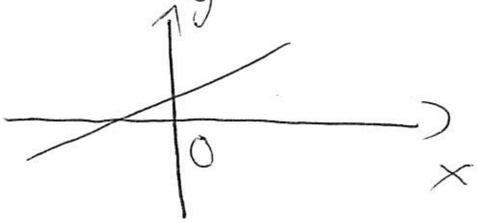
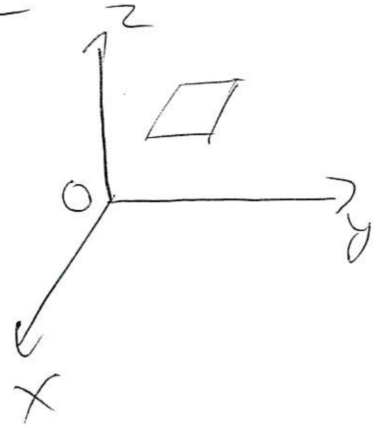
Introduction to linear systems

$ax + by = c$, $a \neq 0$ or $b \neq 0$, is a linear equation in x and y



$ax + by + cz = d$, at least one of a, b, c is non-zero, linear equation in x, y and z



$a_1x_1 + a_2x_2 + \dots + a_nx_n = b$ linear in $x_1, x_2, \dots, x_n$, where at least one $a_i \neq 0$

<u>Definition</u> A linear system is a finite collection of linear equations involving certain variables

Example

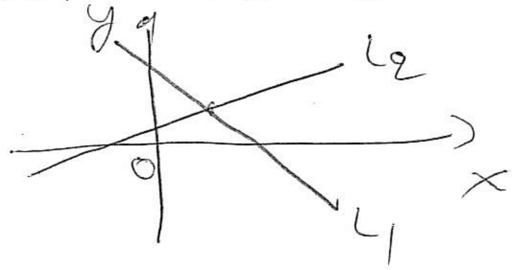
$$\left.\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}\right\}$$ system of 2 equations with 2 unknowns

A solution is a pair (x, y) of values that satisfy both equations simultaneously

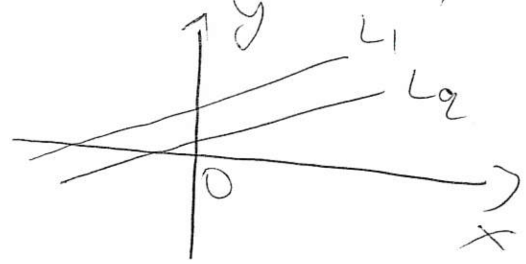
Definition A linear system is consistent if it has at least one solution, and inconsistent if it has no solution

$$a_1 x + b_1 y = c_1 \quad (L_1)$$
$$a_2 x + b_2 y = c_2 \quad (L_2)$$

(1) L_1, L_2 intersect at a single point. The system has exactly one solution

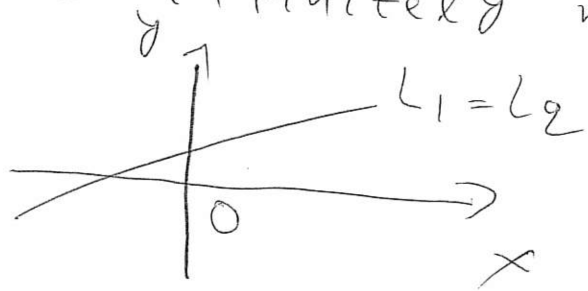


(2) L_1, L_2 are parallel. The system has no solution



(3) L_1, L_2 coincide

The system has infinitely many solutions



<u>Example</u> solve

$$\left.\begin{aligned} x - 2y &= 8 \\ 5x + 3y &= 1 \end{aligned}\right\} \quad \left\{\begin{aligned} x - 2y &= 8 \\ 0x + 13y &= -39 \end{aligned}\right\}$$

$$\begin{aligned} x - 2y &= 8 \\ 13y &= -39 \end{aligned} \quad \left\{\begin{aligned} x &= 8 + 2y = 2 \\ y &= -3 \end{aligned}\right.$$

<u>Example</u> solve

$$\left.\begin{aligned} 2x - y &= 1 \\ 6x - 3y &= 12 \end{aligned}\right\} \quad \left.\begin{aligned} 2x - y &= 1 \\ 0x + 0y &= 9 \end{aligned}\right\} \quad \begin{aligned} &2x - y = 1 \\ &0 = 9 \text{ impossible} \end{aligned}$$

No solution

Example

$$\left.\begin{array}{l}2x+6y=4\\3x+9y=6\end{array}\right\} \quad \left.\begin{array}{l}x+3y=2\\x+3y=2\end{array}\right\} \quad \begin{array}{l}x+3y=2\\0x+0y=0\end{array}$$

$$\left.\begin{array}{l}x=2-3y\\0=0\end{array}\right\}$$ y is a free variable
$y=t$, where t is a parameter which ranges over the set of all real numbers

The system has infinitely many solutions $(2-3t, t)$, $t \in \mathbb{R}$

$t=1$ $(-1, 1)$ solution

Matrices and Gaussian Elimination

Definition If m and n are positive integers, then an $m \times n$ matrix is a rectangular array of numbers of the form

$$\left.\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}}_{n \text{ columns}}\right\} m \text{ rows}$$

It has m rows and n columns

Example $\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$ 3×4,

$\begin{bmatrix} 1 & 2 \\ -1 & -3 \end{bmatrix}$ 2×2, $\begin{bmatrix} 1 & 1 & -1 & 3 \end{bmatrix}$ 1×4,

