Counting:-

> A paper folding Example

> Rubix cube

2x2x2 - 3674160

3x3x3 - 43252003274489856000 (which is more than the age of the universe)

> factorial :- [taking pictures of people]

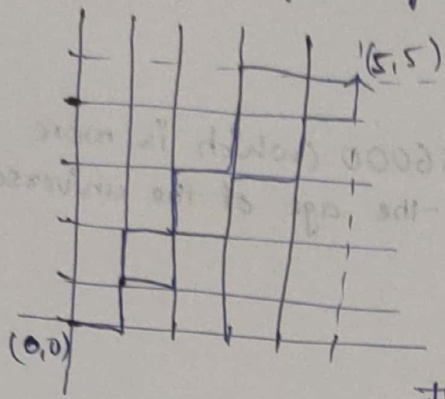
1	1
2	2
3	6
4	24
5	120
.	
10	3628800
20	2432902008176640000 (~~prob~~. more than the age of the universe)

> It is very important for us to understand when my code will stops execution?

> It is very very important thing that have an idea as in how many steps will yous computer take in order to execute a task. For that we need to understand the counting at a very different ~~way~~ level.

> This chapter is all about understanding how to count, where in a situation where it isn't about 1,2,3,4, it's about enumerated all the objects in the system.

> This chapter counting is all about training / explaining you all on how to count. I'll can give you a good illustration of how counting can get very complicated.



Let's take this grid. The question/picture is In how many ways can you reach (5,5) starting from (0,0) by using only rights & ups?

This might be easy. some of you can find the solution by spending some time.

We'll discuss this later. But right now I want to give you a complicated version of the same question.

> What is the total no. of ways starting from (0,0) to (5,5), with the condition, the path shouldn't cross the diagonal from (0,0) to (5,5)?

Rule of Sum & Rule of Product :-

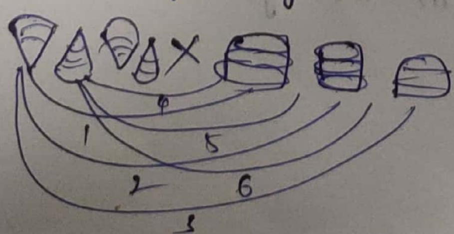
> Assume Ram went to a restaurant, and there are 4 dif. pizzas and 3 dif. burgers. And Ram planned to eat 1 item only. So how many possibilities are there for Ram?

Q1

4 pizza + 3 burger

1 of these $4+3 = \boxed{7 \text{ items.}}$

Q2 Now Ram wants to buy 1 pizza and 1 burger. In how many ways can Ram make his order?



$4 \times 3 = \boxed{12 \text{ ways}}$

$\rightarrow 7+v(\sin 8^{\circ}) \Rightarrow 5^{\circ}$

, So its only common sensical to know when to add and when to multiply.

, This goes by the title (Now tell them the title)

, Now Let's see more examples

Ex1 Alice goes to the library to read a book. There are books of different genres. She goes to the shelf which has 7 science fictions, 5 mystery books and 2 journals. Alice wants to read precisely 1 book. How many choices does she have?

No. of choices $= 7 + 5 + 2 = \boxed{14}$ choices

Ex2 In a town of Germany, 8 newspapers and 4 magazines are printed. Peter wants to subscribe to 1 newspaper and 1 magazine. How many choices does he have?

A newspaper and a magazine

Choices $= 8 \times 4 = \boxed{32}$ choices

Q2 Peter wants to subscribe to 1 newspaper or 1 magazine. In How many ways he can choose?

Choices $= 8 + 4 = 12$

Ex3 8 men and 6 women contest in an election.

a) In how many ways can the people choose a leader?

Choices $= 8+6 = 14$ ways

b) In how many ways can the people choose a man & a woman?

Choices $= 8 \times 6 = 48$ ways

Ex4 How many positive divisors does 2000 have?

* what is a divisor? / factors

Divisor is an integer which divides another number

4 as a number, the divisors are $\{1, 2, 4\}$

$1|4 \quad 2|4 \quad \cancel{3|4} \quad 4|4$

Sol

$2000 = 2\times2\times2\times2\times5\times5\times5$

$= 2^4 \times 5^3$

Now, the divisor of 2000 is in the form of $\boxed{2^a 5^b}$

$0 \le a \le 4 \;, 0 \le b \le 3 \qquad \{a, b\}$

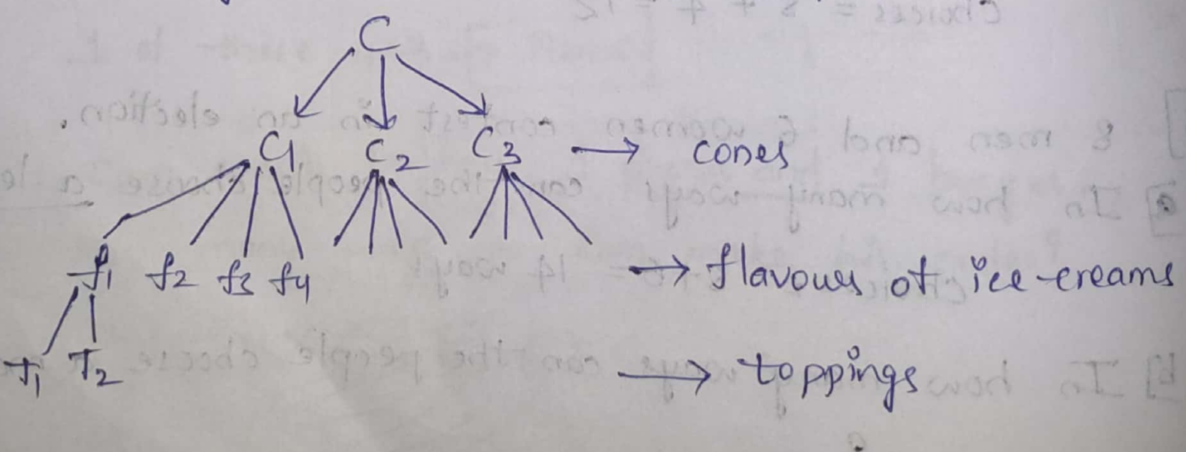
$\{1, 2, 4, \ldots\ldots 2000\}$ 1 a and 1 b

$\quad 5 \times 4 = 20$

possibilities: $a = \{0, 1, 2, 3, 4\}$ (5), $b = \{0, 1, 2, 3\}$ (4)

20 divisors for 2000

Ex5 Ram ~~charlie~~ visits an ice-cream parlour to buy one. He sees that there are 3 cones, 4 ice-creams and 2 toppings. How many choices does Charlie have?



choices $\begin{cases} 3 \text{ cones} \\ 4 \text{ flavours} \\ 2 \text{ toppings} \end{cases}$ No.of choices $= 3 \times 4 \times 2 = 24$

Ex6 In how many ways can Ram draw a face card from a deck?

In a deck of 52 cards, there are 4 suits

the clubs ♣, diamond ♢, hearts ♡, spades ♠

J → 4
Q → 4
K → 4

sums = 12 ways (face cards)

choosing one from 12 is having 12 ways.

choices $= 4+4+4 = 12$

Rule of Sum:

If there are 'n' choices for one event and m choices for another event, both can't be done/occur at the same time, then there are $\boxed{n+m}$ choices for one event.

Rule of Product:-

If there are 'n' choices for one event and 'm' choices for another event, then there are $\boxed{n \times m}$ choices for both these events to occur.

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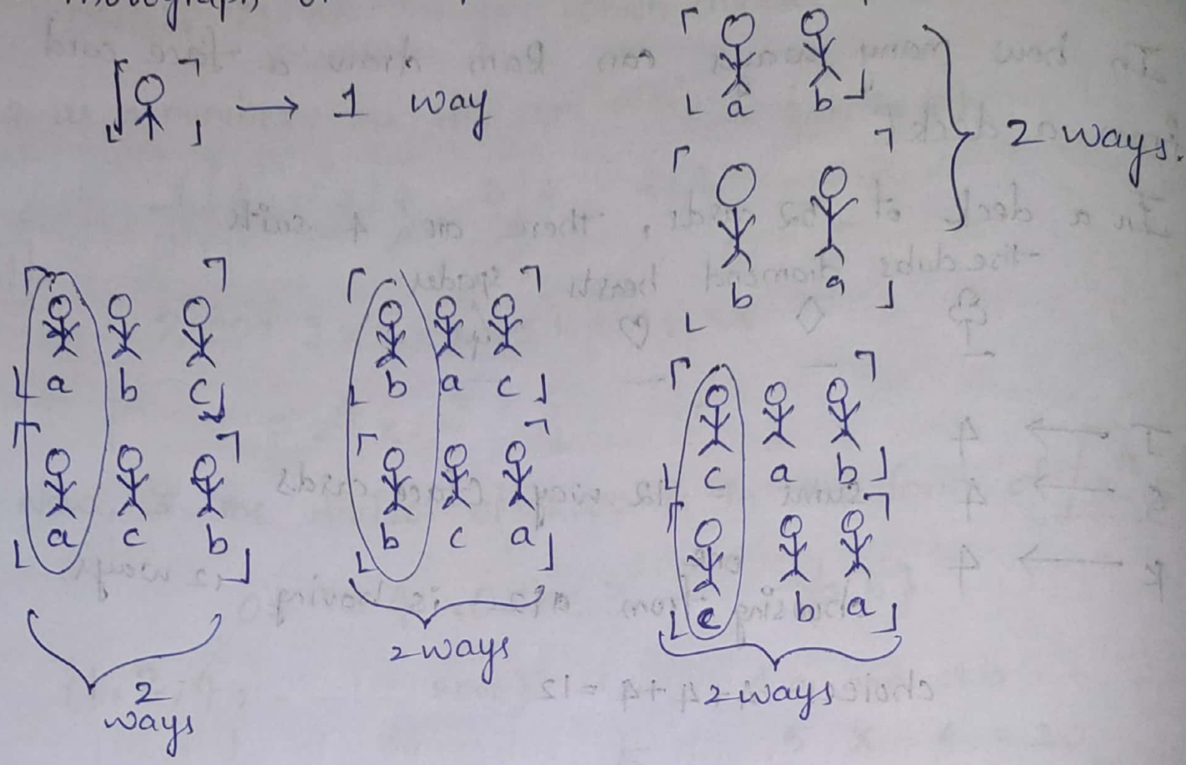
3

Proof of n! :-

Ex:- Photograph of 20 people example.



3 people taking photos
all possible ways are = 3 * 2 ways
= 6 ways.

> There are two ways to write a & b
ab
ba

> If one more new entity has come 'c', now we have 3 positions

c in first place $\boxed{c}$ $\boxed{a\ b}$ → 2 ways
c in 2nd place $\boxed{a}$ $\boxed{c}$ $\boxed{b}$ → 2 ways
c in 3rd place $\boxed{a\ b}$ $\boxed{c}$ → 2 ways

how many times?

→ 2 objects → 2 ways

→ 3 objects → 3 × (no.of ways for 2 objects)

= 3 × 2 = 6

For 4 objects:- , new entity is 'd'

, d is in 1^{st} place ⟶ $\boxed{d}\ \boxed{a\ b\ c}$ ⟶ 6 ways

, d is in 2^{nd} place ⟶ $\boxed{a}\ \boxed{d}\ \boxed{b\ c}$ ⟶ 6 ways.

, d is in 3^{rd} place ⟶ $\boxed{a\ b}\ \boxed{d}\ \boxed{c}$ ⟶ 6 ways

, d is in 4^{th} place ⟶ $\boxed{a\ b\ c}\ \boxed{d}$ ⟶ 6 ways.

Answer is = 4 × (6 ways) = 24 ways.

= 4 × (the no. of ways for 3 objects)

, ~~Answer of~~ No. of possible ways for 5 objects = 5 × (Answer for 4 objects)

= 5 × (4 × (Answer for 3 objects))

= 5 × (4 × (3 × (Answer for 2 objects)))

= 5 × 4 × 3 × 2 = 120

So in general:-

Answer for 'n' objects = n × answer for (n−1) objects

= n × (n−1) × answer for (n−2) objects

= ⋮

= n × (n−1) × ⋯ × 2 × 1

= n!

Observation:-

Ways in which 'n' people can take all possible photographs is $\boxed{n!}$

$$\boxed{n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1}$$

ASTRONOMICAL NUMBERS:-

$1! = 1$
$2! = 2$
$3! = 6$
$4! = 24$
$5! = 120$
$6! = 720$
$\vdots$
$10! = \cancel{3688}\ 3628800$

10! stands for 10 people $a_1, a_2, a_3, \ldots a_{10}$ taking all possible pictures.

Let's assume they will take 1 second to move around.

3628800 seconds
$\approx$ 42 days of time

> This factorial ways probablly work for 1, 2, 3 even 4, 5, and 6, but not beyond that.

More Surprising!!

$15! \rightarrow$ nearly 40000 years of time
!
!

$20! \rightarrow$ Age of the universe
$\sim 2^{60}$ seconds.

These numbers are such a very huge huge numbers.

> Such huge numbers are called as ASTRONOMICAL NUMBERS

> In, computer science we encounter $n!$ a whole lot. Infact some computation involves factorial no.of steps & these problems generally considered as to be very hard or just not feasible.

> Factorial of a small numbes appears to be small, But, even for a two-digit number, it ~~happ~~ is too-too huge that of astronomical in size.

Permutations :-

Let's assume there are 5 people. And now the question is In how many ways can 3 people come forward & take a picture? These 3 people will take ~~all~~ photos in all possible ways.

Here, let's take 5 letters ~~for~~ instead of people, let's say

ABCDE

constraint is, we picture should be taken with 3 people

means, $\boxed{\text{3 from 5}}$, means 2 people should step back.

$\boxed{\text{sol}}$ ABC → for 3 people, as we learned in last class there will be 3! ways to take ~~all~~ pictures!

3	ABC → 3!	2 BCD → 3!	1 CDE → 3!
	ABD → 3!	BCE → 3!	
	ABE → 3!	BDE → 3!	
2	ACD → 3!		
	ACE → 3!		
1	ADE → 3!		

$1+(1+2)+(1+2+3) = 10$ ways.

Now, by this you have exhausted all possibilities.

> 10 possible ways in which 3 people can step forwarding out of 5.

Each of them can take pictures in $3! = 6$ ways.

So, $10 \times 6 = 60$

> No. of ways in which 5 people can decide to take pictures with 3 in each frame = 60 ways

selecting 3 objects from 5

3 from 5

Notation: $^5P_3$ where order is important.

what does that mean?

order is important means, ABC is different with ACB

Ways for 3 from 5 is $\boxed{60 = {}^5P_3}$

$^nP_r \rightarrow$ picking 'r' objects/elements from 'n' objects where order is important.

Total ways to line up 'r' objects out of 'n' objects, [Order is important]

Now, what exactly ^nP_r?

> To understand this better, we'll take another example,

$^{10}P_4 \rightarrow$ pick 4 people out of 10 people in all possible ways

4 people should take pictures in all possible ways, what are the total pictures?

10 people $\rightarrow a_1 a_2 a_3 a_4 a_5 a_6 a_7 a_8 a_9 a_{10}$

$\rightarrow$ 4 positions.

$10 \times 9 \times 8 \times 7 \rightarrow 5040$

$\begin{array}{r} 56 \\ \times 9 \\ \hline 504 \end{array}$ $\begin{array}{r} 4 \\ 5 \end{array}$ 5040

$\boxed{{}^{10}P_4 = 5040}$

> Formula for ^nP_r:

n objects : $a_1\ a_2\ a_3 \cdots a_n$

r positions :

n	$(n-1)$	$(n-2)$	$(n-3)$	....	$(n-r+3)$	$(n-r+2)$	$(n-r+1)$

R

$$^nP_r = n(n-1)(n-2) \times \cdots \times (n-r+1)$$

$$= n(n-1)(n-2) \cdots (n-r+1)\underbrace{(n-r) \cdots 2 \times 1}_{\text{Not in the form}}$$

$$= \frac{n(n-1)(n-2) \cdots (n-r+1) \times (n-r)!}{(n-r)!}$$

$$\boxed{^nP_r = \frac{n!}{(n-r)!}}$$

We will see some permutation examples.

Ex1 How many 3-letter words with/without meaning can be formed from 'LOGARITHMS' if repetition is not allowed?

Ans:- 'LOGARITHMS' → 10 letters

some of words – LOG, LOA, LOR, LOI, LOT, LOH, LOM, ...

picking 3 letters out of 10 letters, $^{10}P_3 = \frac{10!}{(10-3)!}$

$$= \frac{10!}{7!} = \frac{10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}} = 10 \times 9 \times 8 = 720$$

10 ×	9 ×	8

$\Rightarrow 10 \times 9 \times 8 = 720$

Ex2 In how many ways the letters of the word 'LEADER' be arranged!

LEADER → 6 letters.

$= 6!$

Note:- There are 2 E's present in the word

There are 6 letters in the word and 2 repetitions are there.

No. of ways $= \frac{6!}{2!} = \frac{6 \times 5 \times 4 \times 3 \times \cancel{2 \times 1}}{\cancel{2 \times 1}} = 360$

Why did we divide by 2! ?

Let's E_1 and E_2 are considered as distinct, then

$E_1 E_2$, $E_2 E_1$ $\Rightarrow$ EE

$E_1, E_2, E_3 \rightarrow$ distinct. 3!

$E_1 E_2 E_3$, $E_2 E_1 E_3$, $E_3 E_1 E_2$ $\rightarrow 3!$

$E_1 E_3 E_2$, $E_2 E_3 E_1$, $E_3 E_2 E_1$

$\rightarrow$ EEE $\rightarrow$ 1

$\boxed{Ex 3}$ A company has 10 members on its board of directors. In how many ways can they elect

A president

A vice-president

A secretary &

A treasurer. ?

Ans:- 10 members, 4 members to be picked

picking 4 from 4 (order with important)

solution = $^{10}P_4$

$= \frac{10 \times 9 \times 8 \times 7 \times 6!}{(10-4)!} = 10 \times 9 \times 8 \times 7$

$= 5040$

$\boxed{Ex 4}$ In how many ways the word 'HOLIDAY' be arranged such that the letter 'I' will always come to the left of the letter 'L' ?

HOLIDAY $\rightarrow$ 7 letters

The total no. of words with 7 letters is

7!

There are only 2 possibilities for each word that 'I' can be present either left of the letter 'L' or right of the letter 'L'.

So the Answer would become half of the total words.

$$= \frac{7!}{2} = 7\times6\times5\times4\times3 = 2520$$

Ex 5 Find the no. of permutations of the letters of the word 'CLIMATE' such that the vowels occurs in odd places.

CLIMATE → distinct letters.

★ vowels should be in odd places.

