Exercise Determine the number of solutions of the system

$$x_1 + 2x_2 - 3x_3 = 4$$
$$4x_1 + x_2 + 2x_3 = 6$$
$$x_1 + 2x_2 + (k^2-19)x_3 = k$$

depending on the parameter $k \in \mathbb{R}$

Don't write the solution set

Hint Use elementary row operations to reduce the augmented matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 4 & 1 & 2 & 6 \\ 1 & 2 & k^2-19 & k \end{bmatrix}$$

to get the matrix

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & k^2-16 & k-4 \end{bmatrix} = E$$

Cases

(1) $K \neq \pm 4$. Consider

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & K^2-16 & K-4 \end{bmatrix} = E$$

The leading entries are 1, -7 and K^2-16. There are 3 leading variables, namely x_1, x_2 and x_3, and no free variable. The system has unique solution

(2) $K=-4$. Then

$$E = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & -8 \end{bmatrix}$$

and its last row is $[0 \ 0 \ 0 \ -8]$ (note that $-8 \neq 0$). The system is inconsistent (it has no solution)

(3) $K=4$. Then

$$E = \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading entries are 1 and -7.

There are two leading variables, namely x_1 and x_2, and one free variable, namely x_3.

The system has infinitely many solutions

Exercise Determine the number of solutions of the system

$$
\begin{aligned}
x_1 + kx_2 + (1+4k)x_3 &= 1+4k \\
2x_1 + (k+1)x_2 + (2+7k)x_3 &= 1+7k \\
3x_1 + (k+2)x_2 + (3+9k)x_3 &= 1+9k
\end{aligned}
$$

depending on the parameter k

Don't write the solution set.

Hint Use elementary row operations to reduce the augmented matrix

$$
\begin{bmatrix}
1 & k & 1+4k & 1+4k \\
2 & k+1 & 2+7k & 1+7k \\
3 & k+2 & 3+9k & 1+9k
\end{bmatrix}
$$

to get the matrix

$$E = \begin{bmatrix} 1 & k & 4k+1 & 4k+1 \\ 0 & 1-k & -k & -k-1 \\ 0 & 0 & -k & -k \end{bmatrix}$$

Cases

(1) $k \neq 0$ and $k \neq 1$. Then

$$E = \begin{bmatrix} 1 & k & 4k+1 & 4k+1 \\ 0 & 1-k & -k & -k-1 \\ 0 & 0 & -k & -k \end{bmatrix}$$

The leading entries are 1, $1-k$ and $-k$. There are 3 leading variables, namely x_1, x_2 and x_3, and no free variable. The system has unique solution

(2) $k = 0$. Then

$$E = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading entries are 1 and 1. There are 2 leading variables, x_1, x_2, and one free variable, namely x_3

The system has infinitely many solutions

(3) $k = 1$. Then

$$E = \begin{bmatrix} 1 & 1 & 5 & 5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & -1 \end{bmatrix} \xrightarrow{(-1)R_2 + R_3}$$

$$\begin{bmatrix} 1 & 1 & 5 & 5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The last row of

$$\begin{bmatrix} 1 & 1 & 5 & 5 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is $[0 \ 0 \ 0 \ 1]$ (note that $1 \neq 0$)

The system is inconsistent