2.1

Найти

(а) $\frac{\partial z}{\partial u}$ и $\frac{\partial^2 z}{\partial u^2}$, если $z = f(x,y)$, $x = \frac{u}{v}$, $y = uv$

$$\boxed{\frac{\partial z}{\partial u}} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{1}{v}\cdot\frac{\partial f}{\partial x} + v\cdot\frac{\partial f}{\partial y}$$

$$\boxed{\frac{\partial^2 z}{\partial u^2}} = \frac{\partial}{\partial u}\left(\frac{1}{v}\cdot\frac{\partial f}{\partial x} + v\cdot\frac{\partial f}{\partial y}\right) = \frac{1}{v}\cdot\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial x}\right) +$$

$$+ v\cdot\frac{\partial}{\partial u}\left(\frac{\partial f}{\partial y}\right) = \frac{1}{v}\cdot\left(\frac{\partial^2 f}{\partial x^2}\cdot\cancel{\frac{\partial x}{\partial u}}^{\frac{1}{v}} + \underline{\frac{\partial^2 f}{\partial x\partial y}}\cdot\cancel{\frac{\partial y}{\partial u}}^{u}\right) +$$

$$+ u\cdot\left(\frac{\partial^2 f}{\partial y^2}\cdot\cancel{\frac{\partial y}{\partial u}}^{u} + \underline{\frac{\partial^2 f}{\partial y\partial x}}\cdot\cancel{\frac{\partial x}{\partial u}}^{\frac{1}{v}}\right) = \frac{1}{v^2}\cdot\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y} +$$

$$+ v^2\frac{\partial^2 f}{\partial y^2}$$

Ответ: $\frac{\partial z}{\partial u} = \frac{1}{v}\cdot\frac{\partial f}{\partial x} + v\cdot\frac{\partial f}{\partial y}$; $\frac{\partial^2 z}{\partial u^2} = \frac{1}{v^2}\cdot\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y} + v^2\frac{\partial^2 f}{\partial y^2}$.

(б) $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$. Найти $\frac{\partial^2 z}{\partial x\partial y}$

Пусть $f = 2^{\frac{x}{z}} + 2^{\frac{y}{z}} - 8$, тогда

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

№3.3

Найти угол между градиентами скалярных полей $u = \frac{x^3}{3} + 6y^3 + 3\sqrt{6}\,z^6$, $v = \frac{yz^2}{x^2}$ в точке $M(\sqrt{2}; \frac{1}{\sqrt{2}}; \frac{1}{\sqrt{3}})$.

1) $$\begin{cases} \frac{\partial u}{\partial x} = x^2 \\ \frac{\partial u}{\partial y} = 18y^2 \\ \frac{\partial u}{\partial z} = 18\sqrt{6}\,z^5 \end{cases} \qquad \left.\frac{\partial u}{\partial x}\right|_M = 2 \qquad \left.\frac{\partial u}{\partial y}\right|_M = 9 \qquad \left.\frac{\partial u}{\partial z}\right|_M = 2\sqrt{2}$$

$\Rightarrow \operatorname{grad} u = (2; 9; 2\sqrt{2})$

2) $$\begin{cases} \frac{\partial v}{\partial x} = -2\frac{yz^2}{x^3} \\ \frac{\partial v}{\partial y} = \frac{z^2}{x^2} \\ \frac{\partial v}{\partial z} = \frac{2yz}{x^2} \end{cases}$$

$$\left.\frac{\partial v}{\partial x}\right|_M = -\cancel{2}\cdot\frac{1}{\sqrt{2}\cdot\cancel{2}\sqrt{2}\cdot 3} = -\frac{1}{6}$$

$$\left.\frac{\partial v}{\partial y}\right|_M = \frac{\cancel{2}\,1}{3\cdot 2} = \frac{1}{6}$$

$$\left.\frac{\partial v}{\partial z}\right|_M = \cancel{2}\cdot\frac{1}{\sqrt{2}\cdot\sqrt{3}\cdot\cancel{2}} = \frac{1}{\sqrt{6}}$$