Vowels → I A E

consonants → CLMT

1	2	3	4	5	6	7
1		2		3		4

$${}^4C_3 \times {}^4P_3 \times 4!$$

4 odd places → 3 vowels → so picking 3 slots from 4.

$${}^4P_3$$

After picking 3 slots, 4 slots remain. And the total words with 4 letters are → 4!

$$\text{answer} = {}^4P_3 \times 4! = \frac{4!}{1!} \times 4! = 24\times24 = 576$$

Ex 6 In How many ways, the letters of 'MATHEMATICS' be arranged so that the vowels come together?

vowels must come together. MATHEMATICS — 11 letters

vowels → A E A I

consonants → 7 letters.

$$8\times\frac{7!}{2!} \cdot \cancel{8!} \times \frac{4!}{2!} = \boxed{\frac{8!}{\underset{M}{2!}\times\underset{T}{2!}} \times \frac{4!}{2!}}$$

5

Combinations:-

> Now, we're going to learn another important topic.
> Let's take our 5 friends taking picture example, in the same way as 3 of them coming forward and clicking photos.
But thing is, what if everyone thinks order is not important?

> Que) Given 5 friends, in how many ways can 3 people come forward, taking pictures w/o worrying about the order?

In permutation example,

ABC→3! ABD→3! ABE→3!

$^5P_3$ → means picking 3 people with order respected

ABC, ACB, BAC, BCA, dif.

If 3 people decides,

Hey, all 3 of us are in the picture, what is the need of order?

ABC

ABCDE choosing/picking 3 people from 5 without order respect

1 ABC ≡ ACB ≡ BAC ≡ BCA ≡ CAB ≡ CBA → order not important.
2 ABD 7 BCD 10 CDE
3 ABE 8 BCE
4 ACD 9 BDE
5 ACE
6 ADE

1, 1+2
↳ 3, 6, 10, 15, 21, 28

10 possibilities

> Here, we are not taking 3! combinations of the each combination because the order is not important.

> If $\frac{^5P_3}{3!}$ = No. of ways in which choosing 3 from 5.

3 people coming forward not worrying about the order.

This goes by the name "COMBINATIONS":

$\boxed{{}^{n}C_r}$ No. of ways of choosing 'r' objects from 'n' objects not worrying about the order.

$$\boxed{{}^{n}C_r = \frac{{}^{n}P_r}{r!}}$$

The example with 10 people, 4 people taking pictures. (not worrying about the order?)

$$\frac{{}^{10}P_4}{4!} = {}^{10}C_4$$

Because no 4 people will try all possible 4! combinations of pictures.

$$\therefore \boxed{{}^{n}C_r = \frac{n!}{r!\,(n-r)!}}$$

${}^{n}C_0 \rightarrow$ Choosing '0' items out of 'n' items.

$${}^{n}C_0 = \frac{n!}{0!\,(n-0)!} = 1$$

$${}^{n}C_0 = 1$$

So there is precisely 1 way to choose '0' objects from 'n' objects. But this statement might be a little very weird. But generally $\boxed{{}^{n}C_0 = 1}$

Now, choosing (${}^{n}C_n$) 'n' objects from 'n' objects (without the order).

$${}^{n}C_n = \frac{n!}{n!\,(n-n)!} = \frac{\cancel{n!}}{\cancel{n!}\,0!} = 1 \quad \boxed{{}^{n}C_n = 1}$$

So there is only one way to choose all the objects.

Now let's know the some basic formulas with $\binom{n}{r}$.

① $\boxed{\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}}$ This is one formulae. Now we try to prove it.

$${}^{n}C_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$$

$$\frac{n!}{r!\,(n-r)!} = \frac{(n-1)!}{r!\,(n-1-r)!} + \frac{(n-1)!}{(r-1)!\,(n-\cancel{1}-r+\cancel{1})!}$$

$$\frac{n!}{\cancel{r!}\,(n-r)!} = \frac{(n-1)!}{\cancel{r!}\,(n-r-1)!} + \frac{(n-1)!\,r}{r!\,(n-r)!}$$

$$\frac{n\cancel{(n-1)!}}{(n-r)!} = \frac{\cancel{(n-1)!}}{(n-1-r)!} + \frac{\cancel{(n-1)!}\cdot r}{(n-r)!}$$

$$\frac{n}{(n-r)!} = \frac{(n-r)}{(n-r)(n-r-1)!} + \frac{r}{(n-r)!}$$

$$(n-r)!$$

$$n = n-r+r$$

$$n = n$$

So, Therefore $\boxed{{}^{n}C_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}}$

2) $\binom{n}{r} = \binom{n}{n-r}$

$$\frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-r)!\,(\cancel{n}-\cancel{n}+r)!}$$

$$\frac{n!}{r!\,(n-r)!} = \frac{n!}{(n-r)!\,r!}$$

$$\boxed{\therefore\ {}^{n}C_r = {}^{n}C_{n-r}}$$

Some basic examples:-

1) what is the value of ${}^{n}C_5$ when $n=9$.

$${}^{9}C_5 = \frac{9!}{5!\,(9-5)!} = \frac{9!}{5!\,4!} = \frac{9\times \cancel{8}^{2}\times 7\times \cancel{6}\times \cancel{5!}}{\cancel{5!}\times \cancel{4}\times \cancel{3}\times \cancel{2}\times 1}$$

$$= 9\times 2\times 7 = 126$$

2) $${}^{11}C_3 = \frac{11\times \cancel{10}^{5}\times \cancel{9}^{3}\times \cancel{8!}}{\cancel{3!}\times \cancel{8!}} = 11\times 15 = 165$$

$${}^{11}C_3 = {}^{11}C_8 = 165$$

3] what is $^6C_2 + {^6C_1} = ?$

$$\boxed{{^{n-1}C_1} + {^{n-1}C_{r-1}} = {^nC_r}}$$

$${^6C_2} + {^6C_1} = {^7C_2}$$

$${^7C_2} = \frac{7!}{2!\,5!} = \frac{7\times 6}{2} = 21$$

4] what is $^5C_3 + {^5C_2}$?

$${^5C_3} + {^5C_2} = {^6C_3} \qquad n=6,\ r=3$$

$${^6C_3} = \frac{6!}{3!\,3!} = \frac{\cancel{6}\times 5\times 4}{\cancel{3!}} = 20$$

5] In a cricket championship, there are 21 ~~members~~ matches. If each team plays one match with every other team, what are the number of teams?

21 matches, Each team plays with every other team.
↓
equivalent to choosing 2 teams from all the teams.

So $^nC_2 = 21$

$$\frac{n!}{2!\,(n-2)!} = 21 \qquad n(n-1) = 42$$

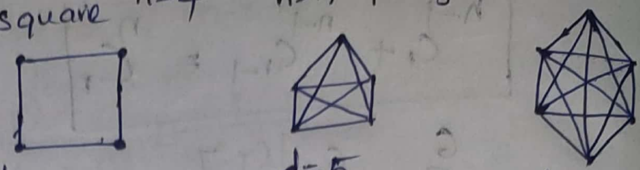
$$\frac{n\times n-1}{2} = 21 \qquad 7\times 6 = 42$$

$$\boxed{n=7}$$

No. of teams = 7

7] Find a formula for counting the no. of diagonals in a n-gon.

Ans] what is n-gone? square n=4 n=5 pentagone n=6



d=2 d=5 d=9

choosing 2 vertices for the edge and removing 'n' edges from total.

↓ border/outline.

$$\text{Total no. of diagonals} = \binom{n}{2} - n$$

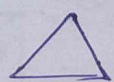
$$= \frac{n!}{2!\,(n-2)!} - n$$

$$= \frac{n(n-1)}{2!} - n = \frac{n^2 - n - 2n}{2}$$

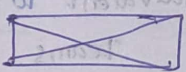
$$= \frac{n(n-3)}{2}$$

The formula is

$\frac{n(n-3)}{2}$ → Now we test this formula,

for $n=3$,  there are no diagonals.

$$\frac{n(n-3)}{2} = \frac{3(3-3)}{2} = 0. ✓$$

for $n=4$, 

$$d = \frac{n(n-3)}{2} = \frac{4(4-3)}{2} = 2 ✓$$

8] A question papers consists of 10 questions divided into parts A and B. Each part consists of 5 questions. A candidate has to answer 6 questions in all of which atleast 2 should be part of A and 2 should be from parts B. In how many ways one can select questions?

10

A B

$\binom{5}{2}$ $\binom{5}{2}$

${}^5C_2 * {}^5C_2 \times {}^6C_2$ ~~wrong~~

	A	B
possibilities:	2	4 ✓
	3	3
	4	2

A	B
5	5
2	4
3	3
4	2
1 ✗	5 ✗
5 ✗	1 ✗
0 ✗	6 ✗

Rule of product

$^5C_2 \times {^5C_4} = 10 \times 5 = 50$

$^5C_3 \times {^5C_3} = 10 \times 10 = 100$ (Addition by Rule of sum.)

$^5C_4 \times {^5C_2} = 5 \times 10 = 50$

200

Now, we learn the both concepts of Permutation & Combinations. But the important thing is that to know/understand the difference between

Permutation	& Combination.
order matters	Order doesn't matter
Arrangement	Selection/ Choosing

COMBINATIONS with Repetition:-

An ice-cream vendor sells 3 flavors:

· Vanilla Chocolate Mango

Assume 10 kids visit the shop, they want one ice-cream each

Possibilities of selling the ice-creams are -

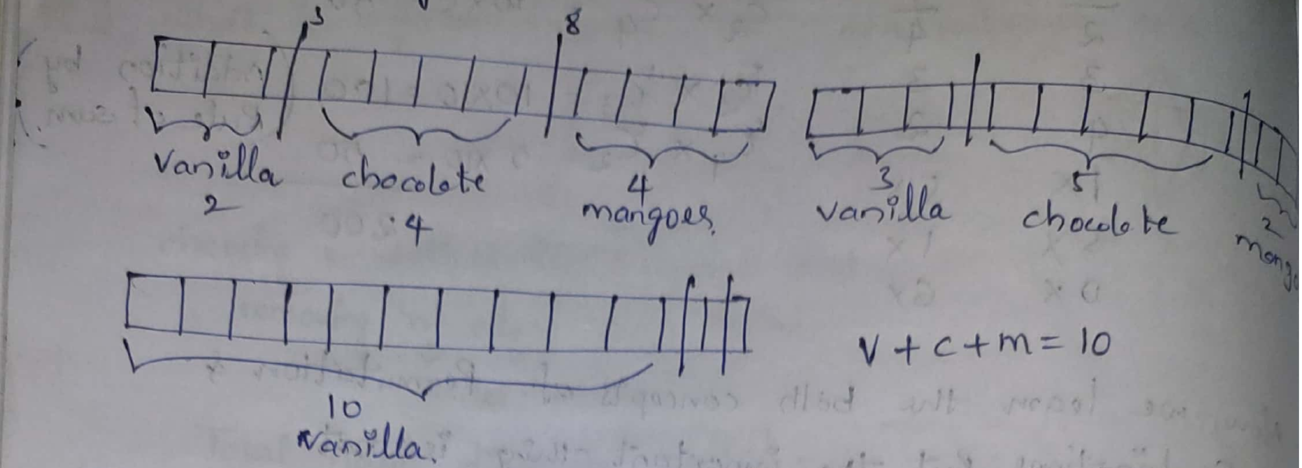
- 10 vanilla
- 10 chocolates
- 10 Mangoes
- 3 vanilla, 2 chocolate, 5 mangoes

Q: In how many ways can the vendor sell 10 ice-creams of 3 flavors?

Note: 10 kids can buy ice-creams in so many ways. this question, see it in the perspective of the vendor.

In how many ways can he sell 10 ice-creams with 3 flavors?

> Think in a way that, take some containers.



$V + C + M = 10$

> The total no. of ways in which the shopkeeper can sell 10 ice-creams with 3 dif. flavors.

⇓ is same as

★ choosing 2 ~~stick~~ containers positions from total 12 containers.

In how many ways can we put 2 sticks in 12 containers separating 10 containers into 3 categories?

★ You should understand this very clearly, deeply. This is important problem, & it appears in dif. forms in dif. places. You should quickly recognize the question & you should know the answer.

$\boxed{\text{Ans}}$: Now you got the answer that, choosing 2 containers from 12, to separate 10 containers into 3 catagories.

So, $10 + \underset{}{\overset{\nearrow \text{catagories}}{3}} - 1 \text{ Choose } \underset{\downarrow\ \text{catagories.}}{3-1} = {}^{12}C_2$

Generally, n – No. of ice-creams to sell

r – No. of flavors

Total no. of ways in which you can sell 'n' ~~ice-cre~~ objects of 'r' catagories

$$= \binom{n+r-1}{r-1}$$

In how many ways can we write 100 as sum of 4 numbers, $a+b+c+d = 100$? , a, b, c, d are non-negative

1 possible: $30+20+30+20 = 100$ $\qquad 100+0+0+0 = 100$

2 - $30+30+20+20 = 100$ $\qquad 0+100+0+0 = 100$

$0+0+100+0 = 100$

100 slots & 3 separators.

103 slots, 3 slots to separate 100 slots into 4 catagories.

$\binom{100+3}{3}$ $\qquad n = 100$, $r = \cancel{3}4$ $\qquad \binom{n+r-1}{r-1}$

$$= \binom{103}{3} = \frac{103 \times \cancel{102}^{\cancel{34}\,17} \times 101}{\cancel{3} \times \cancel{2} \times 1} = 103 \times 101 \times 17$$

Q) Assume there is a jar, it contains 7 different colors of candys? In how many ways can we fill this  with VIBGYOR. In how many ways can we fill this candies of 7 colors?

This is same as, In how many ways can we write 100 as

$$100 = a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_6 + a_7 \; ?$$

This is, choosing 6 slots from 106 slots, to separate 100 slots into 7 different categories.

Here, $n = 100$, $r = 7$

Total no. of ways we can fill the jar is

$$= \binom{106}{6} = \frac{106 \times 105 \times 104 \times 103 \times 102 \times 101}{6 \times 5 \times 4 \times 3 \times 2}$$

<u>Advice</u>:- Don't see the formula always, try to solve it w/o the formula, and try to get used to the formula.

BINOMIAL THEOREM

To understand Binomial concept, first we go step by step, very basic way.

> what is $(a+b)^2 = a^2+2ab+b^2$

Interesting aspects of math starts ~~with~~ from a very elementary example

> Motivation : A Question.

How do we get the expansion of $(a+b)^2$?

$$(a+b)(a+b) = a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

what if we take $(a+b)^3$?

cell

$$\underbrace{(a+b)}_{\text{cell}}(a+b)(a+b) = a^3 + a^2b + a^2b + ab^2 + \dots$$

Question: How is the multiplication done here?

Answer: Pick exactly one element from ~~one~~ individual cells

> What are the total possible ways in which you can pick one element per cell and obtain a term on the right-hand side?

$$(a+b)(a+b)(a+b) = a^3 + 3a^2b + 3ab^2 + b^3$$

once thrice once.

> what if we take $(a+b)^5$? $(a+b)^n$

$(a+b)^2$ — easy

$(a+b)^3$ — easy

$(a+b)^5$ — we can

$(a+b)^n$ — ?

Visualise: $(a+b)^{10}$

$\underbrace{(a+b)(a+b)(a+b)+\dots(a+b)}_{10 \text{ times}} = 1a^{10} + (?)a^9 b' + (?)a^8 b^2$

$1a^{10}$: picking all a's in only 1 way

$(?)a^9 b$: picking '1' 'b' from 10 cells, 9 a's from rest cells. (in 10 ways)

$+ (?)a^8 b^2$ → No. of ways of picking 2 b's from 10 cells. (or) No. of ways of picking 8 a's from 10 cells.

$\binom{10}{2} = \binom{10}{8}$

$+ \binom{10}{3} a^7 b^3 + \binom{10}{4} a^6 b^4 + \binom{10}{5} + a^5 b^5$

$+ \binom{10}{6} a^4 b^6 + \binom{10}{7} a^3 b^7 + \binom{10}{8} a^2 b^8 + \binom{10}{9} a^1 b^9$

$+ \binom{10}{10} a^0 b^{10}$

Theorem:

Let 'a' and 'b' be variables, & let 'n' be a non-negative integer, then

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j$$

> Now, we know how to do with 'n' terms

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

$\binom{n}{1} a^{n-1} b$: picking 1 b from 'n'

- Binomial Theorem Applications:

1) $\left(1+\frac{1}{n}\right)^n = \binom{n}{0} 1^n \left(\frac{1}{n}\right)^0 + \binom{n}{1} 1^{n-1}\left(\frac{1}{n}\right)^1 + \binom{n}{2} 1^{n-2}\left(\frac{1}{n}\right)^2 + \dots \binom{n}{n} 1^0 \left(\frac{1}{n}\right)^n$

$= \binom{n}{0} 1 \cdot 1 + n \cdot 1 \cdot \frac{1}{n} + \frac{n(n-1)}{2} \times 1 \cdot \frac{1}{n \cdot n} + \frac{n(n-1)(n-2)}{3 \cdot 2 \cdot 1} \frac{1}{n^3} + \dots \binom{n}{n}\left(\frac{1}{n}\right)^n$

$= 1 + 1 + \frac{n-1}{2n} + \frac{(n-1)(n-2)}{6n^2} + \dots \binom{n}{n} \frac{1}{n^n}$

$= \frac{1}{0!} + \frac{1}{1!} + \frac{n(1-\frac{1}{n})}{2n} + \frac{n(1-\frac{1}{n})n(1-\frac{2}{n})}{3!\, n^2} + \frac{n(1-\frac{1}{n})n(1-\frac{2}{n})n(1-\frac{3}{n})}{4!\, n^3}$

$+ \dots + \frac{(1-\frac{1}{n})(1-\frac{2}{n})\dots(1-\frac{n-1}{n})}{n!}$

n is sufficiently large.

So, $\frac{x}{n} \to$ very close to zero

$= \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots + \frac{1}{n!} = \sum_{k=0}^{\infty} \frac{1}{k!}$

EULER'S Number:

$$\boxed{\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{1}{k!}}$$

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!}} \rightarrow (e) \text{ Euler's Number.}$$

2] Application :-

Derivative of x^n involves binomial theorem

$$\boxed{\frac{d}{dx}(x^n) = n.x^{n-1}}$$

This derivation ~~proving~~ involves the binomial theorem.

