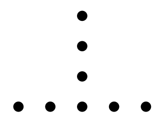
Math 132-04: Discrete Mathematics
Problem session 1

1. How many triangles have their vertices contained in the set of dots below? How many quadrilaterals?



2. How many anagrams (i.e., reorderings of letters) are there of the word ANAGRAM?

3. How many anagrams of MISSISSIPPI have no consecutive Ss?

4. Show that for any $n \geq 1$, we have
$$\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \ldots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \ldots.$$
Here, we note that $\binom{n}{k} = 0$ for $k > n$, so both sides of the equation are finite.

5. Show that for any $n, m, r \in \mathbb{N}$ with $r \leq n$ and $r \leq m$, we have
$$\binom{n+m}{r} = \sum_{i=0}^{r} \binom{n}{i} \cdot \binom{m}{r-i}.$$
This result is known as Vandermonde's Identity. Try to argue using sets instead of playing with formulas (it will be much easier).

6. We define a *multiset* to be a set in which we are allowed to have repeated elements. If we denote a multiset by angular brackets $\langle\ \rangle$, we then have $A = \langle 1, 1, 2, 3 \rangle$ and $B = \langle 1, 2, 2, 3, 3, 3 \rangle$ are different multisets, even though the distinct elements that appear in A and B are the same set $\{1, 2, 3\}$. This definition can be made precise by saying that the elements of a multiset are pairs (x, n), where x is an element and n a natural number that indicates its multiplicity. This is similar to the `Counter` datatype in Python, for instance.

   Let $\left(\!\binom{n}{k}\!\right)$ denote the number of k-element multisubsets drawn fro a set of size n. If $X = \{1, 2, 3\}$, then the 3-element multisubsets of X are
$$\langle 1,1,1\rangle, \langle 1,1,2\rangle, \langle 1,1,3\rangle, \langle 1,2,2\rangle, \langle 1,2,3\rangle, \langle 1,3,3\rangle, \langle 2,2,2\rangle, \langle 2,2,3\rangle, \langle 2,3,3\rangle, \langle 3,3,3\rangle.$$
   Note that the order of the elements still doesn't matter, so $\langle 1, 1, 2 \rangle = \langle 1, 2, 1 \rangle$.

   (a) Show:
   - $\left(\!\binom{n}{0}\!\right) = 1$ for all $n \in \mathbb{N}$.
   - $\left(\!\binom{0}{k}\!\right) = 0$ for all $k \in \mathbb{N}_{\geq 1}$.
   - $\left(\!\binom{n}{1}\!\right) = n$ for all $n \in \mathbb{N}$.
   - $\left(\!\binom{1}{k}\!\right) = 1$ for all $k \in \mathbb{N}$.
   - $\left(\!\binom{n}{2}\!\right) = \binom{n}{2} + n$ for all $n \in \mathbb{N}$.

   (b) Verify the recursive relation
$$\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n-1}{k}\!\right) + \left(\!\binom{n}{k-1}\!\right)$$
   for all $(n, k) \neq (0, 0)$. Can you use this to see Pascal's triangle in an array of the $\left(\!\binom{n}{k}\!\right)$?

   (c) What counting problem from class does $\left(\!\binom{n}{k}\!\right)$ solve? Can you find a closed-form expression of $\left(\!\binom{n}{k}\!\right)$ in terms of n and k?