# Chapter 2
# Essential Dictionary I- Part 2-2

## Functions

# Functions

Functions are everywhere. Whenever a process transforms a mathematical object into another object, there is a function in the background. ‘Function’ is arguably the most used word in mathematics.

- A **function** consists of two sets together with a rule that assign to *each* element of the first set a *unique* element of the second set. The first set is called the **domain** of the function and the second set is called the **co-domain**.
- A function whose domain is a set $A$ may be called a function **over** $A$ or a function **defined on** $A$.
- The terms **map** or **mapping** are synonymous with function.

# Functions

- A function is usually denoted by a single letter or symbol, such as $f$.
- If $x$ is an element of the domain of a function $f$, then the **value of $f$ at $x$**, denoted by $f(x)$

$$f\colon A \longrightarrow B \qquad x \mapsto f(x) \tag{2.16}$$

indicates that $f$ is a function with domain $A$ and co-domain $B$ that **maps** $x \in A$ **to** $f(x) \in B$.

The symbol $x$ is the **variable** or (the **argument**) of the function.

- The symbols '$\longrightarrow$' and '$\mapsto$' have different meanings, and should not be confused.
- The function

$$I_A\colon A \longrightarrow A \qquad x \mapsto x$$

is called the **identity** (**function**) on $A$.

When explicit reference to the set $A$ is unnecessary, the identity is also denoted by Id or 1.

# Functions

- In Definition (2.16) the symbols used for the function's name and variable are inessential; the two expressions

$$f\colon \mathbb{R}\backslash\{0\}\longrightarrow\mathbb{R} \quad x\mapsto\frac{1}{x}, \qquad\qquad x\colon \mathbb{R}\backslash\{0\}\longrightarrow\mathbb{R} \qquad f\mapsto\frac{1}{f}$$

define exactly the same function.

- Let us use the word 'function' in short expressions. These are function definitions:

1. *The integer function that squares its argument.*

2. *The function that returns 1 if its argument is rational, and 0 otherwise.*

3. *The function that counts the number of primes smaller than a given real number.*

# Functions

Functions of several variables are defined over cartesian products of sets. For example, the function

$$f\colon \mathbb{Z}\times\mathbb{Z}\to\mathbb{N} \qquad (x,y)\mapsto \gcd(x,y)$$

depends on two integer arguments, and hence is defined over the cartesian product of two copies of the integers. This definition requires a value for $\gcd(0,0)$,which normally is taken to be zero.

Let $f\colon A\to B$ be a function. The set

$$\{(x,f(x))\in A\times B\colon x\in A\} \tag{2.17}$$

is called the **graph** of $f$.

A function is completely specified by three sets: domain, co-domain and graph.

We can now reformulate the definition of a function, replacing the imprecise term ‘rule’ with the precise term ‘graph’. We write a formal definition.

# Functions

*Definition*. A **function** $f$ is a triple $(X,Y,G)$ of non-empty sets. The sets $X$ and $Y$ are arbitrary, while $G$ is a subset of $X \times Y$ with the property that for every $x \in X$ there is a unique pair $(x,y) \in G$. The quantity $y$ is called the **value of the function at** $x$, denoted by $f(x)$.

We see that, besides sets, the definition of a function requires the constructs of ordered pair and triple. It turns out that these quantities can be defined solely in terms of sets (see Exercise 2.14). So, to define functions, all we need are sets after all.

# Functions

Given a function $f: A \to B$, and a subset $X \subset A$, the set

$$f(X) \stackrel{\text{def}}{=} \{f(x) \colon x \in X\} \qquad (2.18)$$

is called the **image of $X$ under $f$**. The assignment operator gives meaning to the symbolic expression $f(x)$, which otherwise would be meaningless, since we stipulated that the argument of a function is an element of the domain, not a subset of it. Thus $\sin(\mathbb{R})$ is the closed interval $[-1,1]$.