3] 2^n

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k$$

if $x=1=y$, then,

$$(1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}.1^k$$

$$\boxed{2^n = \sum_{k=0}^{n}\binom{n}{k}}$$ 2^n is the sum of binomial co-efficients

$$2^n = \binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\dots+\binom{n}{n}$$

→ Sum of choosing k objects from n objects, where $0 \le k \le n$

> Properties of BINOMIAL THEOREM:

$(x+1)^n = ?$

$$(1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k.1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}x^k$$

$$\boxed{(x+1)^n = \sum_{k=0}^{n}\binom{n}{k}x^k}$$

To evaluate / know / calculate particular term from binomial series;

$$(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \cdots + \binom{n}{n} x^0 y^n$$

$\binom{n}{0} x^n y^0$ ↓ 1st term, $\binom{n}{1} x^{n-1} y^1$ ↓ 2nd term, $\binom{n}{n} x^0 y^n$ ↓ $(n+1)^{th}$ term

The r^{th} term in binomial series $= \binom{n}{r-1} x^{n-r+1} y^{r-1}$

$$\boxed{n^{th} \text{ term} = \binom{n}{n-1} x^1 y^{n-1}}$$

3rd term $= \binom{n}{2} x^{n-2} y^2$

$$\boxed{T_r = \binom{n}{r-1} x^{n-r+1} y^{r-1}}$$

> Middle Term in the expression of $(a+b)^n$:

- If 'n' is even, $\frac{n+2}{2}^{th}$ term is the middle term.

$$T_{\frac{n+2}{2}} = \binom{n}{\frac{n+2}{2}-1} a^{n-\frac{n+2}{2}+1} b^{\frac{n+2-2}{2}}$$

$$\boxed{T_{\frac{n+2}{2}} = \binom{n}{n/2} a^{n/2} b^{n/2}}$$

for ~~even~~ any n, there will be '$n+1$' terms in the binomial expression.

- If n is odd, there will be even no. of terms in the expression, so there exists 2 middle terms for odd 'n'.

$\left(\frac{n+1}{2}\right)^{th}$ term & $\left(\frac{n+3}{2}\right)^{th}$ term are the middle terms.

$$\boxed{T_{\frac{n+1}{2}} = \binom{n}{\frac{n-1}{2}} a^{n+1/2} b^{\frac{n-1}{2}}}$$

$$\boxed{T_{\frac{n+3}{2}} = \binom{n}{\frac{n+1}{2}} a^{\frac{n-1}{2}} b^{\frac{n+1}{2}}}$$

> The largest coefficient in the expansion of $(a+b)^n$ is the coefficient of middle term.

GATE

2022

1) The no. of arrangements of 6 identical balls in 3 identical bins is ______

Combinations with repetition

$$\binom{6+3-1}{3-1} = \binom{8}{2} = 28$$

2020

2) The no. of Permutations of the characters in LILAC so that no character appears in its original position. If the 2 L's are indistinguishable is ______

> There are 3 choices for the 1st slot, & then 2 for 3rd slots.

IAC → choose 2 for 1st, 3rd slots.

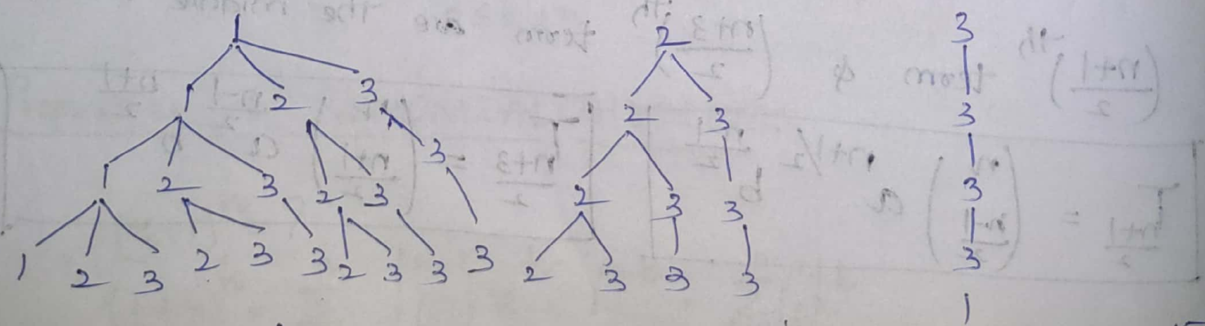
3 2 1 2 __ remaining one from IAC & 2 L's

The remaining IAC the choices are 2, for the L, is distinguishable, only one choices

$3 \times 2 \times 1 \times 2 = 12$

2015

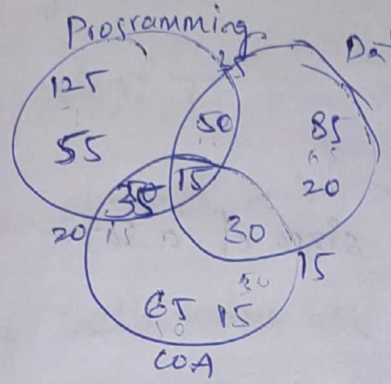
3) The no. of 4 digit number having their digits in non-decreasing order from (left-right) by the digits (1, 2, 3) ______



10 4 $10+4+1 = 15$

2004
200 - total students

9)
125 - Programming language course
85 - Data structures
65 - Computer Organization
50 - both programming & ~~computer organization~~ DS



$$55+20+15-35-20-15+15$$

$$125+85+65-50-35-30+15$$

$$200-175=25$$

2003

9) m identical balls are to be placed in 'n' distinct bags. You are given that $m \geq kn$, where k is a natural no. ≥ 1. In how many ways can the balls be placed in the bags if each bag must contain atleast k balls?

$$\binom{m-kn+n-1}{n-1}$$

2016

9) x^{12} in $(x^3+x^4+x^5+x^6+\ldots)^3$ is ___

3

$$\left(x^3(1+x+x^2+x^3)\right)^3$$

$$x^9\left(1+(x+x^2+x^3)\right)^3$$

$$x^9\, x\left[(1+x+x^2)^3 + 3x(1+x+x^2)^2 + 3(x+x^2+x^3)+1\right] \quad 3x^3$$

$$x(1+x+x^2)^3$$

$$x\left[(x+x^2)^3 + 3(x+x^2)^2 + 3(x+x^2)+1\right]$$

$$x^3+3x^4+3x^5+x^6$$

SET Theory:-

What is SET? How to denote?

sets, elements, subsets

> Specifying sets

> Subsets (Properties)

1. $A \subseteq A$
2. $A \subseteq B$ & $B \subseteq A$ then $A = B$
3. $A \subseteq B$ & $B \subseteq C$, then $A \subseteq C$

> Empty set, Univeral set $\rightarrow$ The sizee of a set

$$\phi \subseteq A \subseteq U$$

> Disjoint Sets : $A \cap B = \phi$

> Set Operations [Union, Intersection, disjoint Union

$A \subseteq A \cup B$ $A \cap B \subseteq A$ ~~to A+ disjoint~~

$B \subseteq A \cup B$ $A \cap B \subseteq B$

> If $A \subseteq B$ & $A \cap B = A$, then $\boxed{A \cup B = B}$

$\{x \mid x \in A \text{ \& } x \notin B\}$ $\{x \mid (x \in A \text{ or } x \in B) \text{ and } x \notin A \cap B\}$

> Complements, Differences, Symmetric. Differences.

$A^c, A', \bar{A}$ $A \setminus B$ or $A - B$ $A \oplus B$, $A \Delta B$ $\rightarrow$ DeMorgan's Law.

> Power sets:- $|P(S)| = 2^{|S|}$

> Counting in Sets

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \setminus B) = n(A) - n(A \cap B)$$

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$$

> Sets of sets objects;

Logic:-

→ Proposition (statement). Declarative $1+1=3$ - F $2+2=4$ - T

→ Compound Propositions.
primitive proposition

→ Basic logical operators. Conjunction (and, but) $P.q$ $P \& q$ $P \wedge q$, Disjunction p or q $P+q$ $P \vee q$

- conjunction:- TT

Negation $\neg p$ p', $\sim p$, $\bar{p}$

> Precedence $(\neg > \wedge > \vee) > \rightarrow > \leftrightarrow$

$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

> Finding Truth value by truth table. $(p \vee q) \wedge \sim p$

$\boxed{p \rightarrow q \equiv \sim p \vee q}$

T	1
F	0

> Tautologies & Contradictions.
$P \vee \sim P$ $P \wedge \sim P$

> Logical Equivalence.

$$P(p,q,\ldots) \equiv Q(p,q,\ldots)$$

$$\sim(P \wedge Q) \equiv \sim P \vee \sim Q$$

- Algebra of Propositions:- what is algebra?
Algebra is nothing playing with objects and operation

1. Idempotent law: $p \vee p \equiv p$ | $p \wedge p \equiv p$
2. Associative law: $(p \vee q) \vee r \equiv p \vee (q \vee r)$ | $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
3. Commutative: $p \vee q \equiv q \vee r$ | $p \wedge q \equiv q \wedge p$
4. Distributive: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$
5. Identity: $P \vee F = P$ | $P \wedge T \equiv P$
Domination: $P \vee T = T$ | $P \wedge F = F$
6. Involution: $\sim\sim p = p$
7. Complement: $p \vee \sim p = T$, $\sim T = F$ | $p \wedge \sim p = F$, $\sim F = T$
8) DeMorgan's: $\sim(p \vee q) \equiv \sim p \wedge \sim q$ | $\sim(p \wedge q) \equiv \sim p \vee \sim q$
9) Absorption: $\cancel{p \vee \sim p \equiv T}$ $p \vee (p \wedge q) \equiv p$ | $\cancel{p \wedge \sim p = F}$ $p \wedge (p \vee q) \equiv p$

> Conditional & Biconditional.

If p then q

$p \rightarrow q$ (h, c)

p implies q

T (F) F

P if and only if q

$P \leftrightarrow q$ (p & q both are T)

T (T) T

F (T) F

You can take flight if & only if you buy a ticket

Exclusive OR (Δ)

TF (T)

FT (T)

$P \rightarrow q \equiv \sim p \vee q$ if n even, n+1 & n-1 are odd

Logical Equivalence with algebra :- ① $\sim(P \rightarrow q) \equiv \cancel{\sim p \vee q}$ $P \wedge \sim q$

by Truth table ① $\cancel{(p \vee \sim q) \rightarrow (p \wedge q)}$

② $\sim(P \vee (\sim p \wedge q)) \equiv \sim p \wedge \sim q$

③ $(p \wedge q) \rightarrow (p \vee q) \equiv T$

④ $p \rightarrow q \equiv \sim q \rightarrow \sim p$

⑤ $(p \rightarrow q) \wedge (p \rightarrow r) \equiv p \rightarrow (q \wedge r)$

$\boxed{\text{Q}}$:- p: It is freezing

q: It is snowing.

a) It is freezing & snowing $p \wedge q$

b) It is freezing but not snowing $p \wedge \sim q$

c) It isn't freezing & it isn't snowing $\sim p \wedge \sim q$

d) It is either freezing or snowing or both $p \vee q$

e) If it is freezing, it also snowing $p \rightarrow q$

f) Either freezing or snowing but is not snowing if it is freezing

$(p \oplus q) \wedge \sim (p \rightarrow \sim q)$

g) It is freezing is necessary and sufficient for it is snowing

$p \rightarrow q$

> Propositional Satisfiability:-

Logical Equivalences:

$p \to q \equiv \sim p \vee q$

$(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$

$(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$ $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$

- $p \leftrightarrow q = (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \equiv \sim p \leftrightarrow \sim q$

$p \leftrightarrow q \equiv (p \wedge q) \vee (\sim p \wedge \sim q)$

$\sim (p \leftrightarrow q) \equiv p \leftrightarrow \sim q \equiv \sim p \leftrightarrow q$

p: It is rained

q: The weather is good

$p \to q$: When it rains, the weather is good.

Converse: when the weather is good, it implies that it has rained

Conditional $p \to q$	Converse $q \to p$	Inverse $\sim p \to \sim q$
T	T	T
F	T	T
T	F	F
T	T	T

$q \to p$: Converse of $p \to q$

$\neg p \to \neg q$: Inverse of $p \to q$

$\neg q \to \sim p$: Contra positive of $p \to q$

$$\boxed{p \to q \equiv \sim q \to \sim p}$$

> XOR:-

p	q	$p \veebar q$
0	0	0
0	1	1
1	0	1
1	1	0

Exclusive OR

p	q	r	$p \veebar q$	$p \veebar q \veebar r$
0	0	0	0	0
0	0	1	0	1
0	1	0	1	1
0	1	1	1	0
1	0	0	1	1
1	0	1	1	0
1	1	0	0	0
1	1	1	0	1

Q] Truth table for $p \veebar (p \vee q)$

p	q	$p \vee q$	$p \veebar p \vee q$
0	0	0	0
0	1	1	1
1	0	1	0
1	1	1	0

Q] The ~~the~~ value ~~is~~ of $p \wedge (p \Rightarrow q)$ is ______ ?

GATE

p	q	$p \Rightarrow q$	$p \wedge (p \Rightarrow q) \Leftrightarrow p \wedge q$	
0	0	1	0	F
0	1	1	0	F
1	0	0	0	F
1	1	1	1	T

a] false ii] q iii] true iv] $p \vee q$ v] $\neg q \vee p$

The no. of expressions given above are logically implied by

$p \wedge (p \Rightarrow q)$ is ______

Let's assume $p \wedge (p \Rightarrow q)$ is 'x'.

i] false (F)

x	F	$x \Rightarrow F$
T	F	F
F	F	T
F	F	T
F	F	T

x is not logically implies 'F'

ii] q

~~x is not~~

x	q	$x \Rightarrow q$
T	T	T
F	F	T
F	T	T
F	F	T

x is logically implies q

iii)

x	T	$p \vee q$	$\sim q \vee p$	$x \to T$	$x \to p \vee q$	$x \to \sim q \vee p$
T	T	T	T	T	T	T
F	T	T	T	F	F	T
F	T	T	F	F	F	T
F	T	F	F	F	T	T

4 are logically implied by $p \wedge (p \to q)$.

(not now)

8) p: $x \in \{8, 9, 10, 11, 12\}$

q: x is a composite number

r: x is a perfect square

s: x is a prime number.

The integers $x \geq 2$ which satisfies $\neg((p \to q) \wedge (\neg r \vee \neg s))$ is _____?

Simplify the proposition.

$= \neg(p \to q) \vee \neg(\neg r \vee \neg s)$

$= \neg(\neg p \vee q) \vee \sim(r \wedge s)$

$= (p \wedge \sim q) \vee (r \wedge s)$

$p \to q$: gives the set $x \in \{8, 9, 10, 12\}$

$\neg r$: $x \in \{8, 10, 11, 12\}$

$\neg s$: $x \in \{8, 9, 10, 12\}$

$\neg r \vee \neg s$: $x \in \{8, 9, 10, 11, 12\}$

$(p \to q \wedge (\neg r \vee \neg s))$: $x \in \{8, 9, 10, 12\}$

$\sim x$: 11

Q] F1 : $p \to \neg p$ → satisfiable

F2 : $(p \to \sim p) \vee (\sim p \to T)$ → satisfiable

p	~p	(p → ~p)	v	(~p → T)	
T	F	F	T	T	T
F	T	T	T	T	T

Q] I stay only if you go.

If $\underset{p}{\underline{\text{you go}}}$, then $\underset{q}{\underline{\text{I'll stay}}}$

$p \to q$ converse is : $q \to p$

If I'll stay, then you'll go

Q] i) If $\underset{p}{\underline{\text{it is cold}}}$, $\underset{q}{\underline{\text{he wears a hat}}}$

$\boxed{p \to q \equiv \sim p \vee q}$

It is not cold or he wears a hat

ii) If $\underset{p}{\underline{\text{productivity increases}}}$, then $\underset{q}{\underline{\text{wages rise}}}$

Productivity doesn't increase or wages rise.

converse : $q \to p$ If wages rise, productivity increases.

Inverse : $\sim p \to \sim q$ If productivity doesn't increase, then wages doesn't rise.

Contra Positive : $\sim q \to \sim p$ If wages doesn't rise, productivity doesn't increase.

Q) i) If $\underbrace{\text{Erik is a poet}}_{p}$, then $\underbrace{\text{he is poor}}_{q}$.

Given $p \to q$:

Contrapositive of $p \to q$ is $\sim q \to \sim p$

$\sim q \to \sim p$: If ~~Erik~~ Erik is not poor, then he isn't a poet.

ii) Only if Marc studies well, he pass the test.

$\Updownarrow$

$p \to q$: If $\underbrace{\text{Marc passes the test}}_{p}$, then only $\underbrace{\text{he studied well}}_{q}$.

Contra positive: If Marc doesn't study well, he won't pass the test.
$\sim q \to \sim p$

Q) Negate each statement.

a) $\underbrace{\text{she works}}_{p}$ and $\underbrace{\text{she will not earn money}}_{\sim q}$.

Given statement: $p \wedge \sim q$

Negate the proposition

$$\sim(p \wedge \sim q) = \sim p \vee q \quad [\, p \to q \equiv \sim p \vee q \,]$$

$$\equiv p \to q$$

If she works, then she will earn money.

b) $\underbrace{\text{He swims}}_{p}$ if and only if $\underbrace{\text{water is warm}}_{q}$.

$\Leftrightarrow / \leftrightarrow$

Given $p \leftrightarrow q$. $\boxed{\sim(p \leftrightarrow q) \equiv \sim p \leftrightarrow q \equiv p \leftrightarrow \sim q}$

He doesn't swim, if and only if water is warm

He swim, if and only if, water isn't warm

c) If $\underbrace{\text{it snows}}_{p}$, then $\underbrace{\text{they don't drive the car}}_{\sim q}$.

Given $p \to \sim q$ $\boxed{\sim(p \to q) \equiv p \wedge \sim q}$

Q) Logical Equivalence

p and q said to be equivalent if $p \leftrightarrow q$ is tautology.

Ex) $(p \wedge q) \rightarrow (p \vee q) \equiv T$

$(p\wedge q) \rightarrow (p \vee q) \equiv \neg[(p \wedge q) \vee (p \vee q)]$ [$p \rightarrow q \equiv \sim p \vee q$]

$\equiv (\sim p \vee \sim q) \vee p \vee q$ [By DeMorgan's law]

$\equiv (\sim p \vee p) \vee (\sim q \vee q)$ [Associative and commutative]

$\equiv T \vee T$ [~~$p \vee T \equiv T$~~] [Negation law / Complement]

$\equiv T$ [$p \vee T \equiv T$] Domination law

Q) i) $(p \wedge q) \rightarrow p \equiv T$

$(p\wedge q) \rightarrow p \equiv \sim(p \wedge q) \vee p$ [$p \rightarrow q \equiv \sim p \vee q$]

$\equiv (\sim p \vee \sim q) \vee p$ [DeMorgan's law]

$\equiv \sim p \vee \sim q \vee p$ [Associative law]

[$p \vee q$ $\equiv (\sim p \vee p) \vee \sim q$ [commutative]

$\equiv T \vee \sim q$ [$p \vee T \equiv T$ Domination]

$\equiv T$

ii) b: $p \rightarrow (p \vee q) \equiv T$

Logic operators:-

$\sim$	not	$\sim p$ T	p F	
$\wedge$	and, but	$p \wedge q$ T	p T	q T
$\vee$	or	$p \vee q$ F	p F	q F
$\rightarrow$	implication	$p \rightarrow q$ F	p T	q F
$\leftrightarrow$	iff	$p \leftrightarrow q$ T	p T F	q T F

$\rightarrow$ True can't imply False.

