Bilkent Calculus I Exams
1989 – 2016

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Fall 2016 Midterm I

1. Consider the functions

$$f(x) = \sqrt{x^2 + a} + bx \qquad \text{and} \qquad g(x) = x^3 + cx^2 + d$$

where a, b, c, d are nonzero real constants.

a. Give values to a, b, c, d by filling in the boxes below so that the resulting functions satisfy $f(1) = 0$ and $g(1) = 0$. No explanation is required in this part.

$a = \boxed{}$ $\qquad b = \boxed{}$ $\qquad c = \boxed{}$ $\qquad d = \boxed{}$

b. Compute the limit $\lim\limits_{x \to 1} \dfrac{f(x)}{g(x)}$ where a, b, c, d have the values given in **Part a**.

2. In each of the following, if the given statement is true for all functions f that are defined on the entire real line, then mark the ☐ to the left of TRUE with a ✗; otherwise, mark the ☐ to the left of FALSE with a ✗ and give a counterexample. No explanation is required.

a. If f is continuous on $(-\infty, \infty)$, then $f'(\pi)$ exists.

☐ TRUE

☐ FALSE, because it does not hold for $f(x) =$

b. If $f(-1) = -1$ and $f(1) = 1$, then $f(c) = 0$ for some c in $(-1, 1)$.

☐ TRUE

☐ FALSE, because it does not hold for $f(x) =$

c. If $f''(x) = -f(x)$ for all x, then $f(x) = \sin x$ or $f(x) = \cos x$.

☐ TRUE

☐ FALSE, because it does not hold for $f(x) =$

d. If f is differentiable on $(-\infty, \infty)$, then $f'(\pi)$ exists.

☐ TRUE

☐ FALSE, because it does not hold for $f(x) =$

e. If $\lim_{x\to 0} |f(x)| = 1$, then $\lim_{x\to 0} f(x) = 1$ or $\lim_{x\to 0} f(x) = -1$.

☐ TRUE

☐ FALSE, because it does not hold for $f(x) =$

3. Suppose that f is a twice-differentiable function satisfying the following conditions:

- $y = x/2 + 1$ is an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = 4$.
- $y = x/4 - 3$ is an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = 8$.
- $y = -x/3 + 2$ is an equation for the tangent line to the graph of $y = f(x)$ at the point with $x = 12$.

Consider the function $g(x) = (f(x^3))^2$.

a. Find an equation for the tangent line to the graph of $y = g(x)$ at the point with $x = 2$.

b. Suppose that $g''(2) = 0$. Find $f''(8)$.

4. Air is escaping from a balloon that is hanging at the end of a 75 cm long string attached to a nail on a wall. Assume that at all times the balloon has the shape of a sphere which is tangent to the wall, and the string extends along a line which passes through the center of the sphere and lies in a vertical plane perpendicular to the wall.

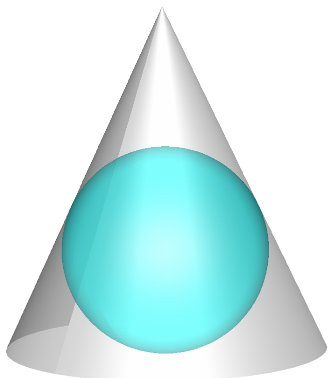
Suppose that at a certain moment the volume of the balloon is decreasing at a rate of 1800π cm^3/s and its surface area is decreasing at a rate of 240π cm^2/s. Determine how fast the angle between the string and the wall is changing at this moment. Express your answer in units of degrees per second.

Fall 2016 Midterm II

1a. Sketch the graph of $y = 1/(x^2 + 1)$ by computing y' and y'', and determining their signs; finding the critical points, the inflections points, the intercepts, and the asymptotes; and clearly labeling them in the picture.

1b. Show that $\left|\dfrac{1}{a^2+1} - \dfrac{1}{b^2+1}\right| \le \dfrac{3\sqrt{3}}{8}\,|a-b|$ for all real numbers a, b.

2a. Find the dimensions of the cone with the smallest possible lateral surface area that surrounds a sphere of radius 1.



2b. Having solved **Part 2a** correctly, after the exam you run into a friend who mistakenly minimized the total surface area $A = \pi r\sqrt{r^2 + h^2} + \pi r^2$ instead of the lateral surface area $A = \pi r\sqrt{r^2 + h^2}$.

Let h_y and r_y be respectively the height and the radius you found, and let h_f and r_f be the ones your friend did. Assuming that your friend also solved the corresponding problem correctly, which one of the following is true? No explanation is required.

☐ $h_y > h_f$ and $r_y > r_f$	☐ $h_y > h_f$ and $r_y = r_f$	☐ $h_y > h_f$ and $r_y < r_f$
☐ $h_y = h_f$ and $r_y > r_f$	☐ $h_y = h_f$ and $r_y = r_f$	☐ $h_y = h_f$ and $r_y < r_f$
☐ $h_y < h_f$ and $r_y > r_f$	☐ $h_y < h_f$ and $r_y = r_f$	☐ $h_y < h_f$ and $r_y < r_f$

3a. Determine $f(5)$ if f is a continuous function that satisfies $\displaystyle\int_0^{4x+\sin(\pi x)} f(t)\,dt = x^2$ for all x.

3b. Determine $f(5)$ if f is a function that satisfies $\displaystyle\int_0^{f(x)} t^2\,dt = 4x+\sin(\pi x)$ for all x.

4. Consider the region R between the graph of $y = x\sin(x^3)$ and the x-axis for $0 \leq x \leq \pi^{1/3}$.

a. Find the volume V of the solid generated by revolving R about the x-axis.

b. Find the volume W of the solid generated by revolving R about the y-axis.

Fall 2016 Final Exam

1. Find $\displaystyle\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(2,1)}$ if y is a differentiable function of x satisfying the equation $1+\ln(2x-3y) = x^2-3y^2$.

2. In each of the following, indicate whether the given statement is True or False by marking the corresponding ☐ with a ✗, and then **explain** why it is true or false.

a. $\displaystyle\int \sin^3 x\,dx = \frac{\sin^4 x}{4} + C$ ☐ True ☐ False

b. $\displaystyle\int \sin^3 x\,dx = \frac{\sin^4 x}{4\cos x} + C$ ☐ True ☐ False

c. $\displaystyle\int \sin^3 x\,dx = \frac{1}{6}\cos x\cos 2x - \frac{5}{6}\cos x + C$ ☐ True ☐ False

d. $\displaystyle\int \frac{dx}{x^2+1} = \frac{\ln(x^2+1)}{2x} + C$ ☐ True ☐ False

e. $\displaystyle\int \frac{dx}{x^2+1} = \arcsin\left(\frac{x}{\sqrt{x^2+1}}\right) + C$ ☐ True ☐ False

3. Some students believe that Bilkent Math 101 exams get more difficult as time passes. This is in fact true. The difficulty level $H(t)$ of these exams satisfies the differential equation

$$\frac{dH}{dt} = H^a$$

with the initial condition $H(0) = 1$, where t is time measured from Fall 1986 in academic years and a is a constant whose value is a secret.

There is a quatrain in Nostradamus's *Les Propheties* which can be interpreted to be about Bilkent Math 101 exams.

a. According to one interpretation, the exams will become infinitely difficult in Fall 2021. Accepting this interpretation, find the difficulty level of the exams in Fall 2016.

b. According to another interpretation, the exams will be twice as difficult in Fall 2021 as they were in Fall 1986. Show that a must satisfy $-8 < a < -7$ if this is the case.

4. Evaluate the limit $\displaystyle\lim_{x\to 0} \frac{e^{x-\frac{1}{2}x^2} - \arctan x - 1}{x^4}$.

5. Evaluate the following integrals.

a. $\displaystyle\int_0^1 x^5 \sqrt[4]{1-x^3}\, dx$

b. $\displaystyle\int \sin 2x \, \tan^2 x \, dx$

Fall 2015 Midterm I

1. Suppose you are told that a function f satisfies the condition:

$$x^3 - 3x - 3 \le f(x) \le 9x^2 - 27x + 17 \qquad \text{for} \quad |x-2| < 1$$

a. Is it possible to determine the value of $\lim_{x\to 2} f(x)$ using only this information?

b. At this point you peek at your neighbor's paper, and see that YES is marked in **Part 1c** and then the solution starts with the sentence:

Taking derivatives of all sides, we obtain

$$(x^3 - 3x - 3)' \le f'(x) \le (9x^2 - 27x + 17)'$$

for $|x-2| < 1$.

You immediately realize that this cannot be true, because

$$\frac{d}{dx}(x^3 - 3x - 3) = \boxed{} \qquad \text{and} \qquad \frac{d}{dx}(9x^2 - 27x + 17) = \boxed{},$$

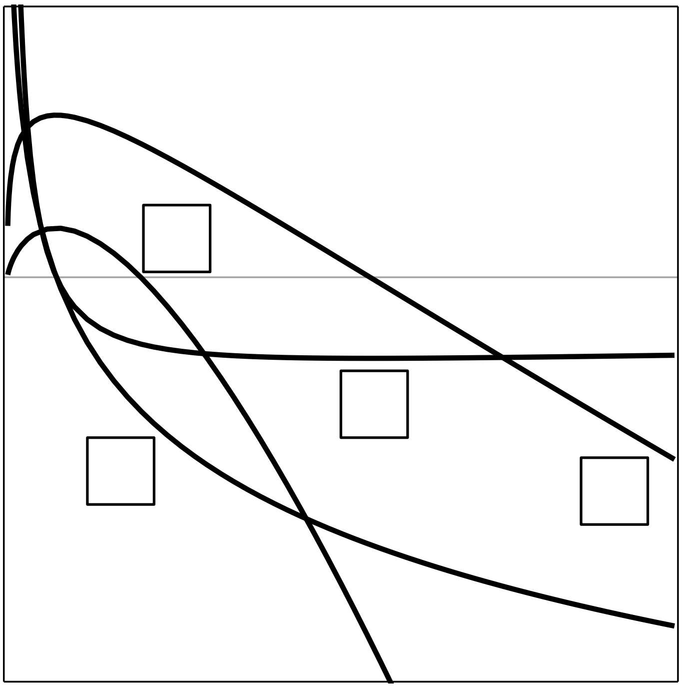
and if we take $a = \boxed{}$, then $|a-2| < 1$ but

$$\frac{d}{dx}(x^3 - 3x - 3)\bigg|_{x=a} = \boxed{} > \boxed{} = \frac{d}{dx}(9x^2 - 27x + 17)\bigg|_{x=a}.$$

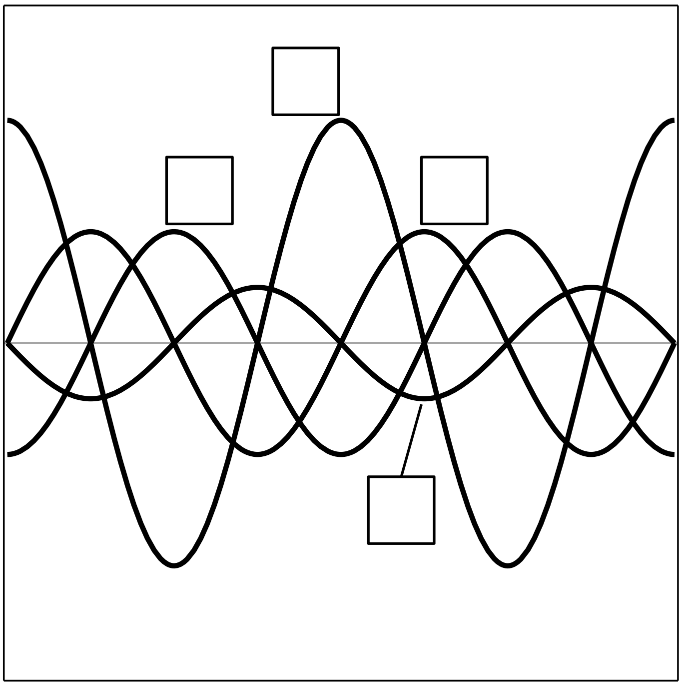
c. Is it possible to determine the value of $f'(2)$ using only this information?

2 . In each of the following figures, the graphs of two functions f and g together with their derivatives f' and g' are shown. Identify each by filling in the boxes with f, g, f' and g'.

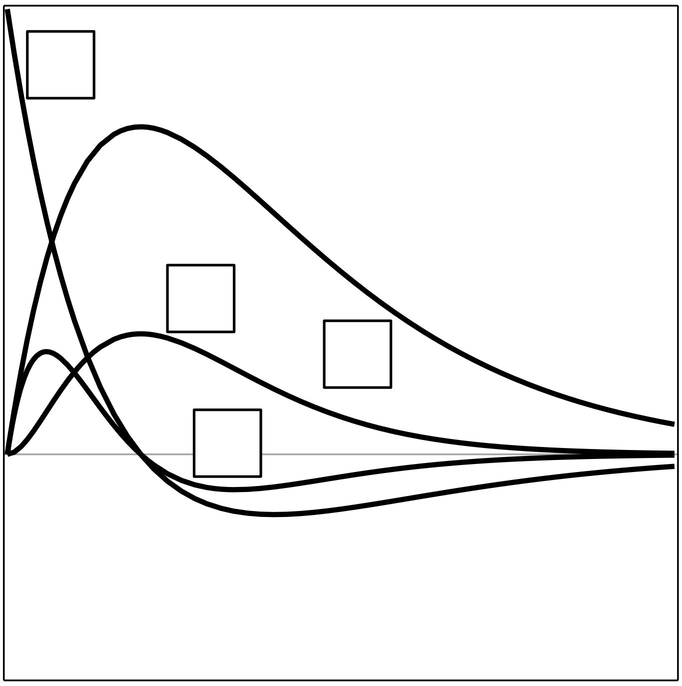
a.



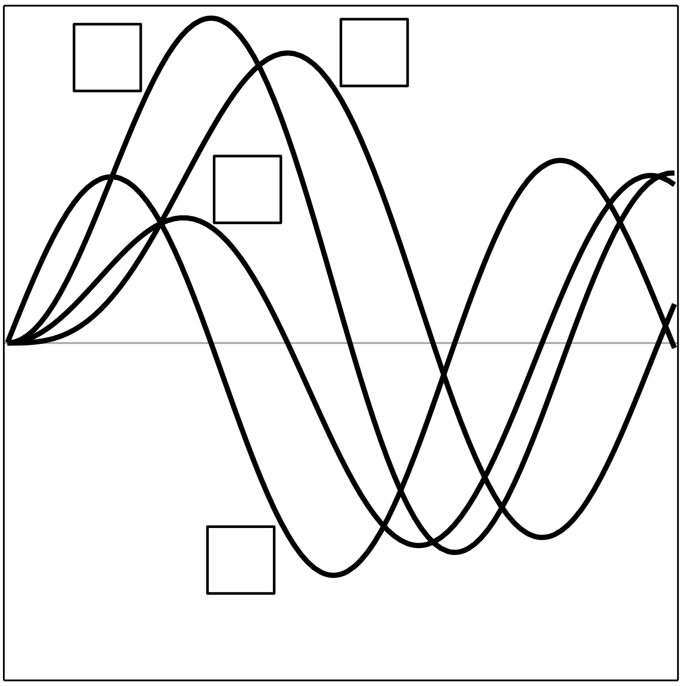
b.



c.



d.



3a. Compute $\dfrac{d}{dx}\sin(x+\sin(x+\sin x))$.

3b. Find an equation for the tangent line to the graph of $y=\tan\left(\dfrac{\pi}{\sqrt{25-x^2}}\right)$ at the point with $x=3$.

4. At a certain moment the dimensions of a cylinder are changing as follows:

- The radius is decreasing at a rate of 3 cm/s.
- The height is increasing at a rate of 15 cm/s.
- The volume is increasing at a rate of 2π cm^3/s.
- The surface area is increasing at a rate of 7π cm^2/s.

Find the radius at this moment.

Fall 2015 Midterm II

1. A function f that is continuous on $(-\infty,\infty)$ and twice-differentiable for $x\neq A$ satisfies the following conditions:

① $f(-\sqrt{5}-1)=B$, $f(0)=-2$, $f(\sqrt{5}-1)=C$, $f(4)=D$, $f(A)=0$ where $B<D$

② The line $y=x-2$ is a slant asymptote of the graph of $y=f(x)$ both as $x\to-\infty$ and as $x\to\infty$

③ $\lim\limits_{x\to A} f'(x)=\infty$

④ $f'(x)>0$ for $x<0$, and for $x>4$ and $x\neq A$; $f'(x)<0$ for $0<x<4$

⑤ $f''(x)>0$ for $x<-\sqrt{5}-1$ and for $\sqrt{5}-1<x<A$; $f''(x)<0$ for $-\sqrt{5}-1<x<\sqrt{5}-1$ and for $x>A$

a. Sketch the graph of $y=f(x)$. Make sure to clearly show all important features.

b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x)=\sqrt[3]{ax^3+bx^2+cx+d}$ satisfies the conditions ①-⑤ for suitable real numbers

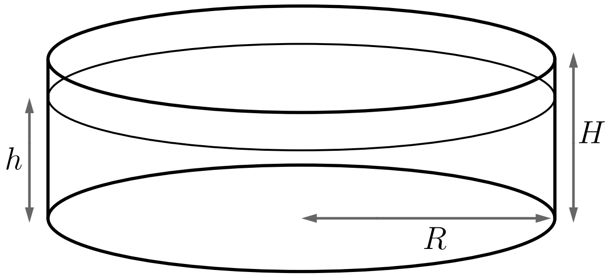
A,B,C,D if the constants a, b, c and d are chosen as

$a=\boxed{}$, $b=\boxed{}$, $c=\boxed{}$ and $d=\boxed{}$.

2. A pool, like the one in front of the Faculty of Science Building A, loses water from its sides and its bottom due to seepage, and from its top due to evaporation. For a pool with radius R and depth H in meters, the rate of this loss in m^3/hour is given by an expression of the form

$$aR^2 + bR^2h + cRh^2$$

where h is the depth of the water in meters, and a, b, c are constants independent of R, H and h. Due to this loss, water must be pumped into the pool to keep it at the same level even when the drains are closed.



Suppose that $a = 1/300$ m/hour and $b = c = 1/150$ 1/hour. Find the dimensions of the pool with a volume of 45π m^3 which will require the water to be pumped at the slowest rate to keep it completely full.

3a. Find the average value of $|x|$ on the interval $[-1, 2]$.

3b. Suppose that a continuous function f satisfies the equation

$$f(x) = x^2 - x + (1 - x)\int_0^x t^2 f(t)\,dt + x\int_x^1 (t - t^2)f(t)\,dt$$

for all x. Express $f''(1/2)$ in terms of $A = f(1/2)$.

4. An island has the shape of a 10 hm×10 hm square and its landscape consists of a mountain whose height h at a horizontal distance x from the shore is given by $h = x^2$ where both h and x are measured in hectometers (=hm). Let V be the volume of the mountain.

a. Express V as an integral with respect to h by considering cutting the mountain into slices as shown in Figure a.

b. Express V as an integral with respect to x by considering cutting the mountain into shells as shown in Figure b.

c. Compute V.

[The figures are on the next page.]

