Fourier Transform :

⇒ Fourier transform is a mathematical tool used for frequency analysis of signal. Fourier transform is applicable for periodic and non-periodic signals.

* conditions for existence of fourier transform (Dirichlet conditions) :

(1) Signal should have finite number of maxima and minima over finite interval

(2) signal should have finite number of discontinuities over any finite interval.

(3) Signal should be absolutely integrable

$$\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$$

⇒ These conditions are sufficient but not necessary.

$$x(t) \xrightarrow{FT} X(j\omega) \,/\, X(\omega) \,/\, X(F)$$

$$\boxed{X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

F.T.

$$\boxed{x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega}$$ I.F.T

* Properties of Fourier transforms:

1. Linearity
2. Conjugation
3. Area under $x(t)$
4. Area under $X(j\omega)$
5. Time reversal
6. Time scaling
7. Time shifting
8. Frequency shifting
9. Convolution in Time.
10. Multiplication in Time
11. Differentiation in Time
12. Integration in Time
13. Differentiation in Frequency
14. Modulation
15. Parseval's energy theorem.
16. Duality property.

① Linearity :

$$x_1(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X_1(j\omega)$$

$$x_2(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X_2(j\omega)$$

$$\alpha x_1(t) + \beta x_2(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} \alpha X_1(j\omega) + \beta x_2(j\omega)$$

proof:

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(j\omega) = \int_{-\infty}^{\infty} \left[\alpha x_1(t) + \beta x_2(t)\right] e^{-j\omega t}\, dt$$

$$\boxed{X(j\omega) = \alpha X_1(j\omega) + \beta X_2(j\omega)}$$

② Conjugation :-

$$x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(j\omega)$$

$$x^*(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X^*(-j\omega)$$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow X^*(j\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{j\omega t}\, dt$$

$$\Rightarrow X^*(-j\omega) = \int_{-\infty}^{\infty} x^*(t)\, e^{-j\omega t}\, dt$$

$$\boxed{\therefore X^*(-j\omega) = \text{F.T. of } x^*(t)}$$

③ Area under $x(t)$

$$\text{Area under } x(t) = \int_{-\infty}^{\infty} x(t)\, dt$$

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t}\, dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

$$\boxed{\text{Area under } x(t) = X(j\omega)\Big|_{\omega=0}}$$

④ Area under $X(j\omega)$

$$\text{Area under } X(j\omega) = \int_{-\infty}^{\infty} X(j\omega)\, d\omega$$

$$\therefore x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t}\, d\omega$$

$$\Rightarrow x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{0}\, d\omega$$

$$\Rightarrow 2\pi \cdot x(0) = \int_{-\infty}^{\infty} X(j\omega)\, d\omega$$

$$\boxed{\text{Area under } X(j\omega) = 2\pi x(t)\Big|_{t=0}}$$

⑤ Time Reversal:

$$x(t) \rightleftharpoons X(j\omega)$$

$$x(-t) \rightleftharpoons X(-j\omega)$$

proof:

$$\because X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow X(-j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{j\omega t}\, dt$$

Let $t = -\tau$, $\Rightarrow dt = -d\tau$

$\Rightarrow$ At $t = -\infty$; $\tau = \infty$

$\Rightarrow$ At $t = \infty$; $\tau = -\infty$

$$\Rightarrow X(-j\omega) = \int_{\infty}^{-\infty} x(-\tau)\, e^{-j\omega\tau}\, -d\tau$$

$$\Rightarrow x(-j\omega) = \int_{-\infty}^{\infty} x(-\tau)\, e^{-j\omega\tau}\, d\tau$$

$$\boxed{\therefore X(-j\omega) = \text{F.T. of } x(-t)}$$

⑥ Time scaling:

$$x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(j\omega)$$

$$x(at) \underset{IFT}{\overset{FT}{\rightleftharpoons}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore X'(j\omega) = \int_{-\infty}^{\infty} x(at) \cdot e^{-j\omega t}\, dt$$

Let $at = \tau \Rightarrow dt = \frac{d\tau}{a}$

At $t = -\infty,\ \tau = -\infty$

At $t = \infty,\ \tau = \infty$

$$\Rightarrow X'(j\omega) = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau$$

$$\boxed{\therefore X'(j\omega) = \frac{1}{a} X\left(\frac{j\omega}{a}\right)}$$

⑦ Time shifting:

$$x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(j\omega)$$

$$x(t-t_0) \underset{IFT}{\overset{FT}{\rightleftharpoons}} e^{-j\omega t_0} X(j\omega)$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore X'(j\omega) = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$$

Let $(t-t_0) = \tau \Rightarrow dt = d\tau$

At $t = -\infty,\ \tau = -\infty$

At $t = \infty,\ \tau = \infty$

$$\Rightarrow X'(j\omega) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega(\tau + t_0)}\, d\tau$$

$$\Rightarrow X'(j\omega) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{-j\omega\tau} \cdot d\tau \times e^{-j\omega t_0}$$

$$\boxed{\therefore X'(j\omega) = e^{-j\omega t_0}\, X(j\omega)}$$

8. Frequency shifting

$$x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(\omega)$$

$$e^{j\omega_0 t}\, x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(\omega - \omega_0)$$

$$\therefore X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore X'(\omega) = \int_{-\infty}^{\infty} e^{j\omega_0 t}\, x(t) \cdot e^{-j\omega t}\, dt$$

$$\Rightarrow X'(\omega) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j(\omega - \omega_0)t}\, dt$$

$$\boxed{X'(\omega) = X(\omega - \omega_0)}$$

Similarly,

$$e^{-j\omega_0 t} x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(\omega+\omega_0)$$

⑨ Convolution in Time :

$$x_1(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X_1(\omega)$$

$$x_2(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X_2(\omega)$$

$$x_1(t) * x_2(t) = x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(\omega) = X_1(\omega) \cdot X_2(\omega)$$

$$\therefore X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Here,

$$\therefore X(\omega) = \int_{-\infty}^{\infty} \left[x_1(t) * x_2(t)\right] e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau\right] e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau) \frac{e^{j\omega\tau}}{e^{j\omega\tau}}\, d\tau\, e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau} d\tau \cdot \int_{-\infty}^{\infty} x_2(t-\tau)\, e^{-j(t-\tau)\omega}\, dt$$

~~Let~~ $t-\tau = \beta \Rightarrow dt = d\beta$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} x_1(\tau)\, e^{-j\omega\tau}\, d\tau \cdot \int_{-\infty}^{\infty} x_2(\beta) \cdot e^{-j\omega\beta}\, d\beta$$

$$\boxed{X(\omega) = X_1(\omega) \cdot X_2(\omega)}$$

(10) Multiplication in Time:

$$x_1(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X_1(\omega)$$

$$x_2(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X_2(\omega)$$

$$x_1(t) \cdot x_2(t) = x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(\omega) = \frac{1}{2\pi}[X_1(\omega) * X_2(\omega)]$$

$$\therefore \; x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Here,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2\pi}\left[X_1(\omega) * X_2(\omega)\right] e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t) = \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x_1(\lambda)\, x_2(\omega-\lambda)\, d\lambda\right] e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1(\lambda)\, x_2(\omega-\lambda)\, \frac{e^{-jt\lambda}}{e^{-jt\lambda}} (d\lambda)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} x_1(\lambda)\, e^{+jt\lambda}\, d\lambda \int_{-\infty}^{\infty} x_2(\omega-\lambda)\, e^{+j(\omega-\lambda)t}\, d\omega$$

$$\text{Let } \omega - \lambda = \beta \Rightarrow d\omega = d\beta$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x_1(\lambda)\, e^{+jt\lambda}\, d\lambda \cdot \frac{1}{2\pi}\int_{-\infty}^{\infty} x_2(\beta)\, e^{jt\beta}\, d\beta$$

$$\boxed{x(t) = x_1(t) \cdot x_2(t)}$$

(11) Differentiation in Time:

$$x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(\omega)$$

$$\frac{d}{dt} x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} j\omega X(\omega)$$

we know that,

$$\therefore x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) \cdot j\omega \cdot e^{j\omega t}\, d\omega$$

$$\therefore \frac{dx(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[j\omega X(\omega) \right] \cdot e^{j\omega t}\, d\omega$$

$\Rightarrow$ Similarly,

$$\frac{d^k x(t)}{dt^k} \underset{IFT}{\overset{FT}{\rightleftharpoons}} (j\omega)^k X(\omega)$$

(12) Differentiation in Frequency:

$$x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(\omega)$$

$$-jt\, x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} \frac{d}{d\omega} X(\omega)$$

$$\therefore \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow \quad \frac{d(X(\omega))}{d\omega} = \int_{-\infty}^{\infty} x(t)\, -jt\, e^{-j\omega t}\, dt$$

$$\therefore \quad \frac{d\, X(\omega)}{d\omega} = \int_{-\infty}^{\infty} \left[-jt\, x(t)\right] e^{-j\omega t}\, dt$$

Similarly,

$$\Rightarrow \quad \frac{d^k X(\omega)}{d\omega^k} \underset{FT}{\overset{IFT}{\rightleftharpoons}} (-jt)^k x(t)$$

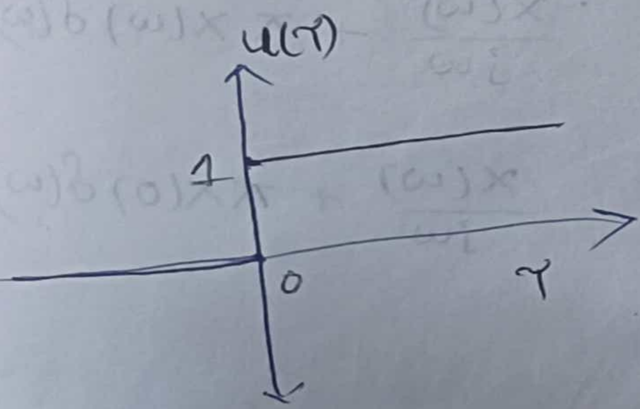
(13) Integration in Time:

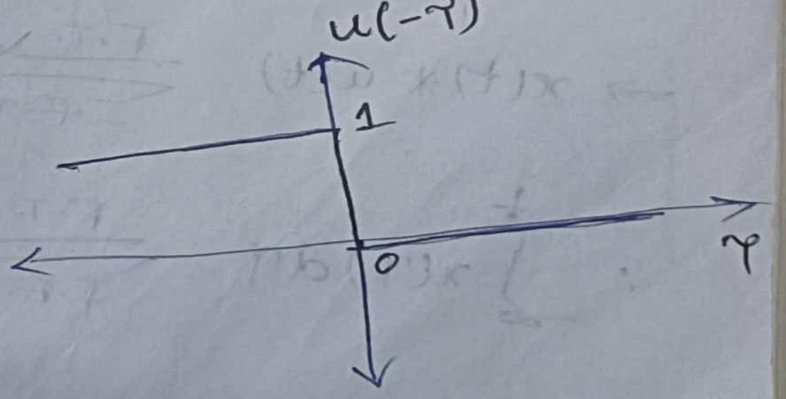
$$x(t) \underset{I.F.T}{\overset{F.T.}{\rightleftharpoons}} X(\omega)$$

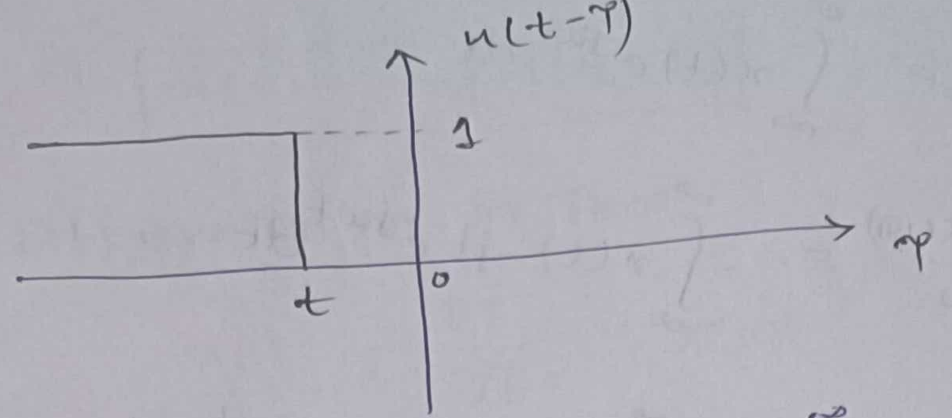
$$\int_{-\infty}^{t} x(\tau)\, d\tau \underset{IFT}{\overset{F.T.}{\rightleftharpoons}} \frac{X(\omega)}{j\omega} + \pi X(0)\, \delta(\omega)$$

we know that,

$$x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau) \cdot u(t-\tau)\, d\tau$$







$$\therefore x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau + \int_{t}^{\infty} x(\tau)\cdot 0 \cdot d\tau$$

$$\Rightarrow x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

$$\therefore \int_{-\infty}^{t} x(\tau)\, d\tau = x(t) * u(t)$$

$$\therefore u(t) * x(t) \underset{\text{I.F.T.}}{\overset{\text{F.T.}}{\rightleftharpoons}} X(\omega) \cdot FT\{u(t)\}$$

$$F.T.\{u(t)\} = \left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$\Rightarrow u(t) * x(t) \underset{\text{I.F.T}}{\overset{\text{F.T.}}{\rightleftharpoons}} X(\omega) \cdot \left[\frac{1}{j\omega} + \pi\delta(\omega)\right.$$

$$\Rightarrow x(t) * u(t) \underset{\text{I.F.T}}{\overset{\text{F.T.}}{\rightleftharpoons}} \frac{X(\omega)}{j\omega} + \pi X(\omega)\delta(\omega)$$

$$\therefore \int_{-\infty}^{t} x(\tau)\, d\tau \underset{\text{I.F.T.}}{\overset{\text{F.T.}}{\rightleftharpoons}} \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

