# *Chebyshev's Inequality*

## Chebyshev's Inequality

If $X$ is any random variable, then for any $b > 0$ we have

$$\mathbb{P}\Big(|X - \mathbb{E}[X]| \geq b\Big) \leq \frac{\mathrm{Var}(X)}{b^2}.$$

*Proof.* Let $X$ be any random variable and define

$$Y = (X - \mathbb{E}[X])^2.$$

Since $Y$ is nonnegative, we can apply Markov's inequality for any $b > 0$:

$$\mathbb{P}(Y \geq b^2) \leq \frac{\mathbb{E}[Y]}{b^2}.$$

But

$$\mathbb{E}[Y] = \mathbb{E}\Big[(X - \mathbb{E}[X])^2\Big] = \mathrm{Var}(X),$$

and

$$\mathbb{P}(Y \geq b^2) = \mathbb{P}\Big((X - \mathbb{E}[X])^2 \geq b^2\Big) = \mathbb{P}\Big(|X - \mathbb{E}[X]| \geq b\Big).$$

Thus,

$$\mathbb{P}\Big(|X - \mathbb{E}[X]| \geq b\Big) \leq \frac{\mathrm{Var}(X)}{b^2}.$$

□

Chebyshev's inequality states that the difference between X and E[X] is somehow limited by Var(X). This is intuitively expected, as the variance shows on average how far we are from the mean.

### Exercise

Let $X \sim \mathrm{Binomial}(n, p)$. Using Chebyshev's inequality, find an upper bound on $\mathbb{P}(X \geq \alpha n)$, where $p < \alpha < 1$. Evaluate the bound for $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$.

**Solution:**

$$\mathbb{P}(X \geq \alpha n) = \mathbb{P}(X - np \geq \alpha n - np) \leq \mathbb{P}(|X - np| \geq n(\alpha - p)) \leq \frac{\mathrm{Var}(X)}{n^2(\alpha - p)^2}.$$

Since $\mathrm{Var}(X) = np(1-p)$ for a binomial variable:

$$\mathbb{P}(X \geq \alpha n) \leq \frac{np(1-p)}{n^2(\alpha - p)^2} = \frac{p(1-p)}{n(\alpha - p)^2}.$$

For $p = \frac{1}{2}$ and $\alpha = \frac{3}{4}$:

$$\mathbb{P}\left(X \geq \frac{3n}{4}\right) \leq \frac{(1/2)(1-1/2)}{n(3/4 - 1/2)^2} = \frac{4}{n}.$$