Math 132: Discrete Mathematics
Problem session 5

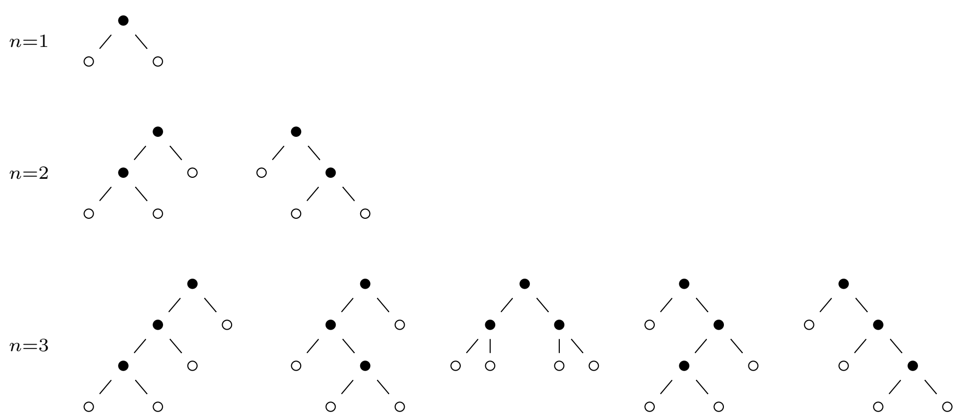
1. Give a combinatorial argument that $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

2. Show that for $n \in \mathbb{N}$, $\sum_{i=0}^{n} 2^i \cdot \binom{n}{i} = 3^n$.

3. Suppose you have 6 distinct jars and 3n indistinct balls. How many ways can you distribute the balls in the jars such that the first jar contains a multiple of 3 balls?

4. Let $(a_1, a_2, \ldots, a_n)$ be a sequence of n nonnegative integers such that $\sum_{i=1}^{n} a_i = n$ and $\sum_{i=1}^{j} a_i \geq j$ for all $1 \leq j \leq n$. For instance, if $n = 2$, we have the sequences $(1,1)$ and $(2,0)$, while if $n = 3$ the possible sequences are $(1,1,1)$, $(1,2,0)$, $(2,0,1)$, $(2,1,0)$, and $(3,0,0)$. Show that the number of such sequences is b_n for all n.

5. Show that b_n is the number of rooted binary trees, each node having 0 or 2 children, with n internal nodes. The first few cases are (internal nodes are black):



6. How many 10 digit numbers can be formed using exactly 4 digits from $\{1, 2, 3, \ldots, 0\}$?

7. Show that if $n \in \mathbb{N}$, then $3 | n^3 + 2n$.

8. Show that if a and b are relatively prime natural numbers that both divide c, then $ab|c$. Show this fails if a and b are not relatively prime.

9. How many divisors of $176820688 = 2^4 \cdot 11^3 \cdot 19^2 \cdot 23$ are there?

10. Let $\veebar$ be the logical operator XOR, where $p \veebar q = 1$ if and only exactly one of p and q are true. Show that $\veebar$ can be expressed in terms of $\neg$, $\vee$, and $\wedge$.

11. Similar to the last problem, show:
    - Every instance of $\leftrightarrow$ can be replaced with an expression involving $\neg$, $\vee$, $\wedge$, and $\rightarrow$.
    - Every instance of $\rightarrow$ can be replaced with an expression involving $\neg$, $\vee$, or $\wedge$.
    - Every instance of $\vee$ can be replaced with an expression involving $\neg$ and $\wedge$.
    - Every instance of $\wedge$ can be replaced with an expression involving $\neg$ and $\vee$.

12. Let $\barwedge$ be the logical operator NAND, where $p \barwedge q = 1$ if and only if both p and q are false. Show that every logical expression can be written using only $\barwedge$.