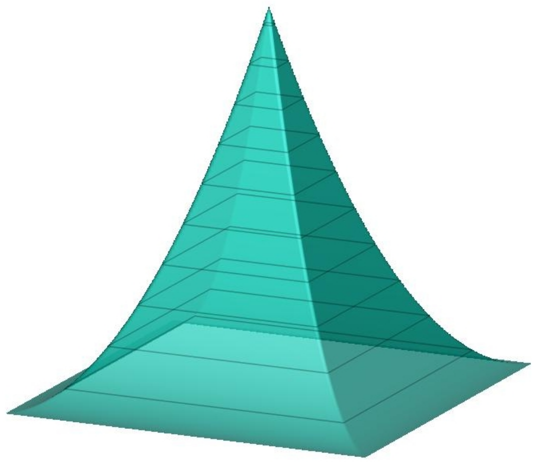


Figure a:

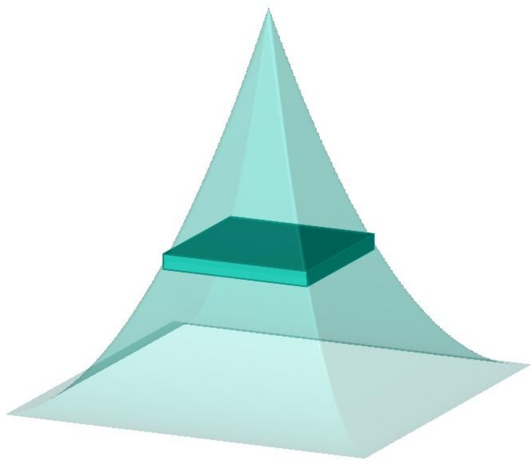
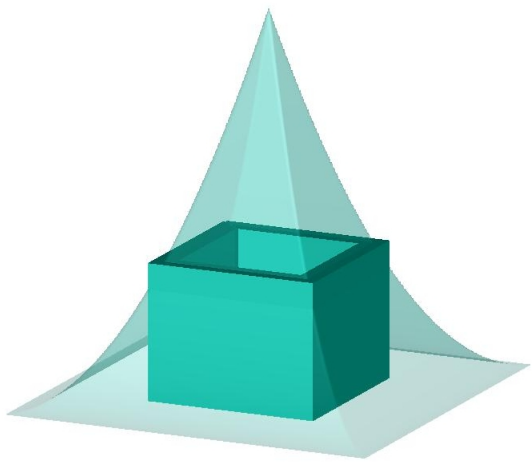


Figure b:



Fall 2015 Final

1. Let $f(x) = x^5 - 3x^3 - x + 4$.

a. Compute $f'(x)$.

b. Find the x-coordinates of all points on the graph of $y = f(x)$ where the tangent line is horizontal.

c. Show that the equation $f(x) = 0$ has at most three real solutions.

d. Show that the equation $f(x) = 0$ has at least three real solutions.

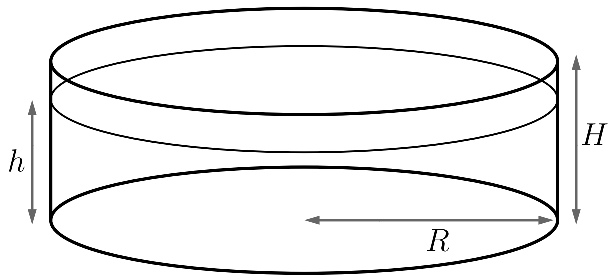
2. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(\sqrt{e}\,,\,1/e)}$ if y is a differentiable function of x satisfying the equation:

$$\ln(xy) = (\ln x)(\ln y)$$

3. A pool, like the one in front of the Faculty of Science Building A, loses water from its sides and its bottom due to seepage. For a pool with radius R and depth H in meters, the rate of this loss in m^3/hour is given by

$$\frac{dV}{dt} = -aR^2h - bRh^2$$

where V is the volume of the water in cubic meters, t is the time in hours, h is the depth of the water in meters, and a and b are constants independent of R, H and h.



Consider a pool with $H = 1$ m, $R = 6$ m and $a = b = \pi/500$ 1/hour.

a. Find the depth of the water h as a function of time t if $h = 1$ m when $t = 0$ hour.

b. The term aR^2h on the right side of the equation represents the rate of loss due to seepage through the bottom of the pool. Find the total volume of the water that seeps through the bottom while the initially full pool completely empties.

4. Evaluate the limit $\displaystyle\lim_{x\to\infty}\left(x\left(x\left(x\left(e^{1/x}-1\right)-1\right)-\frac{1}{2}\right)\right)$.

5. Evaluate the following integrals.

a. $\displaystyle\int \frac{\tan^2 x}{1+\sin x}\,dx$

b. $\displaystyle\int_1^8 \sqrt{4 - \frac{3}{x^2} + \frac{1}{x^3}}\,dx$

Fall 2014 Midterm I

1. In each of the following, if the given statement is true for all functions $f(x)$ that are defined for all $x \neq 0$, then mark the ☐ to the left of True with a ✗; otherwise, mark the ☐ to the left of False with a ✗ and give a counterexample.

a. $\lim_{x\to 0} f(1/x)$ does not exist.

☐ True

☐ False, because it does not hold for $f(x) =$ $\boxed{}$

b. $\lim_{x\to 0} xf(1/x)$ exists.

☐ True

☐ False, because it does not hold for $f(x) =$ $\boxed{}$

c. If $-x^2 < f(x) < x^2$ for all $x \neq 0$, then $\lim_{x\to 0} f(x) = 0$.

☐ True

☐ False, because it does not hold for $f(x) =$ $\boxed{}$

d. If $-1 \leq f(x) \leq 1$ for all $x \neq 0$, then $-1 \leq \lim_{x\to 0} f(x) \leq 1$.

☐ True

☐ False, because it does not hold for $f(x) =$ $\boxed{}$

e. If $\lim_{x\to 0} \dfrac{f(x)}{x}$ does not exist, then $\lim_{x\to 0} f(x) \neq 0$.

☐ True

☐ False , because it does not hold for $f(x) =$

2 . Consider the function $f(x) = \sin\left(\dfrac{\pi}{\sin(\pi/x)}\right)$.

a . Find an equation of the tangent line to the graph of $y = f(x)$ at the point with $x = 6$.

b . Choose one of the options by putting a ✓ in the box to the left of it, and then explain your reasoning in one sentence.

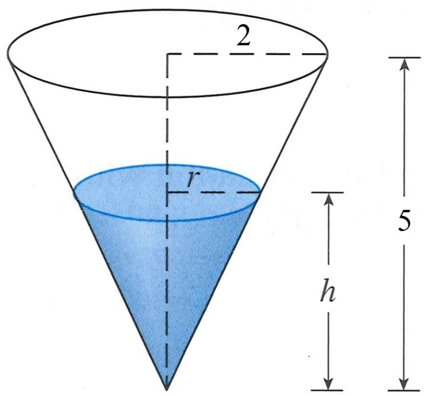
I expect $f(101\pi)$ to be ☐ positive ☐ zero ☐ negative , because:

3 . Consider the curve defined by the equation $x^4 + y^4 = 4xy + 52$.

a . Find all points on the curve at which the tangent line is horizontal.

b . Find $\dfrac{d^2y}{dx^2}$ at all points where the tangent line is horizontal.

4 . A water tank has the shape of an upside-down cone with radius 2 m and height 5 m. The water is running out of the tank through a small hole at the bottom. Assume that the speed of the water flowing through the hole is proportional to the square root of the depth of the water in the tank.



a . In this part, suppose that the water is running out at a rate of 3 m^3/min when the depth of the water in the tank is 4 m. Find the rate at which the water level is changing at this moment.

b . In this part, suppose that the water level is falling at a rate of 1/3 m/min when the tank is full. Find the rate at which the water level is changing when the depth of the water in the tank is 4 m.

Fall 2014 Midterm II

1 . A twice-differentiable function f on $(-\infty, \infty)$ satisfies the following conditions:

① $f(-2) = A$, $f(-1) = B$, $f(0) = C$, $f(1/2) = -1$, $f(D) = 0$ where $A < C$

② $\lim\limits_{x\to-\infty} f(x) = -1$, $\lim\limits_{x\to\infty} f(x) = 1$

③ $f'(x) < 0$ for $x < -1$, $f'(x) > 0$ for $x > -1$, ④ $f''(x) < 0$ for $x < -2$ and for $x > 1/2$, $f''(x) > 0$ for $-2 < x < 1/2$

a. Sketch the graph of $y = f(x)$. Make sure to clearly show all important features.

b. Fill in the boxes to make the following a true statement.

The function $f(x) = \dfrac{ax+b}{\sqrt{x^2+c}}$ satisfies the conditions ①-④ for suitable real numbers A, B, C, D if

the constants a, b and c are chosen as

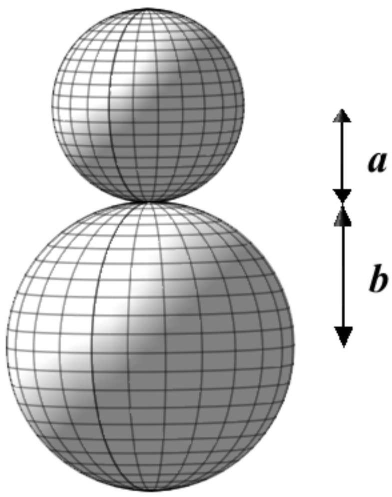
$$a = \boxed{}\,, \quad b = \boxed{} \quad \text{and} \quad c = \boxed{}\,.$$

2. A snowman is an anthropomorphic sculpture made from snow as well as some pieces of coal, a carrot, a hat and a scarf. For the purposes of this question, we consider a snowman to consist of a spherical head of radius a and a spherical body of radius b, and we also assume that the snow does not melt and its density does not change while it is being sculpted.

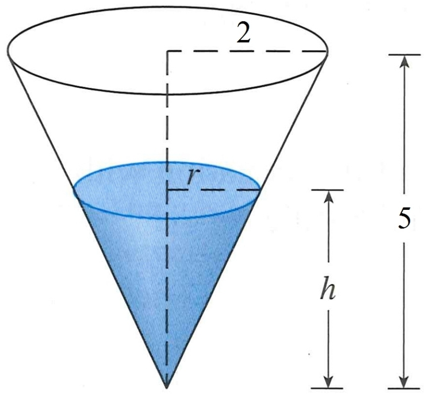
The research shows that the cuteness $\mathcal{K}$ of a snowman is given by

$$\mathcal{K} = \begin{cases} \left(\dfrac{a}{b}\right)^2 \left(1 - \dfrac{a}{b}\right)(a^2+ab+b^2) & \text{if } 0 \le a < b, \\ 0 & \text{if } 0 < b \le a. \end{cases}$$

Find the dimensions of the cutest snowman that can be built with $4\pi/3$ m^3 of snow.



3. A water tank has the shape of an upside-down cone with radius 2 m and height 5 m. The water is running out of the tank through a small hole at the bottom. Assume that the speed of the water flowing through the hole is proportional to the square root of the depth of the water in the tank.



If it takes 3 minutes for the depth of the water to decrease from 5 m to 4 m, find how long it takes for the full tank to completely drain.

4. Evaluate the following integrals.

a. $\displaystyle\int_0^1 \frac{dx}{(1+\sqrt{x})^4}$

b. $\displaystyle\int x^3\sqrt{x^2+1}\,dx$

Fall 2014 Final

1. Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0}\frac{2\sin x - \sin 2x}{x^3}$

b. $\displaystyle\lim_{x\to 0}\frac{x - \tan x}{e^x - e^{-x} - 2x}$

2. In each of the following, if the given statement is true for all functions f that are continuous on the entire real line, then mark the ☐ to the left of True with a ✗; otherwise, mark the ☐ to the left of False with a ✗ and give a counterexample.

a. $\displaystyle\int_0^1 |f(x)|\,dx = \left|\int_0^1 f(x)\,dx\right|$

☐ True

☐ False, because it does not hold for $f(x) =$

b. $\displaystyle\frac{d}{dx}\int_0^x f(t)\,dt = f(x) - f(0)$ for all x

☐ True

☐ False, because it does not hold for $f(x) =$

c. $\dfrac{d}{dx}\displaystyle\int_0^1 f(t)\,dt = f(1) - f(0)$

☐ True

☐ False, because it does not hold for $f(x) =$

d. $\dfrac{d}{dt}\displaystyle\int_0^x f(t)\,dt = f(x) - f(0)$ for all x

☐ True

☐ False, because it does not hold for $f(x) =$

e. $\displaystyle\int_0^1 f(x)^2\,dx = \frac{f(1)^3 - f(0)^3}{3}$

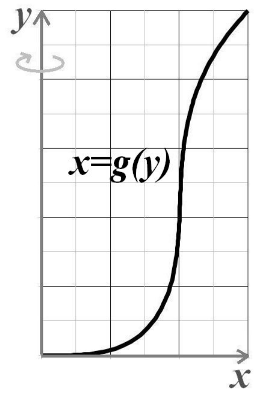
☐ True

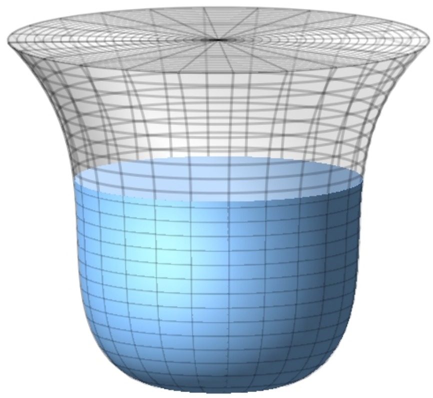
☐ False, because it does not hold for $f(x) =$

f. $\displaystyle\int f(x)^2\,dx = \frac{f(x)^3}{3f'(x)} + C$ for all x for which $f'(x) \neq 0$

☐ True

☐ False, because it does not hold for $f(x) =$



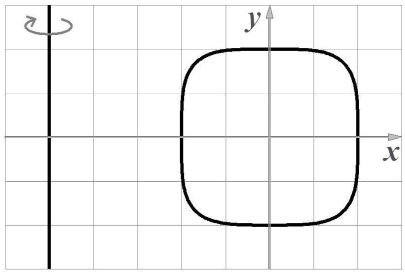


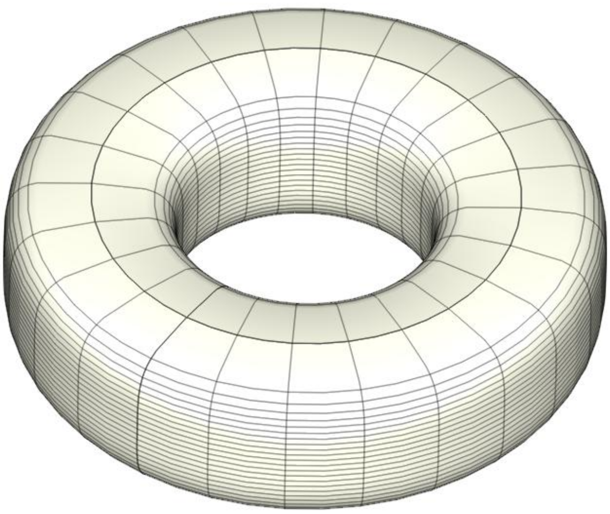
3. The side surface of a water tank has the shape generated by revolving the graph of a continuous nonnegative function $x = g(y)$ for $0 \le y \le 5$ with $g(0) = 0$ and $g(5) = 3$ about the y-axis where all units are in meters. Assume that:

① As water runs out of a small hole at the bottom of tank, the speed of the water flowing through the hole at any moment is proportional to the square root of the depth of the water in the tank at that moment.

② The function g is chosen in such a way that the depth of the water changes at a constant rate at all times.

Find the volume of the tank.





[These figures belong to the question on the next page.]

4. Let V be the volume of the water-dropper shown in the figure which has the shape obtained by revolving the curve $x^4 + y^4 = 1$ about the line $x = -5/2$ where all units are in centimeters.

a. Express V as an integral using the cylindrical shells method.

$$V = 2\pi \int_{\square}^{\square} \boxed{}\; dx$$

b. Express V as an integral using the washer method.

$$V = \pi \int_{\square}^{\square} \boxed{}\; dy$$

c. Show that the improper integral $\displaystyle\int_0^1 u^{-3/4}(1-u)^{1/4}\,du$ converges.

d. Express V in terms of $A = \displaystyle\int_0^1 u^{-3/4}(1-u)^{1/4}\,du$.

5. Evaluate the following integrals.

a. $\displaystyle\int \cos 2x \, \cos^3 x \, dx$

b. $\displaystyle\int_0^{\pi/4} \frac{x \sin x}{\cos^3 x}\, dx$

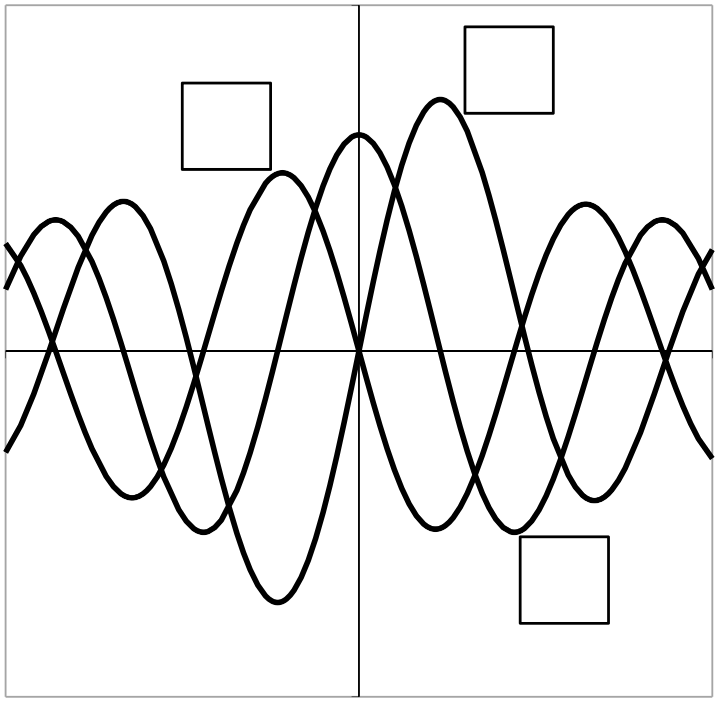
Fall 2013 Midterm I

1a. Determine the constants a, b, c if $\displaystyle\lim_{x\to 1} \frac{\sqrt{ax^2 + bx + c} - 3}{(x-1)^2} = 5$.

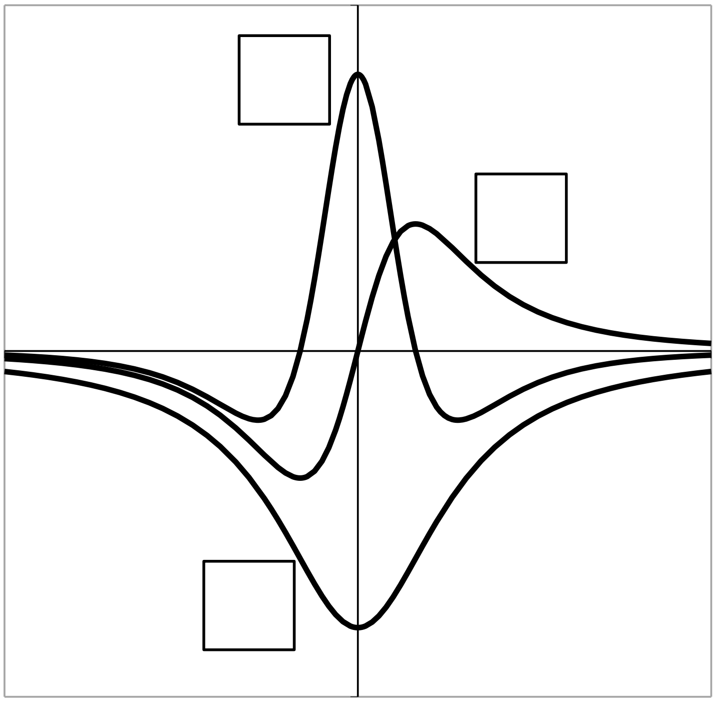
1b. Evaluate $\displaystyle\left.\frac{dy}{dx}\right|_{x=1/6}$ if $y = \sin(3\pi \sin(2\pi \sin(\pi x)))$.

2. In each of the following figures, the graphs of a function f together with its first and second derivatives f' and f'' are shown. Identify each by filling in the boxes with f, f' and f''.

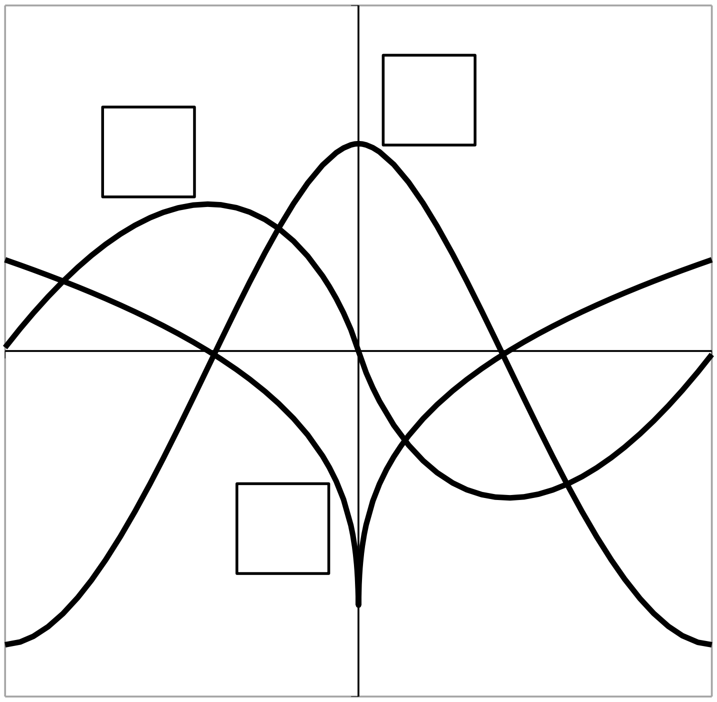
a.