$p \rightarrow q \rightarrow$ conclusion

$\downarrow$

hypothesis, premise

Q If q is true in $P \rightarrow q$?

p	q	$p \rightarrow q$
T	T	T
F	=	T

if $q = T$, $p \rightarrow q = T$

if $p = F$, $p \rightarrow q = T$

When p is T, $p \rightarrow q$ depends on q.

p	q	$p \rightarrow q$
T	F	F
	T	T

$(p \wedge \neg q)' \equiv \boxed{\sim p \vee q} \equiv p \rightarrow q$

Q No.of propositional functions using 'n' propositional variables.

2^{2^n}

Q No.of lines of truth table in which the proposition $p \wedge q \wedge r$ is true.

No.of lines which leads to $p \wedge q \wedge r$ true is only one assignment (1)

$p=1, q=1, r=1$

Q No.of lines of truth table in which the proposition $p \vee q \vee r$ is true.

$2^3 = 8 - 1 = 7$

$\downarrow$

$p \vee q \vee r$ is false only when $p=0, q=0, r=0$

Q. No. of lines of TT, in which proposition $(p \wedge q \wedge r) \rightarrow s$ is true.

First check, when $(p \wedge q \wedge r) \rightarrow s$ is false

$T \quad F \rightarrow F$

Now, when $p \wedge q \wedge r$ is True.

only in 1 assignment $p=1, q=1, r=1$

There is only 1 line $(p \wedge q \wedge r) \rightarrow s$ is false.

Total lines $= 2^4 = 16$

true lines $= 16 - 1 = 15$

Implication:
- > p implies q
- > If p, then q
- > p only if q → I'll teach only if you interact.
  The inner meaning of this is
  If you don't interact, I'll not teach
  $\sim q \rightarrow \sim p$
- > q if p (q when p)
- > q follows from p

p – antecedent, premise, hypothesis
q – consequent, conclusion
$\sim q \rightarrow$ if $\sim q$, then $\sim p$

- > q unless $\sim p$ ≡ q if not $\sim p$ ≡ q if p
- > p is sufficient condition for q (enough)
- > q is necessary condition for p

If f(x) is differentiable, then it is continous.
is sufficient condition for — necessary condition

Examples:-

x unless $\sim p$: $p \rightarrow x$

l only if m : $l \rightarrow m$

l if m : $m \rightarrow l$

$\sim x$ unless p : $\sim p \rightarrow \sim x$

If it rains, I stay. : $p \rightarrow q$

Given statement : $p \rightarrow q$

Converse : $q \rightarrow p$

Inverse : $\sim p \rightarrow \sim q$

Contrapositive : $\sim q \rightarrow \sim p$

Q) What is converse of "I stay only if you go". (GATE 2001)

(p = I stay, q = you go)

✓ a) I stay if you go b) If you don't go then I don't stay

c) If I stay then you go d) If I don't stay then you go.

Given statement is : $p \to q$

converse $(q \to p)$: If you go, then I stay.

Tautology: The proposition which is always true.

(Valid, T) $\boxed{p \vee \sim p \equiv T}$

Contradiction: A proposition, which is always False

(Absurdity, F) $\boxed{p \wedge \sim p \equiv F}$

Satisfiable: A proposition, which is True for atleast 1 combination or assignment, is called as satisfiable.

Q) S1: Every T is satisfiable — True

S2: Every satisfiable is T — False

S3: Every contradiction is not satisfiable — True

Contigency: A proposition, which is satisfiable but not valid.

(or)

A proposition neither Tautology nor Contradiction

Q) S1: Every contingency is Tautology — False

S2: Every contingency is satisfiable — True

S3: Every satisfiable is contingency — False

Q) The proposition $p \to (q \vee r)$ is ?

a) Tautology b) Contradiction ✓ c) Contingency d) None

p	q	r	$q \vee r$	$p \to (q \vee r)$
T	F	F	F	F
.	.	.	.	T

→ false for one assignment

Q] The proposition $(p \wedge q) \rightarrow (p \vee q)$ is?

✓a] Tautology   b] Contradiction   c] Contingency   d] None

p	q	$p \wedge q$	$p \vee q$	$\overset{T}{(p \wedge q)} \rightarrow \overset{F}{(p \vee q)}$
F	F	T	F ✗	F
T	T	↓	↓	

→ So such case that the given expression is become False

p should T
q should T

If $p \wedge q = T$
$p \vee q = T$

Q] Let p, q, r propositions, and $\overset{T}{(p \rightarrow q)} \rightarrow \overset{F}{r}$ be a contradiction. Then the proposition $(r \rightarrow p) \rightarrow q = ?$

a] Tautology   b] always true when p is false

c] a contradiction   ✓d] always true when q is true.

p	q	r	$\overset{T}{(p \rightarrow q)}$	$\overset{T}{(p \rightarrow q)} \rightarrow \overset{F}{r}$	$\overset{T}{(r \rightarrow p)} \rightarrow q$	
T	T	F		should be contradiction	T	T
F	F	F			T	F
F	T	F			T	T

$(r \rightarrow p) \rightarrow q$

$(F \rightarrow p) \equiv T$

$\boxed{T \rightarrow q} \equiv q$

If 'q' is T, then $T \rightarrow q$ is T

$\boxed{p \rightarrow T \equiv T}$ $\boxed{T \rightarrow q \equiv q}$ $\boxed{F \rightarrow p \equiv T}$

Q] Which is below True, when exactly 2 of p, q, r are TRUE?

✗ a] $((p \leftrightarrow q) \wedge r) \vee (p \wedge q \wedge \sim r)$ ✓b] $(\sim(p \leftrightarrow q) \wedge r) \vee (p \wedge q \wedge \sim r)$

✓ c] $((p \rightarrow q) \wedge r) \vee (p \wedge q \wedge \sim r)$ ✗d] $(\sim(p \leftrightarrow q) \wedge r) \wedge (p \wedge q \wedge \sim r)$

p	q	r	$p \leftrightarrow q$	$p \rightarrow q$	$(p \rightarrow q) \wedge r$
0	1	1	0	1	1
1	0	1	0	1	1
1	1	0	1	1	0

↑ $p \wedge q \wedge \sim r$ is True

3] Exportation law:-

$p \rightarrow (q \rightarrow r) \equiv (p \wedge q) \rightarrow r$

Logical Implication:- An implication which is a Tautology.

Inference :- (An argument)

$(P_1 \wedge P_2 \wedge P_3 \wedge \cdots \wedge P_n) \rightarrow q$

An inference is also an implication, which is Tautology is

Rule of Inference.

Q] $S1: (\sim p \wedge (p \vee q)) \rightarrow q$ $\quad S2: q \rightarrow (\sim p \wedge (p \vee q))$

Let p, and q be 2 propositionals.

	p	q	~p	~p∧(p∨q)
S1:	F	F	T	
S2:	T	T	F	

S1: $\underbrace{F \quad F}_{\times p\vee q \equiv T \times}$ So S1 is logical implicance.

S2: T T F $\quad p \vee q \equiv T \quad q \rightarrow (\sim p \wedge (p \vee q))$ with $q = T$, $\sim p \wedge (p \vee q) = F$, $p = T$, $q = T$

$q \not\rightarrow (\sim p \wedge (p \vee q))$ Not a logical implication.

Quantifiers :-

Form	Meaning
1. $\forall x\, P(x)$	All true
2. $\exists x\, P(x)$	Some true, atleast one true
3. $\sim\forall x\, P(x)$	Not all true
4. $\sim\exists x\, P(x)$	None true
5. $\forall x \sim P(x)$	All false
6. $\exists x \sim P(x)$	some false, atleast one false
7. $\sim\forall x \sim P(x)$	Not all false
8. $\sim\exists x \sim P(x)$	None false

1. $\forall x\, P(x) \equiv \sim\exists x \sim P(x)$
2. $\exists x\, P(x) \equiv \sim\forall x \sim P(x)$
3. $\sim\forall x\, P(x) \equiv \exists x \sim P(x)$
4. $\sim\exists x\, P(x) \equiv \forall x \sim P(x)$

These helps in negating the quantified predicate.

Negating the Predicates and Quantifiers :-

Negation of $\forall x \equiv \exists x$ $\sim\forall x \equiv \exists x$

$\sim\exists x \equiv \forall x$

$\sim[P(x)] \equiv \sim P(x)$

Ex: $\sim[\forall x\, P(x)] \equiv \cancel{\sim\forall}\ \exists x \sim P(x)$

Ex1) $\sim[\forall x \exists y \forall z\, P(x,y,z)]$

$\equiv \exists x \forall y \exists z \sim P(x,y,z)$

Ex2) $\sim[\forall x \forall y \forall z [x+y=z]]$

$= \exists x \sim[\forall y \forall z (x+y=z)]$

$= \exists x \exists y \sim[\forall z (x+y=z)]$

$= \exists x \exists y \exists z \sim[x+y=z]$

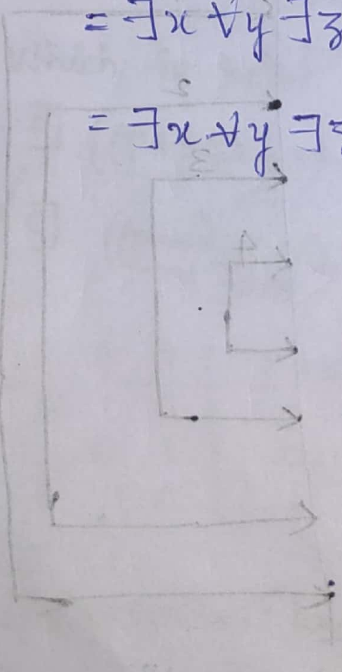
$= \exists x \exists y \exists z\ x+y \neq z.$

Ex3) $\sim\forall x \exists y \forall z [P(x,y,z) \rightarrow Q(x,y,z)]$

$= \exists x \forall y \exists z \sim[\sim P(x,y,z) \vee Q(x,y,z)]$ [Law of Implication

$= \exists x \forall y \exists z (P(x,y,z) \wedge \sim Q(x,y,z))$

Q



Rules of Inferences / Arguments :-

An argument is a sequence of statements / propositions.
All propositions except the final proposition are called as premises.
The final one is called as conclusion.

An argument is valid if final proposition is True, whenever all the premises are True.

An argument is an assertion that given set of propositions $P_1, P_2, \dots P_n$ called premises, yields another proposition Q, called the conclusion. Such argument is denoted

$$P_1, P_2, P_3, \dots P_n \vdash Q$$

> Argument $P_1, P_2, \dots P_n \vdash Q$ is said to be valid if Q is true whenever all the premises $P_1, P_2, P_3, \dots P_n$ are true.

> An argument which is not valid is called *fallacy*.

> An argument $P_1, P_2, P_3, \dots P_n$ is valid iff

$$(P_1 \wedge P_2 \wedge \dots \wedge P_n) \rightarrow Q \text{ is a tautology.}$$

statements $\longrightarrow$ conclusion

Ex: $p \rightarrow q, q \vdash p$

$(p \rightarrow q) \wedge q \rightarrow p$ is a fallacy

$p, p \rightarrow q \vdash q$ (Modus Ponens) (Law of Detachment)

$p \wedge (p \rightarrow q) \rightarrow q$ is T

3 $p \rightarrow q, q \rightarrow r \vdash p \rightarrow r$ (Law of Syllogism)

$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$

$P_1 : p \rightarrow q$
$P_2 : q \rightarrow r$
$\therefore p \rightarrow r$

4
$(\sim p)'$
$(p \vee q)'$
$\therefore (q)'$

$p \Rightarrow$ False
$p \vee q$ — True

$(p \vee q)'$
$\neg q$
$\therefore p$

Scanned with CamScanner

Q If u have current psw, then you can log onto the network

You have current Password

what is the conclusion:-

p: You have current password

q: You can log onto the n/w

$p \wedge (p \to q)$

$$\left.\begin{array}{l} p \to q \\ p \end{array}\right\} \to \text{Hypothesis}$$

$\therefore q \to$ Therefore.

Conclusion: You can log onto the n/w.

Ex If it snows today, then we'll go skiing.
It is snowing today.

p: It ~~is~~ snows today

q: We will go skiing

$p \to q$
p

$\therefore q$ [Modus Ponens] (or) [Law of Detachment]

P If $\sqrt{2} > \frac{3}{2}$ then $(\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$. We know that $\sqrt{2} > \frac{3}{2}$.
Consequently $\left((\sqrt{2})^2 = 2\right) > \left[\left(\frac{3}{2}\right)^2 = \frac{9}{4}\right]$

p: $\sqrt{2} > \frac{3}{2}$

q: $(\sqrt{2})^2 > \left(\frac{3}{2}\right)^2$

$p \to q$, $F \to F \equiv T$
p , F

$\therefore q$ [Law of Detachment Modus Ponens] , F

This argument is valid. ~~bec~~

Q If I take day off, it either rains or snows.
I took Tuesday off or I took Thursday off.
It was sunny on Tuesday.
Con: It didn't snow on Thursday.

p: I take day off

q: It rains

r: It snows

s: Day is tuesday

t: Day is Thursday.

$(p \to q)'$ Modus tollens.
$(\neg q)'$

$\therefore \neg p$

$p \to q$
$q \to r$ Hypothetical syllogism

$\therefore p \to r$

$p \vee q \equiv \sim p \to q$ Disjunctive Syllogism
$\neg p$

$\therefore q$

p Addition Rule

$\therefore p \vee q$ $\quad p \to (p \vee q) = T$

Simplification.
$p \wedge q$ $\quad (p \wedge q) \to p = T$

$\therefore p$

p
q $\quad (p \wedge q) \to p \wedge q$
Conjunction

$\therefore p \wedge q$

$p \vee q \quad \sim p \to q$
$\neg p \vee r \quad p \to r$ Resolution Rule.

$\therefore q \vee r$

$(p \to q)'$ Modus tollens.
$(\neg q)'$
$\therefore \neg p$

$p \to q$
$q \to r$ Hypothetical syllogism
$\therefore p \to r$

$p \vee q \equiv \sim p \to q$ Disjunctive syllogism
$\neg p$
$\therefore q$

p Addition Rule
$\therefore p \vee q$ $\quad p \to (p \vee q) = T$

$p \wedge q$ Simplification.
$\therefore p$ $\quad (p \wedge q) \to p = T$

p
q $\quad (p \wedge q) \to p \wedge q$
$\therefore p \wedge q$ Conjunction

$p \vee q \quad \sim p \to q$
$\neg p \vee r \quad p \to r$ Resolution Rule.
$\therefore q \vee r$

$\sqrt{2}$ is rational $\to$ n.

$\sqrt{2} = \frac{p}{q}$ simplest form $\quad 2 = \frac{p^2}{q^2}$

$2q^2 = p^2$

p and q can't divide further

$(\text{Even})^2 = \text{Even}$
$(\text{Odd})^2 = \text{Odd}$

$(\text{even})^2 =$ Multiple of 4

$p^2 = 2.q^2$
even square.

If p^2 even, p is even
and p^2 is multiple of

$2q^2 = 4.r^2$
$q^2 = 2r^2$
q^2 is even $\Rightarrow$ q is even
q^2 is multiple of

So contradiction. (Not a ~~fact~~

c: $\sqrt{2}$ is rational
false
$\sqrt{2}$ is irrational.

Scanned with CamScanner

p: It is raining.
q: weather is very pleasant.
If it is raining today, the weather is very pleasant.
It is raining today.

Given statements $(p \to q)'$
$$\frac{\begin{array}{c} (p \to q)' \\ p' \end{array}}{\therefore (q)'}$$

Conclusion: The weather is very pleasant today.

$$\frac{\begin{array}{c} (p \to q)' \\ q' \end{array}}{p ? \times} \qquad \frac{\begin{array}{c} (\sim q \to \sim p)' \\ q' \end{array}}{}$$

Q) p, q, r

$$\frac{\begin{array}{c} (p \vee q)' \\ (\sim p \vee r)' \\ (\neg r)' \end{array}}{\therefore q'} \qquad \frac{\begin{array}{c} (p' \wedge q')' \\ (q' \to r)' \end{array}}{\therefore r'} \qquad \frac{\begin{array}{c} (q' \wedge p')' \\ (q' \to r')' \\ (r' \to s')' \end{array}}{\therefore s'}$$

* There are some statements, from which ~~which~~ we can't conclude any decisively.

$$\frac{\begin{array}{c} (p \vee q)' \\ (q \vee r)' \end{array}}{\text{can't conclude anything}}$$

Ex)
$$\frac{\begin{array}{c} (p \to q)' \\ r' \to (\sim q)' \\ r' \end{array}}{\therefore (\sim p)'} \qquad \frac{\begin{array}{c} (p' \wedge q')' \\ (p' \to (r' \wedge q'))' \\ (r' \to (s' \vee t'))' \\ (\neg s)' \end{array}}{\therefore t'} \qquad \frac{\begin{array}{c} (p)' \\ (p' \to q')' \\ (p' \to (q' \to r'))' \end{array}}{\therefore r'}$$

Scanned with CamScanner

Modus Ponens :-

$\sim p \vee s$
$\sim t \vee (s' \wedge r')$
$\sim q \vee r$
$p \vee q \vee t$

∴ Show $(r \vee s)'$ is true.

Assume $(r^0 \vee s^0)^0$

$(p \to q)'$
p'
∴ q

$(\sim p)' \vee s^0$
$(\sim t)' \vee (s^0 \wedge r^0)$
$[(\sim q)' \vee r^0]'$
$(p^0 \vee q^0 \vee t^0)'$ contradict
∴ $r \vee s$

Ex. Show : t' Assume t^0 is false [By contradiction]

$((\sim p)^0 \wedge q')' \to$ Contradiction So
$(r' \to p')$
$((\sim r)^0 \to s^0)'$
$(s^0 \to t^0)'$
∴ t

∴ t
(absurdity)

$(\sim p)' \wedge q'$
$r^0 \to p^0$
$(\sim r)' \to s'$
$s \to t'$
∴ t'

$\sim t \to$ Contradiction [absurdity]

$(\sim t) \equiv$ False

$\boxed{t \equiv \text{True}}$

We deduced ~~son~~ the conclusion from given statements.

Q Jasmine is skiing or it is not snowing
It is snowing or Bart is playing Hockey
C: Jasmine is skiing or Bart is playing hockey.

$p \vee (\sim q)'$
$q^0 \vee r$
$(p \vee r)$

$\sim q \vee p$
$q \vee r$
$(p \vee r)$

$(p \vee (\sim q)) \wedge (q \vee r) \to (p \vee r)$
$0 \quad 1 \wedge r$
$p \vee r$ ✓

$p \vee r$; Jasmine is skiing or Bart is playing hockey.

Relations

. Knowing (who knows whom) let's take 5 people. how many possible combination of knowing each other

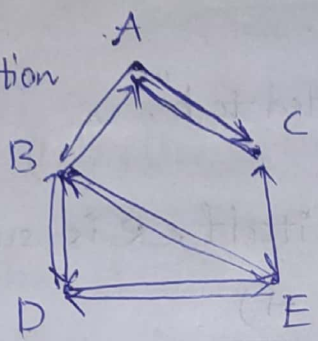
– 2 people

a	b
k	un
k	k
Un	Un
Un	k

, less than, is subset of

The Representation:-

Graphical Representation



$R = \{(A,C), (A,B), (B,D), (B,E) \ldots\}$

Relation on $\{A, B, C, D, E\}$

Elements in the set can be anything.