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ 3×1

Consider a general linear system with m equations and n unknowns

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$ coefficient matrix of the system

$$b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$ column vector

The $m \times (n+1)$ matrix

$$[A \ \ b] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

is the augmented matrix of the system

Example The augmented matrix of

$$2x_1 + 3x_2 - 7x_3 + 4x_4 = 6$$

$$x_2 + 3x_3 - 5x_4 = 0$$

$$-x_1 + 2x_2 - 9x_4 = 17$$

is

$$\begin{bmatrix} 2 & 3 & -7 & 4 & 6 \\ 0 & 1 & 3 & -5 & 0 \\ -1 & 2 & 0 & -9 & 17 \end{bmatrix}$$

Elementary row operations on a matrix A

(1) multiply any (single) row R_p of A by a non-zero constant C. Replace R_p by CR_p. Notation $(C)R_p$

(2) interchange two rows R_p and R_q of A. Replace R_p by R_q and R_q by R_p. Notation $SWAP(R_p, R_q)$

(3) Add a constant multiple of a row R_p of A to a row R_q of A. Replace R_q by $(C)R_p + R_q$. Notation $(C)R_p + R_q$

Example solve

$$x_1 + 2x_2 + 3x_3 = 11$$

$$3x_1 + 8x_2 + 5x_3 = 27$$

$$-x_1 + x_2 + 2x_3 = 2$$

The augmented matrix is

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

we will First substitute $R_2 = [3 \ 8 \ 5 \ 27]$

with $(-3)R_1 + R_2$:

$$\left.\begin{aligned} (-3)R_1 &= [-3 \ -6 \ -9 \ -33] \\ R_2 &= [3 \ 8 \ 5 \ 27] \end{aligned}\right\} (-3)R_1 + R_2 = [0 \ 2 \ -4 \ -6]$$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{(-3)R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

in the last matrix we will substitute

R_3 with $(1)R_1 + R_3$:

$$\left.\begin{aligned} (1)R_1 &= [1 \ 2 \ 3 \ 11] \\ R_3 &= [-1 \ 1 \ 2 \ 2] \end{aligned}\right\} (1)R_1 + R_3 = [0 \ 3 \ 5 \ 13]$$

so

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{(-3)R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{(1)R_1+R_3}$$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix}$$

In the last matrix we substitute R_3 with $\left(-\frac{3}{2}\right)R_2 + R_3 = [0 \quad 0 \quad 11 \quad 22]$

so

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{(-3)R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{(1)R_1+R_3}$$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix} \xrightarrow{\left(-\frac{3}{2}\right)R_2+R_3} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 11 & 22 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_2 + 3x_3 &= 11 \\ 0x_1 + 2x_2 - 4x_3 &= -6 \\ 0x_1 + 0x_2 + 11x_3 &= 22 \end{aligned}\right\} \quad \begin{aligned} x_1 &= 11 - 2x_2 - 3x_3 = 3 \\ x_2 &= 2x_3 - 3 = 1 \\ x_3 &= 2 \end{aligned}$$

Definition The matrices A and B are called row equivalent if we can obtain B by applying row operations to A

Example $A = \begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$ is row equivalent to $B = \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 11 & 22 \end{bmatrix}$

<u>Theorem</u> If the augmented coefficient matrices of two linear systems are row equivalent, then the two systems have the same solution set

<u>Definition</u> A matrix E is called (row) echelon provided it has the following properties:

(1) Any rows consisting entirely of zeros ~~are~~ are at the bottom of the matrix

(2) In each row of E that contains a non-zero element, the 1st non-zero element lies strictly to the right of the 1st non-zero element in the preceding row.

The 1st non-zero element in a row is called the "leading entry"

<u>Example</u> The matrices $\begin{bmatrix} 7 & 1 & -2 & -5 \\ 0 & -2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & 2 & -1 & 3 & 4 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ are echelon

But the matrices $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}$ and

$\begin{bmatrix} 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}$ are not echelon

<u>Gaussian Elimination</u>

(1) Locate the first column of A that contains a non-zero element

(2) If the first entry in this column is zero, interchange the first row of A with a row in which the corresponding entry is non-zero

(3) Now the first entry in our column is non-zero. Replace the entries below it in the same column with zeros by using elementary row operations

(4) Ignore the first row containing the leading entry, and repeat steps

(1), (2), (3) on the remaining submatrix

<u>Example</u>

$$A = \begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

The first column of A which contains a non-zero element is $\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$

The first entry in this column is $1 \neq 0$

I have to replace the entries below 1, namely 3 and -1, by 0

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{(-3)R_1+R_2} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{(1)R_1+R_3}$$

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix}$$

The first column of $\begin{bmatrix} 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix}$ which contains a non-zero element is $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. The first entry in this column is $2 \neq 0$. Replace the entry below 2, namely 3, by 0.

$$\begin{bmatrix} 1 & 2 & 3 & 11 \\ 3 & 8 & 5 & 27 \\ -1 & 1 & 2 & 2 \end{bmatrix} \xrightarrow{(-3)R_1 + R_2} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ -1 & 1 & 2 & 2 \end{bmatrix}$$

$$\xrightarrow{(1)R_1 + R_3} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 3 & 5 & 13 \end{bmatrix} \xrightarrow{\left(-\frac{3}{2}\right)R_2 + R_3} \begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 11 & 22 \end{bmatrix}$$

The matrix $\begin{bmatrix} 1 & 2 & 3 & 11 \\ 0 & 2 & -4 & -6 \\ 0 & 0 & 11 & 22 \end{bmatrix}$ is echelon