Chapter 2
Essential Dictionary I- Part 1

Defining Sets

Defining a set by listing its elements is inadequate for all but the simplest situations. How do we define large or infinite sets? A simple device is to use the **ellipsis** '. . .', which indicates the deliberate omission of certain elements, the identity of which is made clear by the context.

For example, the set $\mathbb{N}$ of **natural numbers** is defined as

$$\mathbb{N} := \{1, 2, 3, \ldots\}.$$

Here the ellipsis represents all the integers greater than 3. Some authors regard 0 as a natural number, so the definition

$$\mathbb{N} := \{0, 1, 2, 3, \ldots\}$$

Is also found in the literature..

The set of **integers**, denoted by Z (from the German *Zahlen*, meaning numbers), can also be defined using ellipses:

$$\mathbb{Z} := \{\ldots, -2, -1, 0, 1, 2, \ldots\} \qquad \text{or} \qquad \mathbb{Z} := \{0, \pm 1, \pm 2, \ldots\}.$$

Defining Sets

To define general sets we need more powerful constructs. A **standard definition** of a set is an expression of the type

$$\{x : x \text{ has } P\} \qquad (2.5)$$

where P is some unambiguous property that things either have or don't have. This expression identifies the set of all objects x that have property P. The colon '**:**' separates out the object's symbolic name from its defining properties. The vertical bar '**|**' or the semicolon '**;**' may be used for the same purpose. Thus the empty set may be defined symbolically as

$$\emptyset \stackrel{\text{def}}{=} \{x : x \neq x\}. \qquad (2.6)$$

The property P is 'x is not equal to x', which is not satisfied by any x. Likewise, the cartesian product $A \times B$ of two sets may be specified as

$$\{x : x = (a, b) \text{ for some } a \in A \text{ and } b \in B\}.$$

The rule 'x has property P' now reads: 'x is of the form (a, b) with $a \in A$ and $b \in B$'. The same set may be defined more concisely as

$$\{(a, b) : a \in A \text{ and } b \in B\}.$$

Defining Sets

This is a variant of the standard definition (2.5), where the type of object being considered (ordered pair) is specified at the outset. This form of standard definition can be very effective.

The set $\mathbb{Q}$ of **rational numbers**—ratios of integers with non-zero denominator—is defined as follows:

$$\mathbb{Q} := \{\tfrac{a}{b} : \; a \in \mathbb{Z}, b \in \mathbb{N}, \gcd(a, b) = 1\} \tag{2.7}$$

The property P is phrased in such a way as to avoid repetition of elements. This is the so-called **reduced form** of rational numbers.

One might think that in the expression for a set we could choose any property P. Unfortunately this doesn't work for a reason known as the *Russell-Zermelo paradox* (1901).

Defining Sets

Consider the set definition

$$W := \{x : x \notin x\} \tag{2.8}$$

in which P is the property of being a set that is not a member of itself. The quantity

$$x = \{3, \{3, \{3, \{3\}\}\}\}$$

has property P and hence belongs to W, whereas

$$x = \{3, \{3, \{3, \{3, \ldots\}\}\}\} \qquad \text{or} \qquad x = \{3, x\}$$

does not have property P and hence does not belong to W. (In the above expression, the nested parentheses must match, so the notation $\{3, \{3, \{3, \{3, \ldots\}$ is incorrect.) Given that W is a set of sets, we ask: does W belong to W? We see that if $W \in W$, then W has property P, that is, $W \notin W$, and vice-versa. Impossible! Thus the standard definition (2.8) does not actually define any set.

Defining Sets

Fortunately, we can define a set in such a way that the definition guarantees the existence of the set. A **Zermelo definition** identifies a set W by describing it as

The set of members of X that have property P

where the **ambient set** X is given beforehand, and P is a property that the members of X either have or do not have. In symbols, this is written as

$$W := \{x \in X : x \text{ has } \mathfrak{P}\}. \tag{2.9}$$

For example, the expression

The set of real numbers strictly between 0 and 1

is a Zermelo definition: the ambient set is the set of real numbers, and we form our set by choosing from it the elements that have the stated property.

Zermelo definitions work because it's a basic principle of mathematics (the so-called *subset axiom*) that for any set X of objects and any property P, there is exactly one set consisting of the objects that are in X and have property P.

Arithmetic

The notation for arithmetical operations is familiar and established. The **sum** and **difference** of two numbers x and y are always written $x + y$ and $x - y$, respectively. By contrast, their **product** may be written in several equivalent ways:

$$xy \qquad x.y \qquad x \times y,$$

and so may their **quotient**:

$$\frac{x}{y} \qquad x/y \qquad x{:}\,y.$$

(The notation $x : y$ is used mostly in elementary texts.) Do not confuse the product dot '·' with the **decimal point** '.', e.g., $3 \cdot 4 = 12$ and $3.4 = 17/5$.

The **reciprocal** of x, defined for $x \neq 0$, is also written in several ways:

$$\frac{1}{x} \qquad 1/x \qquad x^{-1}$$

while the **opposite** of x is $-x$.

Arithmetic

The notation for exponentiation is *x y* , where *x* is the **base** and *y* the **exponent**.

The case of a positive integer exponent is easier, because exponentiation reduces to repeated multiplication:

$$x^n \stackrel{\text{def}}{=} \underbrace{x \ldots x}_{n},\ n \geq 1$$

The assignment operator $\stackrel{\text{def}}{=}$ indicates that this is a definition, giving meaning to the expression on the left. The use of the under-brace is necessary to specify the number of terms in the product, because all terms are identical. Also note the use of the **raised ellipsis** '· · · ' to represent repeated multiplication (or repeated applications of any operator), to be compared with the ordinary ellipsis '. . .', used for sets and sequences (see Sect. 3.1). Thus

$$\underbrace{x \ldots x}_{4}= x.x.x.x \qquad \underbrace{x, \ldots, x}_{4}= x,x,x,x$$

whereas the notation *x . . . x* is incorrect.

.

Arithmetic

In integer arithmetic, the symbol '|' is used for **divisibility**

$3|x$ 3 *divides x* *x is a multiple of* 3*.*

Example. Turn symbols into words:

$$\{x \in Z : x \geq 0,\ 2 \mid x\}.$$

Bad: The set of integers that are greater than or equal to zero, and such that 2 divides them. (*Robotic.*)

good: The set of non-negative even integers.

Arithmetic

A positive divisor of *n*, which is not 1 or *n*, is called a **proper divisor**.

A **prime** is an integer greater than 1 that has no proper divisors.

The acronyms **gcd** and **lcm** are used for **greatest common divisor** and **least common multiple**. (The expression **highest common factor** (hcf)—a variant of gcd which is popular in schools—is seldom used in higher mathematics.)

Two integers are **co-prime** (or **relatively prime**) if their greatest common divisor is 1.