Example Find

$$\begin{vmatrix} 2 & -1 & 3 \\ -2 & 2 & 5 \\ 4 & -2 & 10 \end{vmatrix} \xrightarrow{(1)R_1+R_2}$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 1 & 8 \\ 4 & -2 & 10 \end{vmatrix} \xrightarrow{(-2)R_1+R_3}$$

$$\begin{vmatrix} 2 & -1 & 3 \\ 0 & 1 & 8 \\ 0 & 0 & 4 \end{vmatrix} = (2)(1)(4) = 8$$

Definition The transpose of the $m\times n$ matrix $A=[a_{ij}]$ is the $n\times m$ matrix A^T defined by $A^T=[a_{ji}]$

Example

$$\begin{bmatrix} 2 & 0 & 1 & 1 \\ 3 & 1 & -1 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & 3 \\ 0 & 1 \\ 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Suppose that A and B are matrices of appropriate size and c is a number

Then (a) $(A^T)^T = A$

(b) $(A+B)^T = A^T + B^T$

(c) $(cA)^T = cA^T$

(d) $(AB)^T = B^T A^T$

(6) If A is a square matrix, then $\det(A^T) = \det A$

<u>Example</u> $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$\det A = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 0$

$A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

$\det A^T = 1\cdot\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 7 \\ 6 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 7 \\ 5 & 8 \end{vmatrix} = 0$

<u>Theorem</u> If A, B are $n\times n$ matrices, then $\det(AB) = \det A \cdot \det B$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$

$AB = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

$\det(AB) = (ae+bg)(cf+dh) - (af+bh)(ce+dg)$

$= \cancel{aecf} + aedh + bgcf + \cancel{bgdh} - \cancel{afce} - afdg - bhce - \cancel{bhdg}$

$= ad(eh-fg) + bc(fg-eh) = (ad-bc)(eh-fg)$

$= \det A \cdot \det B$

Theorem If the $n \times n$ matrix A is invertible, then $\det A \neq 0$

Proof $A A^{-1} = I$

$\det(A A^{-1}) = \det I$

$\det A \cdot \det(A^{-1}) = 1$

If $\det A = 0$, then $0 = 1$ impossible.

So $\det A \neq 0$

Note that $\det(A^{-1}) = \dfrac{1}{\det A}$

Definition Let $A = [a_{ij}]$ be an $n \times n$ matrix. The $n \times n$ matrix $\operatorname{adj} A$, called the adjoint of A, is the matrix whose ij-th entry is the cofactor A_{ji} of a_{ji}. Thus

$$\operatorname{adj} A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

$A_{11} = -18$, $A_{12} = 17$, $A_{13} = -6$, $A_{21} = -6$
$A_{22} = -10$, $A_{23} = -2$, $A_{31} = -10$,
$A_{32} = -1$, $A_{33} = 28$

$$\operatorname{adj} A = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

Theorem If $A = [a_{ij}]$ is an $n \times n$ matrix, then $A(\text{adj} A) = (\text{adj} A)A = (\det A) I$

Sketch of the proof

$$A(\text{adj} A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} A_{11} & \cdots & A_{j1} & \cdots & A_{n1} \\ A_{12} & \cdots & A_{j2} & \cdots & A_{n2} \\ \vdots & & \vdots & & \vdots \\ A_{1n} & \cdots & A_{jn} & \cdots & A_{nn} \end{bmatrix}$$

Show that the ij-th element of $A(\text{adj} A)$ is

$$a_{i1}A_{j1} + a_{i2}A_{j2} + \cdots + a_{in}A_{jn} = \begin{cases} \det A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

so $A(\text{adj} A) = \begin{bmatrix} \det A & 0 & \cdots & 0 & 0 \\ 0 & \det A & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \\ 0 & 0 & \cdots & 0 & \det A \end{bmatrix} = (\det A) I$

Similarly $(\text{adj} A)A = (\det A) I$

Theorem The inverse of an invertible matrix A is $A^{-1} = \left(\frac{1}{\det A}\right) \text{adj} A$

Example

$$A = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}, \quad \det A = -94$$

$$A^{-1} = \frac{1}{\det A}(\operatorname{adj} A) = \begin{bmatrix} \frac{18}{94} & \frac{6}{94} & \frac{10}{94} \\ -\frac{17}{94} & \frac{10}{94} & \frac{1}{94} \\ \frac{6}{94} & \frac{2}{94} & -\frac{28}{94} \end{bmatrix}$$

Cramer's Rule

Theorem Consider the system $AX = b$ with n variables and n unknowns, where $A = [a_1\ a_2 \cdots a_n]$ is a matrix with columns $a_1, a_2, \ldots, a_n$ and $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$. If $\det(A) \neq 0$, then the i-th entry of the unique solution $x = (x_1, x_2, \ldots, x_n)$ is

$$x_i = \frac{\det([a_1 \cdots a_{i-1}\ b\ a_{i+1} \cdots a_n])}{\det(A)}$$

where the matrix in the last factor is obtained by replacing the i-th column of A by b

