The weak law of large numbers

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Sample Mean

Definition :

For i.i.d. random variables $X_1, X_2, \ldots, X_n$ The **sample mean**, denoted by $\overline{X}$, is defined as :

$$\overline{X} = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Another common notation for the sample mean is :μ
If the random variables X_i have (CDF) $F_X(x)$, we may write the sample mean as μ to indicate the distribution of the X_i's.

Expectation of the Sample Mean

Note that since the X_i's are random variables, the sample mean is also a random variable.

In particular, we have

$$\mathbb{E}[\overline{X}] = \mathbb{E}\left[\frac{X_1 + X_2 + \cdots + X_n}{n}\right] = \frac{\mathbb{E}[X_1] + \mathbb{E}[X_2] + \cdots + \mathbb{E}[X_n]}{n}.$$

By linearity of expectation, and since $\mathbb{E}[X_i] = \mathbb{E}[X]$ for all i, we obtain

$$\mathbb{E}[\overline{X}] = \frac{n\,\mathbb{E}[X]}{n} = \mathbb{E}[X] = \mu.$$

Variance of the Sample Mean

The variance of the sample mean $\overline{X}$ is given by :

$$\mathrm{Var}(\overline{X}) = \mathrm{Var}\left(\frac{X_1 + X_2 + \cdots + X_n}{n}\right) = \frac{\mathrm{Var}(X_1 + X_2 + \cdots + X_n)}{n^2}.$$

Since the X_i's are independent, we have

$$\mathrm{Var}(X_1+X_2+\cdots+X_n) = \mathrm{Var}(X_1)+\mathrm{Var}(X_2)+\cdots+\mathrm{Var}(X_n) = n\,\mathrm{Var}(X),$$

Since $\mathrm{Var}(X_i) = \mathrm{Var}(X)$ for all i.

$$\mathrm{Var}(\overline{X}) = \frac{n\,\mathrm{Var}(X)}{n^2} = \frac{\mathrm{Var}(X)}{n}.$$

Weak Law of Large Numbers

Weak Law of Large Numbers :

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with a finite expected value

$$\mathbb{E}[X_i] = \mu < \infty.$$

Then, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\Big(|\overline{X} - \mu| \geq \varepsilon\Big) = 0.$$

Proof of the (WLLN)

The proof of the weak law of large numbers is easier if we assume that

$$\mathrm{Var}(X) < \infty.$$

By the, **Chebyshev's inequality**,for all For any $\varepsilon > 0$:

$$\mathbb{P}\Big(|\overline{X} - \mu| \geq \varepsilon\Big) \leq \frac{\mathrm{Var}(\overline{X})}{\varepsilon^2}.$$

Since

$$\mathrm{Var}(\overline{X}) = \frac{\mathrm{Var}(X)}{n}$$

we obtain

$$\mathbb{P}\Big(|\overline{X} - \mu| \geq \varepsilon\Big) \leq \frac{\mathrm{Var}(X)}{n\varepsilon^2}$$

As $n \to \infty$,

$$\lim_{n\to\infty} \mathbb{P}\Big(|\overline{X} - \mu| \geq \varepsilon\Big) = 0.$$

Thank You

Thank you for your attention !