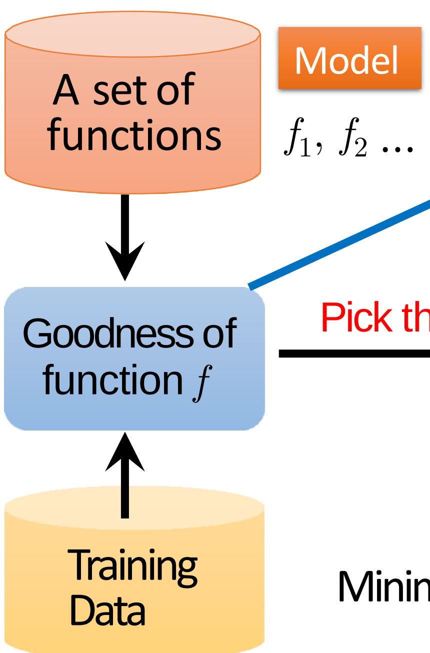
# Recap: last lecture
# Linear Classifier
# More about Lost Functions
# Tips for Gradient Descent
# Logistic Regression (Selflearn)

# Recap: last lecture

Given training data: $\left\{\left(\boldsymbol{x}^{(i)}, y^{(i)}\right)\right\}_{i=1}^{N}$

e.g. $f(\boldsymbol{x}) = \boldsymbol{b} + \boldsymbol{W}\boldsymbol{x}$




Loss Functions

$$l^{(i)} = -\log\left(\exp\left(z_k^{(i)}\right) / \sum_j \exp(z_j^{(i)})\right) \quad \text{softmax}$$

$$l^{(i)} = \sum_{k \neq y^{(i)}} max\left(0, s_k^{(i)} - s_{y^{(i)}}^{(i)} + 1\right) \quad \text{hinge loss}$$

$$L = \frac{1}{N}\sum_{i=1}^{N} l^{(i)} + \lambda R(\boldsymbol{W}) \quad \text{Full loss}$$

Pick the "Best" Function

How?

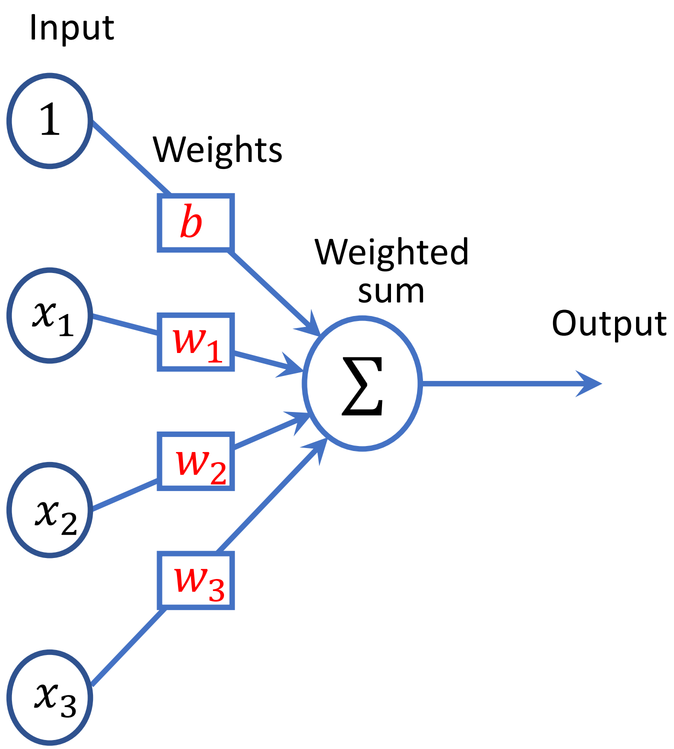
Gradient Descent

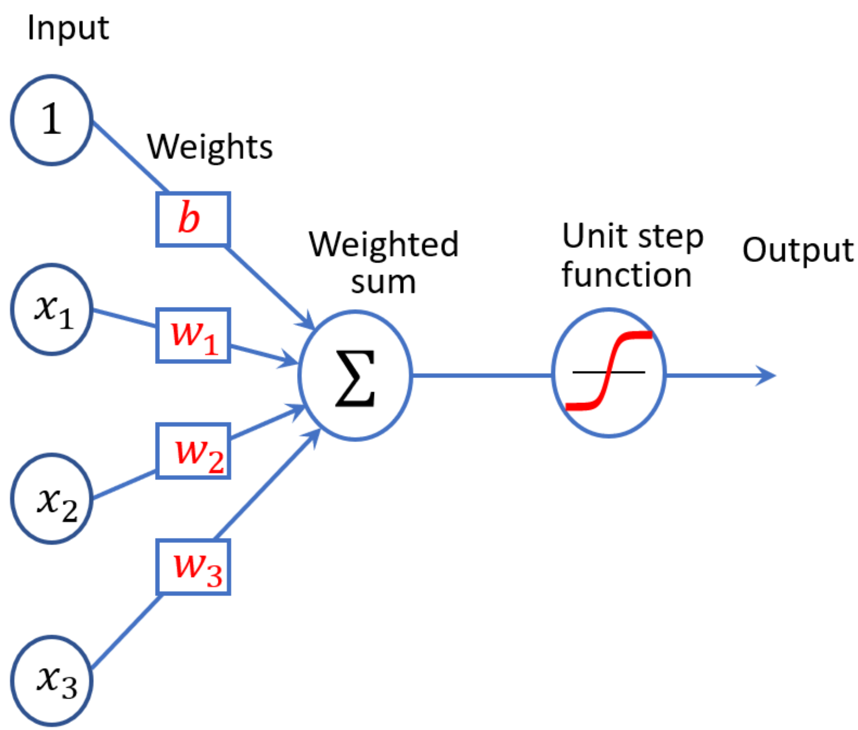
Minimize loss function $L(\boldsymbol{W})$ $\boldsymbol{W}^* = \arg\min_{\boldsymbol{W}} L(\boldsymbol{W})$

# Recap: last lecture

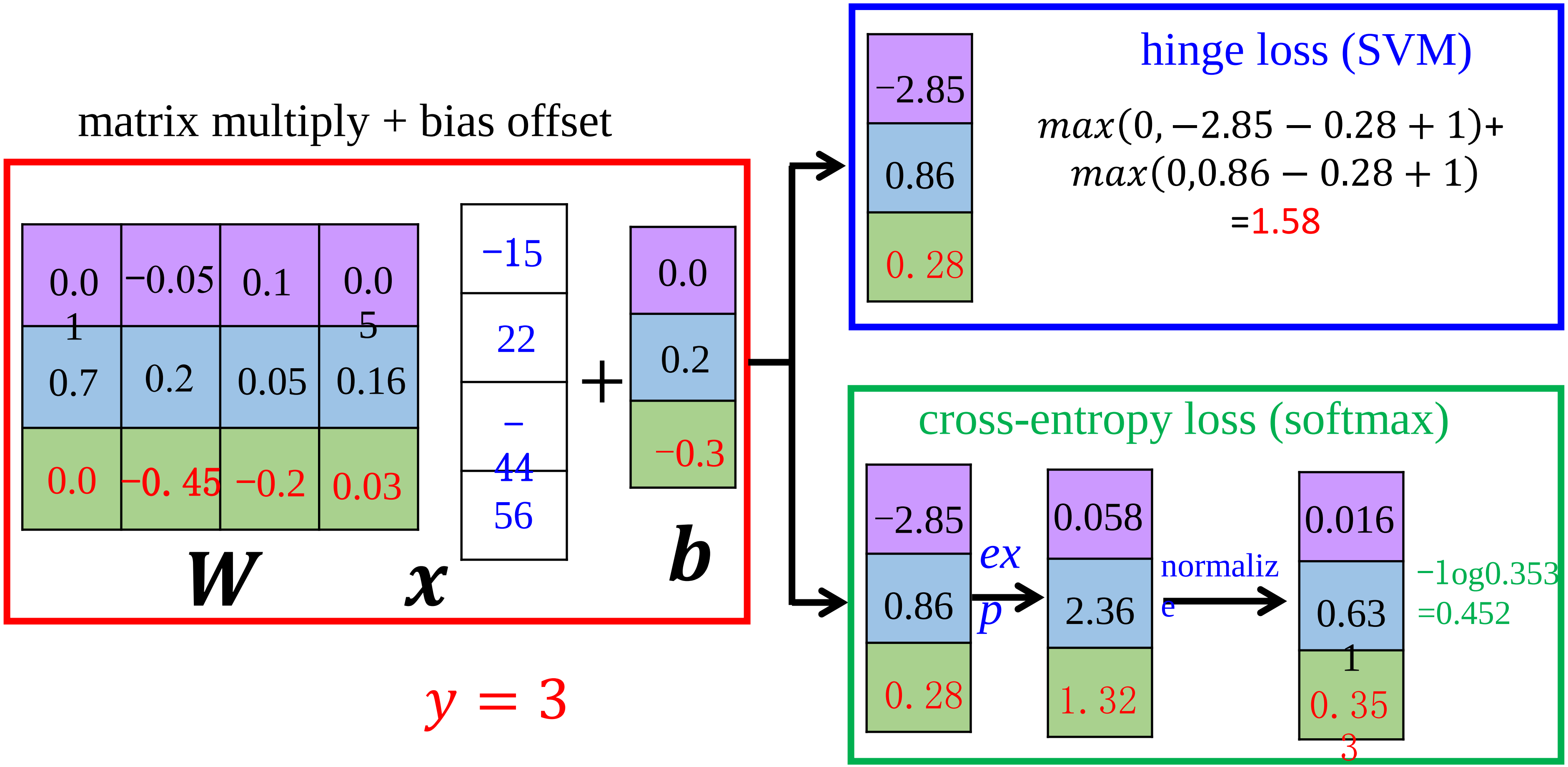
# Recap: last lecture

Linear Regression vs Logistic Regression



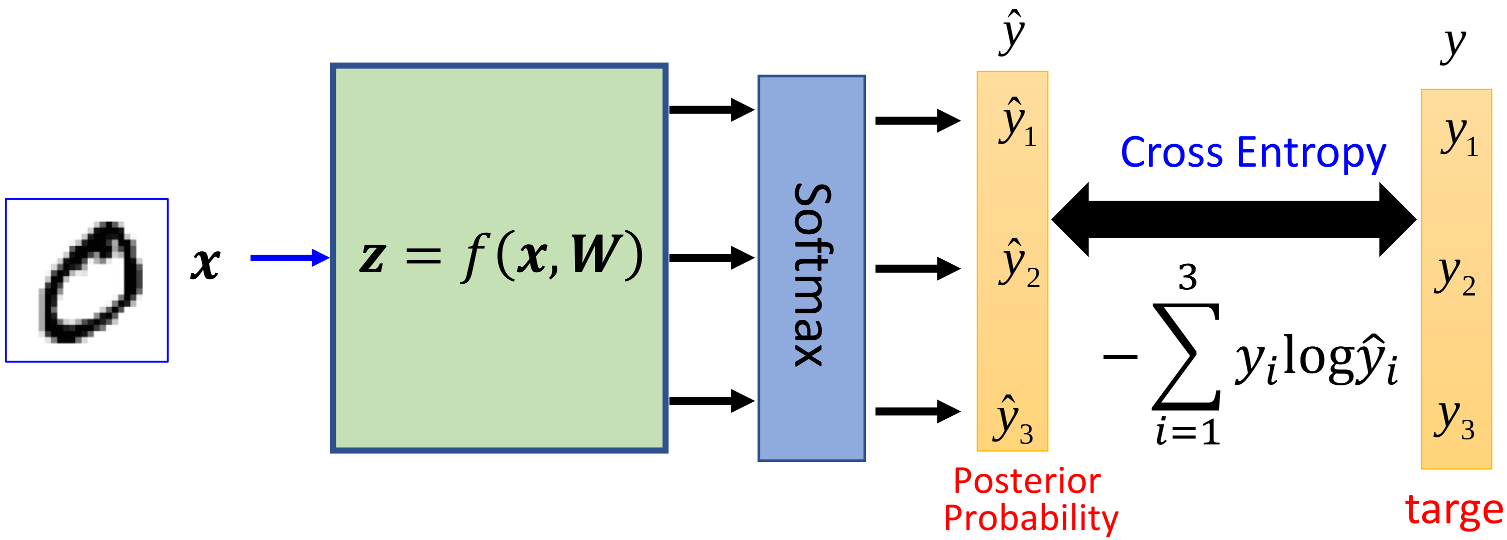



# Recap: last lecture



# Recap: last lecture

## **Softmax Classifier** (Multinomial Logistic Regression)




If $\boldsymbol{x} \in$ class 1

$$y = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$-\log \hat{y}_1$$

If $\boldsymbol{x} \in$ class 2

$$y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$-\log \hat{y}_2$$

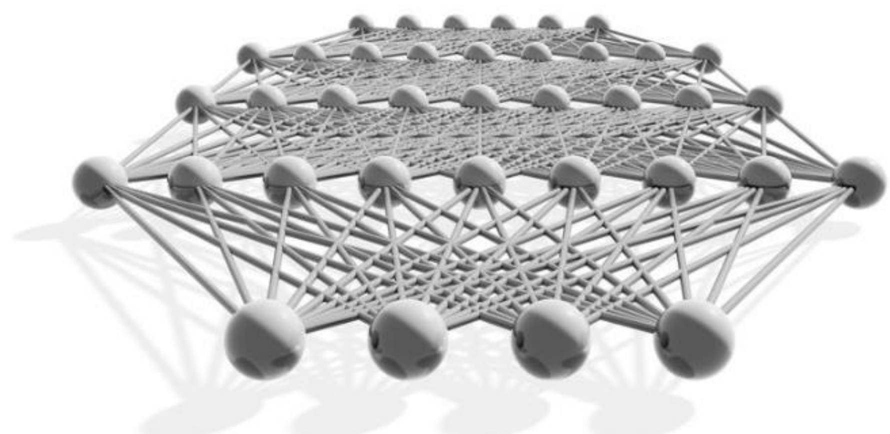
If $\boldsymbol{x} \in$ class 3

$$y = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$-\log \hat{y}_3$$

# Lecture 3

- Neural Networks
- Multilayer Neural Networks
- Backpropagation



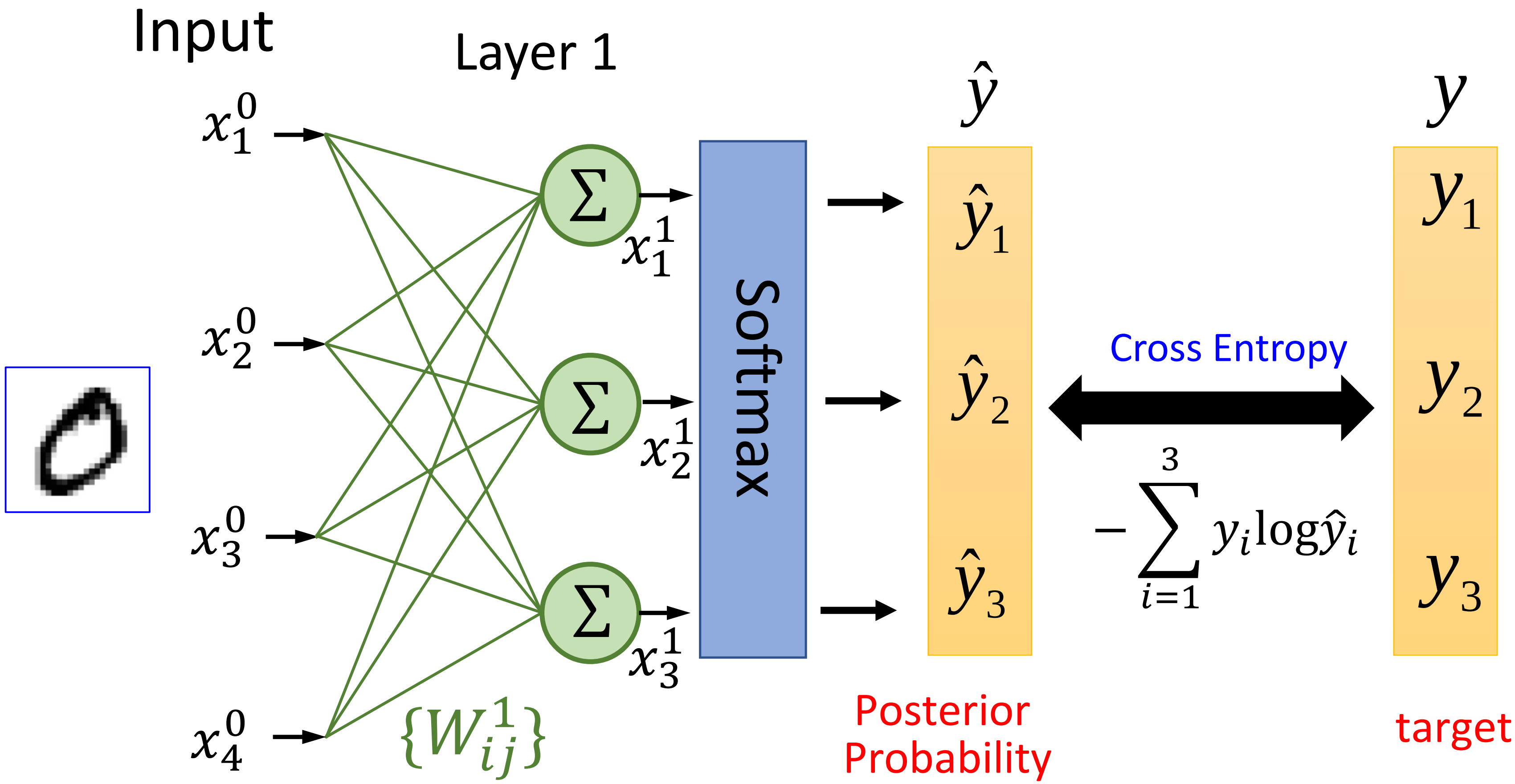
Many slides adapted from Hungyi Lee

# Three Steps for Deep Learning


Deep Learning is so simple. Don't be afraid……

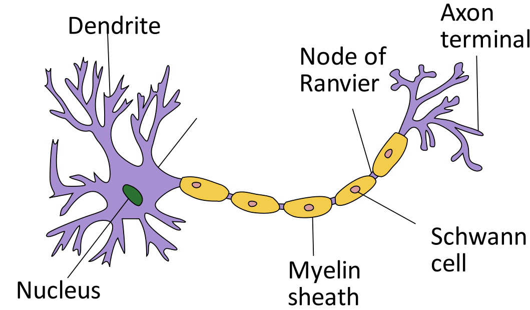
# Recap: last lecture

## **Softmax Classifier** (Multinomial Logistic Regression)

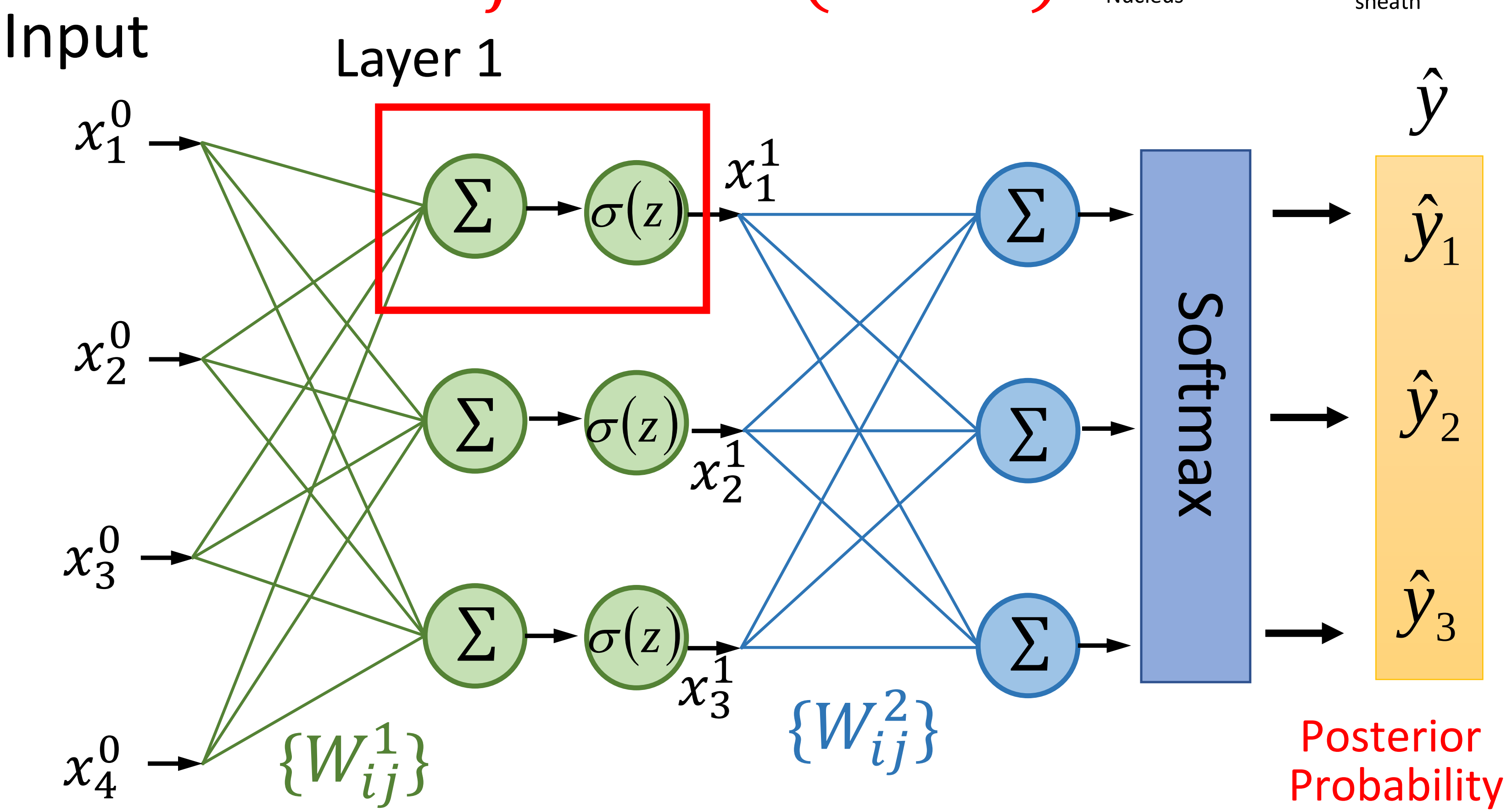



$$f = \mathbf{W}^1 \boldsymbol{x}^0$$

# Neural Network




$$f = \mathbf{W}^1 \sigma(\mathbf{W}^1 \boldsymbol{x}^0)$$

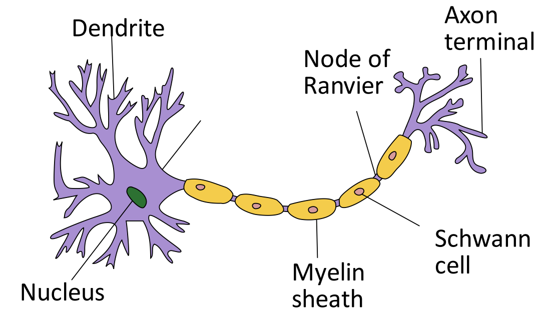


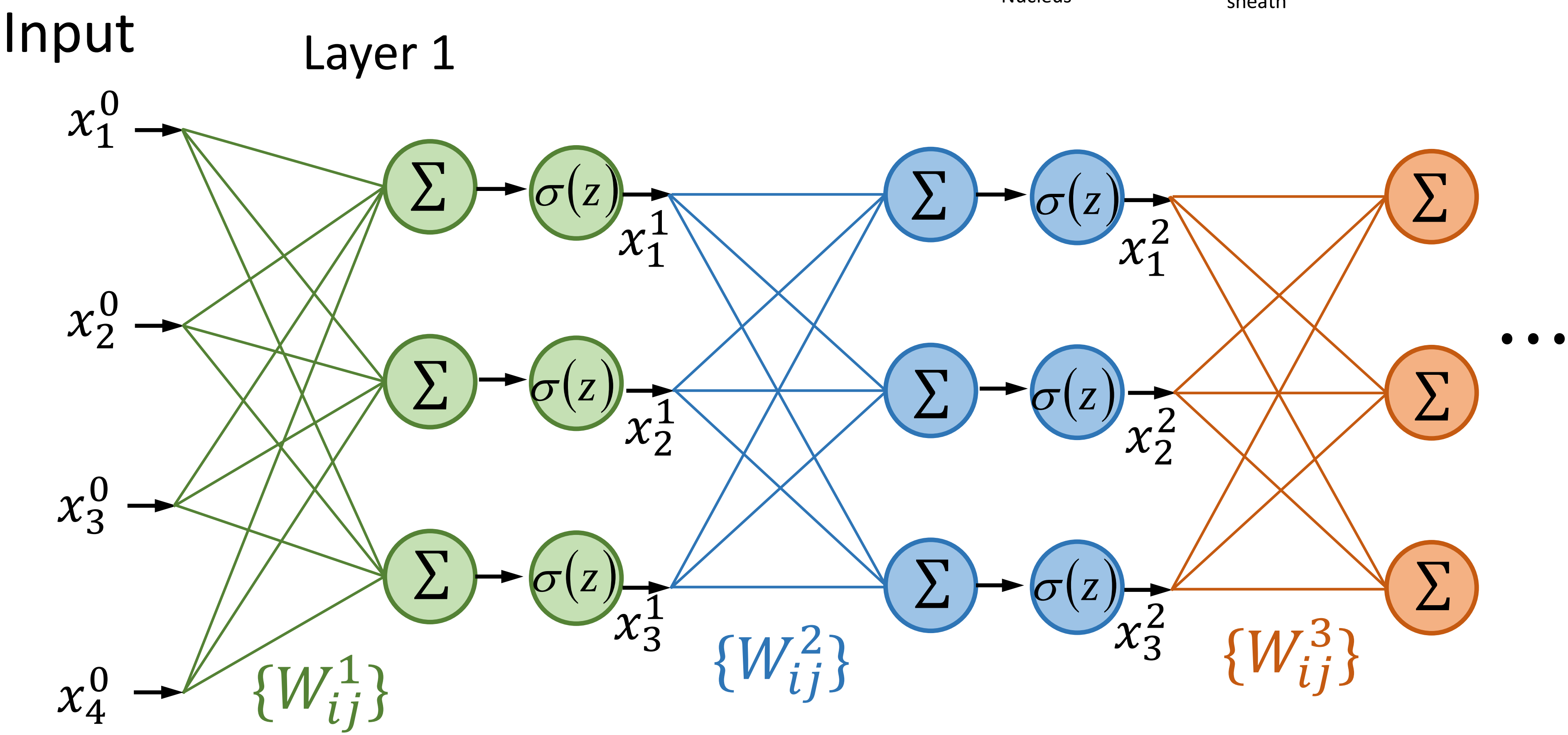

Different connection leads to different network structures

$\sigma(z)$ ***nonlinear function***

Network parameters: all the weights and biases in all the "neurons"

