L}_2 p_0) \end{cases} \tag{27}$$

4 Asymptotic Analysis

The common nonlinear oscillator (Eq. (13)) has been transformed into three different cases, Eqs. (18), (19), and (21), based on three different variable spaces. Thus, their corresponding FPK equations are analyzed, and the solutions with leading order are obtained as follows, respectively.

4.1 Asymptotic Analysis in Space $(r, \boldsymbol{\phi}, \boldsymbol{\theta}, \mathbf{u})$. In this subsection, the solvability condition of the equations $\mathcal{L}_0 p = q$ under variables $(r,\phi,\theta,\mathbf{u})$, just like the three equations in Eq. (27), is given according to Refs. [11] and [12]

$$\int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\theta \int_{R^n} \mathrm{d}\mathbf{u} q^* q = 0 \quad \forall q^* \in \mathrm{Ker}\big(\mathcal{L}_0^*\big) \tag{28}$$

where $\mathcal{L}_0^*$ is the adjoint operator of $\mathcal{L}_0$ in Eq. (23), i.e.,

$$\mathcal{L}_0^*(\cdot) = \omega_0\partial_\phi(\cdot) + \omega_0\partial_\theta(\cdot) + \mathcal{L}_{\mathbf{u}}^*(\cdot) \tag{29}$$

Since the set of the eigenfunctions $\psi_{\mathbf{m}}^*(\mathbf{u})$ is a complete set as discussed in Sec. 2, $q^* \in \mathrm{Ker}\big(\mathcal{L}_0^*\big)$ is expressed as

$$q^*(r,\phi,\theta,\mathbf{u}) = \sum_{m_1=0,\dots,m_n=0}^{\infty} q_{\mathbf{m}}^*(r,\phi,\theta)\psi_{\mathbf{m}}(\mathbf{u}) \tag{30}$$

Substituting Eq. (30) into Eq. (29), the equations governing each coefficient $q_{\mathbf{m}}^*(r,\phi,\theta)$ are given as follows:

$$(\omega_0\partial_\phi + \omega_0\partial_\theta + \lambda_{\mathbf{m}})q_{\mathbf{m}}^*(r,\phi,\theta) = 0 \tag{31}$$

Since $q_{\mathbf{m}}^*(r,\phi,\theta)$ are 2π -periodic in variable ϕ, θ, there is only one periodic solution corresponding to $\lambda_0 = 0$ and $\psi^*(\mathbf{u}) = 1$, i.e., $q_0^*(r,\phi,\theta) = q^*(r)A(\phi-\theta)$, where A is a arbitrary periodic function, and q^* is a arbitrary function in variable r. Thus, the solvability condition (Eq. (28)) is reduced to

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{2\pi} \mathrm{d}\theta \int_{R^n} \mathrm{d}\mathbf{u} q A(\phi-\theta) = 0 \tag{32}$$

Then, the solution of Eq. (22) based on Eqs. (18) and (23) is obtained similarly. As discussed above, the set of the eigenfunctions $\psi_{\mathbf{m}}(\mathbf{u})$ is a complete set, so the solution of the first equation in Eq. (27) $p_0(r,\phi,\theta,\mathbf{u})$ is expanded

$$p_0(r,\phi,\theta,\mathbf{u}) = \sum_{m_1=0,\dots,m_n=0}^{\infty} p_{\mathbf{m}}^{(0)}(r,\phi,\theta)\psi_{\mathbf{m}}(\mathbf{u}) \tag{33}$$

Substituting Eq. (33) into the first equation of Eq. (27), the equation becomes the following equations for each $p_{\mathbf{m}}^{(0)}(r,\phi,\theta)$:

$$(-\omega_0\partial_\phi - \omega_0\partial_\theta + \lambda_{\mathbf{m}})p_{\mathbf{m}}^{(0)}(r,\phi,\theta) = 0 \tag{34}$$

Since $p_{\mathbf{m}}^{(0)}(r,\phi,\theta)$ are 2π -periodic in variables ϕ, θ, there is only one periodic solution corresponding to $\lambda_0 = 0$, i.e., $p_0^0(r,\phi,\theta) = p_0(r)F(\phi-\theta)$, where F is an arbitrary periodic function, and $p_0(r)$ is an arbitrary function in variable r. Then, the solution of the first equation in Eq. (23) is obtained

$$p_0(r,\phi,\theta,\mathbf{u}) = p(r)F(\phi-\theta)\psi_0(\mathbf{u})/N = p(r,\phi-\theta)\psi_0(\mathbf{u})/N \tag{35}$$

where N is a normalization constant to ensure that the integral of $p_0(r,\phi,\theta,\mathbf{u})$ equals 1.

With Eq. (20), it is noted that $p(r,\phi-\theta) = p(r,\vartheta)$ in Eq. (35), then the solutions to the first equation of Eq. (27) in the space $(r,\phi,\theta,\mathbf{u})$ and $(r,\vartheta,\theta,\mathbf{u})$ are the same. Since the last two equations in Eq. (23) are the same as the last two in Eq. (25), which leads to the same result based on Eq. (21) in Sec. 4.3. So, it is concluded that because of the existence of explicit t in a nonlinear stochastic oscillator, it is failed to obtain a one-dimensional process to describe the dynamics of this oscillator approximately by the asymptotic analysis.

