Markov's inequality!

If $X \geq 0$ and $a > 0$, then

$$P(X \geq a) \leqslant \frac{E(X)}{a}$$

Proof: (discret cas)

$x_i \geq 0$ for $1 \leqslant i \leqslant n$

$$E(X) = \sum_{i=1}^{n} x_i P(X = x_i)$$

$$= \underbrace{\sum_{x_i \leqslant a} x_i P(X = x_i)}_{\geq 0} + \sum_{x_i \geq a} x_i P(X = x_i)$$

$$\Rightarrow \boxed{E(X) \geqslant \sum_{x_i \geq a} x_i P(X = x_i)} \quad (1)$$

$x_i \geq a \Rightarrow x_i P(X = x_i) \geq a P(X = x_i)$

$$\Rightarrow \sum_{x_i \geq a} x_i P(X = x_i) \geq \sum_{x_i \geq a} a P(X = x_i)$$

$$\Rightarrow \sum_{x_i \geq a} x_i P(X = x_i) \geq a \underbrace{\sum_{x_i \geq a} P(X = x_i)}_{P(X \geq a)}$$

by (1) : $E(X) \geq a P(X \geq a)$

So : $\boxed{P(X \geq a) \leqslant \frac{E(X)}{a}}$

- (continuous cas) :

$$E(X) = \int_0^{+\infty} x f_X(x)\,dx \geq \int_a^{+\infty} x f_X(x)\,dx \geq a \underbrace{\int_a^{+\infty} f_X(x)\,dx}_{P(X \geq a)}$$

$$\Rightarrow \boxed{P(X \geq a) \leqslant \frac{E(X)}{a}}$$