Solving Equilibrium: Backward Induction

No Author Given

No Institute Given

1 Solving Equilibrium: Backward Induction

To determine the optimal strategy $\langle p^{S^*}, \boldsymbol{p^{T^*}}, pol^{P^*} \rangle$, we use backward induction to derive the solution to the equations. First, we derive the data provider's optimal strategy pol^{P^*} by differentiating the data provider's utility function. Then, we substitute pol^{P^*} into the TSP's utility function, differentiate it, and derive the TSP's optimal strategy $\boldsymbol{p^{T^*}}$. Finally, we substitute $\boldsymbol{p^{T^*}}$ into the service consumer's utility function, differentiate it, and derive the service consumer's optimal strategy p^{S^*}.

Data Provider's Optimal Strategy As illustrated in Fig. 1, for each attribute $1 \leq i \leq m$ in the policy, V_i^P can be partitioned into four disjoint subsets:

$$X_i = V_i^P \cap V_i^{T\setminus R}, \qquad Y_i = V_i^P \cap V_i^{R\setminus T},$$
$$Z_i = V_i^P \cap V_i^{T\cap R}, \qquad W_i = V_i^P \cap \overline{V_i^{T\cap R}}.$$

For clarity, we introduce the short notations $V_i^{T\setminus R}$, $V_i^{R\setminus T}$, and $V_i^{T\cap R}$ to represent the sets $V_i^T \setminus V_i^R$, $V_i^R \setminus V_i^T$, and $V_i^T \cap V_i^R$, respectively.

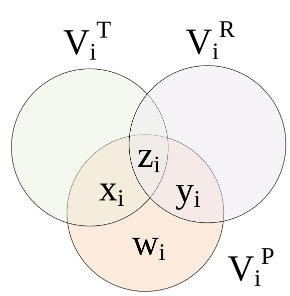

Fig. 1: The relationships among V_i^T, V_i^R, and V_i^P.

Let $x_i = |X_i|$, $y_i = |Y_i|$ and $z_i = |Z_i|$. We can see from Fig. 1 that $|V_i^P \cap V_i^T| = x_i + z_i$, and $|V_i^P \cap V_i^R| = y_i + z_i$. As a result, the data provider's utility can be reformulated as

$$
\begin{aligned}
U_{\mathrm{DP}} &= D\sum_{i=1}^{m} p_i^T \frac{|V_i^P \cap V_i^T|}{|V_i^T|} - \lambda D \sum_{i=1}^{m} \rho_i \left(\frac{|V_i^P \cap V_i^R|}{|V_i^R|}\right)^2 \\
&= D\sum_{i=1}^{m} \left(p_i^T \frac{x_i+z_i}{|V_i^T|} - \lambda\rho_i \frac{(y_i+z_i)^2}{|V_i^R|^2}\right). \qquad (1)
\end{aligned}
$$

By analyzing the monotonicity of U_{DP} with respect to x_i and y_i, and the partial derivation of U_{DP} with respect to z_i, we obtain the following theorem.

Theorem 1. *Let A_C and A_D denote the continuous attributes and discrete attributes, respectively. Let round(x) denote the integer closest to x and $c_{1i} = \frac{|V_i^R|^2}{2\lambda\rho_i|V_i^T|}$. The values of x_i, y_i, and z_i for $1 \le i \le m$ that maximize the data provider's utility are as follows.*

$$x_i^* = |V_i^{T\setminus R}|, \quad y_i^* = 0. \qquad (2)$$

$$z_i^* = \begin{cases} \min\left(p_i^T c_{1i}, |V_i^{T\cap R}|\right), & A_i \in A_C \\ \min\left(\mathit{round}\left(p_i^T c_{1i}\right), |V_i^{T\cap R}|\right), & A_i \in A_D \end{cases} \qquad (3)$$

Proof. From (1), we can see that U_{DP} strictly increases as x_i increases, and decreases as y_i increases. Therefore, they are treated as constants, only z_i is considered as the provider's actual decision variable. Because $x_i = |V_i^P \cap V_i^{T\setminus R}| \le |V_i^{T\setminus R}|$ and $y_i = |V_i^P \cap V_i^{R\setminus T}| \ge 0$, we have that

$$x_i^* = |V_i^{T\setminus R}|, \quad y_i^* = 0.$$

Substituting x_i^* and y_i^* into (1), we are left with

$$U_{\mathrm{DP}} = \sum_{i=1}^{m} D\left(p_i^T \frac{|V_i^{T\setminus R}| + z_i}{|V_i^T|} - \frac{\lambda\rho_i}{|V_i^R|^2} z_i^2\right). \qquad (4)$$

To solve the optimal z_i, we distinguish two cases.

