Last time Revision

To Find the inverse of an $n\times n$ matrix A, we consider the $n\times 2n$ matrix $[A \vdots I]$.
We perform Gauss-Jordan elimination on the matrix $[A \vdots I]$ and the result is $[I \vdots A^{-1}]$. If we can not reduce A to I using elementary row operations, then A is not invertible

Example Find A^{-1} if $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Using elementary row operations we reduce $[A \vdots I]$ to $\left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{array}\right]$

Then $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

Theorem The following properties of an $n \times n$ matrix A are equivalent

(1) A is invertible

(2) A is row equivalent to the $n \times n$ identity matrix I

(3) $AX = 0$ has only the trivial solution

(4) For every n-vector b, the system $AX = b$ has a unique solution

(5) For every n-vector b, the system $AX = b$ is consistent

Example $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Then A is invertible, so the homogeneous system $AX = 0$ has only the trivial solution. Also for every n-vector b, the system has a unique solution $X = A^{-1}b$

Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, then A is not invertible. Thus the homogeneous system $AX = 0$ has infinitely many solutions. Also if $b = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$, then the system

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

has no solution, since from the last equation of

$$x_1 + x_2 + 2x_3 = 4$$
$$x_1 + 2x_2 + x_3 = 1$$
$$0x_1 + 0x_2 + 0x_3 = 1$$

we deduce that $0 = 1$

<u>Definition</u> A matrix with one of the properties of the above theorem is called nonsingular

Powers of square matrices

We define $A^1 = A$, $A^2 = A \cdot A$ and if n is a positive integer, then

$$A^{n+1} = A^n \cdot A$$

We also define $A^0 = I$

If A is invertible and n is positive integer, we define $A^{-n} = (A^{-1})^n$

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$A^{-2} = (A^{-1})^2 = A^{-1}A^{-1} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{11}{2} & -\frac{5}{2} \\ -\frac{15}{4} & \frac{7}{4} \end{bmatrix}$$

Properties (1) If r, s are nonnegative integers, then $A^r A^s = A^{r+s}$ and $(A^r)^s = A^{rs}$

(2) If A is invertible and n is a nonnegative integer, then $(A^n)^{-1} = (A^{-1})^n$

Determinants

Let A be an $n \times n$ matrix.

If $n=1$, namely $A=[a]$, then the determinant is $\det A = a$

Other Notation $\det A = \det(A) = |A|$

If $n=2$, namely $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

then the determinant is

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example $A = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}$, $\det A = 5 \cdot 9 - 6 \cdot 8 = -3$

For higher-order determinants we will give an inductive definition

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad 3 \times 3$$

We define the determinant as follows

$$\det A = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Example

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 1 \cdot (45-48)$$

$$-2(36-42) + 3(32-35) = 0$$

Definition Let A be an $n \times n$ matrix. The determinant of a matrix obtained by eliminating the i-th row and j-th column of A is called the ij-th minor of A, denoted by M_{ij}.

The ij-th cofactor A_{ij} of A is

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$M_{11} = \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3$, $M_{12} = \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = -6$,

$M_{13} = \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$, $A_{11} = 1 \cdot M_{11} = M_{11}$,

$A_{12} = -M_{12}$, $A_{13} = 1 \cdot M_{13} = M_{13}$

$\det A = 1 \cdot A_{11} + 2 \cdot A_{12} + 3 A_{13} = 0$

Definition The determinant $\det A$ of an $n \times n$ matrix $A = [a_{ij}]$ is

$$\det A = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n}.$$

Example

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 7 & 4 & 3 & 5 \\ -6 & 2 & 2 & 4 \end{bmatrix}$$

$$\det A = 2\begin{vmatrix} -1 & 0 & 0 \\ 4 & 3 & 5 \\ 2 & 2 & 4 \end{vmatrix} - (-3)\begin{vmatrix} 0 & -1 & 0 \\ 7 & 4 & 3 \\ -6 & 2 & 2 \end{vmatrix}$$

$$= 2(-1)\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} + 3(1)\begin{vmatrix} 7 & 3 \\ -6 & 2 \end{vmatrix} = 92$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}, \quad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Theorem Let A be an $n \times n$ matrix. The determinant of A can be obtained by expansion along any row or column. The cofactor expansion along the i-th row is $\det A = a_{i1} A_{i1} + \cdots + a_{in} A_{in}$

The cofactor expansion along the j-th column is

$$\det A = a_{1j} A_{1j} + \cdots + a_{nj} A_{nj}$$

Example

$$A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ 0 & -1 & 0 & 0 \\ 7 & 4 & 3 & 5 \\ -6 & 2 & 2 & 4 \end{bmatrix}$$

$$\det A = (-1) \begin{vmatrix} 2 & 0 & -3 \\ 7 & 3 & 5 \\ -6 & 2 & 4 \end{vmatrix} = 92$$

Remark If the square matrix A has either an all-zero row or an all-zero column, then $\det A = 0$

Example $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$

$\det A = 0$

$B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 0 & 4 \end{bmatrix}$, $\det B = 0$

Properties of Determinants

(1) If the $n \times n$ matrix B is obtained from A by multiplying a single row (or a column) of A by the constant K, then $\det B = K \det A$

Example (1) $\begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix}$

$- 4 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 6 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = 2 \left[1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + \right.$

$\left. + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \right] = 2 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2 \cdot 0 = 0$

(2) $\begin{vmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{vmatrix} = 2 \cdot 2 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 14 & 16 & 18 \end{vmatrix}$

$= 2 \cdot 2 \cdot 2 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 8 \cdot 0 = 0$

In general, $\det(KA) = K^n \det A$ for an $n \times n$ matrix A

(2) If the $n \times n$ matrix B is obtained from A by interchanging two rows (or two columns), then $\det B = -\det A$

<u>Example</u> Consider the determinant

$$\begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 1(1-2) - 1(2-1) + 0(4-1)$$

$$= -2$$

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= 1(2-1) - 1(1-2) + 0(1-4)$$

$$= 2$$

(3) If two rows (or two columns) of the $n \times n$ matrix A are identical, then $\det A = 0$

<u>Example</u> Consider the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix}$

Let B be the matrix obtained by interchanging the first two columns.

Then $B = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 1 & 4 \end{bmatrix} = A$, so $\det B = \det A$

But $\det B = -\det A$, so $\det A = -\det A$ and thus $\det A = 0$

(4) If we add a constant multiple of a row (or column) to another row (or column) of A, then $|A|$ does not change

<u>Example</u> $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

$\det A = 3$

Consider $\begin{vmatrix} 2+1 & 1+1 & 3+1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix} =$

$$(2+1)\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - (1+1)\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + (3+1)\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 3\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$+ 1\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= \det A + \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} = \det A + 0 = \det A$$

$$= 3$$

Definition An $n \times n$ matrix $A = [a_{ij}]$ is called

(1) upper triangular if $a_{ij} = 0$ for $i > j$, namely the matrix has only zeros below its principal (main diagonal)

(2) lower triangular if $a_{ij} = 0$ for $i < j$, namely the matrix has only zeros above its main diagonal

(3) triangular if it is either upper triangular or lower triangular

Example

$$\begin{bmatrix} 3 & 11 & 9 & -2 \\ 0 & -2 & 8 & -6 \\ 0 & 0 & 5 & 17 \\ 0 & 0 & 0 & -4 \end{bmatrix} \quad \text{triangular matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 3 & 5 & 2 \end{bmatrix} \quad \text{triangular matrix}$$

5) **Theorem** The determinant of a triangular matrix is equal to the product of its diagonal elements

Remark The determinant of the identity matrix I is 1