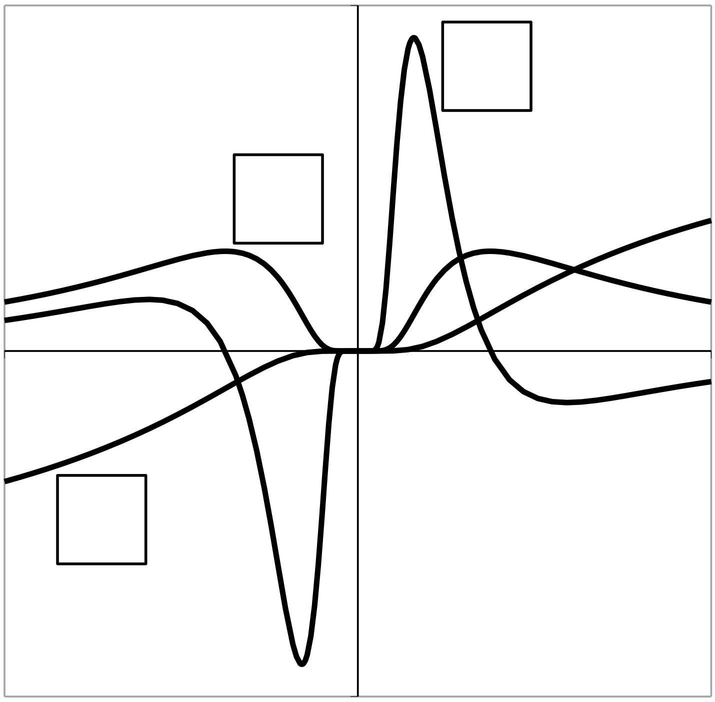
b.



c.



d.



3. A television camera is positioned 900 m from the base of a rocket launching pad. The angle of elevation of the camera has to change at the correct rate in order to keep the rocket in sight. Also, the mechanism for focusing the camera has to take into account the increasing distance from the camera to the rising rocket. Let's assume that the rocket rises vertically and its speed is 250 m/s when it has risen 1200 m.

a. How fast is the distance from the television camera to the rocket changing at that moment?

b. If the television camera is always kept aimed at the rocket, how fast is the camera's angle of elevation changing at that same moment? Express your answer in degrees per second.

4. Consider the following conditions:

① f is differentiable on the entire real line.

② $y = 3x + 7$ is an equation of the tangent line to the graph of $y = f(x)$ at the point with $x = -1$.

③ $y = 5x - 9$ is an equation of the tangent line to the graph of $y = f(x)$ at the point with $x = 2$.

a. Determine the following values for a function f satisfying ①–③.

$f(-1) = \boxed{}$ $f(2) = \boxed{}$ $f'(-1) = \boxed{}$ $f'(2) = \boxed{}$

`In parts (b) and (c), show all your work and explain your reasoning fully and in detail.`

b. Estimate the value of of $f(15/8)$.

c. Indicate whether each of the following statements is True or False by putting a ✗ in the corresponding box. Then if True, prove it; and if False , give a counterexample.

❶ Every function f satisfying ①–③ takes on the value π at some point.

True ☐ False ☐

❷ Derivative f' of every function f satisfying ①–③ takes on the value -1 at some point.

True ☐ False ☐

Fall 2013 Midterm II

1. A continuous function f that is twice-differentiable for $x \neq 0, 3$ satisfies the following conditions:

① $f(0) = 0$, $f(1) = \sqrt[3]{4}$, $f(3) = 0$

② $f'(1) = 0$, $\lim\limits_{x\to 0} f'(x) = \infty$, $\lim\limits_{x\to 3^-} f'(x) = -\infty$, $\lim\limits_{x\to 3^+} f'(x) = \infty$

③ $f'(x) > 0$ for $x < 1$ and $x \neq 0$; $f'(x) < 0$ for $1 < x < 3$; $f'(x) > 0$ for $3 < x$

④ $f''(x) > 0$ for $x < 0$; $f''(x) < 0$ for $0 < x$ and $x \neq 3$

⑤ The line $y = x - 2$ is a slant asymptote of the graph of $y = f(x)$ both as $x \to \infty$ and as $x \to -\infty$

a. Sketch the graph of $y = f(x)$.

b. Fill in the boxes to make the following a true statement.

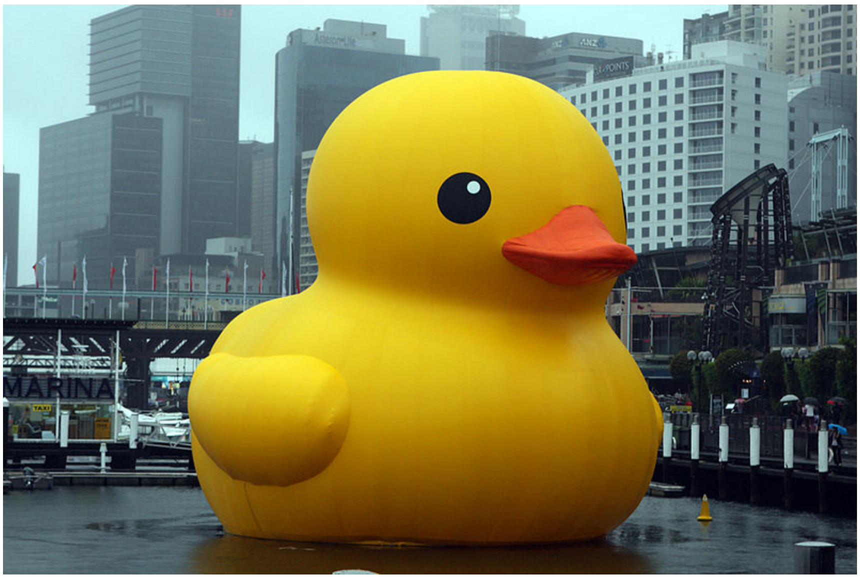
The function $f(x) = x^a(b-x)^c$ satisfies the conditions ①-⑤ if the constants a, b and c are chosen as

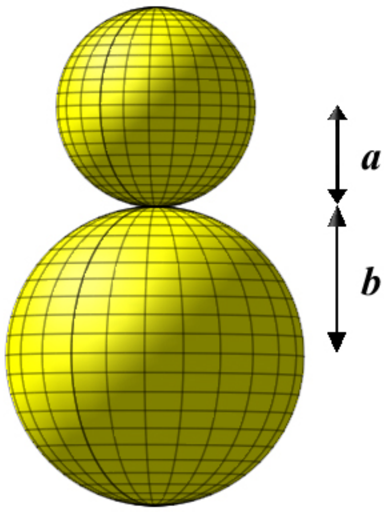
$$a = \boxed{}\,, \quad b = \boxed{} \quad \text{and} \quad c = \boxed{}\,.$$

2. The Rubber Duck is a sculpture designed by Florentijn Hofman and constructed from PVC. For the purposes of this question, we consider the Rubber Duck to consist of a spherical head of radius a and a spherical body of radius b. The research shows that the cuteness $\mathcal{K}$ of the Rubber Duck is given by

$$\mathcal{K} = \begin{cases} \dfrac{a}{b}\left(1 - \dfrac{a}{b}\right)(a+b) & \text{if } 0 \le a < b, \\ 0 & \text{if } 0 < b \le a. \end{cases}$$

Find the dimensions of the cutest Rubber Duck with a total surface area of $400\pi\,\mathrm{m}^2$.





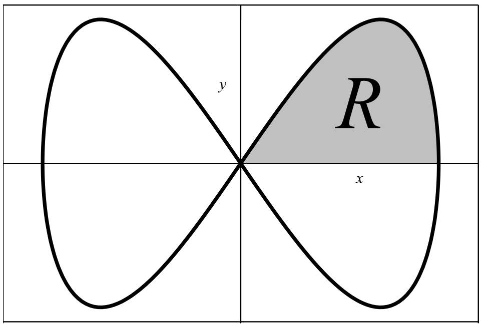
3a. Suppose that f is a continuous function satisfying

$$f(x) = x\int_0^x f(t)\,dt + x^3$$

for all x, and c is a real number such that $f(c) = 1$. Express $f'(c)$ in terms of c only.

3b. Evaluate the integral $\displaystyle\int_1^9 (1+\sqrt{x})\sqrt{1+\frac{2}{\sqrt{x}}}\,dx.$

4. Let R be the region bounded by the curve $y^2 = x^2 - x^4$ and the x-axis in the first quadrant. Let V be the volume obtained by rotating R about the x-axis. Let W be the volume obtained by rotating R about the y-axis.



a. Only three of ❶-❹ will be graded. Indicate the ones you want to be graded by putting a ✓ in the ☐ to the right of it.

❶ ☐ Express V as an integral using the disk method.

$$V = \pi \int_{\square}^{\square} \boxed{} \, dx$$

❷ ☐ Express V as an integral using the cylindrical shells method.

$$V = 2\pi \int_{\square}^{\square} \boxed{} \, dy$$

❸ ☐ Express W as an integral using the washer method.

$$W = \pi \int_{\square}^{\square} \boxed{} \, dy$$

❹ ☐ Express W as an integral using the cylindrical shells method.

$$W = 2\pi \int_{\square}^{\square} \boxed{} \, dx$$

b. Compute V.

c. Compute W.

Fall 2013 Final

1. Consider the curve $y = x + e^{xy}$.

a. Find an equation of the tangent line to the curve at the point $(x, y) = (-1, 0)$.

b. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(-1,0)}$.

2. Suppose that f is a continuous function that is twice-differentiable for $x \neq 0$ and $x \neq 6$ and that satisfies the following conditions:

① $f(0) = 0,\ f(4) = 2\sqrt[3]{4},\ f(6) = 0$

② $f'(4) = 0\,,\ \lim\limits_{x\to 0^-} f'(x) = -\infty\,,\ \lim\limits_{x\to 0^+} f'(x) = \infty\,,\ \lim\limits_{x\to 6} f'(x) = -\infty$

③ $f'(x) < 0$ for $x < 0$; $f'(x) > 0$ for $0 < x < 4$; $f'(x) < 0$ for $4 < x$ and $x \neq 6$

④ $f''(x) < 0$ for $x < 6$ and $x \neq 0$; $f''(x) > 0$ for $6 < x$

⑤ The line $y = -x + 2$ is a slant asymptote of the graph of $y = f(x)$ both as $x \to \infty$ and as $x \to -\infty$

a. Sketch the graph of $y = f(x)$.

b. Fill in the boxes to make the following a true statement.

The function $f(x) = x^a(b-x)^c$ satisfies the conditions ①-⑤ if the constants a, b and c are chosen as

$a = \boxed{}$, $b = \boxed{}$ and $c = \boxed{}$.

3. Consider the limit $\lim\limits_{x\to 0} \dfrac{\int_0^x \sin(xt^3)\,dt}{x^5}$.

a. Fill in the boxes to make the following a true statement.

Both $\boxed{}$ Rule and the $\boxed{}$ Theorem can be used to evaluate the limit $\lim\limits_{x\to 0} \dfrac{\int_0^x \sin(xt^3)\,dt}{x^5}$.

b. Fill in the box to make the following a true statement.

The Fundamental Theorem of Calculus, Part 1 cannot be used directly to compute the derivative $\dfrac{d}{dx}\left(\displaystyle\int_0^x \sin(xt^3)\,dt\right)$, because $\boxed{}$.

c. Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\int_0^x \sin(xt^3)\,dt}{x^5}$.

4a. Evaluate the integral $\displaystyle\int \sin x \,\cos^2 x\, \tan^3 x\, dx$.

4b. Suppose that f is a function that has a continuous second derivative and that satisfies $f(0) = 4$, $f(1) = 3$, $f'(0) = 5$, $f'(1) = 7$, $f''(0) = 8$ and $f''(1) = 11$. Show that:

$$\int_0^1 f(x)f''(x)\,dx \le 1$$

5. In each of the following, if there exists a function f that satisfies the given condition, give an example of such a function; otherwise, just write Does Not Exist inside the box.

a. f is continuous at 0 and f is not differentiable at 0.

$f(x) = \boxed{}$

b. f is continuous on $(-\infty,\infty)$, the absolute maximum value of f is 1, $\lim\limits_{x\to-\infty} f(x) = 0$ and $\lim\limits_{x\to\infty} f(x) = 0$.

$f(x) = \boxed{}$

c. f is continuous on $[0,1]$ and $\displaystyle\left(\int_0^1 f(x)\,dx\right)^2 < \int_0^1 \big(f(x)\big)^2\,dx$.

$f(x) = \boxed{}$

d. f is continuous on $(0,1]$, $\displaystyle\int_0^1 f(x)\,dx = \infty$ and $\displaystyle\int_0^1 \big(f(x)\big)^2\,dx = 1$.

$f(x) = \boxed{}$

e. f is continuous on $(0,1]$, $\displaystyle\int_0^1 f(x)\,dx = 1$ and $\displaystyle\int_0^1 \big(f(x)\big)^2\,dx = \infty$.

$f(x) = \boxed{}$

Fall 2012 Midterm I

1. Consider the unit circle $C_1 : x^2 + y^2 = 1$ and the circle C_2 with radius a and center $(1,0)$ where $0 < a < 2$. (In parts (a), (c), (d), (e), just fill in the answer, no explanation is required; in parts (b) and (f), show all your work!)

a. An equation for the circle C_2 is $\boxed{}$.

b. Find the point of intersection P of the circles C_1 and C_2 that lies above the x-axis.

c. The slope of the line ℓ that passes through the points P and $Q(1,a)$ is

$m = \boxed{}$.

d. An equation of the line ℓ is $\boxed{}$.

e. The x-intercept of the line ℓ is $b = \boxed{}$.

f. Find the limit $\displaystyle\lim_{a\to 0^+} b$.

2. Consider the equation $1 - \dfrac{x^2}{4} = \cos x$.

a. Show that this equation has at least one real solution.

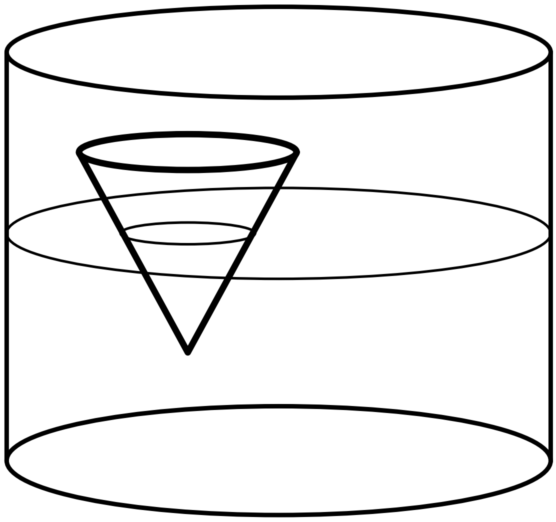
b. Show that this equation has at least two real solutions.

c. Show that this equation has at least three real solutions.

3a. Compute $\dfrac{d}{dx}\cos\sqrt{\sin(\tan(9x))}$. (Do not simplify!)

3b. Find $J''(0)$ if $xJ''(x) + J'(x) + xJ(x) = 0$ for all x, $J(0) = 1$ and $J'''(0)$ exists.

4. A cone of radius 2 cm and height 5 cm is lowered point first into a tall cylinder of radius 7 cm that is partially filled with water. Determine how fast the depth of the water is changing at the moment when the cone is completely submerged if the cone is moving with a speed of 3 cm/s at that moment.



5. In each of the following, if there exists a function f whose domain is the entire real line and that satisfies the given condition, give an example of such a function; otherwise just write Does Not Exist inside the box. No further explanation is required. No partial points will be given.

a. f is continuous at 0 and f is not differentiable at 0.

$f(x) =$

b. $f(x) < 1$ for all x and f has no absolute maximum value.

$f(x) =$

c. $f'(0) = 0$ and f does not have a local extreme value at 0.

$f(x) =$

d. $f(0) = 1$, $f'(0) = -1$, $f(1) = 2$, $f'(1) = 1/2$, and $f''(x) \geq 0$ for all $0 \leq x \leq 1$.

$f(x) =$

e. $f(0) = 1$, $f'(0) = -1$, $f(1) = 2$, $f'(1) = 5$, and $f''(x) \geq 0$ for all $0 \leq x \leq 1$.

$f(x) =$

Fall 2012 Midterm II

1. A twice-differentiable function f satisfies the following conditions:

① $f(0) = 0$; $f(2) = A$, $f(4) = B$, $f(6) = C$, where $A < C$

② $f'(0) = 0$, $f'(4) = 0$; $f''(0) = 0$, $f''(2) = 0$, $f''(6) = 0$

③ $f'(x) < 0$ for $x < 0$; $f'(x) > 0$ for $0 < x < 4$; $f'(x) < 0$ for $x > 4$

④ $f''(x) > 0$ for $x < 2$ and $x \neq 0$; $f''(x) < 0$ for $2 < x < 6$; $f''(x) > 0$ for $x > 6$

⑤ $\lim_{x \to -\infty} f(x) = \infty$, $\lim_{x \to \infty} f(x) = 0$

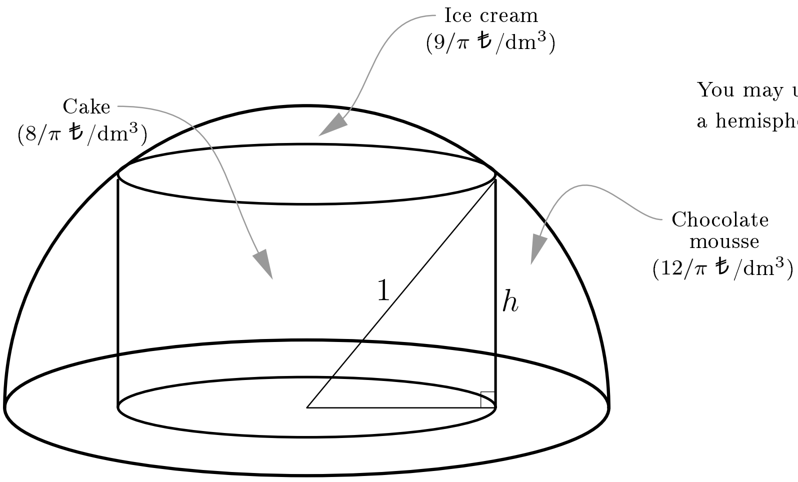
a. Sketch the graph of $y = f(x)$. Make sure to clearly show all important features.

b. Fill in the boxes to make the following a true statement. No explanation is required.

The function $f(x) = x^a b^x$ satisfies the conditions ①-⑤ if the constants a and b are chosen as

$a =$ and $b =$.

2. A dessert in the shape of a hemisphere with radius 1 dm is made by baking a cylindrical cake of height h, and topping it with a spherical cap of ice cream and surrounding it with a hemispherical ring of chocolate mousse as shown in the figure. If the cake costs $8/\pi$ ₺/dm^3, the ice cream costs $9/\pi$ ₺/dm^3 and the chocolate mousse costs $12/\pi$ ₺/dm^3, determine the value of h for (a) the least expensive and (b) the most expensive dessert that can be made.



You may use the fact that the volume of a hemispherical ring of height h is $2\pi h^3/3$.

3. Suppose that f is a twice-differentiable function satisfying $f(0) = -2,\ f'(0) = 11,\ f''(0) = -8,\ f(2) = 5,\ f'(2) = -3,\ f''(2) = 7$; and also suppose that the function

$$g(x) = \frac{1}{x}\int_0^x f(t)\,dt$$

has a critical point at $x = 2$.

a. Find $\lim\limits_{x\to 0} g(x)$.

b. Find $\displaystyle\int_0^2 f(x)\,dx$.

c. Determine whether the critical point of g at $x = 2$ is a local minimum, a local maximum or neither.

4. Let R be the region between the curve $y = \sqrt{8x^2 - x^5}$ and the x-axis in the first quadrant.

a. Find the volume V generated by revolving R about the x-axis.

b. Find the volume W generated by revolving R about the y-axis.

5. In each of the following, if the given statement is true for all functions f that are differentiable on the entire real line, then mark the ☐ to the left of True with a ✗; otherwise, mark the ☐ to the left of False with a ✗ and give a counterexample.

a. If $f(x) \ge x^2$ for all x, then $f'(x) \ge 2x$ for all x.