$\operatorname{grad} v = (-\frac{1}{6}; \frac{1}{6}; \frac{1}{\sqrt{6}})$

3) $$\cos\varphi = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \frac{-\frac{1}{3} + \frac{3}{2} + \frac{2}{\sqrt{3}}}{\sqrt{93}\cdot\frac{2}{9}} = \frac{\frac{9\sqrt{3} - 2\sqrt{3} + 18}{2\cdot 6\sqrt{3}}}{\frac{2\sqrt{93}}{\cancel{9}\,3}} =$$

$$= \frac{(7\sqrt{3}+18)\cdot\cancel{\sqrt{3}}}{2\cancel{\sqrt{93}}^{\sqrt{31}}\cdot 2\cancel{\sqrt{3}}} = \frac{7\sqrt{3}+18}{4\sqrt{31}}$$

Ответ: $\varphi = \arccos\frac{7\sqrt{3}+18}{4\sqrt{31}}$

$$\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\right) = \frac{1}{9\sqrt{3}} \qquad \frac{1}{3} \qquad \frac{1}{3}$$

$4 + 81 + 8 = 93$ $\qquad \frac{1}{36} + \frac{1}{36} + \frac{6}{36} = \frac{4}{18} = \frac{2}{9}$

№3.4

Найти производную ф-ии $u(x,y,z) = x - \ln(z^2+y^2)$ в точке $M(2;1;1)$ по направлению вектора $\vec{l} = -2\vec{i}+\vec{j}-\vec{k}$

1) $\begin{cases} \frac{\partial u}{\partial x} = 1 \\ \frac{\partial u}{\partial y} = -\frac{2y}{z^2+y^2} \\ \frac{\partial u}{\partial z} = -\frac{2z}{z^2+y^2} \end{cases} \Rightarrow \begin{cases} \frac{\partial u}{\partial x}\Big|_M = 1 \\ \frac{\partial u}{\partial y}\Big|_M = -1 \\ \frac{\partial u}{\partial z}\Big|_M = -1 \end{cases}$

2) $\left.\begin{aligned} &\vec{l}\{-2;1;-1\} \\ &|\vec{l}| = \sqrt{4+1+1} = \sqrt{6} \end{aligned}\right\} \Rightarrow \begin{aligned} &\cos\alpha = -\frac{2}{\sqrt{6}} \\ &\cos\beta = \frac{1}{\sqrt{6}} \\ &\cos\gamma = -\frac{1}{\sqrt{6}} \end{aligned}$

3) $\frac{\partial u}{\partial l} = \frac{\partial u}{\partial x}\Big|_M \cdot \cos\alpha + \frac{\partial u}{\partial y}\Big|_M \cdot \cos\beta + \frac{\partial u}{\partial z}\Big|_M \cdot \cos\gamma =$

$= -\frac{2}{\sqrt{6}} - \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} = -\frac{2}{\sqrt{6}}$

Ответ: $-\frac{2}{\sqrt{6}}$.

№3.5

Найти производную функции $u(x,y,z) = (x^2+y^2+z^2)^{\frac{3}{2}}$ в точке $M(1;1;1)$ по направлению вектора $l = \vec{i} - \vec{j} + \vec{k}$.

1) $$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{3}{2}\cdot\sqrt{x^2+y^2+z^2}\cdot 2x = 3x\sqrt{x^2+y^2+z^2} \\ \dfrac{\partial u}{\partial y} = \dfrac{3}{2}\cdot\sqrt{x^2+y^2+z^2}\cdot 2y = 3y\sqrt{x^2+y^2+z^2} \\ \dfrac{\partial u}{\partial z} = 3\quad\sqrt{x^2+y^2+z^2} \end{cases}$$

$$\begin{cases} \left.\dfrac{\partial u}{\partial x}\right|_M = 3\sqrt{3} \\ \left.\dfrac{\partial u}{\partial y}\right|_M = 3\sqrt{3} \\ \left.\dfrac{\partial u}{\partial z}\right|_M = 3\sqrt{3} \end{cases}$$

2) $\vec{l}\{1;-1;1\}$

$|\vec{l}| = \sqrt{3}$

$\cos\alpha = \frac{1}{\sqrt{3}}$ $\cos\beta = \frac{-1}{\sqrt{3}}$ $\cos\gamma = \frac{1}{\sqrt{3}}$

