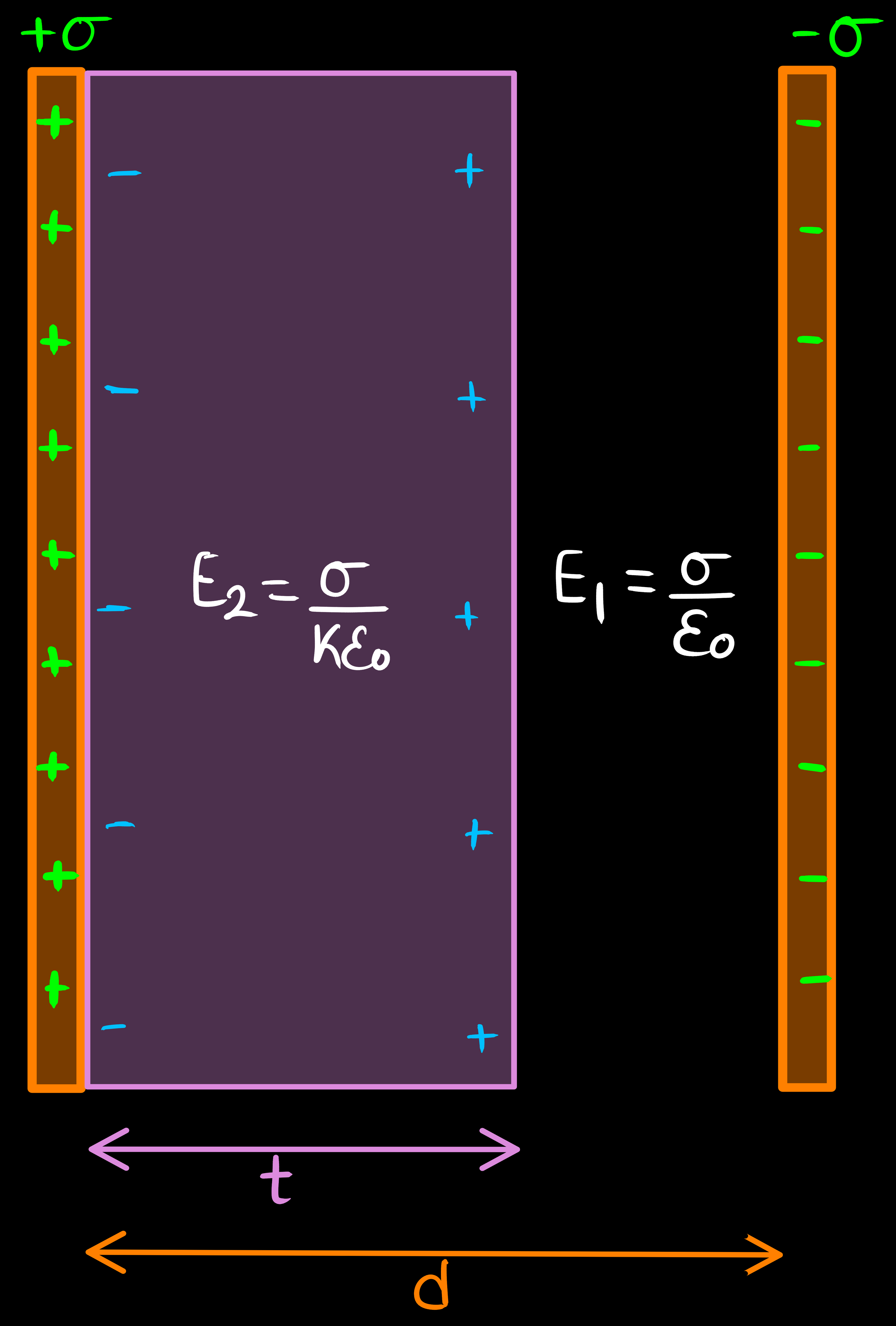
Capacitance of a parallel plate capacitor, partly filled with dielectric

* Consider a parallel plate capacitor having plates surface charge density $\pm\sigma$, Area A and plate separation d.

* Suppose that a dielectric slab of thickness t and dielectric constant K is placed between the plates of the capacitor.



* Electric field in the empty space between the plates -

$$E_1 = \frac{\sigma}{\varepsilon_0}$$

* Electric field inside the dielectric slab -

$$E_2 = \frac{\sigma}{K\varepsilon_0}$$

* The potential difference between the plates -

$$V = E_1(d-t) + E_2 t \qquad \left(\because \; V = -\int E \cdot dr\right)$$

$$V = \frac{\sigma}{\varepsilon_0}(d-t) + \frac{\sigma}{K\varepsilon_0} t$$

$$V = \frac{\sigma}{\varepsilon_0}\left(d - t + \frac{t}{K}\right)$$

Now Capacitance -

$$C = \frac{Q}{V} = \frac{\sigma A}{V}$$

$$C = \frac{\cancel{\sigma} A}{\frac{\cancel{\sigma}}{\varepsilon_0}\left(d - t + \frac{t}{K}\right)}$$

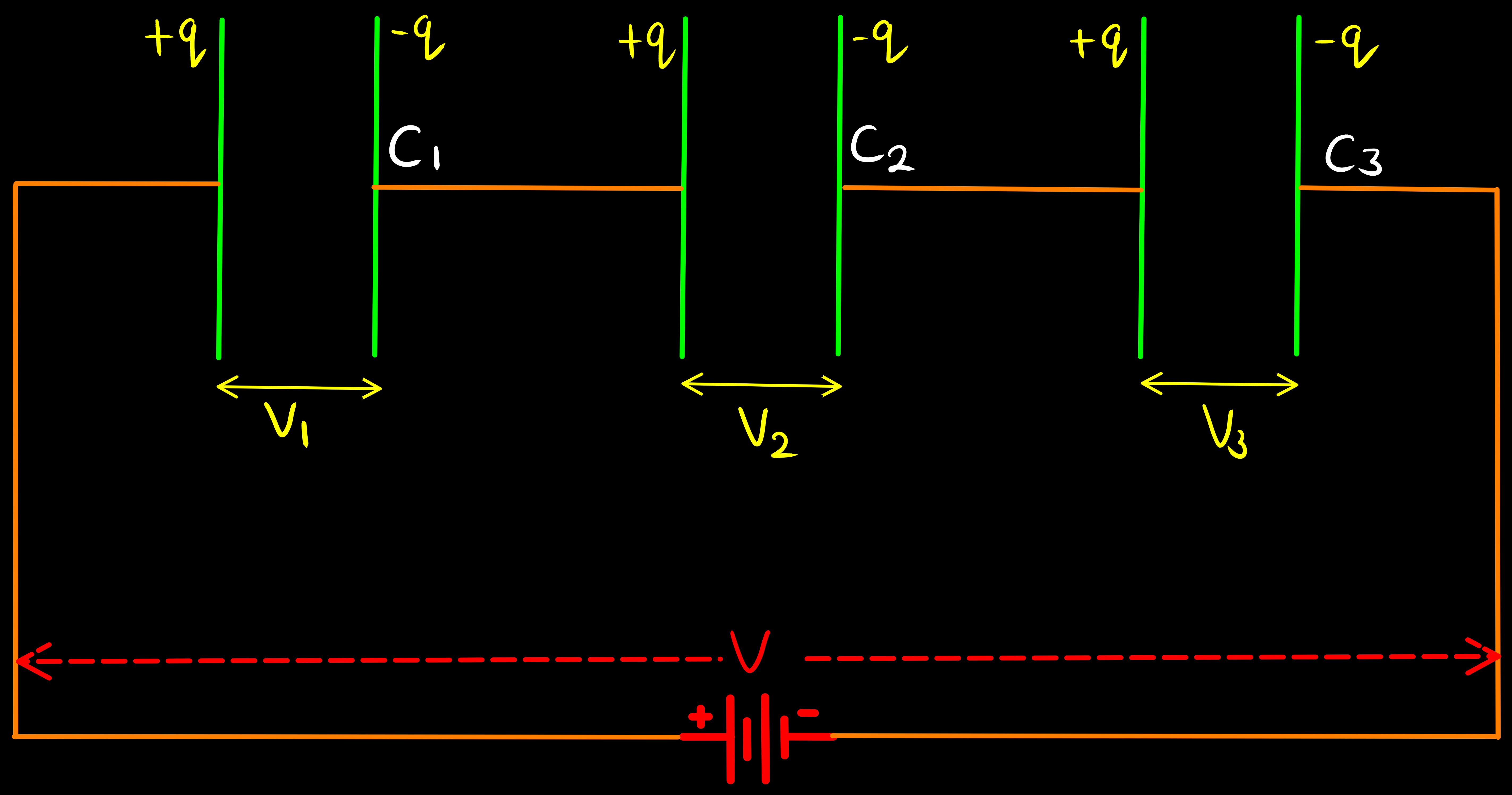
$$\boxed{C = \frac{\varepsilon_0 A}{\left(d - t + \frac{t}{K}\right)}}$$

Combination of capacitors

i) Capacitors in series -

* Consider three capacitors of capacitance C_1, C_2, and C_3, connected in series to a battery of potential difference (V).

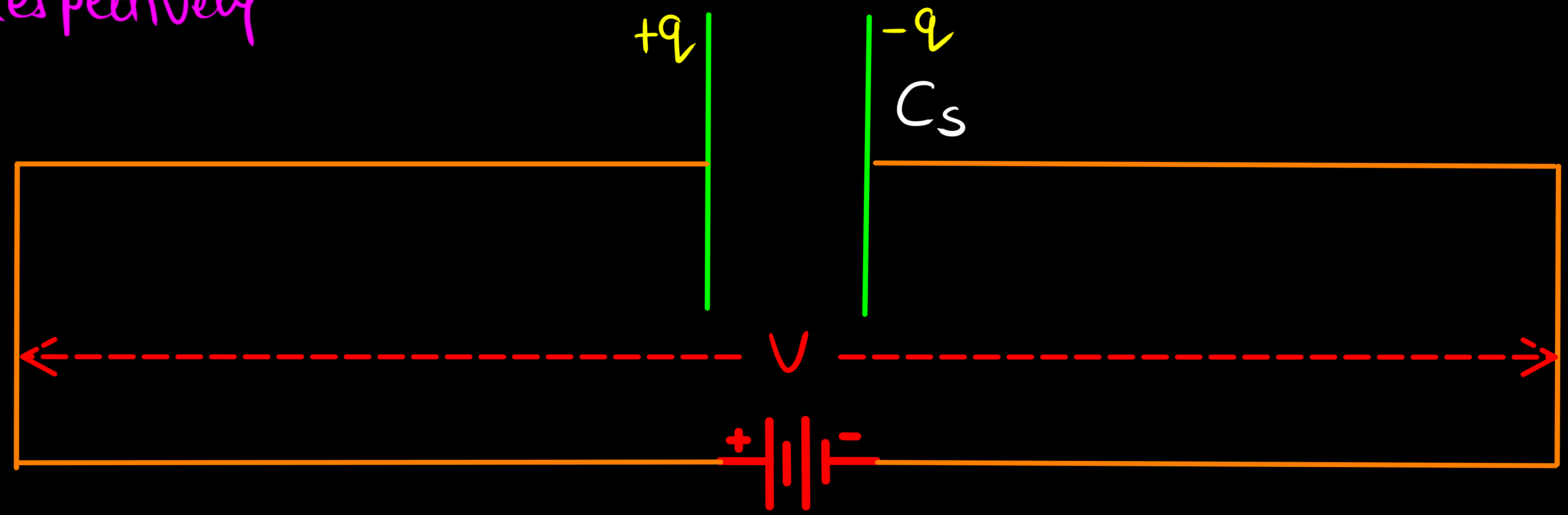
* In series combination charges on the two plates are same on each capacitor.



* The total potential drop (V) across the combination is the sum of the potential drop V_1, V_2, and V_3 across C_1, C_2 and C_3 respectively

$$V = V_1 + V_2 + V_3$$

$$V = \frac{q}{C_1} + \frac{q}{C_2} + \frac{q}{C_3} \quad \text{—(1)}$$



Combination of capacitors

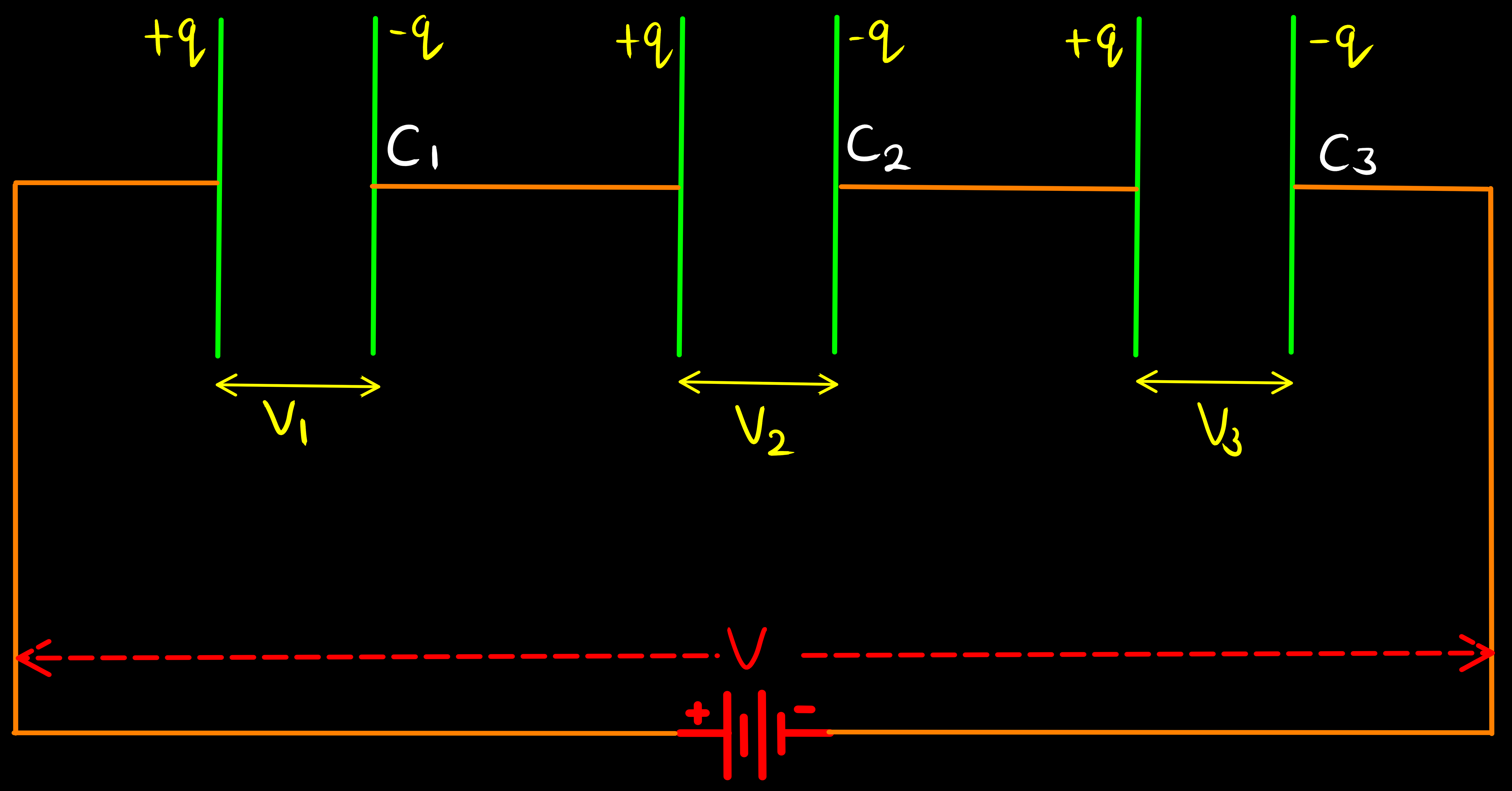
* The effective capacitance of the combination -

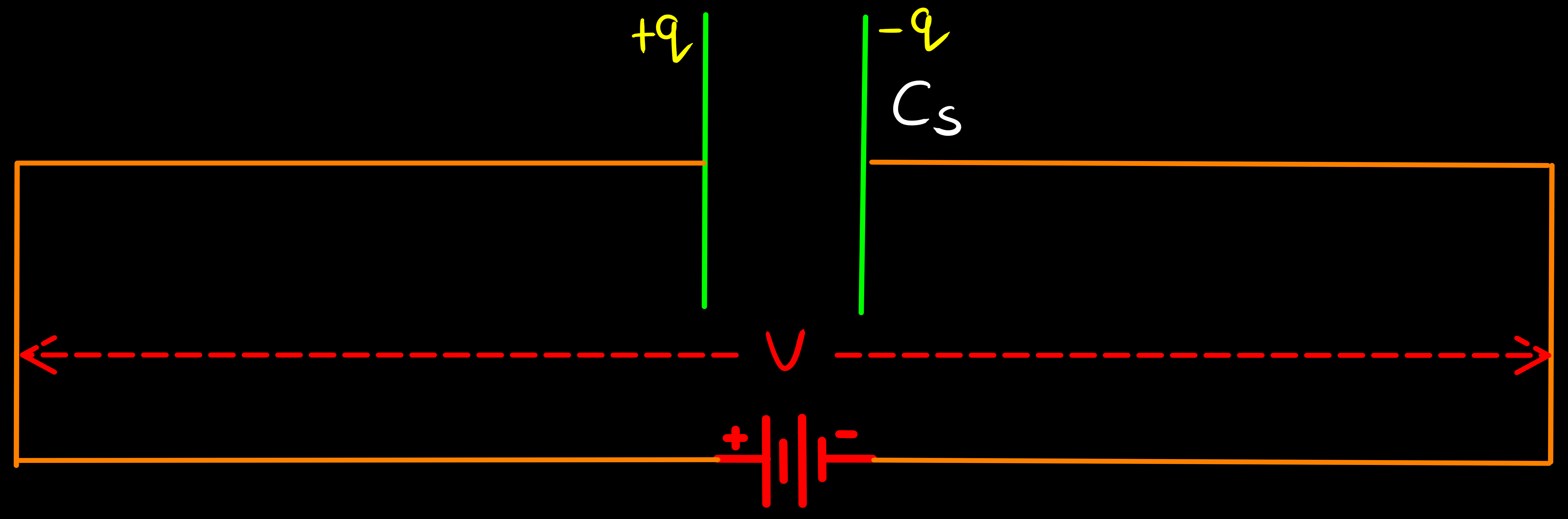
$$V = \frac{q}{C_s} \quad - \text{②}$$

* from eqⁿ ① & ②

$$\frac{\cancel{q}}{C_s} = \frac{\cancel{q}}{C_1} + \frac{\cancel{q}}{C_2} + \frac{\cancel{q}}{C_3}$$

$$\boxed{\frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}}$$





ii) Capacitors in parallel -

* Consider three capacitor of capacitance C_1, C_2 and C_3, connected in parallel with a battery of potential difference (V)

* In parallel combination potential difference is same across each capacitor.

* The charge stored in each capacitor -

$$q_1 = C_1 V \quad , \quad q_2 = C_2 V \quad , \quad q_3 = C_3 V$$

* Total charge supplied by the battery -

$$q = q_1 + q_2 + q_3$$

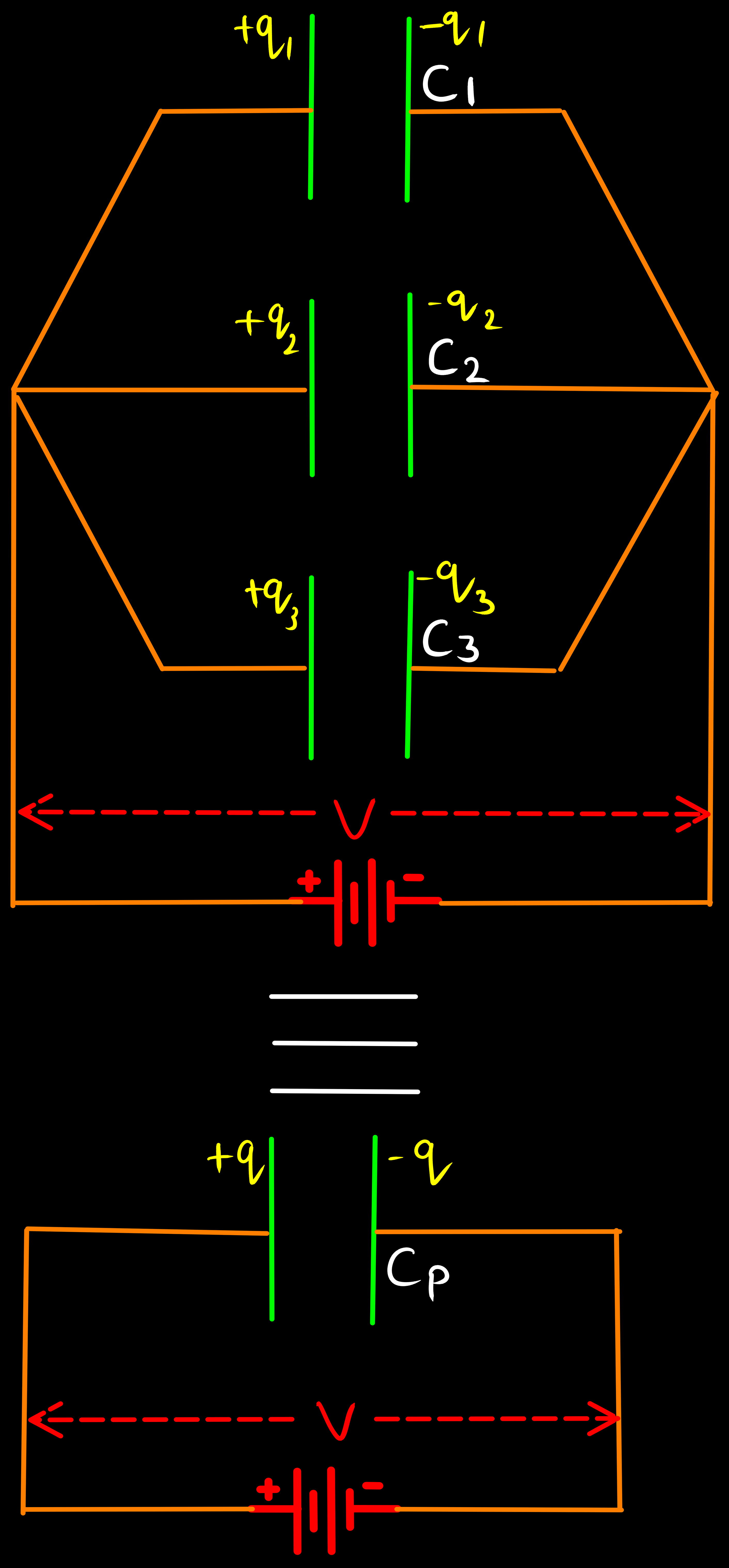
$$q = C_1 V + C_2 V + C_3 V$$

$$q = V[C_1 + C_2 + C_3] \quad - (1)$$

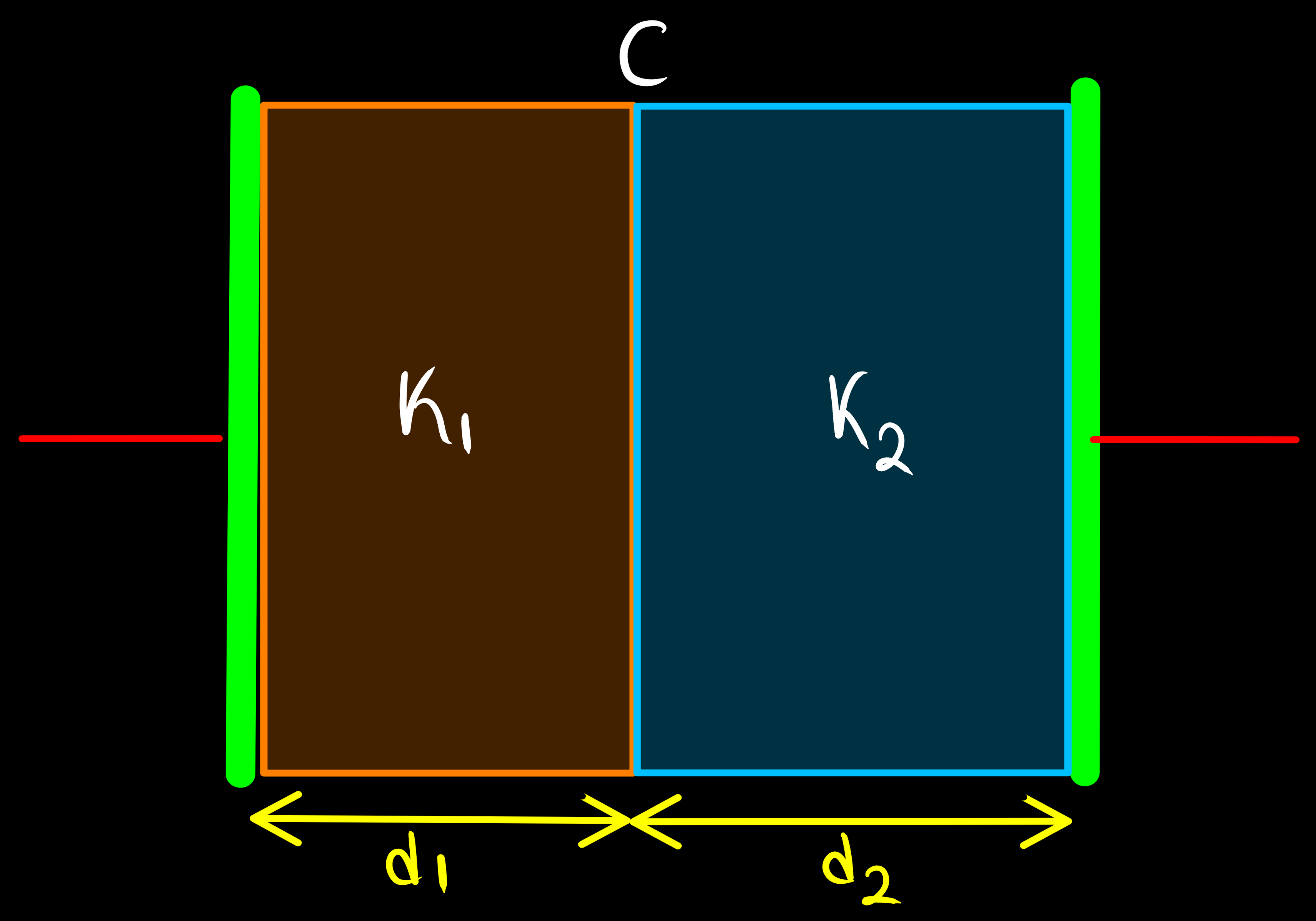
* Also, $q = C_p V \quad - (2)$

* from eqn (1) & (2) $\rightarrow$ $C_p \cancel{V} = \cancel{V}[C_1 + C_2 + C_3]$

$$\boxed{C_p = C_1 + C_2 + C_3}$$



* Some important Concepts-



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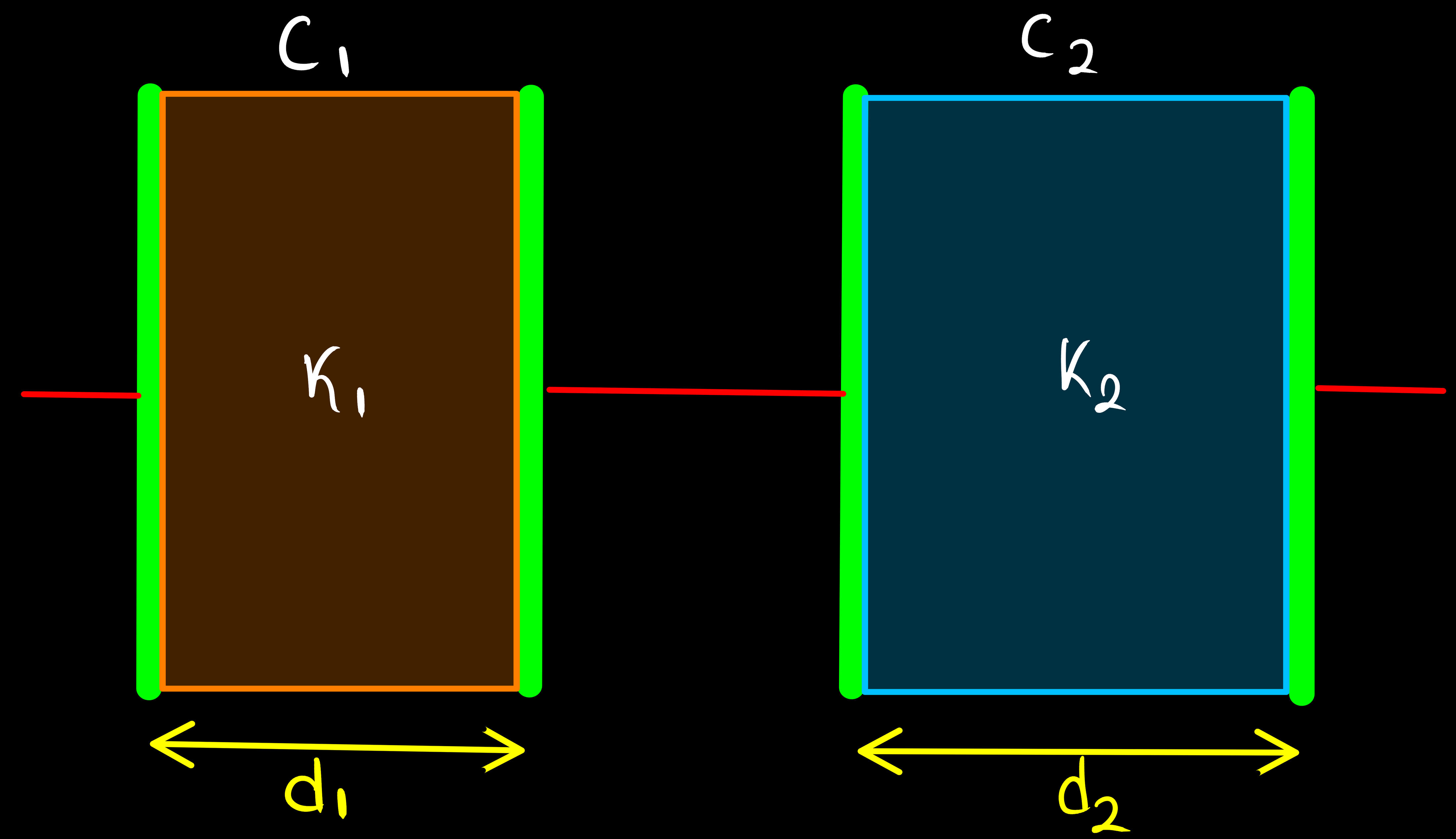


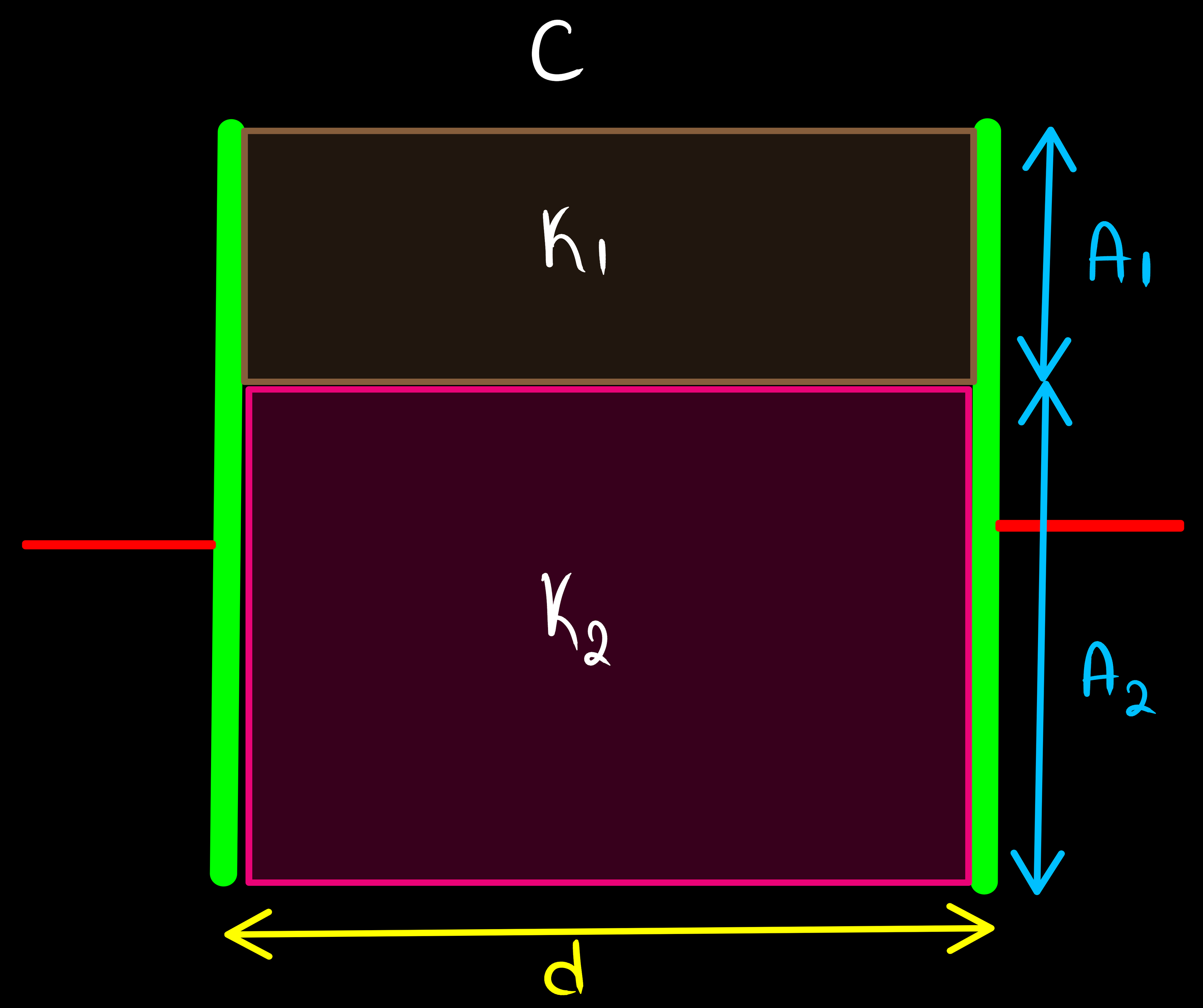
$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C} = \frac{1}{\frac{K_1\varepsilon_0 A}{d_1}} + \frac{1}{\frac{K_2\varepsilon_0 A}{d_2}}$$

$$\frac{1}{C} = \frac{d_1}{K_1\varepsilon_0 A} + \frac{d_2}{K_2\varepsilon_0 A}$$

$$\frac{1}{C} = \frac{1}{\varepsilon_0 A}\left[\frac{d_1}{K_1} + \frac{d_2}{K_2}\right]$$

$$\boxed{C = \frac{\varepsilon_0 A}{\left[\frac{d_1}{K_1} + \frac{d_2}{K_2}\right]}}$$



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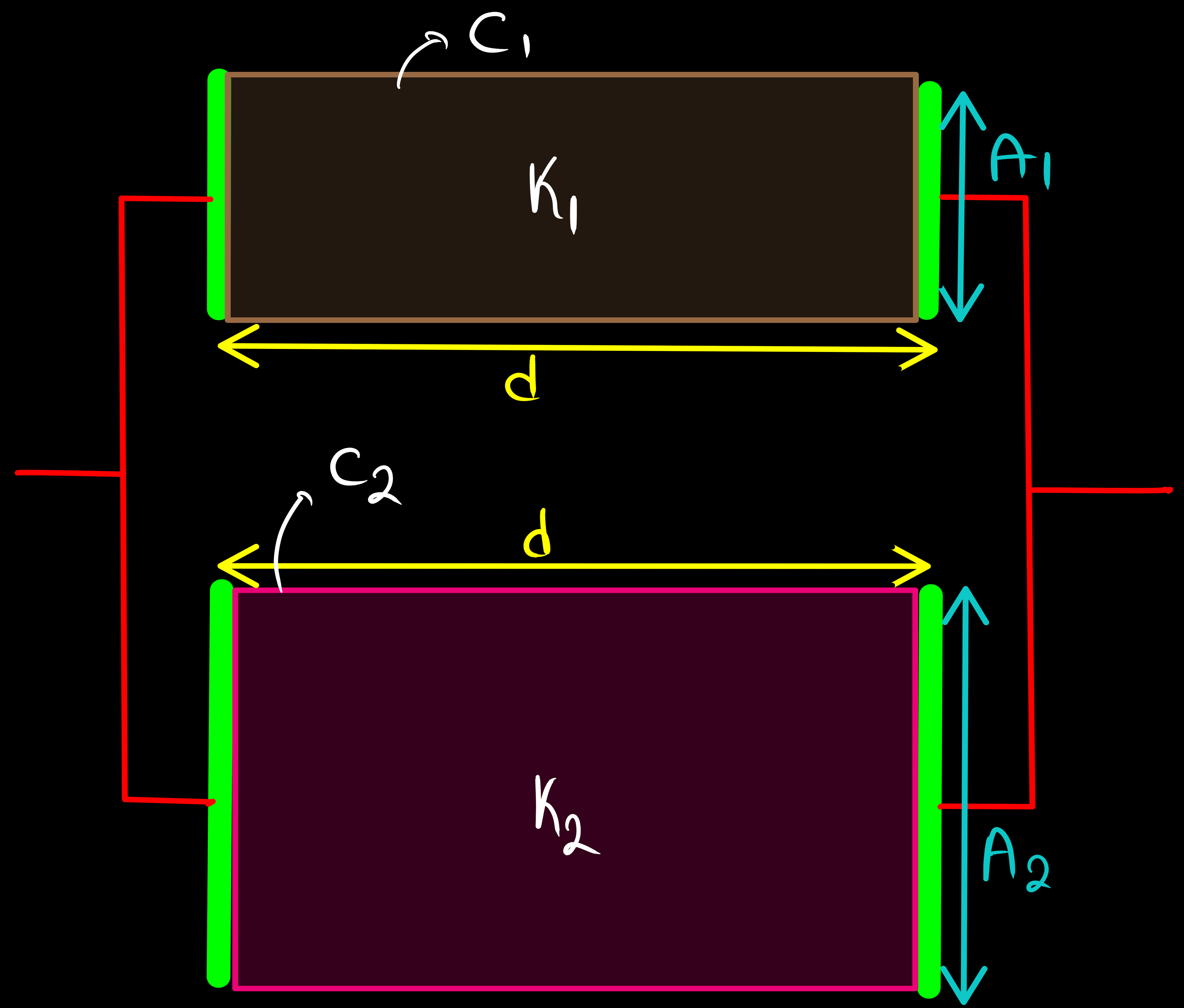


$$C = C_1 + C_2$$

$$C = \frac{K_1\varepsilon_0 A_1}{d} + \frac{K_2\varepsilon_0 A_2}{d}$$

$$\boxed{C = \frac{\varepsilon_0}{d}\left(K_1A_1 + K_2A_2\right)}$$

Energy Stored in a Capacitor

* Let q and V be the charge and potential respectively at an intermediate stage during charging process. Then

$$V = \frac{q}{C}$$

* At this stage the work required to add a very small amount of charge dq is

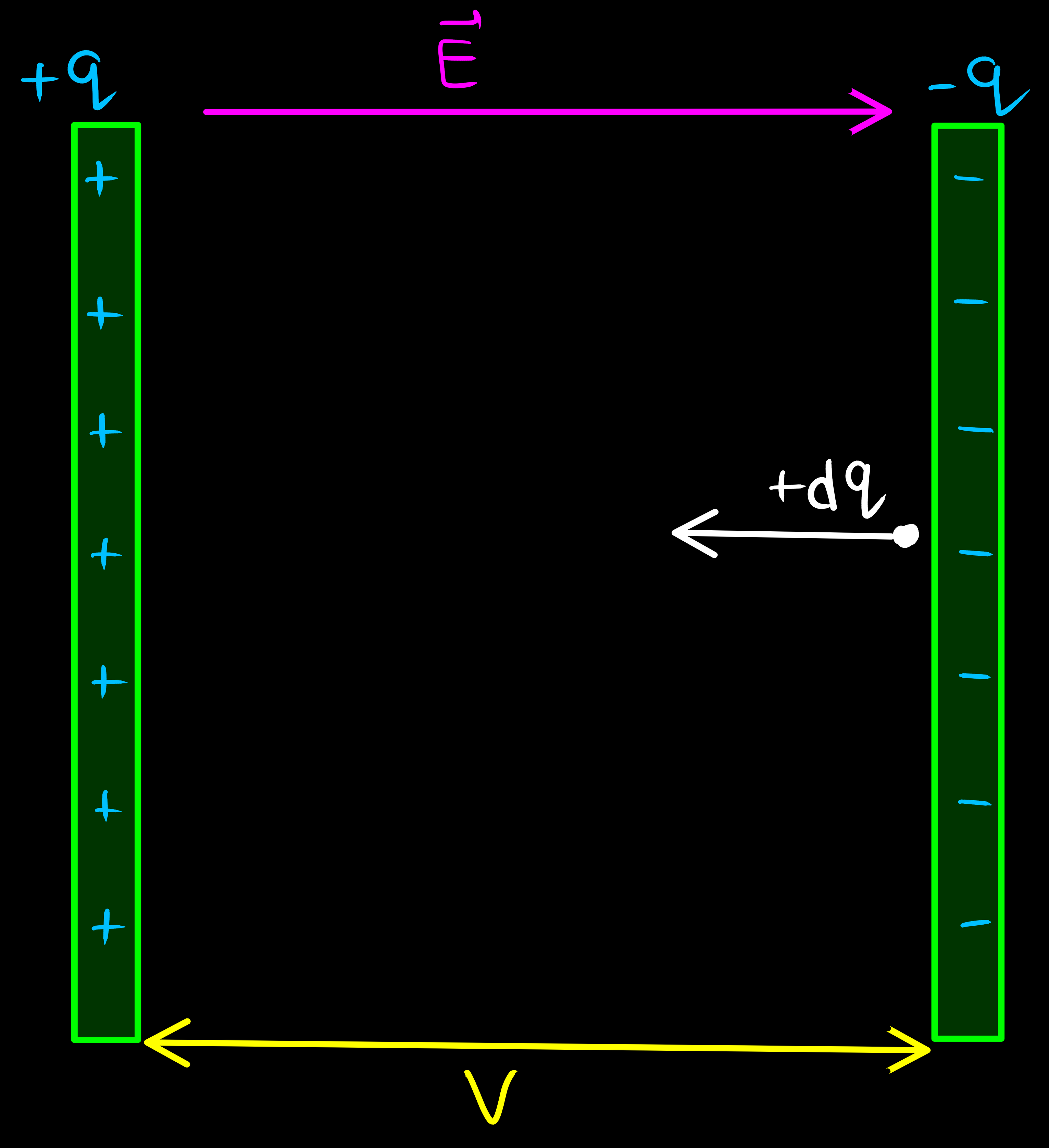
$$dW = V\,dq = \frac{q}{C}\,dq$$

* So, Total work needed to increase the charge q from 0 to final charge Q.

$$W = \int_0^Q \frac{q}{C}\,dq$$

$$W = \frac{1}{C}\int_0^Q q\,dq = \frac{1}{C}\left.\frac{q^2}{2}\right|_0^Q$$

$$W = \frac{1}{2}\frac{Q^2}{C}$$



* This work done by external agency (battery) is stored as Potential Energy of the capacitor.

$$U = \frac{1}{2}\frac{Q^2}{C}$$

Also, $Q = CV$

$$\boxed{U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}CV^2 = \frac{1}{2}QV}$$

* **Energy Density (u)** – It is defined as energy stored per unit volume of the space between the plates of the capacitor.

$$u = \frac{U}{\text{Volume}}$$

$$u = \frac{1}{2}\frac{CV^2}{Ad}$$

for parallel plate capacitor → $C = \frac{\varepsilon_0 A}{d}$

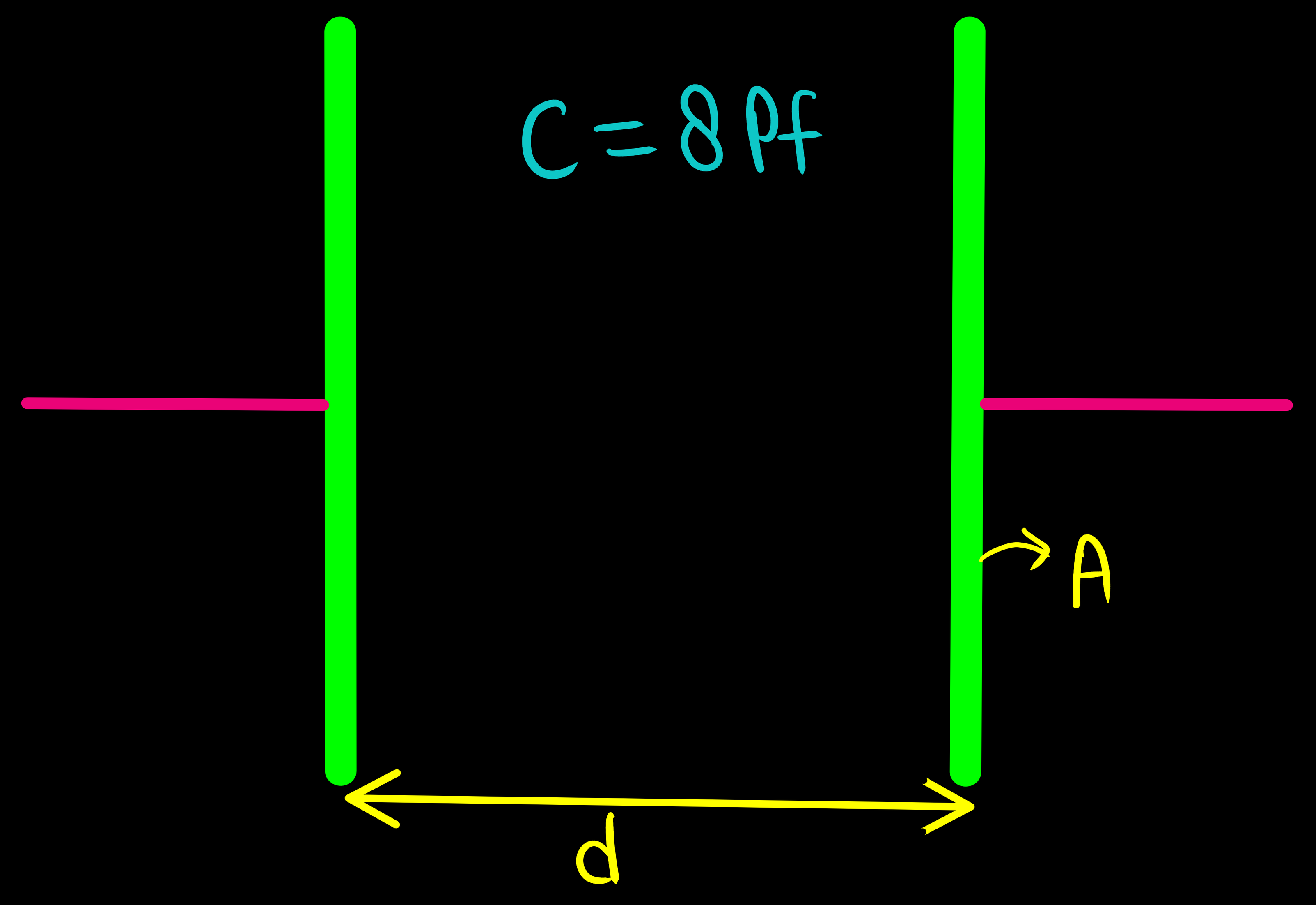
$$u = \frac{1}{2}\frac{\varepsilon_0 \cancel{A} V^2}{d\,\cancel{A}d}$$

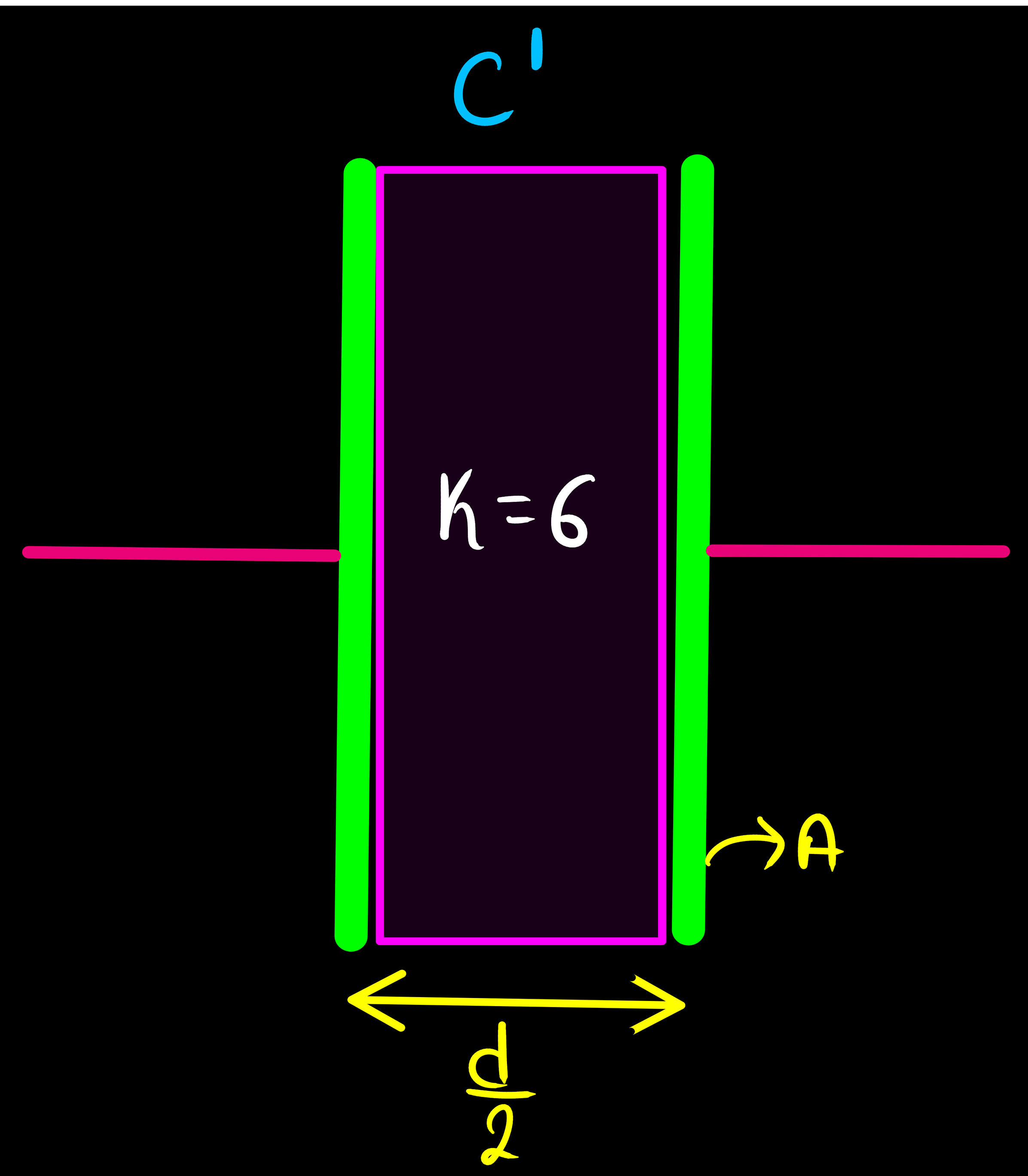
$$u = \frac{1}{2}\varepsilon_0 \times \left(\frac{V}{d}\right)$$

$$\boxed{u = \frac{1}{2}\varepsilon_0 E^2} \qquad \left(\because E = \frac{V}{d}\right)$$

* It is possible to view the potential energy of the capacitor as 'stored' in the electric field between the plates.

2.5 A parallel plate capacitor with air between the plates has a capacitance of 8 pF (1pF = 10^{-12} F). What will be the capacitance if the distance between the plates is reduced by half, and the space between them is filled with a substance of dielectric constant 6?





$$C = \frac{\varepsilon_0 A}{d} = 8\,\text{Pf}$$

$$C' = \frac{K\varepsilon_0 A}{\frac{d}{2}}$$

$$C' = 2K\frac{\varepsilon_0 A}{d}$$

$$\boxed{C' = 2\times 6\times 8\,\text{Pf} = 96\,\text{Pf}}$$

2.8 In a parallel plate capacitor with air between the plates, each plate has an area of $6 \times 10^{-3}\ m^2$ and the distance between the plates is 3 mm. Calculate the capacitance of the capacitor. If this capacitor is connected to a 100 V supply, what is the charge on each plate of the capacitor?

i)

$$C = \frac{\varepsilon_0 A}{d}$$

$$C = \frac{8.854 \times 10^{-12} \times \overset{2}{\cancel{6}} \times \cancel{10^{-3}}}{\cancel{3} \times \cancel{10^{-3}}}$$

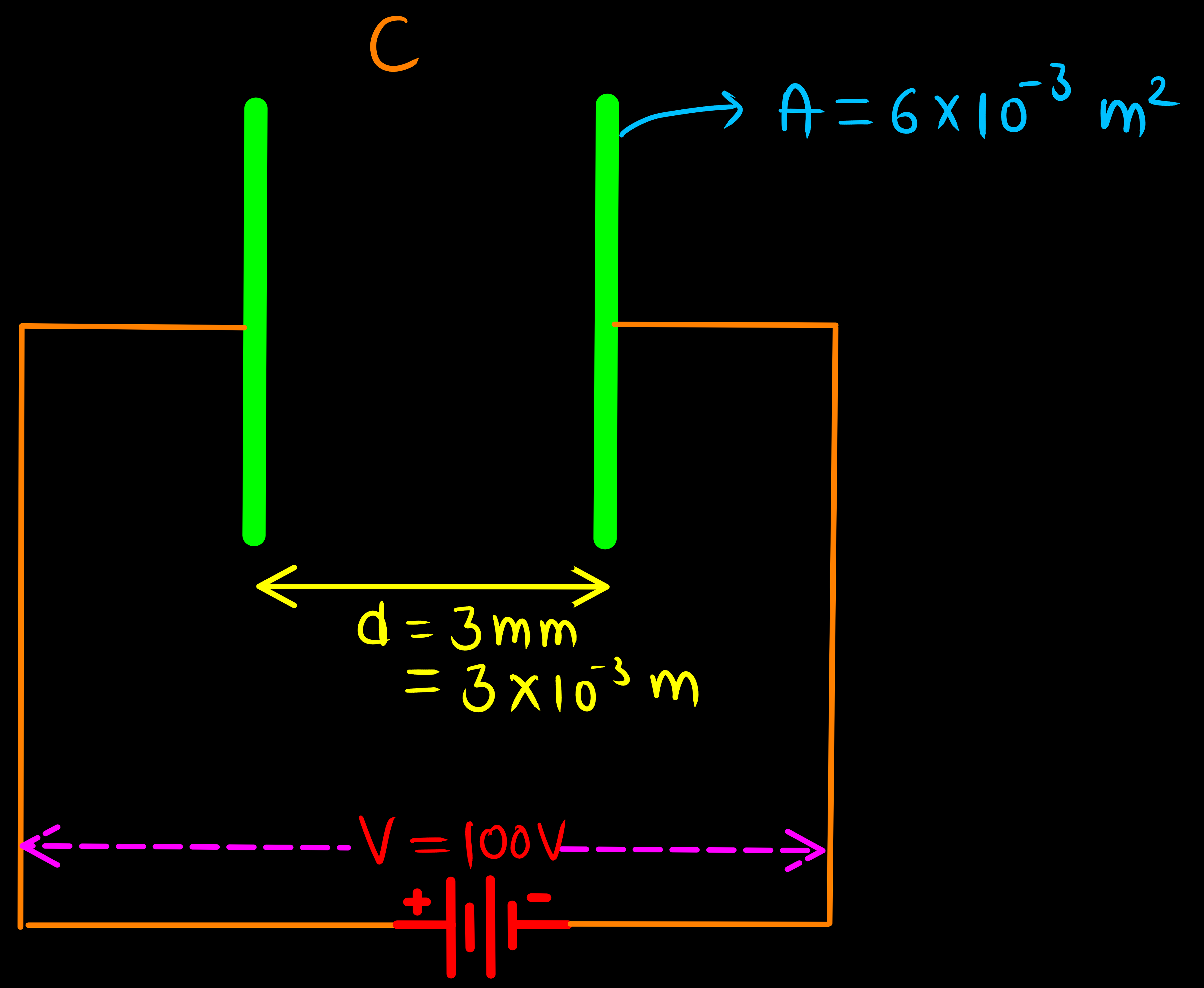
$$\boxed{C = 17.708 \times 10^{-12}\ F}$$

ii)

$$Q = CV$$

$$Q = 17.708 \times 10^{-12} \times 100\ C$$

$$\boxed{Q = 17.708 \times 10^{-10}\ C}$$

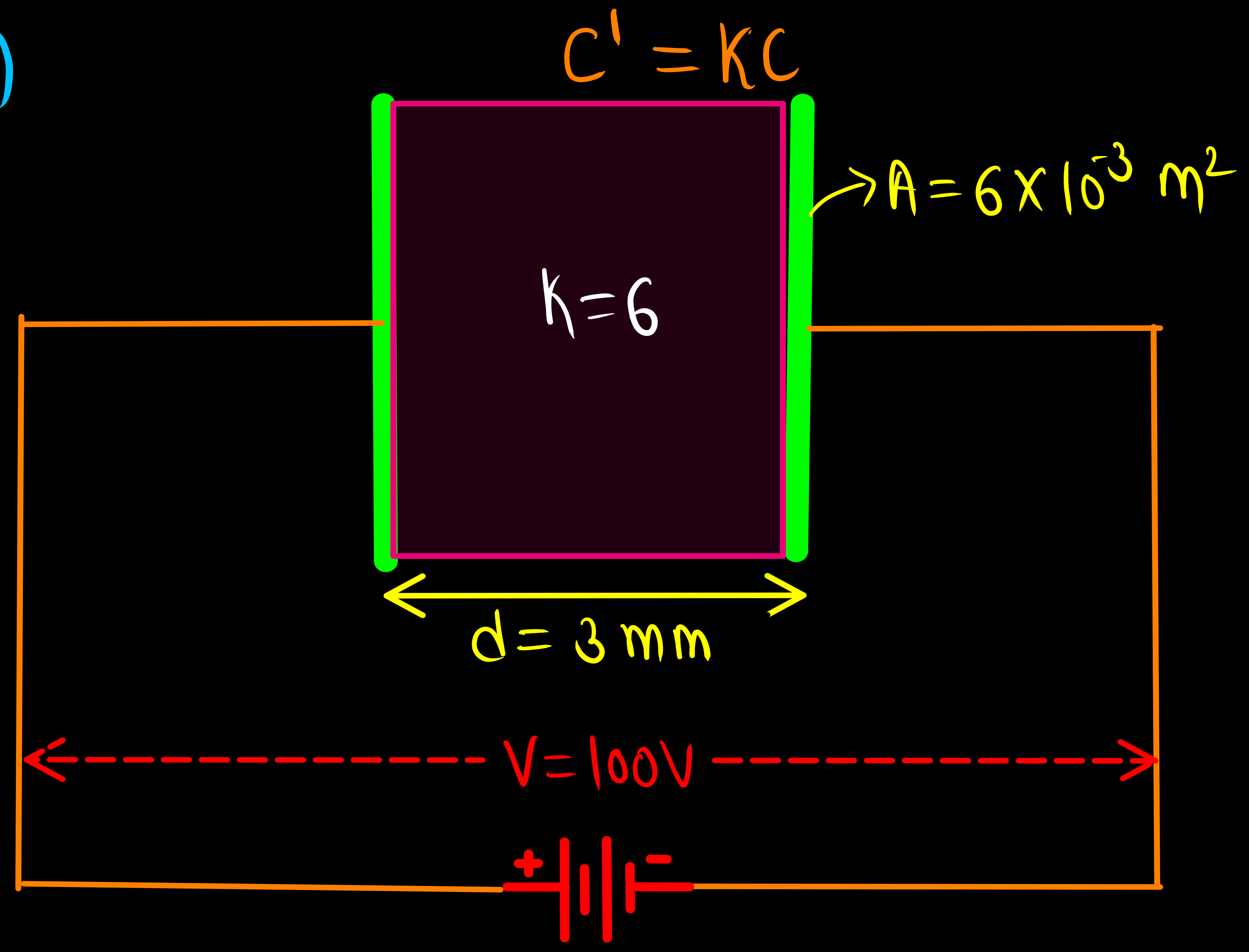


2.9 Explain what would happen if in the capacitor given in Exercise 2.8, a 3 mm thick mica sheet (of dielectric constant = 6) were inserted between the plates,

(a) while the voltage supply remained connected.

(b) after the supply was disconnected.

a)



* when Voltage supply remained connected & dielectric is inserted.

$V' = V$ remains constant

$C' = KC$ Capacitance increases

$Q' = C'V$

$Q' = KCV$

$Q' = KQ$ → Capacitance increases

from previous question $Q = 17.708 \times 10^{-10}\ C$

$Q' = 6 \times 17.708 \times 10^{-10}\ C$

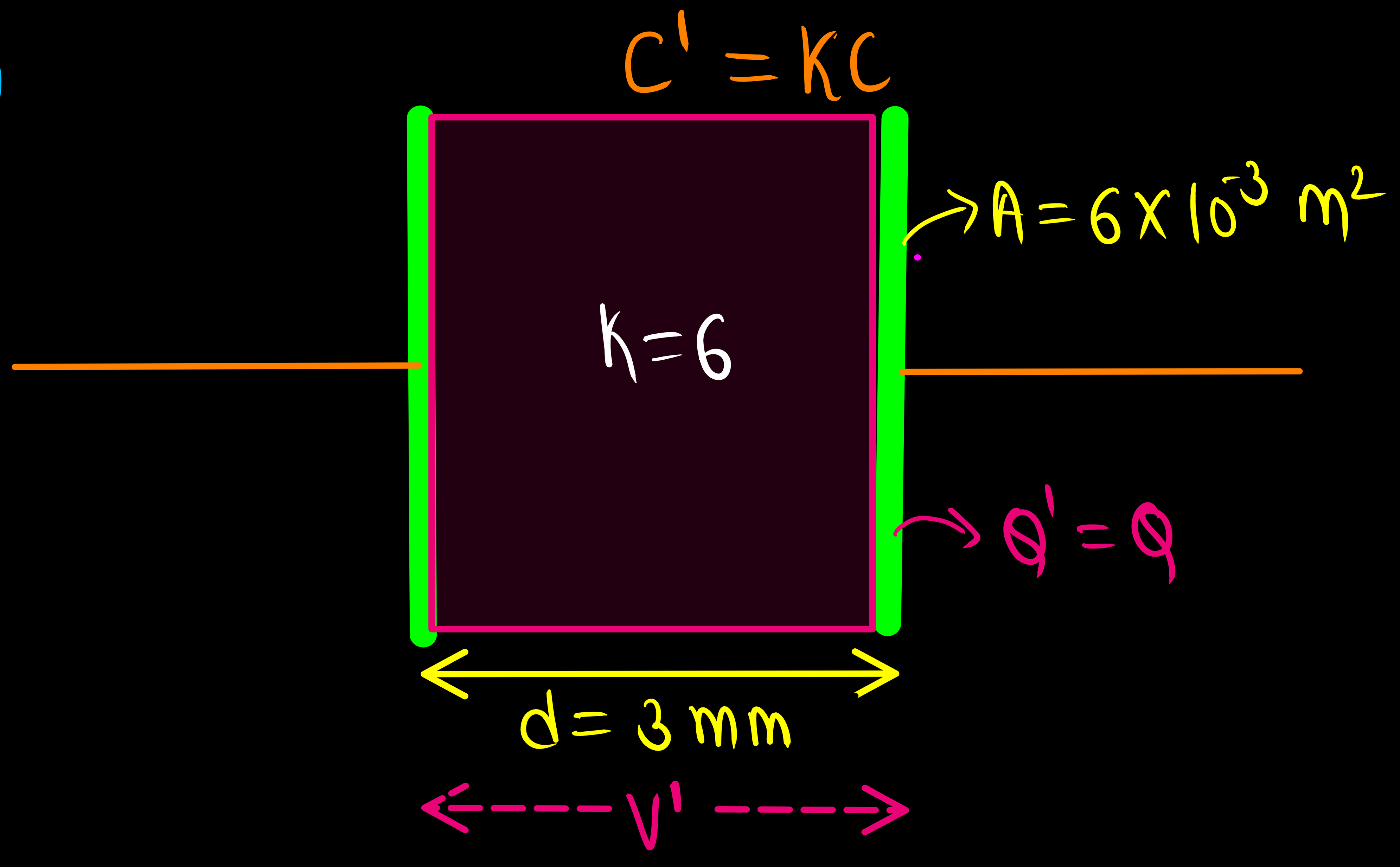
$\boxed{Q' = 106.248 \times 10^{-10}\ C}$

2.9 Explain what would happen if in the capacitor given in Exercise 2.8, a 3 mm thick mica sheet (of dielectric constant = 6) were inserted between the plates,

(a) while the voltage supply remained connected.

(b) after the supply was disconnected.

b)



* When dielectric is inserted after the supply was disconnected.

$Q' = Q$ → Charge on Capacitor plate remains constant

$C' = KC$ → Capacitance increases

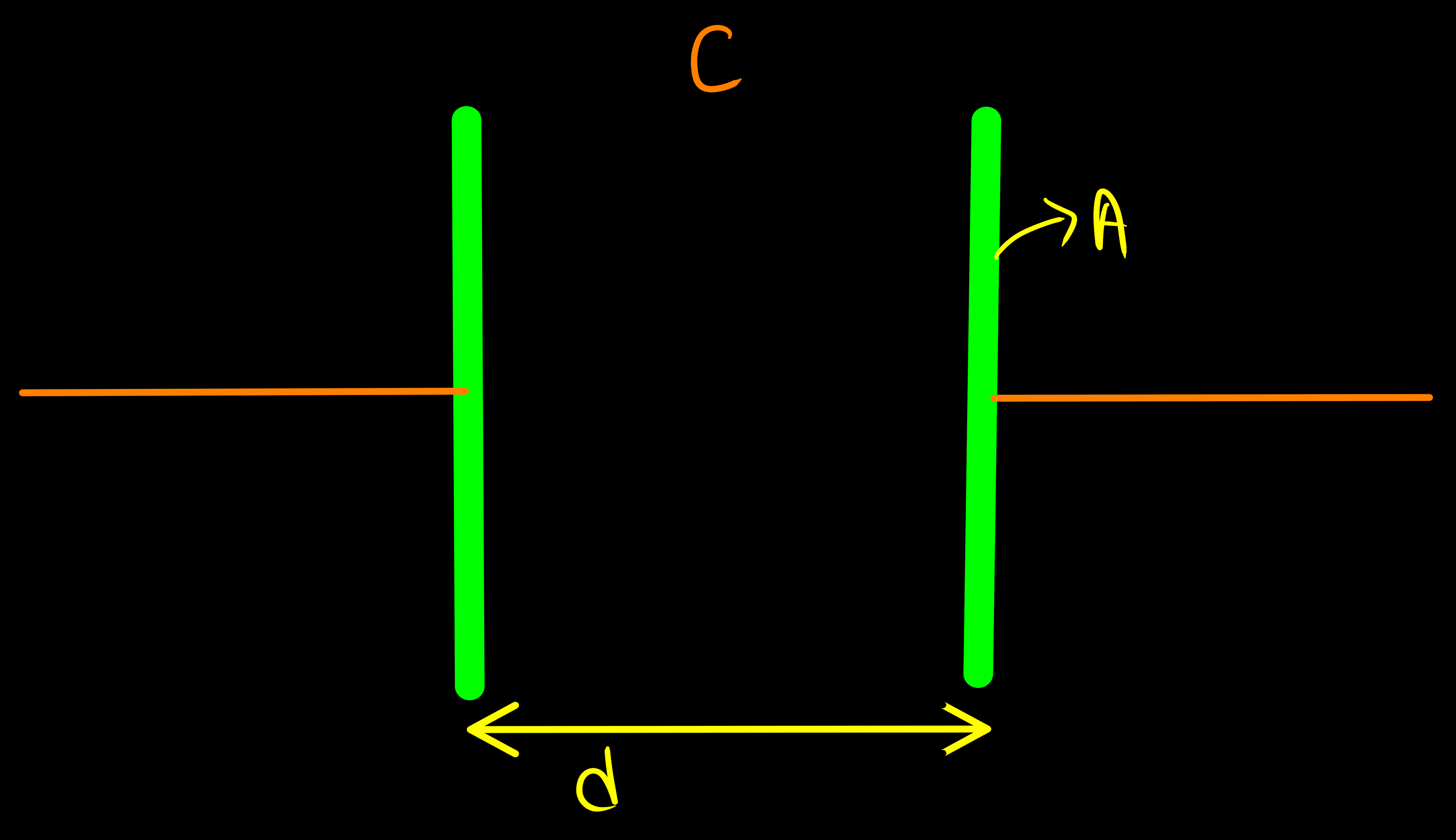
$$V' = \frac{Q'}{C'}$$

$$V' = \frac{Q}{KC}$$

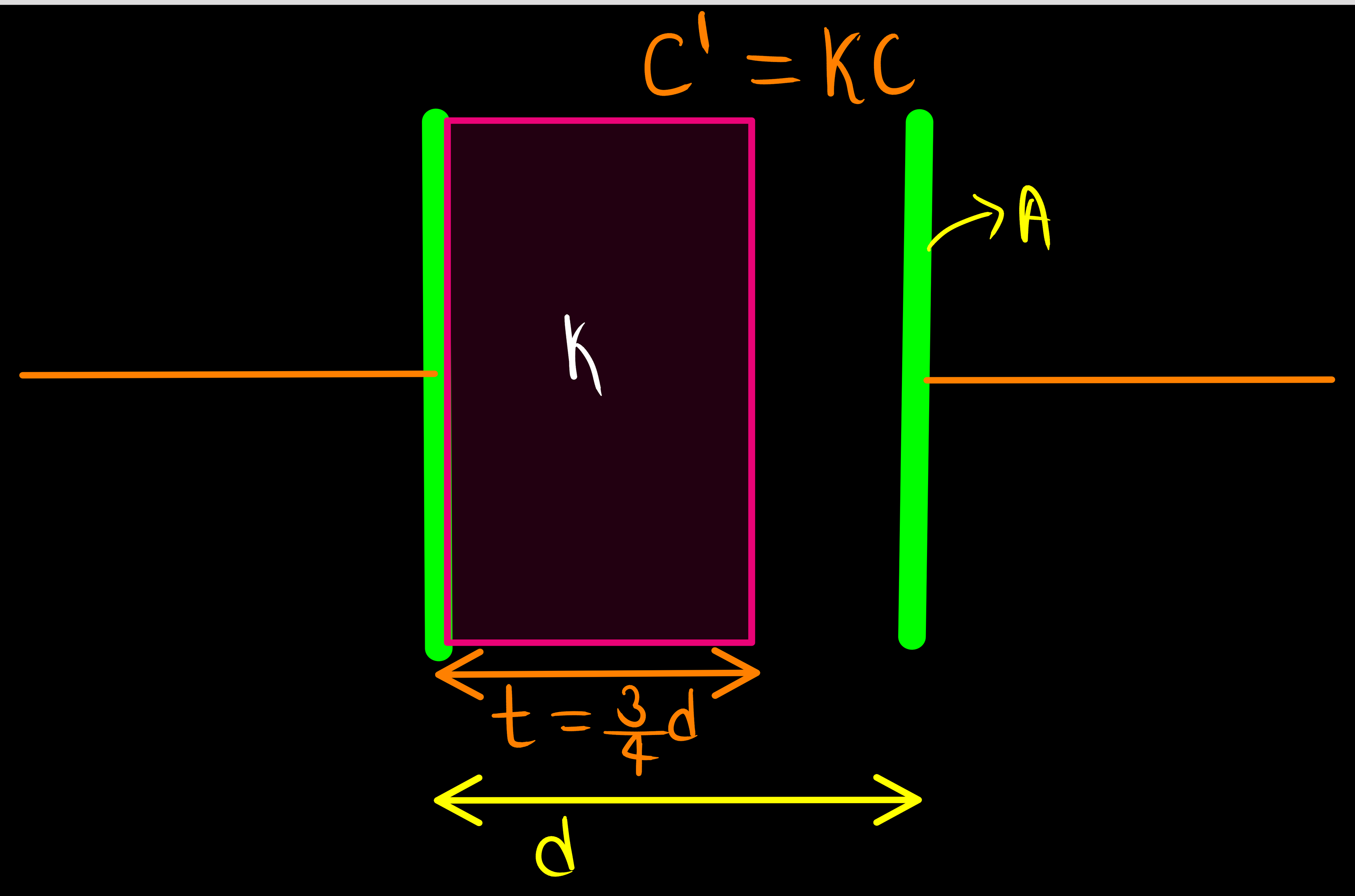
$$V' = \frac{V}{K} \qquad \left(\because \frac{Q}{C} = V\right)$$

↳ Voltage decreases $\frac{1}{K}$ times.

Example 2.8 A slab of material of dielectric constant K has the same area as the plates of a parallel-plate capacitor but has a thickness $(3/4)d$, where d is the separation of the plates. How is the capacitance changed when the slab is inserted between the plates?



$$C = \frac{\varepsilon_0 A}{d}$$



$$C' = \frac{\varepsilon_0 A}{\left(d - t + \frac{t}{K}\right)}$$

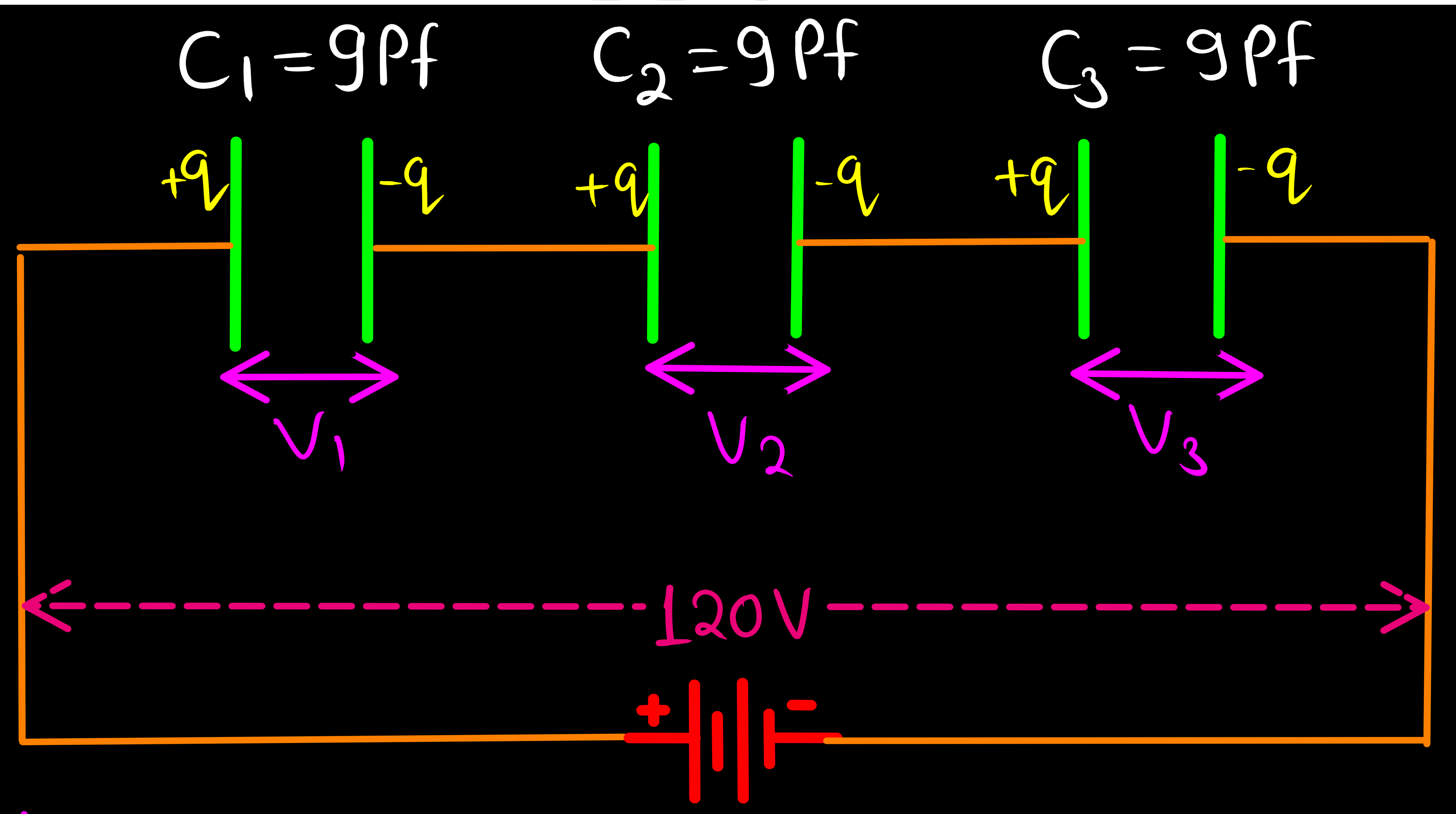
$$C' = \frac{\varepsilon_0 A}{\left(d - \frac{3d}{4} + \frac{3d}{4K}\right)} = \frac{\varepsilon_0 A}{\left(\frac{d}{4} + \frac{3d}{4K}\right)}$$

$$\boxed{C' = \frac{4\varepsilon_0 A}{d\left(1 + \frac{3}{K}\right)} = \frac{4K}{(K+3)} C}$$

2.6 Three capacitors each of capacitance 9 pF are connected in series.

(a) What is the total capacitance of the combination?

(b) What is the potential difference across each capacitor if the combination is connected to a 120 V supply?



*(a)

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$\frac{1}{C} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9}$$

$$\frac{1}{C} = \cancel{\frac{3}{9}}\ \frac{1}{3}$$

$$\boxed{C = 3\,pf}$$

* b) Charge stored by all Capacitors –

$q_1 = q_2 = q_3 = q$ → (In series q → same)

$q = CV = 3 \times 10^{-12} \times 120$

$$\boxed{q = 360 \times 10^{-12}\,C}$$

$$V_1 = \frac{q_1}{C_1} = \frac{q}{C_1}$$

$$V_1 = \frac{360 \times 10^{-12}}{9 \times 10^{-12}}$$

$$\boxed{V_1 = 40\ V}$$

$V_2 = V_3 = V_1 = \underline{\underline{40\ V}}$

2.7 Three capacitors of capacitances 2 pF, 3 pF and 4 pF are connected in parallel.

(a) What is the total capacitance of the combination?

(b) Determine the charge on each capacitor if the combination is connected to a 100 V supply.

a) $C = C_1 + C_2 + C_3$

$C = 2\,pf + 3\,pf + 4\,pf$

$\boxed{C = 9\,pF}$

b) $q_1 = C_1 V$

$q_1 = 2 \times 10^{-12} \times 100$

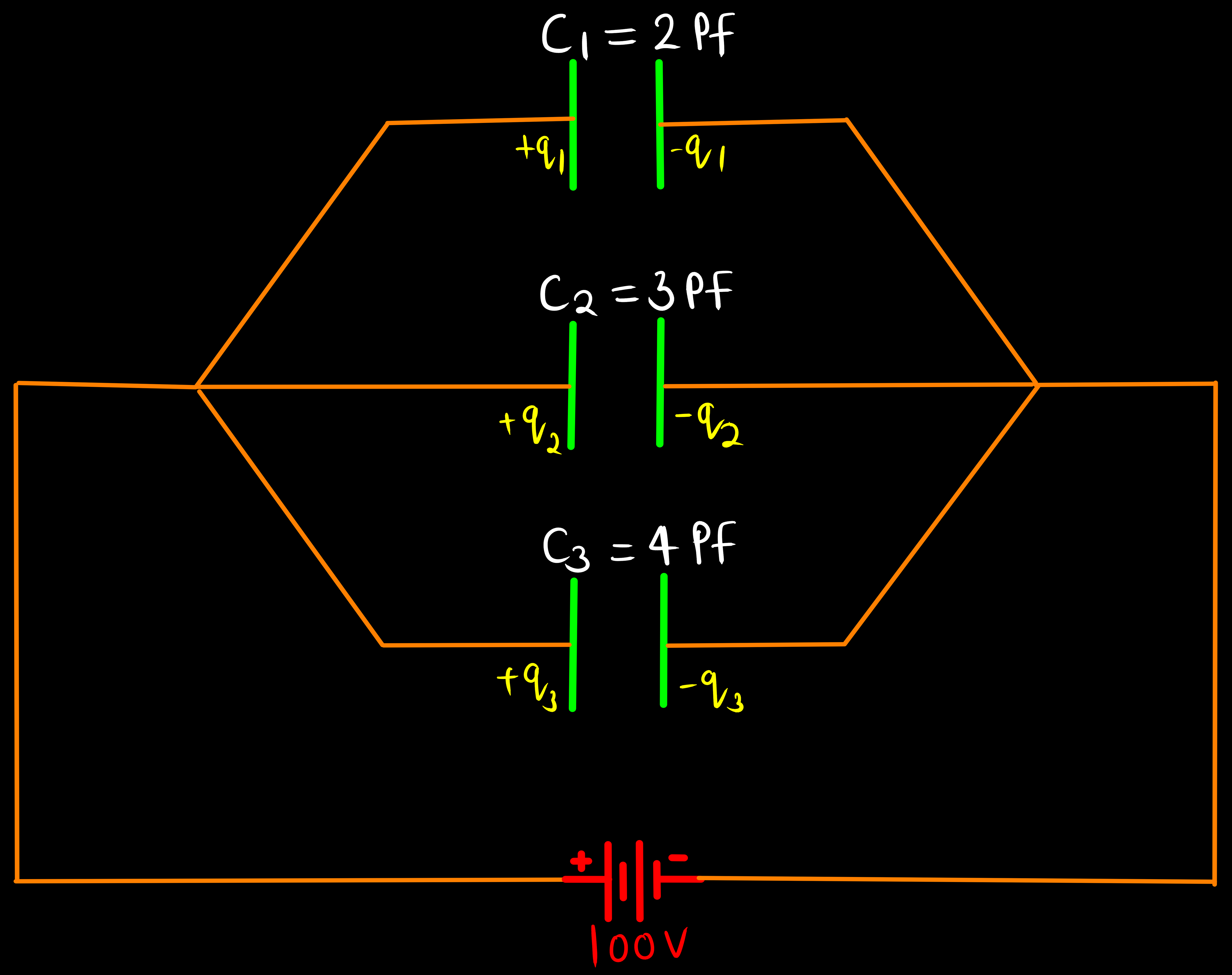
$\boxed{q_1 = 2 \times 10^{-10}\,C}$

$q_2 = C_2 V$

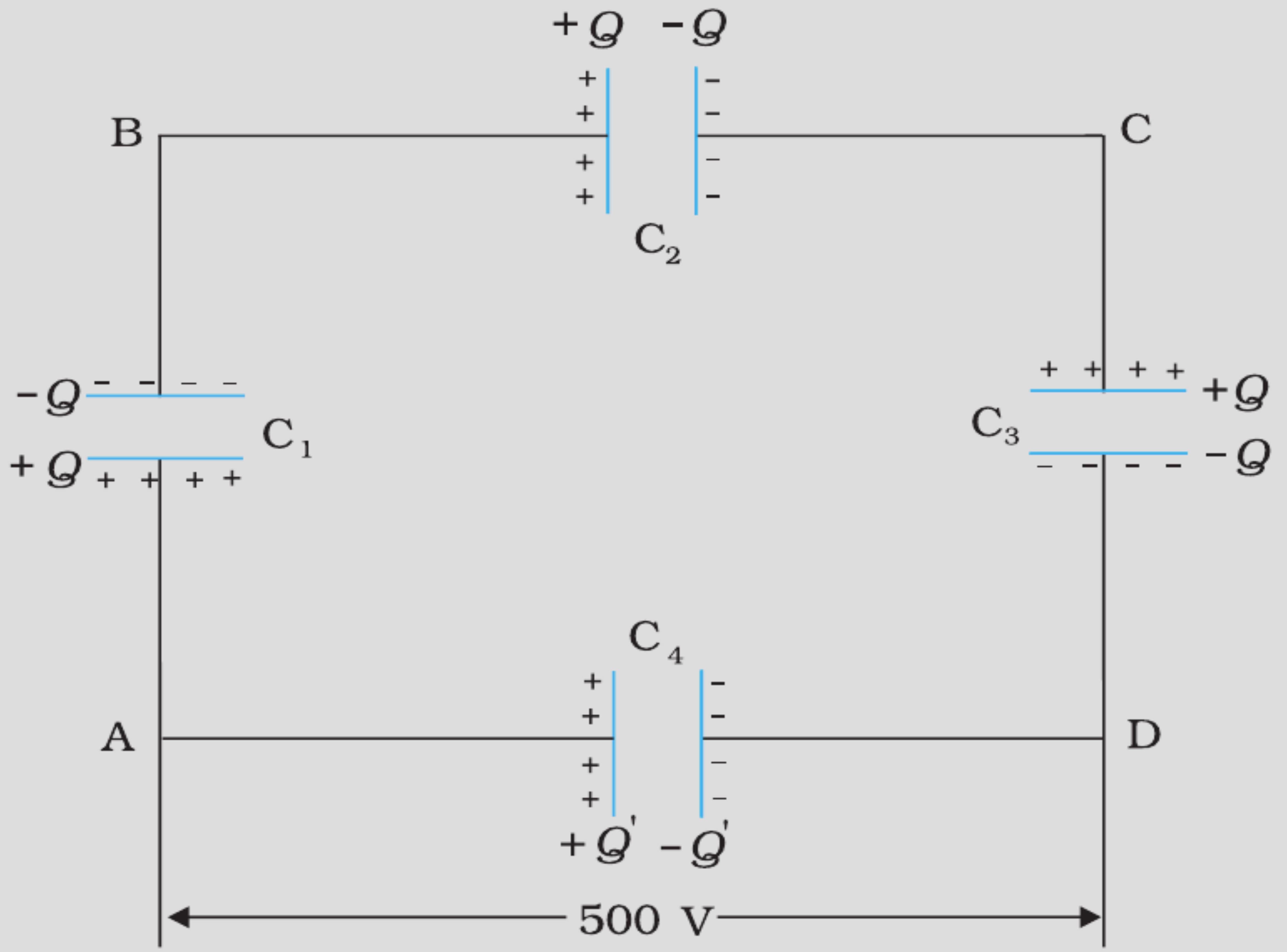
$\boxed{q_2 = 3 \times 10^{-12} \times 100 = 3 \times 10^{-10}\,C}$

$q_3 = C_3 V$

$\boxed{q_3 = 4 \times 10^{-12} \times 100 = 4 \times 10^{-10}\,C}$



Example 2.9 A network of four 10 µF capacitors is connected to a 500 V supply, as shown in Fig. 2.29. Determine (a) the equivalent capacitance of the network and (b) the charge on each capacitor. (Note, the *charge on a capacitor* is the charge on the plate with higher potential, equal and opposite to the charge on the plate with lower potential.)



a) C_1, C_2, and C_3 are in series -

$$\frac{1}{C'} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

$$\frac{1}{C'} = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{3}{10}$$

$$C' = \frac{10}{3}\ \mu F$$

Now, C' and C_4 are in parallel

$$C = C' + C_4$$

$$C = \left(\frac{10}{3} + 10\right) \mu F$$

$$\boxed{C = \frac{40}{3}\ \mu F}$$

b) $q_4 = C_4 V$

$$q_4 = 10 \times 10^{-6} \times 500$$

$$\boxed{q_4 = 5 \times 10^{-3}\ C}$$

$$V_1 + V_2 + V_3 = V$$

$$\frac{q}{C} + \frac{q}{C} + \frac{q}{C} = 500$$

$$\frac{3q}{C} = 500$$

$$q = \frac{500 \times C}{3} = \frac{500 \times 10 \times 10^{-6}}{3}$$

$$q = \frac{5}{3} \times 10^{-3}\ C$$

$$\boxed{q_1 = q_2 = q_3 = \frac{5}{3} \times 10^{-3}\ C}$$

2.10 A 12pF capacitor is connected to a 50V battery. How much electrostatic energy is stored in the capacitor?

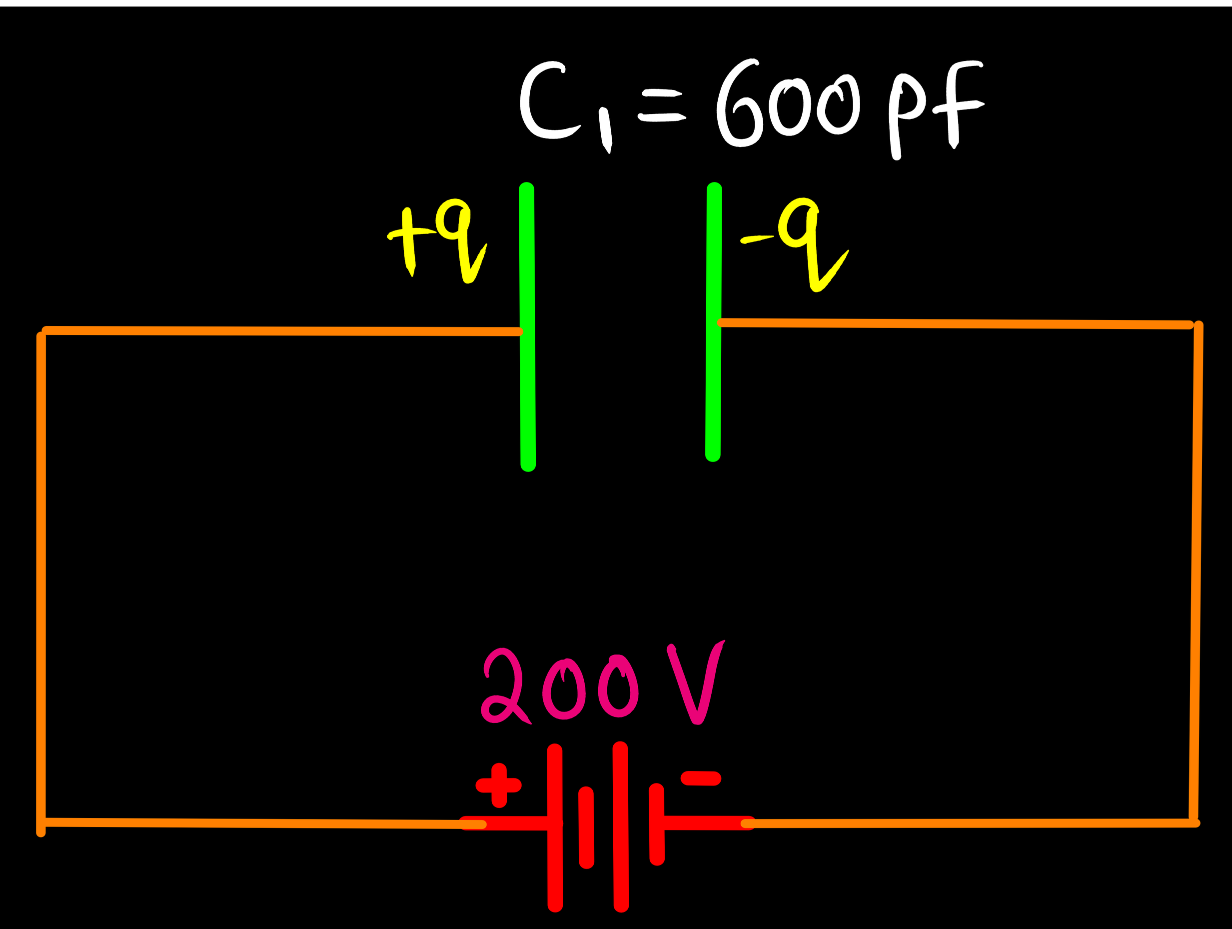
$C = 120\ pF$, $V = 50\ V$

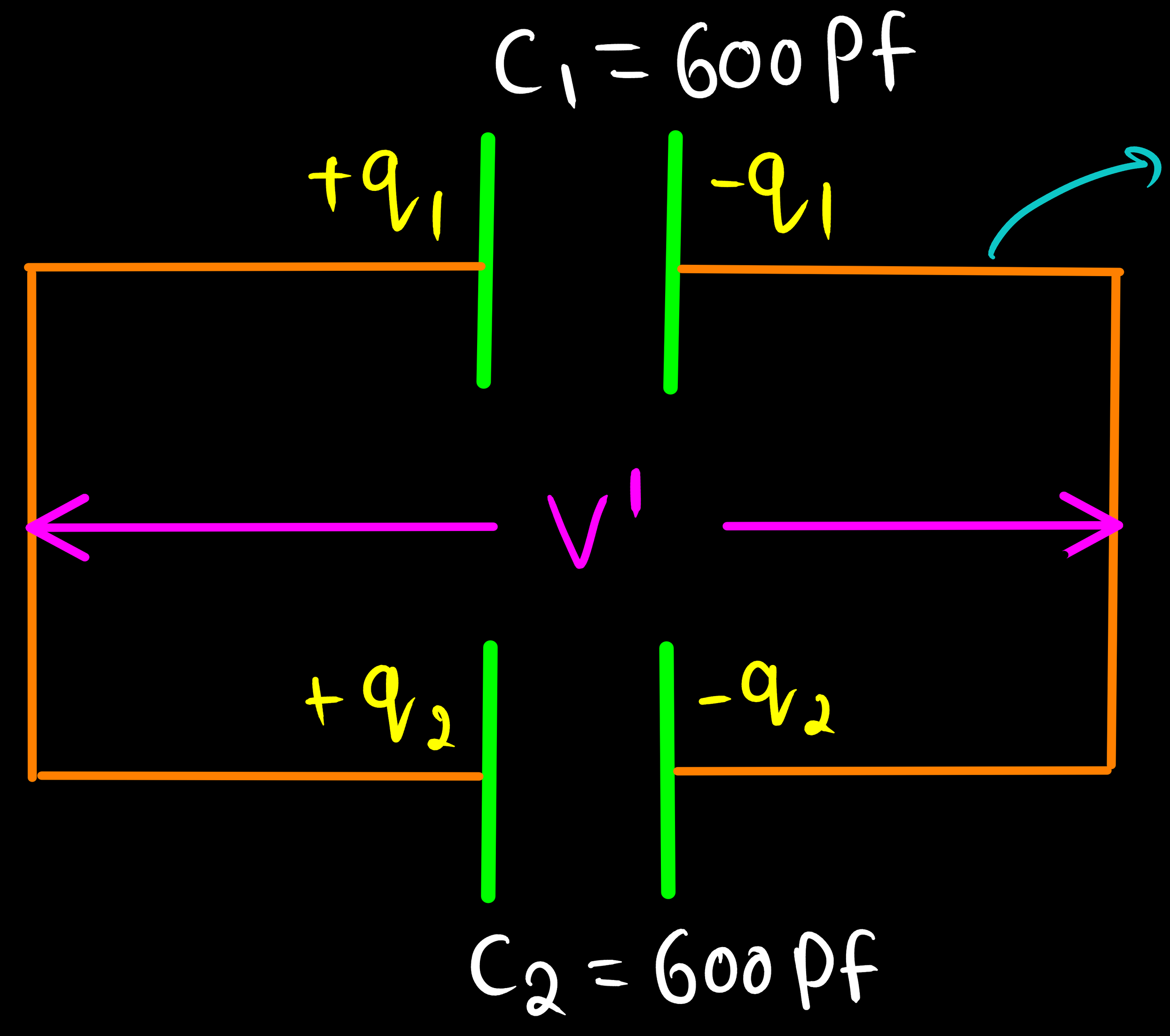
$U = \frac{1}{2} C V^2$ → Energy stored in the capacitor

$U = \frac{1}{2} \times 120 \times 10^{-6} \times 50$

$\boxed{U = 3 \times 10^{-3}\ J}$

2.11 A 600pF capacitor is charged by a 200V supply. It is then disconnected from the supply and is connected to another uncharged 600 pF capacitor. How much electrostatic energy is lost in the process?





Charge will flow from C_1 to C_2 untill both capacitor have common potential.

$q_1 + q_2 = q$

↳ Conservation of charge.

* Initial potential energy

$U_i = \frac{1}{2} C_1 V^2$

$U_i = \frac{1}{\cancel{2}} \times 600 \times 10^{-12} \times \cancel{4}^{2} \times 10^4$

$\boxed{U_i = 12 \times 10^{-6}\ J}$

* $q_1 + q_2 = q$

$\frac{c}{V'} + \frac{c}{V'} = q$

$\frac{2c}{V'} = q$

$V' = \frac{q}{2c} = \frac{V}{2} = \frac{200}{2} = 100V$

$C' = C_1 + C_2 = 1200\ pF$

Final Potential Energy -

$U_f = \frac{1}{2} C' V'^2$

$U_f = \frac{1}{\cancel{2}} \times \cancel{1200}^{600} \times 10^{-12} \times 100 \times 100$

$\boxed{U_f = 6 \times 10^{-6}\ J}$

Energy lost $= U_i - U_f = 12 \times 10^{-6} J - 6 \times 10^{-6} J$

$\boxed{\text{Energy lost} = 6 \times 10^{-6}\ J}$