Case 1: attribute A_i is continuous ($A_i \in A_C$).

Deriving the first-order derivative of (4) with respect to z_i and setting it to zero, we obtain

$$\frac{2\lambda\rho_i D}{|V_i^R|^2}\left(p_i^T c_{1i} - z_i\right) = 0, \qquad (5)$$

where $c_{1i} = \frac{|V_i^R|^2}{2\lambda\rho_i|V_i^T|}$.

Solving (5) for z_i, we obtain the critical point $z_i = p_i^T c_{1i}$.

The second-order derivative of (4) with respect to z_i is

$$\frac{\partial^2 U_{\mathrm{DP}}}{\partial z_i^2} = -\frac{2\lambda\rho_i D}{|V_i^R|^2}. \qquad (6)$$

It is negative, so we can confirm that the critical point corresponds to a maximum.

Because $z_i = |V_i^P \cap V_i^{T\cap R}| \le |V_i^{T\cap R}|$, we clip the critical point to the valid range and obtain that

$$z_i = \min\left(p_i^T c_{1i}, |V_i^{T\cap R}|\right). \tag{7}$$

Case 2: attribute A_i is discrete ($A_i \in A_D$).

In this case, z_i must be an integer. So we round the critical point to the nearest integer first and then clip it to the valid range. The final result is

$$z_i = \min\left(\mathrm{round}\left(p_i^T c_{1i}\right), |V_i^{T\cap R}|\right). \tag{8}$$

In Theorem 1, (2) indicates that $V_i^{P^*}$ should cover $V_i^{T\setminus R}$ and do not intersect with $V_i^R \setminus V_i^T$, respectively. As a result, the range of attribute A_i in the data provider's optimal strategy $V_i^{P^*} = V_i^{T\setminus R} \cup E_i$, where E_i is a subset of $V_i^{T\cap R}$ with size z_i^* determined by (3).

When attribute A_i is discrete, (3) may create two possible outcomes ($\lfloor p_i^T c_{1i} \rfloor$ and $\lceil p_i^T c_{1i} \rceil$) if $\{p_i^T c_{1i}\} = 0.5$ and $p_i^T c_{1i} < |V_i^{T\cap R}|$, leading to a non-uniqueness in the equilibrium. In this case, we use the *Pareto Optimality* criterion to select the best equilibrium. Pareto optimality refers to a situation in which no player can improve their utility by unilaterally changing their strategy, while also considering the utility of others. In this context, we should identify which outcome is better for the TSP and service consumer. By analyzing the first-order derivative of the utility of the TSP and service consumer with respect to z_i, we obtain theorem 2.

Theorem 2. *. In the case that $\{p_i^T c_{1i}\} = 0.5$ and $p_i^T c_{1i} < |V_i^{T\cap R}|$, $\lceil p_i^T c_{1i} \rceil$ results in a higher utility for both the TSP and the service consumer compared to $\lfloor p_i^T c_{1i} \rfloor$.*

According to theorem 2, we obtain the following formula for z_i^* in the case that attributes A_i is discrete. It always creates a unique outcome.

$$z_i^* = \begin{cases} 0, & 0 \le p_i^T c_{1i} < 0.5 \\ \cdots, & \cdots \\ k, & k - 0.5 \le p_i^T c_{1i} < k + 0.5 \\ \cdots, & \cdots \\ |V_i^{T\cap R}|, & p_i^T c_{1i} \ge |V_i^{T\cap R}| - 0.5 \end{cases}. \tag{9}$$

Third-party Service Provider's optimal strategy Substituting the data provider's optimal strategy into the TSP's utility function and removing the terms that do not involve p_i^T, we are left with:

$$\begin{aligned} U_i(p_i^T) = & p^S q_0 \frac{t_1}{t_0} \omega_i \ln\left(\frac{|V_i^{T\setminus R}| + z_i^*(p_i^T)}{|V_i^T|}\right) \\ & - D p_i^T \frac{|V_i^{T\setminus R}| + z_i^*(p_i^T)}{|V_i^T|}. \end{aligned} \tag{10}$$

Theorem 1 tells that $z_i^*(p_i^T)$ has two possible expressions, so we analyze in two cases.