3) $\left.\dfrac{\partial u}{\partial l}\right|_M = 3 - 3 + 3 = 3$

Ответ: 3

№3.6

Найти производную скалярного поля $u = \cos^2 x - 2\sin y + z^2 x y^3$ в точке $M(\frac{\pi}{4}; \frac{\pi}{2}; 0)$ по направлению вектора $\bar{l} = 4\bar{i} + 3\bar{j}$. $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$ $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

1) $\begin{cases} \frac{\partial u}{\partial x} = -2\cos x \sin x + z^2 y^3 \\ \frac{\partial u}{\partial y} = -2\cos y + 3 z^2 x y^2 \\ \frac{\partial u}{\partial z} = 2 z x y^3 \end{cases}$

$\left.\frac{\partial u}{\partial x}\right|_M = -1$

$\left.\frac{\partial u}{\partial y}\right|_M = 0$

$\left.\frac{\partial u}{\partial z}\right|_M = 0$

2) $l\{4; 3; 0\}$ $|\vec{l}| = 5$

$\cos\alpha = \frac{4}{5}$

$\cos\beta = \frac{3}{5}$

$\cos\gamma = 0$

3) $\frac{\partial u}{\partial l} = -\frac{4}{5}$

Ответ: $-\frac{4}{5}$

Исследовать №4.1 на экстремум функцию

$z = e^{3x}(x+y^2+2y)$

1) $\begin{cases} \dfrac{\partial z}{\partial x} = 3e^{3x}(x+y^2+2y) + e^{3x} = 0 \\ \dfrac{\partial z}{\partial y} = 2ye^{3x} + 2e^{3x} = 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} 3x+3y^2+6y+1=0 \\ y+1=0 \end{cases} \Rightarrow \begin{cases} 3x+3-6+1=0 \\ y=-1 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x = \dfrac{2}{3} \\ y=-1 \end{cases} \Rightarrow M=\left(\dfrac{2}{3};-1\right)$ – стационарная точка

2) $\begin{cases} \dfrac{\partial^2 z}{\partial x^2} = 9e^{3x}(x+y^2+2y) + 3e^{3x} + 3e^{3x} = \\ = 6e^{3x} + 9e^{3x}\cdot x + 9e^{3x}y^2 + 18e^{3x}y \\ \dfrac{\partial^2 z}{\partial y^2} = 2e^{3x} \\ \dfrac{\partial^2 z}{\partial x \partial y} = 6ye^{3x} + 6e^{3x} \end{cases} \Rightarrow$ при $M\left(\dfrac{2}{3};-1\right)$:

$+12$
9
21

$\Rightarrow \begin{cases} \dfrac{\partial^2 z}{\partial x^2} = 6e^2 + 6e^2 + 9e^2 - 18e^2 = 3e^2 \\ \dfrac{\partial^2 z}{\partial y^2} = 2e^2 \\ \dfrac{\partial^2 z}{\partial x \partial y} = 6e^2 - 6e^2 = 0 \end{cases}$

Значит $\begin{pmatrix} 3e^2 & 0 \\ 0 & 2e^2 \end{pmatrix}$, $\Delta_1 = 3e^2 > 0$, $\Delta_2 = 6e^2 > 0$ $\Rightarrow$

$\Rightarrow$ экстремум существует, а $M(\frac{2}{3}; -1)$ – минимум функции

Ответ: $M(\frac{2}{3}; -1)$ – min.

4.2

Исследовать на экстремум функцию

$z = x^3 - 3xy + 3y^2 - 5$, $(x>0, y>0)$.

1) $\begin{cases} \frac{\partial z}{\partial x} = 3x^2 - 3y = 0 \\ \frac{\partial z}{\partial y} = 6y - 3x = 0 \end{cases} \Rightarrow \begin{cases} x^2 = y \\ 2y = x \end{cases} \Rightarrow \begin{cases} 4y^2 - y = 0 \\ x = 2y \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} y(4y-1) = 0 \\ x = 2y \end{cases} \Rightarrow \left[\begin{array}{l} y = 0 \text{ (по усл. } y>0) \\ y = \frac{1}{4} \end{array}\right. \Rightarrow \left[\begin{array}{l} x = 0 \text{ (по усл. } x>0) \\ x = \frac{1}{2} \end{array}\right.$