4.2 Asymptotic Analysis in Space $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\theta}, \mathbf{u})$. Similar to Sec. 4.1, the solvability condition (Eq. (28)) is persistent, but $\mathcal{L}_0^*$ changes to the following equation:

$$\mathcal{L}_0^*(\cdot) = \omega_0\partial_\theta(\cdot) + \mathcal{L}_{\mathbf{u}}^*(\cdot) \tag{36}$$

And $q^* \in \mathrm{Ker}\big(\mathcal{L}_0^*\big)$ is expressed as Eq. (30)

$$q^*(a,b\theta,\mathbf{u}) = \sum_{m_1=0,\dots,m_n=0}^{\infty} q_{\mathbf{m}}^*(a,b,\theta)\psi_{\mathbf{m}}(\mathbf{u}) \tag{37}$$

Substituting Eq. (37) into Eq. (36), the equations governing each coefficient $q_{\mathbf{m}}^*(a,b,\theta)$ are given as follows:

$$(\omega_0\partial_\theta + \lambda_{\mathbf{m}})q_{\mathbf{m}}^*(a,b,\theta) = 0 \tag{38}$$

Since $q_{\mathbf{m}}^*(a,b,\theta)$ are 2π -periodic in variable θ, there is only one periodic solution corresponding to $\lambda_0 = 0$ and $\psi^*(\mathbf{u}) = 1$, i.e., $q_0^*(a,b,\theta) = q^*(a,b)$ where q^* is a arbitrary function in variables a, b. Thus, the solvability condition (Eq. (28)) is reduced to the following equation in this case:

$$\int_0^{2\pi} \mathrm{d}\theta \int_{R^n} \mathrm{d}\mathbf{u} q(a,b,\theta,\mathbf{u}) = 0 \tag{39}$$

Then consider the solution of Eq. (22) based on Eqs. (19) and (24) first. As Eq. (33), the solution of the first equation in Eq. (27), $p_0(a,b,\theta,\mathbf{u})$, is expanded

$$p_0(a,b,\theta,\mathbf{u}) = \sum_{m_1=0,\ldots,m_n=0}^{\infty} p_{\mathbf{m}}^{(0)}(a,b,\theta)\psi_{\mathbf{m}}(\mathbf{u}) \tag{40}$$

Substituting Eq. (40) into the first equation of Eq. (27), the equation becomes the following one for each $p_{\mathbf{m}}^{(0)}(r,\phi,\theta)$:

$$(-\omega_0\partial_\theta + \lambda_{\mathbf{m}})p_{\mathbf{m}}^{(0)}(a,b,\theta) = 0 \tag{41}$$

Since $p_{\mathbf{m}}^{(0)}(a,b,\theta)$ is 2π-periodic in variable θ, there is only one periodic solution corresponding to $\lambda_0 = 0$, i.e., $p_0^0(a,b,\theta) = p_0(a,b)$, where $p_0(a,b)$ is an arbitrary function in variables a, b. Then, the solution of the first equation in Eq. (24) is obtained

$$p_0(a,b,\theta,\mathbf{u}) = p(a,b)\psi_0(\mathbf{u})/N \tag{42}$$

where N is a normalization constant to ensure that the integral of $p_0(a,b,\theta,\mathbf{u})$ equals 1.

Considering the second equation in Eq. (27), and substituting Eq. (42) into it, it is rewritten as

$$\mathcal{L}_0 p_1(a,b,\theta,\mathbf{u}) = \frac{f(\mathbf{u})\psi_0(\mathbf{u})}{N\omega_0}\Big\{\sin(\theta)\partial_a\big[h(a,b,\theta)p(a,b)\big] - \cos(\theta)\partial_b\big[h(a,b,\theta)p(a,b)\big]\Big\} \tag{43}$$

Since $\mathrm{E}(f(\mathbf{u})) = 0$, i.e., $\int_{R^n} f(\mathbf{u})\psi_0(\mathbf{u})\mathrm{d}\mathbf{u} = 0$, the solvability condition (Eq. (28)) is satisfied. With Eqs. (7) and (8), the function $f(\mathbf{u})\psi_0(\mathbf{u})$ is expanded as a series of the eigenfunction $\psi_{\mathbf{m}}(\mathbf{u})$

$$f(\mathbf{u})\psi_0(\mathbf{u}) = \sum_{m_1=0,\ldots,m_n=0}^{\infty} E\big[f(\mathbf{u})\psi_{\mathbf{m}}^*(\mathbf{u})\big]\psi_{\mathbf{m}}(\mathbf{u}) \tag{44}$$

Expand $p_1(a,b,\theta,\mathbf{u})$ as p_0 in Eq. (40)

$$p_1(a,b,\theta,\mathbf{u}) = \sum_{m_1=0,\ldots,m_n=0}^{\infty} p_{\mathbf{m}}^{(1)}(a,b,\theta)\phi_{\mathbf{m}}(\mathbf{u}) \tag{45}$$

Substituting Eqs. (45) and (44) into the second equation in Eq. (27), the equation becomes the following equation for each $p_{\mathbf{m}}^{(0)}(a,b,\theta)$:

$$(-\omega_0\partial_\theta + \lambda_{\mathbf{m}})p_{\mathbf{m}}^{(1)}(a,b,\theta) = \frac{E\big[f(\mathbf{u})\psi_{\mathbf{m}}^*(\mathbf{u})\big]}{N\omega_0}\Big\{\sin(\theta)\partial_a\big[h(a,b,\theta)p(a,b)\big] - \cos(\theta)\partial_b\big[h(a,b,\theta)p(a,b)\big]\Big\} \tag{46}$$

Since $\psi_0^*(\mathbf{u}) = 1$, it is easily verified that $\mathrm{E}\big[f(\mathbf{u})\psi_0^*(\mathbf{u})\big] = 0$. Then, the coefficients $p_{\mathbf{m}}^{(1)}(a,b,\theta)$ are obtained

$$p_{\mathbf{m}}^{(1)}(a,b,\theta) = \begin{cases} p_0^{(1)}(a,b), & m = 0 \\ -\dfrac{E\big[f(\mathbf{u})\psi_{\mathbf{m}}^*(\mathbf{u})\big]}{N\omega_0^2}K(a,b,\theta), & m \neq 0 \end{cases} \tag{47}$$

where $K(a,b,\theta) = \exp(\lambda_{\mathbf{m}}\theta/\omega_0)\Big[\int_0^\theta k(a,b,x)\exp(-(\lambda_{\mathbf{m}}x/\omega\ \ 0))\mathrm{d}x + C_{\mathbf{m}}\Big]$, $k(a,b,\theta) = \sin(\theta)\,\partial_a\big[h(a,b,\theta)p(a,b)\big] - \cos(\theta)\partial_b\big[h(a,\ b,\theta)\ p(a,b)\big]$, $C_{\mathbf{m}}$ are the constants to ensure that $p_{\mathbf{m}}^{(1)}$ is periodic in θ. Thus, combining Eqs. (47) and (45), $p_1(a,b,\theta,\mathbf{u})$ is expressed as

$$p_1(a,b,\theta,\mathbf{u}) = p_0^{(1)}(a,b)\psi_0(\mathbf{u}) + \sum_{m_1=0,\ldots,m_n=0}^{\infty} p_{\mathbf{m}}^{(1)}(a,b,\theta)\psi_{\mathbf{m}}(\mathbf{u}) \tag{48}$$

