The Cumulative Distribution Function (CDF)

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Plan

1. Definition of the Cumulative Distribution Function
2. Cases of the Cumulative Distribution Function
   - Discrete Case
   - Continuous Case

General Definition

Definition

Let X be a random variable. The cumulative distribution function (CDF) of X is defined by :

$$F_X(x) = \mathbb{P}(X \leq x), \quad \forall x \in \mathbb{R}.$$

Discrete Case : Definition

Definition

If X is a discrete random variable :

$$\forall x \in \mathbb{R}, \quad F_X(x) = \mathbb{P}(X \leq x) = \sum_{x_i \leq x} \mathbb{P}(X = x_i).$$

The cumulative distribution function is always non-decreasing and satisfies

$$0 \leq F_X(x) \leq 1, \quad \forall x \in \mathbb{R}.$$

Discrete Case : Remark

Remark

If X takes values $x_1 < x_2 < \cdots < x_n$, then :

$$F_X(t) = \begin{cases} 0, & t < x_1, \\ \mathbb{P}(X = x_1), & x_1 \leq t < x_2, \\ \mathbb{P}(X = x_1) + \mathbb{P}(X = x_2), & x_2 \leq t < x_3, \\ \vdots & \\ \mathbb{P}(X = x_1) + \cdots + \mathbb{P}(X = x_n) = 1, & t \geq x_n. \end{cases}$$

Discrete Case : Example 1

Example 1 :

We toss a fair coin twice. We define the random variable X as the number of "heads" obtained in these two tosses.
Determine the cumulative distribution function F_X of X.

Discrete Case : Remark

Remark

If the cumulative distribution function (CDF) of a discrete random variable X is known, we can deduce its probability mass function (PMF) as follows :

$$\mathbb{P}\{X = x_i\} = F_X(x_{i+1}) - F_X(x_i)$$

Discrete Case : Example 2

Example 2 :

Let X be a discrete random variable with cumulative distribution function :

$$F_X(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ \frac{1}{8}, & \text{if } 0 < t \leq 1, \\ \frac{1}{2}, & \text{if } 1 < t \leq 2, \\ \frac{7}{8}, & \text{if } 2 < t \leq 3, \\ 1, & \text{if } t > 3. \end{cases}$$

Determine the probability distribution (law) of X.

Continuous Case : Definition

Definition

If X is a continuous random variable with density f_X :

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt.$$

Continuous Case : Example

Exemple 1 :

Let X be a continuous random variable whose probability density function f is defined as follows :

$$f(x) = \begin{cases} \frac{1}{9}x^2, & \text{for } 0 < x < 3 \\ 0, & \text{otherwise.} \end{cases}$$

Find the cumulative distribution function (CDF) of X.

Continuous Case :Properties of the (CDF)

Properties

1. F_X is continuous and non-decreasing on $\mathbb{R}$.
2. For all $x \in \mathbb{R}$, $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$, where f_X is the probability density function (PDF).
3. $\lim_{x \to -\infty} F_X(x) = 0, \quad \lim_{x \to +\infty} F_X(x) = 1.$
4. $P(a \leq X \leq b) = F_X(b) - F_X(a)$.

Continuous Case :Remark

Remark

For any real numbers a and b with $a \leq b$,

$$\mathbb{P}(a \leq X \leq b) = \mathbb{P}(a \leq X < b) = \mathbb{P}(a < X \leq b) = \mathbb{P}(a < X < b).$$

$$\mathbb{P}(a < X < b) = \int_a^b f(x)\,dx = F(b) - F(a).$$

Example : Random Variable and CDF

Let F be a function defined as follows :

$$F : \mathbb{R} \to [0,1], \quad x \mapsto \begin{cases} 0, & \text{if } x \leq 0, \\ \dfrac{x^3}{27}, & \text{for } 0 < x < 3, \\ 1, & \text{if } x \geq 3. \end{cases}$$

Consider the cumulative distribution function (CDF) of a random variable X.

1. Calculate $P(-1 < X < 0)$.
2. Calculate $P(-1 < X \leq 2)$.
3. Calculate $P(X < 2)$.
4. Calculate $P(X > 3)$.

Thank You

Thank you for your attention !