In maths, we can say,

ARC, ARB, BRA, BRE

BRD, ERB, ERD, ERC,

DRB, DRE, C~~R~~E

50

Matrix Representation:-

	A	B	C	D	E
A	+	1	1	0	0
B	1	+	0	1	1
C	1	0	+	0	0
D	–	1	0	1	1
E	0	1	1	1	1

Any Relation on 'S' is a subset of SXS

Any subset of SXS is a relation on S

Binary Relation:-

Cross product of $A \times B$

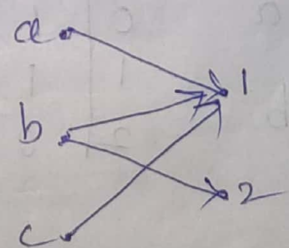
$A = \{a, b, c\}$ $B = \{d, e, f\}$

$ARB = \{(a,d), (a,e), (a,f), (b,d), (b,e), (b,f)$

$cd, ce, cf\}$

$|A \times B| = |A| \times |B|$

$A \times A = \{aa, ab, ac, ba, bb, bc, ca, cb, cc\}$



Binary Relation:- Let A and B are sets. A binary relation or a relation from A to B is subset of $A \times B$.

$\boxed{A \times B = \{(a,b) \mid a \in A, y \in B\}}$

R – relation from A to B.

For each pair (a,b), $a \in A$ and $b \in B$ exactly one of following is true

$\boxed{A \times B \neq B \times A}$

i) $(a,b) \in R$; aRb; "a Related to b".

ii) $(a,b) \notin R$; $a \not R b$; "a isn't Related to b".

If R is a relation from a set A to itself, R is subset of $A^2 = A \times A$. (R is a relation on A)

Ex1 students enrolled in courses
A R B.

The set of all students – A

student enrolled in course – R

set of all courses – B.

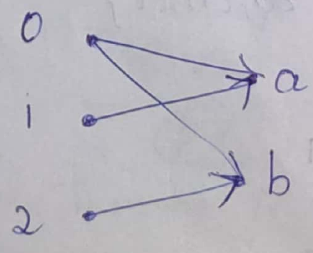
$R = \{(a,b)\}$ — a ↓ student, b → course.

If R is empty, currently, no student is enrolled in any course

Ex2 $A = \{0, 1, 2\}$ $B = \{a, b\}$

$R = \{(0,a), (0,b), (1,a), (2,b)\}$ from A to B.

Representation:-



R	a	b
0	1	1
1	1	0
2	0	1

Ex Let A be the set $\{1,2,3,4\}$, with the Relation

$R = \{(a,b) \mid a \text{ divides } b\}$

a divides b → $a|b$

$A \times A = \{(1,1),(1,2),(1,3),(1,4)$
$(2,1),(2,2),(2,3),(2,4)$
$(3,1),(3,2),(3,3),(3,4)$
$(4,1),(4,2),(4,3),(4,4)\}$

$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}$

Domain: The domain of a relation R is the set of all first elements in ordered pair, belongs to R

Range: The range of a relation R is the set of all second elements in the pair, which belongs to 'R'.

$R = \{(1,y),(1,z),(3,y)\}$ from $A = \{1,2,3\}$ to $B = \{x,y,z\}$

Domain of $R = \{1,3\}$ Range of $R = \{y,z\}$

Cardinality of $A \times B$: $|A \times B| = |A| \times |B| = mn$
m n

Ex $S = \{1,2,3,4,5,6\}$, $R = \{(a,b) \mid a+b \leq 4\}$

$R = \{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\}$

not same.

$\boxed{(a,b) = (c,d) \text{ iff } a=c \text{ and } b=d}$

$(a,b) \neq (b,a)$

Ex Set of lines L, A relation could be $R_1 \{(a,b) \mid a \parallel b\}$

$R_2 = \{(a,b) \mid a \perp b\}$

→ How many possible relations are there on A?

$|A| = n \qquad |A \times A| = n^2$

A relation on A is subset of $A \times A$

= All possible subsets of $(A \times A)$ [Power set concept]

$= \boxed{2^{n^2}}$

$|\underset{m}{A} \times \underset{n}{B}| = |A| \times |B| = mn.$

All relations from A to B = 2^{mn}
No. of ↗

Identity / diagonal Relation:-

R on A = $\{(a,a) \mid a \in A\}$

$R \to \Delta_A$ or Δ

Universal Relation: $A \times A$

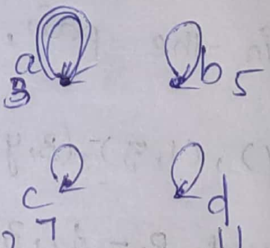
Empty Relation: $\phi, \{\}$

Properties of Relations:-

Reflexive Relation:- If a relation contains all possible (x,x) for all values of x from A.

Identity / Diagonal Relation:

$R = \{(a,a) \; \forall a \in A\}$



R	3	5	7
3	1	0	0
5	0	1	0
7	0	0	1
11	0	0	0

Example: Divides on $A = \{3, 5, 7, 11\}$

$R = \{(3,3), (5,5), (7,7), (11,11)\}$

No. of reflexive relations on set A =

$A = \{a_1, a_2, a_3, \ldots a_n\}$

$|A| = n$

possible reflexive relations are $\Delta_R = \{(a_1,a_1), (a_2,a_2) \ldots (a_n,a_n)\}$

$|\Delta_R| = n^2 - n$

No. of reflexive relations $= 2^{n^2-n} = 2^{n(n-1)}$

Irreflexive Relation:-

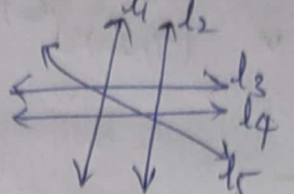
~~R = {~~ $(a,a) \notin R \; \forall a \in A$: $2^{n(n-1)}$

Symmetric Relation:- The possible occurrence in 'R'.

Ex:- Set of all ~~symmetric~~ parallel lines on the set of lines.

R is set of parallel lines on S - set of lines.

$S = \{l_1\ l_2\ l_3\ l_4\ l_5\}$

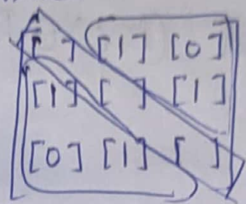


$R_{11} = \{(l_1, l_2), (l_3, l_4)\}$

Ex: Relation of shaking hands ~~of~~ on set $S \rightarrow$ which includes people.

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & 1 \\ b & 1 & 0 & 0 \\ c & 1 & 0 & 0 \end{array}$$

Symmetric Relation.



No. of symmetric Relations on the set S:-

$S = \{a_1, a_2, a_3 \ldots a_n\}$

$1 + 2 + 3 + 4 + \ldots n$

$= \frac{n(n+1)}{2}$ → No. of elements in lower triangle matrix.

No. of Symmetric Relations $= 2^{\frac{n(n+1)}{2}}$ $= \boxed{2^{\left(\frac{n(n+1)}{2}\right)}}$

Q) $R = \{(a,b) \mid (a,b) \in \mathbb{N}, b = a^2\}$ Is R reflexive? Is it Symmetric?

$R = \{(1,1), (2,4), (3,9), (4,16), \ldots\}$

$(1,1) \in R$ ✓

$(2,2) \notin R$ — Not a reflexive relation.

$(3,3) \notin R$

$(2,4) \in R$
$(4,2) \notin R$ } So not a symmetric relation.

$\boxed{R = R'}$

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Q) $R = \{(a,b) \mid a,b \in \mathbb{N},\ a \cdot b = 14\}$ Is R symmetric?

$R = \{(1,14), (2,7), (7,2), (14,1)\}$

$(1,14) \in R$, $(14,1) \in R$ ✓ $\left.\begin{matrix}(2,7)\\(7,2)\end{matrix}\right\} \in R$ Symmetric.

> **Transitive Relation :-**

If 4 is greater than 2, and 2 is greater than 0, then 4 is greater than 0.

If $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$ } If this will be the case, it is called as transitive Relation

Ex:- Not transitive

shaking hands example.

If a shakes hand with b, b shakes hand with c, It doesn't mean that a shakes hands with c.

$(a,b) \in R$ and $(b,c) \in R \not\Rightarrow (a,c) \in R$

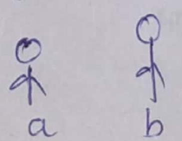
Ex:- Height of people in the class. (transitive)

Anti-Symmetric Relation:- Assumption (No $a=b$)

If $(a,b) \in R$, then $(b,a) \notin R$ unless $a=b$.

If $a=b$, $(a,b) \in R$ ✓ (if $a \neq b$)

Ex:- Height of people in class. (No two people have same height)



$(b,a) \in R$, $(a,b) \notin R$

taller   not taller

Q) $R = \{(a,b) \mid a+b=0\}$ Is R transitive?

$R = \{(0,0), (1,-1), (2,-2), (3,-3), (-1,1), (-2,2), \ldots\}$

R is symmetric, $(1,-1) \in R$, $(-1,1) \in R$ Is $(1,1) \notin R$?

No. So not a transitive.

$R = \{(a,b) \mid \sin a = \sin b\}$

$\sin 0 = 0$, $\sin \pi = 0$

$(0,\pi) \in R$, $(0, 2\pi) \in R$. $(0, n\pi) \in R$.

$(0,0)$ $(\pi,0) \in R$, $(2\pi, 0) \in R$ $(n\pi, 0) \in R$.

(π,π)

$\sin \pi/2 = 1$ $\sin \frac{3\pi}{2} = -1$ $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \notin R$

R is symmetric ✓

$(0,\pi) \in R$ and $(\pi, 0) \in R$

~~But~~ $(0,0) \in R \rightarrow$ So transitive

3. $R = \{(n, n+1) \mid n \in \mathbb{N}\}$ Is R anti-symmetric?

Is R symmetric? Is R transitive?

Is R reflexive? Is

$R = \{(1,2), (2,3), (3,4) \ldots\}$

Reflexive – No

Symmetric – No

Anti-symmetric – Yes $(1,2) \in R$, $(2,1) \notin R$

transitive – No

$(1,2)$ $(2,3)$

$(1,3) \notin R$

4. $R = \{(1,2), (2,1), (3,4)\}$

Symmetric: No $(4,3) \notin R$

Anti-symmetric: $(1,2) \in R$ $(2,1) \in R$

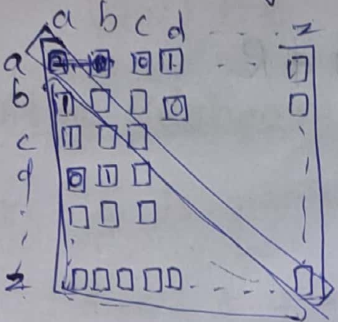
So No.

No.of anti-symmetric Relations:-

$a \neq b$, $R = \{(a,b) \mid a \in A, b \in A\}$

$(b,a) \notin R$

No.of total symmetric Relations $= 2^{\frac{n(n+1)}{2}}$



A reflexive relation is also a symmetric relation.

$$\frac{n(n+1)}{2} - n = \frac{n^2+n-2n}{2} = \frac{n^2-n}{2}$$

$$= \frac{n(n-1)}{2}$$

$|A| = n$, $|A| = n$

No.of anti-symmetric relations $= \cancel{2^{n(n-1)/2}}\ 3^{\frac{n(n-1)}{2}} \times 2^n$

2^n → diagonal element

$(ij)^{th}$ entry	$(ji)^{th}$ entry
0	0 ✓
0	1 ✓
1	0 ✓
1	1 X

3 choices for each pair

No.of pairs $= \frac{n(n-1)}{2}$

$$\boxed{2^n \times 3^{\frac{n(n-1)}{2}}}$$

Q] How to write a program/code to check whether the given relation is Reflexive or not?

Taking i/p of corresponding matrix, and $\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}_{n \times n}$ of size 'n'.

> So every $(i,i)^{th}$ element/entry should be one.

If atleast one of these entries is zero, it isn't a reflexive.

> No need to verify/check other entries, b'coz other may be 1, or may not be 1.

M:- Ref Matrix corresponding to a Relation.

If $I \leq M$, then M is a reflexive relation

$$\begin{array}{c}I\\ \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\end{array} \leq \begin{array}{c}M\\ \begin{bmatrix}1&0&1\\1&1&0\\0&0&1\end{bmatrix}\end{array}$$

Every entry should satisfy the condition with respective entry in M.

Identity Matrix

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \leq \begin{bmatrix}1&\square&\square\\\square&1&\square\\\square&\square&1\end{bmatrix}$$

So $\boxed{I \leq M(R)}$

> How to check given R is symmetric or not?

If the M(R) is symmetric, $\xrightarrow{}$ Transpose.

$\boxed{M(R) = M^T(R)}$

$i = 0 - n$

$j = 0 - (i-1)$

check $R(i,j) == R(j,i)$?

> If $M(R) \neq M^T(R)$, then M(R) is not symmetric.

> How to check given R is anti-symmetric or not?

(ij^{th} entry $\wedge$ ji^{th} entry) $\neq 1$

then we can say M(R) is anti-symmetric

$M(R) \cap M^T(R) \leq I$

$$\begin{array}{c}M\\ \begin{bmatrix}1&0&0\\1&0&1\\1&0&1\end{bmatrix}\end{array} \cap \begin{array}{c}M^T\\ \begin{bmatrix}1&1&1\\0&0&0\\0&1&1\end{bmatrix}\end{array} = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix} \leq I$$

Ex: R_1: Students in the class who have same blood group.

R_2: Students " who were born in the same month.

R_1 is reflexive (a,a) ✓ $\forall a$

symmetric $(a,b)(b,a)$ ✓

transitive $(a,b)(b,c) \Rightarrow (a,c) \in R$ ✓

R_2: Reflexive ✓

Symmetric $(a,b)(b,a)$ ✓

transitive $(a,b)(b,c)(a,c)$ ✓

These R_1 & R_2 are equivalence relations.

Q) $S = \{(a,b) \mid a,b \in \mathbb{N}\}$ $R = \{((a,b),(c,d)) \mid ad = bc\}$

$\frac{a}{b} = \frac{c}{d}$

Is R an equivalence relation?

Reflexive: $\{((1,1),(1,1)), [(1,2),(1,2)], ((1,3),(1,3)) \ldots\}$

$((a,b),(c,d)) \in R \Rightarrow ad = bc \Rightarrow bc = ad$

$((c,d),(a,b)) \in R$ ✓ symmetric ✓

$((1,2),(1,2)) \in R$

$(1,2),(1,2) \in R$

$((1,2),(2,4)) \in R$

$((2,4),(1,2)) \in R$

$((1,2),(1,2)) \in R$ ✓

Transitive: ✓

$((a,b),(c,d)) \in R$ $\quad \frac{a}{b} = \boxed{\frac{c}{d}}$

$((c,d),(e,f)) \in R$ $\quad \boxed{\frac{c}{d}} = \frac{e}{f}$ $\Big\}$ $\frac{a}{b} = \frac{e}{f}$

$((a,b),(e,f)) \in R$ ✓

$\frac{1}{2} = \frac{2}{4}$, $\frac{2}{4} = \frac{3}{6}$ then $\frac{1}{2} = \frac{3}{6}$ ✓

R is an equivalence relation.

Asymmetric Relation:- $\boxed{3^{\frac{n(n-1)}{2}}}$

If aRb, then $b \not R a$ $\quad a,b \in A$

R is a relation on A.

A relation R is called Asymmetric if $(a,b) \in R$, then $(b,a) \notin R$

★ Anti-Symmetric relations is asymmetric relation which allows reflexive property

★ Every Asymmetric relation is an Anti-symmetric Relation

Inverse Relation:- If R is a relation from A to B, then R' is called inverse relation from B to A,

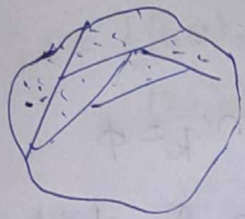
$R' = \{(b,a) \mid (a,b) \in R\}$ → Complementary Relation

Partitions :-

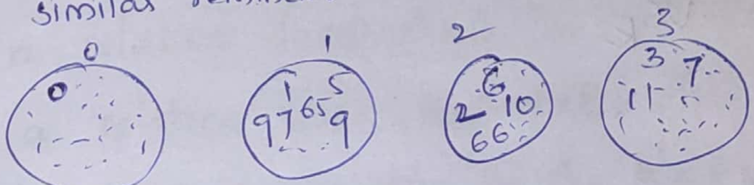
Ex :- The set of people with ~~different~~ same blood group.

R : the people who have same blood group.

Ex :- R is the people who were born in the same month.

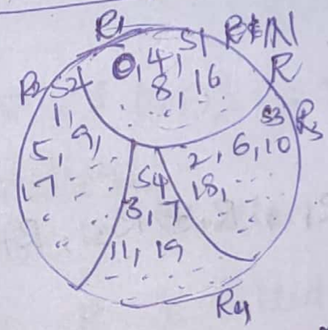


Ex :- The Relation is on the set of integers, ~~of same~~ similar remindor when divides with 4



Ex $R = \{(a,b) \mid a,b \in \mathbb{N}\}$ if $a \equiv b \pmod 4$

$a \bmod 4 \equiv b \bmod 4$

A = 

$\mathbb{N} = S_1 \cup S_2 \cup S_3 \cup S_4$

$S_1 \cap S_2 \cap S_3 \cap S_4 = \phi$

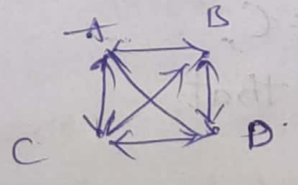
$S_1 S_2 S_3 S_4 \rightarrow$ disjoint clusters

R_1 = reflexive ✓
R_2 symmetric ✓
R_3, R_4 transitive ✓

R is equivalence relation within disjoint clusters.

Across the clusters, there is no relation.

Ex Friends clusters,



If some 'E' knows any one ~~f~~ friend in this group, then 'E' knows everyone in the group.

$A = S_1 \cup S_2 \cup S_3 \ldots S_k$ $S_1 \cap S_2 \cap S_3 \cap \ldots S_k = \phi$

Partitions :- Theorem

> Given a set A with 'n' elements, and an equivalence relation R, then R partitions the set into disjoint subsets.

Converse :-

If there is a partition of A,

$A = S_1 \cup S_2 \cup S_3 \cdots \cup S_k$, and $S_1 \cap S_2 \cap S_3 \cdots \cap S_k = \phi$

then, an equivalence relation R can be defined on A which induces the partition.

$A = S_1 \cup S_2 \cup S_3 \cdots \cup S_k$

$aRb \quad mRn \quad \alpha R \beta$

$aRa \quad mRm \quad \alpha R \alpha$

$bRa \quad nRm \quad \beta R \alpha$

Combining Relations

Any two relations from A to B can be combined in any way two sets can be combined.