(14) Modulation:

$$x(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} X(\omega)$$

$$x(t)\cos\omega_0 t \underset{IFT}{\overset{FT}{\rightleftharpoons}} \frac{X(\omega-\omega_0)+X(\omega+\omega_0)}{2}$$

But,

$$x(t)\cos\omega_0 t = x(t)\left[\frac{e^{j\omega_0 t}+e^{-j\omega_0 t}}{2}\right]$$

$$x(t)\cos\omega_0 t = \underbrace{\frac{x(t)e^{j\omega_0 t}}{2}}_{X_1'(\omega)} + \underbrace{\frac{x(t)e^{-j\omega_0 t}}{2}}_{X_2'(\omega)}$$

Now,

$$X_1'(\omega) = \int_{-\infty}^{\infty} \frac{x(t)e^{j\omega_0 t}}{2} e^{-j\omega t}\,dt$$

$$\Rightarrow X_1^1(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\,dt$$

$$\boxed{X_1'(\omega) = \frac{1}{2}X(\omega-\omega_0)}$$

Similarly,

$$\boxed{X_2'(\omega) = \frac{1}{2}X(\omega+\omega_0)}$$

$$\therefore X'(\omega) = X_1'(\omega) + X_2'(\omega)$$

$$\boxed{X'(\omega) = \frac{1}{2}\left[X(\omega-\omega_0)+X(\omega+\omega_0)\right]}$$

Also,

$$x(t)\sin\omega_0 t \underset{IFT}{\overset{FT}{\rightleftharpoons}} \frac{[X(\omega-\omega_0)-X(\omega+\omega_0)]}{2j}$$

(15) Parseval's Energy theorem:

$$E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$$

$$\therefore \quad E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t)\, x^*(t)\, dt \quad \text{——} \ (1)$$

But,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \quad \text{——} \ (2)$$

$$x^*(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \quad \text{——} \ (3)$$

sub eqn (2) (or) eqn (3) in eqn (1)

$$\Rightarrow E_{x(t)} = \int_{-\infty}^{\infty} x(t) \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) e^{-j\omega t} d\omega \, dt$$

$$\Rightarrow E_{x(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t) \cdot e^{-j\omega t} dt \right] d\omega$$

$$\Rightarrow E_{x(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot X(\omega) \, d\omega$$

$$\boxed{E_{x(t)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega}$$

(16) Duality property:

$$x(t) \xrightarrow[FT]{} X(\omega)$$

replacing by 't'

$$X(t) \xrightarrow[FT]{} 2\pi x(-\omega)$$

$$\therefore x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

Let $t = -t$

$$\Rightarrow x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

$$\Rightarrow 2\pi x(-t) = \int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega$$

$\omega \to t$ and $t \to \omega$

$$\boxed{\therefore 2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t)\, e^{-j\omega t}\, dt}$$

* Fourier transformation of basic signals:-

(1) DC value

(2) Impulse signals

(3) Exponential signals.

(4) Signum function

(5) step signal

(6) complex exponential function signals.

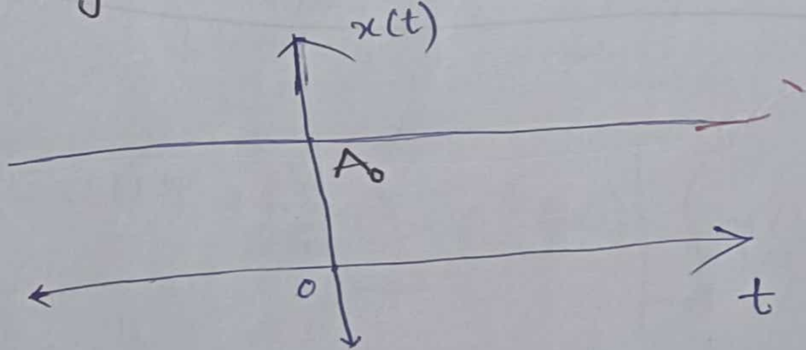
(7) Cos and Sin functions.

(8) Rectangular function

(9) Triangular function

(10) Sinc function

① DC signal:



$$A_0 \underset{IFT}{\overset{FT}{\rightleftharpoons}} 2\pi A_0 \delta(\omega)$$

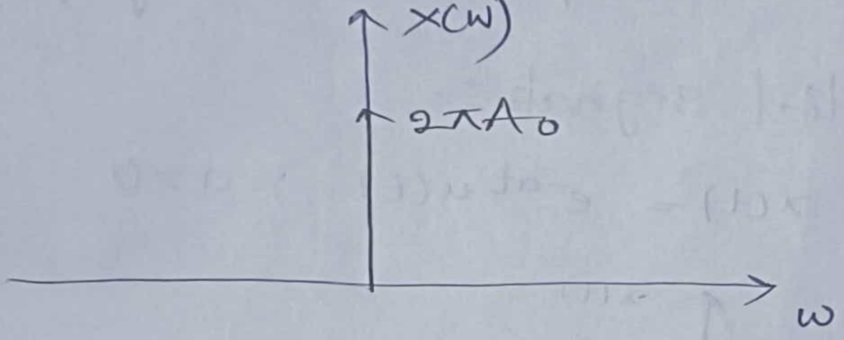
$$\therefore \quad x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi A_0\,\delta(\omega)\cdot e^{j\omega t}\,d\omega$$

$$\Rightarrow x(t) = A_0\int_{-\infty}^{\infty} \delta(\omega)\,e^{j\omega t}\,d\omega$$

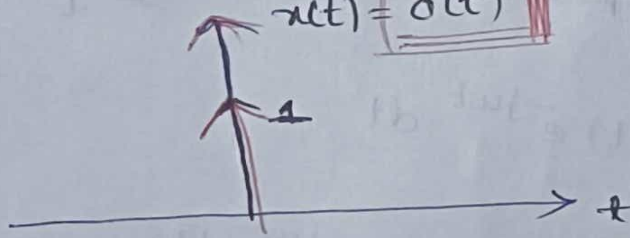
But $\int_{-\infty}^{\infty} x(t)\,\delta(t-t_0)\,dt = x(t_0)$

$$\boxed{\therefore\ x(t) = A_0}$$



② Impulse signal:

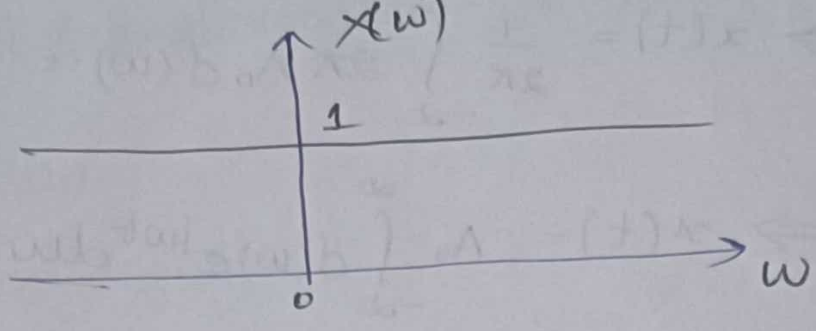
$$\boxed{x(t) = \delta(t)}$$



$$\therefore X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} \delta(t)\cdot e^{-j\omega t}\,dt$$

$$\boxed{X(\omega) = 1}$$



Note:

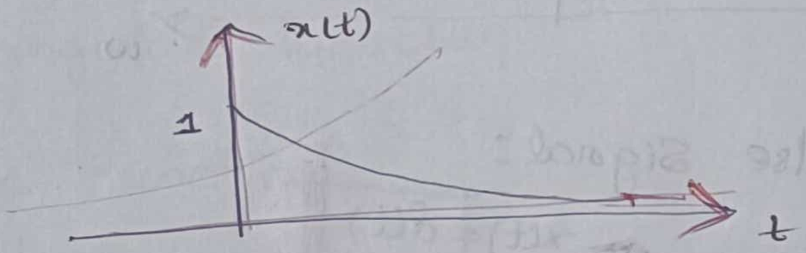
$$\delta(t) \xrightarrow{FT} 1$$

$$1 \xrightarrow{FT} 2\pi\delta(\omega)$$

⇒ A simple example of duality property.

③ Exponential signals :-

(i) $x(t) = e^{-at}u(t)$; $a>0$



$$\therefore \quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} e^{-at}u(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{0}^{\infty} e^{-at}\, e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t}\, dt$$

$$\Rightarrow X(\omega) = \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_{0}^{\infty}$$

$$\Rightarrow X(\omega) = \frac{1}{-(a+j\omega)}\left[0 - 1\right]$$

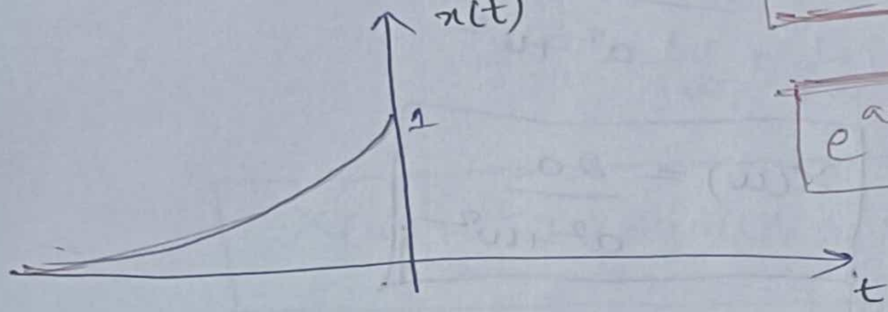
$$\boxed{X(\omega) = \frac{1}{a+j\omega}}$$

$$e^{-at}u(t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} \frac{1}{a+j\omega}$$

(ii) $x(t) = e^{at}.u(-t)\ ;\ a>0$

$\boxed{e^{at}.u(-t);\ a>0}$

$\boxed{e^{at}\ u(-t)\ a>0}$



$$\therefore X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} e^{at}\, u(-t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{0} e^{at} e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{0} e^{(a-j\omega)t}\, dt$$

$$\Rightarrow X(\omega) = \left.\frac{e^{(a-j\omega)t}}{(a-j\omega)}\right]_{-\infty}^{0}$$

$$\boxed{\therefore X(\omega) = \frac{1}{a-j\omega}}$$

$$e^{at}u(-t) \underset{IFT}{\overset{FT}{\rightleftharpoons}} \frac{1}{a-j\omega}$$

(iii) $x(t) = e^{-a|t|}, \quad a > 0$

$$x(t) = \begin{cases} e^{at} & ; t < 0 \\ e^{-at} & ; t > 0 \end{cases}$$

$\therefore \; x(t) = e^{at}u(-t) + e^{-at}u(t)$

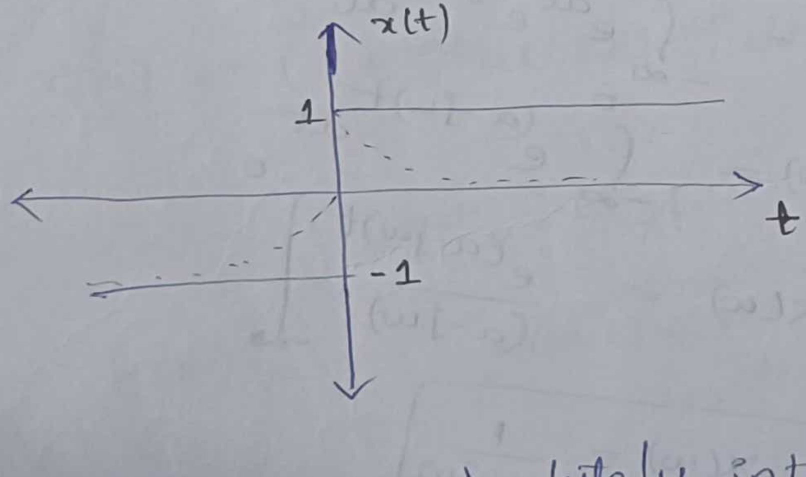
$\Rightarrow X(\omega) = \frac{1}{a-j\omega} + \frac{1}{a+j\omega}$

$\Rightarrow X(\omega) = \frac{(a+j\omega) + (a-j\omega)}{a^2+\omega^2}$

$$\boxed{X(\omega) = \frac{2a}{a^2+\omega^2}}$$

(4) Signum function:

$$x(t) = \operatorname{sgn}(t) = \begin{cases} -1 & ; t < 0 \\ 0 & ; t = 0 \\ 1 & ; t > 0 \end{cases}$$



It is not absolutely integrable.