Clearly, $f(X) \subset B$, and $f(A)$ is the smallest set that can serve as co-domain for $f$. The set $f(A)$ is often called the **image** or the **range** of the function $f$. This term is sometimes used to mean co-domain, which should be avoided.

A **constant** is a function whose image consists of a single point.

# Functions

- A function is said to be **injective** (or **one-to-one**) if distinct points of the domain map to distinct points of the co-domain.
- A function is **surjective** (or **onto**) if $f(A)=B$, that is, if the image coincides with the co-domain.
- A function that is both injective and surjective is said to be **bijective**.
- For any non-empty subset $X$of the domain $A$, we define the **restriction of** $f$ **to** $X$ as

$$f|X\colon X\longrightarrow B \qquad x\mapsto f(x).$$

- Given two functions $f\colon A\longrightarrow B$ and $g\colon B\longrightarrow C$, their **composition** is the function

$$g\circ f\colon A\longrightarrow C \qquad x\mapsto g(f(x)). \tag{2.19}$$

The notation $g\circ f$ reminds us that $f$ acts before $g$. The image $g(f(x))$of $x$ under $g\circ f$ is denoted by $(g\circ f)(x)$,where the parentheses isolate $g\circ f$ as the function's symbolic name.

The hybrid notation $g\circ f(x)$should be avoided.

# Functions

- If $f\colon A \to B$ is a bijective function, then the **inverse** of $f$ is the function $f^{-1}\colon B \to A$ such that

$$f^{-1} \circ f = 1_A, \qquad f \circ f^{-1} = 1_B$$

where $1_{A,B}$ are the identities in the respective sets.

- A function is said to be **invertible** if its inverse exists.
- If $f\colon A \to B$ is injective, then we can always define the inverse of $f$ by restricting its domain to $f(A)$ if necessary.
- In absence of injectivity, it may still be possible to construct an inverse by a suitable restriction of the function. Thus the **arcsine** may be defined by restricting the sine to the interval $[-\pi/2, \pi/2]$.
- Let $f\colon A \to B$ be a function, and let $C$ be a subset of $B$. The set of points

$$f^{-1}(C) \overset{\text{def}}{=} \{x \in A\colon f(x) \in C\} \qquad (2.20)$$

is called the **inverse image** of the set $C$.

# Functions

Since the definition of inverse image does not involve the inverse function, the inverse image exists even if the inverse function does not. These two concepts must be distinguished carefully. When the reciprocal of a function comes into play, things get very confusing, since we now have three unrelated objects represented by closely related notation:

$$f^{-1}(x) \qquad f^{-1}(\{x\}) \qquad f(x)^{-1},$$

The first expression is well-defined if $x$ belongs to the image of $f$ and $f$ is invertible there.

In the second expression there is no condition on $f$, and $x$ need only be an element of the co-domain.

In the third expression the point $x$ must belong to the domain of $f$, and $f(x)$ must be non-zero.

Example:

$sin^{-1}(1)=\frac{\pi}{2},$ $\qquad sin^{-1}(\{1\})=\frac{\pi}{2}+2\pi\mathbb{Z},$ $\qquad \sin(1)^{-1}=\csc(1).$

# Functions

- In the first expression we assume that $\boldsymbol{sin^{-1}}$**=arcsin:** $[-1,1]\to[-\pi/2,\pi/2]$.
- In the third expression the symbol **csc** denotes the co-secant ($\csc(x)=1/\sin(x)$), defined in the domain $\mathbb{R}\setminus\pi\mathbb{Z}$.

With a judicious use of definite and indefinite articles, we can specify a function's type without committing ourselves to a specific object.

1. *The inverse of a trigonometric function.*

2. *The composition of a function with itself.*

3. *An integer-valued bijective function.*

4. *A function which coincides with its own inverse.*

In item 2, we infer that the function maps its domain into itself. Functions of type 3 will be considered in the next section to define cardinality of sets. Functions of type 4 are called **involutions** (e.g., $x\mapsto -x$, over a suitable domain).