NOTES FOR STATISTICAL MECHANICS OF PARTITIONS

1. Lie Algebra of Partitions

(These are just a digital transcription of the written notes of years ago)

1.1. **Example N=2 P:12.** We have seen that for a partition like $P : 1|2$ we can define angular momentum operators with the Casimir satisfying

$$\eta_{11} = \eta_{12} = \eta_{21} = \eta_{22} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1,\, 0) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{1}$$

$$[\eta_{\alpha\beta},\, \eta_{\alpha'\beta'}] = 0 \tag{2}$$

$$\hat{\mathbf{N}} = \eta_{11} + \eta_{22} = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \tag{3}$$

$$\left[\hat{\mathbf{N}},\, \eta_{\alpha\beta}\right] = 0 \tag{4}$$

$$\tau_0 \equiv \frac{1}{2}(\eta_{11} - \eta_{22}) = 0 \tag{5}$$

$$\tau_+ \equiv \eta_{12} = \eta_{21} \equiv \tau_- \tag{6}$$

$$\boldsymbol{J}^2 = \frac{1}{2}(\tau_+\tau_- + \tau_-\tau_+) + \tau_0^2 = \frac{1}{2}\hat{\mathbf{N}} \tag{7}$$

$$\boldsymbol{J} = 0\,!? \tag{8}$$

Thus, for N=2 the Hasse diagram (HD)

$$\begin{array}{ccc} 1\,2 & \leftrightarrow & \boldsymbol{J} = 0\,!? \\ 1\,|\,2 & \leftrightarrow & \boldsymbol{J} = \frac{1}{2} = \begin{cases} m = 1/2 \\ m = -1/2 \end{cases} \end{array}$$

and moving through the HD involves transitions between states with $\boldsymbol{J} = 0$ and $\boldsymbol{J} = 1/2$.

1.2. **Example N=3 P:**1|23**.** We have $4\,\eta_{\alpha\beta}$ independent and $3\,\eta_{\alpha 3}$ which commute with $\eta_{\alpha 2}$. Thus we get a Lie algebra of rank 2 (?).

$$\eta_{11}\,,\, \eta_{12} = \eta_{13}\,,\, \eta_{21} = \eta_{31}\,,\, \eta_{22} = \eta_{33} = \eta_{23} = \eta_{32} \tag{9}$$

$$\hat{\mathbf{N}} = \eta_{11} + \eta_{22} + \eta_{33} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{10}$$

$$\tau_+ \equiv \tau_{12}\ ,\ \tau_- \equiv \eta_{21}\ ,\ \tau_0 \equiv \frac{1}{2}\,(\eta_{11}\ -\ \eta_{22}) \tag{11}$$

$$\tau_0\ =\ \frac{1}{4}\,(2\,\eta_{11}\ -\ \eta_{22}\ -\ \eta_{33}) \tag{12}$$

$$[\tau_0\ ,\ \tau_+]\ =\ [\tau_0\ ,\ \eta_{12}]\ =\ \frac{1}{2}\,(\eta_{12} + \eta_{12})\ =\ \eta_{12} \tag{13}$$

$$\left[\hat{\mathbf{N}}\ ,\ \tau_+\right]\ =\ \left[\hat{\mathbf{N}}\ ,\ \eta_{12}\right]\ =\ \eta_{12}\ -\ \eta_{12}\ -\ \eta_{13}\ =\ -\eta_{13}\ =\ -\eta_{12} \tag{14}$$

Let's rather start with $P : 1|2|3\ \rightarrow\ 9\,\eta_{\alpha\beta}$.

$$\hat{\mathbf{N}}\ =\ \eta_{11}\ +\ \eta_{22}\ +\ \eta_{33} \tag{15}$$

$$\tau_+\ =\ \eta_{12}\ ,\ \tau_-\ =\ \eta_{21} \tag{16}$$

$$\tau_0\ =\ \frac{1}{2}\,(\eta_{11}\ -\ \eta_{22}) \tag{17}$$

$$B_+\ =\ \eta_{13}\ ,\ B_-\ =\ \eta_{23} \tag{18}$$

$$C_+\ =\ \eta_{32}\ ,\ C_-\ =\ \eta_{31} \tag{19}$$

$$M\ =\ \frac{1}{3}\,(\eta_{11}\ +\ \eta_{22} - 2\,\eta_{33}) \tag{20}$$

which gives rise to SU(3).

$$[M\ ,\ \tau_0]\ =\ 0 \tag{21}$$

$$[\tau_0\ ,\ B_\pm]\ =\ \frac{1}{2}\{[\eta_{11}\ ,\ \eta_{.3}]\ -\ [\eta_{22}\ ,\ \eta_{.3}]\}\ =\ \frac{1}{2}\,\eta_{.3}\ =\ \pm\frac{1}{2}B_\pm \tag{22}$$

$$[\tau_0\ ,\ C_\pm]\ =\ \pm\frac{1}{2}\,C_\pm \tag{23}$$

Thus, $B_\pm$ and $C_\pm$ **create** a 1-*particle* and annihilate a 2-*particle*. This commutation relations are then consistent with interpreting

$$\eta_{\alpha\beta}\ \equiv\ \overrightarrow{a}_\alpha \bigotimes \overrightarrow{a}^\dagger_\beta$$

as equivalent to the following bilinear form of creation & annihilation operators in QFT: $a^\dagger_\alpha a_\beta$. That is, $\eta_{\alpha\beta}$ creates an "α" particle/excitation and destroys a "β" one.

We further have thus the B's annihilate a "3-particle" and therefore increase M by $+1$, while the C's create a "3-particle" and change M by -1. Hence, this yields

$$[\tau_0\ ,\ \tau_\pm]\ =\ \pm\tau_{pm}\quad ;\quad [M\ ,\ \tau_\pm]\ =\ 0 \tag{24}$$

$$[M\ ,\ B_{pm}]\ =\ \pm B_{pm}\quad ;\quad [M\ ,\ C_\pm]\ =\ -C_{pm} \tag{25}$$

and the rest

$$[\tau_+ \, , \, \tau_-] \, = \, 2 \, \tau_0 \tag{26}$$

$$[\tau_\pm \, , \, B_\pm] \, = \, [\tau_\pm \, , \, C_\pm] \, = \, 0 \, = \, [C_+ \, , \, C_-] \, = \, [B_+ \, , \, B_-] \tag{27}$$

$$[\tau_\pm \, , \, B_\mp] \, = \, B_\pm \quad ; \quad [\tau_\pm \, , \, C_\mp] \, = \, -C_\pm \tag{28}$$

$$[B_\pm \, , \, C_\mp] \, = \, \frac{1}{2} \, [3 \, M \, \pm \, 2 \, \tau_0] \tag{29}$$