Case 1: Attribute A_i is continuous.

In this case, z_i^* is determined by (7). If $p_i^T > |V_i^{T\cap R}|/c_{1i}$, $z_i^* = |V_i^{T\cap R}|$ is constant. Substituting z_i^* into (10), $U_i(p_i^T)$ simplifies to a linear function of p_i^T with a negative slope. Therefore, the optimal p_i^T lies within $\left[0, |V_i^{T\cap R}|/c_{1i}\right]$ and $z_i^* = p_i^T c_{1i}$. Substituting $z_i^* = p_i^T c_{1i}$ into (10) and analyzing the first-order derivative of $U_i(p_i^T)$ with respect to p_i^T, we obtain the following theorem.

Theorem 3. *$U_i(p_i^T)$ is concave within $\left[0, |V_i^{T\cap R}|/c_{1i}\right]$ and the value of p_i^T that maximizes $U_i(p_i^T)$ is*

$$p_i^{T^*} = \begin{cases} 0, & p^S \le p_i^{B,l} \\ \hat{p_i^T}, & p_i^{B,l} < p^S \le p_i^{B,h} \\ |V_i^{T\cap R}|/c_{1i}, & p^S > p_i^{B,h}, \end{cases} \tag{11}$$

where

$$p_i^{B,l} = \frac{|V_i^{T\setminus R}|^2}{c_{2i}}, \quad p_i^{B,h} = \frac{|V_i^T|(|V_i^T| + |V_i^{T\cap R}|)}{c_{2i}}, \tag{12}$$

$$\hat{p_i^T} = \frac{-3|V_i^{T\setminus R}| + \sqrt{|V_i^{T\setminus R}|^2 + 8p^S c_{2i}}}{4c_{1i}} \tag{13}$$

and $c_{2i} = \frac{q_0}{D}\frac{t_1}{t_0}\omega_i c_{1i}|V_i^T|$.

Proof. The derivative of $U_i(p_i^T)$ with respect to p_i^T is

$$\begin{aligned} \frac{dU_i(p_i^T)}{dp_i^T} =& p^S q_0 \frac{t_1}{t_0} \frac{\omega_i c_{1i}}{|V_i^{T\setminus R}| + p_i^T c_{1i}} \\ & - \frac{D}{|V_i^T|}\left(|V_i^{T\setminus R}| + 2p_i^T c_{1i}\right). \end{aligned} \tag{14}$$

$$= p^S \frac{q_0}{2\lambda}\frac{t_1}{t_0}\frac{\omega_i}{\rho_i}\frac{|V_i^R|^2}{|V_i^T||V_i^{T\setminus R}|} - D\frac{|V_i^{T\setminus R}|^2}{|V_i^T||V_i^{T\setminus R}|} \tag{15}$$

It decreases as p_i^T increases, thus $U_i(p_i^T)$ is concave.

Setting the derivative to 0, we get the following quadratic equation, where $c_{2i} = \frac{q_0}{D}\frac{t_1}{t_0}\omega_i c_{1i}|V_i^T|$.

$$2c_{1i}^2 {p_i^T}^2 + 3|V_i^{T\setminus R}|c_{1i}p_i^T + |V_i^{T\setminus R}|^2 - p^S c_{2i} = 0. \tag{16}$$

Solving for p_i^T, we obtain two critical points.

$$p_i^{T\pm} = \frac{-3|V_i^{T\setminus R}| \pm \sqrt{|V_i^{T\setminus R}|^2 + 8p^S c_{2i}}}{4c_{1i}}. \tag{17}$$

The valid range of p_i^T is $\left[0, |V_i^{T\cap R}|/c_{1i}\right]$. According to (17), p_i^{T-} is always negative and infeasible, and p_i^{T+} may lie outside this range when p^S is too low or too high. Setting p_i^{T+} to 0 and $|V_i^{T\cap R}|/c_{1i}$, we obtain the thresholds given by (12).