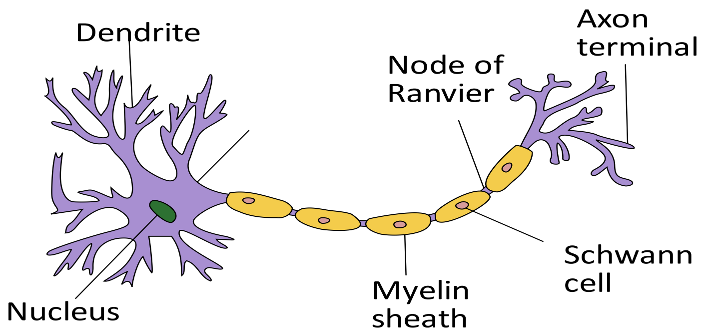
# Neural Network

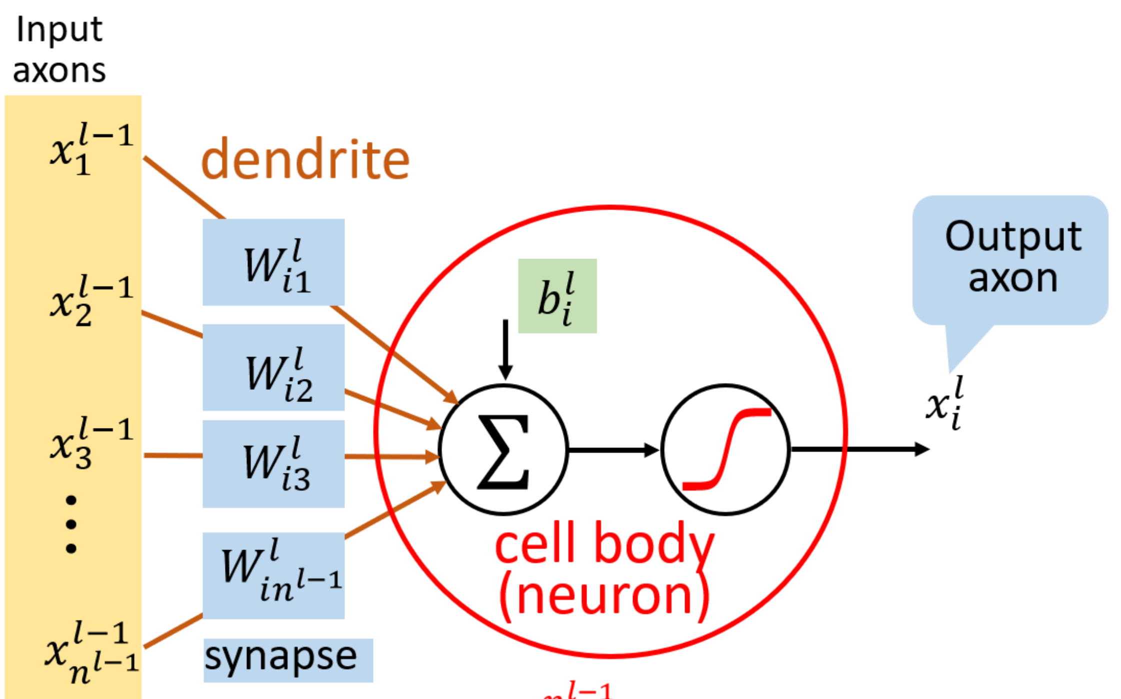



$$f = \mathbf{W}^3 \sigma(\mathbf{W}^1 \sigma(\mathbf{W}^1 \boldsymbol{x}^0))$$

# Brain Evidence



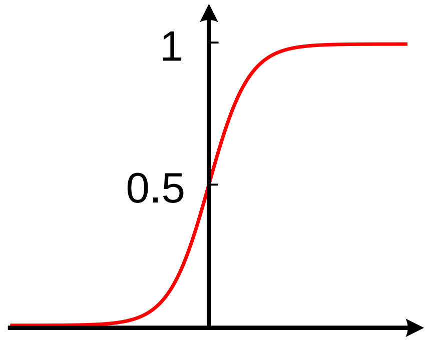




$$x_i^l = \sigma\left(\sum_{j=1}^{n^{l-1}} W_{ij}^l x_j^{l-1} + b_i^l\right)$$

# Activation functions

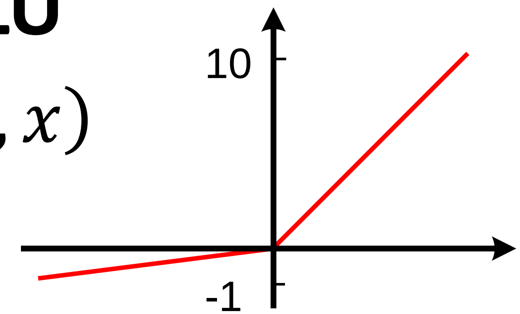
## Sigmoid

$$\sigma(x) = \frac{1}{1+e^{-x}}$$



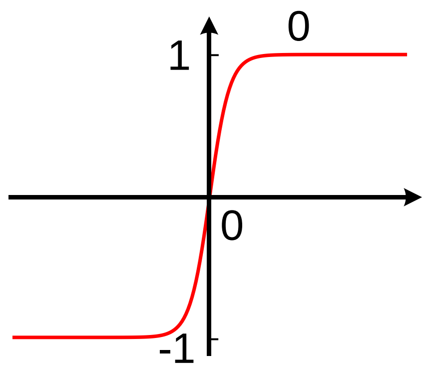

## Leaky ReLU

$$max(0.1x, x)$$




## tanh

$$\tanh(x) = \frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$$

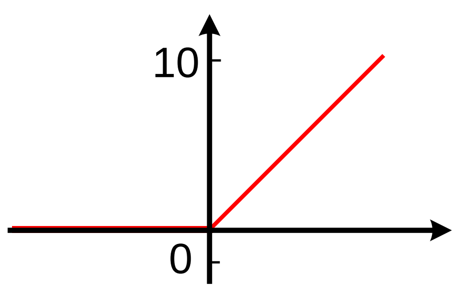



## Maxout

$$max(w_1^T x + b_1, w_2^T x + b_2)$$

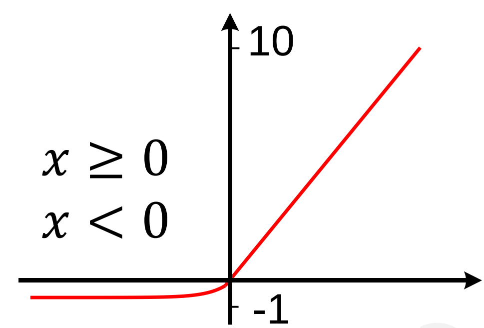
## ReLU

$$max(0, x)$$

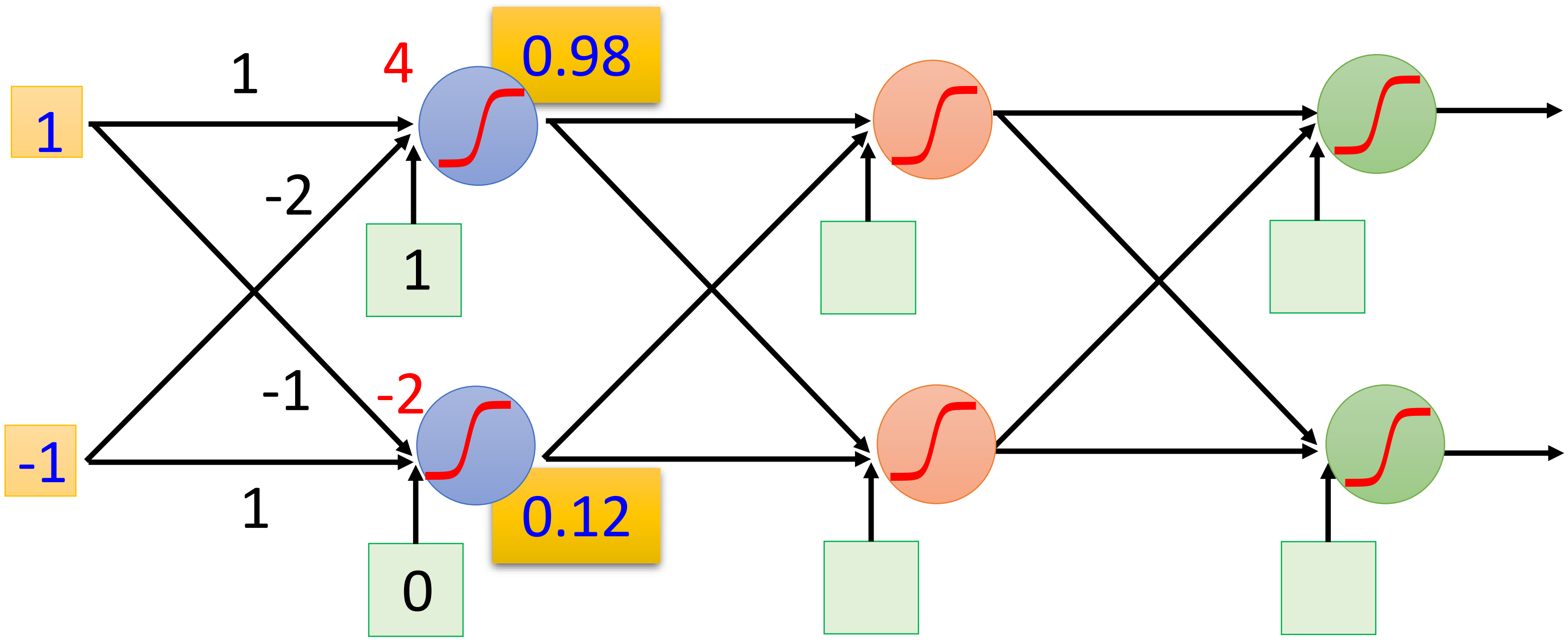



## ELU

$$\begin{cases} x & x \geq 0 \\ \alpha(e^{x}-1) & x < 0 \end{cases}$$

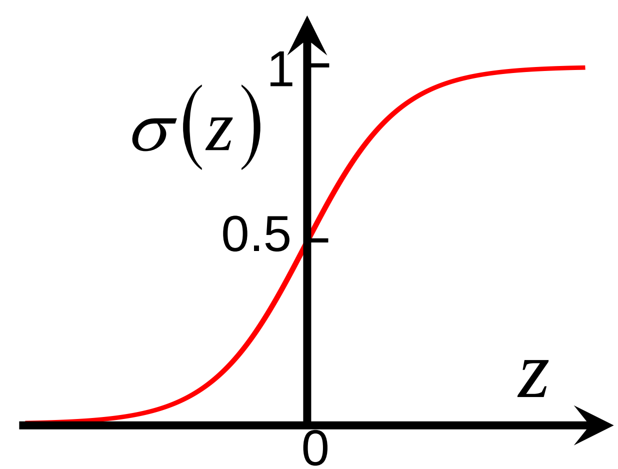


# Fully Connect Feedforward Network

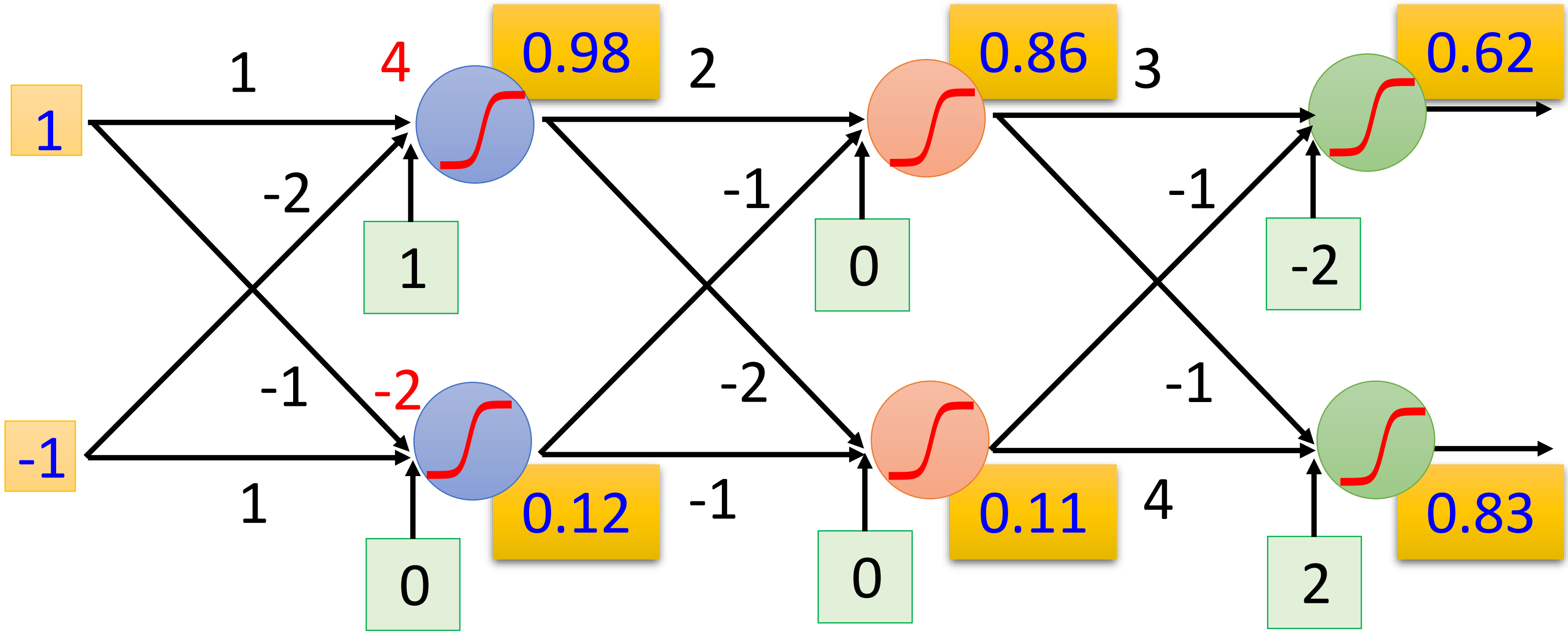



Sigmoid Function

$$\sigma(z) = \frac{1}{1+e^{-z}}$$

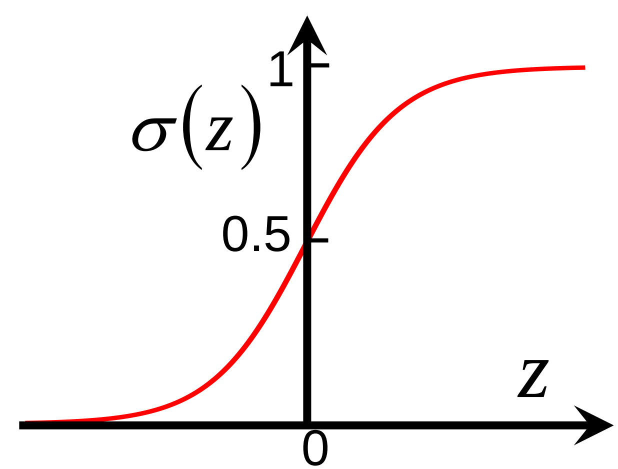


# Fully Connect Feedforward Network




Sigmoid Function

$$\sigma(z) = \frac{1}{1+e^{-z}}$$



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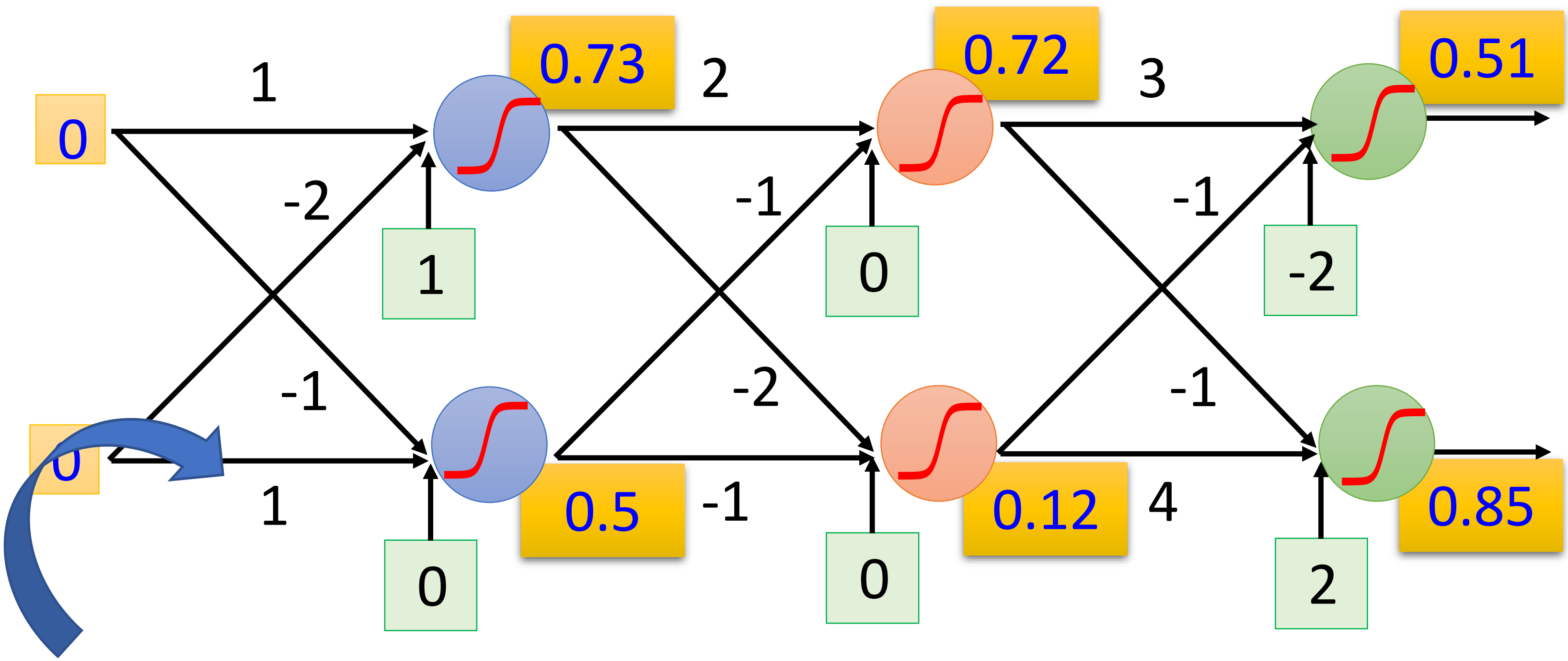


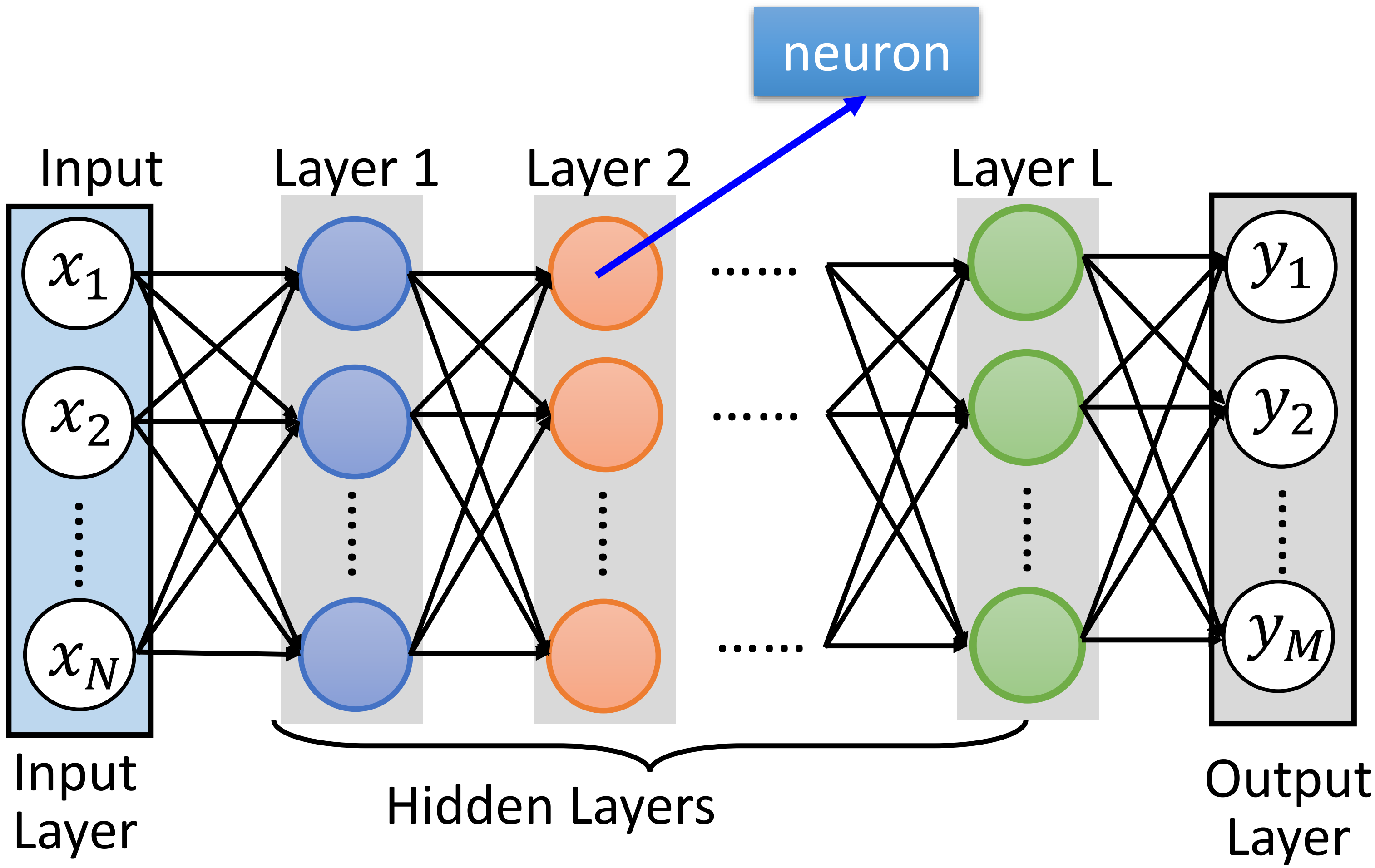

This is a function.

