Computer Science G12

Test 2 Term 3

Date: March 28, 2018

Name:

Each question emphasizes different skills: (K) Knowledge, (T) Thinking, (C) Communication and (A) Application.

The value of each problem is as follows: P1 20% , P2 40%, P3 30% and P4 10%.

1. (20% KTC) Prove that $Q_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} < 1$ by way of finding a closed expression for Q_n. Make sure that your reasoning is clear.

   Answer: $Q_n = 2\,Q_n - Q_n = 1 - \frac{1}{2^n}$. Whence, as it is always that $(1/2)^n > 0$, and therefore $1 - 1/2^n < 1$, we conclude that it will always be $Q_n < 1$ $\Box$

2. (40% KTC) The gist of the **insertion sort** algorithm consists in building a sorted list of numbers by inserting each of them one by one in the right position. Say, the list of numbers R_1, R_2, R_3, … $R_n\, R_{n+1}$ is such that the first n of them are already sorted. How can we sort the full list?

   1. Sketch a reasoning of how to sort the full list using PMI and proving it is correct.
   2. Write a pseudo-code for this algorithm

   Answer: See course notes.

3. (30% KCA) Calculate the closed form of the following sums and prove your result by mathematical induction. Hint: Use the method of shifting and the well-known result $S_n = \sum_{k=1}^{n} k = n(n+1)/2$

   1. $S2_n = \sum_{k=0}^{n} k^2$

      Answer: Consider the sum $S3_n = \sum_{k=1}^{n} k^3$. It is $n^3 = S3_n - S3_{n-1} = \sum_{k=1}^{n}\{k^3 - (k-1)^3\}$. Expanding the last expression we have $n^3 = 3\,S2_n - 3\,S_n + n$. Isolating $S2_n$ from here and simplifying we arrive at the solution $S2_n = n\,(n+1)\,(2n+1)/6$

      Let's prove this result by mathematical induction.

      Base case: It works for $n = 1$. Indeed, by definition $S2_1 = 1$, and the expression we obtained evaluates to $1(1+1)(2 1+1)/6 = 1$. Check!

      Induction step: Let's assume it works for a given n. Let's show that we can then prove it also works for $n+1$. Indeed, for $n+1$ it is by definition $S2_{n+1} = S2_n + (n+1)^2$. Using the assumption, we can rewrite the right-hand side as $n\,(n+1)\,(2n+1)/6 + (n+1)^2$. Simplifying this leads to the result that $S2_{n+1} = (n+1)\,(n+2)\,(2n+3)/6$ $\square$

2. $S3_n = \sum_{k=0}^{n} k^3$

   Answer: Consider the sum $S4_n = \sum_{k=1}^{n} k^4$. It is $n^4 = S4_n - S4_{n-1} = \sum_{k=1}^{n}\{k^4 - (k-1)^4\}$. Expanding the last expression we have $n^4 = 4\,S3_n - 6\,S2_n + 4\,S_n - n$. Isolating $S3_n$ from here and simplifying we arrive at the solution $S3_n = n^2\,(n+1)^2/4$

   The proof by PMI follows the same steps as for the previous case.

4. (10% KTC) Consider the following recurrence relation

$$G_n = G_{n-1} + G_{n-2} + 1 \quad \text{with} \quad G_1 = 1\; G_2 = 1$$

and compare it to the Fibonacci one

$$F_n = F_{n-1} + F_{n-2} \quad F_1 = 1 \quad \text{with} \quad F_2 = 1$$

Because of the extra 1 it seems obvious that $G_n > F_n$. Yet the following *seems* a valid proof that $G_n = F_n - 1$, namely (by the strong PMI): Assume that $G_k = F_k - 1\ \forall\, k \leq n$. We can then prove it holds for $n+1$ as well:

$$G_{n+1} = G_n + G_{n-1} + 1 = F_n - 1 + F_{n-1} - 1 + 1 = F_{n+1} - 1$$

What is **wrong** with this proof?

Answer: The base case $n = 1$ does not satisfy the induction hypothesis $G_n = F_n - 1$. The given argument just proves that IF that hypothesis would work for any given case n^*, THEN it would work for all cases $n > n^*$. However, it so happens that there is no case at all for which it works! Therefore this does not constitute a proof that $G_n = F_n - 1$ $\square$.

Furthermore, the fact that there is a concrete case (the base case $n = 1$) for which the given relation is false means that the statement "*For all values of $n > 0$, it is $G_n = F_n - 1$*" **is actually false**. We would say that the case $n = 1$ is a *counter-example* that falsifies that statement.