$\therefore x(t) = \mathrm{sgn}(t) = u(t) - u(-t)$

$$u(t) = \lim_{a\to 0} e^{-at} u(t)$$

$$u(-t) = \lim_{a\to 0} e^{at} u(-t)$$

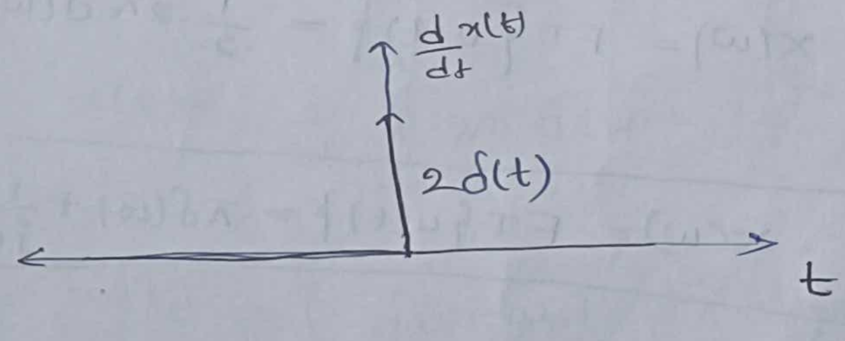
$$\therefore x(t) = \mathrm{sgn}(t) = \lim_{a\to 0} e^{-at} u(t) - \lim_{a\to 0} e^{at} u(-t)$$

$$X(\omega) = FT\{\mathrm{sgn}(t)\} = \lim_{a\to 0} \frac{1}{a+j\omega} - \lim_{a\to 0} \frac{1}{a-j\omega}$$

$$= \frac{1}{j\omega} + \frac{1}{j\omega}$$

$$\boxed{X(\omega) = FT\{\mathrm{sgn}(t)\} = \frac{2}{j\omega}}$$

Also,



$$\frac{d}{dt} x(t) = 2\delta(t)$$

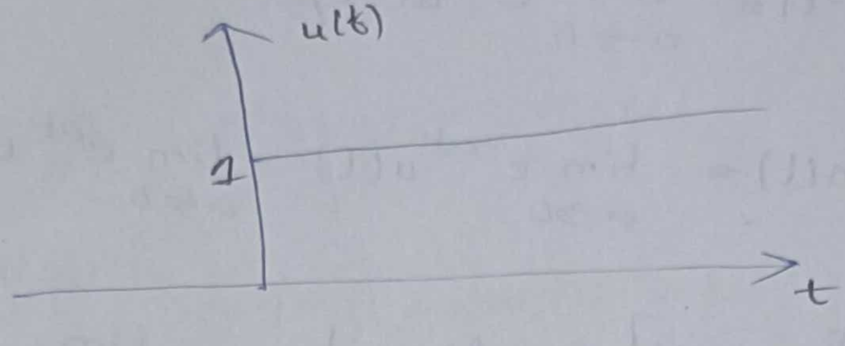
$$\downarrow FT$$

$$\Rightarrow j\omega X(\omega) = 2(1)$$

$$\boxed{X(\omega) = \frac{2}{j\omega}}$$

⑤ step signal:

$$x(t) = u(t) = \begin{cases} 0 ; & t<0 \\ 1 ; & t>0 \end{cases}$$



It is not absolutely integrable.

$$\therefore x(t) = u(t) = \frac{1+\text{sgn}(t)}{2}$$

$$\Rightarrow x(t) = u(t) = \frac{1}{2} + \frac{\text{sgn}(t)}{2}$$

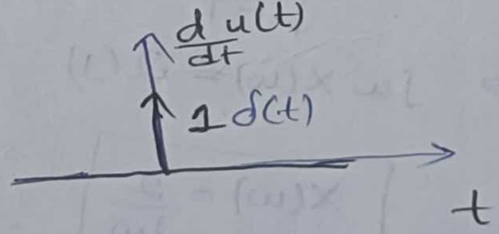
$$\downarrow \text{FT}$$

$$\Rightarrow X(\omega) = \text{FT}\{u(t)\} = \frac{1}{\cancel{2}} \cdot \cancel{2}\pi\,\delta(\omega) + \frac{\cancel{2}}{j\omega} \cdot \frac{1}{\cancel{2}}$$

$$\boxed{\therefore X(\omega) = \text{FT}\{u(t)\} = \pi\delta(\omega) + \frac{1}{j\omega}}$$

Note: Dc value should be considered always.

Eg:



$$\therefore \frac{d}{dt}u(t) = \delta(t)$$

$$\Rightarrow (j\omega)X(\omega) = 1$$

$$\boxed{X(\omega) = \frac{1}{j\omega}} \rightarrow \text{Dc value is missing}$$

DC value $= \frac{1}{2}$

$\Rightarrow$ FT $= \frac{1}{2} \cdot 2\pi\delta(\omega)$

$\Rightarrow$ FT $= \pi\delta(\omega)$

$$\boxed{\therefore\ X(\omega) = \pi\delta(\omega) + \frac{1}{j\omega}}$$

⑥ Complex exponential signal:

$$e^{j\omega_0 t} \underset{IFT}{\overset{F.T.}{\rightleftharpoons}} 2\pi\delta(\omega-\omega_0)$$

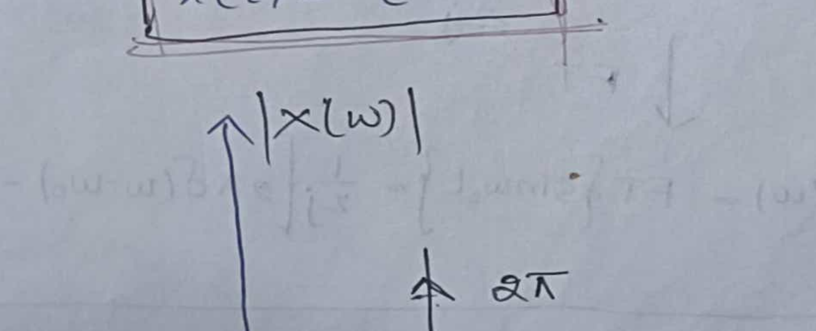
$$e^{-j\omega_0 t} \underset{IFT}{\overset{F.T.}{\rightleftharpoons}} 2\pi\delta(\omega+\omega_0)$$

$$\therefore\ x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow\ x(t) = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi}\,\delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

$$\Rightarrow\ x(t) = \int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega$$

$$\boxed{x(t) = e^{j\omega_0 t}}$$



⑦ Fourier Transform of $\cos\omega_0 t$:

ⓐ

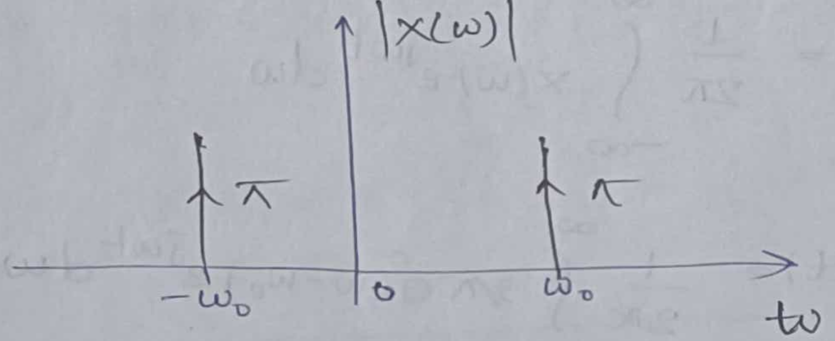
$\therefore x(t) = \cos\omega_0 t$

$$\Rightarrow x(t) = \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$$

$\Big\downarrow$ FT

$$\Rightarrow X(\omega) = FT\{\cos\omega_0 t\} = \frac{1}{2}\left[2\pi\delta(\omega-\omega_0) + 2\pi\delta(\omega+\omega_0)\right]$$

$$\boxed{\therefore X(\omega) = FT\{\cos\omega_0 t\} = \pi\left[\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\right]}$$



ⓑ Fourier transform of $\sin\omega_0 t$.

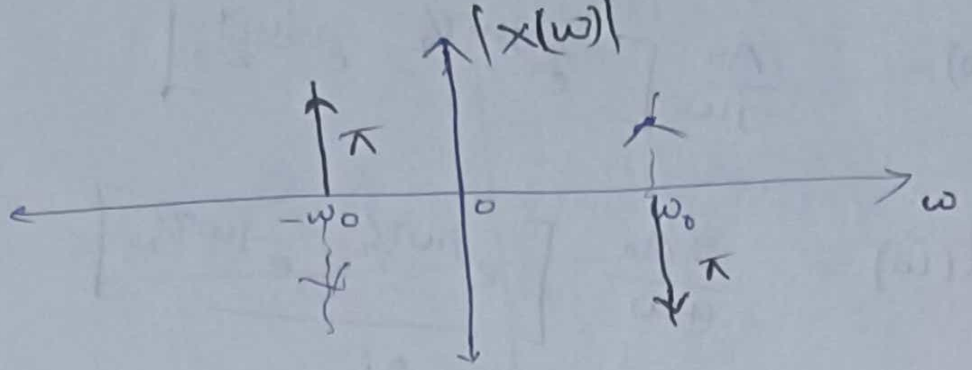
$\therefore x(t) = \sin\omega_0 t$

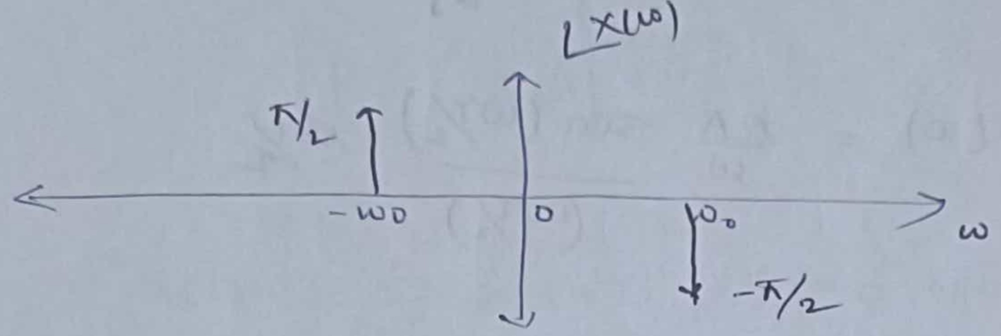
$$\Rightarrow x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}$$

$\Big\downarrow$ FT

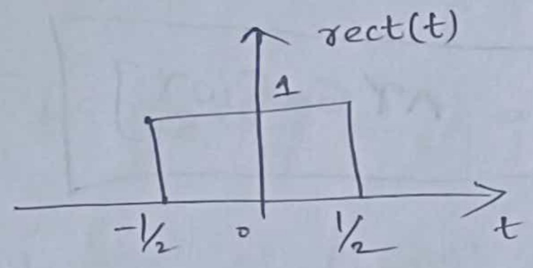
$$\Rightarrow X(\omega) = FT\{\sin\omega_0 t\} = \frac{1}{2j}\left[2\pi\delta(\omega-\omega_0) - 2\pi\delta(\omega+\omega_0)\right]$$

$$\boxed{X(\omega) = FT\{\sin\omega_0 t\} = \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]}$$

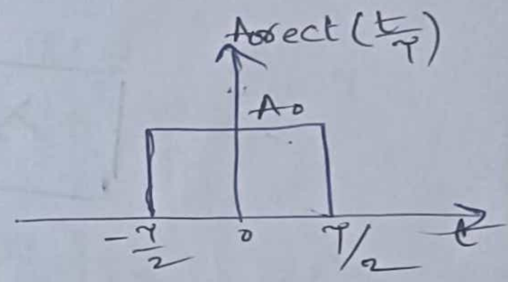




8) Fourier transform of Rectangular function:



$\Rightarrow$



$$x(t) = A_0 \, \text{rect}\left(\frac{t}{\tau}\right) \underset{IFT}{\overset{FT}{\rightleftharpoons}} A_0 \tau \, Sa\left[\frac{\omega\tau}{2}\right]$$

$$\therefore X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$\Rightarrow X(\omega) = \int_{-\tau/2}^{\tau/2} A_0 e^{-j\omega t} dt$$

$$\Rightarrow X(\omega) = \left. \frac{A_0 e^{-j\omega t}}{-j\omega} \right|_{-\tau/2}^{\tau/2}$$

$$\Rightarrow X(\omega) = \frac{A_0}{-j\omega}\left[e^{-j\omega\tau/2} - e^{+\frac{j\omega\tau}{2}} \right]$$

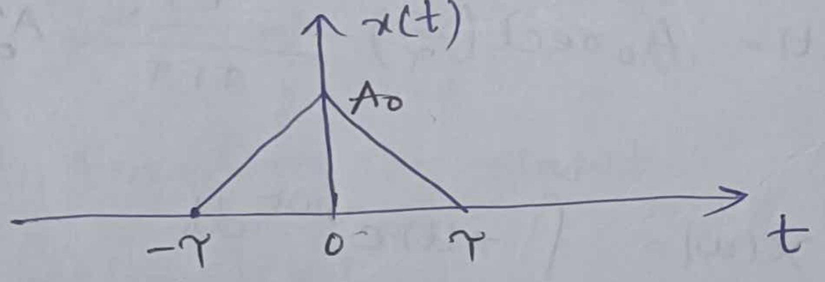
$$\Rightarrow X(\omega) = \frac{2A_0}{(+)\omega}\left[\frac{e^{j\omega\tau/2} - e^{-j\omega\tau/2}}{2j} \right]$$

$$\Rightarrow X(\omega) = \frac{2A}{\omega} \frac{\sin(\omega\tau/2)}{(\omega\tau/2)} \omega\tau/2$$

$$\Rightarrow X(\omega) = A\tau \frac{\sin(\omega\tau/2)}{(\omega\tau/2)}$$

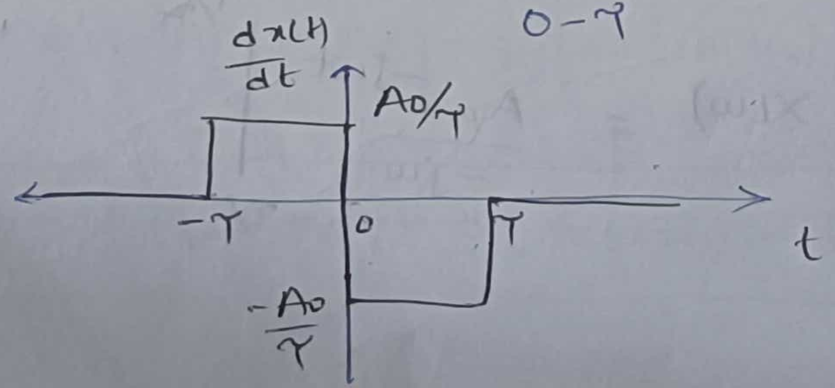
$$\boxed{X(\omega) = A\tau \, Sa\left[\frac{\omega\tau}{2}\right]}$$