☐ True

☐ False, because it does not hold for $f(x) =$ $\boxed{}$

b. $\displaystyle\int_0^1 f(x)f'(x)\,dx \le f(1)^2$

☐ True

☐ False, because it does not hold for $f(x) =$

c. $\displaystyle\int_0^1 f(x)^2\,dx = \left(\int_0^1 f(x)\,dx\right)^2$

☐ True

☐ False, because it does not hold for $f(x) =$

d. $\displaystyle\frac{d}{dx}\int_0^x f(t)\,dt = f(x) - f(0)$

☐ True

☐ False, because it does not hold for $f(x) =$

e. $\displaystyle\frac{d}{dx}\int_0^1 f(x)\,dx = f(1) - f(0)$

☐ True

☐ False, because it does not hold for $f(x) =$

Fall 2012 Final

1. Suppose that y is a differentiable function of x satisfying the equation

$$\int_0^{x+y} e^{-t^2}\,dt = xy\,.$$

a. Find $\displaystyle\left.\frac{dy}{dx}\right|_{(x,y)=(0,0)}$.

b. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(0,0)}$.

2. Evaluate the limit $\displaystyle\lim_{x\to 0}\frac{\cos x - e^{-x^2/2}}{\sin(x^4)}$.

3. A vat with 2000 L of beer contains 4% alcohol by volume. Beer with 6% alcohol is pumped into the vat at a rate of 20 L/min and the mixture is pumped out at the same rate. How long does it take for the alcohol concentration to rise to 5%?

4. Evaluate the following integrals:

a. $\displaystyle\int_{\pi/4}^{\pi/3}\frac{\sqrt{\tan x}}{\sin 2x}\,dx$

b. $\displaystyle\int\frac{x-1}{x^2}\,e^x\,dx$

5. Suppose that f is a differentiable function on $(0,\infty)$ satisfying the following conditions:

① $f(1/x) = x^2 f(x)$ for all $x>0$

② $\displaystyle\int_1^\infty f(x)\ln x\,dx$ is convergent

a. Express $f'(1)$ in terms of $A = f(1)$.

b. Express $\displaystyle\int_0^1 f(x)\ln x\,dx$ in terms of $B = \displaystyle\int_1^\infty f(x)\ln x\,dx$.

Fall 2011 Midterm I

1a. Evaluate the limit $\displaystyle\lim_{x\to 0}\frac{1-\cos(x^2)}{x^2\sin^2 x}$. (Do not use L'Hôpital's Rule!)

1b. Compute $\dfrac{d}{dx}\tan^4(5\cos^2 7x)$. (Do not simplify!)

2a. Use the precise definition of the limit to show that $\displaystyle\lim_{x\to 2}\frac{x}{x-1} = 2$.

2b. We claim that the equation $\dfrac{\tan x}{x} = \dfrac{3}{2}$ has at least one positive root.

i. Consider the following 'proof' of this claim:

> Let $f(x) = 2\tan x - 3x$. Then $f(\pi/3) = 2\sqrt{3} - \pi > 0$ and $f(\pi) = -3\pi < 0$. Therefore there is c in $[\pi/3, \pi]$ such that $f(c) = 0$ by the Intermediate Value Theorem. It follows that c is a positive root of the given equation.

Explain in one sentence what is wrong with this proof.

ii. Give a correct proof of this claim.

3a. A light shines from the top of a pole 5 m high. A ball is dropped from a point 4 m away from the pole. Find how fast the shadow of the ball is moving when the ball is 2 m above the ground and moving at a speed of 9 m/sec.

3b. Compute $\sqrt[3]{63.7}$ approximately using a suitable linearization of the function $f(x) = \sqrt[3]{x}$.

4. Find the absolute maximum and minimum values of $f(x) = x^4 - 10x^2 + 8x$ on the interval $[-1, 3]$.

5. In each of the following if there exists a function f whose domain is the entire real line and that satisfies the given condition, give an example of such a function; otherwise just write Does Not Exist inside the box. No further explanation is required.

a. $f(x)$ is not differentiable at $x = 0$ and $(f(x))^2$ is differentiable at $x = 0$.

$f(x) = \boxed{}$

b. $f'(3) = 5$ and $\lim_{x\to 3} f(x) = \infty$.

$f(x) = \boxed{}$

c. $f(3) = 5$ and $\lim_{x\to 3} f'(x) = \infty$.

$f(x) = \boxed{}$

d. f is not constant and $(f'(x))^2 - f(x)$ is constant.

$f(x) = \boxed{}$

e. $y = 2x + 5$ is the equation of the tangent line to the graph of $y = f(x)$ at the point with $x = 3$ and $f'(3) = 11$.

$f(x) = \boxed{}$

Fall 2011 Midterm II

1. Evaluate the following integrals.

a. $\displaystyle\int \frac{\sin x - \cos x}{\sqrt[3]{\sin x + \cos x}}\, dx$

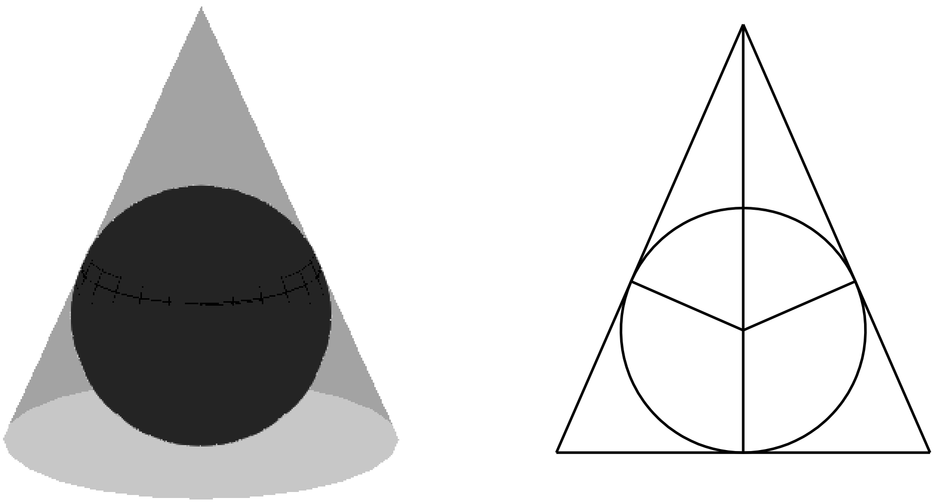
b. $\displaystyle\int_2^5 \frac{x}{\sqrt{x-1}}\, dx$

2a. Suppose that $\dfrac{dy}{dx} = \dfrac{x}{y^2+1}$ for all x and $y(0) = -1$. Find all x for which $y(x) = 0$.

2b. Find the area of the region that lies below the line $y = 2x$ and between the parabolas $y = x^2$ and $y = x^2/2$.

3. Suppose that f is a continuous and positive function on $[0, 5]$, and the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le 5$ is 8. Let $A(c)$ denote the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le c$, and let $B(c)$ denote the area between the graph of $y = f(x)$ and the x-axis for $c \le x \le 5$. Let $R(c) = A(c)/B(c)$. If $R(3) = 1$ and $\left.\dfrac{dR}{dc}\right|_{c=3} = 7$, find $f(3)$.

4. Find the smallest possible volume of a cone that contains a sphere of radius 1.



5. A twice-differentiable function f satisfies the following.

- $f'(x) > 0$ for $x < 0$ and $f'(x) < 0$ for $x > 0$

- $f''(x) > 0$ for $x < -2$ and $x > 1$, and $f''(x) < 0$ for $-2 < x < 1$
- $f(-2) = 2,\ f(0) = 5,\ f(1) = 4$
- $\lim\limits_{x\to-\infty} f(x) = 0$ and $\lim\limits_{x\to\infty} f(x) = 3$

a. Sketch the graph of $y = f(x)$. Make sure to clearly show all important features.

b. Explain in one sentence why f cannot be a polynomial.

c. Explain in one sentence why f cannot be a rational function.

Fall 2011 Final

1. Evaluate the following integrals.

a. $\displaystyle\int \frac{\sin x + \cos x}{\tan x + \cot x}\,dx$

b. $\displaystyle\int_0^{\pi/2} \frac{\sin 2x}{1 + \cos^4 x}\,dx$

2a. Find a formula expressing $\displaystyle\int \frac{dx}{(x^2+1)^{n+1}}$ in terms of $\displaystyle\int \frac{dx}{(x^2+1)^{n}}$ for $n \geq 1$.

2b. Find y'' at $(x, y) = (\pi/6, \pi/3)$ if $y = 2x\sin(x + y)$.

3. Evaluate the limit $\displaystyle\lim_{x\to 0}\left(\frac{\ln(1+x)}{x}\right)^{1/x}$.

4. a. Show that $\displaystyle\int_0^{\infty} t^n e^{-t}\,dt = n\int_0^{\infty} t^{n-1}e^{-t}\,dt$ for all $n \geq 1$.

b. Let R be the region between the graph of $y = \sqrt{x}\,e^{-ax}$ and the x-axis where a is a positive constant. Let V be the volume of the solid generated by revolving R about the x-axis, and let W be the volume of the solid generated by revolving R about the y-axis. Find all values of a for which $V = W$.

$\left[\text{In this question you may use the fact that } \displaystyle\int_0^{\infty} \sqrt{x}\,e^{-x}\,dx = \frac{\sqrt{\pi}}{2}\,.\right]$

5. In each of the following if there exists a function f that satisfies the given condition, give an example of such a function; otherwise just write Does Not Exist inside the box. No explanation is required. No partial credit will be given.

a. f is differentiable on $(-\infty, \infty)$, f has at least two distinct zeros and f' has no zeros.

$f(x) = \boxed{}$

b. f is differentiable on $(-\infty, \infty)$, f has no zeros and f' has at least two distinct zeros.

$f(x) = \boxed{}$

c. f is continuous on $[0, \infty)$, $\displaystyle\int_0^\infty f(x)\,dx = \infty$ and $\displaystyle\int_0^\infty f(x)^2\,dx = 1$.

$f(x) = \boxed{}$

d. f is differentiable on $(-\infty, \infty)$, $f'(x) > 0$ for all x and $\displaystyle\lim_{x\to\infty} f(x) \neq \infty$.

$f(x) = \boxed{}$

e. f is differentiable on $(0, \infty)$, $\displaystyle\lim_{x\to\infty} f'(x) = 0$ and $\displaystyle\lim_{x\to\infty} f(x) = \infty$.

$f(x) = \boxed{}$

Fall 2010 Midterm I

1. Evaluate the following limits. (Do not use L'Hôpital's Rule!)

a. $\displaystyle\lim_{x\to 3} \frac{\cos(\pi x/2)}{x-3}$

b. $\displaystyle\lim_{x\to 1} \frac{x^3 + x - 2}{\sqrt{2x^2 - 1} - \sqrt{x^2 - x + 1}}$

2. Find the x-coordinates of all points P on the curve $y = x^3 - x$ such that the tangent line at P passes through the point $(2, 2)$.

3. A point P is moving in the xy-plane. When P is at $(4, 3)$, its distance to the origin is increasing at a rate of $\sqrt{2}$ cm/sec and its distance to the point $(7, 0)$ is decreasing at a rate of 3 cm/sec. Determine the rate of change of the x-coordinate of P at that moment.

4a. Assume that y is a differentiable function of x satisfying $y = x\,g(x^2 + y^2) - 13$, where g is a function such that $g(5) = 7$, $g'(5) = -2$, and $g''(5) = 12$. Find $\left.\dfrac{dy}{dx}\right|_{(x,y)=(2,1)}$.

4b. Does the equation $x^2 = f(x)$ have a solution for every function f that is continuous on $[-1, 2]$, and has values $f(-1) = 4$ and $f(2) = 1$? If Yes, prove that it does; if No, give an example of a function for which it does not.

5. In each of the following if there exists a function f whose domain is the entire real line and which satisfies the given condition, give an example of such a function; otherwise just write Does Not Exist inside the box. No further explanation is required. No partial credit will be given.

a. f is differentiable, f is not constant and $(f(x))^2 + (f'(x))^2$ is constant.

b. $|f(x)| \le |x|$ for all $x \neq 0$ and $\lim\limits_{x\to 0} f(x)$ does not exist.

c. f is continuous at 0 and $f'(0)$ does not exist.

d. f is differentiable and $f(x) > f'(x) > 0$ for all x.

e. $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ exist and are different.

Fall 2010 Midterm II

1. Evaluate the following integrals.

a. $\displaystyle\int \frac{x^2}{x^3 - 1}\,dx$

b. $\displaystyle\int_0^{2\pi} \sqrt{1 - \cos 2x}\,dx$

2a. Let f be a differentiable function satisfying

$$\int_2^x f(t)\,dt = \sin(f(x)) - \frac{1}{2}$$

for all x in an open interval containing 2. Find $f'(a)$ if $f(a) = \dfrac{\pi}{3}$.

2b. Evaluate the limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{n^2 + k^2}$.

3. Let R be the region lying between the curve $y = \dfrac{e^x}{x}$ and the line $y = x$ for $1 \le x \le 2$.

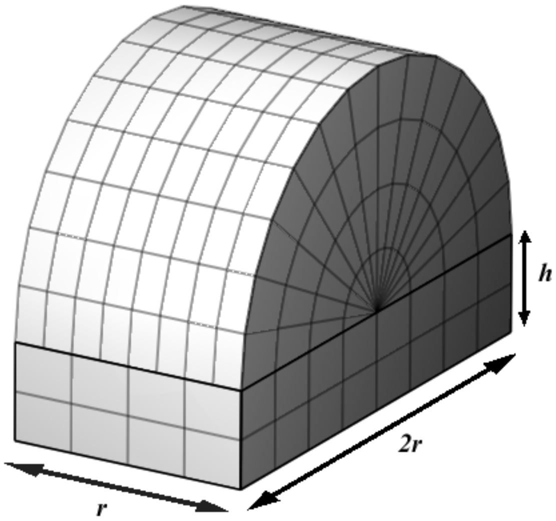
a. Show that $\dfrac{e^x}{x} \ge x$ for $1 \le x \le 2$.

b. Express the volume V of the solid generated by revolving the region R about the x-axis. (Do not compute!)

c. Express the volume W of the solid generated by revolving the region R about the y-axis. (Do not compute!)

d. Compute either □ V or □ W. (Indicate the one you choose by putting a ✗ in the square in front of it.)

4. We want to build a greenhouse that has a half cylinder roof of radius r and height r mounted horizontally on top of four rectangular walls of height h as shown in the figure. We have 200π m^2 of plastic sheet to be used in the construction of this structure. Find the value of r for the greenhouse with the largest possible volume we can build.



5. In each of the following if there exists a function f that is defined and differentiable on the entire real line and that satisfies the given condition, give an example of such a function; otherwise just write Does Not Exist inside the box. No further explanation is required. No partial credit will be given.

a. f has exactly one local maximum, exactly two zeros, and no local minimums.

$f(x) =$

b. f has exactly one local maximum, no zeros, and no local minimums.

$f(x) =$

c. f has no local maximums, no zeros, and no local minimums.

$f(x) = \boxed{}$

d. f has exactly one local maximum, exactly three zeros, and exactly one local minimum.

$f(x) = \boxed{}$

Fall 2010 Final

1a. Find the equation of the tangent line to the curve $y = x^2 2^x$ at $x = 1$.

1b. Evaluate the limit $\displaystyle\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$.

2. Evaluate the following integrals.

a. $\displaystyle\int \frac{dx}{x - \sqrt[3]{x}}$

b. $\displaystyle\int_0^1 \frac{x^2}{\sqrt{1 - x^2}}\, dx$

3a. Let $A = \displaystyle\int_0^\infty e^{-x^2}\, dx$. Express the value of the improper integral $\displaystyle\int_0^\infty x^2 e^{-x^2}\, dx$ in terms of A.

3b. Find all real numbers x that satisfy the equality $x + x^3 + x^5 + \cdots = -1$.

4. Consider the sequence satisfying the conditions $a_1 = 2$ and $a_n = a_{n-1} - \dfrac{1}{a_{n-1}}$ for $n \geq 2$.

a. Compute a_2, a_3, a_4.

b. Exactly one of the following statements is true. Choose the true statement and mark the box in front of it with a ✓.

☐ The sequence $\{a_n\}$ converges.

☐ The sequence $\{a_n\}$ diverges.

c. Prove the statement you chose in part (b).

5. In each of the following indicate whether the given series converges or diverges, and also indicate the best way of determining this by marking the corresponding boxes with a ✗ and filling in the corresponding blank.

a. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\sqrt{n^5-1}}$ ☐ converges ☐ diverges

☐ p-series with $p =$ _____ ☐ Direct Comparison Test with $\sum$ _______

☐ geometric series with $r =$ _____ ☐ Limit Comparison Test with $\sum$ _______

☐ nth Term Test ☐ Ratio Test

☐ Integral Test ☐ Root Test

b. $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)\sin^2 n$ ☐ converges ☐ diverges

☐ p-series with $p =$ _____ ☐ Direct Comparison Test with $\sum$ _______

☐ geometric series with $r =$ _____ ☐ Limit Comparison Test with $\sum$ _______

☐ nth Term Test ☐ Ratio Test

☐ Integral Test ☐ Root Test

c. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$ ☐ converges ☐ diverges

☐ p-series with $p =$ _____ ☐ Direct Comparison Test with $\sum$ _______

☐ geometric series with $r =$ _____ ☐ Limit Comparison Test with $\sum$ _______

☐ nth Term Test ☐ Ratio Test

☐ Integral Test ☐ Root Test

d. $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{\left(2-\dfrac{\sin n}{n}\right)^n}$ ☐ converges ☐ diverges

☐ p-series with $p =$ _____ ☐ Direct Comparison Test with $\sum$ _______

☐ geometric series with $r =$ _____ ☐ Limit Comparison Test with $\sum$ _______

☐ nth Term Test ☐ Ratio Test

☐ Integral Test ☐ Root Test

Fall 2009 Midterm I

1. Consider the function

$$f(x) = \begin{cases} \dfrac{\sqrt{1+x}-1}{x} & \text{if } x \neq 0 \\ k & \text{if } x = 0 \end{cases}$$

where k is a constant.

a. Determine the value of the constant k for which f is continuous at 0.

b. Write the limit defining $f'(0)$. (Do not compute.)

2a. Evaluate the limit $\displaystyle\lim_{x\to 0} \frac{5 - 6\cos x + \cos^2 x}{x \sin x}$. (Do not use L'Hôpital's Rule.)

2b. Evaluate the limit $\displaystyle\lim_{x\to 1} \frac{x^5 + x^3 - 2}{x^3 + x^2 - 2}$. (Do not use L'Hôpital's Rule.)

3. Consider a point P moving along the x-axis, and let Q be a point moving on the curve $y = x^2$ such that the line PQ is normal to the curve $y = x^2$ at the point Q at all times. Determine the rate of change of the x-coordinate of the point Q at the moment when P is at $(3, 0)$ and the x-coordinate of P is increasing at a rate of 1 cm/sec.

4a. Let f be a differentiable function and $g(x) = \big(f(f(x^3) + x)\big)^2$. If $f(1) = 1$, $f(2) = 5$, $f'(1) = -3$, and $g'(1) = 2$, find $f'(2)$.

4b. If $y = \dfrac{\sin(x^2)}{x^3}$, find $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{\pi/2}}$.

5a. Is there a continuous function f on $(0, \infty)$ such that $f(2x)f(x) < 0$ for all $x > 0$? If Yes, give an example; if No, prove that no such function exists.

5b. Is there a continuous function f on $(0, \infty)$ such that $\displaystyle\lim_{x\to\infty} f(x)$ does not exist, but $\displaystyle\lim_{x\to\infty} \big(f(x+1) - f(x)\big)$ exists? If Yes, give an example; if No, prove that no such function exists.

Fall 2009 Midterm II

1. A function f that is continuous on the entire real line and differentiable for $x \neq 0$ satisfies the following conditions: i. $f(0) = 0$; ii. $f(1) = 1$; iii. $f'(x) < 0$ for $x < 0$, $f'(x) > 0$ for $0 < x < 1$, and $f'(x) < 0$ for $x > 1$; iv. $f''(x) < 0$ for $x \neq 0$.

a. Sketch the graph of $y = f(x)$.

b. Give an example of a function in the form $f(x) = ax^b + cx^d$, where a, b, c, d are constants, that satisfies the conditions (i-iv). (You do not need to explain how you found the example, but you have to verify that your example satisfies the conditions.)

2. Find the maximum possible total surface area of a cylinder inscribed in a hemisphere of radius 1.

3. Evaluate the limit $\displaystyle\lim_{x\to 0^+} \frac{\sin x + 2\cos\sqrt{x} - 2}{x^2}$.

4a. If $f(x) = x^2 \displaystyle\int_0^x \sin\left(\frac{\pi}{t^2+1}\right) dt$, express $f'(1)$ in terms of $A = f(1)$.

4b. Find $y(1)$ if y is a differentiable function of x such that $\dfrac{dy}{dx} = xy^2$ and $y(0) = 1$.

5. Evaluate the following integrals.

a. $\displaystyle\int_1^2 \sqrt[3]{\frac{2-x}{x^7}}\, dx$

b. $\displaystyle\int \frac{\sin^2\!\sqrt{x}\ \cos\!\sqrt{x}}{\sqrt{x}}\, dx$

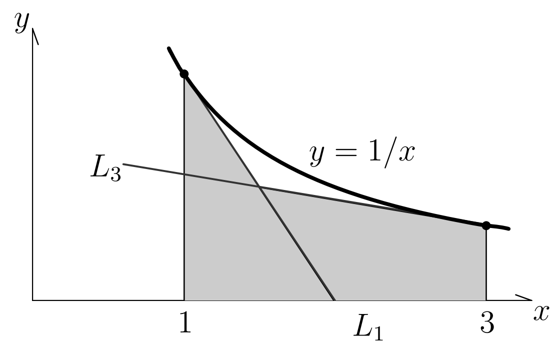
Fall 2009 Final

1. Consider the region bounded by the curve $y = \sin(kx)$ and the x-axis for $0 \le x \le \pi/k$, where k is a positive constant. Let V be the volume of the solid generated by revolving this region about the x-axis, and W be the volume of the solid generated by revolving this region about the y-axis. Find all values of k for which $V = W$.

2a. Let $a > 0$ be a constant. Show that $\dfrac{1}{x} \ge -\dfrac{1}{a^2}(x-a) + \dfrac{1}{a}$ for all $x > 0$.

2b. Express in one sentence what the inequality in part (a) means in geometric terms.

2c. In the figure, the line L_1 and the line L_3 are tangent to the graph of $y = 1/x$ at the points $(1, 1)$ and $(3, 1/3)$, respectively. Compute the area of the shaded region.



2d. Explain in one sentence why part (c) implies that $3 > e$.

3. Evaluate the following limits:

a. $\lim\limits_{x \to 0^+} (e^x - 1)^{1/\ln x}$

b. $\lim\limits_{n \to \infty} \sum\limits_{k=1}^{n} \dfrac{1}{(en+k)\,(\ln(en+k) - \ln n)}$

4a. Express the value of the integral $\displaystyle\int_0^1 \frac{\ln x}{x^2+1}\,dx$ in terms of $A = \displaystyle\int_1^\infty \frac{\ln x}{x^2+1}\,dx$.

4b. Express the value of the integral $\displaystyle\int_0^1 \frac{\ln x}{x^2+1}\,dx$ in terms of $B = \displaystyle\int_0^1 \frac{\arctan x}{x}\,dx$.

5. In each of the following if there exists a function f that satisfies the given conditions, give an example of such a function; otherwise just write Does Not Exist. No further explanation is required. No partial credit will be given.

a. f is differentiable on $(-\infty, \infty)$, and $f'(x) > f(x) > 0$ for all x.

b. f is differentiable on $(0, \infty)$, $f(2x) = -f(x)$ for all $x > 0$, and $f(3) \neq 0$.

c. f'' is continuous on $[0,1]$, $\displaystyle\int_0^1 (f'(x))^2\,dx = 1$, and $\displaystyle\int_0^1 f(x)f''(x)\,dx = -1$.

d. $f'(0)$ exists, but f is not continuous at $x = 0$.

Fall 2008 Midterm I

1. Evaluate the following limits: (Do not use L'Hôpital's Rule.)

a. $\lim\limits_{x \to 9} \dfrac{\sqrt{x} - 3}{x^3 - 82x + 9}$

b. $\lim\limits_{x \to 2} \dfrac{1 + \sin(3\pi/x)}{(x-2)^2}$

2. Consider the limit $\lim\limits_{x \to 2} \dfrac{x\ln(x) - 2\ln(2)}{x-2}$.

a. Interpret this limit as the derivative $f'(a)$ of a function $f(x)$ at a point $x = a$.

b. Compute the limit using the interpretation in part (a).

3. Suppose that $f(1) = 1$, $f'(1) = 2$, $f''(1) = -5$, and that y is a differentiable function satisfying the equation $f(y^2x) = 2yf(x) - 1$. Find $\left.\dfrac{d^2y}{dx^2}\right|_{(x,y)=(1,1)}$.

4a. A differentiable parametric curve $x = f(t)$, $y = g(t)$ satisfies $\dfrac{dy}{dx} = \sqrt{t^2+1}\cos(\pi t)$. If

$$\left.\frac{d^2y}{dx^2}\right|_{t=3/4} = 2\ , \text{ find } \left.\frac{dy}{dt}\right|_{t=3/4} .$$

4b. Show that there is a real number c such that the tangent line to the graph of $u(x) = x^3 - x$ at the point $(c, u(c))$ and the tangent line to the graph of $v(x) = e^{-x^2}$ at the point $(c, v(c))$ are perpendicular to each other.

5. In each of the following if there exists a function f whose domain is the entire real line and that satisfies the given condition, give an example of such a function; otherwise just write Does Not Exist inside the box. No further explanation is required.

a. $\lim\limits_{x\to 0} xf(x) = 0$ and $\lim\limits_{x\to 0} f(x) = \infty$.

b. $\lim\limits_{x\to\infty} f(x) = 0$, but $\lim\limits_{x\to 0} f(1/x)$ does not exist.

c. $\lim\limits_{h\to 0} \dfrac{f(h) - f(-h)}{h}$ exists, but $f'(0)$ does not exist.

d. $f'(0)$ exists, but f is not continuous at $x = 0$.

e. $f(x)f'(x) < 0$ for all x.

Fall 2008 Midterm II

1. Evaluate the following limits:

a. $\lim\limits_{x\to 0} \dfrac{\cos^2(x) - \cos(\sqrt{2}\, x)}{x^4}$

b. $\lim\limits_{x\to\infty} \left(\dfrac{2}{\pi}\arctan x\right)^x$

2a. One morning of a day when the sun will pass directly overhead, the shadow of an 40-m building is 30 m long. At the moment in question, the angle the sun makes with the ground is increasing at the rate of $0.27°/\text{min}$. At what rate and how is the length of the shadow changing?

2b. Find $y(x)$ if $\dfrac{dy}{dx} = e^x - e^{-x}$ and $y(0) = 0$.

3. Sketch the graph of the function $y = e^{-x/2}(x^2 + 4x + 3)$ by computing y' and y'', and determining their signs; finding the critical points, the inflections points, the intercepts, and the asymptotes; and clearly labeling them in the picture.

4. A hangar consisting of two quarter-spheres joined by a half-cylinder, and having a volume of $2000\pi/3$ m^3 will be designed. If the construction cost for the unit area of the spherical parts is $3/4$ of the construction cost of the unit area of the cylindrical part, find the dimensions of the least expensive hangar that can be built.

5. Determine the number of solutions of the equation $e^x = kx$ for each value of the positive constant k.

Fall 2008 Final

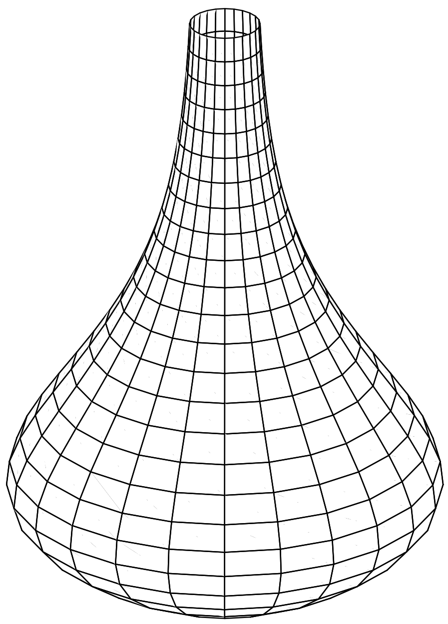
1. Show that $x \geq \ln(1+x) \geq \dfrac{x}{1+x}$ for all $x > -1$.

2. Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0} \frac{x^3 - \sin^3 x}{\sin(x^5)}$

b. $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{2k-1}{(2k-1)^2 + n^2}$

3a. The radius of a vase at a height of h cm from its base is $r(h) = \dfrac{1}{2}(h+3)^2 e^{-h/5} + 2$ cm. If the water is running into the vase at a rate of 150 cm^3/sec, how fast is the level of water changing at the moment when it is 10 cm deep?



3b. Find the area of the surface generated by revolving the curve $y = \sqrt{x}$, $0 \leq x \leq 1$, about the x-axis.

4a. Evaluate the integral $\displaystyle\int_0^{\pi/3} \frac{\sin x\, dx}{2\sin^2 x + 3\cos x}$.

4b. Evaluate the integral $\displaystyle\int_0^1 x\,(f(x)f''(x) + (f'(x))^2)\,dx$ if $f(0) = 3$, $f'(0) = -5$, $f''(0) = 2$, $f(1) = 7$, $f'(1) = 4$, $f''(1) = -1$.

5a. Show that the improper integral $\displaystyle\int_1^{\infty} e^{-a(x+\frac{1}{x})}\,dx$ converges for all positive values of the constant a.

5b. Show that $\displaystyle\int_1^{\infty} e^{-a(x+\frac{1}{x})}\,dx = \frac{e^{-2a}}{a} + \int_0^1 e^{-a(x+\frac{1}{x})}\,dx$ for all $a > 0$.

Fall 2007 Midterm I

1. Define the function f as below

$$f(x) = \begin{cases} \dfrac{\cos x - 1}{x} & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,. \end{cases}$$

a. Is the function f continuous at $x = 0$? (Don't use L'Hôpital's rule.)

b. Find $f'(0)$ if it exists. (Don't use L'Hôpital's rule.)

2a. Define $g(x) = x^2 - \cos x$. Show that there exists a number c in $(0, \pi/2)$ such that $g(c) = -1/2$.

2b. Show that the function $p(x) = x^3 - 3x + 1$ has all its three roots in $[-2, 2]$.

3a. Let a, b and c be constants. Define

$$f(x) = \begin{cases} \cos x & \text{for } x \leq 0\,, \\ a + bx + cx^2 & \text{for } x > 0\,. \end{cases}$$

Find a, b, c so that $f'''(0)$ (the third derivative of f at $x = 0$) exists. Does $f^{(4)}(0)$ (the fourth derivative of f at $x = 0$) exist?

3b. Find the horizontal and vertical asymptotes of the curve

$$y = \frac{3x^2 - 1 + 2\sin x}{x^2 - 1}\quad.$$

4. Let f be a differentiable function on $(-5, 10)$. Assume

$$f\left(\frac{5}{2}\right) = 10, \qquad f\left(\frac{3}{2}\right) = 0, \quad \text{and} \quad f'\left(\frac{3}{2}\right) = 3\,.$$

a. Find $\displaystyle\lim_{x\to 3}\left(3 + f\left(x - \frac{1}{2}\right)\right)$. (Explain!)

b. Find $\displaystyle\lim_{x\to 1}\frac{f\left(x + \frac{1}{2}\right)}{x - 1}$. (Explain!)

5a. Let a curve C be defined through the equation $x + \sin(xy) = y$. Find an equation of the tangent line and an equation of the normal line of C at the point $(0, 0)$.

5b. If one perpendicular side of a right triangle decreases at 1 cm/min and the other perpendicular side increases at 2 cm/min then how fast is the length of the hypotenuse changing at the time where the first perpendicular side is 8 cm and the other one is 4 cm long?

Fall 2007 Midterm II

1a. Evaluate $\displaystyle\lim_{x\to 0} \frac{1}{\sin x} \int_0^{\sin 2x} \cos 5t \, dt$.

1b. A hallway of width 4 ft meets a hallway of width $12\sqrt{3}$ ft at a right angle. Find the length of the longest ladder that can be carried horizontally around the corner from one hallway to the other hallway. (i. While solving this problem, to find the global extrema of a function that is not defined on a closed interval you have to use the first derivative test or the second derivative test.) (ii. Find the exact answer.)

2. Define a function f as follows

$$f(x) = \frac{x^2}{x^2 - 4} .$$

You have to explicitly write the answer to each part.

a. Find the domain of the function f.

b. Find all asymptotes, the x-intercept(s) and the y-intercept(s) of the graph of f.

c. Find $f'(x)$ and $f''(x)$.

d. Find the intervals on which the function f is increasing and decreasing. Identify the function's local extreme values, if any, saying where they are taken on.

e. Find where the graph of f is concave up and where it is concave down. Are there any inflection points on the graph of f?

f. Sketch the graph of f.

3a. Calculate the following limit $\displaystyle\lim_{n\to\infty} \frac{1}{n^3} \sum_{i=1}^{n} i^2 \sin\left(5 + \frac{\pi i^3}{n^3}\right)$.

3b. Calculate $\displaystyle\int \frac{\sin 2x}{3 + 2\cos^2 x} \, dx$.

4. Let f be a continuous function on $[2, 5]$. Assume that

$$f(7 - x) = f(x) \neq 0$$

for all x in [2, 5] and

$$\int_4^5 f(u) \, du = 26 .$$

We define

$$F(x) = \int_4^x f(u)\,du \quad \text{and} \quad G(x) = \int_3^x f(u)\,du$$

for all x in $[2, 5]$.

a. Show that G is an increasing function on $[2, 5]$.

b. Show that there exists only one number c in the interval $[2, 5]$ such that $F(c) = 0$.

c. Calculate $F(2) + G(4)$.

5a. Find the volume of the solid generated by revolving the region bounded by the curves $x = \ln y$, $y = e^{2x}$, and $x = \ln 2$ about the x-axis.

5b. Find the surface area of the solid generated by revolving the curve $y = \sqrt{2x - x^2}$ for $1 \leq x \leq 3/2$.

Fall 2007 Final

1a. Is there any real number that is one less than its cube? Explain and support your answer with well known theorems.

1b. Prove that $\left| xe^{-x^2} + \cos x - \sin x \right| < \dfrac{1 + 2\sqrt{2}}{2}$ for all x in $(-\infty, \infty)$.

2a. Show that $\displaystyle\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{2n-1}{2n} \int_0^{\pi/2} \sin^{2n-2} x\,dx$ where n is a positive integer.

2b. Show that $\displaystyle\int_0^{\pi/2} \sin^6 x\,dx = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{\pi}{2} = \frac{5\pi}{32}$.

2c. Show that $\displaystyle\int_0^{\pi/2} \sin^{2n} x\,dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}\frac{\pi}{2}$ (for any integer $n > 0$).

3a. Find $\displaystyle\lim_{x\to\infty} \left(\frac{x+a}{x-a}\right)^x$ where a is a nonzero real number.

3b. Find $\displaystyle\lim_{n\to\infty} \frac{1}{n^{r+1}} \sum_{k=1}^{n} k^r$ where r is a real number greater than 1.

3c. Find $\displaystyle\lim_{n\to\infty} \frac{1}{n^{r+2}} \sum_{k=1}^{n} k^r$ where r is a real number greater than 1.

4. Evaluate the following integrals.

a. $\displaystyle\int_{\sqrt{e}}^{e} \frac{\sin^{-1}(\ln x)}{x}\,dx$

b. $\displaystyle\int \frac{x-1}{x(x+1)^2}\,dx$

c. $\displaystyle\int_{-\infty}^{0} 2^{3x}\,dx$

5a. Show that the linearization $L(x)$ of e^x about $x = 0$ is $L(x) = x + 1$.

5b. Let $L(x)$ be the linearization of a function f at $x = 2$. Assume that the function f'' is continuous on $(1,4)$ and the graph of $y = f(x)$ is concave up on $(0,5)$. Define a function h as follows: $h(x) = f(x) - L(x)$ for all x in $(1,3)$.

i. Identify the intervals on which h is increasing and decreasing.

ii. Determine the concavity of graph $y = h(x)$.

iii. Find, if any, absolute extreme values of the function h.

Fall 2006 Midterm I

1. Evaluate the following limits. (Do not use L'Hôpital's Rule.)

a. $\displaystyle\lim_{x\to 1} \frac{x^2-4x+3}{\sqrt{x}-1}$

b. $\displaystyle\lim_{x\to\infty} \left((x+2)\sin\left(\frac{2}{x+1}\right)\right)$

2a. Find the equations of the tangent lines to the curve $y = 2x^3 - 3x^2 - 12x + 20$ which are perpendicular to the line $y = 1 - x/24$.

2b. Find $\dfrac{d^2y}{dx^2}$ at $t = \dfrac{\pi}{3}$ if $x = t - \sin t$ and $y = 1 - \cos t$.

3a. Evaluate $g'(0)$ and $g''(0)$ if $g(x) = \sec(x^2 + 5x)$.

3b. If $f(1) = -3$, $f'(1) = 4$ and if $y = f(xy^2 + f(x^2)) + 5$ defines y as a differentiable function of x, determine $\dfrac{dy}{dx}$ at the point $(x,y) = (1,2)$.

4. In each of the following if there exists a function f whose domain is the entire real line and which satisfies the given condition, then give an example of such a function; otherwise just write Does Not Exist inside the box. No further explanation is required.

a. $f(x)$ is not continuous at $x = 4$, but $(f(\sqrt{x}\,))^2$ is continuous at $x = 4$.

b. $f(x)$ is not continuous at $x = 1$, but $(f(x))^2$ is continuous at $x = 1$.

c. $\lim\limits_{x\to 0} f(x)$ does not exist, but $\lim\limits_{x\to 0} f(x^2)$ exists.

d. $\displaystyle\lim_{x\to 0} \frac{f(x)-f(0)}{x}$ exists, but f is not differentiable at 0.

e. f is continuous and $f(x)f(x+1) < 0$ for all x.

5. For each of the following functions determine if there is a positive constant C such that for all $0 < \varepsilon < 1$ and for all x,

$$|x| < C\varepsilon \implies |f(x)| < \varepsilon\,.$$

In particular, if such C exists, find one; if such C does not exist, explain why not.

a. $f(x) = x^2 + 5x$

b. $f(x) = x^{1/3}$

Fall 2006 Midterm II

1a. Evaluate the integral $\displaystyle\int_0^1 t\sin^2(\pi(t^2+1))\,dt\,.$

1b. Find the area of the region enclosed by the curves $y = 2x^3 - x^2 - 5x$ and $y = x^2 - x$.

2a. Evaluate the limit $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n^2}{(n+k)^3}\,.$

2b. Show that the equation $x^5 + x^3 - 3x^2 + 3x + 7 = 0$ has exactly one real root.

3a. Let $f(x) = \dfrac{1}{x}\displaystyle\int_0^x \sin(t^2)\,dt$ for $x > 0$. If $A = f(\sqrt{\pi/2})$, express $f'(\sqrt{\pi/2})$ in terms of A.

3b. Show that if g is a twice differentiable function such that $g(0) = 1$, $g'(0) = -1$, $g(1) = 2$, $g'(1) = 5$, and $g''(x) \geq 0$ for all x, then $g(x) \geq 1/3$ for all $0 \leq x \leq 1$.

4. Find the smallest possible total surface area for a cone circumscribed about a sphere of radius R.

5. Sketch the graph of $y = \dfrac{x^3}{x^2-1}$ by finding y', y''; determining their signs; finding and classifying the critical points, the inflection points and the intercepts; and finding the asymptotes.

Fall 2006 Final

1. Evaluate the following integrals.

a. $\displaystyle\int \frac{1+e^x}{1-e^x}\,dx$

b. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+4}}$

2. Evaluate the following integrals.

a. $\displaystyle\int_{-1/2}^{1/2} \frac{dt}{t+\sqrt{1-t^2}}$

b. $\displaystyle\int_{-1/2}^{1/2} \sqrt{\frac{1-x}{1+x}}\ \arcsin x\, dx$

3. Consider the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - 8x + 12$. Express the following volumes in terms of definite integrals, but do not evaluate.

a. The volume of the solid generated by revolving this region about the y-axis.

b. The volume of the solid generated by revolving this region about the x-axis.

4a. Express $\displaystyle\int_0^1 x^2 f''(x)\, dx$ in terms of $k = \displaystyle\int_0^1 f(x)\, dx$ if $f'(1) = 2f(1)$ and f'' is continuous.

4b. Prove that $\displaystyle\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$.

5a. Find constants A and B such that the limit $\displaystyle\lim_{x\to\infty} x^3\left(A + \frac{B}{x} + \arctan x\right)$ exists.

5b. Let $f(x) = x^{1/x}$ for $x > 0$. Find the absolute maximum value of f.

Fall 2005 Midterm I

1. Evaluate the following limits:

a. $\displaystyle\lim_{x\to\infty} \left(\sqrt[3]{x^3+x^2} - x\right)$

b. $\displaystyle\lim_{x\to 5/2} \frac{\cos \pi x}{8x^3 - 16x^2 - 25}$

2. Assume that $\displaystyle\lim_{x\to 0^+} f(x) = A$ and $\displaystyle\lim_{x\to 0^-} f(x) = B$ where A and B are real numbers. In each of the following, if possible, express the limit in terms of A and B; otherwise, write cannot be done in the box. No further explanation is required.

a. $\displaystyle\lim_{x\to 0^+} f(x^2 - x)$

b. $\displaystyle\lim_{x\to 0^-} f(x^2 - x)$

c. $\displaystyle\lim_{x\to 0^-} \big(f(x^2) - f(x)\big)$

d. $\displaystyle\lim_{x\to\infty} f((x^2 - x)^{-1})$

e. $\displaystyle\lim_{x\to\infty} f(x^{-2} - x^{-1})$

f. $\lim_{x \to 1^-} f(x^{-2} - x^{-1})$

3a. Let f be a differentiable function such that $f(1) = 1$ and the slope of the tangent line to the curve $y = f(xf(xy))^2$ at the point $(1, 1)$ is 3. Find all possible values of $f'(1)$.

3b. Show that if u is differentiable at 0, then the limit

$$\lim_{t \to 0} \frac{u(3t) - u(-2t)}{t}$$

exists.

4a. The angle of elevation from a point 3 m away from the base of a flag pole to its top is $60^\circ \pm 1^\circ$. Estimate the error in the height of the pole calculated from this measurement.

4b. Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

5. Consider the function $f(x) = x^{4/3} - x - x^{1/3}$.

a. Find the absolute maximum value and the absolute minimum value of f on the interval $[-1, 6]$.

b. Determine the number of solutions of the equation $f(x) = 0$.

Fall 2005 Midterm II

1. Find the closest point on the curve $y = x^2$ to the point $(0, 1)$

2. Evaluate the limit: $\displaystyle \lim_{x \to 0} \frac{x \int_0^x \sin(t^2)\, dt}{\sin(x^4)}$.

3. Let a be a positive constant. Consider the region bounded by the parabola $y = a(x - x^2)$ and the x-axis in the first quadrant. Let V be the volume of the solid generated by revolving this region about the x-axis. Let W be the volume of the solid generated by revolving this region about the y-axis. Find all values of a for which $V = W$.

4. Evaluate the following integrals:

a. $\displaystyle \int_0^1 x^5 \sqrt{x^3 + 1}\, dx$

b. $\displaystyle \int \tan^2 x\, dx$

5a. True or false? No explanation is required.

$$\int x^2\,dx = \frac{x^3}{3} + C \qquad \text{True}\square \qquad \text{False}\square$$

$$\int (5x-7)^2\,dx = \frac{(5x-7)^3}{3\cdot 5} + C \qquad \text{True}\square \qquad \text{False}\square$$

$$\int \sin^2 x\,dx = \frac{\sin^3 x}{3\cos x} + C \qquad \text{True}\square \qquad \text{False}\square$$

5b. Determine all differentiable functions f which satisfy

$$\int (f(x))^2\,dx = \frac{(f(x))^3}{3f'(x)} + C \quad .$$

Fall 2005 Final

1. Find y'. (Do not simplify!)

a. $y = (x^2+1)(x^3+1)$

b. $y = \dfrac{x^2+1}{x^3+1}$

2. Evaluate the following limits:

a. $\lim\limits_{x\to\infty} x\,(2^{1/x}-1)$

b. $\lim\limits_{x\to 0} (\cos x)^{1/x^2}$

3. Evaluate the following integrals:

a. $\displaystyle\int_0^1 \arctan x\,dx$

b. $\displaystyle\int_0^{\pi/4} \tan^5 x\,\sec^4 x\,dx$

4. Sketch the graph of $y = \dfrac{\ln x}{x}$ by finding y', y'', determining their signs, finding and classifying the critical points, the inflection points and the intercepts, and finding the asymptotes.

5. Assume that as an ice cube melts it retains its cubical shape, and its volume decreases at a rate that is proportional to its surface area. If the cube loses 1/4 of its volume during the first hour, how long will it take for the entire ice cube to melt?

Fall 2004 Midterm I

1. Evaluate the following limits: (Do not use L'Hôpital's Rule.)

a. $\displaystyle \lim_{x\to 5} \frac{x^3-24x-5}{x^3-2x^2-75}$

b. $\displaystyle \lim_{x\to 8} \frac{1-\cos(x^{1/3}-2)}{(x-8)^2}$

2. Find d^2y/dx^2 at the point $(x,y)=(1,-1)$ if y is a differentiable function of x satisfying

$$y^3+x^2y+x=x^3+2y\ .$$

3a. Show that $\displaystyle \lim_{x\to 3} \frac{1}{x-1}=\frac{1}{2}$ using the ε-δ definition of the limit.

3b. Suppose that for all $0<\varepsilon<1$,

$$|x-1|<\frac{\varepsilon^2}{4} \Longrightarrow |f(x)-3|<\varepsilon$$

and

$$|x-1|<\frac{\varepsilon}{35} \Longrightarrow |g(x)-4|<\varepsilon\ .$$

Find a real number $\delta>0$ such that

$$|x-1|<\delta \Longrightarrow |f(x)+g(x)-7|<\frac{1}{5}\ .$$

4a. Show that the equation $x^4+1=7x^3$ has at least 2 real roots.

4b. Find $\dfrac{dy}{dx}$ if $y=\sin^7(\cos(x^5))$.

5. In each of the following if there exists a function f whose domain is the entire real line and which satisfies the given condition, then give an example of such a function; otherwise just write Does Not Exist. No further explanation is required.

a. $\displaystyle \lim_{x\to 1} f\left(\frac{x-1}{x+1}\right) \neq f(0)$

b. $y=f(x)$ is not differentiable at $x=-1$, but $y=(f(x))^2$ is differentiable at $x=-1$.

c. $y=f(x)$ is not continuous at $x=5$, but it is differentiable at $x=5$.

d. $\displaystyle \lim_{x\to 0} f(x)=2$ and $f(0.000000000000001)=-1$.

e. $f(2)=3$, $f(4)=-1$ and $f(x)\neq 0$ for all x.

Fall 2004 Midterm II

1. Sketch the graph of $y = x^3 - 6x^2 + 5$ by computing y' and y'', determining their signs, finding the local extreme points, the inflection points and the intercepts.

2a. Let $x = \displaystyle\int_{1/2}^{y} \frac{dt}{\sqrt{t - t^3}}$. Find $\dfrac{dy}{dx}$ at a point where $\dfrac{d^2y}{dx^2} = 0$.

2b. A ball of radius 5 cm is dropped directly under a light bulb which is 3 m above the floor. If at the moment the ball is 35 cm away from the bulb it is moving at a speed of 2 m/sec, how fast is the radius of the shadow of the ball on the floor changing?

3. Evaluate the following integrals:

a. $\displaystyle\int_{3}^{4} \sqrt{x^4 - 4x^3 + 4x^2 - 1}\, dx$

b. $\displaystyle\int_{0}^{\pi} \sin^3 x \cos^2 x\, dx$

c. $\displaystyle\int x(x+1)^{1/3}\, dx$

d. $\displaystyle\int \frac{\tan\theta}{\sqrt{\sec\theta}}\, d\theta$

4. A cylinder is drawn inside a sphere of radius 1. Find the maximum possible area of the cylinder.

5a. Find f if $f''(x) = 1$ for all x, $f(0) = 1$ and $f(2) = 2$.

5b. Show that if $g''(x) \le 0$ for all x, $g(0) = 0$ and $g(2) = 0$, then $g(x) \ge 0$ for all $0 \le x \le 2$.

5c. Show that if $h''(x) \le 1$ for all x, $h(0) = 1$ and $h(2) = 2$, then $h(x) \ge \dfrac{7}{8}$ for all $0 \le x \le 2$.

Fall 2004 Final

1. Evaluate the following integrals.

a. $\displaystyle\int_{-\ln 2}^{0} \frac{dx}{e^x - 2e^{-x} - 1}$

b. $\displaystyle\int x \tan^2 x\, dx$

2. Evaluate the following integrals:

b. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{dx}{\sin x - 2\cos x + 3}$

b. $\displaystyle\int \sqrt{1+e^x}\,dx$

3a. Find the linearization of $f(x) = \arcsin x = \sin^{-1} x$ centered at $x = 1/2$.

3b. Evaluate the limit $\displaystyle\lim_{x\to 0} \left(\frac{(1+x)^{1/x}}{e}\right)^{1/x}$.

4. A solid cylinder whose axis is the x-axis intersects another solid cylinder whose axis is the y-axis. Both cylinders have radius 2 units. Find the volume of the part of the intersection that lies in the first octant.

5. Suppose that the function f is continuous, increasing and $f(x) \geq 0$ on $[0, \infty)$.

a. Write down an integral expressing the volume $V(a)$ of the solid generated by revolving the region between the graph $y = f(x)$ and the x-axis for $0 \leq x \leq a$ about the x-axis where a is a constant.

b. Write down an integral expressing the volume $W(a)$ of the solid generated by revolving the region between the graph $y = f(x)$ and the line $y = f(a)$ for $0 \leq x \leq a$ about the y-axis where a is a constant.

c. Find $f'(3)$ if $V(a) = W(a)$ for all $a \geq 0$ and $f(3) = 2$.

d. Find $f(x)$ for all $x \geq 0$ if $V(a) = W(a)$ for all $a \geq 0$ and $f(3) = 2$.

Fall 2003 Midterm I

1. For a real number t, let L_t be the line passing through the point $(-1, 0)$ with slope t, and let M_t be the line passing through the point $(1, 0)$ and perpendicular to L_t.

a. Find the intersection point P_t of the lines L_t and M_t.

b. Describe the curve traced by P_t for $-\infty < t < \infty$ using Cartesian coordinates.

2. Determine all values of the constant a for which the function $f(x) = e^x + ae^{-x}$ is one-to-one. For these values of a, find $f^{-1}(x)$ and the domain of f^{-1}.

3. Find the following limits:

a. $\displaystyle\lim_{h\to 0} \frac{\dfrac{1}{(x+h)^2} - \dfrac{1}{x^2}}{h}$

b. $\displaystyle\lim_{x\to 9} \frac{x^3 - 729}{x - \sqrt{x} - 6}$

4a. Find the limit $\displaystyle\lim_{x\to 0^+} \frac{1 - \sqrt{\cos x}}{(1 - \cos\sqrt{x})\sin x}$

4b. Prove that if $|f(x)| < 7$ for $0 < |x-2| < 1$ and $\lim_{x\to 2} g(x) = 0$, then $\lim_{x\to 2}(f(x)g(x)) = 0$

5. Show that $\lim_{x\to 1}(x^2 + x) = 2$ using the ε-δ definition of the limit.

Fall 2003 Midterm II

1. Find the values of h, k and $r > 0$ that make the circle $(x-h)^2 + (y-k)^2 = r^2$ tangent to the parabola $y = x^2 + 1$ at the point $(1, 2)$ and that also make the second derivatives d^2y/dx^2 have the same value on both curves there.

2. A particle moves along the curve $2x^3 + 2y^3 = 9xy$. If the distance of the particle to the origin is increasing at a rate of 3 m/sec at the moment it passes through the point $(2, 1)$, how fast is its x-coordinate changing at this moment?

3a. Find y' if $y = \tan^3(x^2 + \cos(5x))$. (Do not simplify!)

3b. Assume that f is continuous on $[0, 2]$ and $f(0) = f(2) = 0$. Show that $f(c+1) = f(c)$ for some c.

4. A cone is constructed from a disk of radius a by removing a sector AOC of arc length x and then connecting the edges OA and OC. Find the maximum possible volume of the cone.

5. Graph $y = \dfrac{x-1}{\sqrt{2x^2+1}}$ by computing y' and y'' and determining their signs and the shape of the graph, finding the critical points, the inflection points and the intercepts, and finding the asymptotes.

Fall 2003 Final

1. Find the volume of the solid generated by revolving the region bounded by the curve $y = \ln x$, the x-axis and the line $x = e$ about

   a. the x-axis.

   b. the y-axis.

2. Evaluate the following integrals:

   a. $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

   b. $\displaystyle\int \frac{dx}{e^x - e^{-x}}$

3. Evaluate the following limits:

a. $\lim_{x\to 1}\left(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\right)$

b. $\lim_{x\to\infty}\left(\dfrac{2}{\pi}\arctan 3x\right)^x$

4. Find $\dfrac{dy}{dx}$ at $(x,y)=(1,2)$ if $x^y+y^x=3$.

5a. Show that $|\arctan a-\arctan b|\le|a-b|$ for all a,b.

5b. Find $f''(1)$ if $f(x)=x\displaystyle\int_0^x\frac{t^2}{\sqrt{1+t^4}}\,dt$.

Fall 2002 Midterm I

1) a) Find $\lim_{x\to 0^+}\dfrac{\sqrt{\sin 2x}-\sqrt{\sin x}}{\sqrt{x}}$.

b. Use the ε–δ definition of the limit to show that $\lim_{x\to 0}\left(\sin(\sqrt[3]{x})\sin(1/x)\right)=0$.

2. Let f be a differentiable function such that $xf(x)+f(x^2)=2$ for all $x>0$.

a. Find $f'(1)$.

b. Show that if $f(x_0)=0$ for some $x_0>1$, then there is exists $x_1>x_0$ such that $f(x_1)=0$.

3. Find the derivatives of the following functions:

a. $y=\tan^5(\pi x^3)$

b. $y=\sqrt[3]{\dfrac{x}{x^2+1}}$

4. Find y'' at the point $(x,y)=(\pi/2,0)$ if y is a differentiable function of x satisfying the equation $y^2=x\cos(x+y)$.

5. Points A and B move along the x- and y-axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to the line AB remains constant. How fast is OA changing, and is it increasing, or decreasing, when $OB=2r$ and B is moving toward O at the rate of $0.3r$ m/sec?

Fall 2002 Midterm II

1. Graph $y = \dfrac{x^2-4}{x+1}$ by computing y' and y'' and determining their signs and the shape of the graph, finding the critical points, the inflection points and the intercepts, and finding the asymptotes.

2. Consider the area of the region between the parabola $y = x^2$ and the line $y = c$ for $0 \le x \le 1$ as a function of c. Find the absolute maximum and the absolute minimum values of this function on the interval $0 \le c \le 1$.

3a. Evaluate the limit $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n}{(n+k)^2}$.

3b. Evaluate $y' + y''$ at $x = \pi/4$ if f is a differentiable function with $f'(1/\sqrt{2}) = 4$ and $y = \displaystyle\int_0^{\sin x} f(t)\, dt$.

4. Evaluate the following integrals:

a. $\displaystyle\int \frac{x^3}{\sqrt{x^2-9}}\, dx$

b. $\displaystyle\int_{\pi^2/36}^{\pi^2/4} \frac{\cos(\sqrt{\theta})}{\sqrt{\theta}\sin^2(\sqrt{\theta})}\, d\theta$

5. Find the area of the region enclosed by the curves $y^4 = 2y^3 + x$ and $(y-1)^2 = x + 1$.

Fall 2002 Final

1. Let c be a positive constant. Find the point on the curve $y = \sqrt{x}$ closest to the point $(c, 0)$.

2. Find the volume generated by revolving the region enclosed by the curves $y = \dfrac{1}{\sqrt{3+x^2}}$ and $y = \dfrac{x^2}{2}$ about

a. the x-axis

b. the y-axis

3a. Find $h''(1)$ if $h(x) = (g(x^2))^3$ and $g^{(n)}(1) = n + 1$ for $n \ge 0$.

3b. Find $f'(1)$ if $\displaystyle\int_1^x f(t)\, dt = x + \ln(f(x))$ for all x .

4. Evaluate the following limits:

a. $\displaystyle\lim_{x\to\infty} \left(1 + \frac{2}{3x}\right)^x$

b. $\displaystyle\lim_{x\to 1} \frac{\int_1^x t^t\,dt}{x^{x+1}-1}$

5. Evaluate the following integrals:

a. $\displaystyle\int \frac{(1+e^x)^2}{1+e^{2x}}\,dx$

b. $\displaystyle\int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

Fall 2001 Midterm I

1. Find the following limits: (Do not use L'Hôpital's Rule.)

a. $\displaystyle\lim_{x\to -2} \frac{x^2+5x+6}{x^2+x-2}$

b. $\displaystyle\lim_{x\to 0} \frac{\sqrt{2-\cos x}-1}{\sin^2 \pi x}$

2. Find the derivatives of the following functions:

a. $\displaystyle y = \left(\frac{\sin x}{1+\cos x}\right)^2$

b. $y = \sqrt{x}\sin(\sqrt{x})$

3. Find y'' at the point $(1,1)$ if y is a differentiable function of x satisfying the equation $x^3+y^3 = 6xy-4$.

4. a. Let A and B the points of intersection of the tangent line to the curve $y = 1/x$ at a point $P(x_0, y_0)$ with the y- and x-axes, respectively. Find the length of the line segment AB as a function of x_0.

b. Suppose that the point P is moving along the curve $y = 1/x$. If at the moment P passes through the point $(2, 1/2)$, the length of the line segment AB is increasing at a rate of 4 mm/sec, how fast is x_0 changing?

5. Graph $y = 4x^3 - x^4$ by computing y' and y'', finding the critical points and the inflection points, determining the signs of the derivatives, determining the shape of the graph and finding the intercepts.

Fall 2001 Midterm I

1. An 8 feet high wall is 27 feet away from a building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

2. Suppose that u is a continuous function and a is a real constant satisfying

$$\int_0^x u(t)\, e^{a(t^2-x^2)}\, dt = \sin x$$

for all x. Find a if $u'(\pi/2) = 5$.

3. Find the volumes of the solids generated by revolving each of the following regions about the x-axis.

a. The region lying between $y = \cos x$ and $y = \sin x$ for $0 \le x \le \pi/3$.

b. The region bounded on the left by $x = 3y^2 - 2$, on the right by $x = y^2$, and below by the x-axis.

4. Find the following integrals:

a. $\displaystyle\int \frac{x}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}\, dx$

b. $\displaystyle\int_0^{\ln 3} \frac{1}{e^x+2}\, dx$

5. Let $g(x) = x^{-5}$. Find a function f such that $f(1) = 1$ and

$$\frac{d}{dx}\left(f(x)g(x)\right) = \frac{d}{dx}\left(f(x)\right) \cdot \frac{d}{dx}\left(g(x)\right)$$

for all $x > 0$.

Fall 2001 Final

1. Find the following limits.

a. $\displaystyle\lim_{x\to 0^+} \left(\sin(x^2)\right)^{1/\ln x}$

b. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x^3} \int_0^x \frac{e^{t^4} - e^{-t^4}}{t^2}\, dt \right)$

2. Evaluate the following integrals.

a. $\displaystyle\int x^2 (\ln x)^2\, dx$

b. $\displaystyle\int \frac{1}{(x^2+4x+5)^2}\, dx$

3. Evaluate the improper integral $\displaystyle\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}\, dx$.

4. Graph the function $f(x) = xe^{-x}$ by finding the intervals on which it is increasing, decreasing, concave up, concave down; its critical points and inflections points, and its asymptotes.

5. Let $f(x) = x^{1/x}$ for $x > 0$.

a. Find the absolute maximum value of $f(x)$.

b. Show that $\sqrt{2} \leq \int_2^3 f(x)\,dx \leq \sqrt{3}$.

Fall 2000 Midterm I

1. Evaluate the following limits. (Do not use L'Hôpital's Rule.)

a. $\lim\limits_{x\to 0} \dfrac{x}{\sqrt{1+\tan 3x} - 1}$

b. $\lim\limits_{x\to 0} \dfrac{\sin 4x - 2\sin 2x}{x^3}$

2. Let

$$f(x) = \begin{cases} 1 + (x-1)\sin\left(\dfrac{1}{x-1}\right) & \text{if } x > 1, \\ |x|/x & \text{if } 0 \neq |x| \leq 1 \\ -1 + \sin\left(\dfrac{1}{x+1}\right) & \text{if } x < -1. \end{cases}$$

Determine if each of the following limits exists. Explain your reasoning in detail.

a. $\lim\limits_{x\to 1} f(x)$

b. $\lim\limits_{x\to 0} f(x)$

c. $\lim\limits_{x\to -1} f(x)$

3. Find the equation of the tangent line of the curve $y = \sqrt{x}$ which is perpendicular to the line $2x + 3y = 5$.

4. A particle moves along the curve $y = x^3 + 1$ in such a way that its distance from the origin increases at a rate of 7 units per second. Find the rate of change of the x-coordinate of the particle when it passes through the point $(1, 2)$.

