SCHOOL OF SOFTWARE, TSINGHUA UNIVERSITY

Optimization Method

homework 10

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November 23, 2015

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I. PROBLEM 1

Give a function

$$f(x) = \frac{x_1 + x_2}{3 + x_1^2 + x_2^2 + x_1 x_2}$$

Please find minimum point.
Solution:

$$\begin{cases} \dfrac{\partial f}{\partial x_1} = \dfrac{-x_1^2 - 2x_1 x_2 + 3}{(3 + x_1^2 + x_2^2 + x_1 x_2)^2} = 0 \\ \dfrac{\partial f}{\partial x_2} = \dfrac{-x_2^2 - 2x_1 x_2 + 3}{(3 + x_1^2 + x_2^2 + x_1 x_2)^2} = 0 \end{cases}$$

we can get arrest points $x^{(1)} = (1,1), x^{(2)} = (-1,-1)$.

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{-18x_1 - 12x_2 + 2x_1^3 - 2x_2^3 + 6x_1^2 x_2}{(3 + x_1^2 + x_2^2 + x_1 x_2)^3}$$
$$\frac{\partial^2 f}{\partial x_2^2} = \frac{-12x_1 - 18x_2 - 2x_1^3 + 2x_2^3 + 6x_1 x_2^2}{(3 + x_1^2 + x_2^2 + x_1 x_2)^3}$$
$$\frac{\partial^2 f}{\partial x_1 x_2} = \frac{-12x_1 - 12x_2 + 6x_1^2 x_2 + 6x_1 x_2^2}{(3 + x_1^2 + x_2^2 + x_1 x_2)^3}$$

$\nabla^2 f(x^{(1)}) = \begin{bmatrix} -\frac{1}{9} & -\frac{1}{18} \\ -\frac{1}{18} & -\frac{1}{9} \end{bmatrix}$ $\quad \nabla^2 f(x^{(2)}) = \begin{bmatrix} \frac{1}{9} & \frac{1}{18} \\ \frac{1}{18} & \frac{1}{9} \end{bmatrix}$
Since $\nabla^2 f(x^{(1)})$ is a negative definite matrix and $\nabla^2 f(x^{(2)})$ is a positive definite matrix, $(-1,-1)$ is $f(x)'s$ minimum point.

II. PROBLEM 2

Given a non-linear programming problem.

$$\begin{aligned} \min \quad & (x_1 - \frac{9}{4})^2 + (x_2 - 2)^2 \\ \text{s.t.} \quad & -x_1^2 + x_2 \geq 0 \\ & x_1 + x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Judge the following point whether they are optimized solution or not.

$$x^{(1)} = \begin{bmatrix} \frac{3}{2} \\ \frac{9}{4} \end{bmatrix}, \quad x^{(2)} = \begin{bmatrix} \frac{9}{4} \\ 2 \end{bmatrix}, \quad x^{(3)} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Solution:
we can rewrite it into a convex programming:

$$\begin{aligned} \min \quad & (x_1 - \frac{9}{4})^2 + (x_2 - 2)^2 \\ \text{s.t.} \quad & -x_1^2 + x_2 \geq 0 \\ & 6 - x_1 - x_2 \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

so we just need to check $x^{(1)}$, $x^{(2)}$, $x^{(3)}$ if they are KKT point.
for $x^{(1)}$, its KKT condition:

$$\begin{cases} 2(x_1 - \frac{9}{4}) + 2w_1 x_1 = 0 \\ 2(x_2 - 2) - w_1 = 0 \\ w_1 \geq 0 \end{cases}$$

we can get $w_1 = \frac{1}{2}$, $x^{(1)}$ is a optimized solution, optimized value is $\frac{5}{8}$.
for $x^{(2)}$, we can easily find $x^{(2)}$ is not a optimized solution.
for $x^{(3)}$, its KKT condition:

$$\begin{cases} 2(x_1 - \frac{9}{4}) - w_3 = 0 \\ 2(x_2 - 2) = 0 \\ w_3 \geq 0 \end{cases}$$

This equation has no solution, so $x^{(3)}$ is not a optimized solution.

III. Problem 3

Please compute the minimum distance between base point $x^{(0)} = (0,0)^T$ and convex set

$$S = x | x_1 + x_2 \geq 4, 2x_1 + x_2 \geq 5$$

Solution:
we can rewrite original problem into a convex programming:

$$\begin{aligned} \min \quad & x_1^2 + x_2^2 \\ \text{s.t.} \quad & x_1 + x_2 - 4 \geq 0 \\ & 2x_1 + x_2 - 5 \geq 0 \\ & x_1, x_2 \geq 0 \end{aligned}$$

KKT condition is as follow:

$$\begin{cases} 2x_1 - w_1 - 2w_2 = 0 \\ 2x_2 - w_1 - w_2 = 0 \\ w_1(x_1 + x_2 - 4) = 0 \\ w_2(2x_1 + x_2 - 5) = 0 \\ w_1, w_2 \geq 0 \\ x_1 + x_2 - 4 \geq 0 \\ 2x_1 + x_2 - 5 \geq 0 \end{cases}$$

we can get the solution $\overline{x} = (2,2)^T$, minimum distance is $2\sqrt{2}$.