Math 132: Discrete Mathematics
Problem session 2

1. Let $f : X \to Y$ be a function. f is *injective* (or "into" or "one-to-one") if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X$. f is *surjective* (or "onto") if for all $y \in Y$, there is some $x \in X$ with $f(x) = y$. f is *bijective* if its is both injective and surjective.

   (a) Show that f is injective if and only if there is some function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$, i.e., $g(f(x)) = x$ for all $x \in X$.

   (b) (Subtle) Show that the previous is actually false in one special case, and correct the statement so that it becomes true.

   (c) Show that if there is a function $g : Y \to X$ such that $f \circ g = \mathrm{id}_Y$, then f is surjective.

   (d) (Subtle) Show that the converse of the previous statement is true. What operation are you assuming you can do?

   (e) Show that f is a bijection if and only if there is a function $g : Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$.

2. How many ordered quadruples (x_1, x_2, x_3, x_4) of integers such that $-5 \leq x_i \leq 5$ sum to 0?

3. How many ways can k indistinguishable balls be placed in $n > 1$ distinguishable buckets such that at least one bucket is empty?

4. Recall that the nth Catalan number b_n was defined as the number of paths from $(0, 0)$ to (n, n) in the plane that consist only of steps $(i, j) \mapsto (i + 1, j)$ or $(i, j) \mapsto (i, j + 1)$ and which never go above the line $y = x$.

   (a) Give a combinatorial argument that Catalan numbers satisfy the recurrence relation

   $$b_0 = 1, \qquad b_{n+1} = \sum_{i=0}^{n} b_i \cdot b_{n-i}.$$

   (b) Show that we can identify b_n as the number of ways to arrange n pairs of parentheses () coherently. In the case $n = 3$, the coherent arrangements are:

   $$((())),\ (()()),\ (())(),\ ()(()),\ ()()(),$$

   while an incoherent arrangement would be for example ())((), as the second close-parenthesis occurs before the second open-parenthesis.

   (c) Show that we can identify b_n with the number of ways of parenthesizing $n+1$ variables into pairs. The first few cases are:

   $n = 1$: $\{\ (x_1 x_2)\ \}$
   $n = 2$: $\{\ ((x_1 x_2)x_3),\ (x_1(x_2 x_3))\ \}$
   $n = 3$: $\{\ (((x_1 x_2)x_3)x_4),\ ((x_1(x_2 x_3))x_4),\ (x_1((x_2 x_3)x_4)),\ (x_1(x_2(x_3 x_4))), ((x_1 x_2)(x_3 x_4))\ \}$

   (d) Show that b_n can be identified with the number *triangulations* of a convex $(n + 2)$-gon. (Note that we take the convention that a 2-gon has a single triangulation, so that $b_0 = 1$.) In other words, in a shape with $n + 2$ sides, draw $n - 1$ line segments connecting the vertices of the shape in such a way that you end up with n triangles. For $n = 3$, the $(3 + 2)$-gon triangulations are: