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Optimization Method

homework 12

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Contents

I Problem 1

For the following non-linear programming problem.

$$\begin{aligned} \min \quad & x_2 \\ \text{s.t.} \quad & -x_1^2-(x_2-4)^2+16 \geq 0 \\ & (x_1-2)^2+(x_2-3)^2-13=0 \end{aligned}$$

Judge whether they are local optimal solution or not

$$x^{(1)}=\begin{pmatrix}0\\0\end{pmatrix}, \quad x^{(2)}=\begin{pmatrix}\frac{16}{5}\\\frac{32}{5}\end{pmatrix}, \quad x^{(3)}=\begin{pmatrix}2\\3+\sqrt{13}\end{pmatrix}$$

Solution: $f(x)=x_2$, $g(x)=-x_1^2-(x_2-4)^2+16$, $h(x)=(x_1-2)^2+(x_2-3)^2-13$

$$\nabla f(x)=\begin{bmatrix}0\\1\end{bmatrix} \nabla g(x)=\begin{bmatrix}-2x_1\\-2(x_2-4)\end{bmatrix} \nabla h(x)=\begin{bmatrix}2(x_1-2)\\2(x_2-3)\end{bmatrix}$$

Lagrange function $L(x,w,v)=x_2-w[-x_1^2-(x_2-4)^2+16]-v[(x_1-2)^2+(x_2-3)^2-13]$
$\nabla_x^2 L(x,w,v)=\begin{bmatrix}2(w-v) & 0\\ 0 & 2(w-v)\end{bmatrix}$

For $x^{(1)}=\begin{pmatrix}0\\0\end{pmatrix}$

$$\nabla f(x)=\begin{bmatrix}0\\1\end{bmatrix} \nabla g(x)=\begin{bmatrix}0\\8\end{bmatrix} \nabla h(x)=\begin{bmatrix}-4\\-6\end{bmatrix}$$

K-T-T condition is

$$\begin{cases} 4v=0 \\ 1-8w+6v=0 \\ w\geq 0 \end{cases}$$

when $w=\frac{1}{8}$, $v=0$, the K-T-T condition holds.

$$\begin{cases} \nabla g(x^{(1)})^T d=0 \\ \nabla h(x^{(1)})^T d=0 \end{cases}$$

we get $d=0$ and $G=\emptyset$, so $x^{(1)}$ is an optimal solution.

For $x^{(2)}=\begin{pmatrix}\frac{16}{5}\\\frac{32}{5}\end{pmatrix}$

$$\nabla f(x)=\begin{bmatrix}0\\1\end{bmatrix} \nabla g(x)=\begin{bmatrix}-\frac{32}{5}\\-\frac{24}{5}\end{bmatrix} \nabla h(x)=\begin{bmatrix}\frac{12}{5}\\\frac{34}{5}\end{bmatrix}$$

K-T-T condition is

$$\begin{cases} \frac{32}{5}w - \frac{12}{5}v = 0 \\ 1 + \frac{24}{5}w - \frac{34}{5}v = 0 \\ w \geq 0 \end{cases}$$

when $w = \frac{3}{40}$, $v = \frac{1}{5}$, the K-T-T condition holds.

$$\begin{cases} \nabla g(x^{(2)})^T d = 0 \\ \nabla h(x^{(2)})^T d = 0 \end{cases}$$

we get $d = 0$ and $G = \emptyset$, so $x^{(2)}$ is an optimal solution.