Input vector, output vector

$$f\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}0.62\\0.83\end{bmatrix} \quad f\left(\begin{bmatrix}0\\0\end{bmatrix}\right)=\begin{bmatrix}0.51\\0.85\end{bmatrix}$$

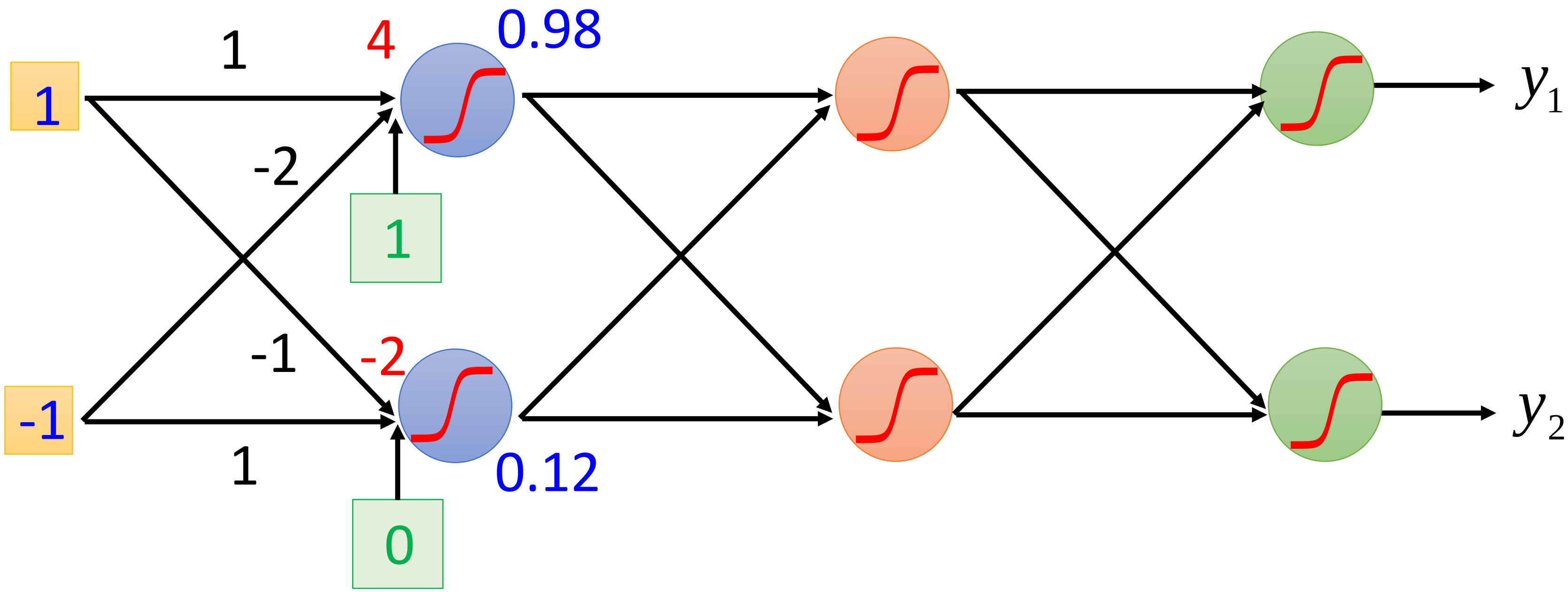
Given network structure, define ***a function set***

# Fully Connect Feedforward Network



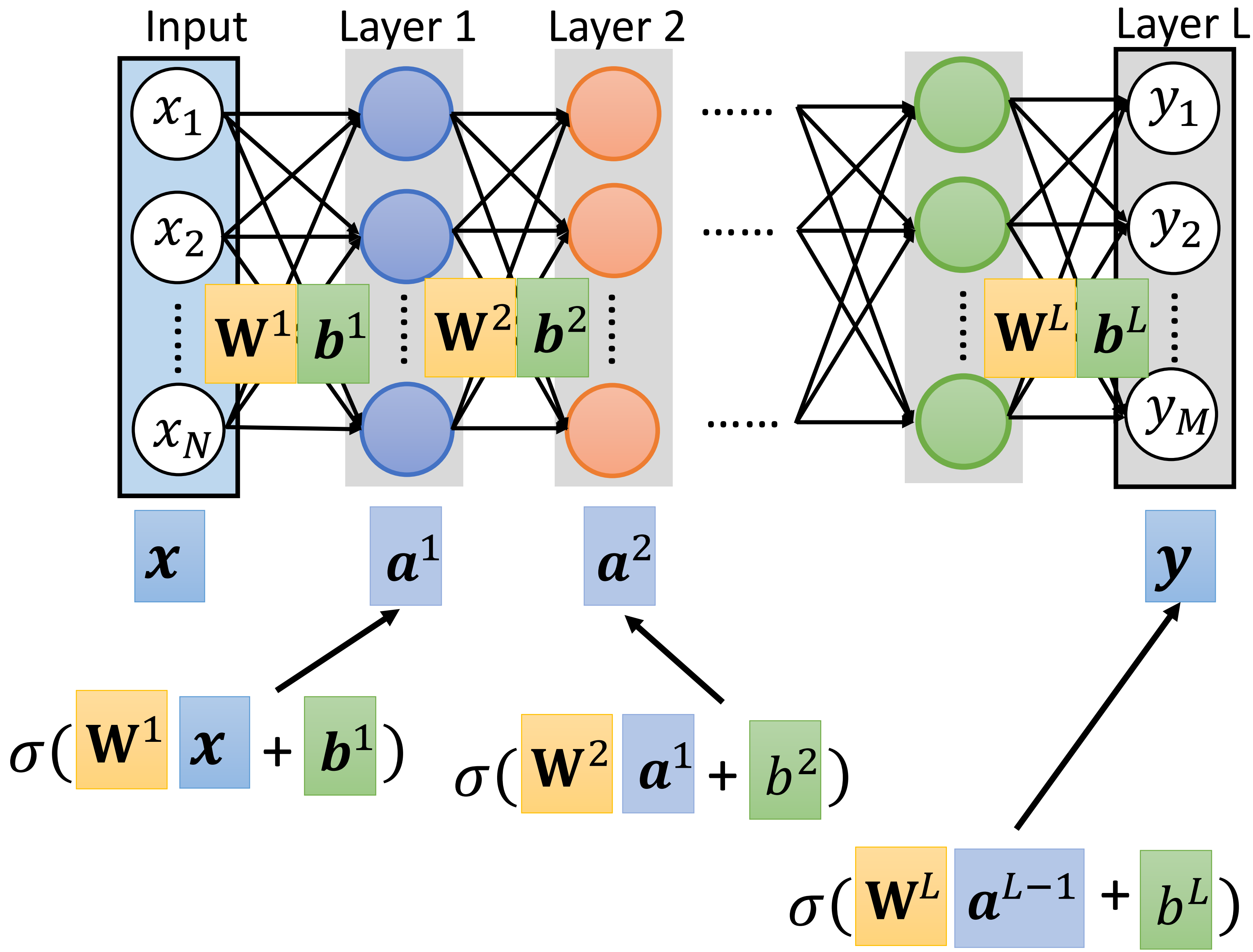

- Each hidden layer can have a different number of neurons.

# Matrix Operation

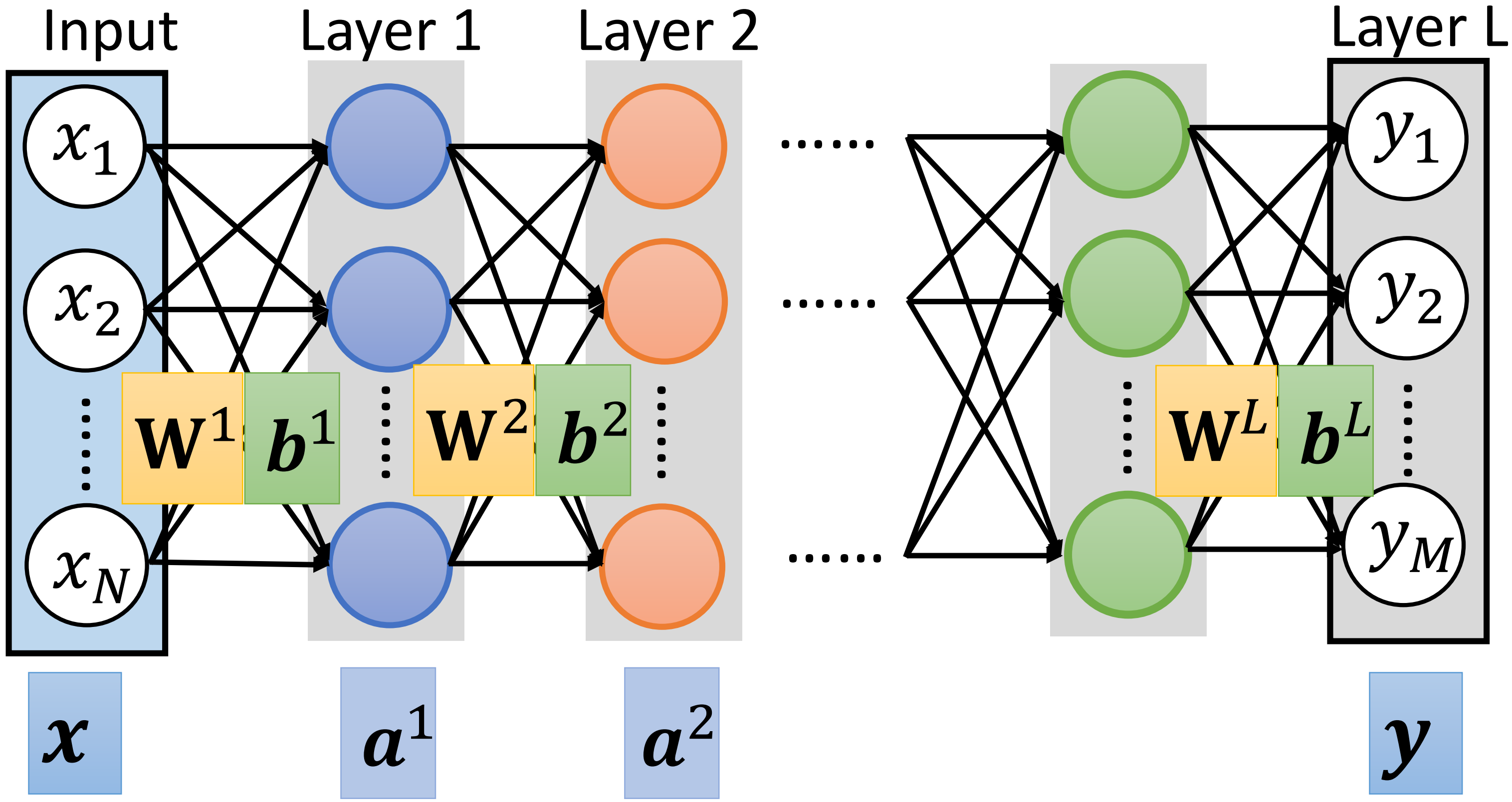



$$\sigma\Big(\underbrace{\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\begin{bmatrix} 4 \\ -2 \end{bmatrix}}\Big) = \begin{bmatrix} 0.98 \\ 0.12 \end{bmatrix}$$

# Neural Network



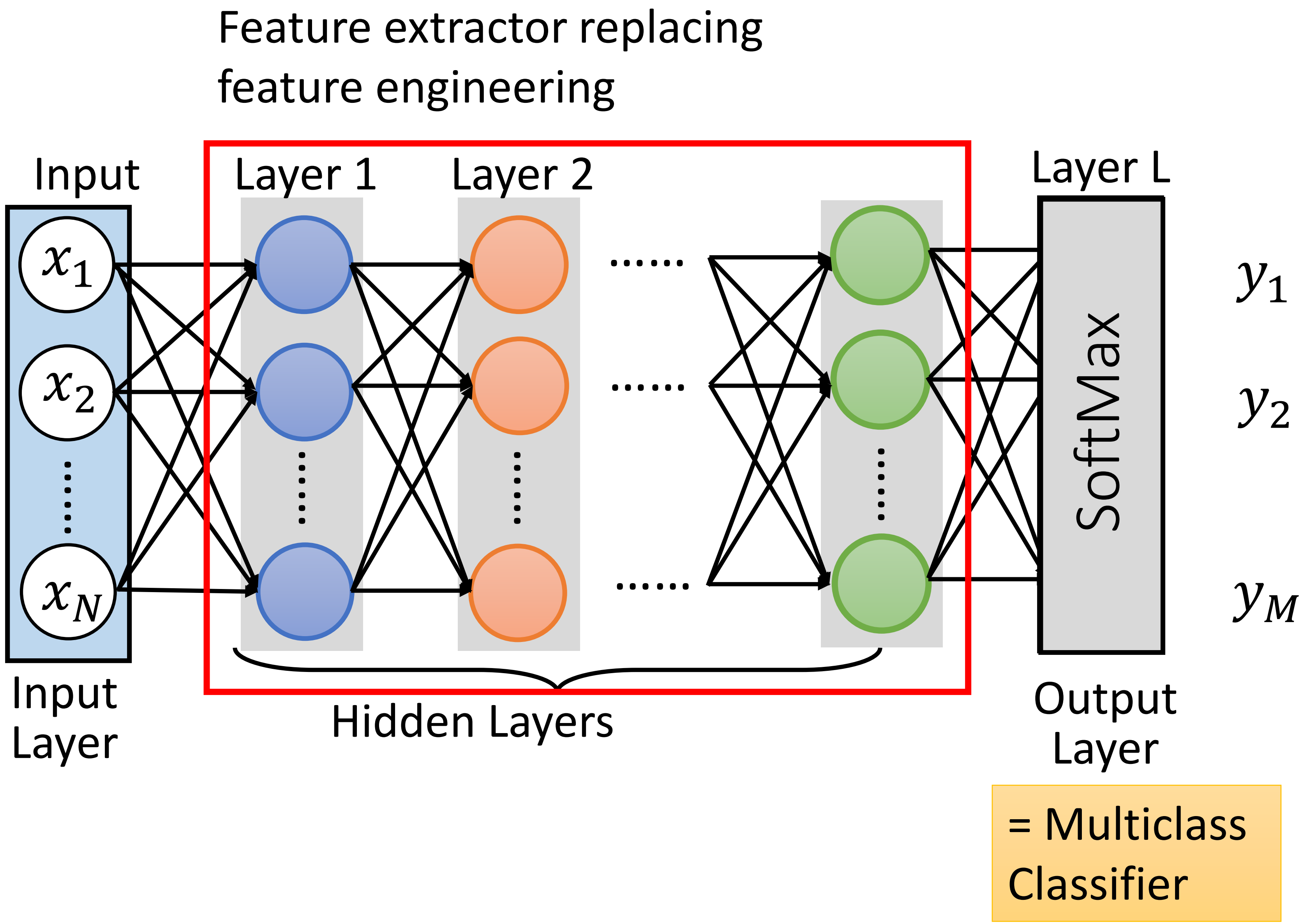
# Neural Network




$\boldsymbol{y} = f(\boldsymbol{x})$ Using parallel computing techniques to speed up matrix operation

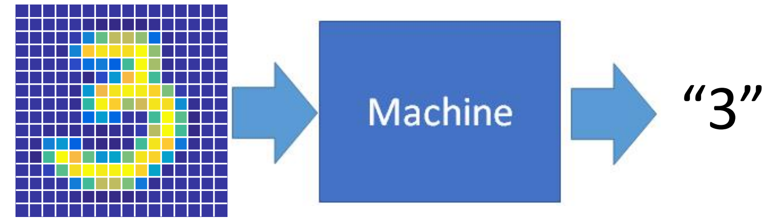
$$\sigma(\mathbf{W}^L \cdots \sigma(\mathbf{W}^2\, \sigma(\mathbf{W}^1 \boldsymbol{x} + \boldsymbol{b}^1) + b^2) \ldots + b^L)$$

# Output Layer as MultiClass Classifier

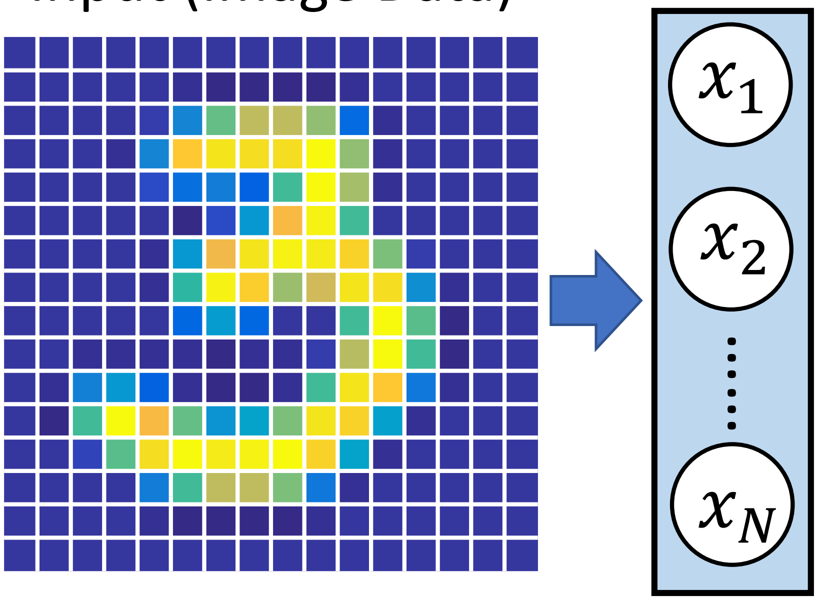


# Example Application

## Handwriting Digit Recognition



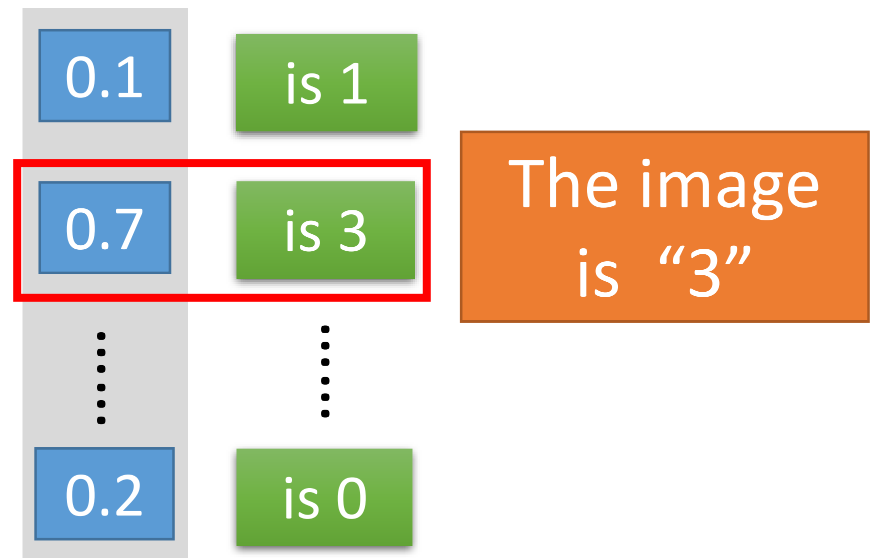

Input (Image Data)




16 x 16 = 256

N=256

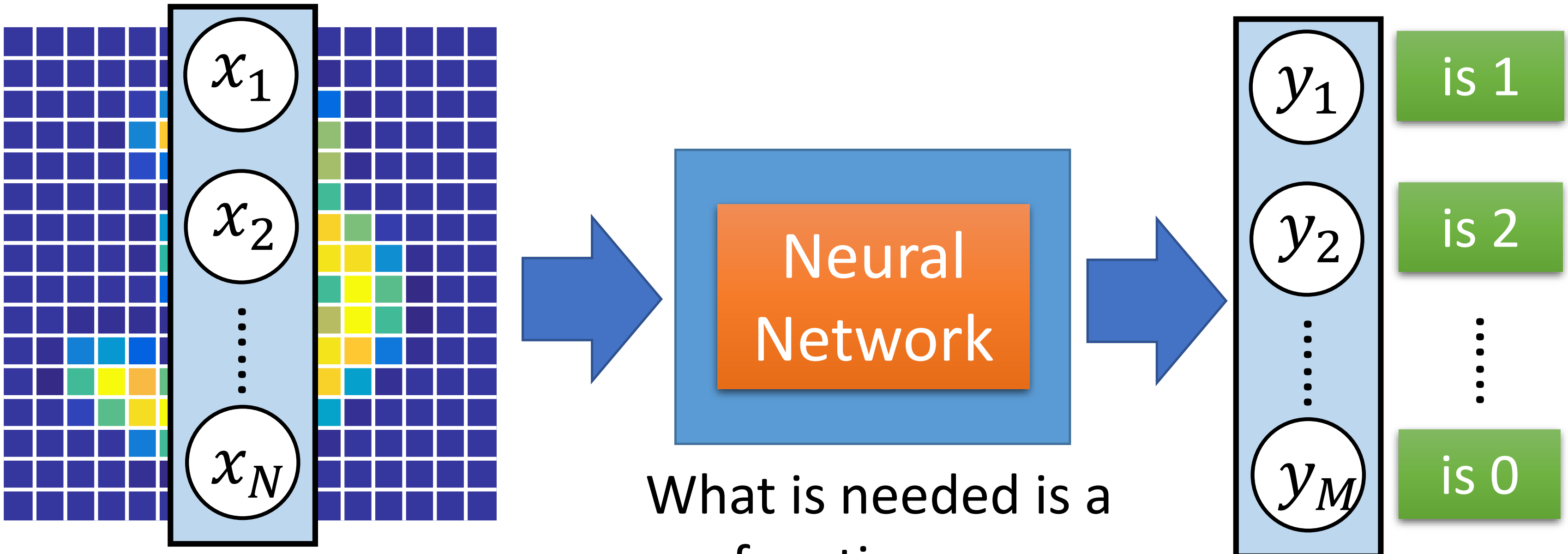
Output (Dim Fixed)




Each dimension represents the confidence of a digit.