In order to ensure that the integral of $p_1(a,b,\theta,\mathbf{u})$ vanishes, the following condition is imposed:

$$\int_0^\infty\int_0^\infty p_0^{(1)}(a,b)\mathrm{d}a\mathrm{d}b = 0, \qquad \int_0^\infty\int_0^\infty\int_0^{2\pi} p_0^{(1)}(a,b,\theta)\mathrm{d}a\mathrm{d}b\mathrm{d}\theta = 0 \tag{49}$$

Finally, considering the third equation in Eq. (27), the solvability condition of it is derived by using Eq. (39)

$$\int_0^{2\pi}\int_{R^n}(\mathcal{L}_1 p_1 + \mathcal{L}_2 p_0)\mathrm{d}\theta\mathrm{d}\mathbf{u} = 0 \tag{50}$$

Substituting Eqs. (42), (47), and (48) into Eq. (50) and considering the $\int_{R^n} f(\mathbf{u})\psi_0(\mathbf{u})\mathrm{d}\mathbf{u} = 0$ and $\int_{R^n}\psi_0(\mathbf{u}) = 1$, the solvability condition (Eq. (50)) becomes

$$\begin{aligned}\int_0^{2\pi}\mathrm{d}\theta\Bigg\{&\Big[-\partial_a\big(A_1(a,b,\theta)p(a,b)\big) + \partial_b\big(A_2(a,b,\theta)p(a,b)\big)\Big]\\ &-[(\partial_a(B_1(a,b,\theta)) - \partial_b(B_2(a,b,\theta)))K(a,b,\theta)\\ &\quad + B_1(a,b,\theta)Ka(a,b,\theta) - B_2(a,b,\theta))Kb(a,b,\theta)]\\ &\times\sum_{m_1=0,\ldots,m_n=0,m\neq 0}^{\infty}\frac{\mathrm{E}\big[f(\mathbf{u})\tilde{\psi}_{\mathbf{m}}(\mathbf{u})\big]\mathrm{E}\big[f(\mathbf{u})\psi_{\mathbf{m}}^*(\mathbf{u})\big]}{\omega_0^2}\Bigg\} = 0\end{aligned} \tag{51}$$

where the coefficients are given below

$$\begin{cases} A_1(a,b,\theta) = \sin(\theta)g(a,b,\theta), \quad A_2(a,b,\theta) = \cos(\theta)g(a,b,\theta) \\ B_1(a,b,\theta) = h(a,b,\theta)\sin(\theta), \quad B_2(a,b,\theta) = h(a,b,\theta)\cos(\theta) \\ Ka(a,b,\theta) = \exp\left(\dfrac{\lambda_{\mathbf{m}}\theta}{\omega_0}\right)\left[\displaystyle\int_0^\theta \partial_a k(a,b,x)\exp\left(-\dfrac{\lambda_{\mathbf{m}}x}{\omega_0}\right)\mathrm{d}x\right] \\ Kb(a,b,\theta) = \exp\left(\dfrac{\lambda_{\mathbf{m}}\theta}{\omega_0}\right)\left[\displaystyle\int_0^\theta \partial_a k(a,b,x)\exp\left(-\dfrac{\lambda_{\mathbf{m}}x}{\omega_0}\right)\mathrm{d}x\right] \end{cases} \tag{52}$$

In Sec. 5, through an example, Eq. (51) is reduced to a two-dimensional FPK-style partial differential equation, and the term including summation is expressed as the power spectral density function $S(\omega)$ and $\Phi(\omega)$ via Eqs. (11) and (12).

4.3 Asymptotic Analysis in Space $(r,\vartheta,\theta,\mathbf{u})$. Since Eq. (25) is similar to Eq. (24), the asymptotic analysis is the same as in Sec. 4.2. The solvability condition is similar to Eq. (39)

$$\int_0^{2\pi}\mathrm{d}\theta\int_{R^n}\mathrm{d}\mathbf{u}q(r,\vartheta,\theta,\mathbf{u}) = 0 \tag{53}$$

The solution to the first equation of Eq. (27) is obtained in the same way

$$p_0(r,\vartheta,\theta,\mathbf{u}) = p(r,\vartheta)\psi_0(\mathbf{u})/N \tag{54}$$

which is the same as Eq. (35).

The following process is the same as in Sec. 4.2 to obtain the final result which is similar to Eq. (51) with some different coefficients:

$$\int_0^{2\pi} \mathrm{d}\theta \Bigg\{ \left[-\partial_a\big(A_1(r,\vartheta,\theta)p(r,\vartheta)\big) + \partial_b\big(A_2(r,\vartheta,\theta)p(r,\vartheta)\big)\right] \\ -[(\partial_r(B_1(r,\vartheta,\theta)) - \partial_\vartheta(B_2(r,\vartheta,\theta)))K(r,\vartheta,\theta) \\ +B_1(r,\vartheta,\theta)Ka(r,\vartheta,\theta) - B_2(r,\vartheta,\theta))Kb(r,\vartheta,\theta)] \\ \times \sum_{m_1=0,\ldots,m_n=0,m\neq 0}^{\infty} \frac{E\left[f(\mathbf{u})\tilde{\psi}_{\mathbf{m}}(\mathbf{u})\right]E\left[f(\mathbf{u})\psi^*_{\mathbf{m}}(\mathbf{u})\right]}{\omega_0^2} \Bigg\} = 0 \tag{55}$$