5. Assume that the function f satisfies the equation $f'(x) = f(x/2)$ for all x, and $f(0) = 1$.

a. Show that if $f(x_0) = 0$ for some $x_0 > 0$, then there is x_1 such that $0 < x_1 < x_0$ and $f(x_1) = 0$.

b. Find $f'''(0)$.

6. Graph the function $y = x^{4/3} + 4x^{1/3}$ by computing y' and y'', determining their signs, and finding the extreme points, the inflection points and x-intercepts.

Fall 2000 Midterm II

1. A wire of length L is bent to form the shape where the sides of the square have length a and the radius of the semicircle is r, and then revolved about the line containing the fourth side of the square to sweep out the surface of a sphere and a cylinder. Find r and a which give the maximum surface area.

(1. This question requires no integration: you can use the surface area formulas for these geometric objects. 2. Note that the bottom and the top of the cylinder as well as its side are included in the surface area.)

2. a. Evaluate the limit $\displaystyle\lim_{x\to 0^+} \int_0^x \frac{1}{(x-t)^4 + x^4}\, dt$.

b. If $f(x) = \displaystyle\int_2^{\sqrt{x}} \sqrt{9+t^4}\, dt$, find $f'(4)$.

3. Evaluate the following integrals:

a. $\displaystyle\int_0^{\pi/2} \frac{\sin 2x}{(a\sin^2 x + b\cos^2 x)^2}\, dx$ where $a, b > 0$.

b. $\displaystyle\int (1 + x^{-3/4})^{1/3}\, dx$.

4. a. If $y' = \dfrac{\sin^3(1/x)\cos(1/x)}{x^2}$ and $y(3/\pi) = 1$, find $y(4/\pi)$.

b. Evaluate the limit $\displaystyle\lim_{n\to\infty} \left(\frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{4n-3} + \frac{1}{4n-1} \right)$.

5. a. Find the volume of the solid obtained by revolving the region between the curve $y = x^2 - x^3$ and the x-axis about the y-axis.

b. Find the area of the smaller piece if a sphere with radius 3 is cut into two by a plane passing 2 units away from its center.

6. Show that $\displaystyle\int_0^{\pi} \frac{t\,\sin^2 t}{1+t^2}\, dt \le \frac{\pi}{4}$.

Fall 2000 Final

1. Consider the rectangles bounded by the axes and the lines perpendicular to the axes through a point on the curve $y = (\ln x)^4/x^3$ for $x \geq 1$. Find the maximum value of the area of these rectangles if it exists.

2. Sketch the graph of the function $y = (x^2 - 2x)e^x$ by computing y', y'', determining the shape of the graph, finding the critical points, the inflection points, and the x-intercepts, and the asymptotes.

3. a. Find the volume of the solid obtained by revolving the region between the graph of $f(x) = \begin{cases} \tan x/x & \text{if } 0 < x \leq \pi/4 \\ 1 & \text{if } x = 0 \end{cases}$ and the x-axis about the y-axis.

b. Evaluate the improper integral $\displaystyle\int_0^\infty \frac{dx}{a + be^x}$.

4. Evaluate the integrals:

a. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos x}{(9 + 2\sin x)(5 + \sin x)}\, dx$

b. $\displaystyle\int (\arcsin x)^2\, dx$

5. Evaluate the following limits:

a. $\displaystyle\lim_{x\to\infty} x \int_0^1 e^{-t^2/x^2} \ln\left(1 + \frac{t}{x}\right) dt$

b. $\displaystyle\lim_{x\to 0} (\cos x)^{1/x^2}$

6. Let f be a continuous function and define

$$g(x) = \int_{-1}^{1} f(t)|x - t|\, dt$$

for all x. Express $g''(x)$ in terms of $f(x)$ for x in $(-1, 1)$.

Fall 1999 Midterm I

1. Evaluate the following limits: (You are not allowed to use the L'Hôpital's Rule.)

a. $\displaystyle\lim_{x\to 0} \frac{\tan x - \sin x}{x(\cos^3 2x - 1)}$

b. $\displaystyle\lim_{x\to 2} \frac{\sin(x - 2)}{\sqrt{x + 3} - \sqrt{2x + 1}}$

2. Find $f'(8)$ if the function $f(x)$ satisfies the following properties:

- $f(ab) = f(a) + f(b)$ for all $a, b > 0$.
- $\lim_{x\to 1} \dfrac{f(x)}{x-1} = 2$.

3. Sketch the graph of $y = x^{-1/3} - x^{-2/3}$ by computing y' and y'', determining their signs, finding the critical and the inflection points, indicating the intervals on which the function is increasing, decreasing, concave up, concave down, and finding the asymptotes.

4. Find y''' at $(x, y) = (1, 0)$ if $\sin(xy) + x = 1$.

5. a. The length of the hypotenuse of a right triangle is constant at 5 cm, and the length of one of its sides is increasing at a rate of 2 cm/sec. Find the rate of change of the area of the triangle, when this side is 4 cm long.

b. Show that for any twice differentiable function $f(x)$ on the entire real line such that $f(0) = 0, f(1) = 1, f(2) = 4$, there exists a point c in $(0, 2)$ such that $f''(c) = 2$.

Fall 1999 Midterm II

1. A 36π m^3 container with cylindrical side and hemispherical ends is going to be constructed. The construction of the cylindrical side surface costs 3×10^9 TL/m^2 and the construction of the hemispheres costs 10×10^9 TL/m^2. Find the minimum cost of constructing this container.

2. Show that the function $f(x) = \displaystyle\int_{e^{-x}}^{e^x} \frac{\sin t}{t}\, dt$ has a critical point at $x = a$ where a is the positive real number satisfying $e^a + e^{-a} = 2\pi$ and determine whether this is a local maximum or minimum.

3. a. Find the volume of the solid generated by revolving the region between the curve $y = x^3$ and the x–axis for $0 \le x \le 1$ about the y–axis.

b. Find the area of the surface generated by revolving the curve $y = x^3$ for $0 \le x \le 1$ about the x–axis.

4. a. Evaluate the integral $\displaystyle\int_1^8 \frac{1}{x + 5\sqrt[3]{x}}\, dx$.

b. Show that $\dfrac{5\pi}{24} \le \displaystyle\int_{\pi/3}^{\pi/2} \sqrt{1 + \sin^4 x}\, dx \le \frac{\pi}{3\sqrt{2}}$.

5. Evaluate the limit

$$\lim_{x\to\infty} \Big(1 + \ln\big(\cos(a/x)\big)\Big)^{x^2}$$

where a is a constant.

Fall 1999 Final

1. a. Let

$$f(x) = \begin{cases} \dfrac{1}{x^2} \displaystyle\int_0^{|x|} \sin t^2 \, dt & \text{if } x \neq 0 \, , \\ \\ 0 & \text{if } x = 0 \, . \end{cases}$$

Does $f'(0)$ exist? Explain.

b. Suppose $g(x)$ and $h(x)$ are differentiable functions on $[0, 1]$ and $g(0) = 4, g(1) = 1, h(0) = 3, h(1) = 0$. Show that there exists a point c in $(0, 1)$ such that $g'(c) = h'(c)$.

2. a. Find constants a and b such that $\lim\limits_{x\to\infty} (xe^{1/x} - (ax + b)) = 0$.

b. Sketch the graph of $y = xe^{1/x}$.

3. Find $\displaystyle\int_{\pi/2}^{3\pi/2} f(x) \, dx$ if $f'(x) = \dfrac{\cos x}{x}$, $f(\pi/2) = a$ and $f(3\pi/2) = b$.

4. Evaluate the following integrals:

a. $\displaystyle\int \frac{1}{2x^2 + 5x + 2} \, dx$

b. $\displaystyle\int x \tan^2 x \, dx$

5. Evaluate the integral $\displaystyle\int_{-\infty}^{\infty} \frac{1}{e^x + ae^{-x}} \, dx$ where a is a positive constant.

Fall 1997 Midterm I

1. Assume the functions f and g have the following properties: for every ε and x,

$$\text{if } 0 < |x - 2| < \frac{3\varepsilon}{2} \text{ then } |f(x) - 3| < \varepsilon \, ,$$
$$\text{and if } 0 < |x - 2| < \varepsilon + \sqrt{\varepsilon} \text{ then } |g(x) - 4| < \varepsilon \, .$$

Find a number $\delta > 0$ such that for all x,

$$\text{if } 0 < |x - 2| < \delta \text{ then } |f(x) + g(x) - 7| < \frac{1}{50} \, .$$

2a. Evaluate the limit $\lim\limits_{x\to 0} \dfrac{\cos^3 3x - \cos^3 2x}{x^2}$.

2b. Find $\dfrac{d^2y}{dx^2}$ by implicit differentiation at the point $(1,-1)$: $x\cos^2(\pi y)+y^3=0$.

3. Graph the function $f(x)=\dfrac{x^3-x^2+3}{x^2-3}$ by computing f' and f'', determining their signs, finding the critical points and inflection points, indicating the intervals on which the function is increasing, decreasing, concave up, down, and finding the asymptotes.

4. A sphere of radius r is inscribed into a cone. Find the radius and the height of the cone (in terms of r) if its volume is to be minimum.

5. A point A is moving in the counterclockwise direction along the ellipse $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$ so that the ray OA has constant angular speed of 6 radians per second (where O is the origin). Find the rate of change of the area of the triangle formed by OA, the x-axis and the perpendicular from A to the x-axis at a moment when A is $(-12/5, 12/5)$.

Fall 1997 Midterm II

1. Let the function f be defined as follows:

$$f(x)=\begin{cases}\int_0^x t\sin(1/t)\,dt & \text{if } x\neq 0\,,\\ 0 & \text{if } x=0\,.\end{cases}$$

Find $f'(0)$.

2. a. Evaluate the limit: $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)-x+x^2/2}{x^3}$.

b. Find $\dfrac{dy}{dx}$ where $y=(\ln x)^{\ln x}$.

3. Let C_1 be the graph of $y=x^m$, C be the graph of $y=kx^m$, and C_2 be the graph of $y=f(x)$ where $m>0$ and $k>1$ are constants, and f is an increasing continuous function with $f(0)=0$. Find f if C bisects C_1 and C_2 in the following sense: for every (a,b) on C, the area of the region between C and C_1 for $0\le x\le a$ is equal to the area of the region between C and C_2 for $0\le y\le b$.

4. The top and the bottom of a truncated rectangular pyramid of height h have sides of length a and b, and A and B, respectively. Find the volume.

5. The plane region between the parabolas $y=4-x^2$ and $y=8-2x^2$ has density $\delta(x,y)=3y$. Find the coordinates of the center of mass.

Fall 1997 Final

1. Evaluate the following limits:

a. $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$.

b. $\lim_{n\to\infty}\displaystyle\int_0^n \frac{(1+\pi/x)^x}{n}\,dx$.

2. Evaluate the following integrals. You may not use any reduction formulas or formulas from the integration tables.

a. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2+2x}}$.

b. $\displaystyle\int_0^\infty \frac{x^2+x}{(x^2+1)^2}\,dx$.

3. Let R be the region in the xy-plane bounded by the curve $y = 1/x^{3/2}$, the x-axis, the lines $x = u$ and $x = u^2$ for $u \geq 1$. Assume that R is revolved about the x-axis and a solid is generated. Let $V(u)$ denote the volume of this solid. Find the maximum value of $V(u)$ for $u \geq 1$.

4. An irregularly shaped water container is 5 units deep. It has a flat bottom and cross-sectional areas $A(z) = (z^3 + 2z + 15)^{1/3}$ units^3 where z is the depth measured from the bottom and the cross-sections are taken by planes parallel to the bottom. Initially the container is full of water. Then the water is pumped out of the container at the constant rate of 0.3 units^3/sec. How fast is the depth of the water decreasing when water is 2 units deep?

5. a. Determine whether the following improper integral converges or diverges:

$$\int_0^\infty \sqrt{x^2+1}e^{-x}\,dx$$

b. Let $p > q > 0$ be constants. Find all possible values of p and q for which the following improper integral is convergent:

$$\int_0^\infty \frac{dx}{x^p + x^q}$$

Fall 1996 Midterm I

1. Show that $(1+x)^\alpha < 1 + \alpha x$ for all $x > 0$ and $0 < \alpha < 1$.

2. Find $(f \circ g \circ h)'(0)$ if $f(x) = \cos x$, $g(y) = \dfrac{\pi}{6}y^2 + y - 1$ and $h(z) = \dfrac{1}{\sqrt{1+z+z^2}}$.

3. Consider the curve $y = \dfrac{3x^2+1}{2x^2+5}$. Find the points where the slope of this curve assumes its absolute maximum and absolute minimum.

4. A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be use if the volume is fixed and the cost of construction is to be kept to a minimum.

5. Let

$$f(x) = \begin{cases} 2x + x^2 \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Find $f'(x)$ for all x. Is f' continuous at $x = 0$? Explain.

Fall 1996 Midterm II

1. Find $y'(1)$ if $x^y + y^x = 3$.

2. Find the area of the surface of revolution obtained by rotating the curve $y = \frac{1}{2}\left(\frac{1}{1-\alpha}\,x^{1-\alpha} - \frac{1}{1+\alpha}\,x^{1+\alpha}\right)$, $0 \leq x \leq 1$, about the x-axis where $0 < \alpha < 1$.

3. Evaluate the integral $\displaystyle\int_0^1 \frac{x^2+1}{x^2-x-6}\,dx\,.$

4. Evaluate the limit $\displaystyle\lim_{n\to\infty}\left(\ln n - \frac{1}{n}\sum_{k=n+1}^{2n} \ln k\right).$

5. Evaluate the limit $\displaystyle\lim_{x\to\infty}\left(x\int_0^1 \frac{e^{-xt}}{1+t^2}\,dt\right).$

Fall 1996 Final

1. If $f(x) = x\displaystyle\int_e^x \frac{dt}{\ln t}$ for $x > 1$, find $f''(e)$.

2. For $0 \leq t \leq 1$, let $A(t)$ denote the area of the triangle bounded by the x-axis, the y-axis and the tangent line to the curve $y = \ln x$ at $(t, \ln t)$. Find the maximum value of $A(t)$.

3. Sketch the graph of $y = x(\ln x)^2$.

4. [missing]

5. Evaluate the improper integral $\displaystyle\int_1^\infty \left(\frac{1}{x} + \tan^{-1} x - \frac{\pi}{2}\right) dx.$

Fall 1995 Midterm I

1. Let $A(0, t^2/\sqrt{3})$ and $B(4, 0)$ where the time $0 \le t < \infty$ is measured in seconds. Let θ be the measure of the angle OBA. At what times is θ changing the fastest and the slowest?

2. Find y'' at $x = 0$ if $\sin xy + x^2 + y = \pi$.

3. Find the constant k for which the limit

$$\lim_{x\to 0} \frac{\sin(x + kx^3) - x}{x^5}$$

exists, and find the value of the limit then for this k.

4. You are drawing up plans for the piping that will connect a drilling rig 30 kilometers offshore to a refinery on shore 35 kilometers down the cost. How far away from the refinery should the pipeline reach the coast to give the least expensive connection if underwater pipe costs 5 BTL per kilometer and land based pipe costs 4 BTL per kilometer. (1 BTL=1 billion Turkish Liras.)

5. Graph the function $y = \dfrac{3x^3 - 2x}{x^2 - 1}$ by finding its asymptotes, local minimums and maximums, inflection points and intercepts.

Fall 1995 Midterm II

1. Let D_n denote the region that is under the curve $y = \dfrac{1}{x^2\sqrt{x^2 - 1}}$, above the x-axis and between the lines $x = 1 + \dfrac{1}{n}$ and $x = n$, where n is a positive integer. Revolve the region D_n around the y-axis and denote the volume of the resulting solid by V_n. Find $\lim\limits_{n\to\infty} V_n$.

2. For which value of α, $\displaystyle\int_{\pi/6}^{\pi/3} (\log_\alpha 2)(\log_2 \sin x)' \tan x \, dx = \frac{\pi}{6}$?

3. Evaluate the limit $\displaystyle\lim_{x\to\infty} x \int_0^x e^{t^2 - x^2} \, dt$.

4. Consider the region inside the circle $x^2 + (y - 2)^2 = 1$ and between the lines $x = -1/2$ and $x = 1/2$. Find the surface area of the solid obtained by revolving this region around the x-axis.

5. Graph the function $y = e^{1/x}$ by finding its

a. asymptotes and $\lim\limits_{x\to 0^-} y'$,

b. local minimums and maximums,

c. inflection points,

d. making a table of the relevant values.

Fall 1995 Final

1a. Explain whether $\displaystyle\int_{-\infty}^{\infty} \frac{x^2}{e^x + e^{-x}}\, dx$ converges or diverges.

1b. Evaluate $\displaystyle\int_{10}^{\infty} \frac{dx}{x(\ln x)(\ln \ln x)}$.

2a. Find $\displaystyle\lim_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$.

2b. Given that $t \to \infty$, simplify $(t - t^{1/2} + O(1)) \cdot (t^{1/2} + 1 + O(t^{-1/2}))$.

2c. Explain briefly whether true or false: As $n \to \infty$,

$$\frac{1}{n+1} - \frac{1}{n+2} = o(n^{-3/2}) \ .$$

3. According to Newton's law of cooling, the rate at which the temperature of an object is changing at any given time is proportional to the difference between its temperature and the temperature of its environment (which is assumed to be constant). A pan of water at 25°C was put in a refrigerator. Ten minutes later, the temperature of the water was 13°; 20 minutes after that, it was 4°. Use Newton's law of cooling to calculate how cold the refrigerator was.

4. Evaluate $\displaystyle\int_{1/2}^{2} \frac{dx}{x^3 + 6x + 7}$.

5. The region bounded by the curve $y = (\arctan x)^2$, the x-axis and the line $x = 1$ is revolved about the y-axis to generate a solid. Find the volume of this solid.

Fall 1994 Midterm I

1. The graph of f is as shown. Draw the graph of the function $g(x) = \dfrac{f(x)}{1 + f(x)}$. Indicate the asymptotes, extrema and the inflection points.

[The picture shows a graph $y = f(x)$ such that $f(-4) = 0$, $f(-3) = -1$, $f(-2) = -2$, $f(-1) = -1$, $f(0) = 0$, $f(2) = 2$, $f(3) = 0$, $f(7/2) = -1$; f is increasing on $[-2, 2]$ and decreasing on $(-\infty, -2]$ and $[2, \infty)$. It is not clear from the picture if f is piecewise-linear.]

2. Find all positive integer values of n for which $\cos(x^{2/n})$ is differentiable at $x = 0$.

3. Estimate $\sin 29°$ by using the quadratic approximation. Give an upper bound for the error.

4. How many real solutions to $x^3 + 3x - 5 = 0$ are there? Apply Newton's Method two steps to locate any root.

5. A straight line passing through the point $(1/2, 1)$ divides the rectangular region with corners at $(0,0), (2,0), (2,4), (0,4)$ into two subregions. Among all triangular subregions with the origin as one vertex that are formed by such a division, find the ones which have maximum and minimum areas.

Fall 1994 Midterm II

1a. Given that $g(x) = \displaystyle\int_1^{f^{-1}(x)} \sqrt{1+t^3}\, dt$ and $f(3) = 2$, $f'(3) = 5$, find $g'(2)$.

1b. Find the volume of the solid described as follows: The base of the solid is the region between the curve $y = 3\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are squares.

2a. Find $\displaystyle\lim_{n\to\infty} \left(\frac{1}{5n^2+1} + \frac{1}{5n^2+2} + \cdots + \frac{1}{6n^2} \right)$.

2b. Find $\displaystyle\lim_{x\to\infty} \frac{\sqrt{\ln x}}{x} \int_2^x \frac{dy}{\sqrt{\ln y^2}}$.

3. Find the centroid of the quarter ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, $x, y \geq 0$.

4. Solve the initial value problem:

$$\frac{1}{y+2}\frac{dy}{dx} = x\cos^2 x^2, \qquad y = e^2 - 2 \text{ when } x = 0\,.$$

5a. Estimate $\ln 0.6$ by quadratic approximation.