# Example Application

- Handwriting Digit Recognition

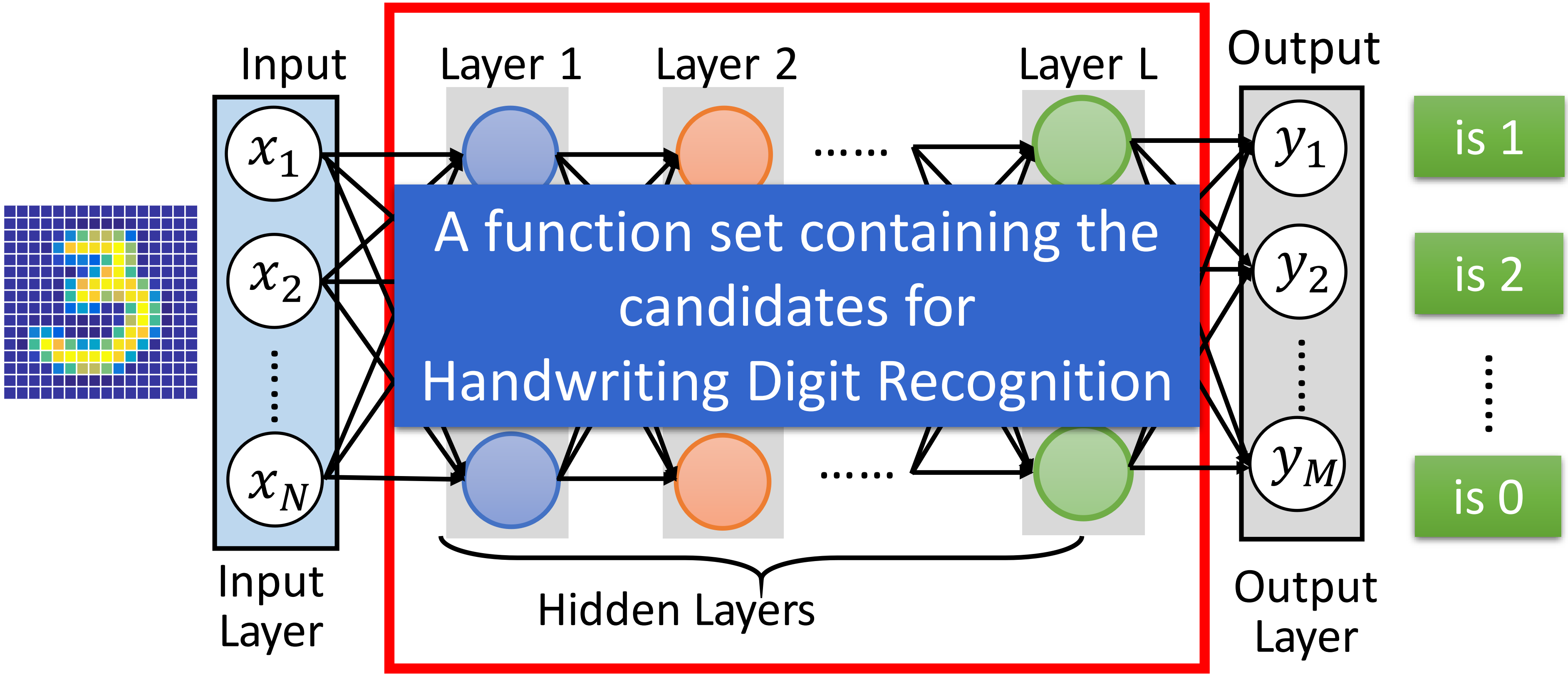



Input: Vector of 256 dimension
$N$=256

What is needed is a function ……

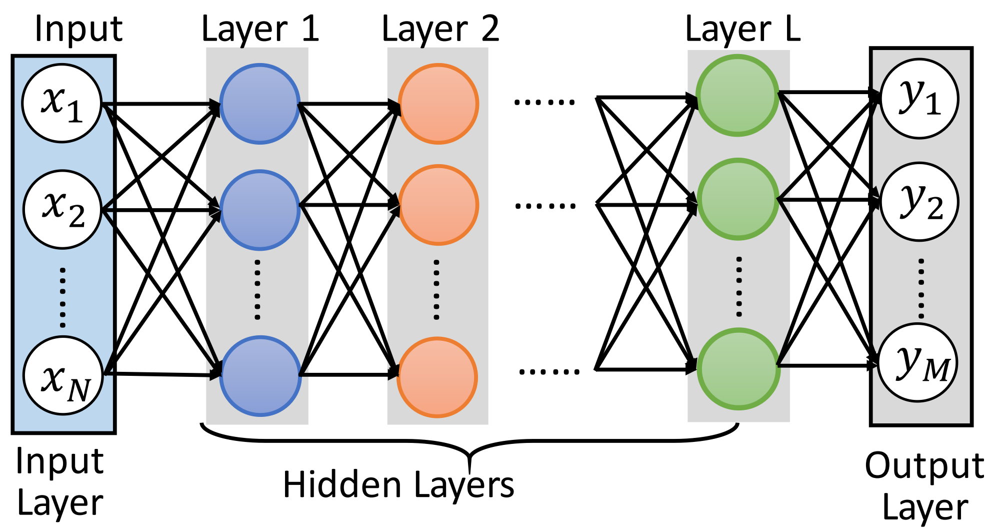
Output: Vector of 10 dimension
$M$=10 classes

# Example Application




You need to decide the network structure to let a good function in your function set.

# FAQ




- Q1: How many layers? How many neurons for each layer?

Trial and Error + Intuition

- Q2: Can the structure be automatically determined?
    - *e.g.* automatic Network Architecture Search (NAS)
- Q3: Can we design the network structure?

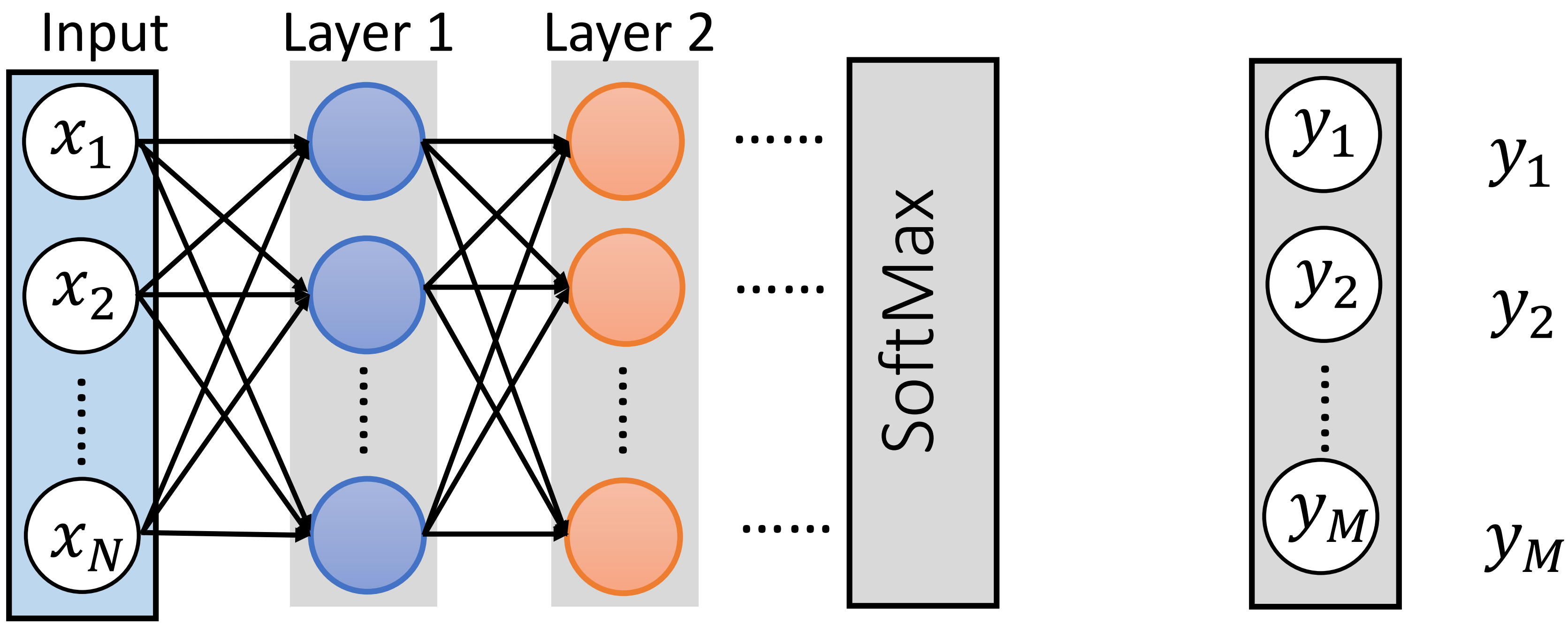
Convolutional Neural Network (CNN)

# Three Steps for Deep Learning

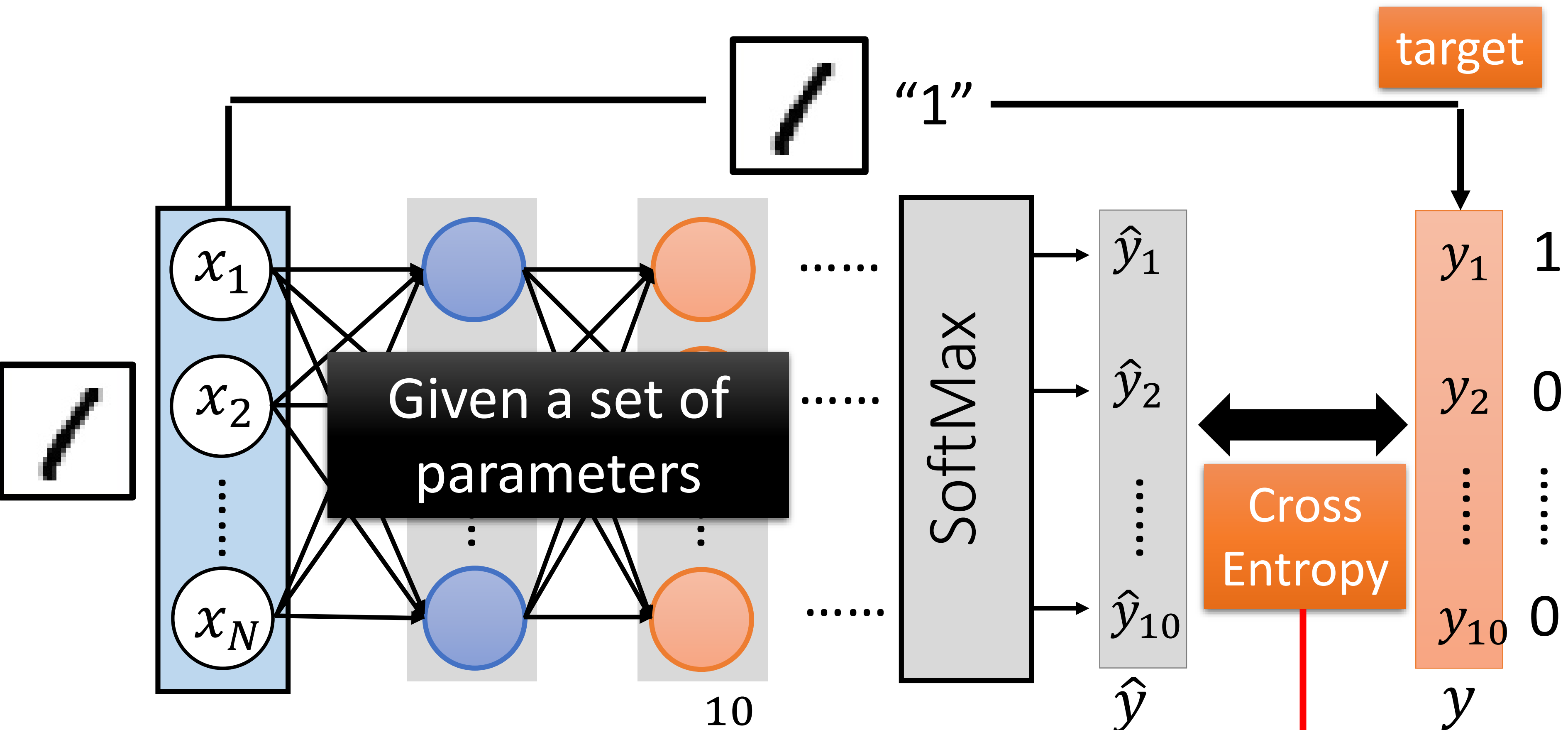
Step 1: Neural Network → Step 2: Goodness of Function → Step 3: Pick the Best Function

Deep Learning is so simple. Don’t be afraid……

# Output Layer as MultiClass Classifier



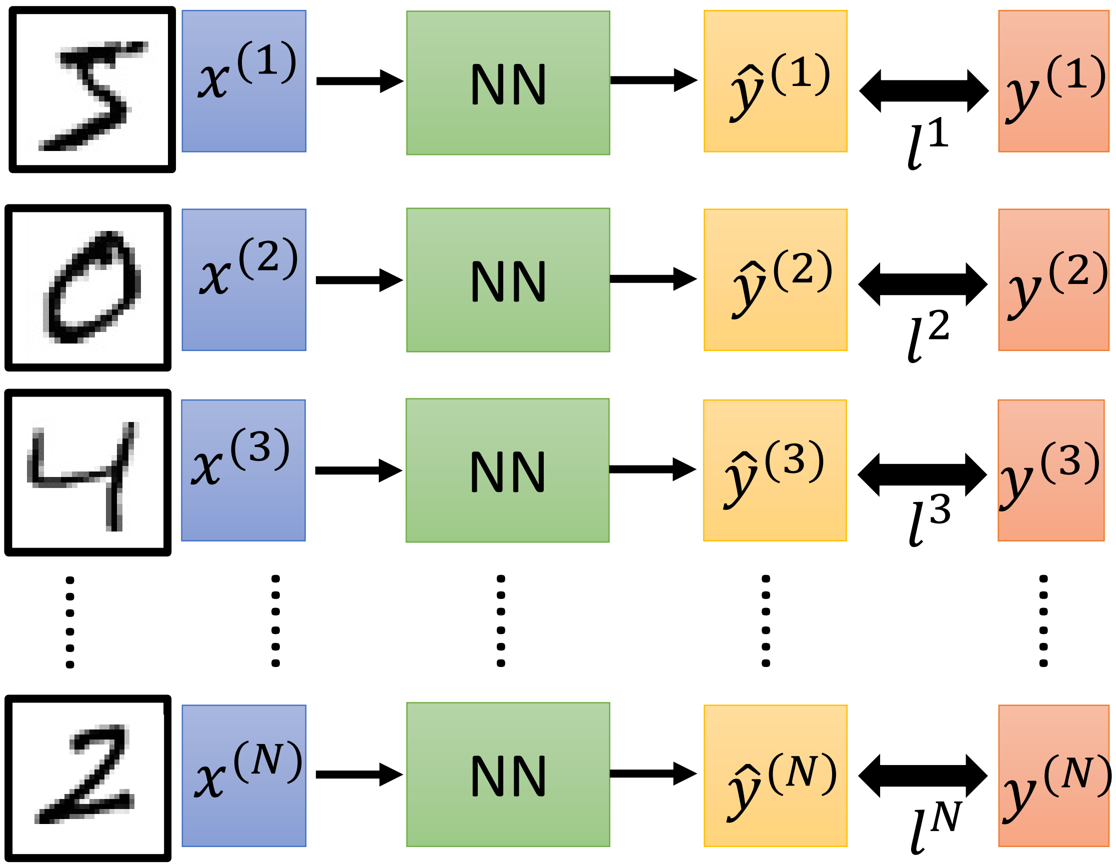
# Loss for an Example




$$l(y, \hat{y}) = -\sum_{i=1}^{10} y_i ln\hat{y}_i$$

# Total Loss

For all training data …




Total Loss:

$$\boxed{L = \sum_{n=1}^{N} l^n}$$

Find ***a function in function set*** that minimizes total loss L

Find ***the network parameters $\theta^*$*** that minimize total loss L

# Three Steps for Deep Learning

Step 1: Neural Network → Step 2: Goodness of Function → Step 3: Pick the Best Function

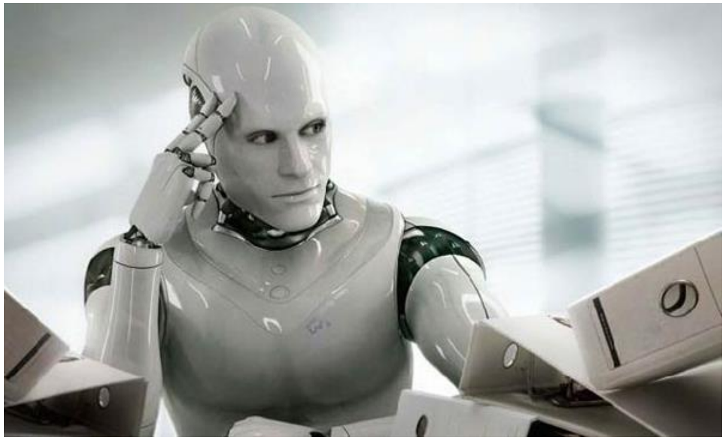
Deep Learning is so simple. Don’t be afraid……

# Gradient Descent

This is the “learning” of machines in deep learning ……

Even alpha go using this approach.

People image ……

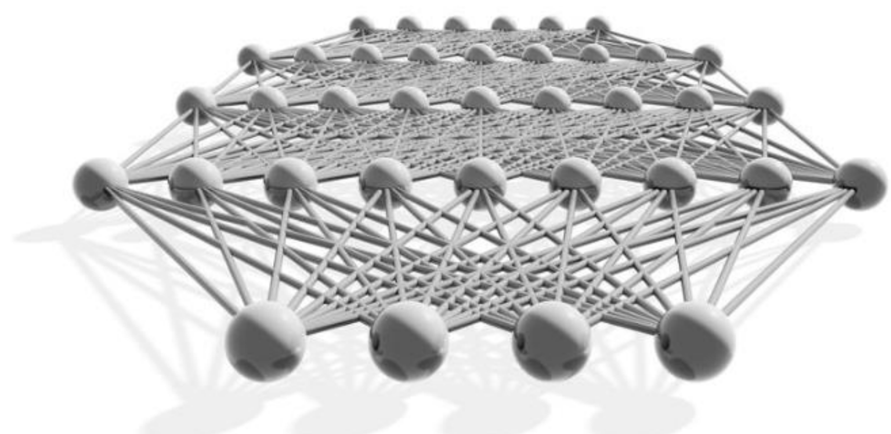


Actually …..

I hope you are not too disappointed.

# Lecture 3

- Neural Networks
- Multilayer Neural Networks
- Backpropagation



Many slides adapted from Hungyi Lee

# Backpropagation for Fully Connect Feedforward Network

# Gradient Descent

Network parameters $\theta = \{w_1, w_2, \cdots, b_1, b_2, \cdots\}$

Starting Parameters $\theta^{(0)} \longrightarrow \theta^{(1)} \longrightarrow \theta^{(2)} \longrightarrow$ ......

$$\nabla L(\theta) = \begin{bmatrix} \partial C(\theta)/\partial w_1 \\ \partial C(\theta)/\partial w_2 \\ \vdots \\ \partial C(\theta)/\partial b_1 \\ \partial C(\theta)/\partial b_2 \\ \vdots \end{bmatrix}$$

$Compute\ \nabla C\left(\theta^{(0)}\right)$ $\quad \theta^{(1)} = \theta^{(0)} - \eta \nabla C\left(\theta^{(0)}\right)$

$Compute\ \nabla C\left(\theta^{(1)}\right)$ $\quad \theta^{(2)} = \theta^{(1)} - \eta \nabla C\left(\theta^{(1)}\right)$

Millions of parameters ......

To compute the gradients efficiently, we use ***backpropagation***.

# Chain Rule

**Case 1**

$$y = g(x) \qquad z = h(y)$$

$$\Delta x \rightarrow \Delta y \rightarrow \Delta z \qquad\qquad \frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$$

**Case 2**

$$x = g(s) \qquad y = h(s) \qquad z = k(x, y)$$

$$\begin{tikzcd} & \Delta x \arrow[dr] & \\ \Delta s \arrow[ur] \arrow[dr] & & \Delta z \\ & \Delta y \arrow[ur] & \end{tikzcd}$$

$$\frac{dz}{ds} = \frac{\partial z}{\partial x}\frac{dx}{ds} + \frac{\partial z}{\partial y}\frac{dy}{ds}$$

# Computational Graph

$$f(x, y, z) = (x + y)z$$

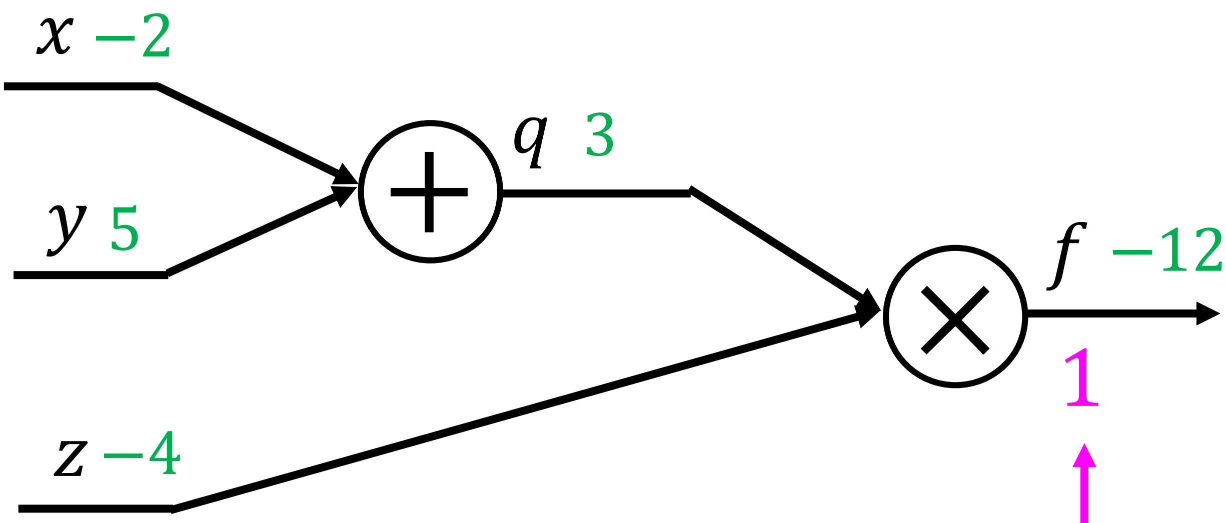
$$q = (x + y)$$

$$f = qz$$

$e.g.,\ x = -2$

$y = 5$

$z = -4$




$$\frac{\partial f}{\partial f}$$

Want: $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$

$$q = (x + y) \longrightarrow \frac{\partial q}{\partial x} = 1, \frac{\partial q}{\partial y} = 1$$

$$f = qz \longrightarrow \frac{\partial f}{\partial q} = z, \frac{\partial f}{\partial z} = q$$

# Computational Graph

$f(x, y, z) = (x + y)z$

$q = (x + y)$

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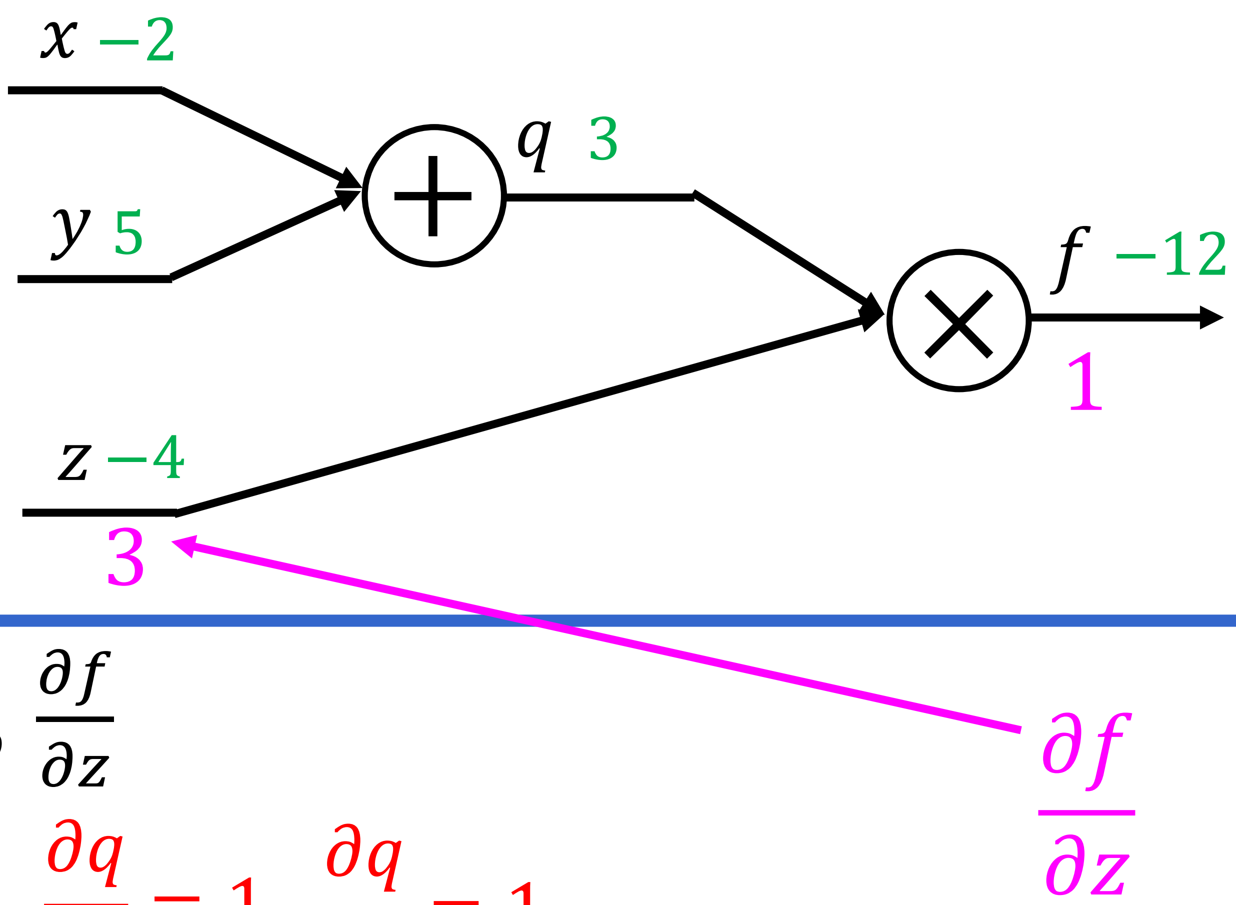


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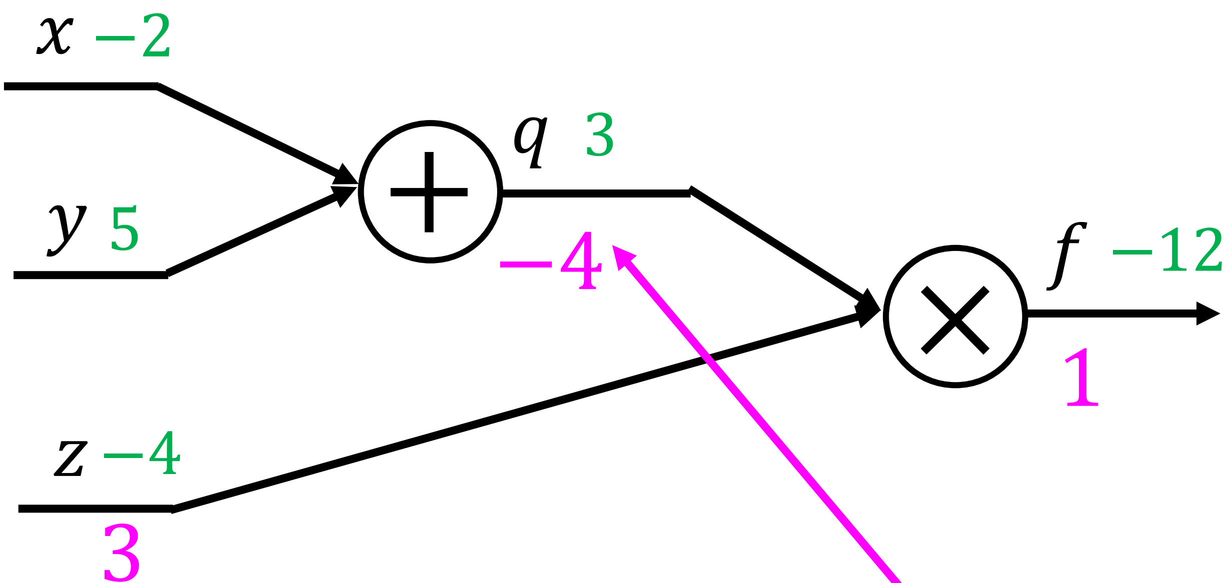


$$\frac{\partial f}{\partial q}$$

Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

$$q = (x + y) \longrightarrow \frac{\partial q}{\partial x} = 1, \frac{\partial q}{\partial y} = 1$$

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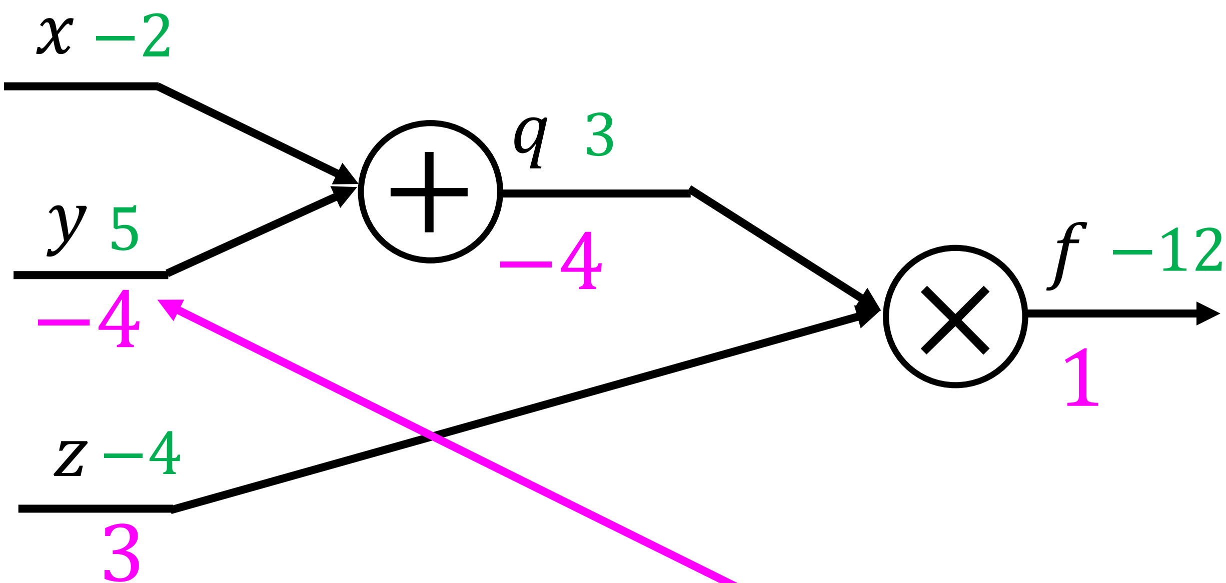


Want: $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial y}$$

$$q = (x + y) \longrightarrow \frac{\partial q}{\partial x} = 1, \; \frac{\partial q}{\partial y} = 1$$

$$f = qz \longrightarrow \frac{\partial f}{\partial q} = z, \; \frac{\partial f}{\partial z} = q$$

Chain Rule

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial q}\frac{\partial q}{\partial y}$$

# Computational Graph

$f(x, y, z) = (x + y)z$

$q = (x + y)$

$f = qz$

e.g., $x = -2$

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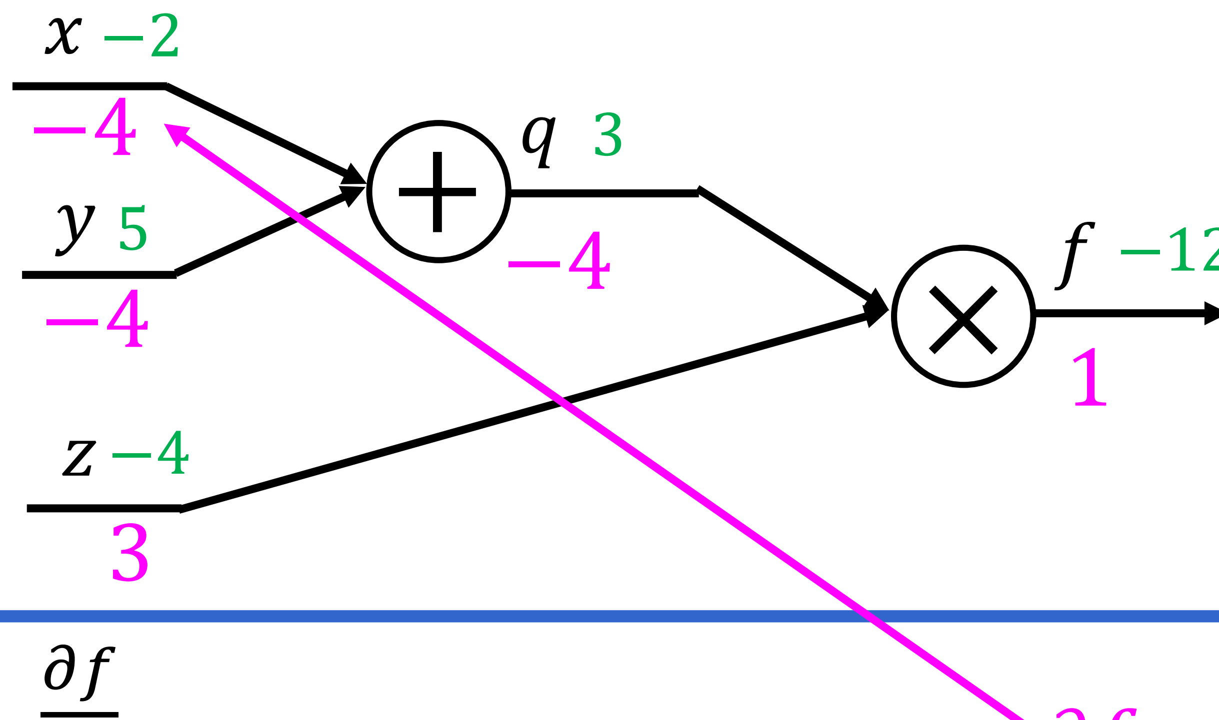


Want: $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$

$$\frac{\partial f}{\partial y}$$

$q = (x + y) \longrightarrow \frac{\partial q}{\partial x} = 1, \frac{\partial q}{\partial y} = 1$

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Chain Rule

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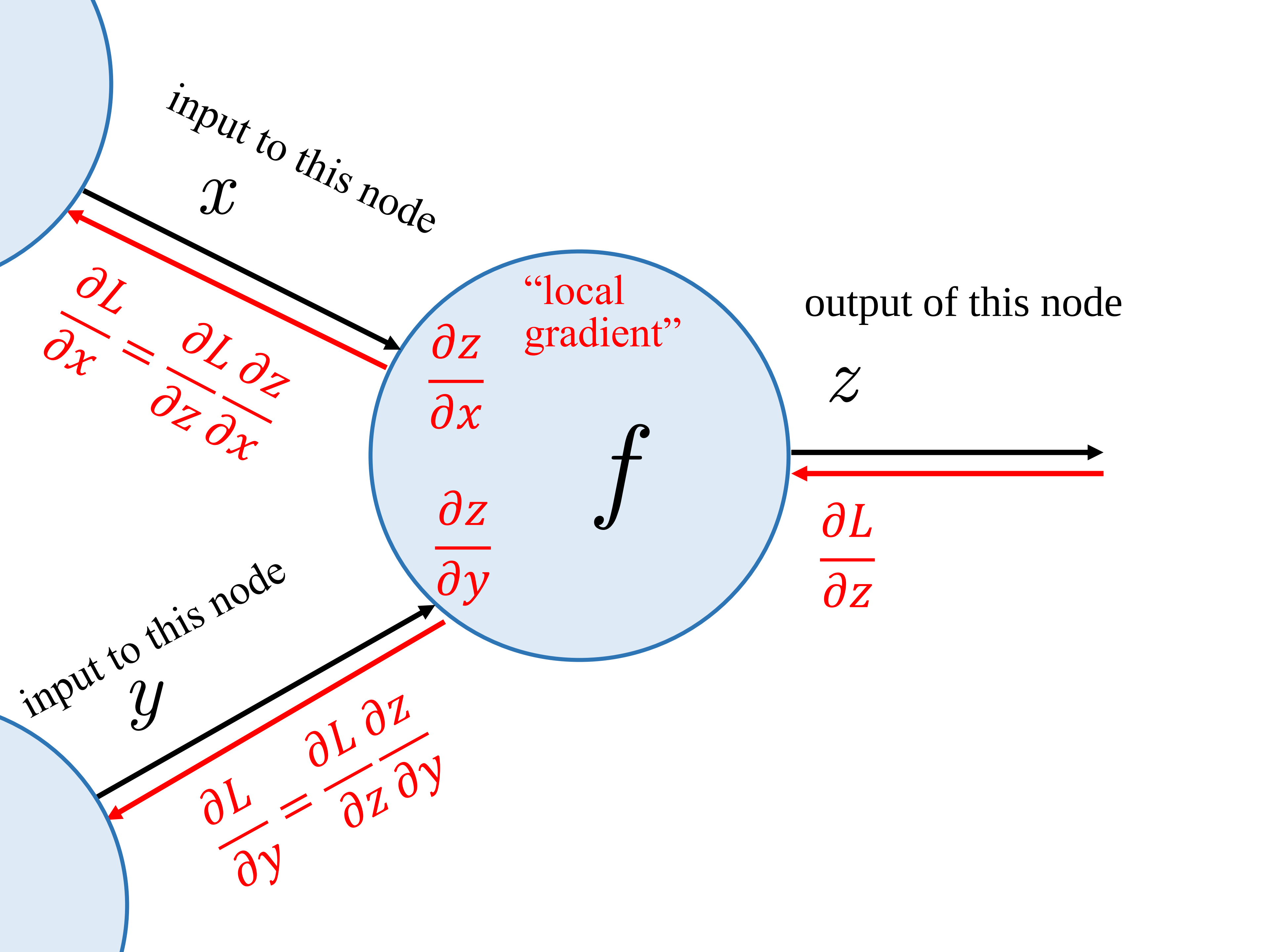
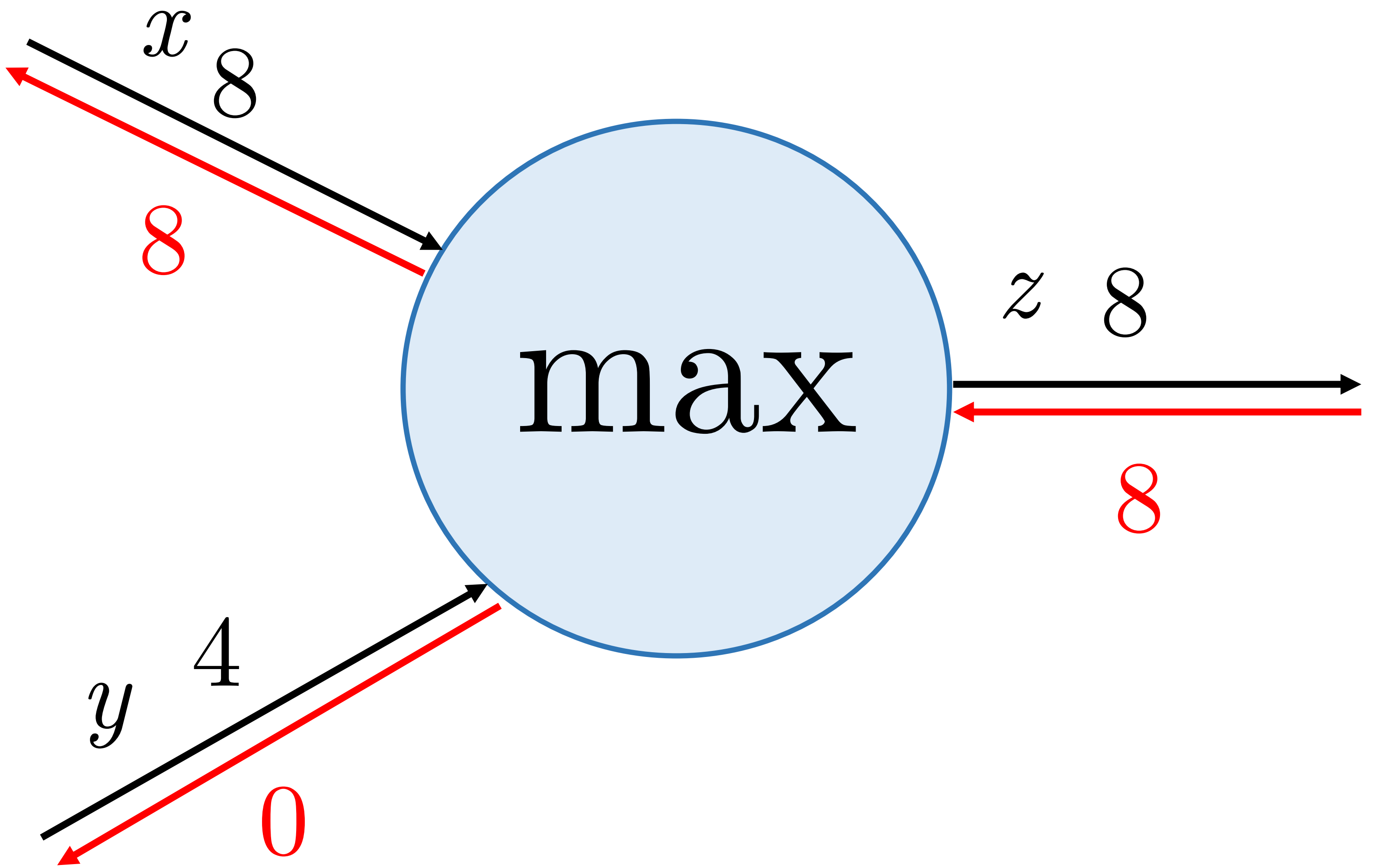
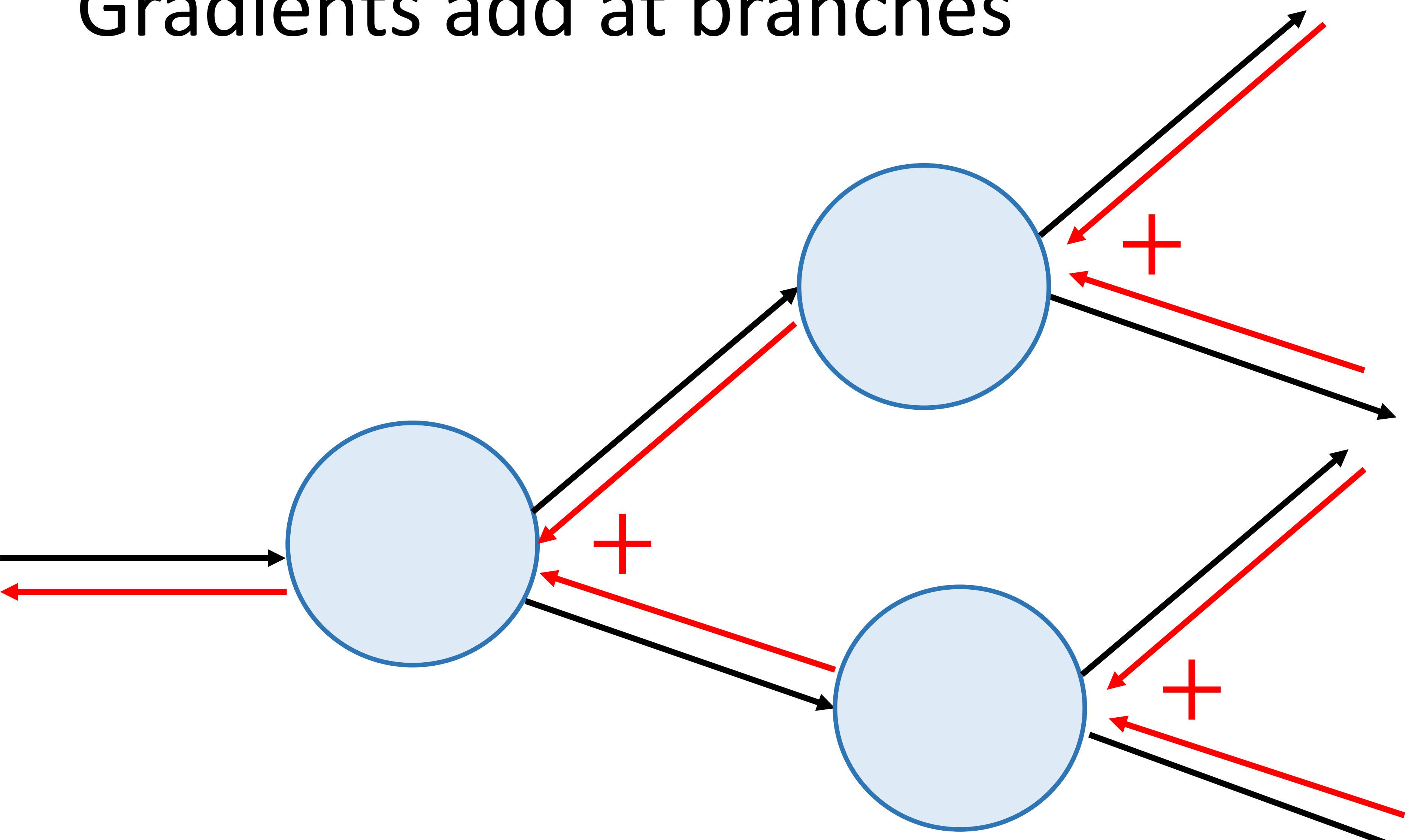
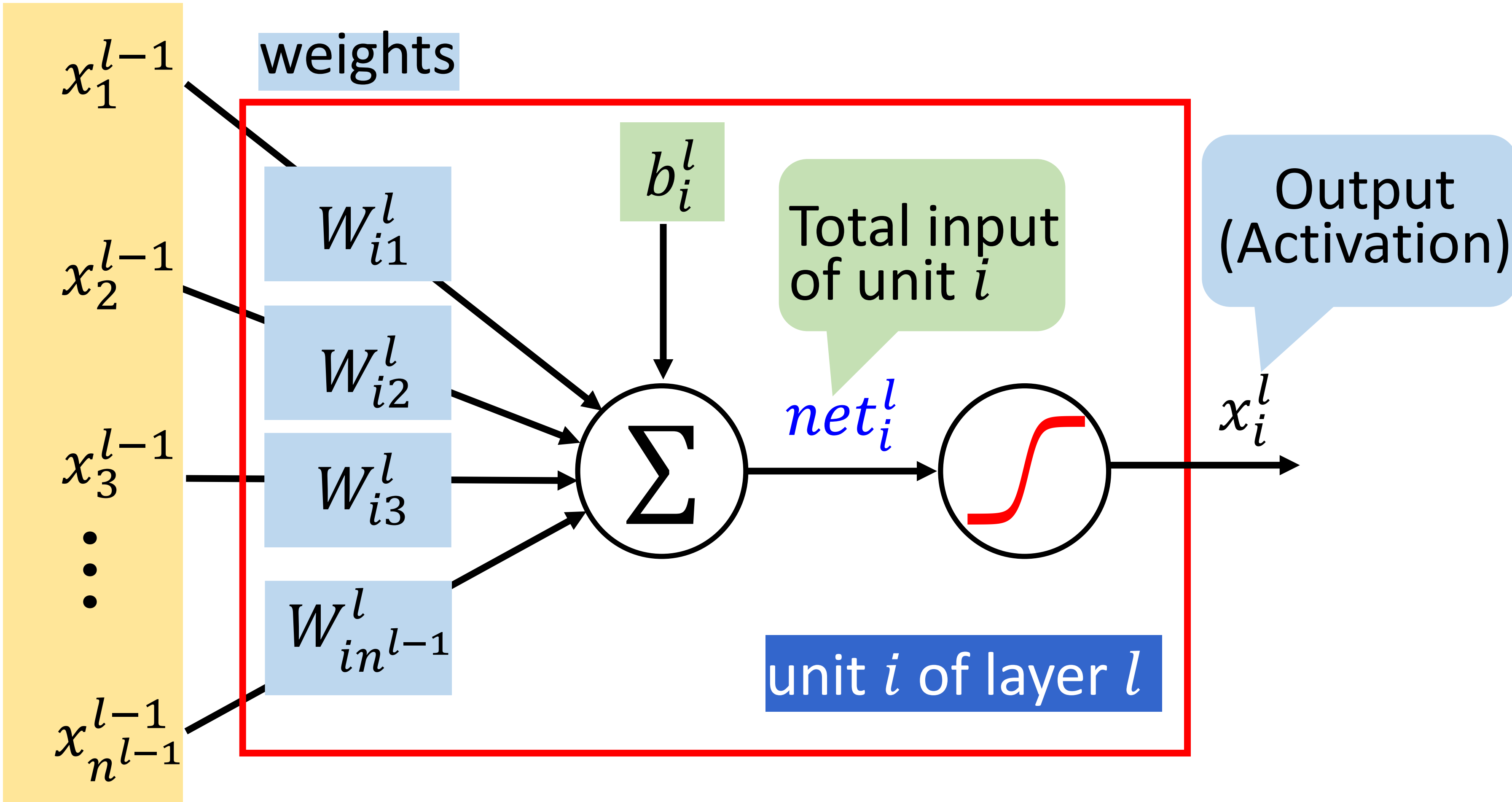

input to this node
$x$
$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial z}\frac{\partial z}{\partial x}$
"local gradient"
$\frac{\partial z}{\partial x}$
$\frac{\partial z}{\partial y}$
$f$
output of this node
$z$
$\frac{\partial L}{\partial z}$
input to this node
$y$
$\frac{\partial L}{\partial y} = \frac{\partial L}{\partial z}\frac{\partial z}{\partial y}$



