Chapter 2
Essential Dictionary I- Part 1

Sets

In writing mathematics we use words and symbols to describe facts. We need to explain the meanings of words and symbols, and to state and prove the facts.

We’ll be concerned with facts later. In this chapter and the next we list mathematical words with accompanying notation. This is our essential mathematical dictionary.

It contains some 200 entries, organized around few fundamental terms: **set**, **function**, **sequence**, **equation**. As we introduce new words, we use them in short phrases and sentences.

Dictionaries are not meant to be read through, so don’t be surprised if you find the exposition demanding. Take it in small doses. The last section of this chapter deals with advanced terminology and may be skipped on first reading.

Sets

A **set** is a collection of *well-defined*, *unordered*, *distinct* objects. (This is the so-called ‘naive definition’ of a set, due to Cantor.) These objects are called the **elements** of a set, and a set is determined by its elements. We may write

The set of all odd integers

The set of vertices of a pentagon

The set of differentiable real functions

In simple cases, a set can be defined by listing its elements, separated by commas, enclosed within curly brackets. The expression

$$\{1, 2, 3\}$$

denotes the set whose elements are the integers 1, 2 and 3.

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It is customary to ignore repeated set elements: {2, 1, 3, 1, 3} = {2, 1, 3}. This convention, adopted by computer algebra systems, simplifies the definition of sets.

If repeated elements are allowed and not collapsed, then we speak of a **multiset**: {2, 1, 3, 1, 3}.

The **multiplicity** of an element of a multiset is the number of times the element occurs. Reference to multiplicity usually signals that there is a multiset in the background:

Every quadratic equation has two complex solutions, counting multiplicities.

The set {} with no elements is called the **empty set**, denoted by the symbol ∅.

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To assign a symbol to a mathematical object, we use an **assignment statement** (or **definition**), which has the following syntax:

$$A := \{1, 2, 3\}.$$

This expression assigns the symbolic name A to the set $\{1, 2, 3\}$, and now we may use the former in place of the latter.

The symbol ‘:=’ denotes the **assignment operator**. It reads ‘*becomes*’, or ‘*is defined to be*’, rather than ‘*is equal to*’, to underline the difference between assignment and equality (in computer algebra, the symbols = and **:=** are not interchangeable at all!). So we can’t write $\{1, 2, 3\} := A$, because the left operand of an assignment operator must be a symbol or a symbolic expression.

There are other symbols for the assignment operator, namely

$$\stackrel{\text{def}}{=} \qquad \stackrel{\nabla}{=}, \tag{2.2}$$

which make an even stronger point.

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To indicate that x is an element of a set A, we write

$x \in A$ *x is an element of A* *x belongs to A.*

The symbol $\notin$ is used to negate membership. Thus

$\{7, 5\} \in \{5, \{5, 7\}\}$ $7 \notin \{5, \{5, 7\}\}$.

A **subset** B of a set A is a set whose elements all belong to A. We write

$B \subset A$ *B is a subset of A* *B is contained in A*

and we use $\not\subset$ to negate set inclusion. For example

$\{3, 1\} \subset \{1, 2, 3\}$ $\emptyset \subset \{1\}$ $\{2, 3\} \not\subset \{2, \{2, 3\}\}$.

Every set has at least two subsets: itself and the empty set. Sometimes these are referred to as the **trivial** subsets.

Every other subset—if any—is called a **proper subset**.

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The **cardinality** of a set is the number of its elements, denoted by #:

$$\#\{7, -1, 0\} = 3 \qquad\qquad \#A = n.$$

The absolute value symbol $|\cdot|$ is also used to denote cardinality: $|A| = n$.

.A set is **finite** if its cardinality is an integer, and **infinite** otherwise.

We write $A \cap B$ for the **intersection** of the sets A and B: this is the set comprising elements that belong to both A and B. If $A \cap B = \emptyset$, we say that A and B are **disjoint**, or have **empty intersection**. The sets $A_1, A_2, \ldots$ are **pairwise disjoint** if $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

We write $A \cup B$ for the **union** of A and B, which is the set comprising elements that belong to A or to B (or to both A and B).

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We write $A \setminus B$ for the (**set**) **difference** of A and B, which is the collection of the elements of A that do not belong to B.

The **symmetric difference** of A and B, denoted by $A \Delta B$, is defined as

$$A \Delta B \stackrel{\text{def}}{=} (A \setminus B) \cup (B \setminus A).$$

The following examples illustrate the action of set operators:

$$\{1, 2, 3\} \cap \{3, 4, 5\} = \{3\}$$

$$\{1, 2, 3\} \cup \{3, 4, 5\} = \{1, 2, 3, 4, 5\}$$

$$\{1, 2, 3\} \setminus \{3, 4, 5\} = \{1, 2\}$$

$$\{1, 2, 3\} \,\Delta\, \{3, 4, 5\} = \{1, 2, 4, 5\}.$$

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The above **set operators** are **binary**; they have two sets as **operands**. The identities

$$A \cap B = B \cap A \quad (A \cap B) \cap C = A \cap (B \cap C)$$

express the **commutative** and **associative** properties of the intersection operator. Union and symmetric difference enjoy the same properties, but set difference doesn't.

Let A be a subset of a set X. The **complement** of A (in X) is the set $X \setminus A$, denoted by A' or by A^c. The complement of a set is defined with respect to an **ambient set** X. Reference to the ambient set may be omitted if there is no ambiguity. So we write

The odd integers is the complement of the even integers

since it's clear that the ambient set is the integers.

Sets

ordered pair is an expression of the type (a, b), with a and b arbitrary quantities. Ordered pairs are defined by the property

$$(a, b) = (c, d) \quad \text{if} \quad a = c \quad \text{and} \quad b = d. \tag{2.4}$$

The ordered pair (a, b) should not be confused with the set $\{a, b\}$, since for pairs order is essential and repetition is allowed.

Let A and B be sets. We consider the set of all ordered pairs (a, b), with a in A and b in B. This set is called the **cartesian product** of A and B, and is written as

$$A \times B.$$

Note that A and B need not be distinct; one may write A^2 for $A{\times}A$, A^3 for $A{\times}A{\times}A$, etc. Because the cartesian product is **associative**, the product of more than two sets is defined unambiguously. Also note that the explicit presence of the multiplication operator '×' is needed here, because the expression AB has a different meaning [see Eq. (2.21), Sect. 2.3].