where the coefficients are given below

$$\begin{cases} A_1(r,\vartheta,\theta) = \sin(\vartheta+\theta)g(r,\vartheta,\theta), \\ A_2(r,\vartheta,\theta) = -\cos(\vartheta+\theta)g(r,\vartheta,\theta)/r \\ B_1(r,\vartheta,\theta) = h(r,\vartheta,\theta)\sin(\vartheta+\theta), \\ B_2(r,\vartheta,\theta) = -h(r,\vartheta,\theta)\cos(\vartheta+\theta)/r \\ Ka(r,\vartheta,\theta) = \exp\left(\frac{\lambda_{\mathbf{m}}\theta}{\omega_0}\right)\left[\int_0^\theta \partial_r k(r,\vartheta,x)\exp\left(-\frac{\lambda_{\mathbf{m}}x}{\omega_0}\right)\mathrm{d}x\right] \\ Kb(r,\vartheta,\theta) = \exp\left(\frac{\lambda_{\mathbf{m}}\theta}{\omega_0}\right)\left[\int_0^\theta \partial_\vartheta k(r,\vartheta,x)\exp\left(-\frac{\lambda_{\mathbf{m}}x}{\omega_0}\right)\mathrm{d}x\right] \\ k(r,\vartheta,\theta) = \sin(\vartheta+\theta)\partial_r\left[h(r,\vartheta,\theta)p(r,\vartheta)\right] \\ \qquad + \partial_\vartheta\left[\cos(\vartheta+\theta)h(r,\vartheta,\theta)p(r,\vartheta)\right]/r \end{cases} \tag{56}$$

With these equations above, the leading order of stationary probability density function of a general nonlinear oscillator under external or (and) internal real noise can be solved. Then based on Eq. (1), the exit problem of one-dimensional nonlinear oscillators under external or (and) internal real noises can be examined quantitatively. However, to get the probability density function, a specific form is necessary. An example is presented in Sec. 5 to obtain the specific form of these two equations above.

5 MFPT and an Example

A Duffing oscillator which is excited by a parametric harmonic excitation and a parametric real noise is investigated as an example to get a specific form of equations in Sec. 4 and to evaluate MFPT by using Eq. (1). By the use of the asymptotic analysis described in Sec. 4, two two-dimensional FPK equations governing variables (a,b), (r,ϑ) are obtained. Based on one of these equations, the global properties of this oscillator are analyzed and the MFPT about its exit behavior between limit cycles is obtained under both wide-band and narrow-band real noise excitation.

The Duffing oscillator under parametric excitations is formulated below

$$\ddot{x}(t) + \varepsilon\beta\dot{x}(t) + \omega_0^2 x(t) + \varepsilon\alpha\omega_0^2 x^3(t) \\ = \varepsilon x(t)\omega_0 F\cos(\omega_f t) + \sqrt{\varepsilon}\sigma\omega_0(x(t) - \bar{x}(t))u(t) \tag{57}$$

where $\bar{x}(t)$ is one of the exact solutions of Eq. (57), which can be obtained analytically, and it represents the operative state of this oscillator, thus usually $\bar{x}(t)$ does not vanish. The type of the parametric real noise, $\sqrt{\varepsilon}\sigma\omega_0(x(t)-\bar{x}(t))u(t)$, models a triggerlike device which drives this oscillator to the expected operative state $\bar{x}(t)$ quickly. If there is no noise in Eq. (57), i.e., $\sigma = 0$, the oscillator persists in the initial zero state, i.e., $x(t) = 0$. Then, when the parametric noise is added, i.e., $\sigma \neq 0$, by the driving of the parametric noise, the oscillator moves to the expected state in a short time and persists there. Figure 1 shows the whole process. Obviously, large deviations are observed in this process, thus the global properties of this oscillator are used to characterize this process. So, the existence of this type of parametric noise excitation enormously enhances the global stability of the expected state and weakens the global stability of other states. This effect is discussed in this section. And, MFPT is used to measure the mean time cost of persisting in the attracting domain of zero solution.

The noise $u(t)$ is a real noise and assumed to be the first component of the process $\mathbf{u}(t)$ in Sec. 2. For simplicity, $u(t)$ can be considered as an output of a first-order linear filter system, which is given below

$$\dot{u}(t) = -\mu u(t) + \sqrt{2\rho}\mu\xi(t) \tag{58}$$

where $\xi(t)$ is a Gaussian white noise with autocorrelation function $\langle\xi(t)\xi(t+\tau)\rangle = \delta(\tau)$. The power spectral density function is obtained easily

$$S(\omega) = \frac{2\rho}{1+(\omega/\mu)^2} \tag{59}$$

The parameter μ denotes the half-power bandwidth, and ρ denotes the value of the half-power.

First, an assumption about ω_f and ω is given below

$$m\omega_f = n(\omega_0 + \varepsilon\gamma) \tag{60}$$

where γ is a tuning parameter. Under this assumption, the harmonic resonance $(m = n)$, subharmonic solution $(m < n)$, and superharmonic solution $(m > n)$ are investigated. In this paper, just the case $m = 1$ and $n = 2$ are considered as an example. Then, five exact solutions of Eq. (57) are obtained by the use of multiscale method and the assumption of elliptic orbits (Eq. (14))

$$\begin{cases} x_1(t) = 0 \\ x_{2,3}(t) = r_2\cos(\omega_0 t + \vartheta_{2,3}) = a_{2,3}\cos(\omega_0 t) + b_{2,3}\sin(\omega_0 t) \\ x_{4,5}(t) = r_3\cos(\omega_0 t + \vartheta_{4,5}) = a_{4,5}\cos(\omega_0 t) + b_{4,5}\sin(\omega_0 t) \end{cases} \tag{61}$$

where the coefficients in Eq. (61) are given below

$$\begin{cases} r_1 = 0, \quad r_{2,3}^2 = \frac{8\gamma \pm 2\sqrt{F^2-4\beta^2}}{3\omega_0\alpha}, \quad \sin(2\vartheta_{2,3,4,5}) = -\frac{2\beta}{F}, \\ \quad \cos(2\vartheta_{2,3}) = -\frac{\sqrt{F^2-4\beta^2}}{F}, \quad \cos(2\vartheta_{4,5}) = \frac{\sqrt{F^2-4\beta^2}}{F} \\ a_1 = 0, b_1 = 0, \quad a_{2,3} = r_2\cos(\vartheta_{2,3}), b_{2,3} = -r_2\sin(\vartheta_{2,3}), \\ a_{4,5} = r_3\cos(\vartheta_{4,5}), b_{4,5} = -r_3\sin(\vartheta_{4,5}) \\ \vartheta_{2,4} \in [0,\pi], \quad \vartheta_{3,5} \in [\pi, 2\pi] \end{cases} \tag{62}$$