$x$ 8
8
max
$z$ 8
8
$y$ 4
0

# Gradients add at branches



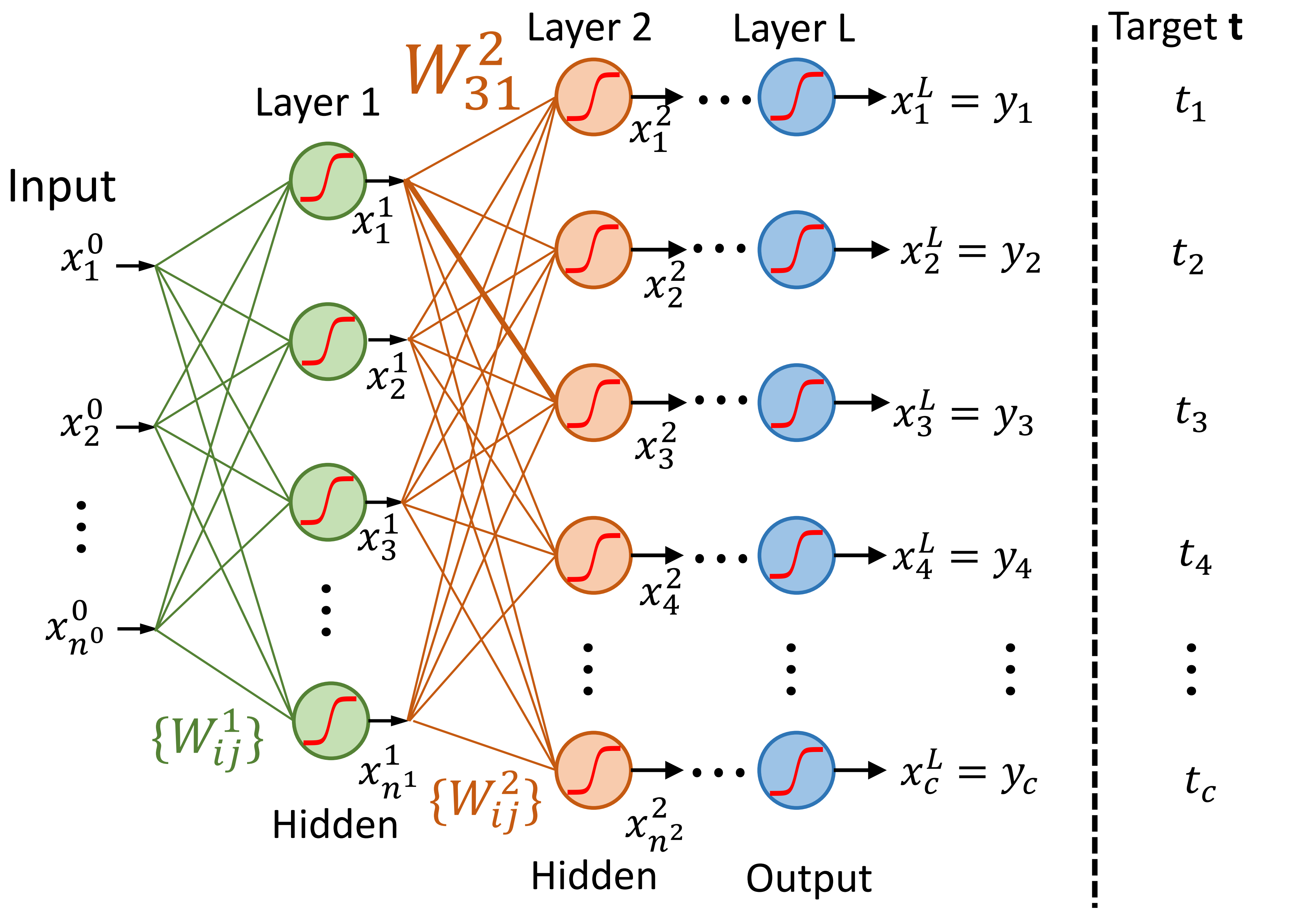



Total input of unit $i$ : $net_i^l = \sum_{j=1}^{n^{l-1}} W_{ij}^{l} x_j^{l-1} + b_i^l$

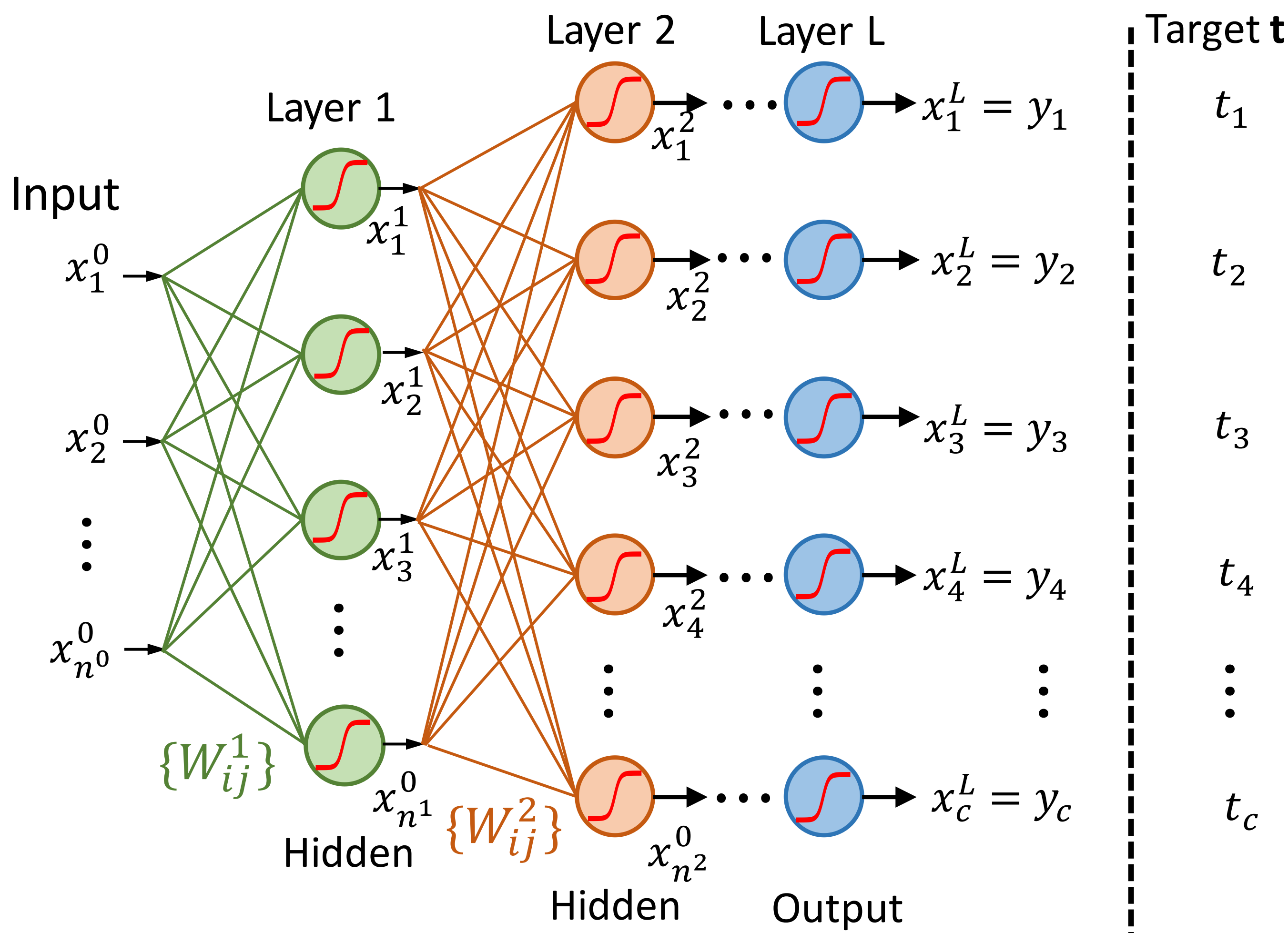
$n^{l-1}$ activations from previous layer as input

Output of unit $i$ : $x_i^l = \sigma\left(net_i^l\right)$

# Fully Connect Feedforward Network



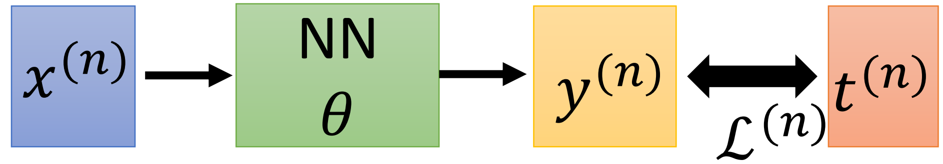
# Fully Connect Feedforward Network




Total input of unit $i$ : $net_i^l = \sum_{j=1}^{n^{l-1}} W_{ij}^l \, x_j^{l-1} + b_i^l$

Output of unit $i$ : $x_i^l = \sigma\left(net_i^l\right)$

# Backpropagation

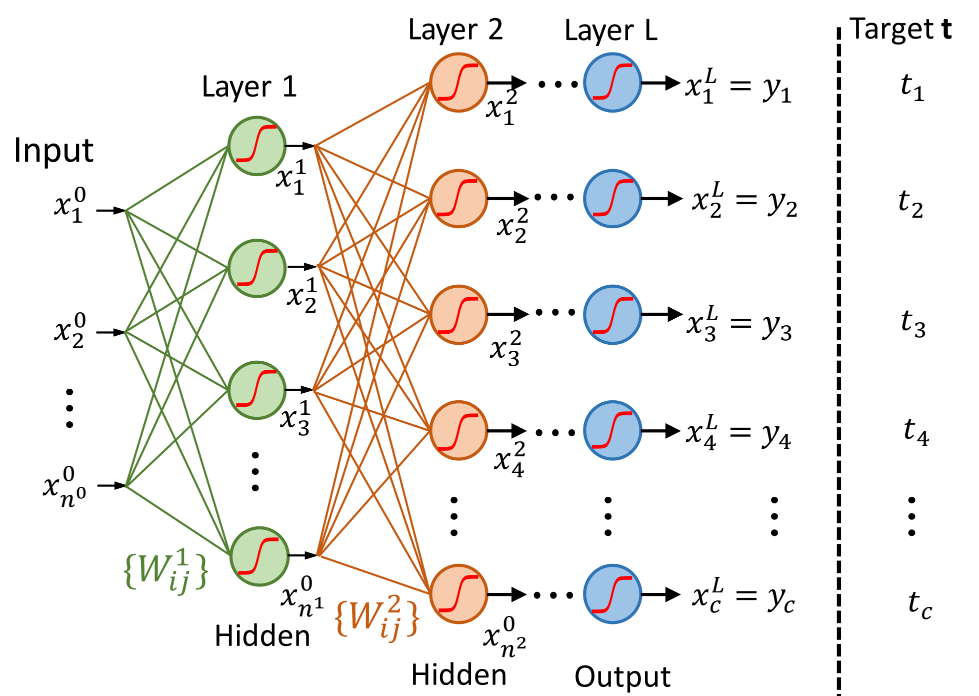



$$\mathcal{C}(\theta) = \sum_{n=1}^{N} \mathcal{L}^{(n)}(\theta) \quad \Rightarrow \quad \frac{\partial \mathcal{C}(\theta)}{\partial w} = \sum_{n=1}^{N} \frac{\partial \mathcal{L}^{(n)}(\theta)}{\partial w}$$

- Choose squared error as training error measurement and sigmoid as activation function at the output layer

$$\mathcal{L}(\theta) = \frac{1}{2} \sum_{k=1}^{c} (t_k - y_k)^2$$

Loss for one training sample



# Gradient Descent

Network parameters $\boxed{\theta = \{W_{ij}^l, b_i^l\} \begin{cases} l = 1, \dots, L \\ i = 1, \dots, n^l \\ j = 1, \dots, n^{l-1} \end{cases}}$

For example: $n^{l-1}$=1000, $n^l$=1000

Then each layer has $10^6$ parameters

Tens of Millions of parameters ……

Need to compute partial derivative with respect to each parameter:

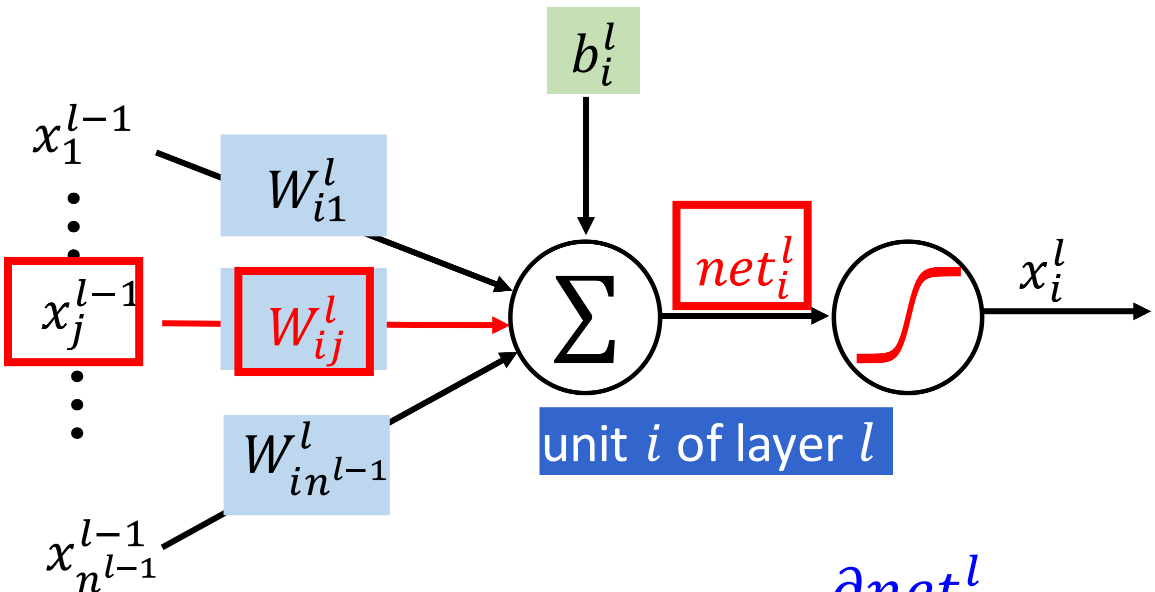
$$W_{ij}^l \leftarrow W_{ij}^l - \eta \frac{\partial \mathcal{L}(\theta)}{\partial W_{ij}^l}$$

$$b_i^l \leftarrow b_i^l - \eta \frac{\partial \mathcal{L}(\theta)}{\partial b_i^l}$$

Compute $\frac{\partial \mathcal{L}}{\partial W_{ij}^l}$ and $\frac{\partial \mathcal{L}}{\partial b_i^l}$

$$net_i^l = \sum_{j=1}^{n^{l-1}} W_{ij}^l x_j^{l-1} + b_i^l$$

- Identify the relation between $W_{ij}^l$ and $net_i^l$, because $W_{ij}^l$ can only affect the network output through $net_i^l$




- Apply chain rule:

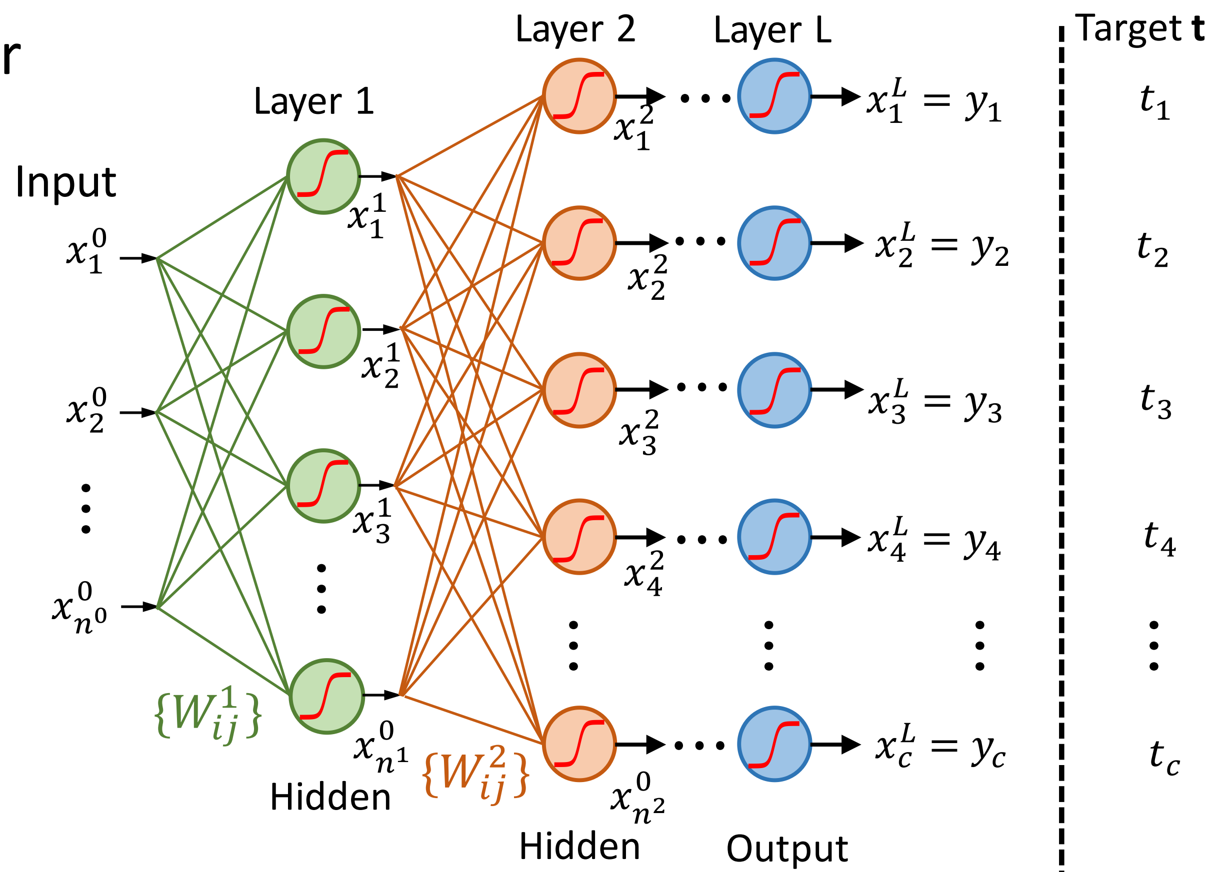
where: $\frac{\partial net_i^l}{\partial W_{ij}^l} = x_j^{l-1}$

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = \frac{\partial net_i^l}{\partial W_{ij}^l} \frac{\partial \mathcal{L}}{\partial net_i^l} = x_j^{l-1} \boxed{\frac{\partial \mathcal{L}}{\partial net_i^l}} \, ?$$

Compute $\frac{\partial \mathcal{L}}{\partial net_i^l}$ $\qquad$ $\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = x_j^{l-1} \frac{\partial \mathcal{L}}{\partial net_i^l}$

- Identify the relation between $net_i^l$ and those neurons connected to it in the immediate downstream (next layer)

- Two cases:
  $l = L$: the output layer
  $l < L$ : hidden layers



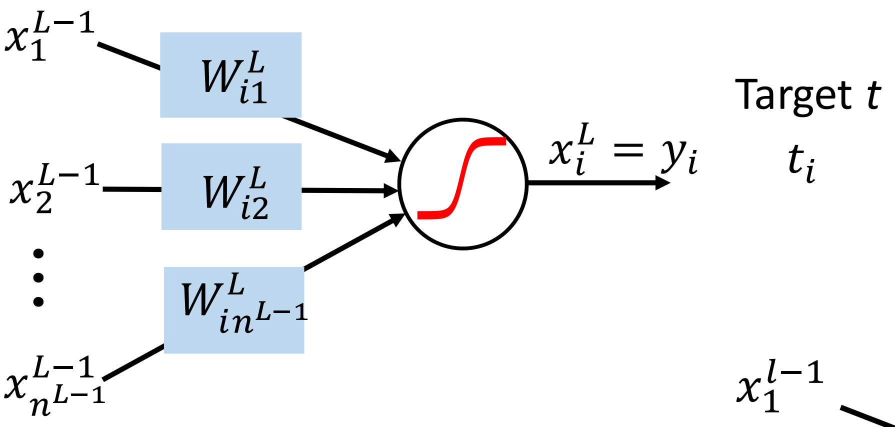
Compute $\dfrac{\partial \mathcal{L}}{\partial net_i^l}$

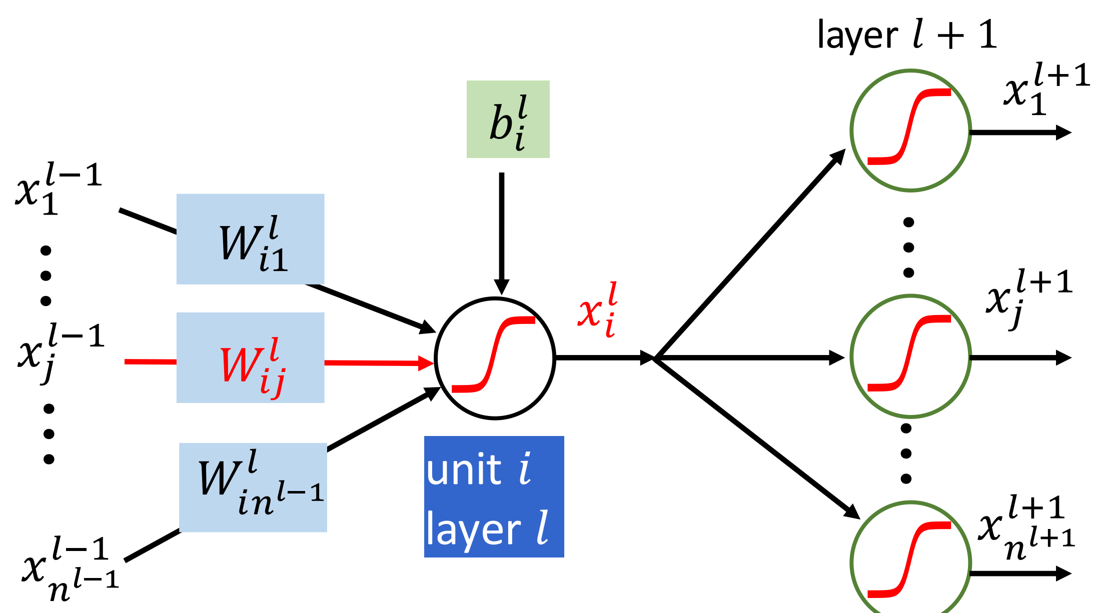
$$\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = x_j^{l-1} \frac{\partial \mathcal{L}}{\partial net_i^l}$$

