SCHOOL OF SOFTWARE, TSINGHUA UNIVERSITY

Optimization Method

homework 1

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CONTENTS

I. PROBLEM 1

Prove the following set S is a convex set.

$$S = \{x|x = Ay, y \geq 0\}$$

A is a $n * m$ matrix, $x \in \mathbb{R}^n, y \in \mathbb{R}^m$.

Proof:

$\forall x^{(1)}, x^{(2)} \in S, \forall \lambda \in [0,1]$, then $x^{(1)} = Ay_1, x^{(2)} = Ay_2, y_1, y_2 \geq 0$.

$$\begin{aligned}
&\lambda x^{(1)} + (1-\lambda)x^{(2)} \\
=&\lambda Ay_1 + (1-\lambda)Ay_2 \\
=&A[\lambda y_1 + (1-\lambda)y_2]
\end{aligned}$$

because $y_1, y_2 \geq 0, \lambda \in [0,1]$, $\lambda y_1 + (1-\lambda)y_2 \geq 0$. Then, $\lambda x^{(1)} + (1-\lambda)x^{(2)} \in S$.
So, S is a convex set.

II. PROBLEM 2

Suppose S is a nonempty convex set of E^n. Prove for all integer $k \geq 2$, if $x^{(1)}, x^{(2)}, \cdots, x^{(k)} \in S$, then

$$\sum_{i=1}^{k} \lambda_i x^{(i)} \in S$$

among it, $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1, \lambda_i \geq 0, i = 1, \cdots, k$.

Proof:

1. Using mathematical induction, when $k = 2$, the above formula holds because of the definition of **convex set**.

2. Suppose when $k = n$, the formula holds. Then when $k = n+1$, we have

$$\begin{aligned}
&\sum_{i=1}^{n+1} \lambda_i x^{(i)} \\
=&\sum_{i=1}^{n} \lambda_i x^{(i)} + \lambda_{n+1} x^{(n+1)} \\
=&\sum_{i=1}^{n} \lambda_i \Big[\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} x^{(i)}\Big] + \lambda_{n+1} x^{(n+1)}
\end{aligned}$$

   Among it, $\sum_{i=1}^{n+1} \lambda_i = 1$, from the hypothesis, we know $\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} x^{(i)} \in S$. Moreover, because $\sum_{i=1}^{n} \lambda_+ \lambda_{n+1} = 1$, therefore $\sum_{i=1}^{n} \lambda_i \big[\sum_{i=1}^{n} \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i} x^{(i)} + \lambda_{n+1} x^{(n+1)}\big] \in S$.
   So, when $k = n+1$, the formula holds.

3. Finally, for all k, $\sum_{i=1}^{k} \lambda_i x^{(i)} \in S$.

III. Problem 3

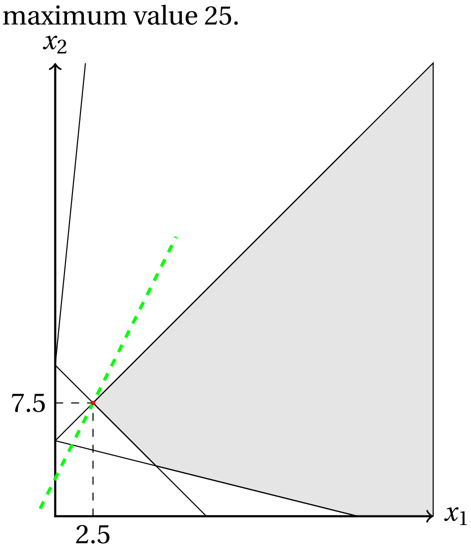
Solve the following linear programming problem using graphical method.

(4)

$$\begin{aligned}
\min \quad & -20x_1 + 10x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 10 \\
& -10x_1 + x_2 \leq 10 \\
& -5x_1 + 5x_2 \leq 25 \\
& x_1 + 4x_2 \geq 20 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Solution:

From the following graph, we can see that when $x_1 = 2.5, x_2 = 7.5$, $-20x_1 + 10x_2$ can reach the maximum value 25.

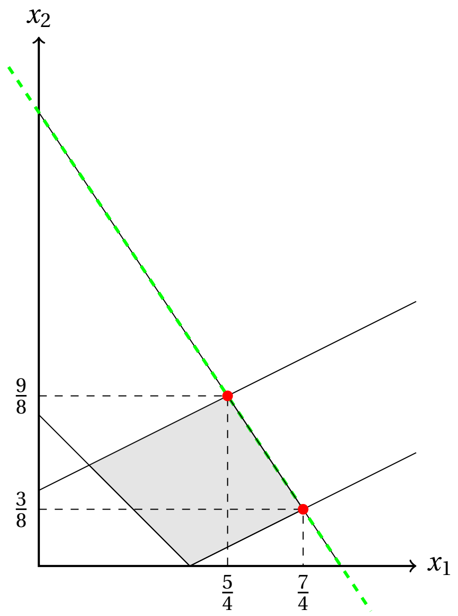


(5)

$$\begin{aligned}
\min \quad & -3x_1 - 2x_2 \\
\text{s.t.} \quad & 3x_1 + 2x_2 \leq 6 \\
& x_1 - 2x_2 \leq 1 \\
& x_1 + x_2 \geq 1 \\
& -x_1 + 2x_2 \leq 1 \\
& x_1, x_2 \geq 0
\end{aligned}$$

Solution:

From the following graph, we can see that from $(\frac{5}{4}, \frac{9}{8})$ to $(\frac{7}{4}, \frac{3}{8})$, $-3x_1 - 2x_2$ reaches the minimum value -6, e.g. $x_1 = \frac{7}{4}, x_2 = \frac{3}{8}$.



x_2
$\frac{9}{8}$
$\frac{3}{8}$
x_1
$\frac{5}{4}$
$\frac{7}{4}$