5b. Estimate $\ln 0.6$ by using Simpson's rule with $n = 4$.

Fall 1994 Final

1a. Evaluate $\displaystyle\int_0^{\ln\sqrt{3}} e^{-x} \arctan(e^x)\, dx$.

1b. Evaluate $\displaystyle\int \frac{dx}{(x^2+2x+2)^{3/2}}$.

2. Evaluate $\displaystyle\int x^5 e^{x^3}\,dx$.

3a. Determine whether $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^3+1}}$ converges or diverges. Give reasons.

3b. For a certain number A the integral $\displaystyle\int_2^\infty \left(\frac{Ax}{x^2+1} - \frac{1}{2x+1}\right) dx$ converges. Find that value of A and the integral.

4. A boat is going in a straight line at 80 m/min when the engine of the boat is shut off. The boat is subject to a deceleration (i.e. negative acceleration) proportional to its speed. One minute later the speed of the boat is reduced to 40 m/min. How far has the boat drifted in that one minute?

5. Consider $y = \operatorname{sech} x$.

a. Plot the graph of the function by considering the asymptotes and the signs of y' and y''.

b. Find the quadratic approximation of $y = \operatorname{sech} x$ about $x = 0$.

c. Find the volume of the solid obtained by revolving the region between $y = \operatorname{sech} x$ and the x-axis about the x-axis.

Fall 1993 Midterm I

1a. Let $f(x) = x + \dfrac{\sin x}{2x - \dfrac{12}{x-1}}$.

i. Find all the points at which f is discontinuous.

ii. Find all discontinuities at which f has a continuous extension.

1b. Let f be a continuous function and assume that for each $0 \le x \le 1$ we have $0 \le f(x) \le 1$. Show that the equation $f(x) = x$ has a solution in the interval $[0, 1]$.

2a. Let $y = t + \sin^2 u^2$ where at $t = 0$, $u = \sqrt{\dfrac{\pi}{3}}$ and $\dfrac{du}{dt} = 1$. Find y and $\dfrac{dy}{dt}$ at $t = 0$.

2b. Consider the equation $\cot \pi x = x$.

i. Show that this equation has exactly one solution r_k in $I_k = (k, k+1)$ for every $k \in \mathbb{Z}$.

ii. Apply Newton's method once to estimate r_k.

3a. Let $f(x) = \sqrt{x}$.

i. Find the quadratic approximation of $f(x)$ for $x \approx 1$.

ii. Determine the accuracy of this approximation when $\dfrac{3}{4} \le x \le \dfrac{5}{4}$.

3b. Show that the equation $x^8 + x - 1 = 0$ has exactly two real roots.

4a. Evaluate $\lim_{x\to\infty} (\sqrt{3x^2 + 2x - 1} - \sqrt{3x^2 - x + 5})$.

4b. Let $f(x) = x^{-3} \sin 3x + Ax^{-2} + B$ where A and B are constants. Determine the values of A and B for which $\lim_{x\to 0} f(x) = 2$.

5. Graph the function $f(x) = \dfrac{x^2}{4 + x|x|}$ by computing f' and f''; determining their signs; finding the critical points and the inflection points; indicating the intervals on which the graph is rising, falling, concave up, down; and finding the asymptotes.

6. If the sum of the volumes of a cube and a sphere is held constant, what ratio of an edge of the cube to the radius of the sphere will make the sum of the surface areas

a. as small as possible?

b. as large as possible?

Fall 1993 Midterm II

1a. Find $\lim_{n\to\infty} \left(\dfrac{1}{n+1} + \dfrac{1}{n+2} + \cdots + \dfrac{1}{3n} \right)$.

1b. Given that $f(1) = 3$, $f(2) = 5$, $f(4) = 7$ and $f(14) = 23$, evaluate $\int_1^2 (x^2 + 1) f'(x^3 + 3x)\, dx$.

2a. Find $\lim_{y\to 0} \dfrac{1}{y} \int_0^y (\cos 2t)^{1/t^2}\, dt$.

2b. If $x^2 y'' = 1$ for all $x > 0$ and $y' = -1$, $y = 1$ when $x = 1$, find y for all $x > 0$.

3a. Estimate $\ln(1+2x)$ about $x = 0$ by using the quadratic approximation and calculate $\ln(1.4)$ approximately.

3b. Estimate $\ln(1.4)$ using Simpson's method by choosing 4 subintervals.

4a. The curve $x = y^{1/2} - y^{3/2}/3$ where $0 \le y \le 1$ is rotated about the y-axis. Find the area of the surface of revolution.

4b. Find the x-coordinate of the centroid of the curve in part (a).

5a. Find the area of the region bounded by the curves $y = \tan x$, $y = 0$, $y = 1$ and $x = \pi/2$.

5b. Find the volume of the solid generated by revolution of the region in part (a) about the x-axis.

6a. Plot the graph of the function $y = x^2 e^x$.

6b. Find $\displaystyle\int \frac{dx}{1+2e^{-3x}}$.

Fall 1993 Final

1. a. Draw the graph of $y = \operatorname{sech} x$ by computing y', y'', and finding the critical points, the inflection points, and the asymptotes.

b. Find the area between the curves $y = \operatorname{sech} x$ and $y = \dfrac{3}{4}\cosh x$.

2. Find the volume generated by revolving the region bounded by $y^2 = x^5 - x^8$ about the y-axis.

3. a. Find the quadratic approximation of the function $f(x) = x^x - \dfrac{1}{2-x}$ around $x = 1$.

b. Estimate the value of x^x for $x = 10/11$.

4. Let $f(x) = \displaystyle\int_1^e t^x (\ln t)^2\, dt$.

a. Compute $f(x)$.

b. Prove that f is continuous at $x = -1$.

5. Evaluate $\displaystyle\int_0^\infty \frac{x\,dx}{(x+1)(x^2+2)}$.

6. a. Find y as a function of x if $x\dfrac{dy}{dx} = \tan y$ and $y = \dfrac{\pi}{6}$ when $x = \dfrac{1}{2}$.

b. Prove that

$$\int_0^x \left(\int_0^u f(t)\,dt \right) du = \int_0^x f(u)(x-u)\,du\ .$$

Fall 1992 Midterm I

1. Let $f(x) = g(x + g(x))g(x)$ where g is such that $g(0) = 2$, $g(1) = 1$, $g(2) = -5$, $g'(0) = -3$, $g'(1) = 7$, $g'(2) = 4$. Find $f'(0)$.

2. Find the normal to the curve $y = 4x^2 + \sin(\pi \tan(x^2))$ at the point where $x = \dfrac{\sqrt{\pi}}{2}$.

3a. Consider $f(x) = \sqrt{x^2 + x} - \sqrt{x^2 - x}$.

i. Find the domain of the function f.
ii. Evaluate $\lim\limits_{x\to-\infty} f(x)$.

3b. Evaluate $\lim\limits_{x\to 0} \dfrac{\sin\alpha x - \tan\alpha x}{x^3}$ where α is a constant.

4. a. Find the values of α, β, γ for which the function

$$f(x) = \begin{cases} 2 & \text{if } x = 0, \\ -x^2 + 4x + \alpha & \text{if } \ 0 < x < 1, \\ \beta x + \gamma & \text{if } \ 1 \le x \le 3. \end{cases}$$

is differentiable on $[0, 3]$.

b. Find the local maxima and minima of f for these values of α, β, γ.

5. A 3 m long wire is used for making a circle and an equilateral triangle. How should the wire be distributed between the two shapes to maximize the sum of the enclosed areas?

6. Graph the function $f(x) = \dfrac{x}{x^3+1}$ by computing f' and f''; determining their signs; finding the critical points and the inflection points; indicating the intervals on which the graph is rising, falling, concave up, down; and finding the asymptotes.

Fall 1992 Midterm II

1. a. Find the quadratic approximation at $x = 0$ of

$$f(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \ .$$

b. Find the error in this approximation for $|x| \le 1$.

2. a. A solid is generated by revolving the region bounded by the graph of $y = f(x)$, the x-axis, the lines $x = 1$ and $x = a$ about the x-axis. Its volume, for all $a \ge 1$, is $a^2 - a$. Find $f(x)$.

b. A surface is generated by revolving the curve $y = \dfrac{\sqrt{x}}{3}(3 - x)$, $0 \le x \le 3$, about the x-axis. Find its area.

3. a. Using the Mean Value Theorem for Definite Integrals, find upper and lower bounds for the value of $\displaystyle\int_0^1 \frac{dx}{x^3+1}$.

b. Obtain better bounds by considering

$$\int_0^1 \frac{dx}{x^3+1} = \int_0^{1/2} \frac{dx}{x^3+1} + \int_{1/2}^1 \frac{dx}{x^3+1}$$

and using the same theorem for the integrals on the right.

4. Evaluate:

a. $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x}\right)^{1/x^2}$

b. $\lim\limits_{x\to 0}\dfrac{x}{5^x-3^x}$

5. Evaluate:

a. $\displaystyle\int_0^{\pi/2}\frac{\sin x\,dx}{3+\cos^2 x}$

b. $\displaystyle\int_e^{e^2}\frac{dx}{x\ln x}$

6. a. Show that the difference of the functions $f(x)=\sin^{-1}\left(\dfrac{x^2-1}{x^2+1}\right)$ and $g(x)=2\tan^{-1}x$ is a constant for $x\geq 0$.

b. Find this constant.

Fall 1992 Final

1. Let f and g be continuous on $[a,b]$ and that $f(a)\leq g(a)$ and $f(b)\geq g(b)$. Show that there exists a point c in $[a,b]$ such that $f(c)=g(c)$.

2. a. Given that

$$\int_0^{\pi}(f(x)+f''(x))\sin x\,dx=A\ \text{ and }\ f(\pi)=B\ ,$$

find $f(0)$ in terms of A and B.

b. Let

$$f(x)=\int_0^x\frac{dt}{\sqrt{1+t^3}}\ \text{ for } x\geq 0\ ,$$

and let $g=f^{-1}$. Show that g'' is proportional to g^2.

3. a. Show that

$$\Gamma(x)=\int_0^{\infty}e^{-x}t^{x-1}\,dt$$

converges for all $x>0$.

b. Show that $\Gamma(x+1)=x\Gamma(x)$ for $x>0$.

4. a. Given

$$f(x) = \begin{cases} \dfrac{g(x)}{x} & \text{for } x \neq 0\,, \\ 0 & \text{for } x = 0\,, \end{cases}$$

and $g(0) = 0$, $g'(0) = 0$, $g''(0) = 17$, find $f'(0)$.

b. Evaluate the limit $\lim\limits_{x \to 0^+} x^{\arcsin x}$.

5. Evaluate the integral $\displaystyle\int \ln(\sqrt{x} + \sqrt{1+x})\, dx$.

6. A garden is to be designed in the shape of a circular sector with radius R and the central angle θ. The garden is to have fixed area A. For what values of R and θ will the length of the fencing around the perimeter be minimized?

Fall 1991 Midterm I

1. Use the ε-δ definition to show that $\lim\limits_{x \to 2}(x^2 + 1) = 5$.

2. Let

$$f(x) = \begin{cases} g(x)\cos(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

where g is a differentiable function with $g(0) = g'(0) = 0$. Show that f is differentiable at $x = 0$ and $f'(0) = 0$.

3. Find the local minimum, local maximum and the inflection points of the function

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 1, \\ (2-x)^3 & \text{if } x > 1. \end{cases}$$

4. Sketch the graph of $y = \dfrac{2 + x - x^2}{(x-1)^2}$.

5. Let $y = \sin x + \cos x$ and

$$z = \sin\left(\frac{y^3}{\sin\left(\dfrac{y^3}{\sin y}\right)}\right) .$$

Find the derivative of z with respect to x at the point $\dfrac{\pi}{4}$.

6. An isosceles triangle is inscribed into a circle of radius r. If the top angle of the triangle is required to lie between 0 and $\dfrac{\pi}{2}$, find the largest and the smallest values possible of the perimeter of the triangle.

Fall 1991 Midterm II

1. Evaluate the following limits:

a. $\lim_{x\to 0} \dfrac{\int_0^x \sin t^3\, dt}{\sin x^4}$

b. $\lim_{x\to 0} (e^x + x)^{1/x}$

2. Evaluate the following integrals:

a. $\displaystyle\int \frac{x^{49}}{1+x^{100}}\, dx$

b. $\lim_{x\to 0} \dfrac{dx}{\sqrt{e^{2x}-1}}$

3. Let $0 < a < b$. Show that

$$\frac{b-a}{1+b^2} < \tan^{-1} b - \tan^{-1} a < \frac{b-a}{1+a^2}\ .$$

4. Find the area of the surface generated by revolving the curve $x = t^2$, $y = t^3/3 - t$, $-\sqrt{3} \le t \le \sqrt{3}$, about the y-axis.

5. The region between the graphs of $y = x^2$ and $y = \dfrac{1}{2} - x^2$ is revolved about the y-axis to form a lens. Compute the volume of the lens.

6. A point $P(t)$ is moving along the x-axis in the positive direction at a speed of 5 cm/sec, and is at the origin at time $t = 0$. Find the rate of change, at time $t = 0$, of the area under the curve $y = (\tanh^2 x + \cos^3 x)^4$ and above the interval $[P(t), 100]$.

Fall 1991 Final

1. Discuss the continuity of the following functions in their domain of definition:

a. $f(x) = \begin{cases} \dfrac{1}{1+e^{1/x}} & \text{if } x \ne 0\,, \\ 1 & \text{if } x = 0\,. \end{cases}$

b. $f(x) = \begin{cases} \dfrac{x-|x|}{x} & \text{if } x < 0\,, \\ 2 & \text{if } x = 0\,. \end{cases}$

3. Evaluate

$$\lim_{n\to\infty} \left(\ln \sqrt[n]{1+\frac{1}{n}} + \ln \sqrt[n]{1+\frac{2}{n}} + \cdots \ln \sqrt[n]{1+\frac{n}{n}} \right)$$

by relating it to a definite integral.

4. Evaluate:

a. $\displaystyle\int x^3 e^{x^2}\, dx$

b. $\displaystyle\int \frac{\sqrt{x}}{\sqrt{1-x^3}}\, dx$

5. Evaluate $\displaystyle\int \frac{x}{\sqrt{x^2+2x+5}}\, dx.$

6. Evaluate $\displaystyle\int \frac{x\, dx}{(x+1)^2(x^2+1)}\ .$

7. Find the volume of the solid generated by revolving the region R bounded by $y = x^3 - 8x$ and $y = -x^2 + 4x$ for $0 \leq x \leq 3$ about the line $x = -1$.

8. The continuously differentiable curve $y = f(x)$ passes through the point $(0, 0)$, and for every $x > 0$ the length of the curve from $(0, 0)$ to $(x, f(x))$ is $e^x + f(x)$. Find $f(x)$.

9. a. Evaluate $\displaystyle\lim_{x\to\infty} \left(1 - \frac{1}{x} + \frac{1}{x\sqrt{x}}\right)^{2x}.$

b. Differentiate $\log_x(x^2+5)$ with respect to x.

Fall 1990 Midterm I

1. Discuss the existence of the limit where for any real number a, denotes the greatest integer in a.

2. A function f satisfies the following condition at a point $x = x_0$: There are numbers $M > 0$, $\beta > 0$ and α such that

$$0 < |x - x_0| < \beta \Longrightarrow |f(x) - f(x_0)| < M|x - x_0|^\alpha\ .$$

Then show that

a. if $\alpha > 0$, then f is continuous at $x = x_0$.

b. if $\alpha > 1$, then f is differentiable at $x = x_0$.

3. Find the equation of the line through the point $(3, 0)$ that is normal to the parabola $y = x^2$.

4. Let $f(x) = \sqrt{x}$. Find a formula for the nth derivative $f^{(n)}$ and prove your formula using induction.

5. Let $g(x) = 1 + \sqrt{x}$ and $(f \circ g)(x) = 3 + 2\sqrt{x} + x$. Find $f'(2)$ by

a. by first solving for f and then differentiating,

b. by the chain rule.

6. Consider the equation $2 \sin x = x$.

a. Show that the equation above has a solution $c \in \left(\dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$.

b. Can we find this solution by using Picard's method? If your answer is YES, find an interval I such that every $x_0 \in I$ is a valid initial value for Picard's method.

Fall 1990 Midterm II

1. A rectangle of fixed perimeter P is rotated about one of its sides to generate a cylinder. Of all such possible rectangles, find the dimensions of the one which generates the cylinder of greatest volume.

2. For $a > 1$ show that $\dfrac{a-1}{a} \le \ln a \le a - 1$.

3. Evaluate:

a. $\displaystyle\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$

b. $\displaystyle\lim_{x \to 0^+} \frac{\ln(\tan 2x)}{\ln(\tan 3x)}$

4. Find the volume of the solid generated by revolving the region bounded by the graphs of $y = 4 - x^2$ and $y = 0$ about the line $x = 3$.

5. Show that the length of the sine curve $y = \sin x$ for $0 \le x \le \pi$ is equal to half of the circumference of the ellipse $2x^2 + y^2 = 2$.

6. Evaluate:

a. $\displaystyle\int e^{(x+e^x)}\, dx$

b. $\displaystyle\int_0^4 \frac{2dx}{\sqrt{x}(4+\sqrt{x})}$

Fall 1990 Final

1. When y is given by the following relation find y'' in terms of x and y. Simplify your answer.

$$\ln(x-y) = x+y$$

2. Calculate the following limit: $\lim\limits_{x\to 0}\left(\dfrac{3}{x^4}+\dfrac{1}{x^2}-\dfrac{3\tan x}{x^5}\right)$.

3. Find the following antiderivative: $\displaystyle\int \sqrt{4+3^x}\,dx$.

4. Find the following antiderivative: $\displaystyle\int \ln(x+\sqrt{1+x^2})\,dx$.

5. Find the area of the surface obtained by revolving $y=\sin x$ around the x-axis for $0\le x\le\pi$.

6. A rectangle with sides of length a and b is inscribed in a isosceles triangle in such a way that one of the sides with length b is contained in the base of the triangle. Find the minimum possible area of triangle.

7. A point$P$ at the origin of the xy-plane is attached by a string of length a to a mass M located at the point $(a,0)$. The point P then moves up the y-axis, dragging the mass M by the string. As it moves, the mass traces a differentiable curve $y=f(x)$. Find the equation of the curve $y=f(x)$.

8. At points of the curve $y=2\sqrt{x}$, lines of length $h=y$ are drawn perpendicular to the coordinate plane. Find the area of the surface formed by these lines from $(0,0)$ to $(3,2\sqrt{3})$.

Fall 1989 Midterm

1. Let $f(x)$ be a differentiable function on $(0,\infty)$ and assume that $\lim\limits_{x\to 0}\dfrac{f(x)}{Ax^4+Bx^3}=C$, where A, B, C are constants and $A\neq 0$.

   a. Find $\lim\limits_{x\to 0}\dfrac{f(x)}{x^2}$.

   b. Let

$$H(x)=\begin{cases}\dfrac{f(x)}{x} & \text{if } x>0,\\ 0 & \text{if } x=0,\\ -\dfrac{f(-x)}{x} & \text{if } x<0.\end{cases}$$

Show that $H'(0)$ exists, and find its value.

2. Let $f_1(x)$, $f_2(x)$ and $f_3(x)$ be differentiable functions on $\mathbb{R}$ such that $f_i(m)=4-i+m$ and $f_i'(m)=i+m$ where $i=1,2,3$ and m is an integer. Let $F(x)=(f_1\circ f_2\circ f_3)(x)$. Find $F'(1)$.

Remark: The data above is enough to solve this problem. However if you want to know if such functions do exist, here are three functions satisfying these conditions: $f_i(x) = 4 - i + x + \dfrac{i-1}{2\pi}\sin 2\pi x + \dfrac{1}{4\pi}\sin 2\pi x^2$, $i = 1, 2, 3$. Note that you should solve the problem without using this example.

3. Sketch the graph of $y = \dfrac{x^3 - x^2 + 4}{x - 1}$.

4. Let $P(x)$ be a polynomial of degree n, and assume that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Show that for all $x \in \mathbb{R}$,

$$P(x) + P'(x) + P''(x) + \cdots + P^{(n)}(x) \geq 0 \ .$$

5. Assume that in an analogue clock the minute arm is twice as long as the hour arm. Find the first time of the day when tips of the minute arm and the hour arm are moving away from each other the fastest.

Remark: You may assume that the arms of the clock lie in the same plane.