The optimal p_i^T equals p_i^{T+} if and only if p^S is between the two thresholds. Otherwise, it is 0 or $|V_i^{T\cap R}|/c_{1i}$, depending on whether p^S is below the lower threshold $p_i^{B,l}$ or above the higher threshold $p_i^{B,h}$.

Case 2: Attribute A_i is discrete.
In this case, z_i^* must be an integer. As shown in (9), it increases from 0 to $|V_i^{T\cap R}|$ in discrete steps as p_i^T increases. Each integer corresponds to a specific interval of p_i^T. Within each interval, z_i^* is constant, making $U_i(p_i^T)$ a linear function of p_i^T with a negative slope. As a result, the maximum utility occurs at the left endpoint of the interval. Denote these left endpoints as $p_i^{T^*}(0), \ldots, p_i^{T^*}(|V_i^{T\cap R}|)$, i.e., $p_i^{T^*}(0) = 0$ and $p_i^{T^*}(k) = \frac{k-0.5}{c_{1i}}$ for $k > 0$.

Substituting $p_i^{T^*}(k)$ into (10), we obtain that

– for $k = 0$,

$$U_i(k) = p^S q_0 \frac{t_1}{t_0} \omega_i \ln \frac{|V_i^{T\setminus R}|}{|V_i^T|} \tag{18}$$

– for $k \geq 1$,

$$\begin{aligned} U_i(k) =& p^S q_0 \frac{t_1}{t_0} \omega_i \ln \frac{|V_i^{T\setminus R}| + k}{|V_i^T|} \\ & - D \frac{(k-0.5)\left(|V_i^{T\setminus R}| + k\right)}{|V_i^T|} \end{aligned} \tag{19}$$

Let k^* be the value of k that maximizes $U_i(k)$, which can be identified by calculating $U_i(k)$ for each $0 \leq k \leq |V_i^{T\cap R}|$, then the optimal p_i^T is $p_i^{T^*}(k^*)$.

The following theorem provides another way to identify k^*. It tells that k^* follows a piecewise function of p^S and can be determined by comparing p^S against a series of thresholds.

Theorem 4. *Define $p_i^S(k)$ as the value of p^S that makes $U_i(k-1) = U_i(k)$.*

(i) $p_i^S(k)$ strictly increases with k.
(ii) $k^* = k$ if $p_i^S(k) < p^S < p_i^S(k+1)$.
(iii) $k^* = k - 1$ or k if $p^S = p_i^S(k)$.

Proof. Solving the equation $U_i(k-1) = U_i(k)$ for p^S, we have that

$$p_i^S(k) = \begin{cases} 0, & k = 0 \\ \frac{1}{c_{2i}} \frac{0.5\left|V_i^{T\setminus R}\right| + 0.5}{\ln\left(1 + \left|V_i^{T\setminus R}\right|\right) - \ln\left|V_i^{T\setminus R}\right|}, & k = 1 \\ \frac{1}{c_{2i}} \frac{2k + \left|V_i^{T\setminus R}\right| - 1.5}{\ln\left(\left|V_i^{T\setminus R}\right| + k\right) - \ln\left(\left|V_i^{T\setminus R}\right| + k - 1\right)}, & k > 1. \end{cases} \tag{20}$$

For $k = 0$, $p_i^S(k) < p_i^S(k+1)$ holds because $p_i^S(0) = 0$ and $p_i^S(1)$ is positive. For $k \geq 1$, $p_i^S(k) < p_i^S(k+1)$ also holds, because the numerator of $p_i^S(k+1)$ is greater than that of $p_i^S(k)$ and the denominator of $p_i^S(k+1)$ is less than that of $p_i^S(k)$. Therefore, $p_i^S(k)$ strictly increases with k.

Setting up the inequality $U_i(k-1) > U_i(k)$ and solving for p^S, we can obtain that $p^S > p_i^S(k)$.

Consider the case that $p_i^S(k) < p^S < p_i^S(k+1)$. Because $p_i^S(k)$ increases with k, $p^S > p_i^S(t)$ for $t \leq k$ and $p^S < p_i^S(t)$ for $t > k$, resulting that $U_i(t-1) < U_i(t)$ for $t \leq k$ and $U_i(t-1) > U_i(t)$ for $t > k$, respectively. That is,

$$U_i(0) < U_i(1) < \cdots < U_i(k) > \cdots > U_i(|V_i^{T\cap R}|),$$

indicating that $k^* = k$.

