SCHOOL OF SOFTWARE, TSINGHUA UNIVERSITY

Optimization Method

homework 13

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CONTENTS

I. PROBLEM 1

Algorithm mapping is defined as follows:

$$f(x) = (6 + x_1 + x_2)^2 + (2 - 3x_1 - 3x_2 - x_1 x_2)^2$$

Solve the Newton direction and the steepest descent direction at point $\hat{x} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$.

Solution:

$$\frac{\partial f}{\partial x_1} = 2(10x_1 + 8x_2 + 6x_1 x_2 + 3x_2^2 + x_1 x_2^2), \quad \frac{\partial f}{\partial x_2} = 2(8x_1 + 10x_2 + +3x_1^2 + 6x_1 x_2 + x_1^2 x_2)$$

$$\frac{\partial^2 f}{\partial x_1^2} = 2(10 + 6x_2 + x_2^2), \quad \frac{\partial^2 f}{\partial x_2^2} = 2(10 + 6x_1 + x_1^2), \quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = 2(8 + 6x_1 + 6x_2 + 2x_1 x_2)$$

for $\hat{x} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$, the steepest descent direction

$$d = -\nabla f(\hat{x}) = \begin{bmatrix} 344 \\ -56 \end{bmatrix}$$

the Newton direction

$$d = -\nabla^2 f(\hat{x})^{-1} \nabla f(\hat{x}) = \begin{bmatrix} 164 & -56 \\ -56 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 344 \\ -56 \end{bmatrix} = \begin{bmatrix} \frac{22}{31} \\ \frac{-126}{31} \end{bmatrix}$$

II. PROBLEM 2

Suppose we have function

$$f(x) = \frac{1}{2} x^T A x + b^T x + c$$

A is positive definite matrix, suppose $x^{(1)} (\neq \bar{x})$ can denote as

$$x^{(1)} = \bar{x} + \mu p$$

$\bar{x}$ is $f(x)$'s minimal point, p is A's eigenvalue λ eigenvector, prove that:

1. $\nabla f(x^{(1)}) = \mu \lambda p$
2. Doing one-dimension search from $x^{(1)}$ along the steepest descent direction, we can get the minimal point $\bar{x}$ in one step.

Proof:

1. Since A is positive definite matrix, $\bar{x} = -A^{-1}b$.

$$\begin{aligned}\nabla f(x^{(1)}) &= Ax^{(1)} + b \\ &= A(\bar{x} + \mu p) + b \\ &= A(-A^{-1}b) + \mu A p + b \\ &= -b + \mu\lambda p + b \\ &= \mu\lambda p\end{aligned}$$

2. From point $x^{(1)}$, conduct the steepest descent method.

$$\begin{aligned}x^{(2)} &= x^{(1)} - \alpha\nabla f(x^{(1)}) \\ &= \bar{x} + \mu p - \alpha\mu\lambda p \\ &= \bar{x} + \mu p(1 - \alpha\lambda)\end{aligned}$$

   Thus, we choose $\alpha = \frac{1}{\lambda}$, we can get the minimal $\bar{x}$ in one step.

III. PROBLEM 3

Suppose A is n order real symmetric positive definite matrix, Prove that A has n orthogonal eigenvector $p^{(1)}, p^{(2)}, \cdots, p^{(n)}$ is conjugate with A.
Proof:
we have $Ap^{(i)} = \lambda_i p^{(i)}$, $i = 1, \cdots, n$. And when $i \neq j$, ${p^{(i)}}^T p^{(j)} = 0$. Then

$$ {p^{(i)}}^T A p^{(j)} = \lambda_j {p^{(i)}}^T p^{(j)} = 0, i \neq j $$

So $p^{(1)}, p^{(2)}, \cdots, p^{(n)}$ is conjugate with A.

IV. PROBLEM 4

Suppose A is n order symmetric positive definite matrix, non-zero vector $p^{(1)}, p^{(2)}, \cdots, p^{(n)} \in \mathbf{E}^n$ is conjugate with A. Prove that:

1. $x = \sum_{i=1}^n \frac{{p^{(i)}}^T A x}{{p^{(i)}}^T A p^{(i)}} p^{(i)}, \quad \forall x \in \mathbf{E}^n$
2. $A^{-1} = \sum \frac{p^{(i)} {p^{(i)}}^T}{{p^{(i)}}^T A p^{(i)}}$

Proof:

1. we can know that $p^{(1)}, p^{(2)}, \cdots, p^{(n)}$ are linearly independent. Thus, they can be a base. $\forall x \in \mathbf{E}^n$, we have

$$ x = \sum_{i=1}^n \lambda_i p^{(i)} \qquad \text{(iv.1)} $$

multiply $p^{(i)^T}A$ for above equation,we have

$$p^{(i)^T}Ax = \lambda_i p^{(i)^T}Ap^{(i)}$$

$$\lambda_i = \frac{p^{(i)^T}Ax}{p^{(i)^T}Ap^{(i)}}$$

Replace λ_i of iv.1, we get

$$x = \sum_{i=1}^{n} \frac{p^{(i)^T}Ax}{p^{(i)^T}Ap^{(i)}} p^{(i)}$$

2. Suppose $A^{-1} = (\beta_1, \cdots, \beta_n)$, from (1) we know

$$\beta_i = \sum_{i=1}^{n} \frac{p^{(i)^T}A\beta_i}{p^{(i)^T}Ap^{(i)}} p^{(i)}$$

So

$$\begin{aligned}
(\beta_1, \cdots, \beta_n) &= \sum_{i=1}^{n} \frac{p^{(i)^T}A(\beta_1, \cdots, \beta_n)}{p^{(i)^T}Ap^{(i)}} p^{(i)} \\
&= \sum_{i=1}^{n} \frac{p^{(i)^T}AA^{-1}}{p^{(i)^T}Ap^{(i)}} p^{(i)} \\
&= \sum_{i=1}^{n} \frac{p^{(i)^T}p^{(i)}}{p^{(i)^T}Ap^{(i)}}
\end{aligned}$$

So

$$A^{-1} = \sum \frac{p^{(i)}p^{(i)^T}}{p^{(i)^T}Ap^{(i)}}$$

V. Problem 5

Suppose we have a linear programming problem

$$\begin{aligned}
\min \quad & \frac{1}{2}x^T Ax \\
\text{s.t.} \quad & x \geq b
\end{aligned}$$

A is a n order symmetric positive definite matrix. Suppose $\bar{x}$ is optimal solution, prove that $\bar{x}$ and $\bar{x} - b$ is conjugate with A.

Solution:

This problem is convex programming problem and $\bar{x}$ is K-T-T point. We have

$$\begin{cases}
A\bar{x} - w^T = 0 \\
w(\bar{x} - b) = 0 \\
w \geq 0
\end{cases}$$

So we get $w = \bar{x}^T A$.

$$\bar{x}^T A(\bar{x} - b) = w(\bar{x} - b) = 0$$

Thus, $\bar{x}$ and $\bar{x} - b$ is conjugate with A.