R_1, R_2 on $A \times B$.

$R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$, $\cancel{R_2 - R_1}$, $R_1 \oplus R_2$, $R_1 \Delta R_2$.

Composite of R and S:

Let R ~~and S~~ be a relation from A to B,

, S be a relation from B to C.

The composite of R and S is the relation from A to C, where $(a, c) \in S \circ R$, $a \in A$, $c \in C$,

And there exist $b \in B$ such that

$(a,b) \in R$ & $(b,c) \in S$.

$S \circ R$ is from A to C

$S \circ R = \{(a,c) \mid (a,b) \in R \;\&\; (b,c) \in S, a \in A, b \in B, c \in C\}$

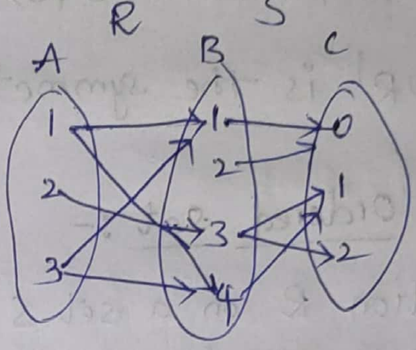
Ex $A=\{1,2,3\}$, $B=\{1,2,3,4\}$, $C=\{0,1,2\}$

with $R=\{(1,1),(1,4),(2,3),(3,1),(3,4)\}$ from A to B

$S=\{(1,0),(2,0),(3,1),(3,2),(4,1)\}$ from B to C

$S\circ R = ?$

$S\circ R=\{(1,0),(1,1),(2,1),$
$(2,2),(3,0),(3,1)\}$



Composite of R and R:-

R is a relation from A×B

R is a relation from ~~B C~~ A×B.

Let R is a relation on A, $R^1=R$, $R^2=R\circ R$

~~$R\circ R=\{$~~ $R^3=R^2\circ R$ $R^{n+1}=R^n\circ R$ for $n=1,2,\ldots$

If $A=\{1,2,3,4\}$, $R=\{(1,1),(1,2),(2,3),(3,3),(3,4)\}$

Find R^2? $R=\{(1,1),(1,2),(2,3),(3,3),(3,4)\}$

$R^2=R\circ R=\{(1,1),(1,2),(1,3),(2,3),(2,4),(3,3),(3,4)\}$

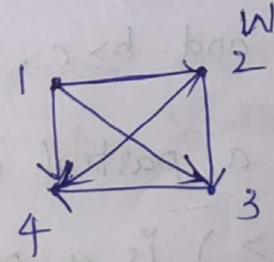
Closures of Relations:-

Transitive closure:- Smallest transitive relation containing the relation R is called transitive closure of R.

Ex:- $A=\{1,2,3,4\}$ $R=\{(1,2)(2,3)(3,4)\}$

Warshall Algo.

$R^+=\{(1,2),(1,3),(1,4),(2,3),(2,4)$
$(3,4)\}$



Reflexive closure:- Smallest possible reflexive relation containing R is called reflexive ~~relation~~.closure of R.

Ex $R=\{(1,1),(1,2),(2,1),(3,2)\}$ $A=\{1,2,3\}$

$R\cup\Delta=\{(1,1),(2,2),(3,3),(1,2),(2,1),(3,2)\}$

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Symmetric closure:- The smallest ~~poss~~ symmetric relation ~~we can~~ which contains R, is called symmetric closure of R

$R \cup R' = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,1), (1,3)\}$

$R = \{(1,1)\ (1,2), (2,2), (2,3), (3,1)\ (3,2)\}$

$R \cup R'$ is the symmetric closure of R.

Partially Ordered set:-

> A relation R on a set S is called ~~po~~ a partial ordering or partially ordered ~~set~~ if it is reflexive, anti-symmetric and transitive.

> A set ~~with~~ S together with a partial order R is called a partially ordered set or poset, denoted by (S, R).

> Members of S are called elements of the poset.

Ex:- Relation $\geq$ is a partial ordering on set of integers.

$Z \rightarrow$ set of integers.

$R \rightarrow \geq$

for every $a \geq a$, so R is reflexive

If $a \geq b$. and $b \geq a$ then $a = b$, R is anti-symmetric

If $a \geq b$ and $b \geq c$, ~~the~~ imply that $a \geq c$.

So, $\geq$ is a partial order on the set Z.

$(Z, \geq)$ is a poset.

Ex1:- $(Z^+, |)$ is a poset. ✓

$\forall a, a|a$, so $|$ is reflexive

> If $a|b$, $b|a$ then $a = b$ ✓ anti-symmetric.

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Famous Examples for posets:-

$(Z, \leq)$ $(Z, \geq)$

$(Z^+, |)$ $(Z^+, \text{m.of})$

$(P(A), \subseteq)$ $(P(A), \supseteq)$

$(D_n, |)$ $(D_n, \text{m.of})$

$\downarrow$

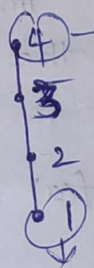
set of positive divisors

$\underbrace{\qquad\qquad}$

Duals of each others.

Hasse diagram :- $\langle P, \preccurlyeq \rangle$ be a poset. Every element of P is a vertex

Construct Hasse diagram for $(\{1,2,3,4\}, \leq)$



$\rightarrow$ Maximal element, greatest

2 cover 1

3 doesn't cover 1

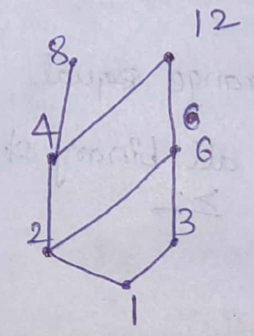
4 doesn't cover 1,2

Minimal, least

3 cover 2

4 cover 3

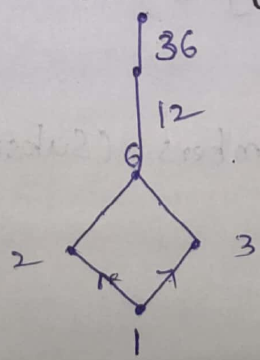
Ex2 Draw Hasse Diagram for the poset $(\{1,2,3,4,6,8,12\}, |)$



$\rightarrow$ No.of edges = 8

* Infinite poset need not have max and min elements.

Ex3 Let $X = \{1, 2, 3, 6, 12, 36\}$ R is a partial order on X, where xRy if x devides y. No.of edges in Hasse diagram of $(X, R) = ?$



No.of edges = 6.

Functions:-

Let A and B be non-empty sets. A function 'f' from A to B is an assignment of exactly one element of B to each element of A.

$f(a)=b \rightarrow$ unique element of B assigned by 'f' to the a of A.

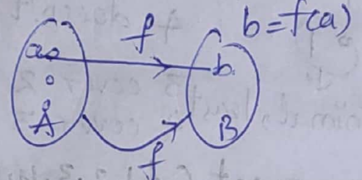
$f: A \rightarrow B$

> function is a mapping from A to B.

> A is the domain of 'f'. B is the co-domain of 'f'.

> Range is set of all elements of B mapped from A
images of A

> If $f(a)=b$, b is the image of a, and a is a preimage of b.



> f is a function that assigns the last two bits of a bitstring of length 2 or greater.

$f(11010) = 10$
$f(10111) = 11$
$f(10100) = 00$
$f(01001) = 01$

Co-domain and range equal.
Domain: Set of all binary strings of length ≥ 2

> $f: Z \rightarrow Z$, $f(x)=x^2$

Domain of f is set of all integers.

Codomain: set of all integers

Range: Set of all perfect numbers. (Subset of co-domain is mapped from A)

A function is real-valued if its codomain is set of real numbers.

" " integer-valued " " " integers.

Def:- Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then f_1+f_2 and $f_1 \cdot f_2$ are also functions from A to R defined for all $x \in A$ by

$$(f_1+f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x)$$

Ex: $f_1 : R \to R$, $f_2 : R \to R$, $f_1(x) = x^2$, $f_2(x) = x - x^2$

$$(f_1+f_2)(x) = f_1(x) + f_2(x) = x^2 + x - x^2 = x$$
$$(f_1 \cdot f_2)(x) = f_1(x) \cdot f_2(x) = x^2(x-x^2) = x \cdot x^2 - x^4$$
$$= x^3(1-x)$$

Relation Vs Function:

Every function is a relation.

~~Eve~~ But not every relation is a function.

$f(x) = \sqrt{x}$ ← relation, but not a function

$f : R_+ \to R$

Def: $f : A \to B$

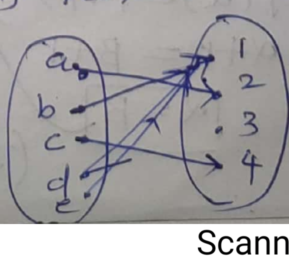
Let $S \subset A$. The image of S under function f is the subset of B that consists of images of elements of S.

$$f(S) = \{ t \mid \exists s \in S, f(s) = t \}$$
$$f(S) = \{ f(s) \mid s \in S \}$$

Ex: $f : A = \{a, b, c, d, e\}$ & $B = \{1, 2, 3, 4\}$ $f(a) = 2$

$S \subset A = \{b, c, d\}$

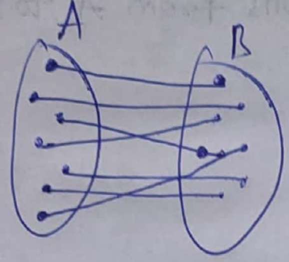
$\boxed{f(S) = \{1, 4\}} \subset B$



One-to-One Function:-

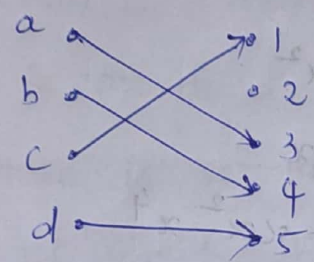
> A function f is one-one or an injunction, iff $f(a) \equiv f(b)$ implies that $a=b \ \forall a, \forall b$ in the domain of f.

> A function is said to be injective if it is one-one.



one element of A should map maximum one element of B.

$$\forall a \forall b \ (f(a)=f(b)) \rightarrow (a=b)$$

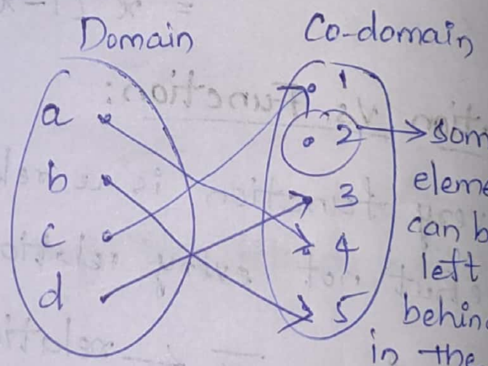


$\Rightarrow$ one-one

$$\forall a \forall b \ (a \neq b) \rightarrow (f(a) \neq f(b))$$

Ex: f from $\{a,b,c,d\}$ to $\{1,2,3,4,5\}$ $f(a)=4, f(b)=5, f(c)=1$ and $f(d)=3$? one-one

Yes, it is one-one



some elements can be left behind in the co-domain.

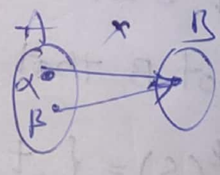
Ex $f(x)=x^3 : Z_+ \rightarrow Z_+$

There are no 2 elements from domain mapping to ~~same~~ one element in co-domain

$\therefore$ It is one-one.

> But the elements in the domain should have a unique element in the co-domain.

Proof:- of one-one function? $\boxed{f(x) = x+1}$ $A \rightarrow B$



~~if~~ Assume: $\alpha \neq \beta$

- Assume that $\exists \alpha, \exists \beta$, such that $f(\alpha)=f(\beta)$

$f(\alpha)=f(\beta)$ contradicts that $\alpha \neq \beta$

$\alpha+1 = \beta+1$

$\boxed{\alpha = \beta}$

So our assumption is wrong

$(f(\alpha)=f(\beta)) \rightarrow \boxed{\alpha=\beta}$

$f: \mathbb{N} \to \mathbb{N}$: $f(x) = 3x$

$f(1) = 3$
$f(2) = 6$
$f(3) = 9$
⋮
f

$x_1, x_2 \in \mathbb{N}$ $\quad f(x_1) = 3x_1 \quad f(x_2) = 3x_2$

Assume $x_1 \neq x_2$ and $f(x_1) \neq f(x_2)$, to prove $f(x)$ one-one

$f(x_1) = f(x_2)$

$\cancel{3}x_1 = \cancel{3}x_2$

$x_1 = x_2 \rightarrow x_1$ and x_2 are forced to be equal.

Means, it is contradicting the assumption $x_1 \neq x_2$

So, $f(x)$ is one-one

If function is one-one function, the co-domain may have some left-oves elements. $\quad |A| \leq |B|$

Range of $f \subseteq$ Co-domain of f.

On-to function:- A function f from A to B is onto or surjection, iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called surjective if it is onto.

$\forall b, b \in B, \exists a \; f(a) = b$

Codomain of f = Range of f.

$\boxed{\text{for all } b \in B, \exists x \text{ such that } f(x) = y}$ - onto.

Proof:- $f: R_+ \to R_+$ defined as $f(x) = |\sqrt{x}|$

$\forall y \in R_+ \;\exists$ there exists $y^2 \in R_+$

$y^2 \to y$

Every element has a pre-image.

Hence, f is onto.

$\boxed{Ex}$ $f: N \to N$, $f(x) = x$ (Identity function)

$f(1) = 1$ $f(2) = 2$, $f(3) = 3$, ...

$\forall y \in N, \exists y \xrightarrow{\forall} y$

pre-image and image is same,

For each image there is a pre-image.

Hence, f is onto., it is one-one also

$\boxed{Ex-7}$ $f: Z \to Z$ $f(x) = |x|$

$f(0) = 0$

$f(1) = 1$ $f(-1) = 1$

$f(2) = 2$ $f(-2) = 2$

$f(3) = 3$ $f(-3) = 3$

Range $= \{0, 1, 2, 3, 4, ---\}$

$(N \cup \{0\}) \neq Z$

the -ve integers are not included.

Hence, the function f is not a onto function.

$\boxed{Ex-4}$ $f: W \to W$, $f(x) = \begin{cases} x-1, & x \text{ is odd} \\ x+1, & x \text{ is even} \end{cases}$

$f(0) = 1$ $f(1) = 0$

$f(2) = 3$ $f(3) = 2$

$f(4) = 5$ $f(5) = 4$

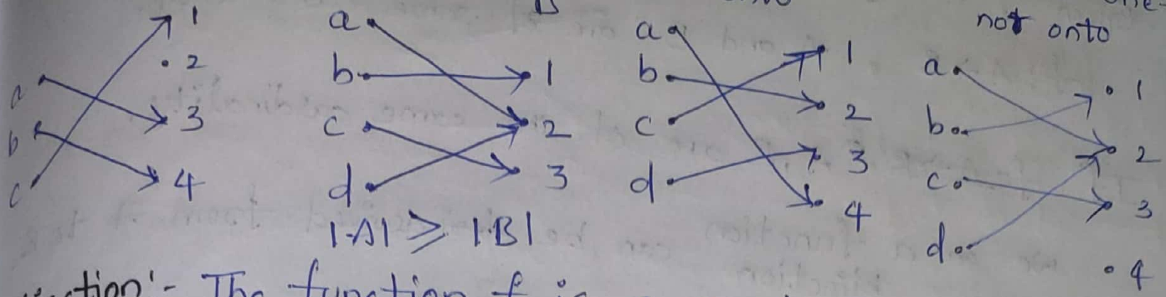
$y \in N$

$y = x+1$ or $x-1$

All the images of $f(x)$ are mapped by the 'W' element

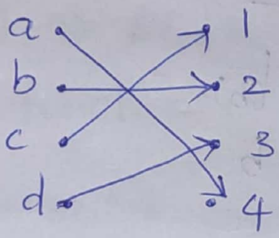
so, it is onto, it is one-one function.

one-one not onto | Onto, not one-one | one-one, onto | Neither one-one not onto



Bijection:- The function f is a one-to-one correspondence or a <u>bijection</u>, if its both one-to-one and onto.

Ex:-



It is one-one onto. } So it is bijection.

Inverse Function:-

If $f: A \to B$, f is from A to B. $f(a) = b$, $a \in A$, $b \in B$

$f^{-1}: B \to A$ exist iff f: is one-one and onto function.

$f^{-1}(b) = a$, $a \in A$ $b \in B$

Ex Bijection.

$f: \{\overset{M_1}{Jan}, \overset{M_2}{Feb}, \overset{M_3}{Mar}, \overset{M_4}{Apr}, May, Jun, July, Aug, Sep, Oct, Nov, Dec\}$.
$\longrightarrow \{1, 2, 3, \ldots 12\}$

$\boxed{f(M_i) = i} \to$ one-one and onto

So, this is a bijection

Ex $f(x) = x^2$ $f: Z \to Z$ (Not a one-one, onto) so not a bijection.

$f: A \to B$

Bijection; one-one, onto

$|A| \leq |B|$ and $|A| \geq |B|$ $\Rightarrow |A| = |B|$

$|Domain| = |Co\text{-}domain|$

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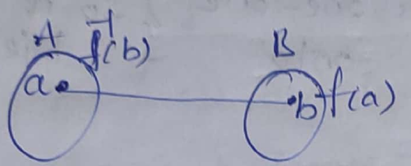
> Given f is bijection, Then we can prove that $f: A \to B$, A and B are of same cardinality.

> If ~~$f: A \to B$~~, A, B are of the same cardinality, ~~we can~~ a function bijection can be ~~deri~~ derived from A to B.

> A ~~one-one~~ function is called invertible if f is one-one.

> A function is called not invertible if f is not one-one.

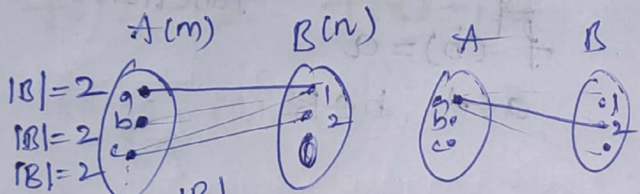


$a = f^{-1}(b)$

$f(a) = b$.

Counting :-

Q No. of functions from A to B



	a	b	c	
	1	1	1	$\to f_1$
	1	1	2	$\to f_2$
	1	2	1	$\to f_3$
	1	2	2	$\to f_4$
	2	1	1	$\to f_5$
	2	1	2	$\to f_6$
	2	2	1	$\to f_7$
	2	2	2	$\to f_8$.

$\underbrace{2 \times 2 \times 2 \times \cdots 2}_{n} = 2^{n}$

$\Downarrow$

$2^{|A|}$

$\Downarrow$

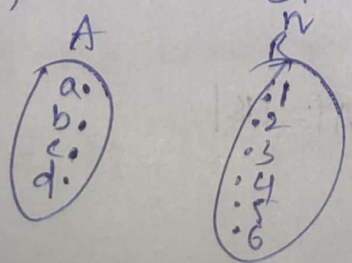
$(n^m) \Leftarrow (|B|)^{|A|}$

Q No. of possible one-one functions from A to B.