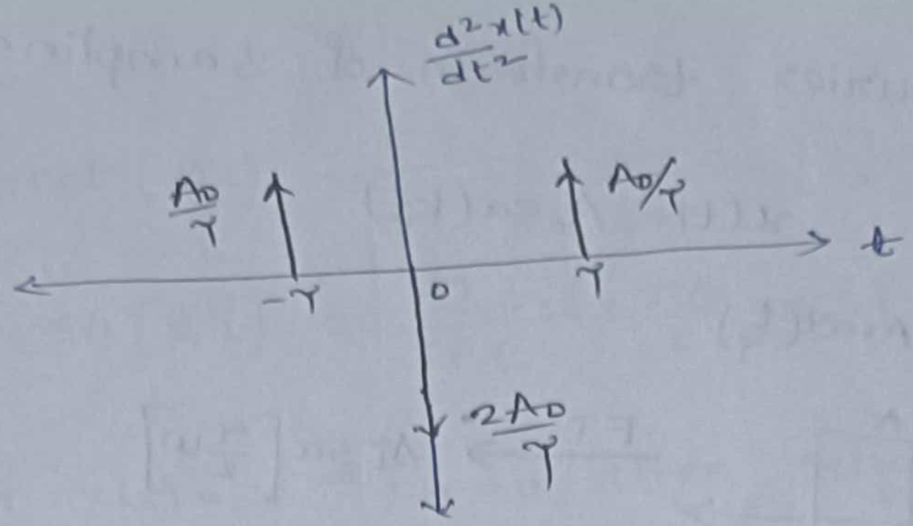
⑨ Fourier transforms of Triangular function:



For $-\tau < t < 0$, slope $= \frac{A_0 - 0}{0 + \tau} = \frac{A_0}{\tau}$

For $0 < t < \tau$, slope $= \frac{A_0 - 0}{0 - \tau} = -\frac{A_0}{\tau}$





$$\therefore \frac{d^2}{dt^2}x(t) = \frac{A_0}{\tau}\delta(t+\tau) - \frac{2A_0}{\tau}\delta(t) + \frac{A_0}{\tau}\delta(t-\tau)$$

$$\Big\downarrow \text{F.T.}$$

$$\Rightarrow (j\omega)^2 X(\omega) = \frac{A_0}{\tau}e^{j\omega\tau} - \frac{2A_0}{\tau} + \frac{A_0}{\tau}e^{-j\omega\tau}$$

$$\Rightarrow -\omega^2 X(\omega) = \frac{A_0}{\tau}\left(\frac{e^{j\omega\tau}+e^{-j\omega\tau}}{2}\right)\times 2 - \frac{2A_0}{\tau}$$

$$\Rightarrow -\omega^2 X(\omega) = \frac{2A_0}{\tau}\left[\cos\omega\tau - 1\right]$$

$$\Rightarrow X(\omega) = \frac{2A_0}{\omega^2\tau}(1-\cos\omega\tau)$$

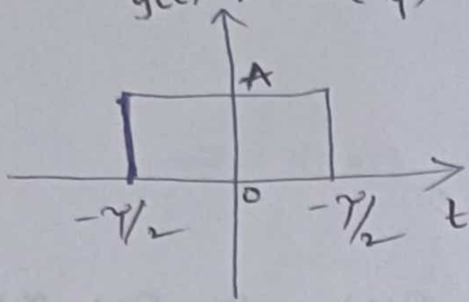
$$\Rightarrow X(\omega) = \frac{\cancel{2}A_0}{\omega^2\tau}\frac{\left[\cancel{2}\sin^2(\omega\tau/\cancel{2})\right]\times\frac{(\omega\tau)^2}{4}}{\left(\frac{\omega\tau}{2}\right)^2}$$

$$\boxed{X(\omega) = A_0\tau\, Sa^2\left(\frac{\omega\tau}{2}\right)}$$

(10) Fourier transform of sampling function

$$x(t) = A_0\, sa(Kt)$$

$y(t) = A\,rect\left(\frac{t}{\tau}\right)$



$$\xrightarrow{FT} A\tau\, sa\left[\frac{\tau}{2}\omega\right]$$

$$\Big\downarrow \text{ 'ω' by 't'}$$

$$A\tau\, sa\left[\frac{\tau}{2}t\right] \xrightarrow{F.T.} 2\pi y(-\omega)$$

on comparing,

$$A\tau = A_0 \quad \text{and} \quad \frac{\tau}{2} = K$$

$$\boxed{\therefore \frac{A_0}{2K} = A} \quad \& \quad \Rightarrow \boxed{\tau = 2K}$$

so, we have,

$$\therefore A\tau\, sa\left[\frac{\tau}{2}t\right] \xrightarrow{FT} 2\pi A\, rect\left(-\frac{\omega}{\tau}\right)$$

$$\therefore A_0\, sa[kt] \longrightarrow \frac{\cancel{2}\pi A_0}{\cancel{2}K} rect\left(\frac{-\omega}{2K}\right)$$

$$\boxed{A_0\, sa[Kt) \xrightarrow{F.T.} \frac{\pi A_0}{K} rect\left(\frac{\omega}{2K}\right)}$$

Note:

$$A\,\mathrm{rect}\left(\frac{t}{\tau}\right) \underset{IFT}{\overset{F.T}{\rightleftharpoons}} A\tau\, sa\left[\frac{\tau}{2}\omega\right]$$

$$A_0\, sa[Kt] \underset{IFT}{\overset{FT}{\rightleftharpoons}} \frac{\pi A_0}{K}\,\mathrm{rect}\left(\frac{\omega}{2K}\right)$$

Eg: If $x(t) = e^{-at^2}$; $a > 0$, then $X(\omega) = ?$

Sol:
$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$$

$$\therefore\quad X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} e^{-at^2} e^{-j\omega t}\, dt$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} e^{-(at^2 + j\omega t)}\, dt$$

But, $at^2 + j\omega t = (\sqrt{a}t)^2 + 2(\sqrt{a}t)\dfrac{j\omega t}{2(\sqrt{a}t)}$

$$= (\sqrt{a}t)^2 + 2(\sqrt{a}t)\frac{j\omega t}{2\sqrt{a}t}$$

$$= (\sqrt{a}t)^2 + 2(\sqrt{a}t)\left(\frac{j\omega}{2\sqrt{a}}\right) + \frac{\omega^2}{4a} - \frac{\omega^2}{4a}$$

$$at^2 + j\omega t \Rightarrow \left(\sqrt{a}t + \frac{j\omega}{2\sqrt{a}}\right)^2 + \frac{\omega^2}{4a}$$

$$\Rightarrow X(\omega) = \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}t + \frac{j\omega}{2\sqrt{a}}\right)^2} e^{-\frac{\omega^2}{4a}}\, dt$$

$$\Rightarrow X(\omega) = e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-\left(\sqrt{a}t + \frac{j\omega}{2\sqrt{a}}\right)^2} dt$$

Let $\sqrt{a}t + \frac{j\omega}{2\sqrt{a}} = u$

$$\Rightarrow \sqrt{a}\, dt = du$$

$$\Rightarrow X(\omega) = e^{-\omega^2/4a} \int_{-\infty}^{\infty} e^{-u^2} \frac{du}{\sqrt{a}}$$

$$\Rightarrow X(\omega) = \frac{e^{-\omega^2/4a}}{\sqrt{a}} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$\boxed{X(\omega) = \sqrt{\pi/a}\; e^{-\omega^2/4a}}$$

* If fourier transform of $x(t)$ is $X(\omega)$, then find F.T. of $y(t) = x(2t-3)$?

Sol:- $y(t) = x(2(t - 3/2))$

$$\Rightarrow Y(\omega) = e^{-j\omega 3/2} \cdot \frac{1}{2} X(\omega/2)$$

$$\boxed{Y(\omega) = \frac{1}{2} e^{-j\omega 3/2} X(\omega/2)}$$

* If $x(t) = \frac{1}{a+jt}$, then $X(\omega) = ?$

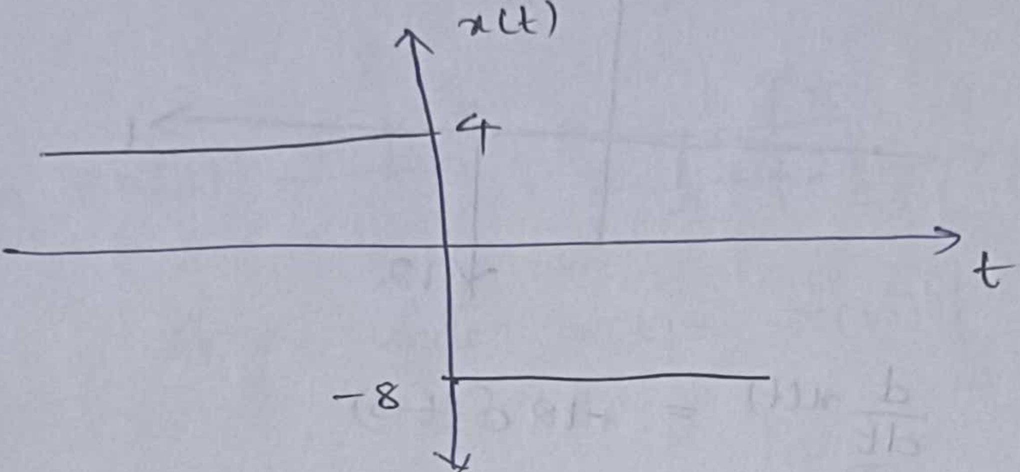
Sol:- $y(t) = e^{-at} u(t)$ (Say)

$e^{-at}\cdot u(t) \xrightarrow{F.T.} \frac{1}{a+j\omega}$

$\downarrow$ 'ω by t'

$\frac{1}{a+jt} \xrightarrow{F.T} 2\pi e^{a\omega} u(-\omega)$

*



Sol:

$$x(t) = 4u(-t) - 8u(t)$$

$$u(t) \longrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

$$u(-t) \longrightarrow \pi\delta(-\omega) - \frac{1}{j\omega}$$

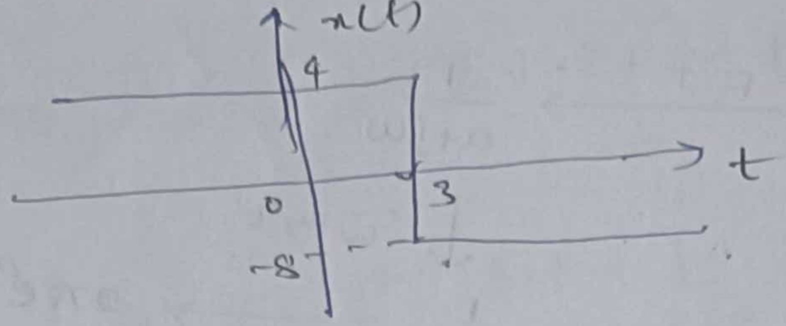
$$x(t) \xrightarrow{F.T.} X(\omega)$$

$$X(\omega) = 4\pi\delta(-\omega) - \frac{4}{j\omega} - 8\pi\delta(+\omega) - \frac{8}{j\omega}$$

$$\boxed{\therefore X(\omega) = 4\pi\delta(\omega) - 8\pi\delta(-\omega) + \frac{12}{j\omega}}$$

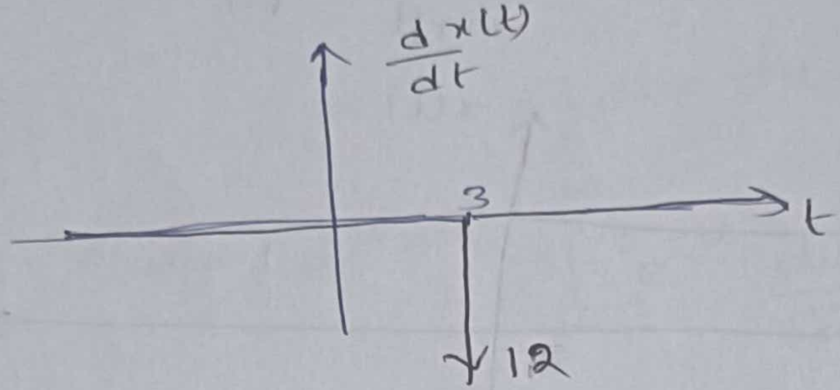
(or)

$$X(\omega) = -4\pi\delta(\omega) - \frac{12}{j\omega}$$

* 

$X(\omega) = ?$

Sol:

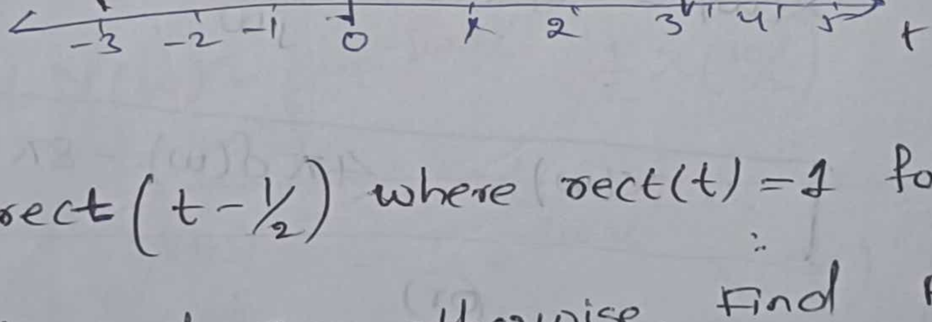


$$\frac{d}{dt} x(t) = -12\,\delta(t-3)$$

$$\therefore \; j\omega X(\omega) = -12 e^{-3j\omega}$$

$$\therefore \; X(\omega) = \frac{-12 e^{-3j\omega}}{j\omega} - 4\pi\delta(\omega)$$

(DC value)

* 

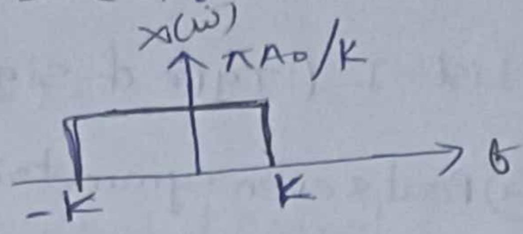
$X(\omega) = ?$

* $x(t) = \text{rect}\left(t - \frac{1}{2}\right)$ where $\text{rect}(t) = 1$ for $-\frac{1}{2} \le t \le \frac{1}{2}$ and zero otherwise. Find F.T. of $x(t) + x(-t)$ is: ________

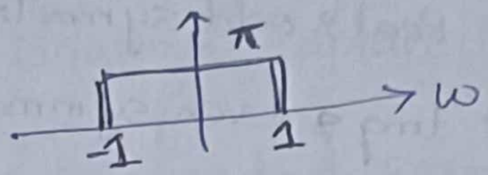
* $x(t) = u(2t) \longrightarrow X(\omega) = ?$ [Hint $\delta(at) = \frac{1}{|a|}\delta(t)$]

* Calculate area of $x(t) = Sa(t)$?

