Introduction to Differential Equations and Linear Algebra

APPM 2360: PROJECT 3

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1. Introduction

Dear Mr. President,

As we are sure you are aware, in the past 35 days, life as we know it nearly ended due to an outbreak of disease that turned infected specimens into zombies. In the midst of chaos and turmoil, you were probably not updated on the means by which we narrowly salvaged the human race. You deserve to know not only how humankind managed to continue to hold dominion over the earth, but also made intergalactic allies. Just when we had begun to gain hope when we calculated that the antidote would indeed save mankind, we realized we would have to inject the zombies with the antidote in order for it to be effective. We realized that humans could not get close enough to the zombies to inject them without getting bitten. Then, just when mankind felt ultimate despair that we were doomed, a new hope arrived. Within the past few year, the legendary Force awakened. When we thought that the storied Jedi were extinct, they showed up to save Earth from the zombies. Luke and Rey arrived on day 12 of the outbreak and killed more zombies per day than we could have ever hoped to. In fact, every day, the Jedi grew stronger. Luke and Rey derived their strength from the hope and optimism of the good humans. In an effort to destroy humankind, the evil Kylo Ren and his legions from the First Order arrived. He added to the number of zombies each day, deriving his strength from hopelessness and evil humans. Of course, rebellions are built on hope. Humans have prevailed and the Jedi managed to help us defeat the zombies. We continue to kill the zombies as they are revived, but now we can kill them faster than they are revived. Earth's population has drastically dwindled and we have been left shaken. In the following technical report, you will find the exact specifications of the zombie outbreak, our efforts to contain the outbreak, and the events following the arrival of our intergalactic allies and enemies.

Sincerely,
Matthew Niemiec and Brooke Shade, Differential Equations Masters

2. Analysis of the Outbreak

I. Initial Outlook

When the first infected person turned into a zombie, panic set in. We had no idea how fast this disease could spread and how difficult it would be to control. But we wanted to begin by implementing preventative measures and quickly did some calculations to determine our hopes of survival. The following equations model the spread of the disease where S represents number of susceptible, but not yet infected, persons. Z represents the number of zombies.

$$\frac{dS}{dt} = -\beta ZS \tag{1}$$

$$\frac{dZ}{dt} = \beta ZS \ + \gamma(N_0 - S - Z) - \alpha ZS \tag{2}$$

In this model, α represents the average zombie kill rate per human per day, γ represents the revival rate of zombies because of course, zombies are undead. Finally, β represents the average infection rate per zombie per day and N_0 represents the initial population of humans.

We wanted to determine the fate of mankind, or the mathematical solutions that the populations would tend towards or away from, otherwise known as the equilibrium solutions. To find this, we needed to determine the values for S and Z that would make $\frac{dS}{dt}$ and $\frac{dZ}{dt}$ zero. These are called nullclines. In other words, we wanted to know the number of susceptible people and zombies that would make it such that the number of humans and zombies were *not* changing over time. The places where $\frac{dS}{dt} = 0$ and $\frac{dZ}{dt} = 0$ intersect are the equilibrium points of the model. The equilibrium points are $(0,\, N_0)$ and $(N_0,\, 0)$ [1], or the places where the population is entirely human or entirely zombie. This is grim news for mankind, it means that either humans live or zombies live. We went a step further to determine the stability of these equilibrium points using the following Jacobian:

$$J = \begin{bmatrix} -\beta Z & -\beta S \\ \beta Z - \alpha Z - \gamma & \beta S - \alpha S - \gamma \end{bmatrix}$$

[1]See the Appendix for the work shown in this problem

By evaluating the Jacobian at the equilibira, we determined whether (0, N_0) and (N_0, 0) are likely to happen depending on other constants such as α, γ, and β. After intense calculations, we determined that (0, N_0) is a stable solution. The eigenvalues for this equilibrium point are:

$$-\gamma \tag{3}$$

$$-\beta N \tag{4}$$

Thus both eigenvalues are always negative because γ, β, and N are positive values. Therefore, (0, N_0) is stable, the fate of mankind will tend towards all humans being turned into zombies. For (N_0, 0), the eigenvalues are not as simple:

$$\frac{\beta N - \alpha N - \gamma + \sqrt{(\beta N + \alpha N + \gamma)^2 - 4\beta\alpha N^2}}{2} \tag{5}$$

$$\frac{\beta N - \alpha N - \gamma - \sqrt{(\beta N + \alpha N + \gamma)^2 - 4\beta\alpha N^2}}{2} \tag{6}$$

Equation (6) will be positive if βN is greater than αN + γ. It doesn't matter what is inside of the radical because if it is positive, it contributes to the positive term, β. If it is negative, then it is an imaginary number and is not considered when determining stability. All of this simply means that if the rate of infection is greater than the sum of the rate that humans kill the zombies and the revival rate of zombies, the solution is unstable. An unstable solution means that solutions will tend away from it. So it can only ever be possible to achieve a scenario where (N_0, 0) is a reality and all of the humans live if we were to choose a β, α, and γ such that βN is less than αN + γ. For example if we could achieve an α that is very large, it would mean that humans are killing the zombies very quickly and (N_0, 0) could be a stable solution and humankind could survive.

1: Numerical Analysis

After some time, we finally have values that we can use to determine a quantifiable outlook for the zombie apocalypse. We know that α, the average zombie kill rate per human per day, is 0.00002 per humanoid per day. β, the average infection rate per zombie per day, is 0.00003 per humanoid per day. The revival rate of zombies, γ, is 0.000025 per day. Finally, the initial population size is 60,000 humanoids, 59,999 of which are humans with 1 zombie. Provided below is a model which shows the relative populations of species over a 35 day span.

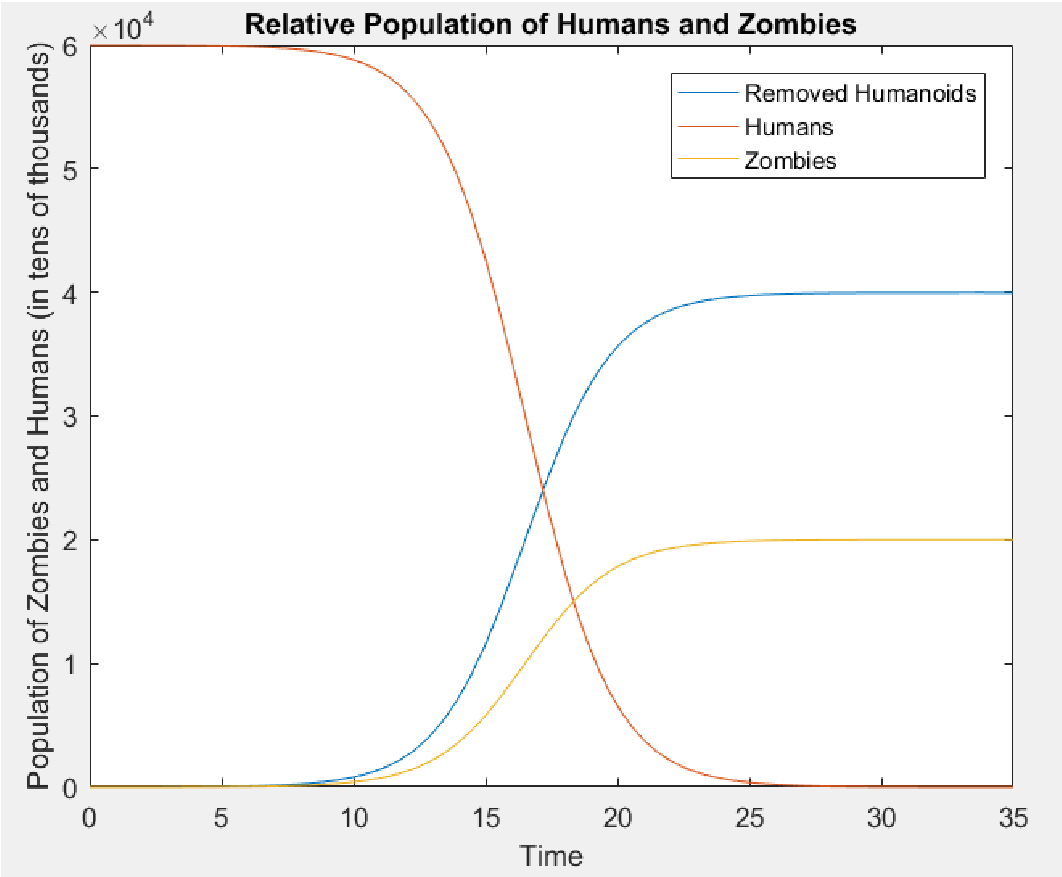

Figure 1: The population of zombies and humans over 35 days

This figure was made possible by code provided by excellent faculty at the University of Colorado. It is through their efforts in such a dangerous time that we have been able to do incredible math even in the apocalypse. The code provided uses a function called 'ode45' which is a first-order differential equation solver. After combining our models for $\frac{dS}{dt}$ and $\frac{dZ}{dt}$ into one differential equation, 'ode45' takes the differential equation and uses very small steps approximately solve the differential equation and model the symbolic equation as a numeric graph. Then, this is plotted on a graph as three lines showing population over time of 'Humans', 'Zombies', and the 'Removed Humanoids' which keeps track of the dead zombies and humans that have not yet been revived as zombies.

As you can see from the model, at 35 days, the graph shows neither equilibrium solution; it is neither all humans nor all zombies since there is a large population of removed humanoids still. If we look back at the stability, we know that $(0, N_0)$, all zombies and zero humans, is a stable solution. But what about (N_0, 0), all humans and no zombies? Obviously from the graph, by 35 days, humans will all be dead or zombified. However, we knew all along that (N_0, 0) was an unstable solution since βN $-$ αN $-$ γ is positive, specifically 1.8-1.2-0.000025, which is about 0.6. Humans were doomed from the start. So why doesn't the graph show all zombies and no humans nor removed humanoids? The answer is that it will reflect this equilibrium as humanoids are transformed into zombies. It is simply that this process is happening very slowly, specifically 0.000025 removed humanoids become zombies per day.

II. An Antidote!

Scientists on the East Coast managed to develop an antidote with no known side effects which returns zombies back to humans at at rate of ρ zombies per day. This change is reflected in our differential equations as follows:

$$\frac{dS}{dt} = -\beta ZS + \rho Z \tag{7}$$

$$\frac{dZ}{dt} = \beta ZS \ + \gamma(N_0 - S - Z) - \alpha ZS - \rho Z \tag{8}$$

With the same values for α, β, γ, and N_0, we see a much more hopeful model for the populations of zombies, humans, and removed humanoids. Although the amount of humans sharply decline, they will overtake the zombies and be able to kill the zombies and the revived zombies faster than the disease can spread and the removed humanoids can turn into zombies:

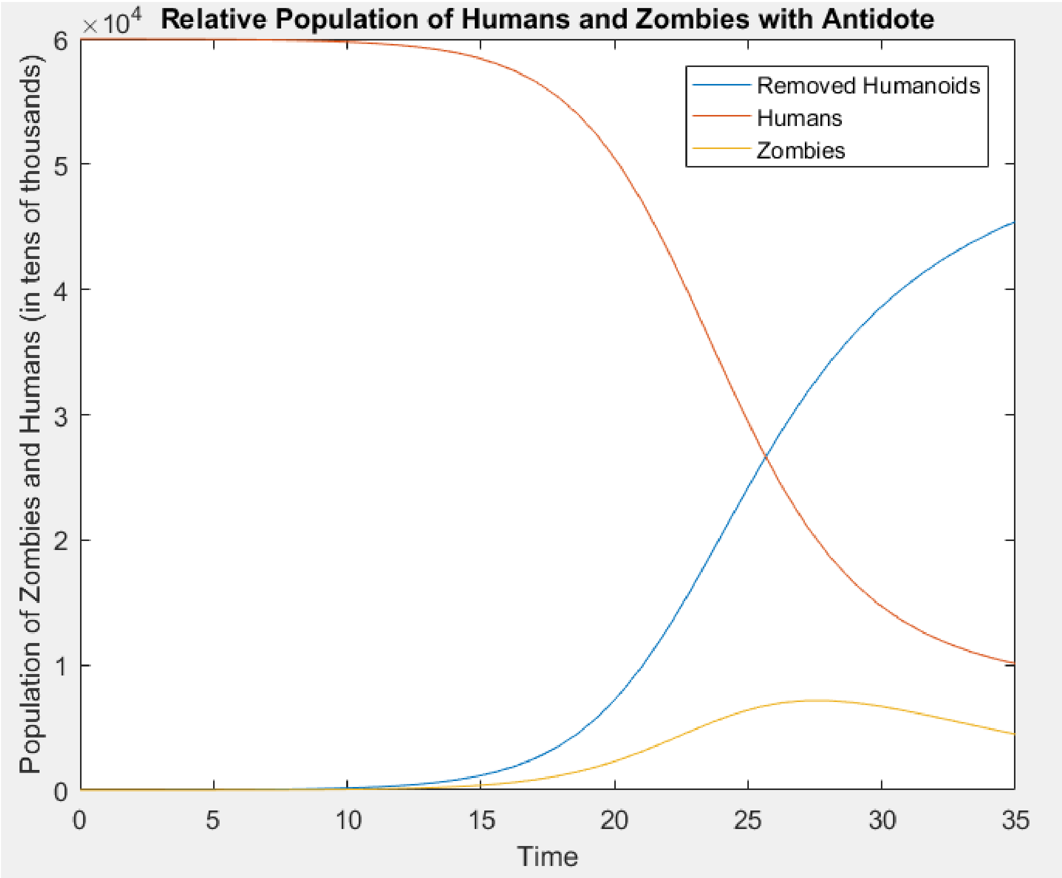

Figure 2: The population of zombies and humans over 35 days with antidote

Unfortunately, it is highly improbable that this is a viable solution because it makes a few grave assumptions. First, this model assumes that the antidote is fully developed before the outbreak begins. While this might be true for the East Coast, it is not true for the rest of America, so this model doesn't take into consideration the rest of the zombies taking over our country. Furthermore, this model assumes that humans, when encountering a zombie, will be able to inject it with antidote instead of being infected first. It seems improbable that if we were not able to kill the zombies faster than they infected us that we would be able to cure them any faster. Finally, this assumes that the antidote is not a finite resource, that can be distributed to the humans fast enough, and that it is a permanent and instantaneous solution.

Mathematically, this antidote provides two equilibrium points. The first equilibrium point reflects all humans and no zombies, which we already determined is unstable; it cannot be a reality. The second equilibrium point is about 6,667 humans and about 10 zombies. This is a stable solution. So if the antidote was a realistic solution and if we could neglect the previously stated problems with it, then about 6,667 could survive with about 10 zombies and about 53,323 removed humanoids over time.

~~IV~~ III. A New Hope

Just when we lost hope in our survival because of the improbabilities of an effective antidote for all of America, not just the East Coast, salvation came in the form of the Force™. On day 12 of the outbreak, Luke and Rey arrived to save all of mankind. Because of the high concentration of midichlorians in their blood, the Jedi are immune to the zombie disease. They were able to kill an impressive number of zombies per day since not only did they get stronger each day, but derived strength from the hope of the optimistic humans, which was about 3/4 of the humans. Unfortunately, as usual, right behind Luke and Rey, Kylo Ren and his legions from the First Order followed. To Kylo, the humans have always been a pesky race and he wanted to thwart Luke and Rey and allow the zombies to take over. Kylo can revive the zombies by deriving his power proportional to the number of evil, pessimistic humans on Earth, which is 1/4 of humans. The following complex equation models the change made to the population model at day 12:

$$K(S,t) = -0.0084(t-12)(\frac{3S}{4}) + 0.00005(\frac{S}{4}) \tag{9}$$

The first term reflects the average rate which Jedis are killing zombies which has a coefficient of 0.0084 per day and is proportional to time (in days) and 3/4 of the number of humans (those with hope). In other words, Luke and Rey get stronger each day and can kill the zombies by a factor of the number of days they have been on Earth times the amount of hopeful humans. The second term reflects the rate at which Kylo Ren turns humans into zombies which has a coefficient of 0.00005 per day and is proportional to 1/4 of the number of humans (those without hope). Below is a model of the populations when our intergalactic allies and enemies arrive:

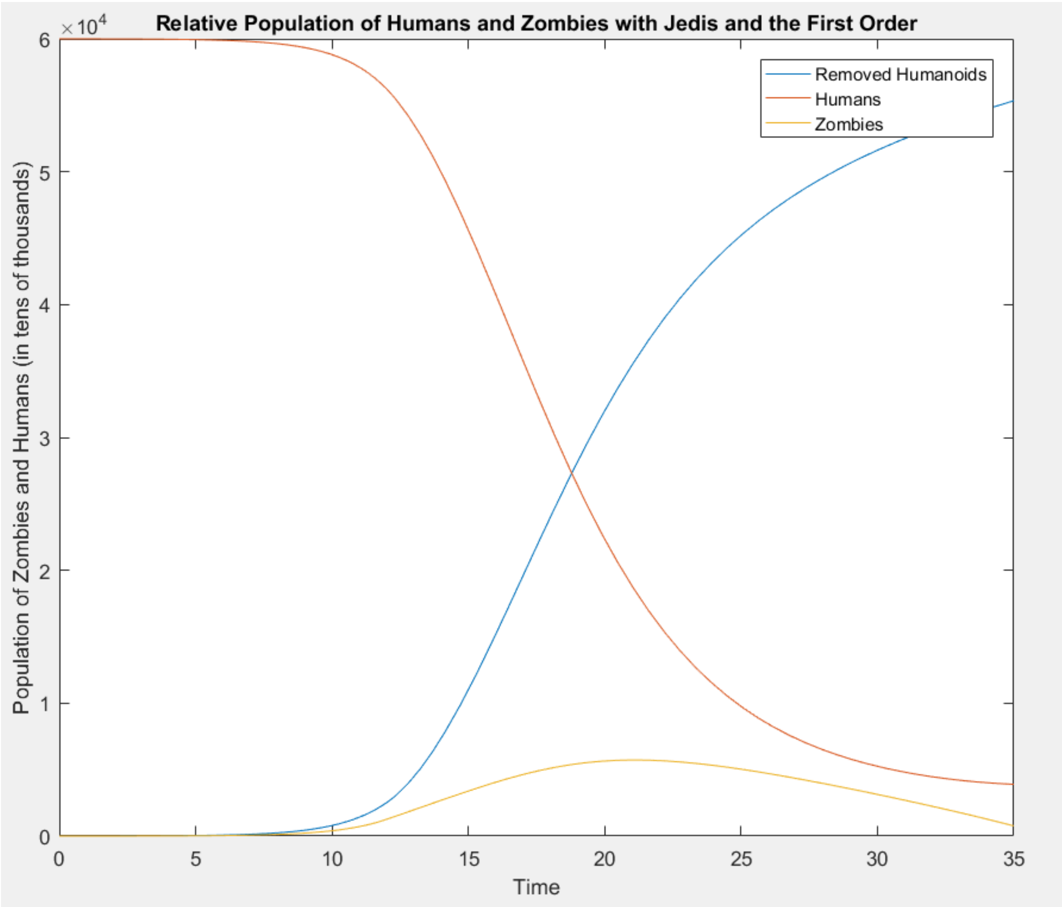

Figure 3: The population of zombies and humans over 35 days with Jedis arriving on day 12 of the outbreak

So with the help of Jedis, the human race can survive! Although the majority of humans will be die, so will the zombies. And what is more, the humans will then be able to kill the zombies as they are revived. At day 12, the zombies stop increasing so quickly. Up until day 12, this is the same model where humanity was doomed. But with the help of the Jedi and the 3/4 of the human population that still has hope, the zombies can be defeated.

3. Conclusion

Mr. President,
We are incredibly delighted to inform you that the human race is not dead. To summarize our findings, because of the rate at which the disease spread, the rate at which dead zombies were revived, and the unfortunately slow rate at which humans killed zombies, the human race was doomed to die. A glimmer of hope appeared when a possible antidote was discovered. While in theory, this antidote could have saved the East Coast if we somehow quarantined the entire East Coast, the rest of the world was doomed. This doesn't begin to encapsulate the other issues with a cure that must be injected at close range and has yet to best tested for side effects. We owe our lives to the Jedi, who, despite the efforts of Kylo Ren and the First Order, ended the zombie epidemic. Their strength grew as time progressed, but also depended upon the hope of mankind. We are not here to tell you how to do your job, Mr. President, but biologically, if this disease is viral, it could mutate. We implore you to acquire your best chemical engineers and biologists to create new antidotes and a better means of administering them. The Jedi have the rest of the universe to save, they might not be able to help us next time. Finally, our message to you and the rest of mankind is to have hope - it saved us.
Sincerely,
Master Brooke and Padawan Matthew

4. Appendix

Project3.m

```
clc; clear;

% APPM 2360 Project 3
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%%% Section 4 %%%%

%%% Part A - Find the Nullclines %%%

% dS/dt= -BZS
% Nullcines when Z and S are zero

% dZ/dt= BZS + y(N0-S-Z)-aZS
% Nullclines when (0, N0) and (N0, 0)

%%% Part B - Find the Jacobian, evaluate equilibrium pts %%%
syms  S B Z y a N p K

dsdt = -B*Z*S;
dzdt = (B*Z*S) + y*(N-S-Z)-(a*Z*S);

eqpts = solve(dsdt==0,dzdt==0,S,Z);
eqpts = [eqpts.S,eqpts.Z];
disp('The equilibrium points are: ')
disp(eqpts, '\n')

J = jacobian([dsdt;dzdt],[S,Z]);
disp('The Jacobian is: ')
disp(J, '\n')

```

```
eq1 = subs(J, [S, Z], [0, N]);
disp('The Jacobian evaluated at the first equilibrium point is')
disp(eq1, '\n')

eq2 = subs(J, [S, Z], [N, 0]);
disp('The Jacobian evaluated at the second equilibrium point is')
disp(eq2, '\n')

%%% Part C- find eigenvalues for J at eq1 and eq2 %%%

eig1 = eig(eq1);
eig2 = eig(eq2);

disp('The eigenvalues of the Jacobian at the first equilibrium is')
disp(eig1, '\n')
disp('The eigenvalues of the Jacobian at the second equilibrium is')
disp(eig2, '\n')

%%% Part D - Stability %%%

% a, B, N and y are always positive values since they represent physical
% phenomena

% N = initial population at t=0
% alpha = zombies killed by humans
% beta = rate of transmission of disease
% gamma(y) = reciprocal of duration of infection (proportional to rate of
% R = Revival of infected specimens having been 'removed')

```

```
% Both eigenvalues for equilibrium pt 1 (0,N) are stable as they
    are always
% negative

subs(eig2, [N, B, y, a], [10, 10, 10, 1]);

%%% Part E- solving numerically %%%

% Assign parameter values
a = 0.00002; B = 0.00003; gamma = 0.000025; N = 60000;
% Set length of simulation
tspan = [0 35];
% Set initial conditions
x0 = [59999; 1];
% Solve system
[t,x] = ode45(@(t,x) szr(t,x,a,B,gamma,N), tspan, x0);

R = N - x(:, 1) - x(:, 2);
plot(t, R)
hold on;
plot(t,x(:,1))
plot(t,x(:,2))
title('Relative Population of Humans and Zombies');
xlabel('Time');
ylabel('Population of Zombies and Humans (in tens of thousands)'
    );
legend('Removed Humanoids','Humans','Zombies');

%%%%%% Part 5- How Can We Fix This Zombie Thing??? %%%%%%
%%% 5.1 An antidote %%%

%%% Part A- modeling a temporary cure %%%

%%% Part B- Graphing the Zombies/Humans with antidote%%%

```

```
a = 0.00002; B = 0.00003; gamma = 0.000025; N = 60000; p=0.2;
tspan = [0 35];
x0 = [59999; 1];
[t,x] = ode45(@(t,x) antidote(t,x,a,B,gamma,N,p), tspan, x0);

R = N - x(:, 1) - x(:, 2);
figure
plot(t, R)
hold on;
plot(t,x(:,1))
plot(t,x(:,2))
title('Relative Population of Humans and Zombies with Antidote');
xlabel('Time');
ylabel('Population of Zombies and Humans (in tens of thousands)');
legend('Removed Humanoids','Humans','Zombies');

%%% Part C %%%

% Talked about in the write-up

%%% Part D- Equilibrium points %%%

dsdt = (-0.00003*Z*S)+0.2*Z;
dzdt =(0.00003*Z*S) + 0.000025*(60000-S-Z)-(0.00002*Z*S)-0.2*Z;
eqpts = solve(dsdt==0,dzdt==0,S,Z);
eqpts = [eqpts.S,eqpts.Z];
disp('The equilibrium points of the anitdote Jacobian are')
disp(eqpts, '\n')

J = jacobian([dsdt;dzdt],[S,Z]);

eq3 = subs(J, [S, Z], [eqpts(1, 1), eqpts(1, 2)]);

```

```
% The following is unstable because at least one of these
    eigenvalues is positive
eig3 = eig(eq3);
disp('The eigenvalues of the antidote Jacobian''s first
    equilibrium is')
disp(double(eig3))


eq4 = subs(J, [S, Z], [eqpts(2, 1), eqpts(2, 2)]);
eig4 = eig(eq4);
disp('The eigenvalues of the antidote Jacobian''s second
    equilibrium is')
disp(double(eig4))

% This is a stable solution because both eigevalues are negative

%%%% 5.2 - A Time Dependent Modification %%%%

% Since the force has recently awakened and our Jedi friends
% have discovered that we, here on earth, need some serious
% help to defeat the zombies. Since Jedis have a high
% concentration of midichlorians in their bloodstream,
% they are immune to the zombie disease. We get the help of
% Rey and Luke, cos they're the last jedi, they can kill a
% small fraction zombies per day. They derive their strength
% from the good people of earth. With Luke's training, Rey
% gets stronger every day, so the number of zombies killed
% is proportional to days on earth. But no good deed goes
% unpunished. Kylo Ren has caught wind of what Rey and Luke
% are doing and he wants thwart their efforts and let the
% zombies take over the earth. The humans have always been
% a pesky race anyways. So Kylo Ren and his legions show up
% and revive a small fraction of the number of zombies Kylo
% derives his power from the bad people on earth. We assume
% the ratio of good to bad people remains constant. The
```

```
% fate of the universe rests in our ability to model these
% functions in MATLAB.

a = 0.00002; B = 0.00003; gamma = 0.000025; N = 60000;
tspan = [0 12];
x0 = [59999; 1];
[t,x] = ode45(@(t,x) szr(t,x,a,B,gamma,N), tspan, x0);

tspan= [12 35];
[t1,x1]=ode45(@(t,x) jedis(t,x,a,B,gamma,N),tspan, [x(end,1),x(
    end,2)]);

R1 = N - x(:, 1) - x(:, 2);
R2 = N - x1(:, 1) - x1(:, 2);
figure
plot([t.' t1.'], [R1.' R2.'])
hold on;
plot([t.' t1.'],[x(:,1).' x1(:,1).'])
plot([t.' t1.'],[x(:,2).', x1(:,2).'])
title('Relative Population of Humans and Zombies with Jedis and
    the First Order');
xlabel('Time');
ylabel('Population of Zombies and Humans (in tens of thousands)'
    );
legend('Removed Humanoids','Humans','Zombies');
```

szr.m

```
function dydt = szr(t, x, a, B, gamma, N)
% Evaluates the right hand side of the SZR model for equations S
    (t) and Z(t).
% Here, y(t) = [S(t); Z(t)], so x(1) = S(t) and x(2) = Z(t).
dydt = [-B*x(1)*x(2);
B*x(1)*x(2) + gamma*(N - x(1) - x(2)) - a*x(1)*x(2)];
end
```

antidote.m

```
% The same as the szr function, but with the antidote
function dydt = szr2(t, x, a, B, gamma, N,p)
    % Evaluates the right hand side of the SZR model for
        equations S(t) and Z(t).
    % Here, y(t) = [S(t); Z(t)], so x(1) = S(t) and x(2) = Z(t).
    dydt = [-B*x(1)*x(2)+(p*x(2));
    B*x(1)*x(2) + gamma*(N - x(1) - x(2)) - a*x(1)*x(2)-(p*x(2))
        ];
end
```

jedis.m

```
%on day 12, the jedis show up. They derive their power from the
% good people of earth (approx 3/4 of those still alive) and
% kill 0.0084 times the number of days they've been on earth
% since the jedis get stronger each day. However, Kylo Ren shows
% up and derives his power from the evil people on earth
% (approx 1/4) and revives 0.00005 zombies per day. The fate of
% Earth rests in the hands of Differential Equations students

function dydt = jedis2(t, x, a, B, gamma, N)
    % Evaluates the right hand side of the SZR model for
        equations S(t) and Z(t).
    % Here, y(t) = [S(t); Z(t)], so x(1) = S(t) and x(2) = Z(t).
    dydt = [-B*x(1)*x(2);
```

```
    B*x(1)*x(2) + gamma*(N - x(1) - x(2)) - a*x(1)*x(2)-0.0084*(
        t-12)*((3*x(1))/4)+0.00005*(x(1)/4)];
end
```

Work for Problems

4a

$$\frac{dS}{dt} = -\beta ZS \tag{10}$$
$$\frac{dZ}{dt} = \beta ZS \ + \gamma(N_0 - S - Z) - \alpha ZS \tag{11}$$
$$\tag{12}$$

Finding the first nullcline, we get

$$\frac{dS}{dt} = 0 = -\beta ZS \tag{13}$$

Which means that $\frac{dS}{dt}$ is 0 when either $Z = 0$ or $S = 0$. To get the second we get

$$\frac{dZ}{dt} = 0 = (\beta - \alpha)ZS + \gamma(N_0 - S - Z) \tag{14}$$
$$\tag{15}$$

Here the first term is 0 when either Z or S is 0. Then the second term is 0 when the non-zero variable is equal to N_0. Both of these satisfy the condition for the first equation to be 0, which means the the equilibrium points are at $(S, Z) = (0, N_0); (N_0, 0)$.

Bibliography

[1] Differential Equations and Linear Algebra Farlow Hall McDill West *2nd Edition.* Pearson Prentice Hall

[2] The faculty and TAs at the University of Colorado at Boulder