Linear systems

Last time we saw that using elementary row operations the augmented matrix $[A \ b]$ can be reduced to a more simple form, namely an echelon form (Gaussian Elimination)

Definition Suppose that the augmented matrix $[A \quad b]$ is in echelon form. The variables which correspond to columns containing leading entries are called leading variables. All the other variables are called free variables

Example Solve the system

$$x_4 + x_5 = 1$$

$$x_1 + x_2 + x_3 + 2x_4 + 2x_5 = 3$$

$$x_1 + x_2 + x_3 + 2x_4 + 3x_5 = 2$$

The augmented matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix}$$

The first column of the matrix that contains a nonzero element

is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. The First entry is 0.
so

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \xrightarrow{SWAP(R_1, R_2)}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix}$$

we will substitute R_3 with $(-1)R_1 + R_3$:

$$\left.\begin{aligned} (-1)R_1 &= [-1 \ -1 \ -1 \ -2 \ -2 \ -3] \\ R_3 &= [1 \ 1 \ 1 \ 2 \ 3 \ 2] \end{aligned}\right\} \text{so}$$

$$(-1)R_1 + R_3 = [0 \ 0 \ 0 \ 0 \ 1 \ -1]$$

Thus

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 3 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \xrightarrow{SWAP(R_1, R_2)}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 3 & 2 \end{bmatrix} \xrightarrow{(-1)R_1 + R_3}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \text{ echelon matrix}$$

The leading variables are

x_1, x_4, x_5.

The free variables are x_2, x_3

$x_2 = s, \quad x_3 = t$ where s and

t are arbitrary parameters

$$\left.\begin{aligned} x_1 + x_2 + x_3 + 2x_4 + 2x_5 &= 3 \\ x_4 + x_5 &= 1 \\ x_5 &= -1 \end{aligned}\right\} \quad \begin{aligned} x_4 &= 1 - x_5 = 2 \\ x_5 &= -1 \end{aligned}$$

$x_1 = 3 - x_2 - x_3 - 4 + 2 = 1 - x_2 - x_3 = 1 - s - t$

$x_4 = 2$

$x_5 = -1$

The system has infinitely many solutions of the form $(1-s-t, s, t, 2, -1)$

Example Solve the system

$$x_1 + x_2 = 2$$
$$3x_1 - x_2 = 6$$
$$x_1 - x_2 = 0$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 6 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{(-3)R_1 + R_2}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 0 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{(-1)R_1 + R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 0 \\ 0 & -2 & -2 \end{bmatrix}$$

$$\xrightarrow{(-\frac{1}{2})R_2 + R_3} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -4 & 0 \\ 0 & 0 & -2 \end{bmatrix} \text{ echelon}$$

$$\left.\begin{aligned} x_1 + x_2 &= 2 \\ -4x_2 &= 0 \\ 0x_1 + 0x_2 &= -2 \end{aligned}\right\} \quad \begin{aligned} x_1 + x_2 &= 2 \\ x_2 &= 0 \\ 0 &= -2 \text{ impossible} \end{aligned}$$

The system is inconsistent, namely it has no solution

Remark If an echelon form of the augmented matrix $[A \; b]$ has a row of the form $[0 \; 0 \; \ldots \; 0 \; r]$, where $r \neq 0$, then the system is inconsistent

Reduced Row-Echelon Matrices

Definition A matrix E is called reduced echelon matrix if it satisfies the following conditions

(1) E is an echelon matrix

(2) Each leading entry of E is 1

(3) Each leading entry of E is the only nonzero element in its column

Example The matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ are reduced echelon matrices

The matrices $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are not reduced echelon matrices.

Gauss – Jordan Elimination

(1) First transform the matrix A into echelon form by using Gaussian Elimination

(2) Then divide each element of each nonzero row by its leading entry

(3) Finally, use each leading 1 to "clear out" any remaining nonzero elements in its column

Theorem Every matrix is row equivalent to one and only one reduced echelon matrix

__Example__ Find the reduced echelon Form of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow{(-3)R_1 + R_2}$$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 2 & 7 & 9 & 23 \end{bmatrix} \xrightarrow{(-2)R_1 + R_3}$$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 3 & 7 & 15 \end{bmatrix} \xrightarrow{\left(-\frac{3}{2}\right)R_2 + R_3} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The last matrix is in echelon Form

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 2 & 4 & 8 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{\left(\frac{1}{2}\right)R_2} \begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\xrightarrow{(-2)R_2 + R_1} \begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{(-2)R_3 + R_2}$$

$$\begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \xrightarrow{(3)R_3 + R_1} \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

The reduced echelon Form of

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{bmatrix}$$ is

$$B = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

<u>Example</u> Use Gauss - Jordan elimination to solve the system

$$x_1 + x_2 + x_3 + x_4 = 12$$
$$x_2 - x_3 + 4x_4 = 5$$
$$3x_1 + 2x_2 + 4x_3 - x_4 = 31$$

The augmented matrix of the system is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 3 & 2 & 4 & -1 & 31 \end{bmatrix} \xrightarrow{(-3)R_1 + R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & -1 & 1 & -4 & -5 \end{bmatrix}$$

$$\xrightarrow{(1)R_2 + R_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 12 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{(-1)R_2 + R_1} \begin{bmatrix} 1 & 0 & 2 & -3 & 7 \\ 0 & 1 & -1 & 4 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 + 2x_3 - 3x_4 &= 7 \\ x_2 - x_3 + 4x_4 &= 5 \\ 0 &= 0 \end{aligned}\right\} \quad \begin{aligned} x_1 &= 7 - 2s + 3t \\ x_2 &= 5 + s - 4t \\ x_3 &= s, \ x_4 = t, \end{aligned}$$

$s \in \mathbb{R}$, $t \in \mathbb{R}$

Theorem A linear system of equations has either

(1) a unique solution, or

(2) no solution, or

(3) infinitely many solutions

Definition A homogeneous system with m equations and n unknowns is a linear system of the form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = 0$$

A homogeneous system will always have the trivial solution, which is when all variables are equal to 0, namely $x_1 = x_2 = \cdots = x_n = 0$

Conclusion A homogeneous system will have either one solution (the trivial solution) or infinitely many solutions

Example Consider the system

$$x_1 + 3x_2 + 2x_3 + x_4 = 0$$
$$2x_1 + 7x_2 + 4x_3 = 0$$
$$2x_1 + 6x_2 + 5x_3 + 4x_4 = 0$$

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & 2 & 1 & 0 \\ 2 & 7 & 4 & 0 & 0 \\ 2 & 6 & 5 & 4 & 0 \end{bmatrix} \xrightarrow{(-2)R_1+R_2} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 2 & 6 & 5 & 4 & 0 \end{bmatrix}$$

$$\xrightarrow{(-2)R_1+R_3} \begin{bmatrix} 1 & 3 & 2 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{(-3)R_2+R_1}$$

$$\begin{bmatrix} 1 & 0 & 2 & 7 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix} \xrightarrow{(-2)R_3+R_1} \begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1 &= -3x_4 = -3s \\ x_2 &= 2x_4 = 2s \\ x_3 &= -2x_4 = -2s \\ x_4 &= s \in \mathbb{R} \end{aligned}\right\} \text{ infinitely many solutions}$$

Theorem If $m<n$, then the homogeneous system has infinitely many solutions

Remark If $m<n$ and the system is not homogeneous, then the system has no solutions or has infinitely many solutions

Example The system

$$x_1 + x_2 + x_3 = 0$$
$$x_1 + x_2 + x_3 = 1$$

has no solution