Dragon Acadmemy 2017-2018

Computer Science G12

Remember I'm following the text I posted on msantos.sdf.org/G12/ .

These notes will be updated further in the coming days and can be found on that same site, both in Markdown as well as PDF formats.

Notes Tue 17 Oct 2017

Boolean Algebra

Connectives and their truth tables

And, or, not, conditional, biconditional, xor: $\wedge$, $\vee$, $\urcorner$, $\rightarrow$, $\leftrightarrow$, $\oplus$

Truth values: True/False, 0/1, T/$\perp$

Alternative notations

While symbols listed above are the ones that are common in logic, some computer science texts, or rather computer languages, may use other symbols for them.

- Negation: NOT p, !p, $\bar{p}$, p'
- Conjunction: p AND q, $p \cdot q$, $\quad p\,q$
- Disjunction: p OR q, $p + q$

Boolean Laws

1. Commutativity
2. Associativity
3. Distributivity
4. Idempotence
5. De Morgan Laws

Notes of Tue Oct 31 2017.

Logic riddles:

1. A, B, C either Knights (allways tell truth) or Knaves (allways lie). You ask A, but she whispers and you don't understand it. B then says: "She

said she is a knave", to which C immediately replies "That's a lie". What can you say about A,B, and C being either a knight or a knave?
2. A: "Exactly one of us is a knave", B: "Exactly two of us is a knave", C: "all of us are knaves". Same question.

More on logic riddles can be found in e.g.

- The notes I'm basing this unit on. See link above.
- The site Knight and Knaves
- Logicians have a peculiar sense of humor as reflected in the title of research papers like "*The Hardest Logic Puzzle Ever*" by George Boolos or "*How to solve the hardest logic puzzle ever in two questions*" by Gabriel Uzquiano. (And you thought they would only be dealing with *serious stuff*, eh?)

  You can dig further into this topic in the corresponding wikipedia page. Be aware though, these are tricky problems, so don't desperate if you feel at lost -that said, if you find them trivial, let me know and I tell the authors!!

SAT (Boolean SATisfiability) problem:

- Example of a SAT problem: Which combination of truth values of p, q and r makes the following sentence true? ~p ^ (r ^ (q v p))
- Easy understand complexity: Exponential complexity. Unsolved general case. P !=? NP

Propositional Logic as a formal system/language

We will describe again here what we have seen on propositional logic, but his time we will consider the topic as a *(formal) language*.

In more general terms, a language is an example of what's called a *formal system*. Other examples of formal systems are for instance, any *axiomatic system*, e.g., Euclid's geometry could arguably be considered the first axiomatic/formal system we know of.

One distinguishes two parts when formally stating what propositional logic, and by extension, any (formal) language, is: Its **syntax** and its **semantics**.

Note: What does the word semantic mean? where does it come from? Semantics comes from greek σημαντικός (like semanticos) and more or less translates into the word "meaning". Thus, semantic as adjective relates to the meaning in language and logic.

Syntax of Propositional Logic

1. Alphabet of symbols: $\Sigma = \{p_1,\, p_2, \ldots,\, (,\,), \neg, \vee, \wedge, \to, \leftrightarrow, \oplus, \bot, T\}$
2. Definition: **Atom** := Any of the symbols $p_1,\, p_2, \ldots$
   - Note: Atoms are usually denoted by lowercase symbols from a natural language (e.g. english) alphabet like $p, q, r, \ldots$ or variants as shown above $(p_1, p_2 \ldots)$.
3. Definition of a sentence using that alphabet Σ: Sentences are strings containing only characters of the alphabet Σ.
4. Rules defining a *well-formed formula*:
   1. (WFF1) All atoms are wff
   2. (WFF2) if A and B are wff, the so are (A), $\neg A$, $A \vee B$, $A \wedge B$, $A \oplus B$, $A \to B$, $A \leftrightarrow B$.
   3. (WFF3) All wffs are constructed by repeated application of rules (WFF1) and (WFF2) a *finite number of times*.

Definition: All wff which are not an atom are called *compound statements*.

Definition: **Literal** := Any atom p or the symbol $\neg$ immediately followed by an atom p. That is, either p or $\neg p$.

Definition: **Connective** := Any of the symbols of the alphabet that either links together two or more wff or modifies one. Examples: $\neg, \vee, \wedge \ldots$

Definition: **Language based on alphabet** Σ := Is the collection of all wff based on that alphabet.