Conclusion (iii) can be proved similarly and is omitted here.

Service Consumer's optimal strategy As demonstrated in previous sections, $z_i^*(p_i^T)$ and $p_i^{T^*}(p^S)$ are piecewise functions for each $1 \leq i \leq m$, which follows that $z_i^*(p^S)$ is also piecewise. According to (7), (11), and Theorem 4, the expression of $z_i^*(p^S)$ is as follows.

– If attribute A_i is continuous,

$$z_i^*(p^S) = \begin{cases} 0, & p^S \leq p_i^{B,l} \\ \hat{z_i^*}(p^S), & p_i^{B,l} < p^S \leq p_i^{B,h} \\ |V_i^{T\cap R}|, & p^S > p_i^{B,h}, \end{cases} \tag{21}$$

where

$$\hat{z_i^*}(p^S) = \frac{-3|V_i^{T\setminus R}| + \sqrt{|V_i^{T\setminus R}|^2 + 8p^S c_{2i}}}{4}. \tag{22}$$

– If attribute A_i is discrete,

$$z_i^*(p^S) = \begin{cases} 0, & p^S < p_i^B(1) \\ 1, & p_i^B(1) \leq p^S < p_i^B(2) \\ \cdots & \\ |V_i^{T\cap R}|, & p^S \geq p_i^B(|V_i^{T\cap R}|). \end{cases} \tag{23}$$

Corollary 1. *The service consumer's utility $U_{SC}(p^S)$ is piecewise smooth. Let BP denote the breakpoints where $U_{SC}(p^S)$ changes its expression. Then,*

$$BP = \bigcup_{i\in I_C} \{p_i^{B,l}, p_i^{B,h}\} \cup \bigcup_{i\in I_D} \{p_i^B(1), \cdots, p_i^B(V_i^{T\cap R})\}. \tag{24}$$

Given the piecewise nature of $U_{\mathrm{SC}}(p^S)$, we will analyze the optimization within each smooth region between consecutive breakpoints. Derive the first-order derivative of $U'_{\mathrm{SC}}(p^S)$ with respect to p^S.

$$U'_{\mathrm{SC}}(p^S) = \left(f'(Q) - p^S\right) Q'(p^S) - Q \tag{25}$$

$$Q'(p^S) = q_0 \frac{t_1}{t_0} \sum_{i=1}^{m} \frac{\omega_i {z_i^*}'(p^S)}{|V_i^{T\setminus R}| + z_i^*(p^S)} \tag{26}$$

Due to the complexity of $U'_{\mathrm{SC}}(p^S)$ and the piecewise nature of $z_i^*(p^S)$, it is not feasible to get an analytical solution for $U'_{\mathrm{SC}}(p^S) = 0$. Thus, numerical methods are required.

Suppose that the sorted breakpoints are $ep_1 < ep_2 < \cdots < ep_n$. For $0 \leq k \leq n$, Lemma 1 gives two properties of $U'_{\mathrm{SC}}(p^S)$ within the region (ep_k, ep_{k+1}). Theorem 5 provides a guide to finding the root of $U'_{\mathrm{SC}}(p^S) = 0$ within the region (ep_k, ep_{k+1}) and guarantees the uniqueness of the root. Based on this theorem, we design Algorithm 16 to optimize the service consumer's utility. In lines 4-23, it searches for the candidate optimal point with each smooth region of $U'_{\mathrm{SC}}(p^S)$. In line 24, the global maximum is identified by comparing all candidates.