$A_0 Sa(kt) \xrightarrow{F.T.}$



Then $Sa(t) \xrightarrow{F.T.}$



Area under $x(t) = X(\omega)\Big|_{\omega=0}$

Area under $x(t) = \pi$ //

(or)

$$E_{x(t)} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \frac{1}{2\pi}\int_{-1}^{1} \frac{\pi^2}{\cancel{2\pi}} d\omega = \frac{1}{\cancel{2}\pi} \cdot \frac{\pi^2}{\cancel{2\pi}} \cdot \cancel{2}$$

$$\boxed{E_{x(t)} = \pi \, J}$$

* $y(t) = x(t)\cos t$; $\quad Y(\omega) = \begin{cases} 2 & ; |\omega| \le 2 \\ 0 & ; \text{otherwise} \end{cases}$

then $x(t) = ?$

* The Fourier transform of $h(t)$ is $H(j\omega) = \frac{2\cos\omega \sin 2\omega}{\omega}$. The value of $h(0)$ is — ?

* Match list 1 with list 2 and select the correct answer.

List-I (Type of signal)

ⓐ Real & even symmetric

ⓑ Real & odd symmetric

ⓒ Img & even symmetry

ⓓ Img & odd symmetry

List-II (Property of FT)

① Img & even symmetry

② real & even symmetr

③ real & odd symmetr

④ Img & odd symmetr

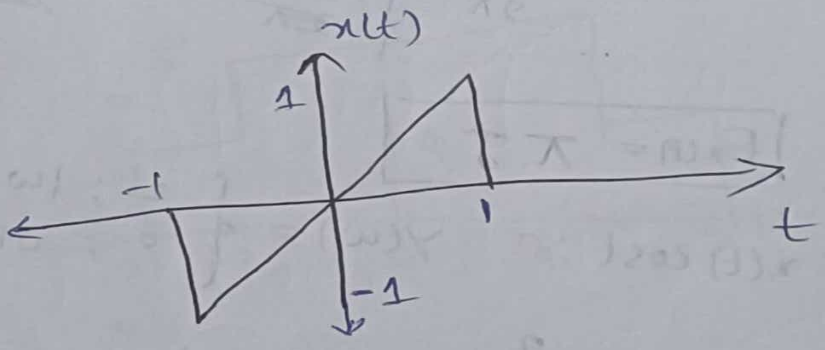
* The signal $x(t)$ is described by

$$x(t) = \begin{cases} 1 & ; -1 \le t \le 1 \\ 0 & ; \text{otherwise} \end{cases}$$

Two of the angular frequencies at which F.T. of $x(t)$ becomes zero.

ⓐ $\pi, 2\pi$ ⓑ $0, \pi$ ⓒ $0.5\pi, 1.5\pi$ ⓓ $2\pi, 2.5\pi$

*



$\rightarrow X(\omega) = ?$

Introduction to Laplace Transforms

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s)$$

$\therefore$ $\boxed{X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt}$ → Two sided signal / Bilateral Laplace transform

$\therefore$ $\boxed{X(s) = \int_{0}^{\infty} x(t) e^{-st} dt}$ → One sided signal / Unilateral Laplace transform

$$\boxed{x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} ds}$$

where, σ = Damping factor (stability)

ω = angular frequency (rad/sec)

* Relation between Laplace Transform and Fourier transform:

$$x(t) \underset{I.L.T}{\overset{L.T.}{\rightleftharpoons}} X(s)$$

$$x(t) \underset{I.F.T}{\overset{F.T.}{\rightleftharpoons}} X(\omega)$$

$$\therefore X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-(\sigma + j\omega)t} dt$$

$$\Rightarrow X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-\sigma t} e^{-j\omega t}\, dt$$

$$\boxed{\therefore X(S) = FT\{x(t) e^{-\sigma t}\}}$$

if $\sigma = 0$,

$$FT\{x(t)\} = X(\omega) = X(s)$$

$$\boxed{\therefore LT\{x(t)\} = FT\{x(t) e^{-\sigma t}\}}$$

* Condition for existance of L.T:

$$LT\{x(t)\} = FT\{x(t) e^{-\sigma t}\}$$

$$\because Y(\omega) = \int_{-\infty}^{\infty} y(t) e^{-j\omega t}\, dt$$

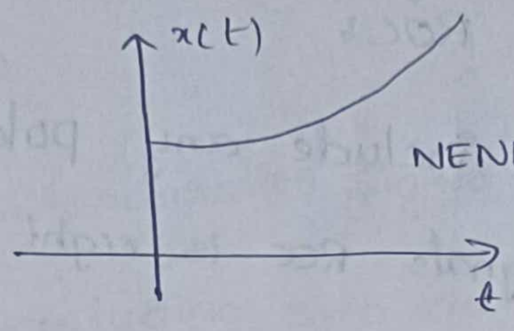
$$\therefore Y(\omega) = \int_{-\infty}^{\infty} [x(t) e^{-\sigma t}] e^{-j\omega t}\, dt$$

$\Rightarrow Y(\omega) = \int_{-\infty}^{\infty} x(t) e^{-\sigma t} e^{-j\omega t}\, dt$ should be absuletly integrable, which indirectly depends on $y(t) = x(t) e^{-\sigma t}$

$$\boxed{\therefore \int_{-\infty}^{\infty} |y(t)|\, dt < \infty}$$

$$\therefore \int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| \, dt < \infty$$ for some range of values of σ. (ie. ROC [Range of Convergance])

Eg: $x(t) = e^{3t} u(t)$



$$\therefore \int_{-\infty}^{\infty} |x(t) e^{-\sigma t}| \, dt < \infty$$

$$\Rightarrow \int_{-\infty}^{\infty} |e^{3t} u(t) e^{-\sigma t}| \, dt = \int_{0}^{\infty} e^{(3-\sigma)t} \, dt$$

$$= \frac{1}{3-\sigma} e^{(3-\sigma)t} \Big|_{0}^{\infty}$$

$$= \frac{1}{3-\sigma} \left[e^{(3-\sigma)\infty} - e^{0} \right]$$

$$= \frac{1}{3-\sigma} \left[e^{(3-\sigma)\infty} - 1 \right]$$

$$\therefore \int_{-\infty}^{\infty} |e^{3t} u(t) e^{-\sigma t}| \, dt = \frac{-1}{3-\sigma} \quad \forall \; \sigma > 3$$

* Region of convergence (ROC):

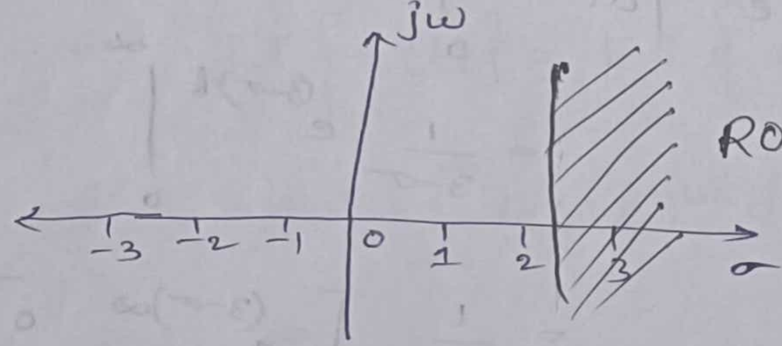
$\Rightarrow$ It is the range of complex variable 's' in s-plane for which Laplace transform is finite (or) convergent.

* Properties of ROC:

(1) ROC does not include any poles.

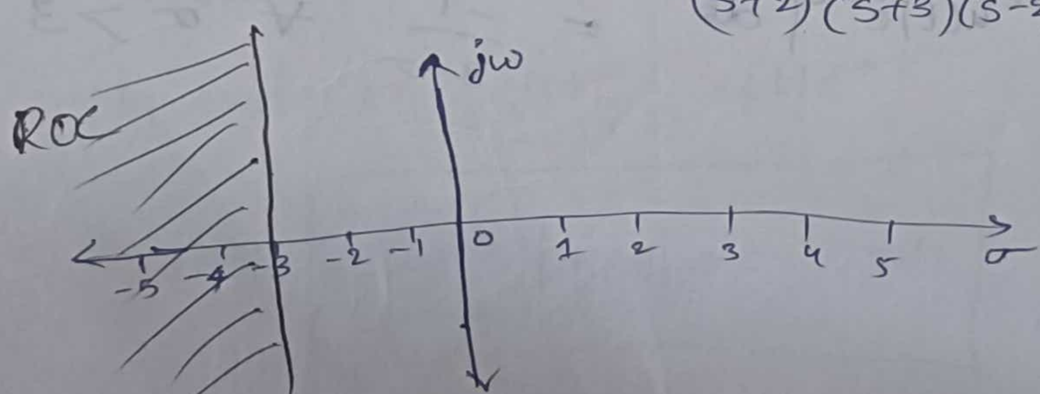
(2) Right sided signals ROC is right side to the right most pole.

If $X(s) = \dfrac{1}{(s+2)(s+3)(s-2)}$



(3) For left sided signals, ROC is left side to the left most pole.

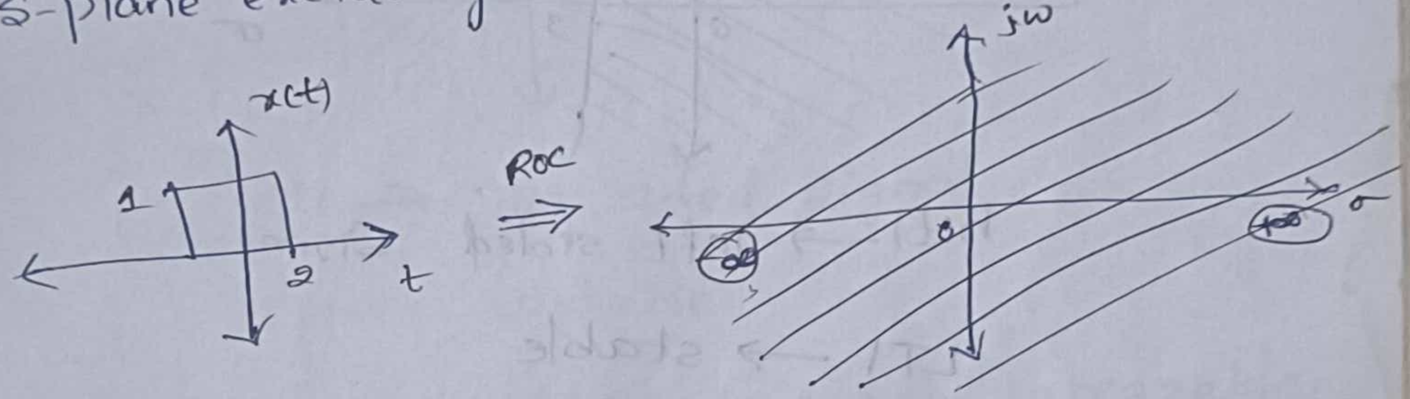
Let $X(s) = \dfrac{1}{(s+2)(s+3)(s-2)}$



(4) For the absolutely integrability of a signal (or) the stability of the system, ROC should include imaginary axis.

(5) For both sided signal, ROC is a strip in the s-plane.

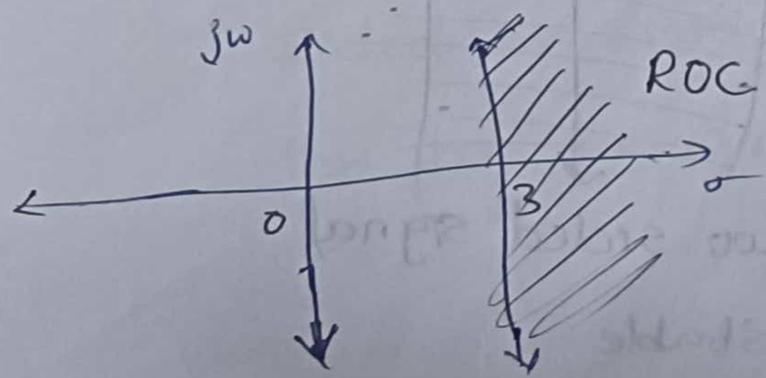
(6) For finite duration signal, ROC is the entire s-plane excluding $s=0$ and $+\infty$ and $-\infty$ (check at $s=0,\infty,-\infty$)



* check the stability of LTI system and comment about the extension of h(t)?