$\Rightarrow M\left(\frac{1}{2}; \frac{1}{4}\right)$ – стационарная точка

2) $\begin{cases} \frac{\partial^2 z}{\partial x^2} = 6x \\ \frac{\partial^2 z}{\partial y^2} = 6 \\ \frac{\partial^2 z}{\partial x \partial y} = -3 \end{cases} \Rightarrow$ При $M\left(\frac{1}{2}; \frac{1}{4}\right)$: $\begin{cases} \frac{\partial z^2}{\partial x^2} = 3 \\ \frac{\partial z^2}{\partial y^2} = 6 \\ \frac{\partial z^2}{\partial x \partial y} = -3 \end{cases} \Rightarrow$

Получаем $\begin{pmatrix} 3 & -3 \\ -3 & 6 \end{pmatrix}$, $\Delta_1 = 3 > 0$, $\Delta_2 = 9 > 0$ $\Rightarrow$ экстремум существует, а $M\left(\frac{1}{2}; \frac{1}{4}\right)$ – минимум функции

Ответ: $M\left(\frac{1}{2}; \frac{1}{4}\right)$ – min.

$2 - \frac{4}{2}$

№4.3

Исследовать на экстремум ф-ию:

$z = y^3 + 3x^2y - 12x - 15y$, $(x > 0, y > 0)$

1) $\begin{cases} \frac{\partial z}{\partial x} = 6xy - 12 = 0 \\ \frac{\partial z}{\partial y} = 3y^2 + 3x^2 - 15 = 0 \end{cases} \Rightarrow \begin{cases} xy = 2 \\ y^2 + x^2 = 5 \end{cases} \Rightarrow \begin{cases} x = \frac{2}{y} \\ y^2 + \left(\frac{2}{y}\right)^2 = \end{cases}$

$\Rightarrow \begin{cases} x = \frac{2}{y} \\ y^2 + \frac{4}{y^2} = 5 \end{cases} \Rightarrow \begin{cases} x = \frac{2}{y} \\ y^4 - 5y^2 + 4 = 0 \quad (1) \end{cases} \Rightarrow$

(1) $y^4 - 5y^2 + 4 = 0$

Пусть $y^2 = t$, тогда:

$$t^2 - 5t + 4 = 0$$

$D = b^2 - 4ac = 25 - 4 \cdot 4 \cdot 1 = 9$

$t_{1,2} = \frac{-b \pm \sqrt{D}}{2a} = \frac{5 \pm 3}{2} \Rightarrow \begin{cases} t = 4 \\ t = -1 \end{cases}$ (не подходит) $\Rightarrow \left[\begin{array}{l} y = 2 \\ y = -2 \end{array} \right.$ (не удов. $y > 0$)

$\Rightarrow \begin{cases} x = \frac{2}{y} \\ y = 2 \end{cases} \Rightarrow M(1; 2)$ – стационарная точка

2) $\begin{cases} \frac{\partial^2 z}{\partial x^2} = 6y \\ \frac{\partial^2 z}{\partial y^2} = 6y \\ \frac{\partial^2 z}{\partial x \partial y} = 6x \end{cases} \Rightarrow$ (при $M(1;2)$) $\begin{cases} \frac{\partial^2 z}{\partial x^2} = 12 \\ \frac{\partial^2 z}{\partial y^2} = 12 \\ \frac{\partial^2 z}{\partial x \partial y} = 6 \end{cases}$

$\begin{array}{r} 144 \\ -\ 36 \\ \hline 108 \end{array}$

$\begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}$

$\Delta_1 = 12 > 0$, $\Delta_2 = 108 > 0 \Rightarrow$

$M(1; 2)$ – min.

№4.4

$z = y^2 \ln x - x^2$ (при $x>0, y>0$)

1) $\begin{cases} \dfrac{\partial z}{\partial x} = \dfrac{y^2}{x} - 2x = 0 \\ \dfrac{\partial z}{\partial y} = 2y \ln x = 0 \end{cases} \Rightarrow \begin{cases} y^2 = 2x^2 \\ 2y \ln x = 0 \end{cases}$

