Inverses of Matrices

I $n\times n$ identity matrix

If A is $m\times n$, then $AI = A$

If B is $n\times r$, then $IB = B$

Definition Let A be an $n\times n$ matrix. Then A is called invertible if there exists an $n\times n$ matrix B such that $AB = BA = I$

Theorem If A is invertible, then there exists exactly one matrix B such that $AB = BA = I$

Proof Let C be a matrix such that $AC = CA = I$. Then

$$C = CI = C(AB) = (CA)B = IB = B$$

The unique inverse of A is denoted by A^{-1}

Example let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. We need to solve $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or

$$\begin{bmatrix} a+2c & b+2d \\ -a+3c & -b+3d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{array}{l} a + 2c = 1 \\ -a + 3c = 0 \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} b + 2d = 0 \\ -b + 3d = 1 \end{array}\right\}$$

The two systems have the same coefficient matrix $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$

Adjoin I and take the matrix

$$\left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ -1 & 3 & 0 & 1 \end{array}\right] \xrightarrow{(1)R_1 + R_2} \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 5 & 1 & 1 \end{array}\right]$$

$$\xrightarrow{(\frac{1}{5})R_2} \left[\begin{array}{cc:cc} 1 & 2 & 1 & 0 \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right] \xrightarrow{(-2)R_2 + R_1}$$

$$\left[\begin{array}{cc:cc} 1 & 0 & \frac{3}{5} & -\frac{2}{5} \\ 0 & 1 & \frac{1}{5} & \frac{1}{5} \end{array}\right]$$

$$a = \frac{3}{5}, \quad b = -\frac{2}{5}, \quad c = \frac{1}{5}, \quad d = \frac{1}{5}$$

Check also that

$$\begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} \frac{3}{5} & -\frac{2}{5} \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

<u>Example</u> Let $A = \begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix}$. Does A have an inverse?

$$\begin{bmatrix} 1 & -3 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a-3c & b-3d \\ -2a+6c & -2b+6d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left.\begin{aligned} a-3c &= 1 \\ -2a+6c &= 0 \end{aligned}\right\}$$

$$\left.\begin{aligned} a-3c &= 1 \\ a-3c &= 0 \end{aligned}\right\} \quad 0 = 1 \quad \text{impossible}$$

The matrix A does not have an inverse

<u>Theorem</u> The matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad-bc \neq 0$, in which case

$$A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

<u>Example</u> Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix}$

we need to solve

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Consider

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -3 & 3 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-2)R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_1+R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_3} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{(1)R_2+R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{(-3)R_3+R_1}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right] \xrightarrow{(-1)R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix}$$

Theorem If the $n \times n$ matrix A is invertible, then for any n-vector b the system $AX = b$ has the unique solution $X = A^{-1}b$

Proof First we prove that $AX = b$ has a solution. Note $A(A^{-1}b) = (AA^{-1})b = Ib = b$, so $X = A^{-1}b$ is a solution of $AX = b$. If X_1 is another solution of $AX = b$, then $AX_1 = b$, so $AX_1 = AX$ and thus

$$A^{-1}(AX_1) = A^{-1}(AX). \text{ So } X_1 = X$$

Example The system

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -3 & 3 \\ 1 & -1 & 1 \end{bmatrix} X = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \text{ has}$$

unique solution $X = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Theorem If the matrices A and B of the same size are invertible, then

(1) A^{-1} is invertible and $(A^{-1})^{-1} = A$

(2) The product AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Proof (2) we have

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1}$$

$= I$. similarly $(B^{-1}A^{-1})(AB) = I$

Definition The $n \times n$ matrix E is called elementary matrix if it can be obtained by performing a single elementary row operation on the $n \times n$ identity matrix I

Example $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{(3)R_1} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = E_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(2)R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{SWAP(R_1, R_2)} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_3$$