Lemma 1. *Equation $f'(Q) - p^S = 0$ has exactly one root. Denote the root by $\overline{p^S}$. Then, for any $0 \leq k \leq n$,*

- *if $ep_k > \overline{p^S}$, $U'_{SC}(p^S) < 0$ for $p^S \in (ep_k, ep_{k+1})$;*
- *if $ep_k < \overline{p^S}$, $U'_{SC}(p^S)$ decreases as p^S increases from ep_k to $\min\left(ep_{k+1}, \overline{p^S}\right)$.*

Proof. For $1 \leq i \leq m$ and $p^S \in (ep_k, ep_{k+1})$, we can see from (23) that $z_i^*(p^S)$ is always constant if attribute A_i is discrete. Otherwise, $z_i^*(p^S)$ is constant or equal to $z_i^*(p^S)$. Because

$$\hat{z_i^*}'(p^S) = \frac{c_{2i}}{\sqrt{\left|V_i^{T\setminus R}\right|^2 + 8p^S c_{2i}}} \tag{27}$$

is positive and decreases as p^S increases, we see that $z_i^*(p^S)$ is monotonically increasing (not strictly) and ${z_i^*}'(p^S)$ is monotonically decreasing (not strictly), indicating that (26) is positive and monotonically decreasing (not strictly), that is, $Q'(p^S) > 0$ and $Q''(p^S) \leq 0$.

Let $g(p^S) = f'(Q) - p^S$, then

$$g'(p^S) = f''(Q)Q'(p^S) - 1 < 0, \tag{28}$$

because $f''(Q) < 0$ and $Q'(p^S) > 0$. As a result, $g(p^S)$ is decreasing. Because $g(0) > 0$ and $g(g(0) \leq 0$, $g(p^S) = 0$ has exactly one root. Denote the root as $\overline{p^S}$.

Substituting $g(p^S) = f'(Q) - p^S$ into (25), we have

$$U'_{\mathrm{SC}}(p^S) = g(p^S)Q'(p^S) - Q. \tag{29}$$

For $p^S \in (ep_k, ep_{k+1})$, if $ep_k > \overline{p^S}$, (29) is negative because $g(p^S) < g(\overline{p^S}) = 0$, $Q'(p^S) > 0$ and $Q > 0$.

Take the derivative of (29) with respect to p^S, we have

$$U''_{\mathrm{SC}}(p^S) = \left(g'(p^S) - 1\right) Q'(p^S) + g(p^S)Q''(p^S) \tag{30}$$

Because $Q'(p^S) \geq 0$, $Q''(p^S) \leq 0$ and $g'(p^S) < 0$ for $p^S \in (ep_k, ep_{k+1})$, and $g(p^S) > 0$ for $p^S < \overline{p^S}$, (30) is negative for $p^S \in (ep_k, \min(ep_{k+1}, \overline{p^S})$ if $ep_k < \overline{p^S}$. As a result, $U'_{\mathrm{SC}}(p^S)$ decreases as p^S increases from ep_k to $\min\left(ep_{k+1}, \overline{p^S}\right)$.

Theorem 5. *For $0 \leq k \leq n$, $U'_{SC}(p^S) = 0$ has exactly one root within (ep_k, ep_{k+1}) if $U'_{SC}(ep_k) > 0 > U'_{SC}(ep_{k+1})$. Otherwise, $U'_{SC}(p^S) \neq 0$ for any $p^S \in (ep_k, ep_{k+1})$. Let $l, r \in (ep_k, ep_{k+1})$ with $l < r$, then the root of $U'_{SC}(p^S) = 0$ is in $[l, r]$ if and only if $Q'(l) \geq 0 \geq Q'(r)$.*

Algorithm 1: Maximize the service consumer's utility

Input: Breakpoints BP, tolerance ϵ
Output: Optimal p^{S^*}

Sort breakpoints: $bp_1 < bp_2 < \cdots < bp_n$;
Initialize candidate set $C \leftarrow \emptyset$;
foreach *interval* $[bp_k, bp_{k+1}]$ **do**
  if $U'_{SC+}(bp_k) \leq 0$ **then** $C \leftarrow C \cup \{bp_k\}$;
  else if $U'_{SC-}(bp_{k+1}) \geq 0$ **then** $C \leftarrow C \cup \{bp_{k+1}\}$;
  else
    $l \leftarrow bp_k$, $r \leftarrow bp_{k+1}$;
    while $r - l > \epsilon$ **do**
      $mid \leftarrow \frac{l+r}{2}$;
      if $U'_{SC}(mid) > 0$ **then** $l \leftarrow mid$;
      else $r \leftarrow mid$;
    end
    $C \leftarrow C \cup \left\{\frac{l+r}{2}\right\}$;
  end
end
return $p^{S^*} = \arg\max_{p \in C} U_{SC}(p)$;