- Identify the relation between $net_i^l$ and those neurons connected to it in the immediate downstream (next layer)

- Case 1
  $l = L$: the output units

- Case 2:
  $l < L$ : hidden units



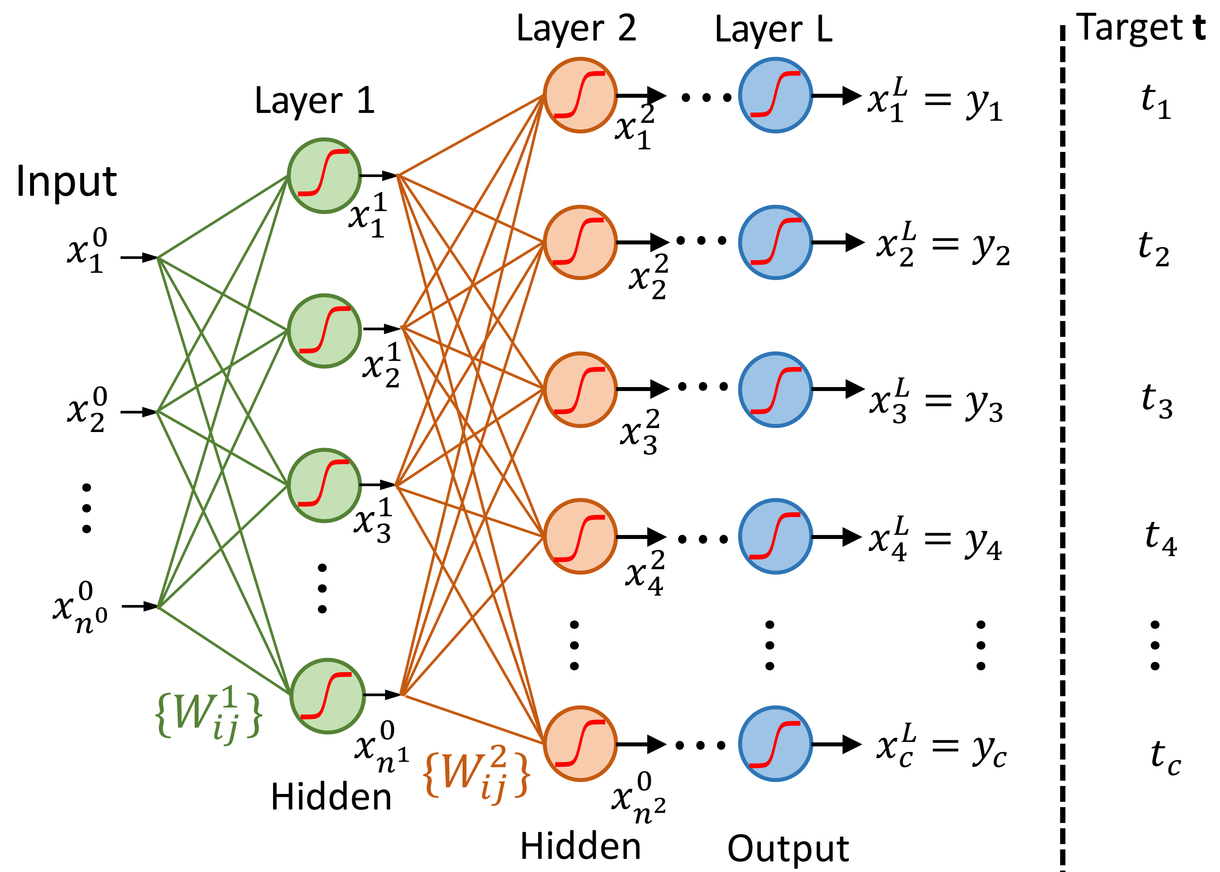



Compute $\dfrac{\partial \mathcal{L}}{\partial net_i^l}$

- Sensitivity of unit $i$ at layer $l$

Describe how the overall error changes with the unit's net activation:

$$\delta_i^l = -\frac{\partial \mathcal{L}}{\partial net_i^l}$$



Compute $\dfrac{\partial \mathcal{L}}{\partial net_i^l}$

$$\mathcal{L}(\theta) = \frac{1}{2}\sum_{k=1}^{c}(t_k - y_k)^2$$

$$x_i^L = \sigma(net_i^L)$$

$$x_i^L = y_i$$

- Case 1
  $l = L$: the output units

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^L} = x_j^{L-1}\frac{\partial \mathcal{L}}{\partial net_i^L}$$

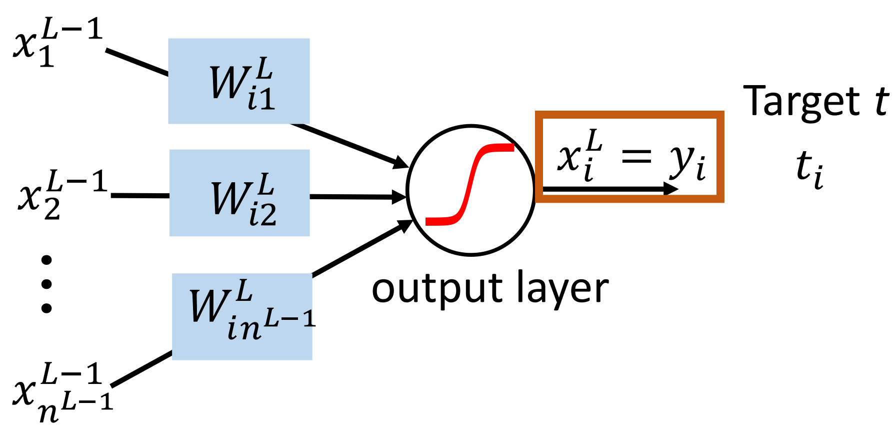
- Chain Rule

$$\frac{\partial \mathcal{L}}{\partial net_i^L} = \frac{\partial x_i^L}{\partial net_i^L}\frac{\partial \mathcal{L}}{\partial x_i^L} = \sigma'(net_i^L)\frac{\partial \mathcal{L}}{\partial y_i} = -(t_i - y_i)\sigma'(net_i^L)$$

Suppose:

$$\delta_i^L = (t_i - y_i)\sigma'(net_i^L)$$

Then: $\dfrac{\partial \mathcal{L}}{\partial W_{ij}^L} = -\delta_i^L x_j^{L-1}$



Compute $\dfrac{\partial \mathcal{L}}{\partial net_i^l}$

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = x_j^{l-1} \frac{\partial \mathcal{L}}{\partial net_i^l}$$

$$x_i^l = \sigma(net_i^l)$$

$$\delta_i^l = -\frac{\partial \mathcal{L}}{\partial net_i^l}$$

- Case 2
  $l < L$: hidden units

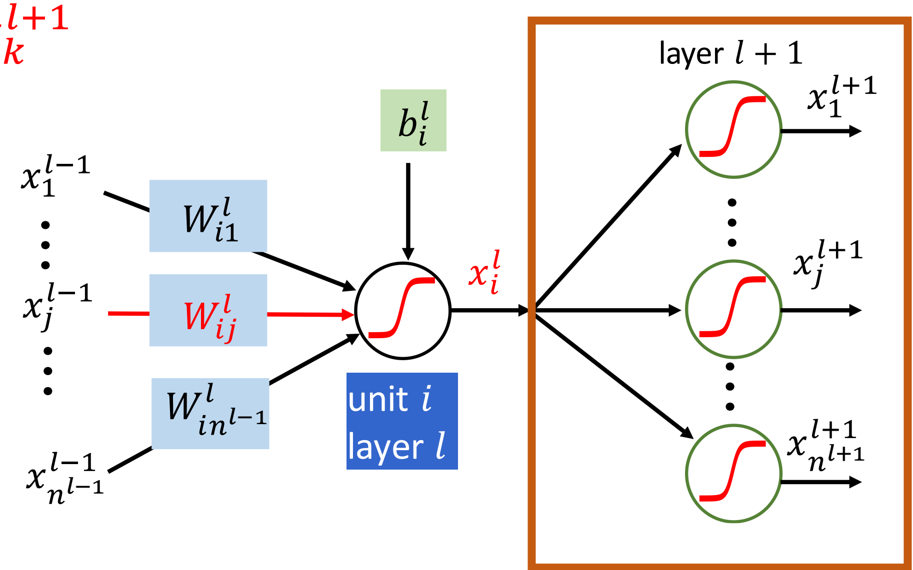
- Identify the relation between $net_i^l$ and those neurons connected to it in the immediate downstream (next layer)

- Chain Rule

$$\frac{\partial \mathcal{L}}{\partial net_i^l} = \sum_{k \in downstream(net_i^l)} \frac{\partial net_k^{l+1}}{\partial net_k^l} \frac{\partial \mathcal{L}}{\partial net_k^{l+1}}$$

$$= \sum_{k=1}^{n^{l+1}} \frac{\partial net_k^{l+1}}{\partial net_i^l} \frac{\partial \mathcal{L}}{\partial net_k^{l+1}}$$

$$\delta_i^l = \sum_{k=1}^{n^{l+1}} \frac{\partial net_k^{l+1}}{\partial net_i^l} (\delta_k^{l+1})$$



Compute $\dfrac{\partial \mathcal{L}}{\partial net_i^l}$

- Case 2
  $l < L$: hidden units

$$\boxed{\delta_i^l} = \sum_{k=1}^{n^{l+1}} \boxed{\frac{\partial net_k^{l+1}}{\partial net_i^l}} \delta_k^{l+1}$$

$$= \sum_{k=1}^{n^{l+1}} \frac{\partial net_k^{l+1}}{\partial x_i^l} \frac{\partial x_i^l}{\partial net_i^l} \delta_k^{l+1}$$

$$= \sum_{k=1}^{n^{l+1}} W_{ki}^{l+1} \sigma'(net_i^l) \delta_k^{l+1}$$

$$= \sigma'(net_i^l) \sum_{k=1}^{n^{l+1}} W_{ki}^{l+1} \boxed{\delta_k^{l+1}}$$

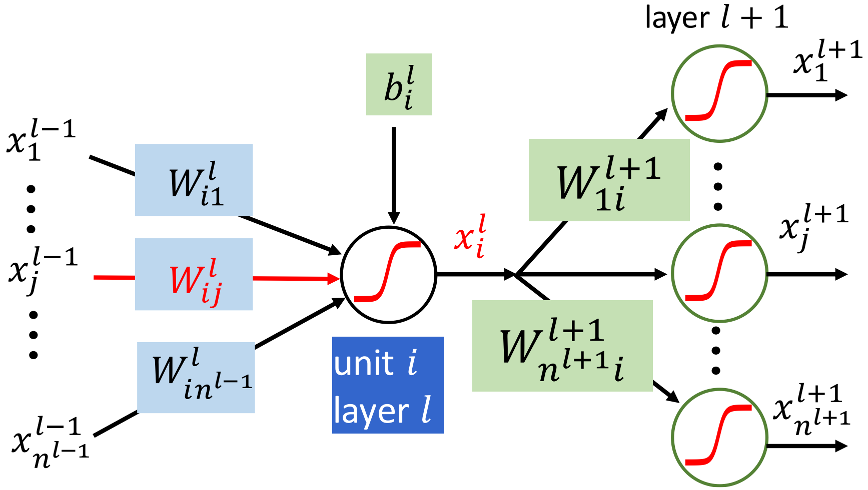
$$\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = -x_j^{l-1} \delta_i^l$$

$$\boxed{\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = x_j^{l-1} \frac{\partial \mathcal{L}}{\partial net_i^l}}$$

$$x_i^l = \sigma(net_i^l)$$

$\sigma$ is the activation function

$$net_k^{l+1} = \sum_{i=1}^{n^l} W_{ki}^{l+1} x_i^l + b_k^{l+1}$$



# Summary: Error Propagation

$$\delta_i^l = -\frac{\partial \mathcal{L}}{\partial net_i^l}$$

$\sigma$ is the activation function

- Case 1

  $l = L$: the output units

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^L} = -\delta_i^L x_j^{L-1} \qquad \delta_i^L = \sigma'(net_i^L)\underline{(t_i - y_i)}$$

error of output layer

- Case 2

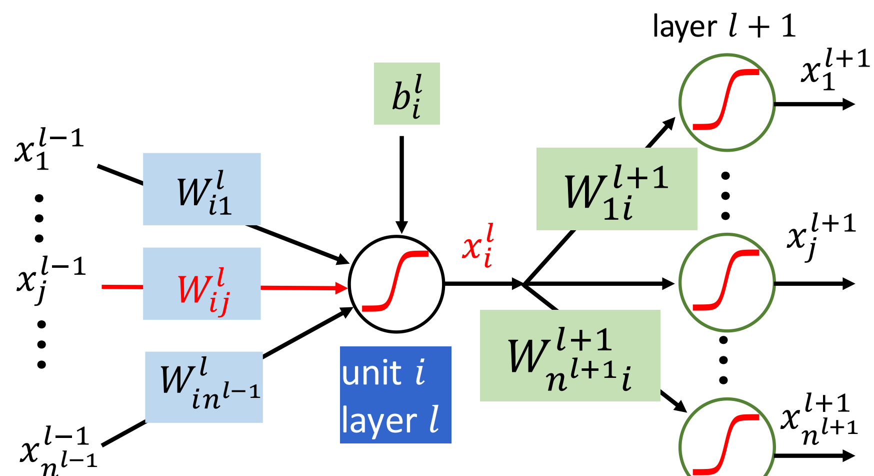
  $l < L$: hidden units

error backpropagated

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = -\delta_i^l x_j^{l-1} \qquad \delta_i^l = \sigma'(net_i^l)\underline{\sum_{k=1}^{n^{l+1}} W_{ki}^{l+1}\delta_k^{l+1}}$$

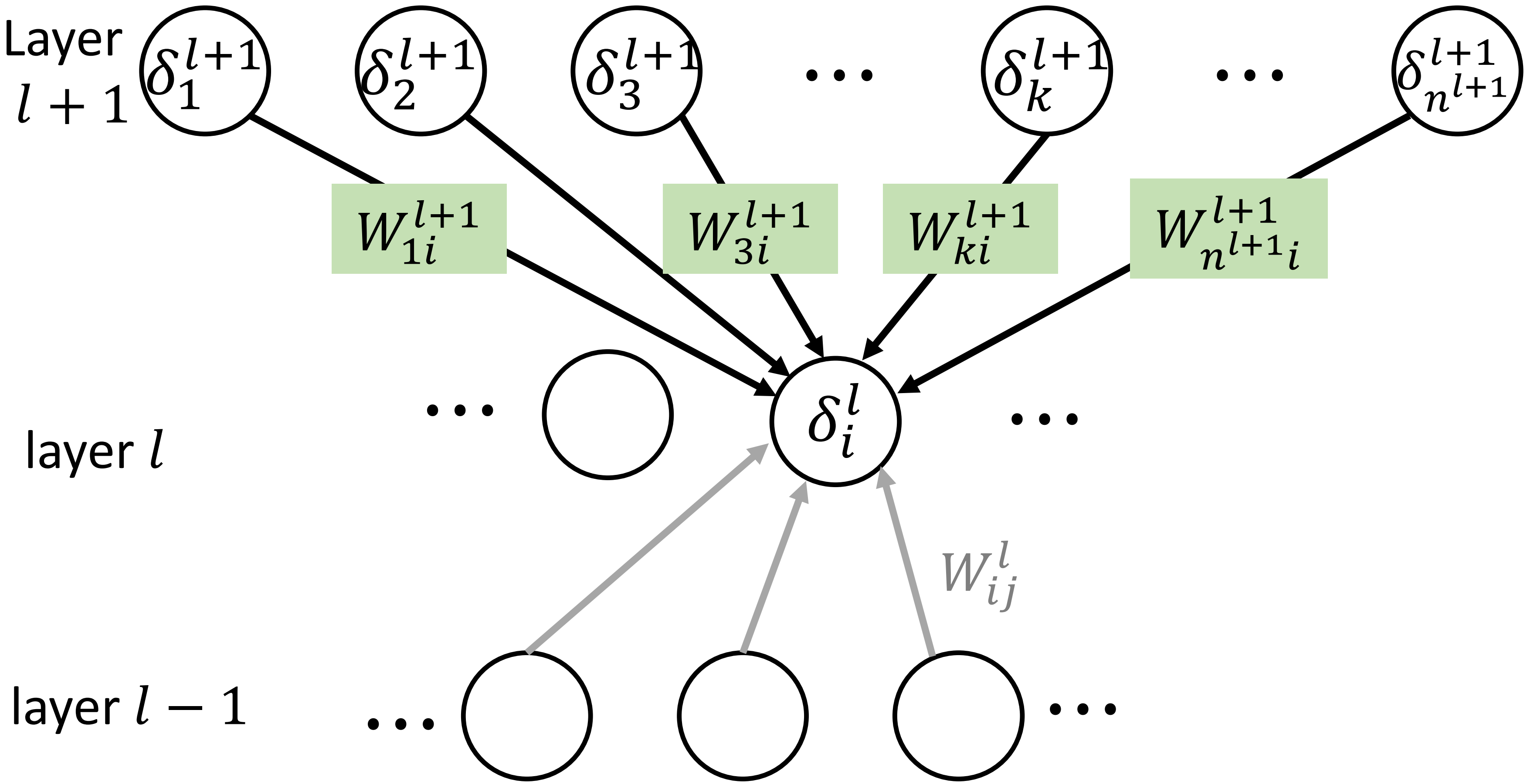
$\delta_i^l$ : an "error term" that measures how much that unit is "responsible" for any errors in the output.

Case 1 and case 2 can be unified.



# Illustration of Error Propagation

$$\delta_i^l = \sigma'(net_i^l) \sum_{k=1}^{n^{l+1}} W_{ki}^{l+1} \delta_k^{l+1}$$




- The sensitivity at a hidden unit is proportional to the weighted sum of the sensitivities at the output units.
- The output unit sensitivities are thus propagated “back” to the hidden units.

# BP: Summary

$$net_i^l = \sum_{j=1}^{n^{l-1}} W_{ij}^l x_j^{l-1} + b_i^l$$

$$\boxed{x_i^l} = \sigma\left(net_i^l\right)$$

- $l = L$: the output units

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^L} = -\boxed{\delta_i^L}\boxed{x_j^{L-1}}$$

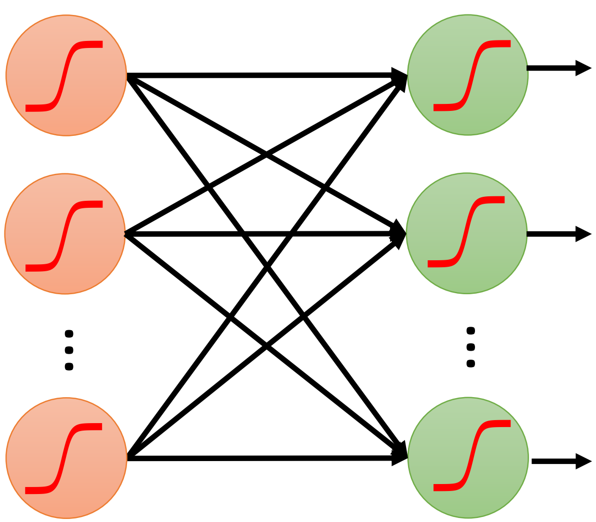
$$\boxed{\delta_i^L} = (t_i - y_i)\sigma'\left(net_i^L\right)$$

- $0 < l < L$: hidden units

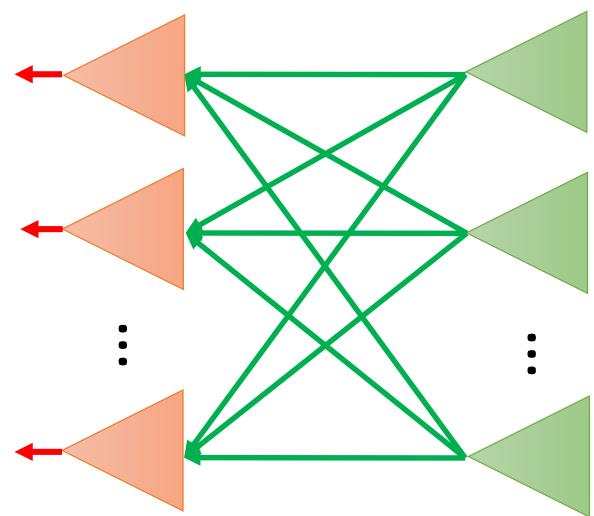
$$\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = -\boxed{\delta_i^l}\boxed{x_j^{l-1}}$$

$$\boxed{\delta_i^l} = \sigma'\left(net_i^l\right) \sum_{k=1}^{n^{l+1}} W_{ki}^{l+1} \boxed{\delta_k^{l+1}}$$

**_Forward Pass_**



**_Backward Pass_**



$$\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = -\delta_i^l x_j^{l-1}$$

# BP: Summary

$$net_i^l = \sum_{j=1}^{n^{l-1}} W_{ij}^l x_j^{l-1} + b_i^l$$

$$x_i^l = \sigma\left(net_i^l\right)$$

- $l = L$: the output units

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^L} = -\delta_i^L x_j^{L-1}$$

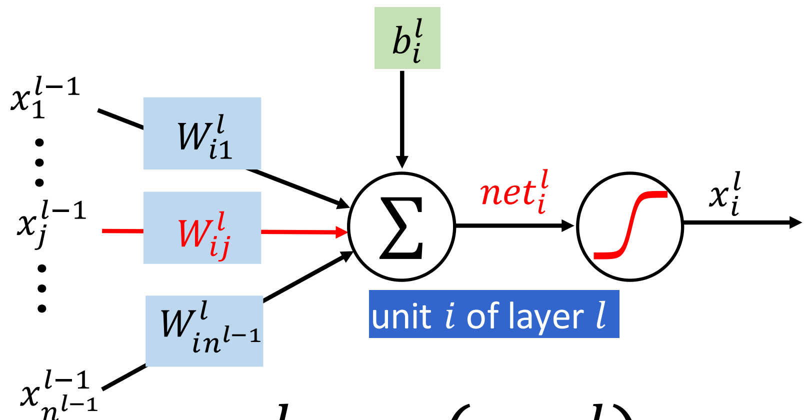
$$\delta_i^L = (t_i - y_i)\boxed{\sigma'\left(net_i^L\right)}$$

- $0 < l < L$: hidden units

$$\frac{\partial \mathcal{L}}{\partial W_{ij}^l} = -\delta_i^l x_j^{l-1}$$

$$\delta_i^l = \boxed{\sigma'\left(net_i^l\right)} \sum_{k=1}^{n^{l+1}} W_{ki}^{l+1} \delta_k^{l+1}$$

- Implementation note




$$x_i^l = \sigma\left(net_i^l\right)$$

Need to compute: $\sigma'\left(net_i^L\right)$

Suppose sigmoid activation function:

