SCHOOL OF SOFTWARE, TSINGHUA UNIVERSITY

Optimization Method

homework 2

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CONTENTS

I. Problem 1

Suppose $S = \{x|Ax \geq b\}$, A is a $m*n$ matrix, $m > n$ and $r(A) = n$. Prove that the necessary and sufficient condition of $x^{(0)}$ is S's pole is that A and b can be divided as follows.

$$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Among them, A_1 has n rows and $r(A) = n$, b_1 is a n-dimensional column vector, subject to $A_1 x^{(0)} = b_1, A_2 x^{(0)} \geq b_2$.

Proof:

- prove the necessity first.
  we know that $x^{(0)}$ is S's pole. let me prove it by contradiction.Divide A, b into the following form at $x^{(0)}$:

  $$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},\ A_1 x^{(0)} = b_1,\ A_2 x^{(0)} > b_2.$$

  $r(A_1) = k < n$, suppose first k columns is linearly independent, we can rewrite $A_1 = [B\ N]$, B is invertible.

  $$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix},\ x_B = B^{-1} b_1 - B^{-1} N x_N.$$

  The solution of $A_1 x = b_1$ is

  $$x = \begin{bmatrix} x_B \\ x_N \end{bmatrix} = \begin{bmatrix} B^{-1} b_1 - B^{-1} N x_N \\ x_N \end{bmatrix}. \tag{i.1}$$

  S's pole

  $$x^{(0)} = \begin{bmatrix} x_B^{(0)} \\ x_N^{(0)} \end{bmatrix} = \begin{bmatrix} B^{-1} b_1 - B^{-1} N x_N^{(0)} \\ x_N^{(0)} \end{bmatrix}. \tag{i.2}$$

  Since $A_2 x^{(0)} > b_2$, $\exists x_N \in N_\delta(x_N^{(0)})$, $A_1 x = b_1$ and $A_2 x > b_2$ holds. On the line passed $x_N^{(0)}$, $\exists x_N^{(1)}, x_N^{(2)} \in N_\delta(x_N^{(0)})$, subject to $\lambda x_N^{(1)} + (1-\lambda) x_N^{(2)} = x_N^{(0)}$, $\lambda \in (0,1)$, substitute this into (i.2), we get

  $$\begin{aligned} x^{(0)} &= \begin{bmatrix} B^{-1} b_1 - B^{-1} N(\lambda x_N^{(1)} + (1-\lambda) x_N^{(2)}) \\ \lambda x_N^{(1)} + (1-\lambda) x_N^{(2)} \end{bmatrix} \\ &= \lambda \begin{bmatrix} B^{-1} b_1 - B^{-1} N x_N^{(1)} \\ x_N^{(1)} \end{bmatrix} + (1-\lambda) \begin{bmatrix} B^{-1} b_1 - B^{-1} N x_N^{(2)} \\ x_N^{(2)} \end{bmatrix}. \end{aligned} \tag{i.3}$$

  Thus, $x^{(0)}$ can be represented as convex combination of two point in S which is contradicted with $x^{(0)}$ is pole.

- Then prove the sufficiency.
  Suppose at point $x^{(0)}$, A, b can be divided as follows:

  $$A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix},\ b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix},\ A_1 x^{(0)} = b_1,\ A_2 x^{(0)} \geq b_2, r(A_1) = n.$$

And suppose $\exists x^{(1)}, x^{(2)} \in S$, subject to

$$\begin{aligned} x^{(0)} &= \lambda x^{(1)} + (1-\lambda) x^{(2)}, \lambda \in (0,1) \\ A_1 x^{(0)} &= \lambda A_1 x^{(1)} + (1-\lambda) A_1 x^{(2)} \end{aligned}$$

Because $A_1 x^{(0)} = b_1$, $A_1 x^{(1)} \geq b_1$, $A_1 x^{(2)} \geq b_1$and $\lambda, 1-\lambda > 0$, substitute it into the above equation.

$$b_1 = A_1 x^{(0)} = \lambda A_1 x^{(1)} + (1-\lambda) A_1 x^{(2)} \geq \lambda b_1 + (1-\lambda) b_1 = b_1$$

so we have

$$\begin{aligned} \lambda A_1 x^{(1)} + (1-\lambda) A_1 x^{(2)} &= \lambda b_1 + (1-\lambda) b_1 \\ \lambda (A_1 x^{(1)} - b_1) + (1-\lambda)(A_1 x^{(2)} - b_1) &= 0 \end{aligned}$$

Since $\lambda, 1-\lambda > 0$, $A_1 x^{(1)} - b_1 \geq 0$, $A_1 x^{(2)} - b_1 \geq 0$, so $A_1 x^{(1)} - b_1 = 0$, $A_1 x^{(2)} - b_1 = 0$, then

$$\begin{aligned} A_1 x^{(0)} = A_1 x^{(1)} = A_1 x^{(2)} = b_1 \\ x^{(0)} = x^{(1)} = x^{(2)} \end{aligned}$$

Thus, $x^{(0)}$ is pole.

II. PROBLEM 2

solve the following LP problem using simplex algorithm.

1.

$$\begin{aligned} \min \quad & 3x_1 - 5x_2 - 2x_3 - x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leq 4 \\ & 4x_1 - x_2 + x_3 + 2x_4 \leq 6 \\ & -x_1 + x_2 + 2x_3 + 3x_4 \leq 12 \\ & x_j \geq 0, j = 1, \cdots, 4 \end{aligned}$$

Solution:
Lead slack variable x_5, x_6, x_7.

$$\begin{aligned} \min \quad & 3x_1 - 5x_2 - 2x_3 - x_4 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 + x_5 = 4 \\ & 4x_1 - x_2 + x_3 + 2x_4 + x_6 = 6 \\ & -x_1 + x_2 + 2x_3 + 3x_4 + x_7 = 12 \\ & x_j \geq 0, j = 1, \cdots, 7 \end{aligned}$$

solve it using simplex algorithm.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	
x_5	1	1	1	0	1	0	0	4
x_6	4	-1	1	2	0	1	0	6
x_7	-1	1	2	3	0	0	1	12
f	-3	5	2	1	0	0	0	0
x_2	1	1	1	0	1	0	0	4
x_6	5	0	2	2	1	1	0	10
x_7	-2	0	1	3	-1	0	1	8
f	8	0	-3	1	-5	0	0	-20
x_2	1	1	1	0	1	0	0	4
x_6	$\frac{19}{3}$	0	$-\frac{4}{3}$	0	2	1	$-\frac{2}{3}$	$\frac{14}{3}$
x_4	$-\frac{2}{3}$	0	$\frac{1}{3}$	1	0	0	$\frac{1}{3}$	$\frac{8}{3}$
f	$-\frac{22}{3}$	0	$-\frac{10}{3}$	0	$-\frac{14}{3}$	0	$-\frac{1}{3}$	$-\frac{68}{3}$

From the above simplex table, we can see that solution $\mathbf{x}$=(0,4,0,$\frac{8}{3}$,0,$\frac{14}{3}$,0)$, f_{min} = -\frac{68}{3}$.

2.

$$\begin{aligned} \min \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & 3x_1 + 3x_2 + x_3 = 30 \\ & 4x_1 - 4x_2 + x_4 = 6 \\ & 2x_1 - x_2 \le 12 \\ & x_j \ge 0, j = 1, \cdots, 4 \end{aligned}$$

Solution:

Lead slack variable x_5.

$$\begin{aligned} \min \quad & -3x_1 - x_2 \\ \text{s.t.} \quad & 3x_1 + 3x_2 + x_3 = 30 \\ & 4x_1 - 4x_2 + x_4 = 6 \\ & 2x_1 - x_2 + x_5 = 12 \\ & x_j \ge 0, j = 1, \cdots, 5 \end{aligned}$$

solve it using simplex algorithm.

	x_1	x_2	x_3	x_4	x_5	
x_3	3	3	1	0	0	30
x_4	4	-4	0	1	0	16
x_5	2	-1	0	0	1	12
f	3	1	0	0	0	0
x_3	0	6	1	0	0	18
x_1	1	-1	0	$-\frac{3}{4}$	0	4
x_5	0	1	0	$\frac{1}{4}$	1	4
f	0	4	0	$-\frac{1}{2}$	0	-12
x_2	0	1	$\frac{1}{6}$	$-\frac{1}{8}$	0	3
x_1	1	0	$\frac{1}{6}$	$\frac{1}{8}$	0	7
x_5	0	0	$-\frac{1}{6}$	$-\frac{3}{8}$	1	1
f	0	0	$-\frac{2}{3}$	$-\frac{1}{4}$	0	-24

From the above simplex table, we can see that solution $\mathbf{x}$=(7,3,0,0,1),$f_{min} = -24$.

III. Problem 3

Suppose when solving LP problem using simplex algorithm

$$\begin{aligned} \min \quad & \mathbf{cx} \\ \text{s.t.} \quad & \mathbf{Ax} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

in one iteration, the check number of variable x_j $z_j - c_j > 0$, and the corresponding column in simplex table $y_j = \mathbf{B}^{-1}\mathbf{p}_j \leq 0$, Prove that

$$\mathbf{d} = \begin{bmatrix} -y_j \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

is the polar direction of feasible region, component 1 correspond to x_j among it. (suppose $\mathbf{B}$ is $\mathbf{A}$'s first m columns)

Solution:

Suppose $\mathbf{A}$ is a $m * n$ matrix, and

$$\mathbf{A} = [\mathbf{p}_1\ \mathbf{p}_2\ \cdots\ \mathbf{p}_m\ \cdots\ \mathbf{p}_n] = [\mathbf{B}\ \mathbf{p}_{m+1}\ \cdots\ \mathbf{p}_n\ ]$$

Since

$$\mathbf{A}\mathbf{d} = \left[\mathbf{B}\ \mathbf{p}_{m+1}\ \cdots\ \mathbf{p}_n\ \right] \begin{bmatrix} -\mathbf{B}^{-1}\mathbf{p}_j \\ 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = -\mathbf{p}_j + \mathbf{p}_j = 0$$

and $\mathbf{d} \geq 0$, $\mathbf{d}$ is the direction of feasible region. Then, we can prove $\mathbf{d}$ is a polar direction by contradiction.
Suppose $\mathbf{d}$ is a positive linear combination of two directions of feasible region, $\mathbf{d_1}$, $\mathbf{d_2}$.

$$\mathbf{d} = \lambda\mathbf{d_1} + \mu\mathbf{d_2} \tag{iii.1}$$

$\lambda, \mu > 0$, $\mathbf{d_1}, \mathbf{d_2} \geq 0$, from $\mathbf{d}$'s form, we can know that

$$\mathbf{d_1} = \begin{bmatrix} -y_j^{(1)} \\ 0 \\ \vdots \\ a_j \\ \vdots \\ 0 \end{bmatrix}, \mathbf{d_2} = \begin{bmatrix} -y_j^{(2)} \\ 0 \\ \vdots \\ b_j \\ \vdots \\ 0 \end{bmatrix}, a_j, b_j > 0.$$

Because $\mathbf{d_1}$ is the direction of feasible region, $\mathbf{A}\mathbf{d_1} = 0$,$\mathbf{d_1} \geq 0$.

$$-\mathbf{B}y_j^{(1)} + a_j p_j = \mathbf{0} \tag{iii.2}$$

In the same way,

$$-\mathbf{B}y_j^{(2)} + b_j p_j = \mathbf{0} \tag{iii.3}$$

from the above two equation, we can get

$$\begin{aligned} \tfrac{1}{a_j}\mathbf{B}y_j^{(1)} &= \tfrac{1}{b_j}\mathbf{B}y_j^{(2)} \\ y_j^{(1)} &= \tfrac{a_j}{b_j}y_j^{(2)} \end{aligned} \tag{iii.4}$$

substitute it into $\mathbf{d_1}$, now we get

$$\mathbf{d_2} = \frac{b_j}{a_j}\mathbf{d_1}, a_j, b_j > 0. \tag{iii.5}$$

$\mathbf{d_1}, \mathbf{d_2}$ are non-zero vectors with same direction, which is contradict with the hypothesis. Thus, $\mathbf{d}$ can not be represented as a positive linear combination of two directions of feasible region. $\mathbf{d}$ is polar direction.