$2y \ln x = 0 \Downarrow$

$y = 0$ (по усл. $y>0$) $\quad \ln x = 0$, $\underline{x = 1}$

$\begin{cases} y = \sqrt{2} \\ x = 1 \end{cases} \Rightarrow$

$x^{-1} = -\dfrac{1}{x^2}$

$\Rightarrow M(1; \sqrt{2})$ – стационарная точка

2) $\begin{cases} \dfrac{\partial^2 z}{\partial x^2} = -\dfrac{y^2}{x^2} - 2 \\ \dfrac{\partial^2 z}{\partial y^2} = 2 \ln x \\ \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{2y}{x} \end{cases} \Rightarrow$ (При $M(1;\sqrt{2})$) $\begin{cases} \dfrac{\partial^2 z}{\partial x^2} = -4 \\ \dfrac{\partial^2 z}{\partial y^2} = 0 \\ \dfrac{\partial^2 z}{\partial x \partial y} = 2\sqrt{2} \end{cases}$

$\begin{pmatrix} -4 & 2\sqrt{2} \\ 2\sqrt{2} & 0 \end{pmatrix}$ $\quad \Delta_1 = -4 < 0,$ $\quad \Delta_2 = -8 < 0$ $\Rightarrow$ данная ф-ия не имеет экстремумов

Ответ: данная ф-ия не имеет экстремумов.

№4.5

$z = e^x(x + y^2 + 2y)$

1) $\begin{cases} \frac{\partial z}{\partial x} = e^x(x+y^2+2y) + e^x = 0 \\ \frac{\partial z}{\partial y} = 2ye^x + 2e^x = 0 \end{cases} \Rightarrow \begin{cases} y^2 + 2y + x + 1 = 0 \\ y = -1 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} 1 - 2 + x + 1 = 0 \\ y = -1 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = -1 \end{cases} \Rightarrow M(0;-1)$ – стац. точка

2) $\begin{cases} \frac{\partial^2 z}{\partial x^2} = e^x(x+y^2+2y) + e^x + e^x = \underset{2}{2e^x} + \underset{0}{e^x x} + \underset{1}{e^x y^2} + \underset{-2}{e^x \cdot 2y} \\ \frac{z}{\partial y^2} = 2 \\ \frac{\partial z}{\partial x \partial y} = 2ye^x + 2e^x \end{cases} \Rightarrow$ При $M(0;-1)$

$\Rightarrow \begin{cases} \frac{\partial^2 z}{\partial x^2} = 1 \\ \frac{\partial^2 z}{\partial y^2} = 2 \\ \frac{\partial^2 z}{\partial x \partial y} = 0 \end{cases}$ Значит $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$; $\Delta_1 = 1 > 0$, $\Delta_2 = 2 > 0$, $\Rightarrow$

$\Rightarrow$ экстремум существует и

$M(0;1)$ – т. минимума

Ответ: $M(0;1)$ – min

№5.1

Исследовать на экстремум ф-ию

$Z = \frac{1}{x} + \frac{1}{y}$ при условии $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{2}$ $(x<0, y<0)$

$(x^{-2})' = -2\frac{1}{x^3}$

$$F = \frac{1}{x} + \frac{1}{y} + \lambda\left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{2}\right)$$

1) $$\begin{cases} \frac{\partial F}{\partial x} = -\frac{1}{x^2} - \frac{2\lambda}{x^3} = 0 \\ \frac{\partial F}{\partial y} = -\frac{1}{y^2} - \frac{2\lambda}{y^3} = 0 \\ \frac{\partial F}{\partial \lambda} = \frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{2} = 0 \end{cases} \Rightarrow \begin{cases} x = -2\lambda \\ y = -2\lambda \\ 2y^2 + 2x^2 - x^2y^2 = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = -2\lambda \\ y = -2\lambda \\ 8\lambda^2 + 8\lambda^2 - 16\lambda^4 = 0 \end{cases} \Rightarrow \begin{cases} x = -2\lambda \\ y = -2\lambda \\ 16\lambda^4 - 16\lambda^2 = 0 \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} x = -2\lambda \\ y = -2\lambda \\ 16\lambda^2(\lambda^2 - 1) = 0 \end{cases}$$

$\lambda = 0$

$x, y = 0$ (по усл. $x, y < 0$)

$\lambda = 1$

$x = -2$
$y = -2$

$\lambda = -1$

$x = 2$
$y = 2$
(по усл. $x, y < 0$)