Proof $X = A^{-1}b$ is the unique solution of $AX = b$. Then x_i equals the i-th entry of $\frac{1}{\det(A)}((\operatorname{adj} A) b)$

But the i-th entry of $\frac{1}{\det(A)}((\operatorname{adj} A) b)$ is

$$\frac{1}{\det(A)} \sum_{l=1}^{n} A_{li} b_l = \frac{1}{\det(A)} \begin{vmatrix} a_{11} & \cdots & b_1 & \cdots & a_{1n} \\ a_{21} & \cdots & b_2 & \cdots & a_{2n} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & b_n & \cdots & a_{nn} \end{vmatrix}$$

Consider the system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

with $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$.

Suppose that $\det(A) \neq 0$. Then the unique solution (x_1, x_2, x_3) of the system is given by

$$x_1 = \frac{1}{\det(A)} \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix},$$

$$x_2 = \frac{1}{\det(A)} \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix},$$

$$x_3 = \frac{1}{\det(A)} \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

Example Solve the system

$$x_1 + 4x_2 + 5x_3 = 2$$
$$4x_1 + 2x_2 + 5x_3 = 3$$
$$-3x_1 + 3x_2 - x_3 = 1$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$\det(A) = 29$

$$\begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 5 \\ 1 & 3 & -1 \end{vmatrix} = 33, \quad \begin{vmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \\ -3 & 1 & -1 \end{vmatrix} = 35,$$

$$\begin{vmatrix} 1 & 4 & 2 \\ 4 & 2 & 3 \\ -3 & 3 & 1 \end{vmatrix} = -23$$

Then the unique solution is given by

$$x_1 = \frac{33}{29}, \quad x_2 = \frac{35}{29}, \quad x_3 = -\frac{23}{29}$$

Vector Spaces

Definition A vector space over $\mathbb{R}$ consists of a nonempty set V whose elements are called vectors and two operations addition $+$ and scalar multiplication $\cdot$ such that

(a) If $u \in V$, $v \in V$, then $u + v \in V$

(1) $u + v = v + u$, for any $u, v \in V$ (commutativity)

(2) $(u+v)+w = u+(v+w)$, for any $u, v, w \in V$ (associativity of addition)

(3) There is a vector $0 \in V$ satisfying $v + 0 = 0 + v = v$ for all $v \in V$ (zero element)

(4) For each $u \in V$, there is a vector $-u \in V$ such that $u + (-u) = 0$ (additive inverse)

(4) If $u \in V$ and $c \in \mathbb{R}$, then $c \cdot u \in V$

(5) $a \cdot (u + v) = a \cdot u + a \cdot v$, for any $a \in \mathbb{R}$ and $u, v \in V$ (distributivity over vector addition)

(6) $(a + b) \cdot u = a \cdot u + b \cdot u$, for any $a, b \in \mathbb{R}$ and $u \in V$ (distributivity over scalar addition)

(7) $a \cdot (b \cdot u) = (a \cdot b) \cdot u$ for any $a, b \in \mathbb{R}$ and $u \in V$ (associativity of scalar multiplication)

(8) $1 \cdot u = u$, for any $u \in V$ (unitarity)

Example The set of all continuous functions on an interval $[a,b]$ is a vector space over $\mathbb{R}$ with

$$(f+g)(x) = f(x) + g(x), \quad x \in [a,b]$$

$$(cf)(x) = cf(x), \quad x \in [a,b]$$

Clearly $f+g$ and cf are continuous on $[a,b]$

Suppose that f, g, h are continuous on $[a,b]$. Then

$(f+g)(x) = f(x)+g(x) = g(x)+f(x) = (g+f)(x)$, $x \in [a,b]$

$((f+g)+h)(x) = (f+g)(x) + h(x) = f(x) + g(x) + h(x) = f(x) + [g(x)+h(x)] = f(x) + (g+h)(x) = (f+(g+h))(x)$, $x \in [a,b]$

Let 0 be the function that maps $x \in [a,b]$ to 0. Then

$$(f+0)(x) = f(x) + 0(x) = f(x) + 0 = f(x),$$

$x \in [a,b]$

$$(f+(-f))(x) = f(x) + (-f)(x) = f(x) - f(x)$$
$$= 0 = 0(x), \quad x \in [a,b]$$

$$(c\cdot(f+g))(x) = c(f+g)(x) = c(f(x)+g(x))$$
$$= cf(x) + cg(x) = (cf+cg)(x), \quad x \in [a,b]$$

$$((c+d)f)(x) = (c+d)f(x) = cf(x) + df(x)$$
$$= (cf+df)(x), \quad x \in [a,b]$$

$$(c\cdot(d\cdot f))(x) = c(df)(x) = cdf(x)$$
$$= ((cd)f)(x), \quad x \in [a,b]$$

$$(1\cdot f)(x) = f(x), \quad x \in [a,b]$$