Sol:- (i) $h(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} H(s)$

where $\delta(t) \longrightarrow \boxed{LTI} \longrightarrow h(t)$

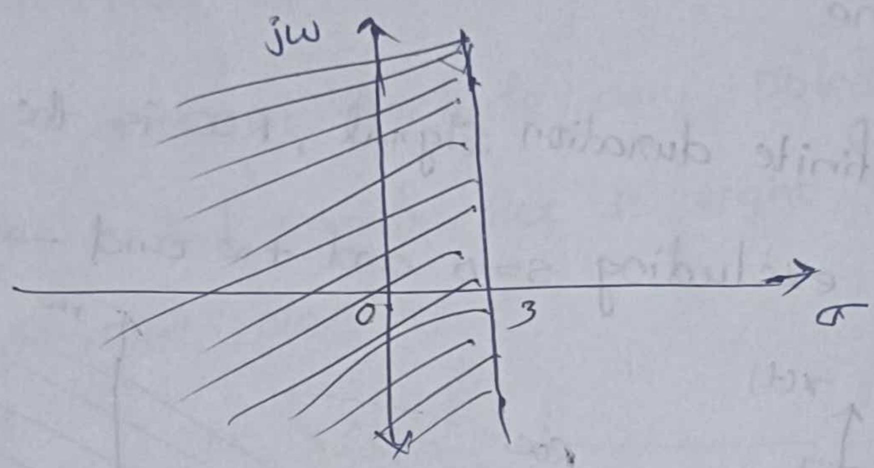


$h(t) \longrightarrow$ Right sided signal

LTI $\longrightarrow$ unstable

$h(t) \longrightarrow$ not absolutely integrable

(ii) $h(t) \overset{LT}{\rightleftharpoons} H(s)$; $\sigma < 1$



$h(t) \longrightarrow$ left sided signal

LTI $\longrightarrow$ stable

$h(t) \longrightarrow$ ~~But~~ absolutely integrable

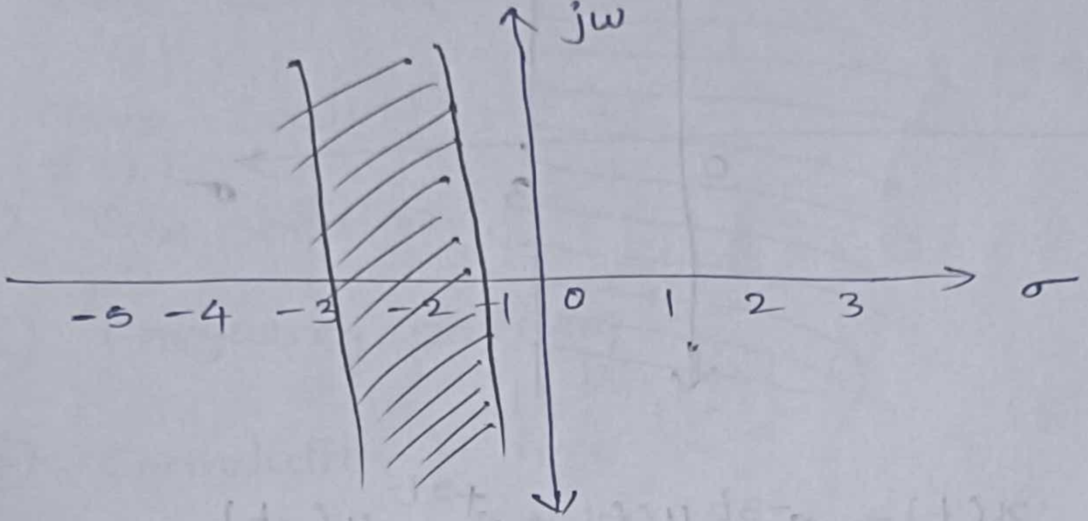
(iii) $h(t) \overset{LT}{\rightleftharpoons} H(s)$; $-3 < \sigma < 2$

$h(t) \longrightarrow$ two sided signal

LTI $\longrightarrow$ stable

$h(t) \rightarrow$ It is absolutely integrable.

(iv) $h(t) \xrightleftharpoons{LT} H(s)$, $-3 < \sigma < -1$



$h(t) \rightarrow$ Two sided signal

LTI $\rightarrow$ Unstable

$h(t) \rightarrow$ Not absolutely integrable.

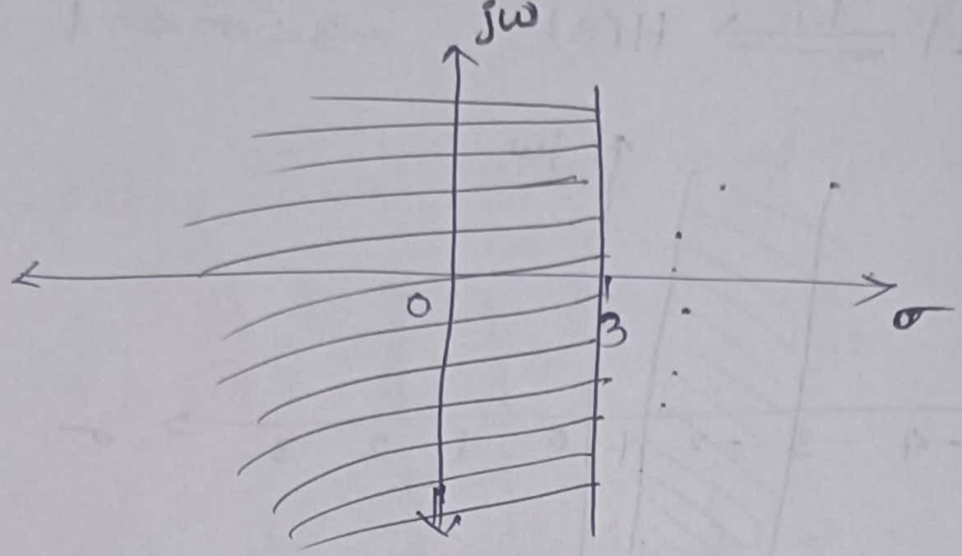
* Short cut for ROC:

step-1: Compare 'σ' with the real part of the coefficient of 't' in power 'e'.

step-2: check if the signal is left sided or Right sided and decide less than (or) greater than.

eg-1: $x(t) = e^{(3+2j)t}\, u(-t-3)$

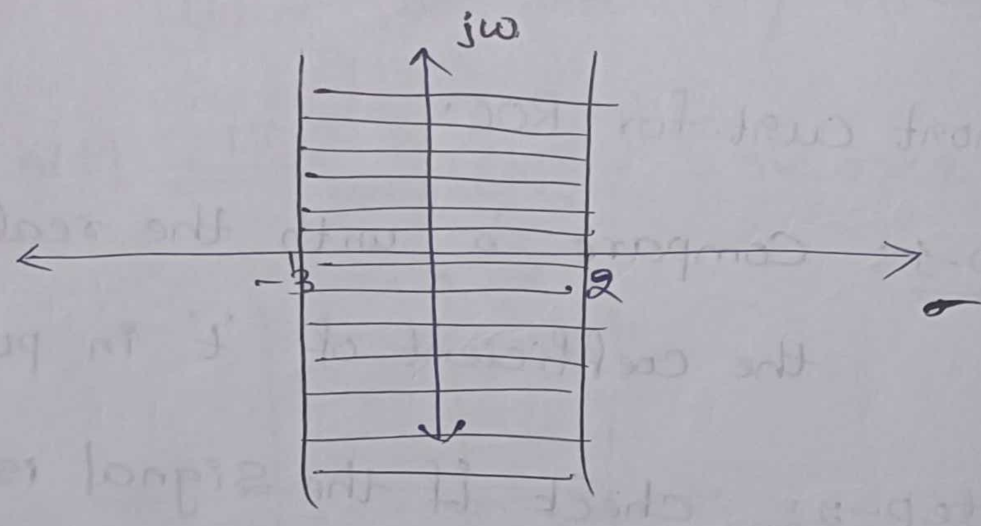
Sol: It is left-sided signal. So, $\sigma < 3$



Eg-2: $x(t) = e^{-3t} u(t) + e^{+2t} u(-t)$

Sol: Right sided + left sided

$\sigma > -3 \quad \cap \quad \sigma < 2$



eg-3: $x(t) = e^{+3t} u(t) + e^{-2t} u(-t)$

Sol: $\sigma > +3$ and $\sigma < -2$

L.T. does not exist.

* properties of Laplace Transforms:

1. Linearity
2. Conjugation
3. Time Reversal
4. Time Scaling
5. Time shifting
6. Frequency shifting
7. Convolution in Time
8. Multiplication in Time
9. Differentiation in Time
10. Integration in Time
11. Differentiation in frequency
12. Integration in frequency
13. Initial value theorem
14. Final value theorem.

① Linearity:

$$x_1(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X_1(s) \; ; \; ROC = R_1$$

$$x_2(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X_2(s) \; ; \; ROC = R_2$$

$$\alpha x_1(t) + \beta x_2(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} \alpha X_1(s) + \beta X_2(s) \; ;$$

$$ROC \supseteq R_1 \cap R_2$$

proof:

$$\therefore \; X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow X(s) = \int_{-\infty}^{\infty} [\alpha x_1(t) + \beta x_2(t)] e^{-st} dt$$

$$\Rightarrow X(s) = \alpha \int_{-\infty}^{\infty} x_1(t) e^{-st} dt + \beta \int_{-\infty}^{\infty} x_2(t) e^{-st} dt$$

$$\boxed{X(s) = \alpha X_1(s) + \beta X_2(s)}$$

② Conjugation:

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s) \; ; \; ROC = R$$

$$x^*(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X^*(s^*) \; ; \; ROC = R$$

proof:

$$\therefore \; X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow X^*(s) = \int_{-\infty}^{\infty} x^*(t)\, e^{-s^* t}\, dt$$

$$\Rightarrow X^*(s^*) = \int_{-\infty}^{\infty} x^*(t)\, e^{(-s^*)^* t}\, dt$$

$$\boxed{\therefore X^*(s^*) = \int_{-\infty}^{\infty} x^*(t)\, e^{-st}\, dt}$$

③ Time Reversal

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s) \;;\; ROC = R$$

$$x(-t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(-s) \;;\; ROC = -R$$

proof:

$$X'(s) = \int_{-\infty}^{\infty} x(-t)\, e^{-st}\, dt$$

$$\Rightarrow X'(s) = \int_{-\infty}^{\infty} x(\tau) \cdot e^{s\tau}\, d\tau$$

$\Big\{$ Since, $-t = \tau \Rightarrow -dt = d\tau$

At $t = -\infty \;;\; \tau = \infty$

$t = \infty \;;\; \tau = -\infty \Big\}$

$$\Rightarrow \boxed{X'(s) = X(-s)}$$

④ Time scaling:

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s) \quad ; ROC = R$$

$$x(at) \underset{ILT}{\overset{LT}{\rightleftharpoons}} \frac{1}{|a|} X\left(\frac{s}{a}\right) \; ; ROC = R$$

$$\left\{ \begin{array}{l} \frac{\sigma}{a} + j\omega ; \\ \sigma + j\omega < R \end{array} \right.$$

Proof:

$$X'(s) = \int_{-\infty}^{\infty} x(at) e^{-st} dt$$

$$\Rightarrow X'(s) = \int_{-\infty}^{\infty} x(\tau) . e^{-s\tau/a} \frac{d\tau}{a}$$

$$\left\{ \text{Since, } \quad at = \tau \Rightarrow dt = d\tau/a \right.$$

$$\left. \begin{array}{l} \text{At } t = -\infty, \quad \tau = -\infty \\ \quad t = \infty, \quad \tau = \infty \end{array} \right\}$$

$$\Rightarrow X'(s) = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau) e^{-(s/a)\tau} d\tau$$

$$\boxed{\therefore X'(s) = \frac{1}{|a|} X(s/a)}$$

⑤ Time shifting:

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s) \quad ; ROC = R$$

$$x(t - t_0) \underset{ILT}{\overset{LT}{\rightleftharpoons}} e^{-st_0} X(s) ; ROC = R$$

$$x(t + t_0) \underset{ILT}{\overset{LT}{\rightleftharpoons}} e^{st_0} X(s) ; ROC = R$$

proof:

$$\therefore X'(s) = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-st}\, dt$$

Let $t - t_0 = \tau \Rightarrow dt = d\tau$

At $t = \infty, \quad \tau = +\infty$

$t = -\infty, \quad \tau = -\infty$

$$\Rightarrow X'(s) = \int_{-\infty}^{\infty} x(\tau)\, e^{-s(\tau + t_0)}\, d\tau$$

$$\Rightarrow X'(s) = e^{-st_0} \int_{-\infty}^{\infty} x(\tau)\, e^{-s\tau}\, d\tau$$

$$\boxed{X'(s) = e^{-st_0} \cdot X(s)}$$

⑥ Frequency shifting:

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s) \quad ; \quad ROC = R$$

$$e^{s_0 t} x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s - s_0) \; ; \; ROC = R + Re\{s_0\}$$

$$e^{-s_0 t} x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s + s_0) \; ; \; ROC = R - Re\{s_0\}$$

Proof:

$$\therefore X'(s) = \int_{-\infty}^{\infty} e^{s_0 t} x(t) \cdot e^{-st}\, dt$$

$$\Rightarrow X'(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-(s - s_0)t}\, dt$$

$$\boxed{X'(s) = X(s - s_0)}$$

⑦ Convolution in Time :

$$x_1(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X_1(S) \; ; \; ROC = R_1$$

$$x_2(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X_2(S) \; ; \; ROC = R_2$$

$$x(t) = x_1(t) * x_2(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X_1(S) \cdot X_2(S) ; ROC \supseteq R_1 \cap R_2$$

Proof :

$$\therefore X'(s) = \int_{-\infty}^{\infty} [x_1(t) * x_2(t)] e^{-st} dt$$

$$\Rightarrow X'(S) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau) d\tau \, e^{-st} dt$$

$$\Rightarrow X'(s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(\tau) \cdot \frac{e^{-s\tau}}{e^{-s\tau}} x_2(t-\tau) d\tau \, e^{-st} dt$$

$$\Rightarrow X'(S) = \int_{-\infty}^{\infty} x_1(\tau) \cdot e^{-s\tau} d\tau \int_{-\infty}^{\infty} x_2(t-\tau) e^{s\tau} e^{-st} dt$$

$$\Rightarrow X'(s) = X_1(S) \cdot \int_{-\infty}^{\infty} x_2(t-\tau) \cdot e^{-s(t-\tau)} dt$$

Let $t - \tau = k \Rightarrow dt = dk$

At $t = -\infty$; $k = -\infty$

$t = \infty$; $k = \infty$

$$\Rightarrow X'(S) = X_1(S) \cdot \int_{-\infty}^{\infty} x_2(k) e^{-sk} dk$$

$$\boxed{\therefore X'(S) = X_1(S) \cdot X_2(S)}$$

⑧ Multiplication in Time :-

$$x_1(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X_1(S) \; ; \; ROC = R_1$$

$$x_2(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X_2(S) \; ; \; ROC = R_2$$

$$x = x_1(t) \cdot x_2(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} \frac{1}{2\pi j} [X_1(S) * X_2(S)] \; ;$$

$$ROC = R_1 \cap R_2$$

proof :-

⑨ Differentiation in Time:

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s) \; ; \; ROC = R$$

$$\therefore \frac{d^n}{dt^n} x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} s^n X(s) \; ; \; ROC \supseteq R \quad \{\text{Two sided L.T}\}$$

$$\therefore \frac{d^n}{dt^n} x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} s^n X(s) - s^{n-1} x(0) - s^{n-2} x'(0) - \dots \; ; \; ROC \supseteq R$$

{Unilateral / one sided L.T.}

Proofs

(i) Two sided L.T.:

$$\therefore X'(s) = \int_{-\infty}^{\infty} \frac{d}{dt} x(t) e^{-st} \, dt$$

$$\therefore x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} X(s) e^{st} \, ds$$

$$\Rightarrow \frac{d}{dt} x(t) = \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} [s \cdot X(s)] e^{st} \, ds$$

$$\therefore \frac{d}{dt} x(t) = s \cdot X(s)$$

Similarly,

$$\boxed{\frac{d^n}{dt^n} x(t) = s^n X(s)}$$

(ii) one sided L.T:

$$\therefore X(s) = \int_0^\infty x(t) e^{-st} dt$$

$$\frac{d}{dt} x(t) \longrightarrow X'(s)$$

$$\therefore X'(s) = \int_0^\infty \frac{dx(t)}{dt} \cdot e^{-st} dt$$

$$\Rightarrow X'(s) = \left[e^{-st} \cdot x(t) - \int e^{-st} (-s) \cdot x(t) dt \right]_0^\infty$$

$$\Rightarrow X'(s) = e^{-st} x(t) \Big|_0^\infty + s \int_0^\infty e^{-st} x(t) dt$$

$$\Rightarrow X'(s) = e^{-\infty} x(\infty) - e^{0} x(0) + s \int_0^\infty x(t) e^{-st} dt$$

$$\Rightarrow X'(s) = -x(0) + sX(s)$$

$$\boxed{X'(s) = s X(s) - x(0)}$$

Similarly, for $\frac{d^n}{dt^n} x(t)$, we get

$$X'(s) = s^n X(s) - s^{n-1} x(0) - s^{n-2} x'(0) -$$
$$s^{n-3} x''(0) \cdots\cdots$$

where $x(0) = x(t)\Big|_{t=0}$

(10) Integration in Time:

(11) Differentiation in Frequency:

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s) \quad ; \quad ROC = R$$

$$t^n x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} (-1)^n \frac{d^n}{ds^n} X(s) \quad ; \quad ROC = R$$

Proof:

$$\therefore \quad X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow \quad \frac{d}{ds} X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} (-t)\, dt$$

$$\Rightarrow \quad -\frac{d}{ds} X(s) = \int_{-\infty}^{\infty} t x(t) e^{-st} dt$$

$$\boxed{-\frac{d}{ds} X(s) = L.T\{t x(t)\}}$$

(12) Integration in Frequency:

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s)$$

$$\frac{x(t)}{t} \underset{ILT}{\overset{LT}{\rightleftharpoons}} \int_{+s}^{\infty} X(s) ds$$

Proof:

$$\therefore \quad X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow \quad \int_{s}^{\infty} X(s)\, ds = \int_{s}^{\infty} \int_{-\infty}^{\infty} x(t) e^{-st} dt\, ds$$

$$\Rightarrow \quad \int_{s}^{\infty} X(s)\, ds = \int_{-\infty}^{\infty} x(t) \int_{s}^{\infty} e^{-st} ds\, dt$$

$$\therefore \int_s^{\infty} X(s)\,ds = \int_{-\infty}^{\infty} \frac{x(t)\,e^{-st}}{t}\,dt$$

(13) Initial value theorem:

$$x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} X(s)$$

$$\frac{d}{dt}x(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} sX(s) - x(0)$$

$$\boxed{x(0) = \lim_{t\to 0} x(t) = \lim_{s\to\infty} sX(s)}$$

$$\therefore X(s) = \int_0^{\infty} x(t)\,e^{-st}\,dt$$

$$\Rightarrow sX(s) - x(0) = \int_0^{\infty} \frac{dx(t)}{dt}\,e^{-st}\,dt$$

$$\Rightarrow \lim_{s\to\infty} [sX(s) - x(0)] = \lim_{s\to\infty} \int_0^{\infty} \frac{dx(t)}{dt}\,e^{-st}\,dt$$

$$\Rightarrow \lim_{s\to\infty} [sX(s) - x(0)] = 0$$

$$\Rightarrow \lim_{s\to\infty} sX(s) = x(0).$$

$$\boxed{\therefore x(0) = \lim_{s\to\infty} sX(s) = \lim_{t\to 0} x(t)}$$

(14) Final value theorem:

$$x(t) \rightleftharpoons X(s)$$

$$\frac{d}{dt}x(t) \rightleftharpoons sX(s) - x(0)$$

$$\boxed{x(\infty) = \lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s)}$$

$$\therefore X(s) = \int_0^\infty x(t)\, e^{-st}\, dt$$

$$\Rightarrow sX(s) - x(0) = \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt$$

$$\Rightarrow \lim_{s\to 0} sX(s) - x(0) = \lim_{s\to 0} \int_0^\infty \frac{dx(t)}{dt} e^{-st}\, dt$$

$$\Rightarrow \lim_{s\to 0} sX(s) - x(0) = \int_0^\infty \frac{dx(t)}{dt}\, dt$$

$$\Rightarrow \lim_{s\to 0} sX(s) - x(0) = x(t)\Big|_0^\infty$$

$$\Rightarrow \lim_{s\to 0} sX(s) - x(0) = x(\infty) - x(0)$$

$$\boxed{\therefore x(\infty) = \lim_{s\to 0} sX(s)}$$

conditions for final value theorem:

⇒ Signal must be causal signal. i.e., $x(t) = 0 \ \forall t < 0$

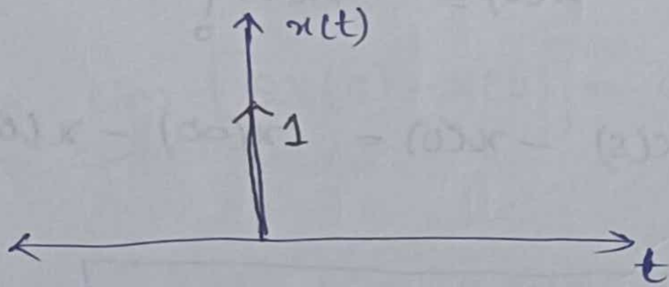
⇒ All the poles of the signal should lie on the left hand side of s-plane.

* Laplace transform of Basic signals:

1. Impulse signal
2. Unit step signal
3. DC value
4. signum function
5. Exponential signal
6. Ramp signal
7. Sin and cos signal

1. Impulse signal:

$$x(t) = \delta(t)$$



$$\therefore \quad X(s) = \int_{-\infty}^{\infty} x(t) \cdot e^{-st} dt$$

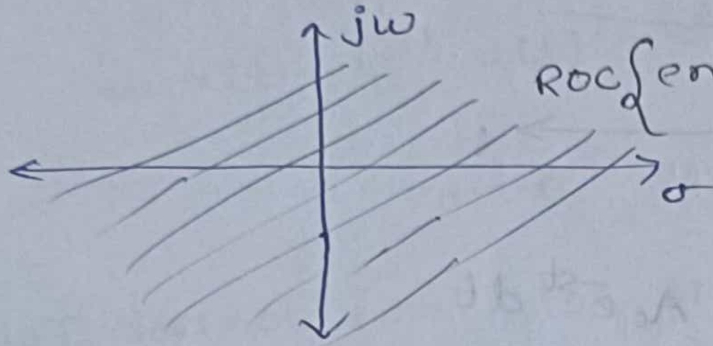
$$\Rightarrow X(s) = \int_{-\infty}^{\infty} \delta(t) \cdot e^{-st} dt$$

$$\Rightarrow X(s) = e^{-s(0)}$$

$\Rightarrow X(s) = e^{-(\sigma + j\omega)0}$

$\boxed{X(s) = 1}$ Independent of 'σ'.

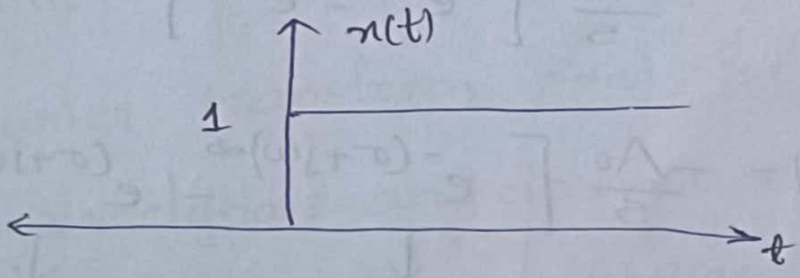
$\therefore \delta(t) \underset{ILT}{\overset{LT}{\rightleftharpoons}} 1$



ROC {entire s-plane}

② Unit step Signal :-

$$x(t) = u(t) = \begin{cases} 0 ; & t < 0 \\ 1 ; & t > 0 \end{cases}$$



$$\therefore X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow X(s) = \int_{0}^{\infty} 1 . e^{-st} dt$$

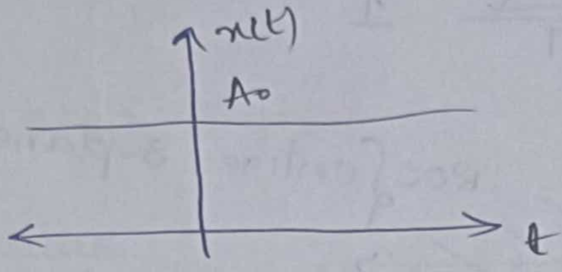
$$\Rightarrow X(s) = \frac{e^{-st}}{-s} \Big|_{0}^{\infty}$$

$$\Rightarrow X(s) = \left[\frac{e^{-s\infty} - e^{0}}{-s} \right] = \left[\frac{e^{-(\sigma + j\omega)\infty} - 1}{-s} \right]$$

$\boxed{X(s) = \frac{1}{s}}$ where ROC : $\sigma > 0$

③ Dc value:

$$A_0 \underset{ILT}{\overset{LT}{\rightleftarrows}} ?$$



$$\therefore X(s) = \int_{-\infty}^{\infty} A_0 e^{-st} dt$$

$$\Rightarrow X(s) = \frac{A_0 e^{-st}}{-s} \Big|_{-\infty}^{\infty}$$

$$\Rightarrow X(s) = \frac{-A_0}{s} \left[e^{-s\infty} - e^{s\infty} \right]$$

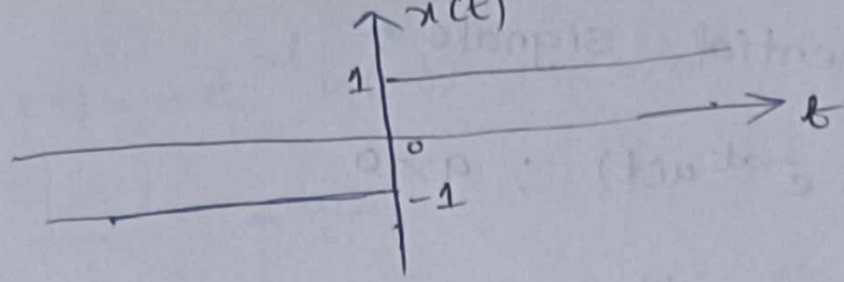
$$\Rightarrow X(s) = \frac{-A_0}{s} \left[\underset{\sigma > 0}{\underset{\downarrow}{e^{-(\sigma + j\omega)\infty}}} - \underset{\sigma < 0}{\underset{\downarrow}{e^{(\sigma + j\omega)\infty}}} \right]$$

$\therefore$ Roc doesnot be there for this signal.

Hence, L-T does not exist for DC values.

④ Signum function:

$$\therefore x(t) = sgn(t) = \begin{cases} -1 & ; t < 0 \\ 0 & ; t = 0 \\ 1 & ; t > 0 \end{cases}$$



$\therefore x(t) = u(t) - u(-t)$

$\Rightarrow x(t) = e^{0t}.u(t) - e^{0t}u(-t)$

$\downarrow \qquad\qquad \downarrow$

$\sigma > 0 \qquad\qquad \sigma < 0$

$\therefore$ L.T does not exist for signum function because there is no common ROC.

Notes

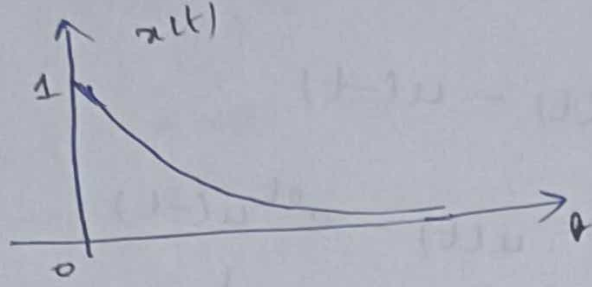
$\Rightarrow$ Fourier series exist for only periodic signals

$\Rightarrow$ Fourier transform exist for all energy and power signals and does not exist for NENP signal except impulse signal since it is absolutely integrable.

$\Rightarrow$ Laplace transform does not exist for double sided power signals.