The existence condition of these solutions given below is assumed to be satisfied

$$F^2 - 4\beta^2 > 0, \quad 4\gamma - \sqrt{F^2-4\beta^2} > 0, \quad 2\beta \leq F \tag{63}$$

The zero solutions $x_1(t)$ and $x_{4,5}(t)$ are locally stable limit cycles, and $x_{2,3}(t)$ are locally unstable limit cycles. It is noted that $x_2(t)$ and $x_3(t)$ ($x_4(t)$ and $x_5(t)$) have the same amplitude but different phase angles.

Now one of these stable limit cycles, $x_5(t)$, is chosen as the operative state $\bar{x}(t)$. Then, the function $g(x(t),\dot{x}(t),t)$ and $h(x(t),\dot{x}(t),t)$ in Eq. (13) are expressed according to Eq. (57)

$$g(x(t),\dot{x}(t),t) = \beta\dot{x}(t) + \alpha\omega_0^2 x^3(t) - x(t)\omega_0 F\cos(\omega_f t), \\ h(x(t),t) = \sigma\omega_0(x(t) - \bar{x}(t)) \tag{64}$$

Using the asymptotic analysis in Sec. 4, Eq. (51) is reduced via computation and the use of Eqs. (11) and (12)

$$-\partial_a\left[(I_1+W_1)p_0(a,b)\right]-\partial_b\left[(I_2+W_2)p_0(a,b)\right]+\partial^2_{aa}(u_1p_0(a,b))+\partial^2_{ab}\left[(u_2+v_1)p_0(a,b)\right]+\partial^2_{bb}(v_2p_0(a,b))=0 \tag{65}$$

where

$$\begin{cases} I_1=-\gamma b+bF/4-\beta a/2+3\alpha\omega_f b^3/16+3\alpha\omega_f a^2b/16\\ I_2=Fa/4-3\alpha\omega_f ab^2/16-3\alpha\omega_f a^3/16+a\gamma-b\beta/2\\ u_1=\dfrac{\sigma^2}{16}\left[-2a_5aS(\omega_f)-2b_5bS(\omega_f)+2b_5^2S(0)+2b^2S(0)+a_5^2S(\omega_f)+a^2S(\omega_f)+b^2S(\omega_f)+b_5^2S(\omega_f)-4bb_5S(0)\right]\\ u_2=\dfrac{\sigma^2}{16}\left[-2a_5b_5S(0)-2abS(0)+2ba_5S(0)+2b_5aS(0)-2bb_5\Phi(\omega_f)-2aa_5\Phi(\omega_f)+b^2\Phi(\omega_f)+a^2\Phi(\omega_f)+a_5^2\Phi(\omega_f)+b_5^2\Phi(\omega_f)\right]\\ v_1=\dfrac{\sigma^2}{16}\left[-2a_5b_5S(0)-2abS(0)+2ba_5S(0)+2b_5aS(0)+2bb_5\Phi(\omega_f)+2aa_5\Phi(\omega_f)-b^2\Phi(\omega_f)-a^2\Phi(\omega_f)-a_5^2\Phi(\omega_f)-b_5^2\Phi(\omega_f)\right]\\ v_2=\dfrac{\sigma^2}{16}\left[-2a_5aS(\omega_f)-2b_5bS(\omega_f)+2a_5^2S(0)+2a^2S(0)+a_5^2S(\omega_f)+a^2S(\omega_f)+b^2S(\omega_f)+b_5^2S(\omega_f)-4aa_5S(0)\right]\\ W_1=\partial_a(u_1)+\partial_b(u_2)\\ W_2=\partial_a(v_1)+\partial_b(v_2) \end{cases} \tag{66}$$

And, Eq. (55) is also reduced via computation and the use of Eqs. (11) and (12)

$$-\partial_r\left[(J_1+Y_1)p_0(r,\vartheta)\right]-\partial_\vartheta\left[(J_2+Y_2)p_0(r,\vartheta)\right]+\partial^2_{rr}(y_1p_0(r,\vartheta))+\partial^2_{r\vartheta}\left[(y_2+z_1)p_0(r,\vartheta)\right]+\partial^2_{\vartheta\vartheta}(z_2p_0(r,\vartheta))=0 \tag{67}$$