For $x^{(3)} = \begin{pmatrix} 2 \\ 3+\sqrt{13} \end{pmatrix}$

$$\nabla f(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \nabla h(x) = \begin{bmatrix} 0 \\ 2\sqrt{13} \end{bmatrix}$$

K-T-T condition is

$$\begin{cases} 1 - 2\sqrt{13}v = 0 \end{cases}$$

when $v = \frac{\sqrt{13}}{26}$, the K-T-T condition holds.

$$\begin{cases} \nabla h(x^{(3)})^T d = 0 \end{cases}$$

we get $d = \begin{bmatrix} d_1 \\ 0 \end{bmatrix}, d_1 \neq 0$ and $G = \{d | d = \begin{bmatrix} d_1 \\ 0 \end{bmatrix}, d_1 \neq 0\}$.

$$\nabla_x^2 L(x^{(3)}, w, v) = \begin{bmatrix} -\frac{1}{\sqrt{13}} & 0 \\ 0 & -\frac{1}{\sqrt{13}} \end{bmatrix}$$

$$d^T \nabla_x^2 L(x^{(3)}, w, v) d = (d_1, 0) \begin{bmatrix} -\frac{1}{\sqrt{13}} & 0 \\ 0 & -\frac{1}{\sqrt{13}} \end{bmatrix} \begin{bmatrix} d_1 \\ 0 \end{bmatrix} = -\frac{1}{\sqrt{13}} d_1^2 < 0$$

so $x^{(3)}$ is not an optimal solution.

II PROBLEM 2

Given a non-linear programming problem.

$$\begin{aligned} \max \quad & b^T x \quad x \in R^n \\ \text{s.t.} \quad & x x^T \leq 1 \end{aligned}$$

$b \neq 0$, prove that $\overline{x} = \frac{b}{||b||}$ holds on necessary condition of optimality.

Proof:
we can rewrite the non-linear programming problem into:

$$\begin{aligned} \min \quad & -b^T x \quad x \in R^n \\ \text{s.t.} \quad & 1 - xx^T \geq 0 \end{aligned}$$

From K-K-T condition, we can know that:

$$\begin{cases} -b + \omega x = 0 \\ \omega(1 - x^T x) = 0 \\ \omega \geq 0 \end{cases}$$

we can get K-T-T point $x = \frac{b}{||b||}$. Besides, we know the above problem is a convex programming problem, K-T-T condition is the sufficient condition of optimal solution.

III Problem 3

Given a non-linear programming problem.

$$\begin{aligned} \min \quad & \frac{1}{2}[(x_1 - 1)^2 + x_2^2] \\ \text{s.t.} \quad & -x_1 + \beta x_2^2 = 0 \end{aligned}$$

Discuss β's span when $\overline{x} = (0,0)^T$ becomes local optimal solution.

Solution:
we can know $f(x) = \frac{1}{2}[(x_1 - 1)^2 + x_2^2]$, $h(x) = -x_1 + \beta x_2^2$,
$\nabla f(x) = \begin{bmatrix} x_1 - 1 \\ x_2 \end{bmatrix}$, $\nabla h(x) = \begin{bmatrix} -1 \\ 2\beta x_2 \end{bmatrix}$, from K-T-T condition, we have

$$\begin{cases} x_1 - 1 + v = 0 \\ x_2 - 2\beta v x_2 = 0 \\ \omega \geq 0 \end{cases}$$

Since $\overline{x} = (0,0)^T$, we get $v = 1$. for Lagrange function

$$L(x, v) = \frac{1}{2}[(x_1 - 1)^2 + x_2^2] - v(-x_1 + \beta x_2^2)$$

at point $\overline{x} = (0,0)^T$,

$$\nabla_x^2 L(\overline{x}, v) = \begin{bmatrix} 1 & 0 \\ 0 & 1 - 2\beta \end{bmatrix}, \nabla h(\overline{x}) = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

Direction set $\overline{G} = \{d | \nabla h(\bar(x))^T d = 0\} = \{(0, d_2)^T | d_2 \in \mathbb{R}\}$,

$$(0, d_2)\begin{bmatrix}1 & 0\\ 0 & 1-2\beta\end{bmatrix}\begin{bmatrix}0\\ d_2\end{bmatrix} = (1-2\beta)d_2^2 > 0$$

we get $\beta < \frac{1}{2}$, $\overline{x}$ is optimal solution. when $\beta = \frac{1}{2}$, original problem becomes non-constraint problem

$$\min \frac{1}{2}(x_1^2 + 1)$$

$\overline{x}$ is still optimal solution.
Above all, when $\beta \leq \frac{1}{2}$, $\overline{x}$ is optimal solution.

