SCHOOL OF SOFTWARE, TSINGHUA UNIVERSITY

Optimization Method

homework 9

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I. PROBLEM 1

Suppose A is a $m*n$ matrix, B is a $n*l$ matrix, $c \in E^n$, Prove that only one of the following two system has solutions:

1. $Ax \le 0, Bx = 0, c^T x > 0, x \in E^n$
2. $A^T y + B^T z = c, y \ge 0, y, z \in E^n$

Proof: since $Bx = 0$ is the same as

$$\begin{cases} Bx \le 0 \\ Bx \ge 0 \end{cases}$$

so the first system can turn into

$$\begin{bmatrix} A \\ B \\ -B \end{bmatrix} x \le 0, c^T x > 0$$

according to Farkas Theorem,

$$\begin{bmatrix} A^T & B^T & -B^T \end{bmatrix} \begin{bmatrix} y \\ u \\ v \end{bmatrix} = c, \begin{bmatrix} y \\ u \\ v \end{bmatrix} > 0$$

has no solutions. we set $z = u - v$, we know that $A^T y + B^T z = c, y \ge 0$ has no solution, or vice versa.

II. PROBLEM 2

Suppose A is a $m*n$ matrix, $c \in E^n$, Prove that only one of the following two system has solutions:

1. $Ax \le 0, x \ge 0, c^T x > 0, x \in E^n$
2. $A^T y \ge 0, y \ge 0, y \in E^n$

Proof:
the first system can turn into

$$\begin{bmatrix} A \\ -I \end{bmatrix} x \le 0, c^T x > 0$$

according to Farkas Theorem,

$$\begin{bmatrix} A^T & -I \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = c, \begin{bmatrix} y \\ u \end{bmatrix} > 0$$

has no solutions. It means $A^T y - u = c, y \ge 0, u \ge 0$ has no solution. Also we have

$$A^T y \ge 0, y \ge 0$$

has no solutions, or vice versa.

III. PROBLEM 3

$f(x_1,x_2)=10-2(x_2-x_1^2)^2$, $S=\{(x_1,x_2)|-11\le x_1\le 1,-1\le x_2\le 1\}$,is $f(x_1,x_2)$ a convex function on S?

Solution:

for each (x_1,x_2),

$$\nabla^2 f(x)=\begin{bmatrix}-24x_1^2+8x_2 & 8x_1\\ 8x_1 & -4\end{bmatrix}$$

obviously, $\nabla^2 f(x)$ is not positive semidefinite for all x, $f(x_1,x_2)$ is not a convex function.

IV. PROBLEM 4

Suppose f is a convex function on E^n, $x^{(1)},x^{(2)},\cdots,x^{(n)}$belongs to E^n,prove that

$$f(\lambda_1 x^{(1)}+\cdots+\lambda_k x^{(k)})\le\lambda_1 f(x^{(1)})+\cdots+\lambda_k f(x^{(k)})$$

$\lambda_1+\lambda_2+\cdots+\lambda_k=1,\lambda_i>0,i=1,2,\cdots,k.$

Proof:

Using mathematical induction,when $k=2$, since $f(x)$ is a convex function on E^n.

$$f(\lambda_1 x^{(1)}+\lambda_2 x^{(2)})\le\lambda_1 f(x^{(1)})+\lambda_2 f(x^{(2)})$$

Suppose when $k=n$ the equation holds, $k=n+1$,

$$f(\lambda_1 x^{(1)}+\cdots+\lambda_{n+1}x^{(n+1)})=f(\sum_1^n\lambda_i(\frac{\lambda_1}{\sum_1^n\lambda_i}x^{(1)}+\cdots+\frac{\lambda_n}{\sum_1^n\lambda_i}x^{(n)})+\lambda_{n+1}x^{(n+1)})$$

Set $\hat{x}=\frac{\lambda_1}{\sum_1^n\lambda_i}x^{(1)}+\cdots+\frac{\lambda_n}{\sum_1^n\lambda_i}x^{(n)}$, since $f(x)$ is convex and $\sum_1^n\lambda_i+\lambda_{n+1}=1$, we have

$$f(\lambda_1 x^{(1)}+\cdots+\lambda_{n+1}x^{(n+1)})\le(\sum_1^n\lambda_i)f(\hat{x})+\lambda_{n+1}f(x^{(n+1)})$$

Moreover, from hypothesis we have

$$f(\hat{x})\le\frac{\lambda_1}{\sum_1^n\lambda_i}f(x^{(1)})+\cdots+\frac{\lambda_n}{\sum_1^n\lambda_i}f(x^{(n)})$$

then we get

$$f(\lambda_1 x^{(1)}+\lambda_2 x^{(2)}+\cdots+\lambda_{n+1}x^{(n+1)})\le\lambda_1 f(x^{(1)})+\lambda_2 f(x^{(2)})+\cdots+\lambda_{n+1}f(x^{(n+1)})$$

thus, the original equation holds.