Where

$$\begin{cases} J_1=r\big(2\beta+F\sin(2\vartheta)\big)/4\\ J_2=F\cos(2\vartheta)/4+\gamma-3\alpha r^2\omega/16\\ y_0=-\dfrac{\sigma^2}{32r}\Big[2r_3^2\cos(2\vartheta-2\vartheta_5)S(0)-2r_3^2\Phi(\omega_f)-2r^2S(\omega_f)-2r_3^2S(\omega_f)-2r_3^2S(0)+rr_3\cos(\vartheta-\vartheta_5)\Phi(\omega_f)\\ \qquad +r_3^2\sin(2\vartheta-2\vartheta_5)\Phi(\omega_f)-rr_3\sin(\vartheta-\vartheta_5)\Phi(\omega_f)+r_3^2\cos(2\vartheta-2\vartheta_5)\Phi(\omega_f)+4r_3r\cos(\vartheta-\vartheta_5)S(\omega_f)\Big]\\ y_1=-ry_0\\ y_2=\dfrac{\sigma^2}{16r}\left[r_3^2\Phi(\omega_f)-r_3^2\sin(2\vartheta-2\vartheta_5)S(0)-r^2\Phi(\omega_f)+2r_3r\sin(\vartheta-\vartheta_3)S(0)\right]\\ z_0=-\dfrac{\sigma^2}{32r^2}\Big[-2r_3^2\sin(2\vartheta-2\vartheta_5)-2r^2\Phi(\omega_f)+r_3r\cos(\vartheta-\vartheta_5)\Phi(\omega_f)+r_3r\sin(\vartheta-\vartheta_5)\Phi(\omega_f)-r_3^2\sin(2\vartheta-2\vartheta_5)\Phi(\omega_f)\\ \qquad +r_3^2\cos(\vartheta-\vartheta_5)\Phi(\omega_f)+4r_3r\sin(\vartheta-\vartheta_5)S(0)\Big]\\ z_1=-rz_0\\ z_2=-\dfrac{\sigma^2}{16r^2}\Big[-2r_3r\sin(\vartheta-\vartheta_5)\Phi(\omega_f)-2r_3r\cos(\vartheta-\vartheta_5)S(\omega_f)+2r^2S(0)r_3^2\cos(2\vartheta-2\vartheta_5)S(0)+r^2S(\omega_f)+r_3^2\Phi(\omega_f)\\ \qquad +r_3^2S(0)-4rr_3\cos(\vartheta-\vartheta_5)S(0)\Big]\\ Y_1=-y_0+\partial_r(y_1)+\partial_\vartheta(y_2)\\ Y_2=-z_0+\partial_r(z_1)+\partial_\vartheta(z_2) \end{cases} \tag{68}$$

Both Eqs. (65) and (67) are two-dimensional FPK-style equations. And, only $S(\omega_f)$, $S(0)$, $\Phi(\omega_f)$, and $\Phi(0)$ are involved in these two equations. In other words, the stationary probability density function is partly determined by only two points in the power spectral density functions $S(\omega)$ and $\Phi(\omega)$. In the case of a real noise defined in Eq. (58), $\Phi(\omega)\equiv 0$ and $S(\omega)$ is given in Eq. (59).

If one fixes $S(\omega_f)\equiv S(0)$, the real noise $u(t)$ defined in Eq. (58) has the same effect of a Gaussian white noise, then Eqs. (65) and (67) are reduced to a result which can be obtained by the use of stochastic averaging method. This can be verified easily.

Based on Eqs. (65) and (67), the influence of real noise on the response and the stationary probability density function can be analyzed by considering the change of the half-power bandwidth μ across the value ω_f. When $\mu>\omega_f$, there is a relation $S(0)\approx S(\omega_f)$, so the effect of this wide-band real noise is similar to the effect of white noise. When $\mu\approx\omega_f$, the effect of

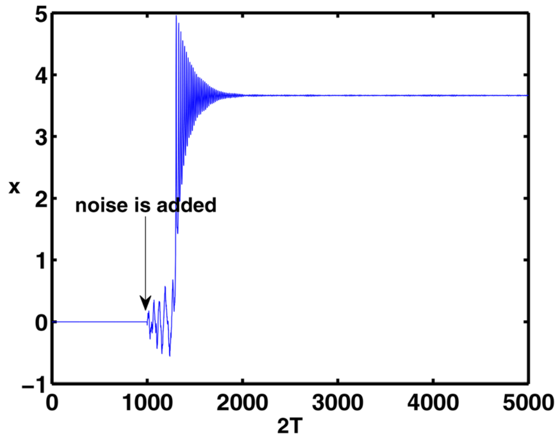

Fig. 1 The time history of x-component of Poincaré points on Poincaré section $\Sigma = \{(x, \dot{x}, t) | t = 0, \bmod 2T\}$, where T stands for the period of the external excitation $T = 2\pi/\omega$

the wide-band real noise is just a little weaker than the effect of white noise. When $\mu < \omega_f$, there is $S(\omega_f) \approx 0$, thus the effect of the real noise is weakest and cannot substitute for the one of white noises. According to these, the real noise driving the oscillator (Eq. (57)) has the same effect as a white noise if $\mu > \omega_f$ is satisfied. These phenomena are verified by the following digital simulation.

In order to study the transitions between limit cycles and zero solution of this oscillator, Eq. (65) with von der Pol variable space is chosen. Because the limit cycles in displacement–velocity variables space are transformed into fixed points in von der Pol variables space. This can facilitate the analysis about the exit behaviors. And, it should be noted that in some other case, the choice of Eq. (67) with amplitude–phase angle variables space is the more convenient one, for example, to analyze the moment Lyapunov exponent and maximal Lyapunov exponent [12,13], because the amplitude and phase angle variables are uncoupled in simplified linearized system. In other words, the choice of variables space should facilitate the analysis of the specific system and research question.

MFPT is used to measure the mean time cost during the process that the oscillator starting from the initial zero state $x_1(t) = 0$ to the expected state $x_5(t)$. Based on the FPK type equation obtained, the MFPT is obtained by the use of WKB approximation (Eq. (69)) and singular perturbation method in the limit of $\sigma^2 \to 0$

$$p(a,b) = w(a,b)\exp\left(-\frac{\Psi(a,b)}{\sigma^2}\right) + O(\sigma^2) \tag{69}$$

To complete this paper, the method to obtain $p(a, b)$ and MFPT T_{MFPT} is briefed. And, details can be found in Refs. [4], [8] [14], and [25]. Substituting Eq. (69) into Eq. (65) and using rays method [26], the functions $w(a, b)$ and $\Psi(a, b)$ are obtained via solving ten ordinary differential equations. Then, Eq. (1) is rewritten as

$$T_{\mathrm{MFPT}} = \frac{\int_D p(a,b)\mathrm{d}a\mathrm{d}b}{\int_{\partial D} \mathbf{P}\mathrm{c}(a,b)\cdot\mathbf{n}(a,b)\mathrm{d}s} \tag{70}$$

Fixing parameters as $\omega_0 = 5$, $\omega_f = 10.3$, $\varepsilon = 0.1$, $\beta = 0.1$, $F = 4$, $\sigma = 0.03$, $\alpha = 0.1$, and $\rho = 0.5$, the stationary probability density function in the attracting domain of zero solution is obtained, and the prefactor function $w(a, b)$ is displayed in Fig. 2. It is shown that asymmetric shape of prefactor $w(a, b)$ in Eq. (69) is caused by the form of the parametric noise. The side with higher value of $w(a, b)$ leads to the higher probability density which means the more probable exit point. And with comparing these two pictures in this figure, the phenomenon of asymmetry is enhanced by the narrow-band noise.