$\Rightarrow M(-2; -2)$ – стационарная точка $\lambda = 1$

2) $\begin{cases} \dfrac{\partial^2 F}{\partial x^2} = \dfrac{2}{x^3} + \dfrac{6}{x^4} \\ \dfrac{\partial^2 F}{\partial y^2} = \dfrac{2}{y^3} + \dfrac{6}{y^4} \\ \dfrac{\partial^2 F}{\partial x \partial y} = 0 \end{cases}$ $\Rightarrow$ При $M(-2;-2)$:
$\lambda = 1$

$\begin{cases} \dfrac{\partial^2 F}{\partial x^2} = -\dfrac{2}{8} + \dfrac{6}{16} = \dfrac{3}{8} - \dfrac{2}{8} = \dfrac{1}{8} \\ \dfrac{\partial^2 F}{\partial y^2} = -\dfrac{2}{8} + \dfrac{6}{16} = \dfrac{1}{8} \\ \dfrac{\partial^2 F}{\partial x \partial y} = 0 \end{cases}$

условной экстремум существует

$\begin{pmatrix} \frac{1}{8} & 0 \\ 0 & \frac{1}{8} \end{pmatrix}$ $\Delta_1 = \frac{1}{8} > 0$, $\Delta_2 = \frac{1}{64} > 0$ $\Rightarrow$ $M(-2;-2)$ – min.

Ответ: $M(-2;-2)$ – точка условного минимума.

№5.2

$z = x + y$ условие $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{2}$ $(x>0, y>0)$

$F = x + y + \lambda\left(\frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{2}\right)$

1) $\begin{cases} \frac{\partial F}{\partial x} = 1 - \frac{2\lambda}{x^3} = 0 \\ \frac{\partial F}{\partial y} = 1 - \frac{2\lambda}{y^3} = 0 \\ \frac{\partial F}{\partial \lambda} = \frac{1}{x^2} + \frac{1}{y^2} - \frac{1}{2} = 0 \end{cases} \Rightarrow \begin{cases} x^3 = 2\lambda \\ y^3 = 2\lambda \\ 2x^2 + 2y^2 - x^2y^2 = 0 \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} x = \sqrt[3]{2\lambda} \\ y = \sqrt[3]{2\lambda} \\ \frac{1}{(2\lambda)^{\frac{2}{3}}} + \frac{1}{(2\lambda)^{\frac{2}{3}}} = \frac{1}{2} \end{cases}$ При $\lambda = 4$ $\Rightarrow \begin{cases} x = 2 \\ y = 2 \\ \lambda = 4 \end{cases} \Rightarrow$

$\lambda = 4$ $\Rightarrow M(2;2)$ – стационарная точка
$\lambda = 4$

2) $\begin{cases} \frac{\partial^2 F}{\partial x^2} = \frac{6\lambda}{y^4} \\ \frac{\partial^2 F}{\partial y^2} = \frac{6\lambda}{y^4} \\ \frac{\partial^2 F}{\partial x \partial y} = 0 \end{cases} \Rightarrow$ При $M(2;2)$, $\lambda = 4$ $\begin{cases} \frac{\partial^2 F}{\partial x^2} = \frac{24}{16} = \frac{3}{2} \\ \frac{\partial^2 F}{\partial y^2} = \frac{24}{16} = \frac{3}{2} \\ \frac{\partial^2 F}{\partial x \partial y} = 0 \end{cases}$

Получим:

$$\begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \quad \Delta_1 = \frac{3}{2} > 0, \quad \Delta_2 = \frac{9}{4} > 0 \Rightarrow$$

условный экстремум существует, и $M(2;2)$ – min

Ответ: $M(2;2)$ – точка условного минимума.

N5.3

$Z = x^2 + y^2$ при $3x + 4y = 12$

$F = x^2 + y^2 + \lambda(3x + 4y - 12)$

1) $$\begin{cases} \frac{\partial F}{\partial x} = 2x + 3\lambda = 0 \\ \frac{\partial F}{\partial y} = 2y + 4\lambda = 0 \\ \frac{\partial F}{\partial \lambda} = 3x + 4y - 12 = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{3}{2}\lambda \\ y = -2\lambda \\ -\frac{9}{2}\lambda - 8\lambda - 12 = 0 \end{cases} \Rightarrow$$