$A = \{a, b, c, d\}$ $\quad B = \{1, 2, 3, 4, 5, 6\}$ $\quad |A| \le |B|$

$|A| = 4$ $\quad |B| = 6$ $\quad m \le n$

$m \quad\quad n$



$\overset{a}{6} \times \overset{b}{5} \times \overset{c}{4} \times \overset{d}{3} = \frac{6!}{2!} = {}^{6}P_{4}$

$= {}^{|B|}P_{|A|} = {}^{n}P_{m}$

If $\overset{n}{|A|} = \overset{n}{|B|}$, and $f: A \to B$,

No. of one-one functions $= {}^nP_n = \frac{n!}{(n-n)!} = \boxed{n!}$

We can count the no. of ~~functions~~ one-one functions only when A and B are finite sets.

Total possible functions which are onto from A to B -

$A = \{a, b, c\} = m \quad B = \{1, 2\} = n$

6 onto functions. ✓

a	b	c	
1	1	1	× onto, 2 is not mapped
1	1	2	onto
1	2	1	onto
1	2	2	onto
2	1	1	onto
2	1	2	onto
2	2	1	onto
2	2	2	× onto, 1 is not mapped

$2! \times 2$

1 2 1, 1 2 2
2 1 1, 2 1 2

$\frac{{}^3C_2 \times 2!}{}$

$3! \quad \boxed{3 \times 2!}$

Principle of Inclusion and Exclusion

$\boxed{2^m - 2}$ $\quad \boxed{{}^3P_2} = {}^{|A|}P_{|B|}$

$\cancel{2[2^{m-1} - 1]}$ Principle of Exclusion and Inclusion. $16 - 1 = 15$

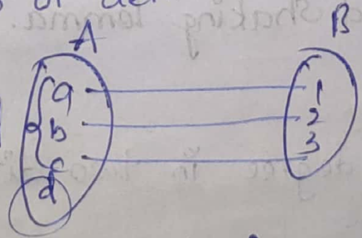
> Total no. of bijections from A to B :-

$|A| = |B|$

$A = \{a, b, c\}\, m \quad B = \{1, 2, 3\}\, n$

$= \boxed{n!}$

Partial Functions :- A partial function f from A to B is an assignment from each element a in subset of A, called domain of definition of 'f'.



$f(d) \to$ undefined

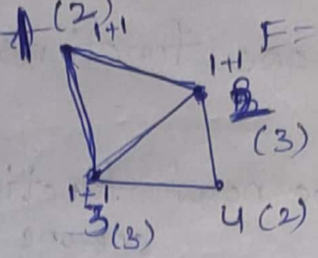
$\{a, b, c\} \subset A$ → domain of definition.

$f(n) = \sqrt{n}$ partial $Z \to R$

Graph:-

$V = \{1,2,3,4\}$

$G = (V, E)$



$E = \{(1,2), (1,3), (2,3), (2,4), (3,4)\}$

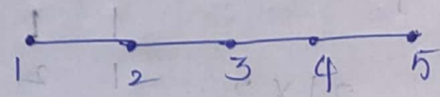
$V \rightarrow$ Set of nodes / vertices

$E \rightarrow$ Set of edges

Given a vertex set, there can be any no. of edge sets.

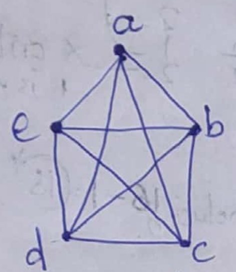
$\{1,2,3,4\} \rightarrow \{2,3,2,3\}$
$\downarrow$
degree-sequence.

Path Graph:-



$\Rightarrow P_5$

$\downarrow$
Path graph $- P_k \rightarrow$ path length



Max degree for any vertex in graph $= n-1$ if $n \rightarrow$ no. of vertices.

$\downarrow$
complete graph $K_5 \rightarrow$ no. of vertices.

> A path graph with 'n' vertices P_n, will always have $(n-1)$ edges

> K_n - a complete graph will have $\frac{n(n-1)}{2}$ edges.

$$(n-1) + (n-2) + (n-3) + \cdots + 1 = \frac{(n-1)(n)}{2} = \frac{n(n-1)}{2}$$

$$^5C_2 = \frac{5 \times 4}{2} \quad \boxed{^nC_2}$$

— Hand shaking lemma.

> $\sum_{v \in V} \deg(v) = 2|E|$

> Every edge contributes a degree in two different vertices.

> We know addition of odd number of odd numbers is odd $\boxed{3+5+9=17}$

- Addition of even number of odd numbers is even

$\boxed{3+7=10}$

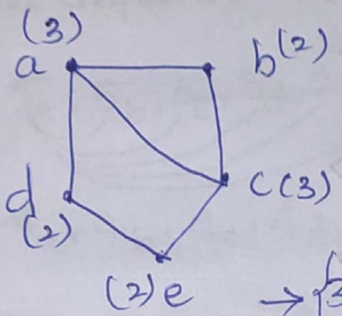
- Addition of any no.of even numbers is always even.

$$\sum_{v \in V} \deg(v) = 2|E|$$

$$\underbrace{d_1+d_2+d_3+\dots d_n}_{\text{all degrees are } \cancel{\text{can be}} \text{ even}} = \underbrace{2|E|}_{\text{even number.}}$$

(or)

There are even no.of odd numbers (degrees)



$\Rightarrow$ There are even no.of odd degree vertices.

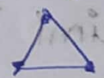
$\rightarrow \{3, 2, 3, 2, 2\} \rightarrow$ degree sequence.

- ★ Every graph ~~b~~ will have even no.of odd degrees.

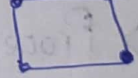
- Can we construct a graph from ~~the~~ given degree sequence?

Given degree sequence, you may not have a graph.

$\{1,1,1\} \rightarrow \times$ no graph

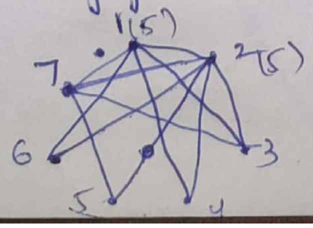
$(2,2,2) \rightarrow$  K_3

$(1,1) \rightarrow$ a ——— b

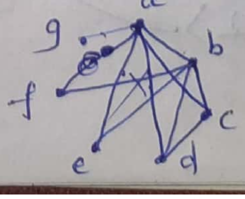
$(2,2,2,2) \rightarrow$ 

Observe $\{1,1,1\} \Rightarrow$ violating the constraint that every graph will have even no.of odd degrees.

- $\langle 5, 5, 5, 5, 2, 2, 2 \rangle$ Can a graph be drawn for this sequence? satisfying the constraint.



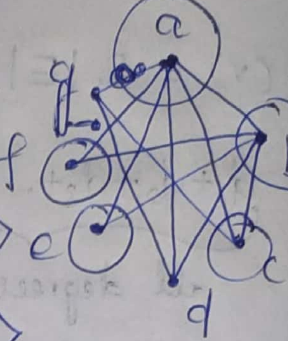
$\langle 5_a, 5_b, 3_c, 3_d, 2_e, 2_f, 2_g \rangle$



Construct graph for degree-sequence
$\langle 5, 5, 3, 3, 2, 2, 2 \rangle$.

1. $S_1 = \langle 5_a, 5_b, 3_c, 3_d, 2_e, 2_f, 2_g \rangle$

2. $S_2 = \langle *_a, 4_b, 2_c, 2_d, 1_e, 1_f, 2_g \rangle$



3. $S_3 = \langle *_a, *_b, 1_c, 1_d, *_e, *_f, 2_g \rangle$

4. $S_4 = \langle *_a, *_b, *_c, 1_d, *_e, *_f, 1_g \rangle$

5. $S_5 = \langle *_a, *_b, *_c, *_d, *_e, *_f, *_g \rangle$

$S_1 = \langle 5_a, 5_b, 3_c, 3_d, 2_e, 2_f, 2_g \rangle$

$S_2 = \langle *_a, 4_b, 2_c, 2_d, 1_e, 1_f, 2_g \rangle$

$S_2' = \langle *_a, 4_b, 2_c, 2_d, 2_g, 1_e, 1_f \rangle$

$S_3 = \langle *_a, *_b, 1_c, 1_d, 1_g, *_e, 1_f \rangle$

$S_4 = \langle *_a, *_b, *_e, *_c, *_d, 1_g, 1_f \rangle$

$S_5 = \langle *_a, *_b, 0_e, 0_c, 0_d, 0_g, 0_f \rangle$

Havel Hakimi Theorem :- ~~By~~ given graph sequence can we derive graph or not

> Process of deriving the graph from a given degree-sequence.

> A degree-sequence is called Graphic if a simple graph can be drawn with given sequence.

$\langle 5, 5, 3, 3, 2, 2, 2 \rangle$ - graphic sequence.

$\langle 2, 2, 2 \rangle$

$\langle 5, 5, 5, 5, 2, 2, 2 \rangle$

Havel Hakimi Theorem:-

A degree sequence $S = d_1 \geq d_2 \geq d_3 \geq \ldots \geq d_n$, where $d_i \leq n-1$, is graphic <u>if and only if</u> the reduced sequence, ~~S~~

$$S' = \{ *, d_2 - 1, d_3 - 1, \ldots d_n \} \text{ is graphic.}$$

- If the last sequence contains all 0's or ~~*~~'s, then it is graphic.

$$\text{Sequence} \xrightarrow{\text{reduce}} \text{Sequence} \xrightarrow{\text{reduce}} \ldots \quad \text{Sequence} \downarrow \text{ contains all zeros.}$$

Q. Degree sequence $= \langle 5, 5, 5, 5, 2, 2, 2 \rangle$

$S = \langle 5_a, 5_b, 5_c, 5_d, 2_e, 2_f, 2_g \rangle$

$S_1 = \langle *_a, 4_b, 4_c, 4_d, 1_e, 1_f, 2_g \rangle$

$S_1' = \langle *_a, 4_b, 4_c, 4_d, 2_g, 1_e, 1_f \rangle$

$S_2 = \langle *_a, *_b, 3_c, 3_d, 1_g, 0_e, 1_f \rangle$

$S_2' = \langle *_a, *_b, 3_c, 3_d, 1_g, 1_f, 0_e \rangle$

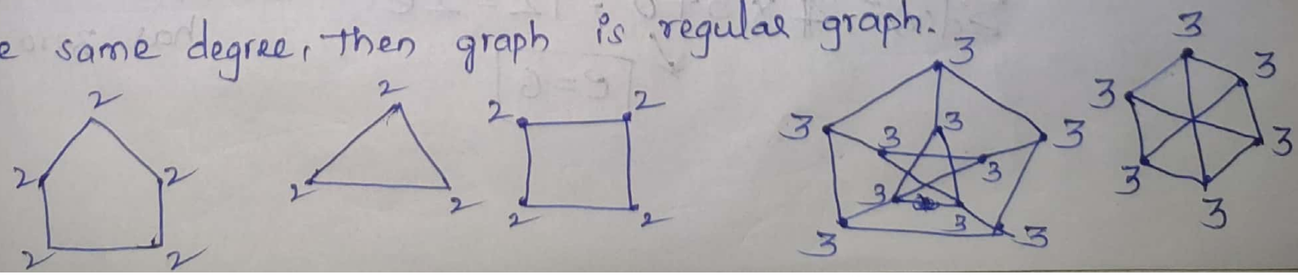
$S_3 = \langle *_a, *_b, *_c, 2_d, 0_g, 0_f, 0_e \rangle$

$S_4 = \langle *_a, *_b, *_c, *_d, -1_g, -1_f, 0_e \rangle$ → Degree can't be negative

$\therefore S_4$ sequence is not graphic.

Hence, S sequence is not graphic.

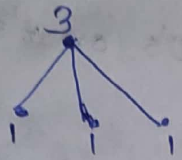
Regular Graph:- For every single vertex of V, if vertex has the same degree, then graph is regular graph.



Irregular graph:- A graph that is not regular..

If you find atleast 2 vertices, such that both of them have dif. degrees, then graph can't be regular.

> Constructing a graph with "n" vertices, such that degree of each vertex is different- Is this possible? NOT possible.



Problems:- G is an undirected graph with 'n' vertices, & 25 edges such that each vertex of G has degree atleast 3. The max. possible value of 'n' is ____

$$\boxed{\delta \leq \frac{2e}{v} \leq \Delta}$$

no. of edges.

min. degree of vertex

no. of vertices

max. degree of vertex

$$\boxed{\delta \leq \frac{2e}{v}} \Rightarrow 3 \leq \frac{2\times 25}{n}$$

$$n \leq \left\lfloor \frac{50}{3} \right\rfloor$$ → max of integer of any float [floor]

$$n \leq \lfloor 16.. \rfloor$$

$$\boxed{n \leq 16}$$

max. possible value of $n = 16$

Q) A Graph has 4 vertices each of degree 3 and an isolated vertex. Then the no. of edges ____

$$\sum d(v) = 4\times 3 + 0 = 12 = 2e$$

Isolated vertex has zero degree.

$$\boxed{e = 6}$$

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What is the no.of vertices in an undirected with 27 edges, 6 vertices of degree 2, 3 vertices of degree 4 and remaining of degree 3?

$$6\times2+3\times4+x\times3 = 2\times27$$

should be even

$$12+12+3x = 54$$

$$3x = 54-24$$

$$3x = 30$$

$$\boxed{x=10}$$

$n = 6+3+10$

$\boxed{n=19}$

Q) Which of these are graphic?

a) 2,2,2,2 b) 7,6,6,4,4,3,2,2 c) 4,3,1,1,1 d) 5,4,3,2,1

a) 2,2,2,2

regular graph

c) $4_a, 3_b, 1_c, 1_d, 1_e \Rightarrow *_a, 2_b, 0_c, 0_d, 0_e \Rightarrow *_a, *_b, -1_c, -1_d, 0_e$

-ve degree not possible

so not graphic

d) 5, 4, 3, 2, 1 Not graphic

having degree = no.of vertices

not possible

b) 7,6,6,4,4,3,2,2 ⇒ *, 5, 5, 3, 3, 2, 1, 1 ⇒ *, *, 4, 2, 2, 1, 1, 0

Graphic *, *, *, *, *, 0, 0, 0 ⇐ *, *, *, 1, 1, 0, 0, 0

Q) No.of labelled simple graphs on 5 vertices having 4 edges.

Max. no.of edges with 5 vertices = $\frac{5\times4}{2} = 10$

No.of labelled graphs having 4 edges on 5 vertices = $\boxed{{}^{10}C_4}$

> Total no.of simple graphs on 'n' labelled vertices $= 2^{n(n-1)/2}$

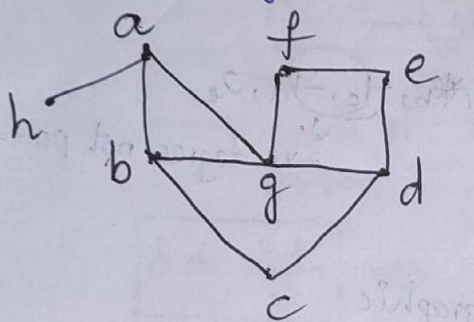
No.of zero edge graphs $\rightarrow {}^{n(n-1)/2}C_0$ No.of 1 edge graphs $\rightarrow {}^{n(n-1)/2}C_1$ No.of 'n' edge graphs ${}^{n(n-1)/2}C_n$

Walk:- Finding a way from vertex 'u' to vertex 'v'. Edges and vertices can be repeated any no.of times.

Trail:- It is a walk from 'u' to 'v', but edges shouldn't be repeated.

Path: It is a walk, but vertices shouldn't be repeated. means, edges shouldn't be repeated.

Ex



	Repetition of Vertices	Repetition of Edges
Walk	✓	✓
Trail	✓	X
Path	X	X

Walk: from a to c

① a-b-c ② a-g-f-e-d-e-f-g-d-c

So many.

Trail: a to e

① a-g-b-c-d-g-f-e.

② a-g-f-e

Path: from a to d

① a-g-d ② a-g-f-e-d

③ a-b-c-d.

★ Every path is a trail and a walk.

★ Every trail is a walk.

Closed trail: a to a

① a-g-f-e-d-g-b-a

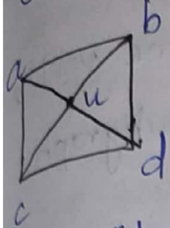
② a-b-g-d-e-f-g-a

$\boxed{\text{Circuit}}$

Closed path:- from a to a (CYCLE)

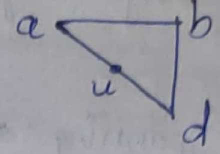
① a-b-g-a ② a-g-f-e-d-c-b-a

③ a-g-d-c-b-a

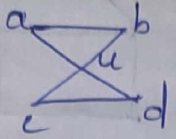


cycle and circuit.

cycle: a-u-d-b-a
(closed path)

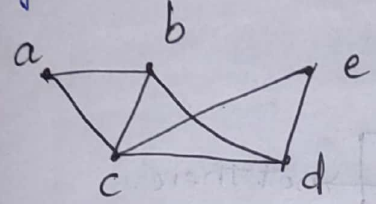


circuit (closed trail): ~~a-b-c~~
a-u-d-c-u-b

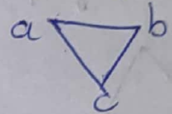


★ Every Cycle is a part of circuit. But a circuit can't be a cycle.

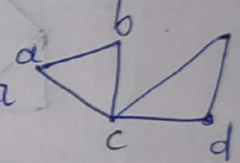
Ex



cycle starting from a:
a-b-c-a



circuit from a:
a-b-c-d-e-c-a



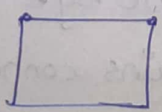
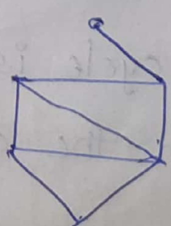
★ If there is a walk from u to v, there is a path from u to v.] always.

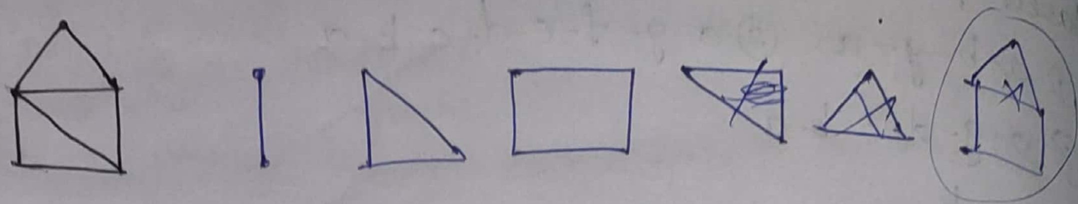
> To know/derive a path from given walk from 'u' to 'v', remove the circuits from the walk sequence.
cycles.

> if walk sequence is $u - x_1 - x_2 \cdots x_2 - x_5 - v$
remove.

path sequence: $u - x_1 - x_2 - x_5 - v$ → path.