By fixing $\sigma = 0.055$, MFPT versus half-power bandwidth μ is displayed in Fig. 3. It is shown that the analytical result obtained from Eq. (70) is close to the simulation one obtained by Monte Carlo simulation as the increasing of the half-power bandwidth, especially, when $\mu > \omega_f$. This verifies the prediction made above based on Eqs. (65) and (67). If $\mu > \omega_f$, the effect of real noise is very similar to the effect of white noise, and if $\mu < \omega_f$, the effect of real noise is much weak and cannot substitute for one of the white noises.

Then, MFPT versus the inverse square of the noise intensity σ with different half-power bandwidths is displayed in Fig. 4. It is shown that the analytical result matches the simulation one well when $\mu > \omega_f$, but when $\mu < \omega_f$ the error becomes greater as the half-power bandwidth μ decreases. And, these results verify again the fact that if $\mu > \omega_f$ is satisfied, the effect of white noise can substitute for the one of real noise. It is concluded that Eqs. (65) and (67) can model the oscillator excited by parametric real noise well, especially when the half-power bandwidth is wide relative to the frequency of harmonic excitations, and these two equations can be used to predict the MFPT to measure the process where the oscillator exits the domain of zero solution to persist in the operative state with both wide and narrow bandwidth of real noise.

6 Conclusions and Discussion

Real noise exists widely in the real life, and it induces a lot of interesting phenomena in the nonlinear systems. In order to study the effect of real noise on the global properties of nonlinear oscillators, the output of a linear filter system is used to assume a real noise. To analyze a general nonlinear oscillator, an explicit parameter t is introduced to the form of nonlinear oscillator, and the parametric excitations are considered. By the spectrum

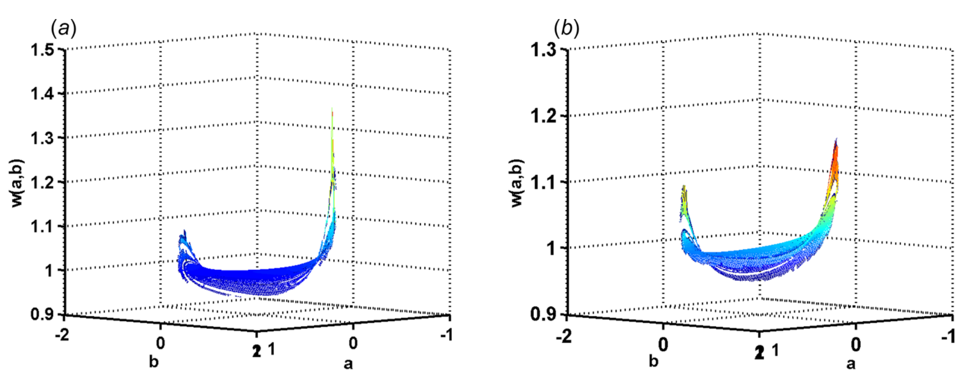

Fig. 2 The stationary probability density function obtained with half-power bandwidth $\mu = 10$ for (*a*) and $\mu = 100$ for (*b*)

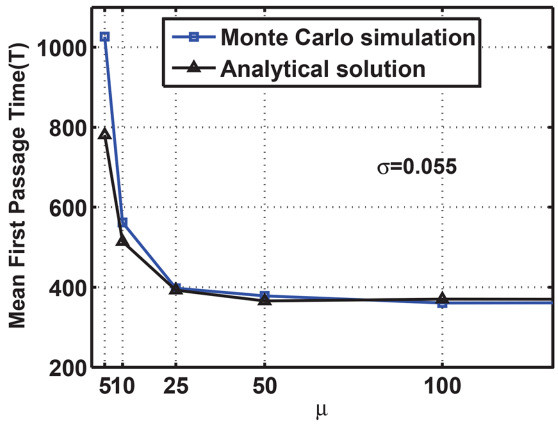

Fig. 3 MFPT versus different half-power bandwidths μ

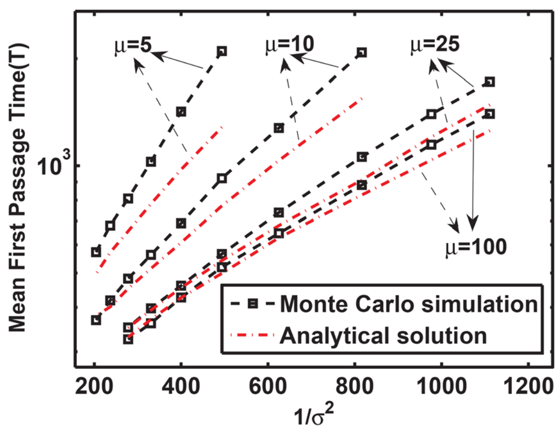

Fig. 4 MFPT versus the inverse square of the noise intensity σ with different half-power bandwidths μ

representations of the FPK operator and its adjoint operator of the linear filter system and the asymptotic analysis of the oscillator, two two-dimensional FPK-type equations governing the von der Pol variables and the amplitude–phase process, respectively, are obtained. Based on these FPK equations, the MFPT about exit behavior can be predicted in the both wide-band real noise and narrow-band real noise case. The detailed balance condition and the strong mixing condition are removed in the process, and that makes considering of the narrow-band real noise possible. One example is presented to estimate the global properties of a specific Duffing oscillator under parametric harmonic and real noise excitations. And, it is shown that the narrow-band real noise can enhance the asymmetry of prefactor function and that if the half-power bandwidth is larger than the frequency of periodic excitation, the effect of white noise can substitute for the one of the real noises.

