Notes on computational inversion for mean field games

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1 Introduction

The PDE system of MFG is

$$\begin{cases} -\partial_t u(x,t) - \nu\Delta u(x,t) + H(x, \nabla_x u(x,t)) = F(x, m(\cdot,t)), \\ \partial_t m(x,t) - \nu\Delta m(x,t) - \nabla_x \cdot (m(x,t)\nabla_p H) = 0, \\ m(\cdot,0) = m_0,\ u(\cdot,T) = G, \end{cases} \tag{1.1}$$

There is tremendous progress in the numerical solutions of mean-field games. Here are a few recent papers that are very interesting [1, 3, 5, 13, 14]. It seems that most nontrivial techniques are mostly developed for specific types of MFGs, utilizing the special structures of the MFGs.

[References to read: The first step would be to review the work documented in [1, 2, 4, 13, 15].]

The references [6, 7, 10, 12, 16] developed methods to solve learning, inversion, or control problems for probabilistic formulations of mean field games. For the moment, we should not invest too much time on those works.

References [9, 8] studied the problem of bridging MFGs with classical traffic flow models, aiming at recovering traffic flow models from MFG models.

Here are some open problems:

- Differentiability of the MFG system with respect to parameters to be optimized over, that is, the sensitivity analysis problem. Is it mathematically correct to compute adjoint problems? This could be a

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- Also related to sensitivity analysis: what are the condition number of the MFG inverse problems? Can we look at the sensitivity matrix and analyze/compute the condition numbers (How ill-conditioned are MFG inverse problems?)

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2 A simple multiscale method for mean field game [11]

2.1 MFG system

$$\begin{cases} u_t - v\Delta u + H(x, \nabla u) = V[m] \\ u(t=0, x) = u_0(x) + V_0[m(t=0)](x) \end{cases}$$

$$\begin{cases} -m_t - v\Delta m - \nabla \cdot (m \nabla_p H(x, \nabla u)) = 0 \\ m(t=T, x) = m_T(x) \end{cases}$$

where u is the cost function and m is the density of agents.

2.2 Numerical Method

2.2.1 Settings

1. Introduce multi-grid setting from level L_0 to L, denote U and M as the whole solution on a given level.

2. Use backward Euler method to approximate

3. Alternate sweeping: Here we denote M_k and U_k as the approximation after k-th iteration. We rewrite discrete version of equation as

$$\begin{cases} F(U, M) = 0 \\ G(U, M) = 0 \end{cases}$$

For $k \geq 2$, obtain U_k from $F(U_k, M_{k-1}) = 0$, then obtain M_k from $G(U_k, M_k) = 0$

2.2.2 Algorithm

The algorithm is stated as follow:

1. Solve equation on the L_0 grid by a given method with initial guess.
2. Do interpolation of the result onto the next level of finer grid as the initial guess.
3. Do alternate sweeping on finer grid to get the result

4. Repeat 2 and 3.

Q: Can initial guess be on U?

2.3 Convergence

Sometimes the convergence criterion is not met, so we use relaxation to impose convergence

$$U \leftarrow \alpha U_{new} + (1-\alpha)U_{old}, \quad M \leftarrow \alpha M_{new} + (1-\alpha)M_{old}$$

where $\alpha \in (0,1]$ is the relaxation factor.

2.4 New second-order finite difference scheme for discretizing the MFG systems

2.4.1 Key Contribution

1. Introduce 2 one-sided second-order difference operators

$$(\mathcal{D}W)_i^- = \frac{3W_i - 4W_{i-1} + W_{i-2}}{2h} \quad (\mathcal{D}W)_i^+ = \frac{-3W_i + 4W_{i+1} - W_{i+2}}{2h}$$

2. The total mass regarding m is conserved in the scheme.

3 Random features for high-dimensional non-local mean field games [1]

3.1 MFG system

$$\begin{cases} -\partial\phi(t,x) + H(t,x,\nabla\phi(t,x)) = \int_{\mathbb{R}^d} K(x,y)d\rho(t,y) \\ \partial_t\rho(t,x) - \nabla\cdot(\rho(t,x)\nabla_p H(t,x,\nabla\phi(t,x))) = 0 \\ \rho(0,x) = \rho_0(x), \quad \phi(T,x) = \psi(x) \end{cases}$$

where an individual faces a cost

$$\phi(t,x) = \inf_{z(t)=x} \int_t^T L(s,z(s),\dot{z}(s)) + \int_{\mathbb{R}^d} K(z(s),y)d\rho(s,y)ds + \psi(z(T))$$

3.2 Key contribution

Build an approximation

$$K(x,y) = K_r(x,y) = \sum_{i,j=1}^{r} k_{ij}\zeta_i(x)\zeta_j(y)$$

where $\{\zeta_i\}_{i=1}^r$ and (k_{ij}) are chosen basis functions and expansions coefficients.

3.3 Dual problem

Introducing unknown coefficient $a(t) = (a_1(t), a_2(t), ..., a_r(t))$, we can rewrite the system as

$$0 = \mathrm{K}^{-1}a(t) - \frac{\delta}{\delta a(t)} \int_{\mathbb{R}^d} \phi_a(0, x) d\rho_0(x)$$

where $\mathrm{K} = (k_{ij})_{i,j=1}^r$ and ϕ_a is the viscosity solution of

$$\begin{cases} -\partial_t \phi(t, x) + H(t, x, \nabla\phi(t, x)) = \sum_{i=1}^r a_i(t)\zeta_i(x) \\ \phi(T, x) = \psi(x) \end{cases}$$

Note1: Assume K is symmetric, then the equation reduces to a optimization problem.

Note2: $\{a_i\}$ contains all the information about the population interaction from the kernel.

Here ϕ_a can be represented as the cost function of the optimal trajectory

$$\phi_a(t, x) = \inf_{z(t)=x} \int_t^T L(s, z(s), \dot{z}(s)) + \sum_{i=1}^r a_i(s)\zeta_i(z(s))ds + \psi(z(T))$$

Eventually we have a saddle-point formulation of the problem.

3.4 Random features

In this paper, we sample from a Gaussian normal distribution

$$p(\omega) = \frac{\sigma^d}{(2\pi)^{\frac{d}{2}}} exp(-\frac{\sigma^2||\omega||^2}{2})$$

Q: Is it comprehensive enough?

3.5 Numerical algorithm

From the saddle point problem

$$\inf_a \sup_{z_x:z_x(0)=x} \frac{1}{2} \int_0^T \sum_{i,j=1}^r (\mathrm{K}^{-1})_{ij} a_i(t)a_j(t)dt - \int_{\mathbb{R}^d} \int_0^T L(s, z_x(s), \dot{z}_x(s)) + \sum_{i=1}^r a_i(s)\zeta_i(z_x(s))ds + \psi(z_x(T))d\rho_0(x)$$

we do discretization in space and time, substitute the objective function by $\mathcal{L}$, the saddle point problem problem reduces to

$$\inf_{\mathrm{a}} \sup_{\mathrm{v}} \mathcal{L}(\mathrm{a}, \mathrm{v})$$

where a is the discretized coefficient and v is the discretized control.

Iterate a and v at every time step

$$\begin{cases} \mathrm{v}^{(k+1)} = \mathrm{v}^{(k)} + h_v \nabla_{\mathrm{v}} \mathcal{L}(\mathrm{a}^k, \mathrm{v}) \\ \bar{\mathrm{v}}^{(k+1)} = 2\mathrm{v}^{(k+1)} - \mathrm{v}^{(k)} \\ \mathrm{a}^{(k+1)} = \arg\min_{\mathrm{a}} \mathcal{L}(\mathrm{a}, \bar{\mathrm{v}}^{(k+1)}) + \frac{||\mathrm{a}-\mathrm{a}^k||^2}{2h_a} \end{cases}$$

4 Inverse and learning problems in MFG: computation

One typical formulation related to transport is:

$$L = \int_{\mathbb{T}^d} \frac{1}{2}\rho v^\top G_M v dx + \mathcal{F}(\rho(\cdot,t)),$$

where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ ($\approx (S^1)^d$? I know this gives a periodic boundary condition and leads to an extra equation, but is there any other reason to consider this instead of regular regions like $[0,1]^d$?). In this case the problem is

$$\begin{aligned}
&\min_{\rho,v} \int_0^T \left(\int_{\mathbb{T}^d} \frac{1}{2}\rho v^\top G_M v dx + \mathcal{F}(\rho(\cdot,t)) \right) dt,\\
&s.t.\ \rho_t + \nabla\cdot(\rho v) = 0,\\
&\qquad \rho(\cdot,0) = \rho_0,\ \ \rho(\cdot,T) = \rho_T,
\end{aligned}$$

which is a PDE-constrained optimization problem. The first term means the kinetic energy, and term of $\mathcal{F}$ means some regularization. The first constraint comes from the continuity of ρ.

Theorem 4.1. *(ρ, v) is the solution iff. it it the solution of the following PDEs:*

$$\begin{aligned}
&\rho_t + \nabla\cdot(\rho v) = 0,\\
&(G_M v)_t + \nabla\left(\frac{1}{2}v^\top G_M v - \frac{\delta}{\delta\rho}\mathcal{F}(\rho)\right) = 0,\\
&\frac{\partial (G_m v)_i}{\partial x_j} = \frac{\partial (G_m v)_j}{\partial x_i},\ \ i \neq j,\\
&\int_0^1 (G_M v)_i(x_1,\cdots,x_{i-1},s,x_{i+1},\cdots,x_d)ds = 0,\ \ i = 1,\cdots,d.
\end{aligned}$$

This PDE system can be obtained by taking derivative (or variation, more exactly) to the Lagrangian function of the problem, after transformation.

Opposite to the forward problem which solves ρ, v w.r.t. given G_M and $\mathcal{F}$, the inverse problem's goal is to reconstruct unknown G_M or $\mathcal{F}$ from observation values $\hat{\rho}, \hat{v}$ (may be noisy). Take the reconstruction of G_M as an example. Suppose $G_M = (f_{ij}(g_0))$ with given f_{ij}, then the inverse problem can be written as ($p = 2$ is a usual choice)

$$\begin{aligned}
&\min_{g_0,\rho,v} \frac{\alpha}{2}||\rho - \hat{\rho}||^2 + \frac{\beta}{2}||v - \hat{v}||^2 + \frac{\alpha_0}{2}(||\rho_0 - \hat{\rho}_0||^2 + ||\rho_T - \hat{\rho}_T||^2) + \frac{\gamma}{p}||\nabla g_0||_p^p,\\
&s.t.\ (\rho, v) \text{ is the solution of corresponding forward problem.}
\end{aligned}$$

Usually ρ and v are observable, since ρ means the density and ρv means the flux.

This inverse problem can be naturally extended to the case of n observations, just replace $||\rho - \hat{\rho}||^2$ with $\sum_{i=1}^n ||\rho_i - \hat{\rho}_i||^2$ and so on. (I suppose there is no essential difference between one observation and n, but maybe for n observations, there are more efficient optimization algorithms to be implemented.) This can also be extended to cases where ρ, v can only be observed on the boundary (which is more ill-posed).

To numerically solve the inverse problem, we need first to discretize it (by finite volume method). A primal-dual method can be implemented after this. Consider the Lagrangian of this constrained optimization problem with primal variables ρ, v, g_0 and dual variables ψ_i (i.e. Lagrangian multipliers). Each step contains: a gradient descent iteration to update primal variables, and the fixing of dual variables.

The paper gives many 1D and 2D numerical results. One main parameter in the inverse problem is γ, which controls the smoothness of g_0. In specific, the larger γ is, the smoother will G_M be. And when γ is small, G_M may have a large deviation from the real value at some points.

(It seems that the convergence is not very good. Maybe is because the constraints are not convex or because of the discretization process?)

Noised observations are also tested. In some cases, the noises significantly influence the numerical results of G_M, but by taking appropriate γ (relatively large), a good fitting of G_M also can be obtained.

5 A mean field game inverse problem [4]

5.1 Optimization problem

$$\begin{aligned} \underset{\rho,\mathbf{v}}{\text{minimize}} \quad & \int_0^T \int_{\mathbb{T}^d} \frac{1}{2}\rho \mathbf{v}^T G_M \mathbf{v} dx + \mathcal{F}(\rho(\cdot,t))dt + \mathcal{G}((\rho(\cdot,T)) \\ \text{subject to} \quad & \rho_t + \nabla \cdot (\rho \mathbf{v}) = 0 \\ & \rho(\cdot,0) = \rho_0 \end{aligned}$$

where $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, ρ is the density distribution, $\mathbf{v}$ is the velocity field, G_M is called the ground metric.

Q: $\mathcal{F}$ and $\mathcal{G}$ are functional that are convex. Is this assumption valid?

5.2 Derivation process

1. Derive equivalence between minimizer of optimization and solution of PDE (Theorem 4)

2. Construct a new optimization problem as the inverse model

Basically the constraints in the inverse model is the PDE system

3. Derive KKT condition for the inverse model.

4. Discrete format of inverse problem compatible with discrete potential MFG

5.3 Primal-Dual Algorithm

The optimization problem of inverse model is reformulated as

$$\underset{\rho,\mathbf{v},\theta}{\text{minimize}} \quad f(\rho,\mathbf{v},\theta)$$
$$\text{s.t.} \quad c_i(\rho,\mathbf{v},\theta)=0, \quad i=1,2,...,r$$

then we can write Lagrangian

$$L(\rho,\mathbf{v},\theta,\{\psi_i\}_{i=1}^{r}) = f(\rho,\mathbf{v},\theta)+\sum_{i=1}^{r} c_i(\rho,\mathbf{v},\theta)\psi_i$$

for $k \geq 1$

$$\begin{cases} \rho^{k+1} = \rho^k - \tau_\rho \frac{\delta}{\delta\rho} L(\rho^k,\mathbf{v}^k,\theta^k,\{\psi_i^k\}_{i=1}^{r}) \\ \mathbf{v}^{k+1} = \mathbf{v}^k - \tau_{\mathbf{v}} \frac{\delta}{\delta\mathbf{v}} L(\rho^k,\mathbf{v}^k,\theta^k,\{\psi_i^k\}_{i=1}^{r}) \\ \theta^{k+1} = \theta^k - \tau_\theta \frac{\delta}{\delta\theta} L(\rho^k,\mathbf{v}^k,\theta^k,\{\psi_i^k\}_{i=1}^{r}) \end{cases}$$

$$\psi_i^{k+1} = \psi_i^k + \sigma c_i(\rho^*,\mathbf{v}^*,\theta^*)$$

where $\rho^* = 2\rho^{k+1}-\rho^k$, $\mathbf{v}^* = 2\mathbf{v}^{k+1}-\mathbf{v}^k$, $\theta^* = 2\theta^{k+1}-\theta^k$.

6 Sharon Di's paper: Adversarial Inverse Reinforcement Learning

First, Introduce a time varying stochastic policy that represents the probability of how rational individual will react facing a situation(state):

$$\pi : \mathcal{S} \to \mathcal{P}(\mathcal{A})$$

Second, adding a entropy regularization term to the reward function enables bounded rationality since agents are allowed to act more than optimal actions.

When Nash Equilibrium is reached, we expect

1. Agent bounded rationality
2. Population consistency: satisfies McKean-Vlasov equation

$$\mu_{t+1}(s') = \sum_{s\in\mathcal{S}} \mu_t(s) \sum_{a\in\mathcal{A}} \pi_t(a|s) P(s'|s,a,\mu_t)$$

Instead of maximising the likelihood of the observed trajectories, we optimise the likelihood with respect to the distribution defined in Theroem 1.

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