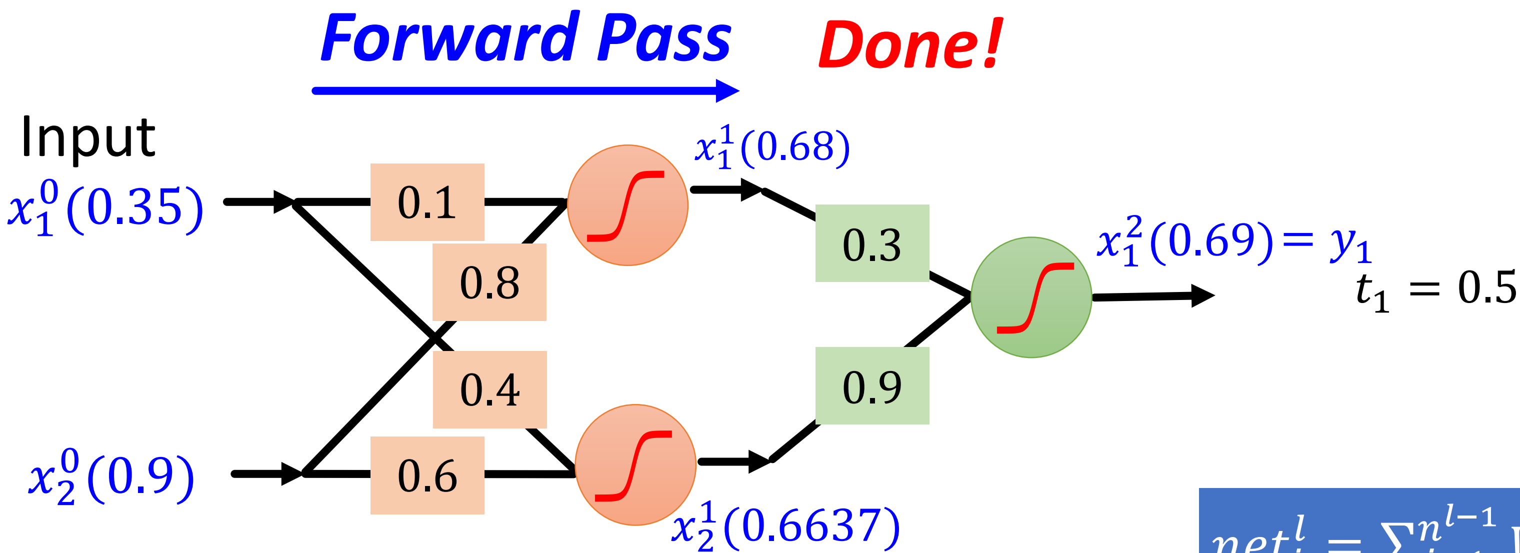
$$\sigma(z) = \frac{1}{1+e^{-z}}$$

$$\sigma'(z) = \sigma(z)\left(1 - \sigma(z)\right)$$

Activations $\{x_i^l\}$ have already been stored away from the forward pass through the network.

$$\sigma'\left(net_i^l\right) = \sigma\left(net_i^l\right)\left(1 - \sigma\left(net_i^l\right)\right) = x_i^l\left(1 - x_i^l\right)$$

# An Example for Backpropagation




$$net_i^l = \sum_{j=1}^{n^{l-1}} W_{ij}^l \, x_j^{l-1}$$

$$x_i^l = \sigma(net_i^l)$$

$$\sigma(x) = \frac{1}{1+exp(-x)}$$

$W_{11}^1 = 0.1$ $\quad W_{21}^1 = 0.4$

$W_{12}^1 = 0.8$ $\quad W_{22}^1 = 0.6$

$$x_1^1 = \frac{1}{1+exp(-(0.1*0.35+0.9*0.8))} = 0.68$$

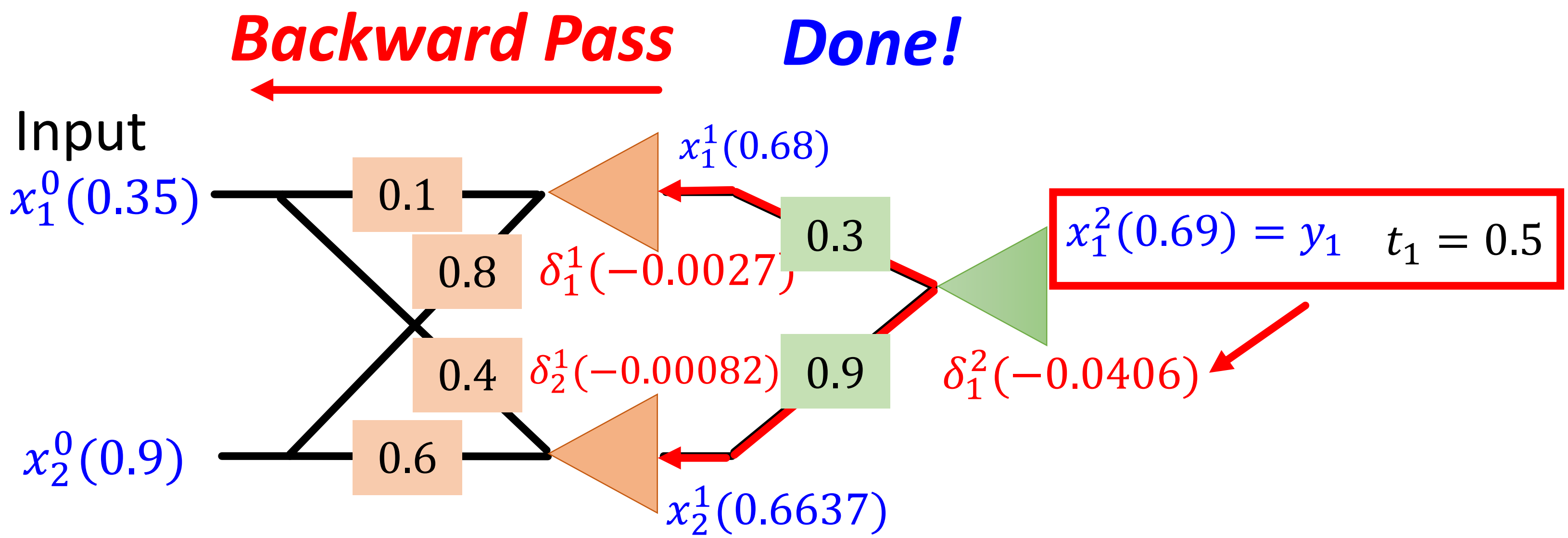
$x_2^1 = 0.6637$

$W_{11}^2 = 0.3$

$W_{12}^2 = 0.9$

$x_1^2 = 0.69 = y_1$

# An Example for Backpropagation




$L = 2$: the output units: $\partial \mathcal{L}/\partial W_{ij}^L = -\delta_i^L x_j^{L-1}$ $\quad$ $\delta_i^L = (t_1 - y_1)\sigma'(net_i^L)$

$$\delta_1^2 = (t_1 - y_1)\sigma'(net_1^2) = (t_1 - y_1)x_1^2(1 - x_1^2)$$

$$= (0.5 - 0.69) * 0.69 * (1 - 0.69) = -0.0406$$

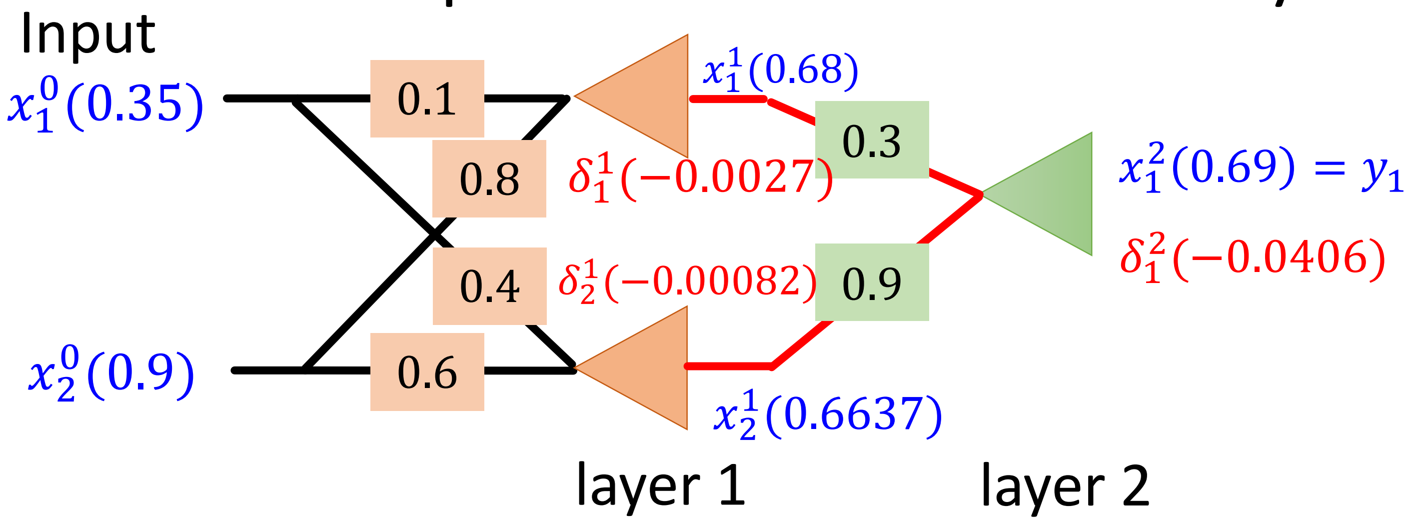
$\sigma'(net_i^l) = x_i^l(1 - x_i^l)$

$0 < l < 2$: hidden units: $\partial \mathcal{L}/\partial W_{ij}^l = -\delta_i^l x_j^{l-1}$ $\quad$ $\delta_i^l = \sigma'(net_i^l) \sum_{k=1}^{n^{l+1}} W_{ki}^{l+1} \delta_k^{l+1}$

$$\delta_1^1 = x_1^1(1 - x_1^1) * W_{11}^2 * \delta_1^2 = 0.68 * (1 - 0.68) * 0.3 * (-0.0406) = -0.0027$$

$$\delta_2^1 = 0.6637 * (1 - 0.6637) * 0.9 * (-0.0406) = -0.0082$$

# An Example for BP: Summary




$1 \leq l \leq 2$: hidden units: $\partial\mathcal{L}/\partial W_{ij}^l = -\delta_i^l x_j^{l-1}$

$W_{11}^1 = 0.1$ $W_{21}^1 = 0.4$ $W_{11}^2 = 0.3$

$W_{12}^1 = 0.8$ $W_{22}^1 = 0.6$ $W_{12}^2 = 0.9$

$$\frac{\partial\mathcal{L}}{\partial W_{11}^1} =$$

$$\frac{\partial\mathcal{L}}{\partial W_{12}^1} =$$

$$\frac{\partial\mathcal{L}}{\partial W_{21}^1} =$$

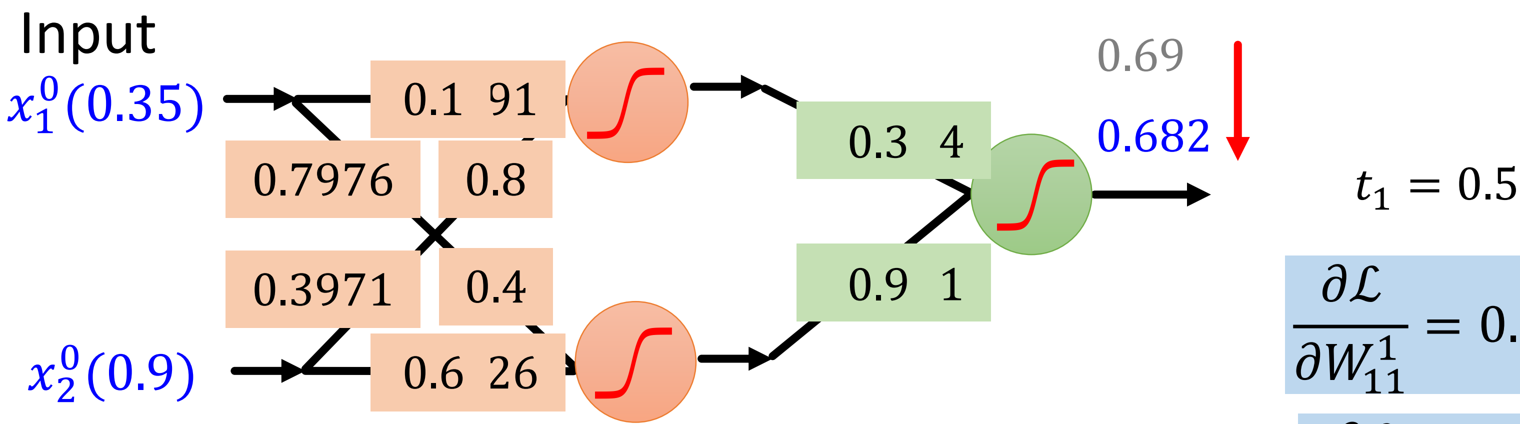
$$\frac{\partial\mathcal{L}}{\partial W_{22}^1} =$$

$$\frac{\partial\mathcal{L}}{\partial W_{11}^2} =$$

$$\frac{\partial\mathcal{L}}{\partial W_{12}^2} =$$

# An Example for Backpropagation

***Weights Update***




$$\frac{\partial \mathcal{L}}{\partial W_{11}^1} = 0.000945$$

$$\frac{\partial \mathcal{L}}{\partial W_{12}^1} = 0.00243$$

$$\frac{\partial \mathcal{L}}{\partial W_{21}^1} = 0.00287$$

$$\frac{\partial \mathcal{L}}{\partial W_{22}^1} = 0.00738$$

$$\frac{\partial \mathcal{L}}{\partial W_{11}^2} = 0.0276$$

$$\frac{\partial \mathcal{L}}{\partial W_{12}^2} = 0.0269$$

Suppose: learning rate $\eta = 1$

$$W_{11}^1 = 0.1 - 0.000945 = 0.0991$$

$$W_{12}^1 = 0.8 - 0.00243 = 0.7976$$

$$W_{21}^1 = 0.4 - 0.00287 = 0.3971$$

$$W_{22}^1 = 0.6 - 0.00738 = 0.5926$$

$$W_{11}^2 = 0.3 - 0.0276 = 0.2724$$

$$W_{12}^2 = 0.9 - 0.0269 = 0.8731$$

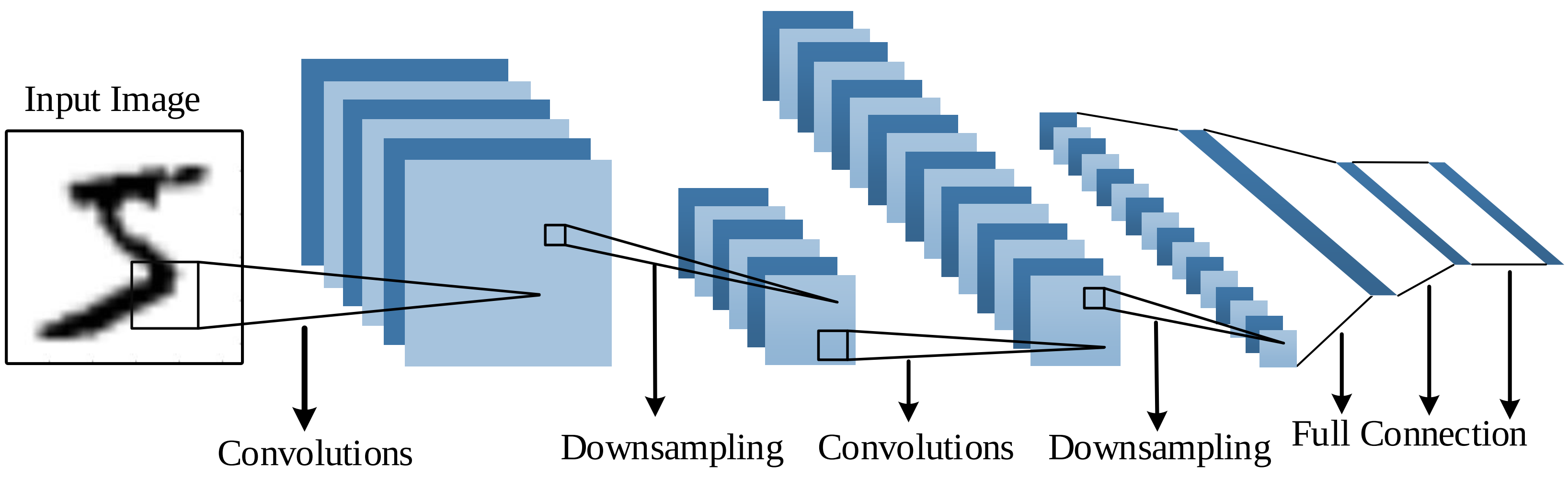
***Just Repeat!***

# Summary for Backpropagation

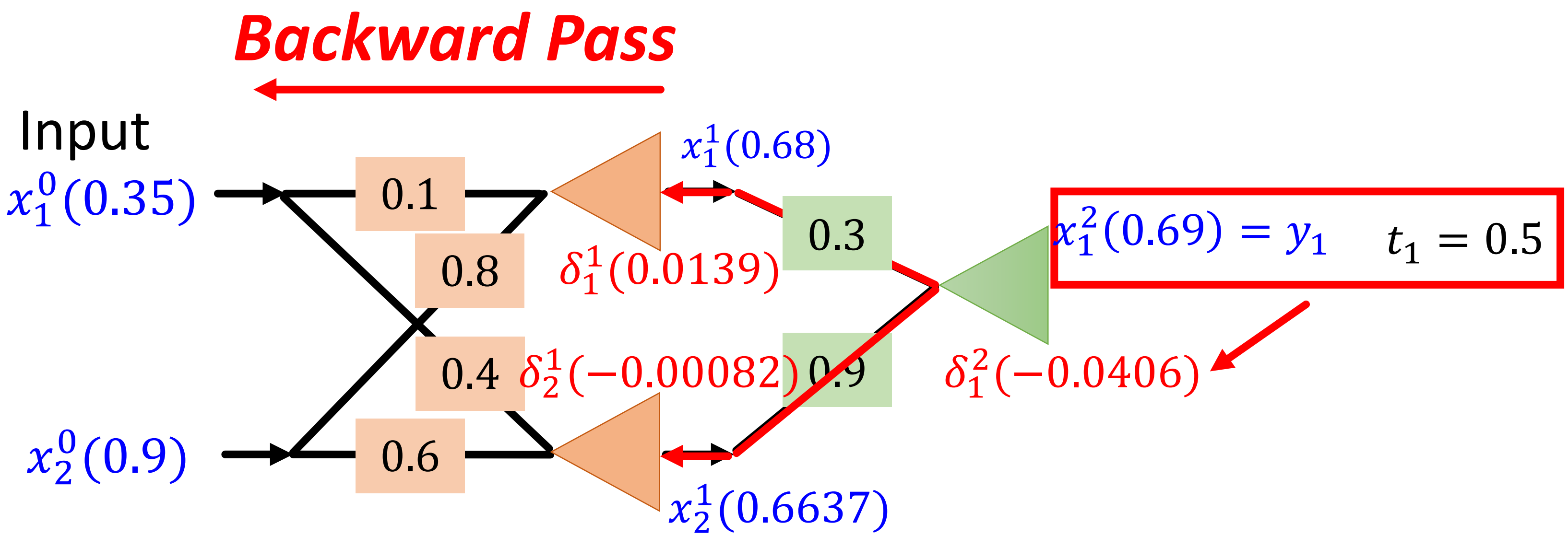
- Neural nets will be very large: impractical to write down gradient formula by hand for all parameters
- Backpropagation = recursive application of the chain rule along a computational graph to compute the gradients of all inputs, parameters, and intermediates
- Implementations maintain a graph structure, where the nodes implement the forward() / backward() API
- Forward: compute result of an operation and save any intermediates needed for gradient computation in memory
- Backward: apply the chain rule to compute the gradient of the loss function with respect to the inputs

# Next Lecture

## Convolutional Neural Network



# An Example for Backpropagation

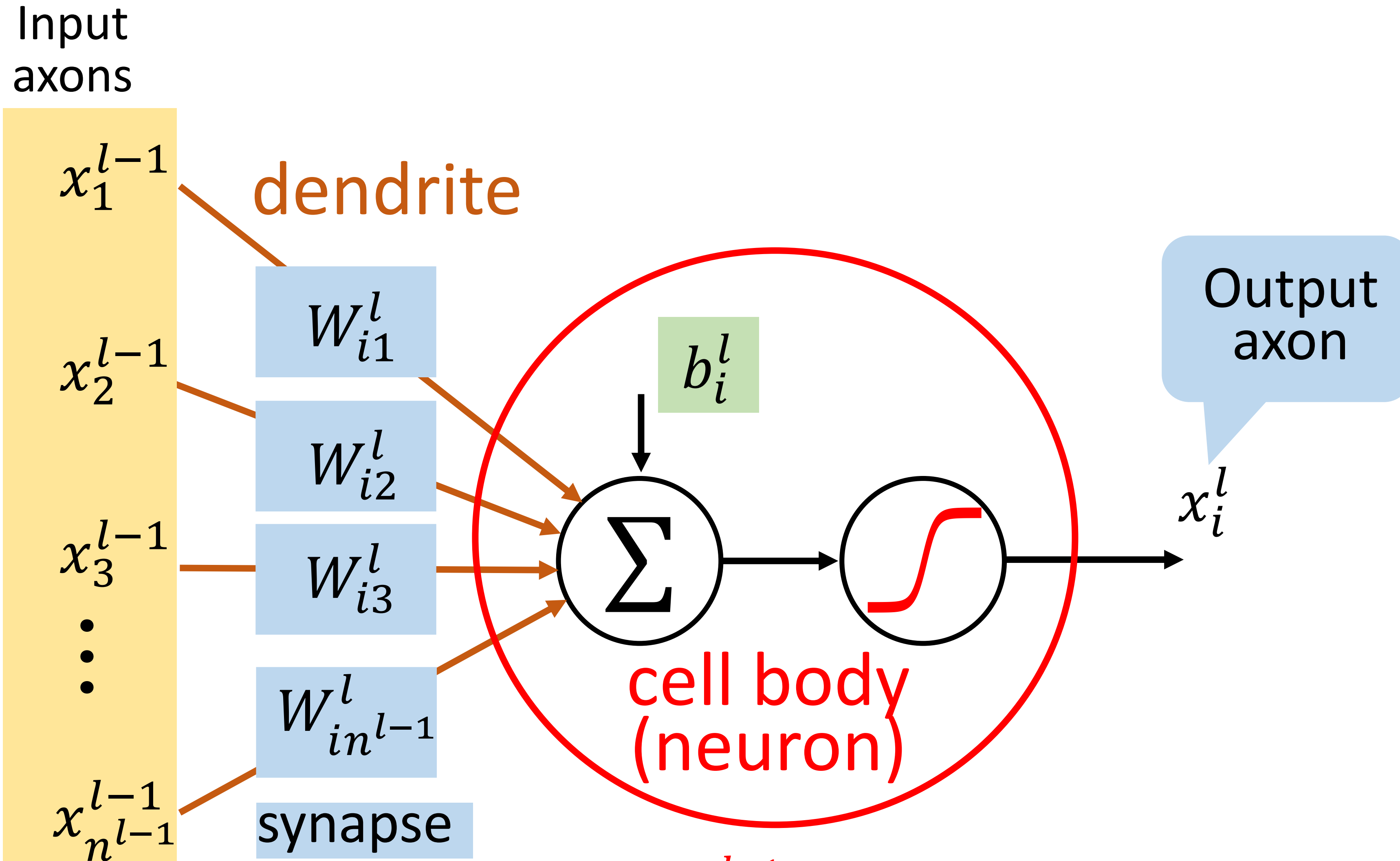



$l$ =2: the output units: $\partial\mathcal{L}/\partial W_{ij}^L = -\delta_i^L x_j^{L-1}$ $\delta_i^L = (t_1 - y_1)\sigma'(net_i^L)$

$0 < l < 2$: hidden units: $\partial\mathcal{L}/\partial W_{ij}^l = -\delta_i^l x_j^{l-1}$ $\delta_i^l = \sigma'(net_i^l)\sum_{k=1}^{n^{l+1}} W_{ki}^{l+1}\delta_k^{l+1}$

$\sigma'(net_i^l) = x_i^l(1 - x_i^l)$

$$\delta_1^2 = (t_1 - y_1)\sigma'(net_1^2) = (t_1 - y_1)x_1^2(1 - x_1^2)$$
$$= (0.5 - 0.69) * 0.69 * (1 - 0.69) = -0.0406$$
$$\delta_1^1 = x_1^1(1 - x_1^1) * W_{11}^2 * \delta_1^2 = 0.68 * (1 - 0.68) * 0.3 * (-0.0406) = -0.0027$$
$$\delta_2^1 = 0.6637 * (1 - 0.6637) * 0.9 * (-0.0406) = -0.0082$$




$$x_i^l = \sigma\left(\sum_{j=1}^{n^{l-1}} W_{ij}^l x_j^{l-1} + b_i^l\right)$$