Although the Duffing oscillator analyzed here has multisteady states, it is just a simple system among nonlinear systems. Some more complicated nonlinear dynamical phenomena, like chaos and homoclinic cycle, are missed to study here. The differences between the effects of real noise and white noise on them still remain to study in the further work.

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References

[1] Bag, B. C., and Hu, C. K., 2006, "Escape Through an Unstable Limit Cycle: Resonant Activation," Phys. Rev. E, **73**(6), pp. 1–4.

[2] Bag, B. C., and Hu, C. K., 2007, "Escape Through an Unstable Limit Cycle Driven by Multiplicative Colored Non-Gaussian and Additive White Gaussian Noises," Phys. Rev. E, **75**(4), pp. 1–4.

[3] Ceriotti, M., Bussi, G., and Parrinello, M., 2009, "Langevin Equation With Colored Noise for Constant-Temperature Molecular Dynamics Simulations," Phys. Rev. Lett., **102**(2), pp. 1–4.

[4] Klosek-Dygas, M., Matkowsky, B. J., and Schuss, Z., 1988, "Colored Noise in Dynamical Systems," SIAM J. Appl. Math., **48**(2), pp. 425–441.

[5] Sri Namachchivaya, N., and Van Roessel, H. J., 1993, "Maximal Lyapunov Exponent and Rotation Numbers for Two Coupled Oscillators Driven by Real Noise," J. Stat. Phys., **71**(3–4), pp. 549–567.

[6] Sri Namachchivaya, N., and Van Roessel, H. J., 2001, "Moment Lyapunov Exponent and Stochastic Stability of Two Coupled Oscillators Driven by Real Noise," ASME J. Appl. Mech., **68**(6), pp. 903–914.

[7] Doyle, M. M., and Sri Namachchivaya, N., 1994, "Almost-Sure Asymptotic Stability of a General Four-Dimensional System Driven by Real Noise," J. Stat. Phys., **75**(3/4), pp. 525–552.

[8] Roy, R., 1994, "Noise Perturbations of a Non-Linear System With Multiple Steady States," Int. J. Non-Linear Mech., **29**(5), pp. 755–773.

[9] Liu, X. B., and Liew, K. M., 2002, "Lyapunov Exponents for Two Nonlinear Systems Driven by Real Noises," Proc. R. Soc. A, **458**(2027), pp. 2705–2719.

[10] Liu, X. B., and Liew, K. M., 2003, "The Lyapunov Exponent for a Codimension Two Bifurcation System That is Driven by a Real Noise," Int. J. Non-Linear Mech., **38**(10), pp. 1495–1511.

[11] Liew, K. M., and Liu, X. B., 2004, "The Maximal Lyapunov Exponent for a Three-Dimensional Stochastic System," ASME J. Appl. Mech., **71**(5), pp. 677–690.

[12] Liu, X. B., and Liew, K. M., 2005, "On the Stability Properties of a Van der Pol Duffing Oscillator That is Driven by a Real Noise," J. Sound Vib., **285**(1–2), pp. 27–49.

[13] Hu, D., Huang, Y., and Liu, X. B., 2012, "Moment Lyapunov Exponent and Stochastic Stability for a Binary Airfoil Driven by an Ergodic Real Noise," Nonlinear Dyn., **70**(3), pp. 1847–1859.

[14] Roy, R. V., 1995, "Noise-Induced Transitions in Weakly Nonlinear Oscillators Near Resonance," ASME J. Appl. Mech., **62**(2), pp. 496–504.

[15] Roy, R. V., and Nauman, E., 1995, "Noise-Induced Effects on a Non-Linear Oscillator," J. Sound Vib., **183**(2), pp. 269–295.

[16] Roy, R., 1994, "Averaging Method for Strongly Non-Linear Oscillators With Periodic Excitations," Int. J. Non-Linear Mech., **29**(5), pp. 737–753.

[17] Roy, R., 1994, "Stochastic Averaging of Oscillators Excited by Colored Gaussian Processes," Int. J. Non-Linear Mech., **29**(4), pp. 463–475.

[18] Chen, L. C., and Zhu, W. Q., 2010, "First Passage Failure of Quasi-Partial Integrable Generalized Hamiltonian Systems," Int. J. Non-Linear Mech., **45**(1), pp. 56–62.

[19] Chen, L. C., and Zhu, W. Q., 2010, "Reliability of Quasi Integrable Generalized Hamiltonian Systems," Probab. Eng. Mech., **25**(1), pp. 61–66.

[20] Chen, L., and Zhu, W. Q., 2010, "First Passage Failure of Quasi Non-Integrable Generalized," Arch. Appl. Mech., **80**(8), pp. 883–893.

[21] Zhu, W. Q., Deng, M. L., and Huang, Z. L., 2002, "First-Passage Failure of Quasi-Integrable Hamiltonian," Appl. Mech., **69**(3), pp. 274–282.

[22] Deng, M. L., and Zhu, W. Q., 2007, "Energy Diffusion Controlled Reaction Rate of Reacting Particle Driven by Broad-Band Noise," Eur. Phys. J. B, **59**(3), pp. 391–397.

[23] Li, X., Liu, Z., and Zhu, W., 2009, "First-Passage Failure of Quasi Linear Systems Subject to Multi-Time-Delayed Feedback Control and Wide-Band Random Excitation," Probab. Eng. Mech., **24**(2), pp. 144–150.

[24] Feng, C. S., Wu, Y. J., and Zhu, W. Q., 2009, "First-Passage Failure of Strongly Non-Linear Oscillators With Time-Delayed Feedback Control Under Combined Harmonic and Wide-Band Noise Excitations," Int. J. Non-Linear Mech., **44**(3), pp. 269–275.

[25] Naeh, T., Kłosek, M. M., Matkowsky, B. J., and Schuss, Z., 1990, "A Direct Approach to the Exit Problem," SIAM J. Appl. Math., **50**(2), pp. 595–627.

[26] Ludwig, D., 1975, "Persistence of Dynamical Systems Under Random Perturbations," SIAM Rev., **17**(4), pp. 605–640.