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Optimization Method
homework 6

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I. PROBLEM 1

Given a linear programming problem:

$$\begin{aligned} \min \quad & -2x_1 - x_2 + x_3 \\ \text{s.t.} \quad & x_1 + x_2 + 2x_3 \le 6 \\ & x_1 + 4x_2 - x_3 \le 4 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

its simplex table is as follow:

	x_1	x_2	x_3	x_4	x_5	
x_3	0	-1	1	$\frac{1}{3}$	-$\frac{1}{3}$	$\frac{2}{3}$
x_1	1	3	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{14}{3}$
	0	-6	0	-$\frac{1}{3}$	-$\frac{5}{3}$	-$\frac{26}{3}$

1. if vector $b = (6,4)^T$ changed into $b' = (2,4)^T$ on the right side, is optimal base still optimal? Please solve the optimal table using old optimal table.
2. if change the coefficient of x_1 in objective function from $c_1 = -2$ into c_1', what range does c_1' belong to, old optimal solution is still optimal for new question?

Solution:

1. compute column vector on the right side

$$\overline{b'} = B^{-1}b' = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$c_B\overline{b'} = (1, -2) \begin{bmatrix} -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix} = -\frac{22}{3}$$

	x_1	x_2	x_3	x_4	x_5	
x_3	0	-1	1	$\frac{1}{3}$	-$\frac{1}{3}$	$\frac{2}{3}$
x_1	1	3	0	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{10}{3}$
	0	-6	0	-$\frac{1}{3}$	-$\frac{5}{3}$	-$\frac{22}{3}$
x_5	0	3	-3	-1	1	2
x_1	1	1	2	1	0	2
	0	-1	-5	-2	0	-4

this solution is $(x_1, x_2, x_3) = (2, 0, 0)$, $f_{min} = -4$.

2. change c_1 to c_1'

$$\begin{cases} z_1' - c_1' = 0 \\ z_2' - c_2' = -6 + 3(c_1' + 2) \le 0 \\ z_3' - c_3' = 0 + 0(c_1' + 2) \le 0 \\ z_4' - c_4' = -\frac{1}{3} + \frac{1}{3}(c_1' + 2) \le 0 \\ z_5' - c_5' = -\frac{5}{3} + \frac{2}{3}(c_1' + 2) \le 0 \end{cases}$$

we get $c_1' \le -1$.

II. Problem 2

Given a linear programming problem:

$$\begin{aligned} \max \quad & -5x_1 + 5x_2 + 13x_3 \\ \text{s.t.} \quad & -x_1 + x_2 + 3x_3 \le 20 \\ & 12x_1 + 4x_2 + 10x_3 \le 90 \\ & x_1, x_2, x_3 \ge 0 \end{aligned}$$

Please solve this problem using simplex method first. Then change the original problem respectively as follow, and solve them using old optimal table.

1. change the coefficient of x_3 in objective function c_3 from 13 into 8
2. change b_1 from 20 to 30
3. change b_2 from 90 to 70
4. change column of A from $(-1,12)^T$ to $(0,5)^T$
5. add constraint condition $2x_1 + 3x_2 + 5x_3 \le 50$

Solution:
Add slack variable x_4, x_5, change it into:

$$\begin{aligned} \max \quad & -5x_1 + 5x_2 + 13x_3 \\ \text{s.t.} \quad & -x_1 + x_2 + 3x_3 + x_4 \le 20 \\ & 12x_1 + 4x_2 + 10x_3 + x_5 \le 90 \\ & x_1, x_2, x_3, x_4, x_5 \ge 0 \end{aligned}$$

solve it using simplex method.

	x_1	x_2	x_3	x_4	x_5	
x_4	-1	1	3	1	0	20
x_5	12	4	10	0	1	90
	5	-5	-13	0	0	0
x_2	-1	1	3	1	0	20
x_5	16	0	-2	-4	1	10
	0	0	2	5	0	100

this solution is $(x_1, x_2, x_3) = (0, 20, 0)$, $f_{max} = 100$.

1. when c_3 changed from 13 to 8, x_3's test number$z_3' - c_3' = (z_3 - c_3) + c_3 - c_3'$=2+13-8=7>0,solution does not change. $(x_1, x_2, x_3) = (0, 20, 0)$, $f_{max} = 100$.

2. change b_1 from 20 to 30

	x_1	x_2	x_3	x_4	x_5	
x_2	-1	1	3	1	0	30
x_5	16	0	-2	-4	1	-30
	0	0	2	5	0	150
x_2	23	1	0	-5	$\frac{3}{2}$	-15
x_3	-8	0	1	2	$-\frac{1}{2}$	15
	16	0	0	1	1	120
x_4	$-\frac{23}{5}$	$-\frac{1}{5}$	0	1	$-\frac{3}{10}$	3
x_3	$\frac{6}{5}$	$\frac{2}{5}$	1	0	$\frac{1}{10}$	9
	$\frac{103}{5}$	$\frac{1}{5}$	0	0	$\frac{13}{10}$	117

the solution is $(x_1, x_2, x_3) = (0, 0, 9)$, $f_{max} = 117$.

3. change b_2 from 90 to 70

	x_1	x_2	x_3	x_4	x_5	
x_2	-1	1	3	1	0	20
x_5	16	0	-2	-4	1	-10
	0	0	2	5	0	100
x_2	23	1	0	-5	$\frac{3}{2}$	5
x_3	-8	0	1	2	$-\frac{1}{2}$	5
	16	0	0	1	1	90

the solution is $(x_1, x_2, x_3) = (0, 5, 5)$, $f_{max} = 90$.

4. change column of A from $(-1, 12)^T$ to $(0, 5)^T$, x_1's test number

$$z_1 - c_1 = c_B B^{-1} p_1 - c_1 = 0 - (-5) = 5 > 0$$

the solution is $(x_1, x_2, x_3) = (0, 20, 0)$, $f_{max} = 100$.

5. add constraint condition $2x_1 + 3x_2 + 5x_3 \leq 50$, add a new slack variable x_6

	x_1	x_2	x_3	x_4	x_5	x_6	
x_2	-1	1	3	1	0	0	20
x_5	16	0	-2	-4	1	0	10
x_6	2	3	5	0	0	1	50
	0	0	2	5	0	0	100
x_2	-1	1	3	1	0	0	20
x_5	16	0	-2	-4	1	0	10
x_6	5	0	-4	-3	0	1	-10
	0	0	2	5	0	0	100
x_2	$\frac{11}{4}$	1	0	$-\frac{5}{4}$	0	$\frac{3}{4}$	$\frac{25}{2}$
x_5	$\frac{27}{2}$	0	0	$-\frac{5}{2}$	1	$-\frac{1}{2}$	15
x_3	$-\frac{5}{4}$	0	1	$\frac{3}{4}$	0	$-\frac{1}{4}$	$\frac{5}{2}$
	$\frac{5}{2}$	0	0	$\frac{7}{2}$	0	$\frac{1}{2}$	95

the solution is $(x_1, x_2, x_3) = (0, \frac{25}{2}, \frac{5}{2})$, $f_{max} = 95$.

III. PROBLEM 3

Given the original linear problem:

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Suppose this problem and its dual problem is feasible, $w^{(0)}$ is an optimal solution of dual problem.

1. if $\mu \neq 0$ multiply the kth equation of original problem, we can get a new problem. Please solve dual problem of this problem.

2. if μ multiply the kth equation of original problem and add it to the rth equation. Please solve dual problem of this problem.

Solution:
suppose A is a $m * n$ matrix. And write

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix}, b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

we can rewrite the original problem into:

$$\begin{aligned} \min \quad & cx \\ \text{s.t.} \quad & A_i x = b_i, i = 1, 2, \cdots, m \\ & x \geq 0 \end{aligned}$$

and its dual problem:

$$\begin{aligned} \max \quad & \sum_{i=1}^{m} b_i w_i \\ \text{s.t.} \quad & \sum_{i=1}^{m} w_i A_i \leq c \end{aligned}$$

1. $\mu \neq 0$ multiply the kth equation of original problem, dual problem is

$$\begin{aligned} \max \quad & b_1 w_1 + b_2 w_2 + \cdots + \mu b_k w_k + \cdots + b_m w_m \\ \text{s.t.} \quad & w_1 A_1 + \cdots + \mu w_k A_k + \cdots + w_m A_m \leq c \end{aligned}$$

   we can see that $\mathbf{w} = (w_1^{(0)}, \cdots, \frac{1}{\mu} w_k^{(0)}, \cdots, w_m^{(0)})$ is a feasible solution, and at this point, the value of objective function equals to the dual one, so this a optimal solution.

2. the original problem after changing:

$$
\begin{aligned}
\min \quad & cx \\
\text{s.t.} \quad & A_1 x = b_1, \\
& \quad \vdots \\
& A_k x = b_k, \\
& \quad \vdots \\
& (A_r + \mu A_k) x = b_r + \mu b_k, \\
& \quad \vdots \\
& A_m x = b_m, \\
& x \geq 0
\end{aligned}
$$

its dual problem:

$$
\begin{aligned}
\max \quad & b_1 w_1 + \cdots + \mu b_k w_k + \cdots + (b_r + \mu b_k) w_r + \cdots + b_m w_m \\
\text{s.t.} \quad & w_1 A_1 + \cdots + \mu w_k A_k + \cdots + w_r (A_r + \mu A_k) + \cdots + w_m A_m \leq c
\end{aligned}
$$

we can see that $\mathbf{w} = (w_1^{(0)}, \cdots, w_k^{(0)} - \mu w_r^{(0)}, \cdots, w_r^{(0)}, \cdots, w_m^{(0)})$ is a feasible solution, and at this point, the value of objective function equals to the dual one, so this a optimal solution.