Linear systems

Last time we saw that every homogeyeous system with more variables than equations has infinitely many solutions

Example The system

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 1$$

is inconsistent

Result Every nonhomogeneous system with more variables than equations either has no solution or has infinitely many solutions

Consider

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$ square matrix

$a_{11}, a_{22}, \ldots, a_{nn}$ Form the principal diagonal of A

An identity matrix, denoted by I, is a square matrix that has ones on its principal diagonal and zero elsewhere

<u>Examele</u> $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Example   Solve

$$x_1 + x_2 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$x_2 + 3x_3 = 0$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{(-1)R_1 + R_2}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \end{bmatrix} \xrightarrow{(-1)R_2 + R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

$$\xrightarrow{\left(\frac{1}{2}\right)R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{(-1)R_2 + R_1} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow{(1)R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{(-1)R_3 + R_2}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$ reduced echelon

$$\left.\begin{aligned} x_1 &= 0 \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}\right\}$$

$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix}$ is row equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

Theorem Let A be an $n \times n$ matrix. Then the homogeneous system with coefficient matrix A has only the trivial solution if and only if A is row equivalent to the $n \times n$ identity matrix

Exercise Determine for what values of k the system

$$3x_1 + 2x_2 = 1$$
$$6x_1 + 4x_2 = k$$

has (a) a unique solution
(b) no solution
(c) infinitely many solutions

Solution The augmented matrix of the system is

$$\begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & k \end{bmatrix} \xrightarrow{(-2)R_1 + R_2} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & k-2 \end{bmatrix}$$

There are 2 cases:

(1) $k \neq 2$ Then $k-2 \neq 0$ and therefore the system is inconsistent

(2) $k = 2$. Then

$$\begin{bmatrix} 3 & 2 & 1 \\ 6 & 4 & k \end{bmatrix} \xrightarrow{(-2)R_1 + R_2} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

x_1 is the leading variable and x_2 is a free variable

$x_2 = s$

$x_1 = \frac{1}{3} - \frac{2s}{3}$

In this case the system has infinitely many solutions

Note that the system does not have unique solution for any k

Matrix Operations

(1) write a linear system of m equations with n unknowns in the form $A \cdot X = b$

(2) Solve $A \cdot X = b$ by using the inverse A^{-1} (here $m = n$).

<u>Recall</u> that an $m \times n$ matrix is a rectangular array of the form

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

<u>Shorthand notation</u> $A = [a_{ij}]$

If $m = n$, then $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$

is a square matrix and $a_{11}, a_{22}, \ldots, a_{nn}$ form the principal diagonal of A

<u>Example</u> $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ square

<u>Definition</u> Two $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are equal if they agree entry by entry, namely $a_{ij} = b_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$. In this case we write $A = B$

Example $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$,

$C = \begin{bmatrix} 3 & 4 & 7 \\ 5 & 6 & 8 \end{bmatrix}$. Clearly $A \neq B$ and also $A \neq C$

Definition Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two $m \times n$ matrices. Then the sum $A + B$ is the matrix

$$A + B = [a_{ij} + b_{ij}]$$

Example $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -7 & 5 \end{bmatrix}$,

$B = \begin{bmatrix} 4 & -3 & 6 \\ 9 & 0 & -2 \end{bmatrix}$

$$A + B = \begin{bmatrix} 7 & -3 & 5 \\ 11 & -7 & 3 \end{bmatrix}$$

Definition If $A = [a_{ij}]$ is a matrix and c is a number, then $cA = [c a_{ij}]$

$(-1)A = -A$, $\quad A + (-B) = A - B$

Example $A = \begin{bmatrix} 3 & 0 & -1 \\ 2 & -7 & 5 \end{bmatrix}$

$2A = \begin{bmatrix} 6 & 0 & -2 \\ 4 & -14 & 10 \end{bmatrix}$, $-A = \begin{bmatrix} -3 & 0 & 1 \\ -2 & 7 & -5 \end{bmatrix}$

Definition A column vector (or vector) is an $n \times 1$ matrix of the form

$$a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

Alternative notation $a = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = (a_1, \dots, a_n)$

Definition A row vector is a $n \times 1$ matrix of the form $a = [a_1 \; a_2 \cdots a_n]$

Definition let $a = [a_1 \; a_2 \dots a_n]$ be a row vector and $b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ be a column vector (both of them have n elements) The product

$$ab = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

Example $a = [3 \quad 0 \quad -1 \quad 7]$

$$b = \begin{bmatrix} 5 \\ 2 \\ -3 \\ 4 \end{bmatrix}$$

$$a \cdot b = 3 \cdot 5 + 0 \cdot 2 + (-1)(-3) + 7 \cdot 4 = 46$$

Definition Let $A = [a_{ij}]$ be an $m \times n$ matrix and $B = [b_{ij}]$ be an $n \times r$ matrix. The product of A and B, denoted $A \cdot B$, is the $m \times r$ matrix $C = [c_{ij}]$ defined by

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

$1 \leq i \leq m$, $1 \leq j \leq r$

$$\begin{array}{cc} & \\ \text{i-row} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \end{array} \begin{array}{c} \text{j-column} \\ \begin{bmatrix} b_{11} & \cdots & b_{1j} & \cdots & b_{1r} \\ b_{21} & \cdots & b_{2j} & \cdots & b_{2r} \\ \vdots & & \vdots & & \vdots \\ b_{n1} & \cdots & b_{nj} & \cdots & b_{nr} \end{bmatrix} \end{array}$$

$$c_{ij} = [a_{i1} \; a_{i2} \cdots a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$$

$$\underset{m \times n}{A} \quad \underset{n \times r}{B} = \underset{m \times r}{A \cdot B}$$

Remark Let $A = [a_1 \; a_2 \cdots a_n]$ where

$a_i = \begin{bmatrix} a_{1i} \\ a_{2i} \\ \vdots \\ a_{mi} \end{bmatrix}$ is the i^{th} column of A

Then $A \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = v_1 a_1 + v_2 a_2 + \cdots + v_n a_n$

Example $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$ 2×4

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$ 4×3

$AB = \begin{bmatrix} 70 & 80 & 90 \\ 158 & 184 & 210 \end{bmatrix}$

Remark In general $AB \neq BA$

$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

$AB = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$

$BA = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \neq AB$

A zero matrix, denoted by O, is a matrix whose entries are all 0

<u>Example</u> $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

<u>Theorem</u> Let A, B and C be matrices of appropriate sizes to make the indicated operations possible and a, b be real numbers. Then

(1) $A+B = B+A$

(2) $A+(B+C) = (A+B)+C$

(3) $A+O = O+A = A$

(4) $a(A+B) = aA+aB$

(5) $(a+b)A = aA+bA$

(6) $a(bA) = (ab)A$

(7) $A(BC) = (AB)C$

(8) $A(B+C) = AB+AC$

(9) $(A+B)C = AC+BC$

(10) $AO = OA = O$

PROOF (8) $A = [a_{ij}]_{m\times n}$

$B = [b_{ij}]\ n\times r$, $C = [c_{ij}]\ n\times r$

$B + C = [b_{ij} + c_{ij}]$

The ij^{th} element of $A(B+C)$ is

$$\sum_{k=1}^{n} a_{ik}(b_{kj} + c_{kj}) = \sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj})$$

The ij^{th} element of $AB + AC$ is

$$\sum_{k=1}^{n} a_{ik}b_{kj} + \sum_{k=1}^{n} a_{ik}c_{kj} = \sum_{k=1}^{n}(a_{ik}b_{kj} + a_{ik}c_{kj})$$

Fact If A is square matrix, then $AI = IA = A$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 7 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

$$AI = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 7 & 6 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 7 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ -1 & 7 & 6 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 7 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

Consider the linear system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$AX = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = b$$

So the system can be written in the form $Ax = b$

A solution of this system is a vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, \ldots, x_n)$

such that $Ax = b$

we call x an n-vector

Example The system

$$3x_1 - 4x_2 + x_3 + 7x_4 = 10$$

$$4x_1 - 5x_3 + 2x_4 = 0$$

$$x_1 + 9x_2 + 2x_3 - 6x_4 = 5$$

is equivalent to

$$\begin{bmatrix} 3 & -4 & 1 & 7 \\ 4 & 0 & -5 & 2 \\ 1 & 9 & 2 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 10 \\ 0 \\ 5 \end{bmatrix}$$