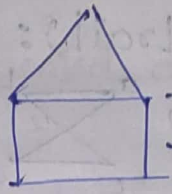
> Subgraph:- Given G', $V' \subset V$, $E' \subset E$, forms the subgraph

 is subgraph of 



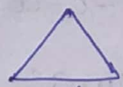
Spanning Subgraph:- A subgraph which is having same vertex set as the original graph is called as spanning subgraph.

→ spanning subgraph
↘ induced subgraph → ✗

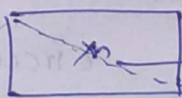


→ spanning
→ induced ✗

Induced subgraph:- A subgraph which has all the edges corresponding to those set of vertices is called an induced subgraph.



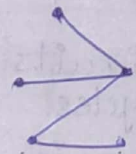
↓
induced subgraph



→ not there.
↓
not induced subgraph

> What is the subgraph which is spanning and induced?
The graph itself.

TREE: Connected $\underbrace{\text{acyclic}}_{\text{no cycles}}$ graph



→ tree ✗
→ graph ✓

In a tree,

No. of edges = No. of vertices − 1

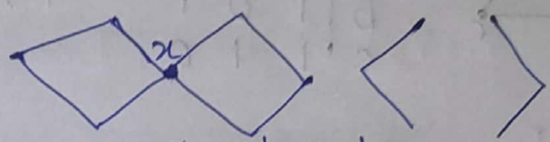
$$\boxed{|E| = |V| - 1}$$

> If in a graph cycle is present, and if one edge is removed from the cycle, still it remains connective.

> Given any connected graph G, with 'n' vertices, it will have the no. of edges $\geq n-1$

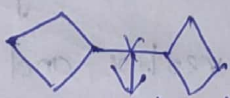
Connected graph:- Given graph G, For every pair of graph, there should exist a path, then it is a connected graph.

> cut vertex:- ~~If~~ In a ~~given~~ connected graph G, If G becomes disconnected by removing a vertex, then this vertex is called as cut vertex.

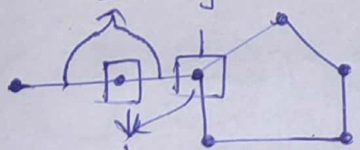


x - cut vertex.

> cut edge:- If one edge is removed from the graph, graph becomes disconnected it is the cut edge.



cut edge.



Directed graph:- liking each other

a likes b

b may not like a.

$a \longrightarrow b$

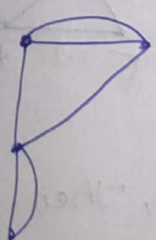
$E = \{(a,b) ..\}$

$\text{msg} \xrightarrow{\text{encryption}} \text{cipher text}$

only incoming calls

$a \longleftarrow b$

Multigraph:- If two vertices connected through multiple edges.

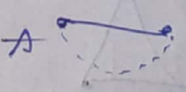


$A \xleftrightarrow{(30)} B$ → weighted edge.

weighted graph.

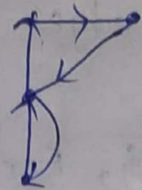
Road/path graph → price to reach some location.

A road network

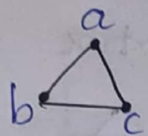


Undirected graph:- Bidirectional graph.

Directed multigraph:-



Q] How to represent a graph ~~to~~ in way that computer can understand?



	a	b	c
a	0	1	1
b	1	0	1
c	1	1	0

0 —: 2 vertices aren't connected.
1 : if vertices connected.

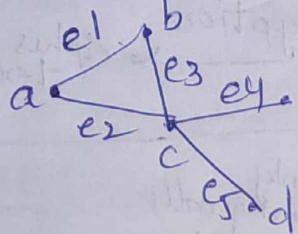
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \text{symmetric matric}$$

> For Every undirected graph,
the matrix representation is symmetric.

> Adjacent matrix:- ~~W~~ A vertex and vertices connected to.

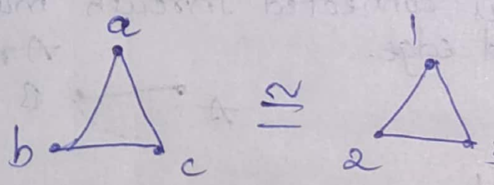
$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & & & & \\ 0 & & & & \\ 1 & \cdot & \cdot & \cdot & \end{bmatrix} \end{array}$$

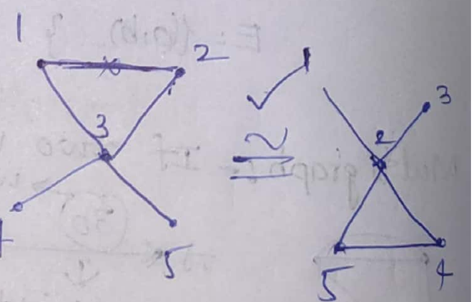
> Incidence Matrix:- A vertex and edges, connectivity.



	e1	e2	e3	e4	e5
a	1	1	0	0	0
b	1	0	1	0	0
c	0	1	1	1	1
d	0	0	0	0	1
e	0	0	0	1	0

Isomorphic graphs:- $\cong$ → symbol

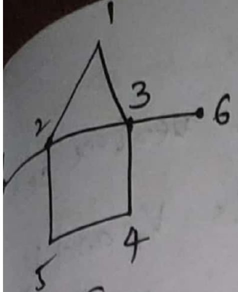




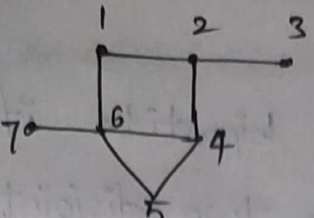
> If two graphs G and H are isomorphic graphs, then

$$|V_G| = |V_H| \; \& \; |E_G| = |E_H|$$

degree. Seq. of G = Deg. Seq. of H



G



H

$|V_G| = 7$

$|E_G| = 8$

$|V_H| = 7$

$|E_H| = 8$

Deg. Seq. (G) =

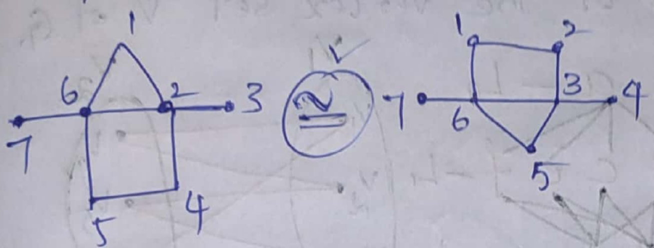
$\langle 4, 4, 2, 2, 2, 1, 1 \rangle$

$\neq$

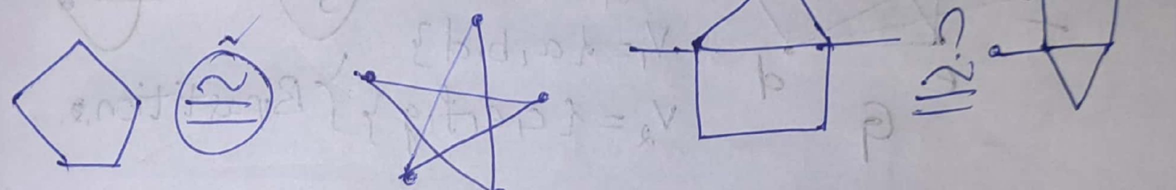
Deg. Seq. (H) =

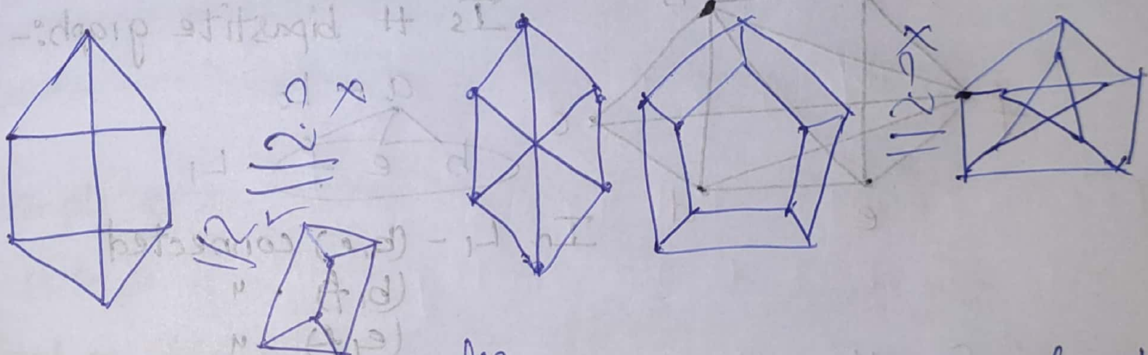
$\langle 4, 3, 3, 2, 2, 1, 1 \rangle$

G and H are not isomorphic.



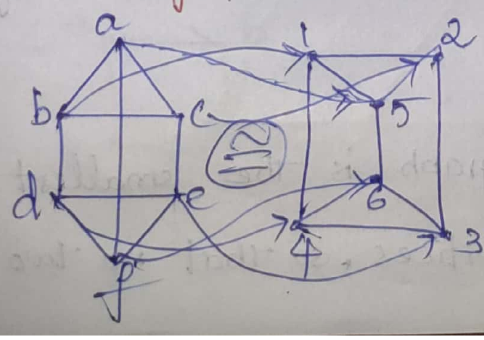
Ex





> Isomorphic graphs, are also checked by onto and one-one function

> If bijective function exists between two vertex sets of graphs then, both graphs are isomorphic graphs.



$V_1 = \{a, b, c, d, e, f\}$

$V_2 = \{1, 2, 3, 4, 5, 6\}$

$f(a) = 5 \quad f(b) = 1 \quad f(c) = 2$

$f(d) = 4 \quad f(e) = 3 \quad f(f) = 6$

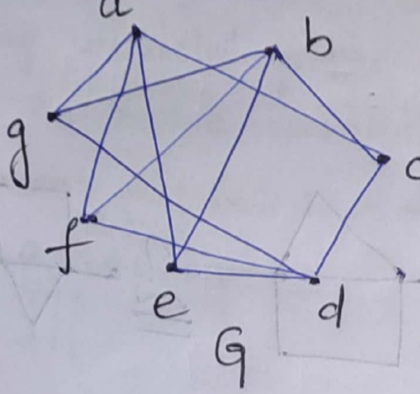
Bipartite Graphs:-

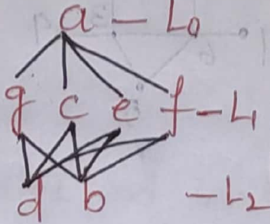
> A simple graph G is called bipartite if its vertex set V ~~is~~ can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2.

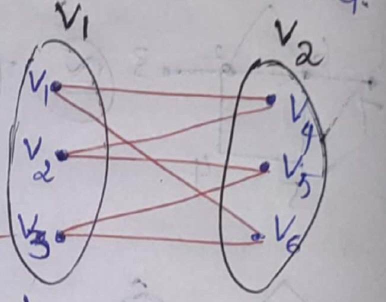
> (No edge in G connects either two vertices in V_1 or two vertices in V_2)

> (V_1, V_2) a bipartition of the vertex set V of G.

Ex:

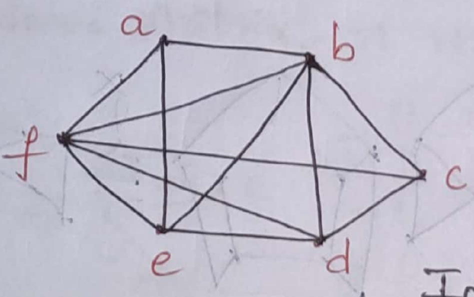




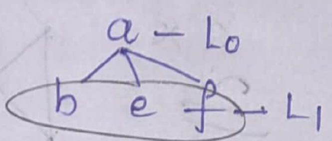


$V_1 = \{a, b, d\}$
$V_2 = \{c, e, f, g\}$ } Bipartition.

H:



Is H bipartite graph:-



In L_1 - (b,e) connected
(b,f) "
(e,f) "

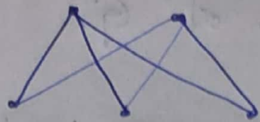
So H graph is not a bipartite graph.

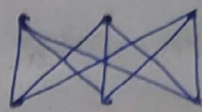
> A simple graph is bipartite iff it is possible to assign ~~two~~ one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

> The chromatic number of a graph is the smallest no. of colors needed to color the vertices, so that no two adjacent

vertices have the same color.

Complete Bipartite Graphs ($K_{m,n}$) :-

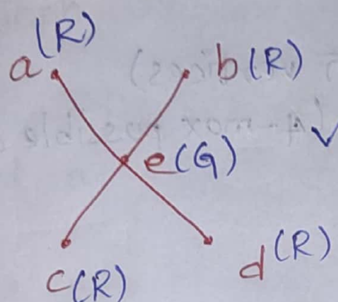
$K_{2,3}$:


$K_{3,3}$:


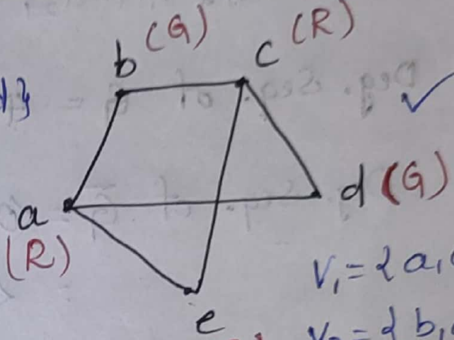
$K_{1,2}$:


$K_{2,4}$:

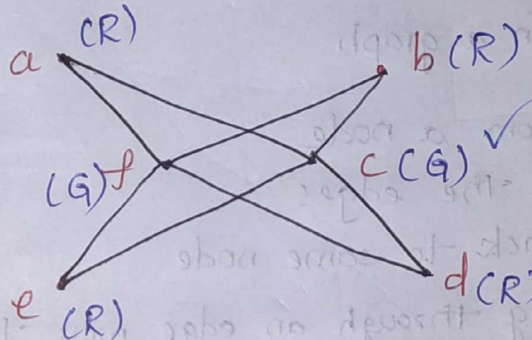

$|E|$ in $K_{m,n} = m \times n$

Ex:-


$V_1 = \{a, b, c, d\}$ ✓
$V_2 = \{e\}$



✓
$V_1 = \{a, c\}$
$V_2 = \{b, d, e\}$



✓ $V_1 = \{a, b, d, e\}$
$V_2 = \{c, f\}$

> If a simple graph G has 'v' vertices and 'e' edges,

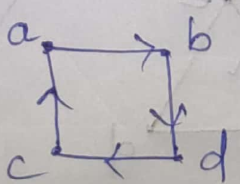
$|E|$ in $\bar{G} = ?$

Total possible edges on 'v' vetrices $= \frac{v(v-1)}{2} = {}^{v}C_2$

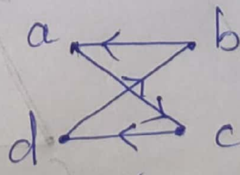
No. of edges in $\bar{G} = \boxed{{}^{v}C_2 - e}$

Converse graph of directed graph :-

$$\boxed{(G^{conv})^{conv} = G}$$



G



G^{conv}

> If the deg. seq. of the simple graph G is

$d_1, d_2, d_3, \ldots\ldots d_n$.

then, what is the deg. seq. of $\overline{G}$:

1) Subtract each deg. from $n-1 \rightarrow$ [∵ '$n-1$' is the max. possible degree.]

2) Arrang deg. Seq. in Non-increasing order

$n-1-d_1,\ n-1-d_2,\ n-1-d_3, \ldots\ldots n-1-d_n$

Ex) Deg. Seq. of G = 4, 3, 3, 2, 2 (5 vertices)

(4-max possible degree)

Deg. Seq. of $\overline{G} = \langle 0, 1, 1, 2, 2 \rangle$

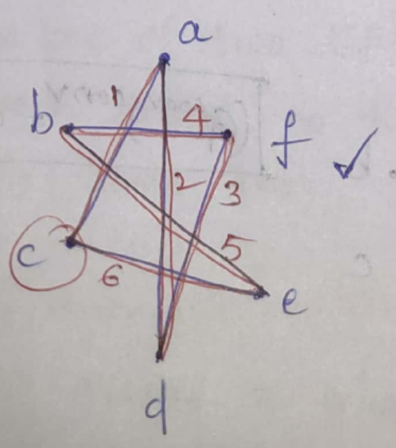
$= \langle 2, 2, 1, 1, 0 \rangle$

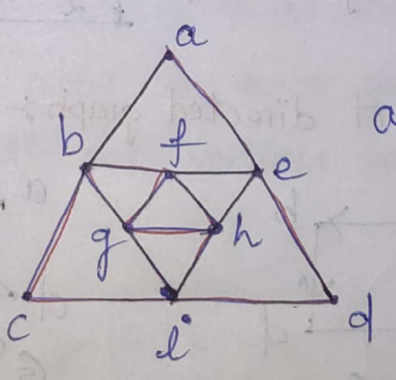
Eulerian Graph :- In a graph

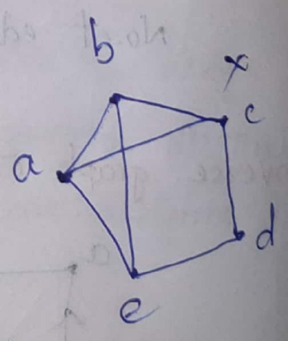
- start from a node
- visit all the edges
- come back to same node
- w/o going through an edge more than once.

Eulerian Circuit :- start from one vertex, traverse all the edges and come back to the starting vertex.

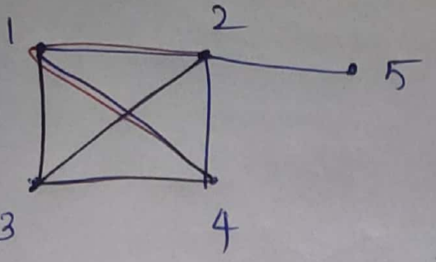
> A graph G is called an Eulerian graph if it contains an Eulerian circuit.





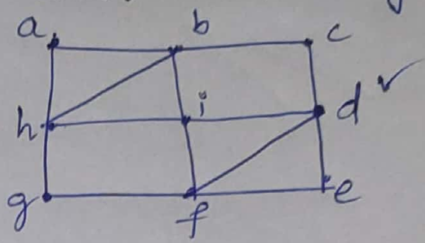


Ex:



Not Eulerian graph

- Degree of Every node should be even. Then, the graph is Eulerian graph.



- Hamiltonian Graphs:-

A graph where one can go through all the vertices, without repeating vertices and edges more than once, is called as Hamiltonian graph.

- A simple path in a graph G that passes through every vertex exactly once is called a Hamiltonian path.
- A simple circuit in a graph G that passes through every vertex exactly once is called a Hamiltonian circuit.

- Every K_n, $n \geq 3$, is a Hamiltonian graph.
- Every C_n, $n \geq 3$, is " " "
- Every W_n, $n \geq 3$, is a hamiltonian graph.