IV PROBLEM 4

Given a non-linear programming problem.

$$\begin{aligned} \min \quad & (x_1 - 1)^2 + (x_2 + 1)^2 \\ \text{s.t.} \quad & -x_1 + x_2 - 1 \geq 0 \end{aligned}$$

1. Solve this problem using graph method and optimality condition.
   Solution:
   we have $f(x) = (x_1-1)^2 + (x_2+1)^2$, $g(x) = -x_1 + x_2 - 1$, $\nabla f(x) = \begin{bmatrix}2(x_1-1)\\ 2(x_2+1)\end{bmatrix}$, $\nabla g(x) = \begin{bmatrix}-1\\ 1\end{bmatrix}$
   optimal condition is as follows:

$$\begin{cases} 2(x_1 - 1) + w = 0 \\ 2(x_2 + 1) - w = 0 \\ \omega(-x_1 + x_2 - 1) = 0 \\ \omega \geq 0 \\ -x_1 + x_2 - 1 \geq 0 \end{cases}$$

   we know that when $x_1 = -\frac{1}{2}$,$x_2 = \frac{1}{2}$,$\omega = 3$, optimal value $f_{min} = \frac{9}{2}$.

2. Write out its dural problem.
   Solution:
   Lagrange function

$$L(\omega) = (x_1 - 1)^2 + (x_2 + 1)^2 - \omega(-x_1 + x_2 - 1)$$

   Objective function of dural problem is

$$\theta(\omega) = inf\{(x_1 - 1)^2 + (x_2 + 1)^2 - \omega(-x_1 + x_2 - 1) | x \in \mathbb{R}\}$$

when $\omega \geq 0$, $\theta(\omega) = -\frac{1}{2}\omega^2 + 3\omega$, the dural problem is

$$\begin{aligned} \max \quad & -\frac{1}{2}\omega^2 + 3\omega \\ \text{s.t.} \quad & w \geq 0 \end{aligned}$$

3. Solve the dural problem.

$$\begin{cases} -w + 3 + w_1 = 0 \\ w w_1 = 0 \\ w_1 \geq 0 \\ w \geq 0 \end{cases}$$

we get $w = 3, w_1 = 0$, the optimal value is $\theta_{max} = \frac{9}{2}$

V Problem 5

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = 0 \\ & x^T x \leq \gamma^2 \end{aligned}$$

A is a $m * n (m < n)$ matrix, $\text{rank}(A) = m$, $c \in R^n$ and $c \neq 0$, γ is positive, What is the optimal solution and optimal value of objective function.

Proof:
Since $f(x)$ is a linear function, $g(x)$ is a concave function and $h(x)$ is a linear function, if the K-T-T condition holds at point $\overline{x}$, $\overline{x}$ is a optimal solution.
we can see that the feasible region is a convex set, f_{min} exists at the edge of region. So we can rewrite the problem into

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = 0 \\ & \gamma^2 - x^T x = 0 \end{aligned}$$

the K-T-T condition is

$$\begin{cases} c - A^T v + 2v_{m+1} x = 0 \\ Ax = 0 \\ \gamma^2 - x^T x = 0 \end{cases}$$

$v = (v_1, v_2, \cdots, v_m)^T$ and v_{m+1} is a K-T-T multiplier, then we can get the following result: when $c \neq A^T v$

$$v = (AA^T)^{-1} Ac, v_{m+1} = -\frac{f_{min}}{2\gamma^2}$$

$$f_{min} = -r\sqrt{c^T (c - A^T v)}$$

$$x = \frac{\gamma^2}{f_{min}}(c - A^T v)$$

when $c = A^T v$, the solution is nonunique, the optimal value is $f_{min} = 0$