$-9\lambda - 16\lambda - 24 = 0$

$$\Rightarrow \begin{cases} x = -\frac{3}{2}\lambda \\ y = -2\lambda \\ -25\lambda = 24 \end{cases} \Rightarrow \begin{cases} x = \frac{36}{25} \\ y = \frac{48}{25} \\ \lambda = -\frac{24}{25} \end{cases} \Rightarrow M = \left(\frac{36}{25}; \frac{48}{25}\right)$$

$\lambda = -\frac{24}{25}$

2) $$\begin{cases} \frac{\partial^2 F}{\partial x^2} = 2 \\ \frac{\partial^2 F}{\partial y^2} = 2 \\ \frac{\partial^2 F}{\partial x \partial y} = 0 \end{cases} \Rightarrow \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \Delta_1 = 2 > 0, \quad \Delta_2 = 4 > 0 \Rightarrow$$

$M\left(\frac{36}{25}; \frac{48}{25}\right)$ – точка условного минимума

Ответ: $M\left(\frac{36}{25}; \frac{48}{25}\right)$ – точка условного минимума.

№5.4

$u = 2x + z - 3y$ при $x^2 + y^2 + z^2 = 4$

$F = 2x + z - 3y + \lambda(x^2 + y^2 + z^2 - 4)$

1) $\begin{cases} \frac{\partial F}{\partial x} = 2 + 2\lambda x = 0 \\ \frac{\partial F}{\partial y} = -3 + 2\lambda y = 0 \\ \frac{\partial F}{\partial} = 1 + 2\cancel{\lambda z} = 0 \\ \frac{\partial F}{\partial \lambda} = x^2 + y^2 + z^2 - 4 = 0 \end{cases} \Rightarrow \begin{cases} x = -\frac{1}{\lambda} \\ y = \frac{3}{2\lambda} \\ z = -\frac{1}{2\lambda} \\ \frac{1}{\lambda^2} + \frac{9}{4\lambda^2} + \frac{1}{4\lambda^2} - 4 = 0 \ (1) \end{cases} \Rightarrow$

(1) $4 + 9 + 1 - 16\lambda^2 = 0$

$\Rightarrow$ $\lambda^2 = \frac{14}{16}$

$\lambda = \frac{\sqrt{14}}{4}$ $\qquad$ $\lambda = -\frac{\sqrt{14}}{4}$

$M_1\left(-\frac{4}{\sqrt{14}}; \frac{6}{\sqrt{14}}; -\frac{2}{\sqrt{14}}\right)$ $\qquad$ $M_2\left(\frac{4}{\sqrt{14}}; -\frac{6}{\sqrt{14}}; \frac{2}{\sqrt{14}}\right)$

2) $\begin{cases} \frac{\partial^2 F}{\partial x^2} = 2\lambda \\ \frac{\partial^2 F}{\partial y^2} = 2\lambda \\ \frac{\partial^2 F}{\partial z^2} = 2\lambda \\ \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial^2 F}{\partial x \partial z} = \frac{\partial^2 F}{\partial y \partial z} = 0 \end{cases} \Rightarrow$

① При M_1:

$$\begin{pmatrix} \frac{\sqrt{14}}{2} & 0 & 0 \\ 0 & \frac{\sqrt{14}}{2} & 0 \\ 0 & 0 & \frac{\sqrt{14}}{2} \end{pmatrix}$$

$M_1\left(-\frac{4}{\sqrt{14}}; \frac{6}{\sqrt{14}}; -\frac{2}{\sqrt{14}}\right)$ – точка условного минимума.

$\Delta_1 > 0,\ \Delta_2 > 0 \Rightarrow$

② При M_2:

$$\begin{pmatrix} -\frac{\sqrt{14}}{2} & 0 & 0 \\ 0 & -\frac{\sqrt{14}}{2} & 0 \\ 0 & 0 & -\frac{\sqrt{14}}{2} \end{pmatrix} \Rightarrow$$

$M_2\left(\frac{4}{\sqrt{14}}; -\frac{6}{\sqrt{14}}; \frac{2}{\sqrt{14}}\right)$ – не является экстремумом

доп исследование?

$\Delta_1 < 0,\ \Delta_2 < 0$

Ответ: $M_1\left(-\frac{4}{\sqrt{14}}; \frac{6}{\sqrt{14}}; -\frac{2}{\sqrt{14}}\right)$ – точка условного минимума.