NOTES FOR STATISTICAL MECHANICS OF PARTITIONS

1. Valuations on Lattices

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1.1. Example: Buff on's needle problem.

1.2. GC. Rota's seminar[1]: Valuations on Lattices.

Definition 1: If $A \subset \Sigma$, for a set of (N_Σ) elements, Σ, the *indicator function*, or simply indicator, of A, denoted as I_A, is the function on Σ given by

$$\begin{aligned} I_A(c) &= 1; \quad c \in A \\ I_A(c) &= 0; \quad c \notin A \end{aligned}$$

Note to self: For partitions, the indicator function would be defined for the set of *blocks* or *clusters*. Hence, if S is the underlying set of N elements whose partitions we are studying,

$$N_\Sigma = \sum_{j=1}^{N} \binom{N}{j} = 2^N - 1$$

and $\Sigma = \mathcal{P}(S) - \emptyset$ the set of all non-empty subsets of S. Any partition P can be viewed as $P = \{P_1, \ldots, P_\mathcal{K}\} \subset \Sigma(S)$, a finite collection of non-empty subsets, $P_i \subset S$ such that they cover S, i.e., $S = \cup_i P_i$. Hence, $\forall\, P \in \Pi(S) \Rightarrow P \subset \Sigma(S)$.

Indicator functions satisfy:

$$\begin{aligned} I_{A \wedge B} &= I_A I_B \\ I_{A \vee B} &= I_A + I_B - I_A I_B = 1 - (1 - I_A)(1 - I_B) \end{aligned}$$

(**TODO**: Check that, indeed, we can substitute $\cap$, $\cup$ for $\wedge$, $\vee$)

Definition 2: An *L-simple* function, or *simple* function, is a finite linear combination

$$f = \sum_{i=1}^{k} \alpha_i \, I_{iP}$$

where $\alpha_i \in \mathbb{R}$ and ${}_iP \in \Pi(S)$ are partitions of a given set S.

Definition 3: A valuation μ on the lattice, Π, of all partitions of a set S is a function

$$\begin{gathered} \mu : \Pi \longrightarrow \mathbb{R} \\ \forall\, P, Q \in \Pi \rightarrow \\ \mu(P \vee Q) = \mu(P) + \mu(Q) - \mu(P \wedge Q) \\ \mu(\bar{0}) = 0 \end{gathered}$$

For distributive lattices $P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$ (and its dual relation).

Iterating

$\mu(P \vee Q \vee R) = \mu(P) + \mu(Q \vee R) - \mu((P \wedge Q) \vee (P \wedge R)) =$
$\mu(P) + \mu(Q) + \mu(R) - \mu(P \wedge Q) - \mu(P \wedge R) - \mu(Q \wedge R) + \mu(P \wedge Q \wedge R)$

$\mu(P \vee Q \vee R \vee T \vee \dots) = \mu(P) + \mu(Q) + \mu(R) + \mu(T) \cdots +$
$-\,\mu(P \wedge Q) - \mu(P \wedge R) - \dots +$
$+\,\mu(P \wedge Q \wedge R) + \mu(P \wedge Q \wedge T) + \dots -$
$-\,\mu(P \wedge Q \wedge R \wedge T)$

Definition 4: A *generating set* of $\Pi(S)$ is a subset $G \subset \Pi(S)$, such that $\forall P \in \Pi \Rightarrow P = \vee_i B_i \quad ; B_i \in G$.

Using the inclusion-exclusion formula for indicators, it can be shown that any simple function can be written as

$$f = \sum_{i=1}^{r} \beta_i I_{B_i}$$

A valuation ν on G can be extended to one μ on $\Pi(S)$ by using the exclusion-inclusion formula. $\forall\, P \in \Pi(S)\ ;\ P = B_1 \vee \dots \vee B_n$

$$\mu(P) = \sum_i \nu(B_i) - \sum_{i<j} \mu(B_i \wedge B_j) + \dots$$

Definition 5: Given a valuation μ on G, and a simple function $f = \alpha_1 I_{B_1} + \dots + \alpha_k I_{B_k}$, with $B_i \in G$, we define the *integral of f wrt μ* as

$$\int f\, d\mu = \sum_{i=1}^{k} \alpha_i\, \mu(B_i)$$

Groemer's Integral Theorem: Let G be a generating set for a lattice L and μ a valuation on G. The following statements are equivalent

(1) μ extends uniquely to a valuation on L.
(2) μ satisfies the inclusion-exclusion identities

$$\mu(B_1 \vee \dots \vee B_n) = \sum_i \mu(B_i) - \sum_{i<j} \mu(B_i \wedge B_j) + \dots$$

(3) μ defines an integral on the vector space of linear combinations of indicator functions of sets in L

Proof: (1)$\Rightarrow$ (2) and (3)$\Rightarrow$(1), trivial; (2)$\Rightarrow$(3), not trivial.

1.3. Valuations on Simplicial Complexes.

1.4. Euler Characteristic.

References

[1] *Introduction to Geometric Probability*, Daniel A. Klain and Gian-Carlo Rota, CUP, 1997.