Definition: A **grammar** is the set of rules that tells us what the wff are.

- The grammar for propositional logic is given by the rules (WFF1), (WFF2) and (WFF3) above. This set of rules is an exampl of *Backus-Knaur Form* (BNF).

The idea of a language as a formal system, i.e., defined by a grammar as a set of transformation rules as stated above, is due to Noam Chomsky.

In more technical terms, the example of grammar above is called a *context-free grammar* and plays an essential role in defining programming languages and compilers, which translate a code into an executable.

- **Solved exercise:** Prove that the sentence $(\neg((p \vee q) \wedge r))$ is a wff.
  - **Proof:**
    1. p, q and r are wff [by WFF1]
    2. $(p \vee q)$ is a wff [by 1 and WFF2]
    3. $((p \vee q) \wedge r)$ is a wff [by 2 and WFF2]
    4. $(\neg((p \vee q) \wedge r))$ is a wff [by 3 and WFF2] . q.e.d.

Semantics of Propositional Logic

The semantics of propositional logic is already well known by us: it consists in specifying the truth tables of all connectives.

Notes of Wed Nov. 7/8 2017.

Parse trees and Truth trees

For details, check out the book reference provided at the very beginning of these notes.

Parse Trees

Eg: Parse tree for $\neg p \to (q \vee r)$

$$
\begin{array}{ccccc}
 & & \rightarrow & & \\
 & \swarrow & & \searrow & \\
\neg & & & & \vee \\
\downarrow & & & & \swarrow\ \searrow \\
p & & & & q \quad r
\end{array}
\tag{1}
$$

Truth Trees

Advantages:

1. **Simplifying** expressions/statements
2. Determining **SAT** (satisfiability) questions
3. Determining *contradictions*/**tautologies**

Eg: (Book notes: pp 51, example 1.10.1)

$$
\begin{array}{ccccc}
 & & \neg p \to (q \wedge r) & & \\
 & \swarrow & & \searrow & \\
\neg(\neg p) & & & & q \wedge r \\
\downarrow & & & & \swarrow\ \searrow \\
p & & & & q \quad r
\end{array}
\tag{2}
$$

See book examples 1.10-2 to 1.10.10.

Homework: Exercices 5 (see book pp. 61)

Example workout: Exercise 1(a) book pp. 61

- **Problem**: Determine wether the following argument is valid or not using a truth tree: $A \equiv p \to q,\ r \to s,\ p \lor r \vDash q \lor s$

  Solution:

  The key is to see that proving such argument is valid is equivalent to proving that the following statement is a tautology

  $\vDash \neg\,((p \to q) \land (r \to s) \land (p \lor r)) \lor (q \lor s)$

  The above line states that the expression to the right of the symbol $\vDash$ is a tautology.

  In order to prove that an expression is a tautology the trick is to prove that its negation is a **contradiction** (*proof by contradiction*), i.e., we need to prove that

  $((p \to q) \land (r \to s) \land (p \lor r))\ \land \neg(q \lor s) \vDash .$

  Let's do that.

$$
\begin{array}{c}
(p \to q) \land (r \to s) \land (p \lor r) \land \neg(q \lor s)\\
\downarrow\\
p \to q\\
r \to s\\
p \lor r\\
\neg(q \lor s)\\
\swarrow \qquad \searrow\\
\neg p \qquad\quad q\\
\swarrow\downarrow \qquad\quad \downarrow\searrow\\
\neg r \qquad s \qquad \neg r \qquad s\\
\swarrow\downarrow \quad \downarrow\searrow \quad \downarrow\searrow \qquad \downarrow\searrow\\
p \quad r \quad p \quad r \quad p \quad r \quad p \quad r\\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow\\
\neg q \; \neg q \; \neg q \; \neg q \; \neg q \; \neg q \; \neg q \; \neg q\\
\neg s \; \neg s \; \neg s \; \neg s \; \neg s \; \neg s \; \neg s \; \neg s\\
\bot \;\; \bot \;\; \bot \;\; \bot \;\; \bot \;\; \bot \;\; \bot \;\; \bot
\end{array} \tag{3}
$$

  Clearly, not all branches end up in a contradiction, e.g., the right-most branch containing $\neg q \land \neg s$.

  Whence, the original statement is what is called a **contingent statement**, i.e., *a statement whose truth value can be either false or true, depending on the truth values of the atoms involved.* $\square$