Example Find the reduced echelon Form of $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{SWAP(R_1, R_2)} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{SWAP(R_1, R_2)} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$E_1 A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow{(-1)R_1 + R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_1 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E_2$$

$$E_2 (E_1 A) = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 3 \end{bmatrix} \xrightarrow{(-2)R_2 + R_3} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-2)R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = E_3$$

$$E_3 (E_2 E_1 A) = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(1)R_2 + R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(1)R_2 + R_1} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_4$$

$$E_4 (E_3 E_2 E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-1)R_3+R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} = E_5$$

$$E_5 (E_4 E_3 E_2 E_1 A) = I$$

<u>Theorem</u> If an elementary row operation is performed on the $m \times n$ matrix A, then the result is the product matrix EA, where E is the elementary matrix obtained by performing the same row operation on the $m \times m$ identity matrix I

<u>Theorem</u> Every elementary matrix is invertible

<u>Proof</u> (1) SWAP (R_i, R_j). Then $E^{-1} = E$ (2) $(C) R_i$. Then E^{-1} is the elementary matrix that multiplies R_i by $\frac{1}{C}$. (3) $(C) R_i + R_j$. Then E^{-1} is the elementary matrix that multiplies R_i by $-C$ and adds it to R_j

Theorem The $n \times n$ matrix A is invertible $\Leftrightarrow$ A is row equivalent to the $n \times n$ identity matrix

Proof ($\Rightarrow$) Consider the system $AX = 0$. Since A is invertible, $X = 0$ is the unique solution of $AX = 0$. Thus A is row equivalent to I

($\Leftarrow$) $E_k E_{k-1} E_{k-2} \cdots E_2 E_1 A = I$

$$E_k^{-1} E_k E_{k-1} E_{k-2} \cdots E_2 E_1 A = E_k^{-1}$$

$$E_{k-1} E_{k-2} \cdots E_2 E_1 A = E_k^{-1}$$

$$E_{k-2} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1}$$

$A = (E_1)^{-1}(E_2)^{-1} \cdots (E_k)^{-1}$ invertible as product of invertible matrices

$$A^{-1} = \left[(E_1)^{-1}(E_2)^{-1} \cdots (E_k)^{-1}\right]^{-1} =$$
$$E_k E_{k-1} \cdots E_2 E_1$$

Example Find A^{-1} If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$\left[\begin{array}{ccc:ccc} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-1)R_2 + R_1}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{(-2)R_1 + R_2}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -3 & 4 & -2 & 3 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{SWAP(R_2, R_3)}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & -3 & 4 & -2 & 3 & 0 \end{array}\right] \xrightarrow{(-1)R_2}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & -3 & 4 & -2 & 3 & 0 \end{array}\right] \xrightarrow{(3)R_2 + R_3}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right] \xrightarrow{(1)R_3 + R_2}$$

$$\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Theorem The Following properties of an $n \times n$ matrix A are equivalent

(1) A is invertible

(2) A is row equivalent to the $n \times n$ identity matrix A

(3) $AX = 0$ has only the trivial solution

(4) For every n-vector b, the system $AX = b$ has a unique solution

(5) For every n-vector b, the system $AX = b$ is consistent

Proof Clearly $(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4)$

$(1) \Rightarrow (5)$ clear

$(5) \Rightarrow (1)$ we construct a matrix B such that $AB = I$. Then we prove that the system $BX = 0$ has only the trivial solution

Then B is invertible and

$$(AB)B^{-1} = B^{-1}, \text{ so } A = B^{-1}$$

is invertible

A matrix with one of the properties of the above theorem is called nonsingular

Powers of square matrices

A is a square matrix.

we define $A^0 = I$, $A^1 = A$, $A^2 = A \cdot A$,

$A^n = \underbrace{A \cdot A \cdots A}_{n\text{-times}}$ when n is a positive integer

If A is invertible, we define

$A^{-n} = (A^{-1})^n$ where n is a positive integer

Example $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix}$$