⑤ Exponential signals :-

(i) $x(t) = e^{-at} u(t)$; $a > 0$



$$\therefore X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\Rightarrow X(s) = \int_{0}^{\infty} e^{-at} \cdot e^{-st} dt$$

$$\Rightarrow X(s) = \int_{0}^{\infty} e^{-(a+s)t} dt$$

$$\Rightarrow X(s) = \left. \frac{e^{-(s+a)t}}{-(s+a)} \right|_{0}^{\infty}$$

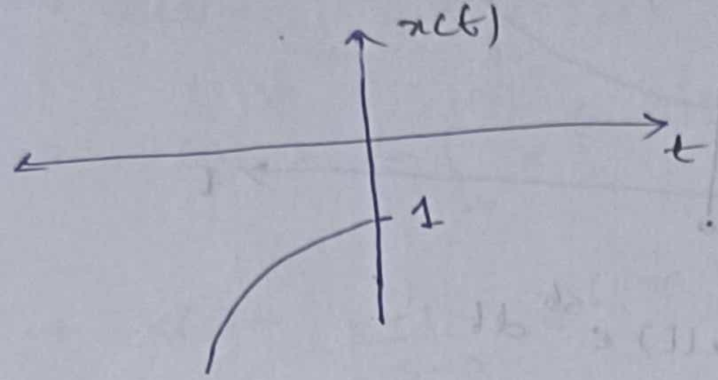
$$\Rightarrow X(s) = \frac{-1}{(s+a)} \left[\frac{e^{-(\sigma + j\omega + a)(\infty)}}{1} - e^{0} \right]$$

To exist, $(\sigma + a) > 0$; $\sigma > -a$

$$\Rightarrow X(s) = \frac{-1}{s+a} [-1]$$

$$\boxed{X(s) = \frac{1}{s+a}} \quad ; \text{ ROC} : \sigma > -a$$

(ii) $x(t) = -e^{-at}u(-t) \; ; \; a > 0$



$$\therefore X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$\Rightarrow X(s) = \int_{-\infty}^{0} -e^{-at}\, e^{-st}\, dt$$

$$\Rightarrow X(s) = \int_{-\infty}^{0} -e^{-(s+a)t}\, dt$$

$$\Rightarrow X(s) = \frac{-1}{-(s+a)} \cdot e^{-(s+a)t} \Big|_{-\infty}^{0}$$

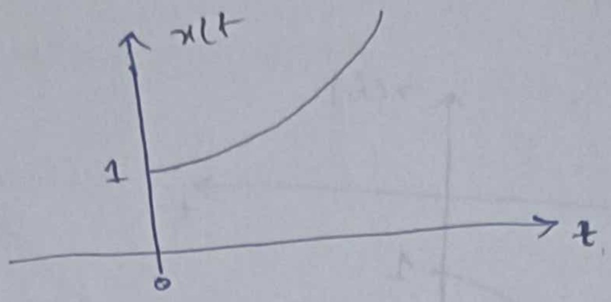
$$\Rightarrow X(s) = \frac{1}{s+a}\left[e^{0} - e^{+(\sigma + j\omega + a)\infty} \right]$$

$$\Rightarrow X(s) = \frac{1}{s+a}\left[1 - e^{(\sigma+a)\infty} \cdot e^{j\omega} \right]$$

Here $\sigma + a < 0$; $\Rightarrow \boxed{\sigma < -a}$

$$\boxed{\therefore X(s) = \frac{1}{s+a}} \; ; \; ROC : \sigma < -a$$

(iii) $e^{at} u(t) = x(t)$; $a > 0$



$$\therefore X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$\Rightarrow X(s) = \int_{0}^{\infty} e^{at} \cdot e^{-st}\, dt$$

$$\Rightarrow X(s) = \int_{0}^{\infty} e^{-(s-a)t}\, dt$$

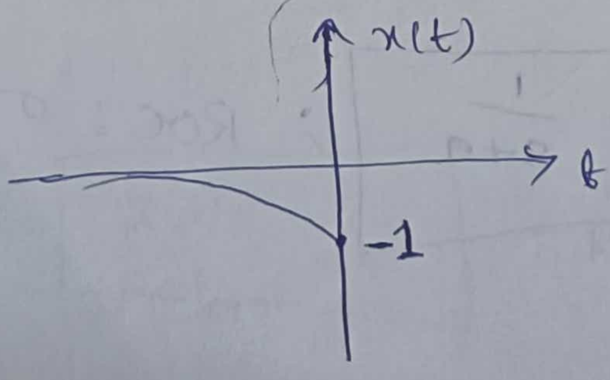
$$\Rightarrow X(s) = \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_{0}^{\infty}$$

$$\Rightarrow X(s) = \frac{-1}{(s-a)} \left[e^{-(\sigma-a)\infty} \cdot e^{-j\omega\infty} - e^{0} \right]$$

$$\Rightarrow X(s) = \frac{-1}{(s-a)} (-1) \quad \text{for } \sigma - a > 0$$

$$\boxed{\therefore X(s) = \frac{1}{s-a}} \quad ROC : \sigma > a$$

(iv) $-e^{at} u(-t) = x(t)$; $a > 0$



$$\therefore X(s) = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt$$

$$\Rightarrow X(s) = \int_{-\infty}^{0} -e^{at} e^{-st}\, dt$$

$$\Rightarrow X(s) = -\int_{-\infty}^{0} e^{-(s-a)t}\, dt$$

$$\Rightarrow X(s) = \frac{+1}{s-a} \cdot e^{-(s-a)t} \Big|_{-\infty}^{0}$$

$$\Rightarrow X(s) = \frac{1}{s-a}\left[e^{0} - e^{(s-a)\infty} \right]$$

$$\Rightarrow X(s) = \frac{1}{s-a}\left[1 - e^{(\sigma-a)\infty} \cdot e^{j\omega\infty} \right]$$

$$\sigma - a < 0 \Rightarrow \sigma < a$$

$$\boxed{\therefore X(s) = \frac{1}{s-a}} \quad ROC: \sigma < a$$

Note:

$$e^{-at} u(t) \rightleftharpoons \frac{1}{s+a} \; ; \; ROC = \sigma > -a$$

$$-e^{-at} u(-t) \rightleftharpoons \frac{1}{s+a} \; ; \; ROC = \sigma < -a$$

$$e^{at} u(t) \rightleftharpoons \frac{1}{s-a} \; ; \; ROC = \sigma > a$$

$$-e^{at} u(-t) \rightleftharpoons \frac{1}{s-a} \; ; \; ROC = \sigma < a$$

⑥ Ramp signal:

$$x(t) \overset{LT}{\rightleftharpoons} X(s)$$

$$t^n x(t) \overset{LT}{\rightleftharpoons} (-1)^n \frac{d^n}{ds^n} X(s)$$

$$t x(t) \overset{LT}{\rightleftharpoons} -1 \frac{d}{ds} X(s)$$

$$\therefore \quad u(t) \overset{LT}{\rightleftharpoons} \frac{1}{s} \; ; \quad ROC = \sigma > 0$$

$$x(t) = r(t) = \begin{cases} 0 & t < 0 \\ t & t > 0 \end{cases} = t\,u(t)$$

$$\Rightarrow \quad t u(t) \overset{LT}{\rightleftharpoons} (-1) \frac{d}{ds} 1/s$$

$$\boxed{\therefore \; t u(t) \overset{LT}{\rightleftharpoons} \frac{1}{s^2} \; ; \quad ROC = \sigma > 0}$$

Note:

$$u(t) \rightleftharpoons 1/s$$

$$t u(t) \rightleftharpoons \frac{1}{s^2}$$

$$t^2 u(t) \rightleftharpoons \frac{2}{s^3}$$

$$t^3 u(t) \rightleftharpoons 6/s^4$$

$$\vdots \qquad \vdots$$

$$t^n u(t) \rightleftharpoons \frac{n!}{s^{n+1}}$$

(7) cos and sin signals:

(i) L.T. of $\cos\omega_0 t . u(t)$:

$$x(t) = \cos\omega_0 t u(t) = \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] u(t)$$

$$\therefore X(S) = \int_{-\infty}^{\infty} \left[\frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}\right] u(t) . e^{-st}\, dt$$

$$\Rightarrow X(S) = \frac{1}{2}\int_{0}^{\infty} \left[e^{-(s-j\omega_0)t} + e^{-(s+j\omega_0)t}\right] dt$$

$$\Rightarrow X(S) = \frac{1}{2}\left[\frac{e^{-(\sigma+j\omega-j\omega_0)t}}{-(s-j\omega_0)} + \frac{e^{-(\sigma+j\omega+j\omega_0)t}}{-(s+j\omega_0)}\right]_0^{\infty}$$

$$\Rightarrow X(S) = \frac{1}{2}\left[\frac{e^{-\sigma(\infty)} . e^{-j(\omega-\omega_0)\infty}}{-(s-j\omega_0)} + \frac{e^{-\sigma(\infty)} . e^{-j(\omega+\omega_0)\infty}}{-(s+j\omega_0)} - \frac{1}{-(s-j\omega_0)} - \frac{\cancel{1}\,1}{-(s+j\omega_0)}\right]$$

Here $\sigma > 0$ for both the two terms to exist.

$$\Rightarrow X(S) = \left[\frac{+1}{s-j\omega_0} + \frac{1}{s+j\omega_0}\right] \times \frac{1}{2}$$

$$\Rightarrow X(S) = \left[\frac{s+j\omega_0 + s-j\omega_0}{s^2+\omega_0^2}\right] \times \frac{1}{2}$$

$$\boxed{\therefore X(S) = \frac{s}{s^2+\omega_0^2} \quad ; \quad ROC: \sigma > 0}$$

(ii) L.T of $\sin\omega_0 t \cdot u(t)$:

$$x(t) = \sin\omega_0 t \cdot u(t) = \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] u(t)$$

$$\therefore X(s) = \int_0^\infty \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] e^{-st}\, dt$$

$$\Rightarrow X(s) = \frac{1}{2j}\int_0^\infty e^{-(s-j\omega_0)t}\, dt - \frac{1}{2j}\int_0^\infty e^{-(s+j\omega_0)t}\, dt$$

$$\Rightarrow X(s) = \frac{1}{2j}\left[\frac{e^{-(s-j\omega_0)t}}{-(s-j\omega_0)} - \frac{e^{-(s+j\omega_0)t}}{-(s+j\omega_0)}\right]_0^\infty$$

$$\Rightarrow X(s) = \frac{1}{2j}\left[\frac{e^{-(\sigma+j\omega-j\omega_0)\infty}}{-(s-j\omega_0)} - \frac{e^{-(\sigma+j\omega-j\omega_0)\infty}}{-(s+j\omega_0)} + \frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right]$$

Here, $\sigma > 0$ for both the terms to exist.

$$\Rightarrow X(s) = \frac{1}{2j}\left[\frac{1}{s-j\omega_0} - \frac{1}{s+j\omega_0}\right]$$

$$\Rightarrow X(s) = \frac{1}{2j}\left[\frac{\cancel{s}+j\omega_0-\cancel{s}+j\omega_0}{s^2+\omega_0^2}\right]$$

$$\boxed{X(s) = \frac{\omega_0}{s^2+\omega_0^2} \;;\; \text{ROC}: \sigma > 0}$$

* Find laplace transform of $x(t) = te^{-at}u(t)$.

sol:

$$e^{-at}u(t) \overset{LT}{\rightleftharpoons} \frac{1}{(s+a)}$$

$$\therefore t \cdot e^{-at}u(t) \overset{LT}{\rightleftharpoons} (-1)^1 \frac{d}{ds}\left(\frac{1}{s+a}\right)$$

$$\therefore te^{-at}u(t) \overset{LT}{\rightleftharpoons} \left(\frac{1}{s+a}\right)^2$$

Note:

(1) $e^{-at}t^n u(t) \overset{LT}{\rightleftharpoons} \frac{n!}{(s+a)^{n+1}}$

(2) $e^{-at}\cos\omega_0 t\, u(t) \overset{LT}{\rightleftharpoons} \frac{(s+a)}{(s+a)^2+\omega_0^2}$

(3) $e^{-at}\sin\omega_0 t\, u(t) \overset{LT}{\rightleftharpoons} \frac{\omega_0}{(s+a)^2+\omega_0^2}$

* Consider $X(s) = \frac{\omega_0}{s^2+\omega_0^2}$. The final value of $x(t)$ would be:

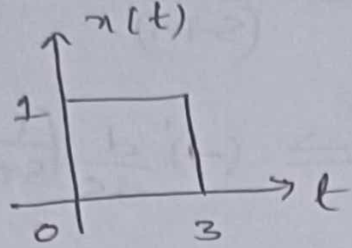
(a) 0 (b) 1 (c) $-1 \le x(t) \le 1$ (d) ∞

* The laplace transform of $I(t)$ is given by $I(s) = \frac{2}{s(s+1)}$. As $t \to \infty$, the final would be 2

* The L.T. of $\left[\frac{1-e^{t}}{t}\right]u(t)$ is:

(a) $\log\left[\frac{s}{s-1}\right]$ (b) $\log\left[\frac{s-1}{s}\right]$ (c) $\log\left[\frac{s-1}{s+1}\right]$ (d) $\log\left[\frac{s+1}{s-1}\right]$

* Find laplace transform and Roc of given signal :



$\Longrightarrow X(s) = ?\ ;\ ROC = ?$

$$X(s) = \frac{1-e^{-3s}}{s}\ ;\ \text{ROC : entire s-plane excluding } s=-\infty$$

* $x(t) = e^{r(t)} \Longrightarrow X(s) = ?,\ ROC = ?$

∴ Laplace transform does not exist because of no common Roc.

* Consider a signal $x(t)$ having laplace transform $X(s) = \log\left[\frac{s+5}{s+6}\right]$. Then $x(t)$ is ___

Sol:- $\left[\frac{e^{-6t} - e^{-5t}}{t}\right] u(t)$.

* $$X(s) = \frac{3s^2+8s+25}{(s+3)(s^2+2s+10)}\ ;\ 2e^{-3t}u(t) + e^{-t}\cos 3t\, u(t)$$

* $$X(s) = \frac{8s^2+11s}{(s+2)(s+1)^3}\ ;\ \left(16-2t-1.5t^2\right)e^{-t} - 10e^{-2t}\Big]u(t)$$

Filters

⇒ Filters is a frequency network which passes a desired range of frequencies and remove unwanted frequencies.

⇒ Filters are of two types. They are:

1. Passive filters
2. Active filters

⇒ Passive filters produce filtering only and contains R, L, C elements whereas active filters produce filtering along with amplification and contains Transistors and op-amps.

⇒ Inductors are not used in active filters because of bigger size

* Passive filters:

(1) RC Low pass filter: