

# Optimization Method

## *homework 11*

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December 3, 2015

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## I. PROBLEM 1

For the following non-linear programming problem.

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & -x_1^2 - (x_2 - 4)^2 + 16 \geq 0 \\ & (x_1 - 2)^2 + (x_2 - 3)^2 - 13 = 0 \end{aligned}$$

Judge whether they are local optimal solution or not

$$x^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x^{(2)} = \begin{pmatrix} \frac{16}{5} \\ \frac{32}{5} \end{pmatrix}, \quad x^{(3)} = \begin{pmatrix} 2 \\ 3 + \sqrt{13} \end{pmatrix}$$

**Solution:**  $f(x) = x_2$ ,  $g(x) = -x_1^2 - (x_2 - 4)^2 + 16$ ,  $h(x) = (x_1 - 2)^2 + (x_2 - 3)^2 - 13$

$$\nabla f(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \nabla g(x) = \begin{bmatrix} -2x_1 \\ -2(x_2 - 4) \end{bmatrix} \quad \nabla h(x) = \begin{bmatrix} 2(x_1 - 2) \\ 2(x_2 - 3) \end{bmatrix}$$

Lagrange function  $L(x, w, v) = x_2 - w[-x_1^2 - (x_2 - 4)^2 + 16] - v[(x_1 - 2)^2 + (x_2 - 3)^2 - 13]$

$$\nabla_x^2 L(x, w, v) = \begin{bmatrix} 2(w - v) & 0 \\ 0 & 2(w - v) \end{bmatrix}$$

For  $x^{(1)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\nabla f(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \nabla g(x) = \begin{bmatrix} 0 \\ 8 \end{bmatrix} \quad \nabla h(x) = \begin{bmatrix} -4 \\ -6 \end{bmatrix}$$

K-T-T condition is

$$\begin{cases} 4v = 0 \\ 1 - 8w + 6v = 0 \\ w \geq 0 \end{cases}$$

when  $w = \frac{1}{8}$ ,  $v = 0$ , the K-T-T condition holds.

$$\begin{cases} \nabla g(x^{(1)})^T d = 0 \\ \nabla h(x^{(1)})^T d = 0 \end{cases}$$

we get  $d = 0$  and  $G = \emptyset$ , so  $x^{(1)}$  is an optimal solution.

For  $x^{(2)} = \begin{pmatrix} \frac{16}{5} \\ \frac{32}{5} \end{pmatrix}$

$$\nabla f(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \nabla g(x) = \begin{bmatrix} -\frac{32}{5} \\ -\frac{24}{5} \end{bmatrix} \quad \nabla h(x) = \begin{bmatrix} \frac{12}{5} \\ \frac{34}{5} \end{bmatrix}$$

K-T-T condition is

$$\begin{cases} \frac{32}{5}w - \frac{12}{5}v = 0 \\ 1 + \frac{24}{5}w - \frac{34}{5}v = 0 \\ w \geq 0 \end{cases}$$

when  $w = \frac{3}{40}$ ,  $v = \frac{1}{5}$ , the K-T-T condition holds.

$$\begin{cases} \nabla g(x^{(2)})^T d = 0 \\ \nabla h(x^{(2)})^T d = 0 \end{cases}$$

we get  $d = 0$  and  $G = \emptyset$ , so  $x^{(2)}$  is an optimal solution.

For  $x^{(3)} = \begin{pmatrix} 2 \\ 3 + \sqrt{13} \end{pmatrix}$

$$\nabla f(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \nabla h(x) = \begin{bmatrix} 0 \\ 2\sqrt{13} \end{bmatrix}$$

K-T-T condition is

$$\begin{cases} 1 - 2\sqrt{13}v = 0 \end{cases}$$

when  $v = \frac{\sqrt{13}}{26}$ , the K-T-T condition holds.

$$\{\nabla h(x^{(3)})^T d = 0\}$$

we get  $d = \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$ ,  $d_1 \neq 0$  and  $G = \{d \mid d = \begin{bmatrix} d_1 \\ 0 \end{bmatrix}, d_1 \neq 0\}$ .

$$\nabla_x^2 L(x^{(3)}, w, v) = \begin{bmatrix} -\frac{1}{\sqrt{13}} & 0 \\ 0 & -\frac{1}{\sqrt{13}} \end{bmatrix}$$

$$d^T \nabla_x^2 L(x^{(3)}, w, v) d = (d_1, 0) \begin{bmatrix} -\frac{1}{\sqrt{13}} & 0 \\ 0 & -\frac{1}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} d_1 \\ 0 \end{bmatrix} = -\frac{1}{\sqrt{13}} d_1^2 < 0$$

so  $x^{(3)}$  is not an optimal solution.

## II. PROBLEM 2

Given a non-linear programming problem.

$$\begin{aligned} \max \quad & b^T x \quad x \in R^n \\ \text{s.t.} \quad & xx^T \leq 1 \end{aligned}$$

$b \neq 0$ , prove that  $\bar{x} = \frac{b}{\|b\|}$  holds on necessary condition of optimality.

**Proof:**

we can rewrite the non-linear programming problem into:

$$\begin{aligned} \min \quad & -b^T x \quad x \in R^n \\ \text{s.t.} \quad & 1 - xx^T \geq 0 \end{aligned}$$

From K-K-T condition, we can know that:

$$\begin{cases} -b + \omega x = 0 \\ \omega(1 - x^T x) = 0 \\ \omega \geq 0 \end{cases}$$

we can get K-T-T point  $x = \frac{b}{\|b\|}$ . Besides, we know the above problem is a convex programming problem, K-T-T condition is the sufficient condition of optimal solution.

### III. PROBLEM 3

Given a non-linear programming problem.

$$\begin{aligned} \min \quad & \frac{1}{2}[(x_1 - 1)^2 + x_2^2] \\ \text{s.t.} \quad & -x_1 + \beta x_2^2 = 0 \end{aligned}$$

Discuss  $\beta$ 's span when  $\bar{x} = (0, 0)^T$  becomes local optimal solution.

**Solution:**

we can know  $f(x) = \frac{1}{2}[(x_1 - 1)^2 + x_2^2]$ ,  $h(x) = -x_1 + \beta x_2^2$ ,

$\nabla f(x) = \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix}$ ,  $\nabla h(x) = \begin{bmatrix} -1 \\ 2\beta x_2 \end{bmatrix}$ , from K-T-T condition, we have

$$\begin{cases} x_1 - 1 + \nu = 0 \\ x_2 - 2\beta \nu x_2 = 0 \\ \nu \geq 0 \end{cases}$$

Since  $\bar{x} = (0, 0)^T$ , we get  $\nu = 1$ . for Lagrange function

$$L(x, \nu) = \frac{1}{2}[(x_1 - 1)^2 + x_2^2] - \nu(-x_1 + \beta x_2^2)$$

at point  $\bar{x} = (0, 0)^T$ ,

$$\nabla_x^2 L(\bar{x}, \nu) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - 2\beta \end{bmatrix}, \nabla h(\bar{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

Direction set  $\bar{G} = \{d | \nabla h(\bar{x})^T d = 0\} = \{(0, d_2)^T | d_2 \in \mathbb{R}\}$ ,

$$(0, d_2) \begin{bmatrix} 1 & 0 \\ 0 & 1 - 2\beta \end{bmatrix} \begin{bmatrix} 0 \\ d_2 \end{bmatrix} = (1 - 2\beta)d_2^2 > 0$$

we get  $\beta < \frac{1}{2}$ ,  $\bar{x}$  is optimal solution. when  $\beta = \frac{1}{2}$ , original problem becomes non-constraint problem

$$\min \frac{1}{2}(x_1^2 + 1)$$

$\bar{x}$  is still optimal solution.

Above all, when  $\beta \leq \frac{1}{2}$ ,  $\bar{x}$  is optimal solution.

#### IV. PROBLEM 4

Given a non-linear programming problem.

$$\begin{aligned} \min \quad & (x_1 - 1)^2 + (x_2 + 1)^2 \\ \text{s.t.} \quad & -x_1 + x_2 - 1 \geq 0 \end{aligned}$$

1. Solve this problem using graph method and optimality condition.

**Solution:**

$$\text{we have } f(x) = (x_1 - 1)^2 + (x_2 + 1)^2, g(x) = -x_1 + x_2 - 1, \nabla f(x) = \begin{bmatrix} 2(x_1 - 1) \\ 2(x_2 + 1) \end{bmatrix}, \nabla g(x) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

optimal condition is as follows:

$$\begin{cases} 2(x_1 - 1) + w = 0 \\ 2(x_2 + 1) - w = 0 \\ \omega(-x_1 + x_2 - 1) = 0 \\ \omega \geq 0 \\ -x_1 + x_2 - 1 \geq 0 \end{cases}$$

we know that when  $x_1 = -\frac{1}{2}, x_2 = \frac{1}{2}, \omega = 3$ , optimal value  $f_{min} = \frac{9}{2}$ .

2. Write out its dural problem.

**Solution:**

Lagrange function

$$L(\omega) = (x_1 - 1)^2 + (x_2 + 1)^2 - \omega(-x_1 + x_2 - 1)$$

Objective function of dural problem is

$$\theta(\omega) = \inf \{(x_1 - 1)^2 + (x_2 + 1)^2 - \omega(-x_1 + x_2 - 1) | x \in \mathbb{R}\}$$

when  $\omega \geq 0$ ,  $\theta(\omega) = -\frac{1}{2}\omega^2 + 3\omega$ , the dual problem is

$$\begin{aligned} \max \quad & -\frac{1}{2}\omega^2 + 3\omega \\ \text{s.t.} \quad & \omega \geq 0 \end{aligned}$$

3. Solve the dual problem.

$$\begin{cases} -w + 3 + w_1 = 0 \\ ww_1 = 0 \\ w_1 \geq 0 \\ w \geq 0 \end{cases}$$

we get  $w = 3, w_1 = 0$ , the optimal value is  $\theta_{max} = \frac{9}{2}$

## V. PROBLEM 5

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = 0 \\ & x^T x \leq \gamma^2 \end{aligned}$$

$A$  is a  $m * n (m < n)$  matrix,  $\text{rank}(A) = m$ ,  $c \in R^n$  and  $c \neq 0$ ,  $\gamma$  is positive, What is the optimal solution and optimal value of objective function.

**Proof:**

Since  $f(x)$  is a linear function,  $g(x)$  is a concave function and  $h(x)$  is a linear function, if the K-T-T condition holds at point  $\bar{x}$ ,  $\bar{x}$  is a optimal solution.

we can see that the feasible region is a convex set,  $f_{min}$  exists at the edge of region. So we can rewrite the problem into

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = 0 \\ & \gamma^2 - x^T x = 0 \end{aligned}$$

the K-T-T condition is

$$\begin{cases} c - A^T v + 2v_{m+1}x = 0 \\ Ax = 0 \\ \gamma^2 - x^T x = 0 \end{cases}$$

$v = (v_1, v_2, \dots, v_m)^T$  and  $v_{m+1}$  is a K-T-T multiplier, then we can get the following result:  
when  $c \neq A^T v$

$$v = (AA^T)^{-1}Ac, v_{m+1} = -\frac{f_{min}}{2\gamma^2}$$

$$f_{min} = -r\sqrt{c^T(c - A^T v)}$$

$$x = \frac{\gamma^2}{f_{min}}(c - A^T v)$$

when  $c = A^T v$ , the solution is nonunique, the optimal value is  $f_{min} = 0$