MAT555E: Statistical Data Analysis for Computational Sciences

Fall22-Lecture 03: Multiple Linear Regression

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Learning Objectives

- Introduction to multiple linear regression
- Least-squares estimation
- Statistical tests
- Model selection criteria
- Implementing a multiple linear regression model with Python statsmodels library

Multiple Linear Regression Model

Multiple linear regression model set-up

- A **regression model** that involves **more than one independent variable** is called a **multiple linear regression (MLR) model**.
- Consider a MLR model with **p independent** variables:

$$Y = \beta_0 + \beta_1 * X_1 + ... + \beta_j * X_j + ... + \beta_p * X_p + \epsilon,$$

- where Y represents the response variable, X_j's $(j = 1, 2, ..., p)$ are independent variables, and with all the assumptions imposed on ϵ in SLR.
- Alternatively,

$$E(Y_i) = \beta_0 + \beta_1 * X_1 + ... + \beta_p * X_p, \quad i = 1, \dots, n.$$

Interpretation of regression coefficients

- The parameters β_j $(j = 0, 1, 2, ..., p)$ are called the **regression coefficients**.
- If the **range of the data** includes $X_1 = X_2 = ... = X_p = 0$, then β_0 is the expected value of Y when $X_1 = X_2 = ... = X_p = 0$. Otherwise, β_0 has **no physical interpretation**.
- The parameter β_j represents the change expected in Y per unit change in X_j **when all of the remaining independent variables are held constant** $(\beta_j = \frac{\partial E(Y_i)}{\partial X_j}, j = 1, 2, ..., p)$.
- For this reason, the parameters β_j $(j = 1, 2, ..., p)$ are often called **partial regression coefficients**.

Marginal and partial effects of regression coefficients

- The effect of β in the linear model with a single predictor X is **usually not the same** as the effect of β of the same variable X in a model with multiple independent variables.
- The effect β is a marginal effect, **ignoring all other potential independent variables**, whereas β is a partial effect, **conditioning on the other independent variables**.

Multiple linear regression model set-up

- Suppose we have given a random sample of size n such that $\{(X_{1i}, \dots, X_{pi}, Y_i)\}_{i=1}^{n}$.
- Then, at individual data point level, we can write down the MLR model as:

$$Y_i = \beta_0 + \beta_1 * X_{i1} + \dots + \beta_p * X_{ip} + \epsilon_i, \quad i = 1, \dots, n.$$

- Alternatively,

$$E(Y_i) = \beta_0 + \beta_1 * X_{i1} + \dots + \beta_p * X_{ip} + \epsilon_i, \quad i = 1, \dots, n.$$

Model geometric interpretation

- In MLR, the equation $E(Y_i) = \beta_0 + \beta_1 * X_{i1} + ... + \beta_p * X_{ip}$ describes a **p-dimensional regression hyperplane** in the (p+1)-dimensional space of $Y, X_1, \dots, X_p$.
- The parameter β_0 is the **intercept** of the **p-dimensional hyperplane**.

> In linear algebra: the equation above defines a **p-dimensional hyperplane** where β_0 is the off-set from the origin and $\langle \beta_1, \dots, \beta_p \rangle$ is the vector normal to the hyperplane.

Recall

- In **SLR model**, the equation $E(Y_i) = \beta_0 + \beta_1 * X_{i1}$ describes a **regression line** in the **X-Y plane** and the parameter β_0 is the **y-intercept**.
- But, we never know the true values of β_0 and β_1.
- So, in SLR, actually, we are trying to **estimate the regression line** equation $\widehat{E(Y_i)} = \widehat{Y_i} = \hat{\beta}_0 + \hat{\beta}_1 * X_{i1}$ or
- In other words, we are trying to **fit the equation** $\widehat{Y_i} = \hat{\beta}_0 + \hat{\beta}_1 * X_{i1}$ to the data, which is called as **fitted regression line**, **least-squares line** etc.

Revisiting Advertising Data

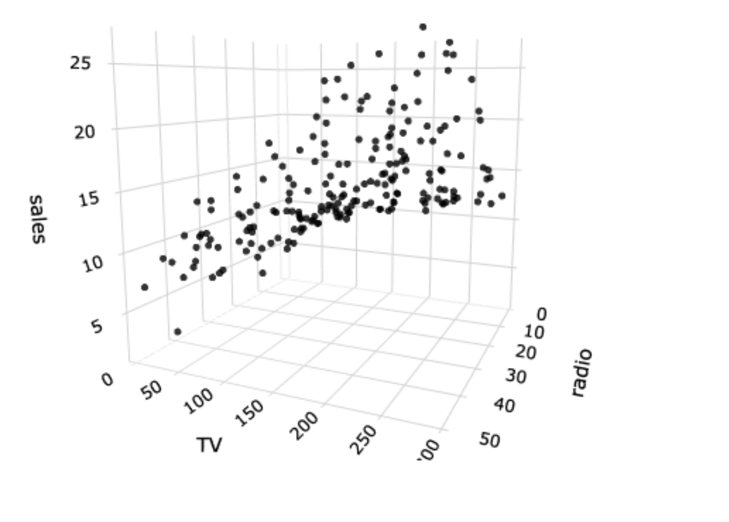
Revisiting advertising data

- Consider the advertising data which consists of `sales` of a product in 200 different markets, along with advertising budgets for the product in each of those markets for three different media: `TV`, `radio`, and `newspaper`.
- Now assume a **multiple linear regression model** for sales with **two predictors**: `TV` and `radio` such that:

$$sales_i = \beta_0 + \beta_1 * TV_i + \beta_2 * radio_i + \epsilon_i$$

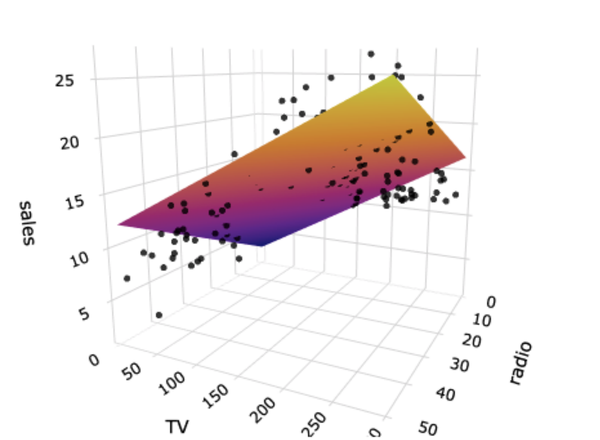
- for $i = 1, 2, \ldots, 200$ and $\epsilon_i \sim N(0, \sigma^2)$.

3D-scatter plot of advertising data



Fitting a regression plane

- Since we have **two predictors**, now, we are trying to **fit a regression plane**, $\widehat{sales}_i = \hat{\beta}_0 + \hat{\beta}_1 * TV_i + \hat{\beta}_2 * radio_i$ to the advertising data.



Multiple Linear Regression Model Continued

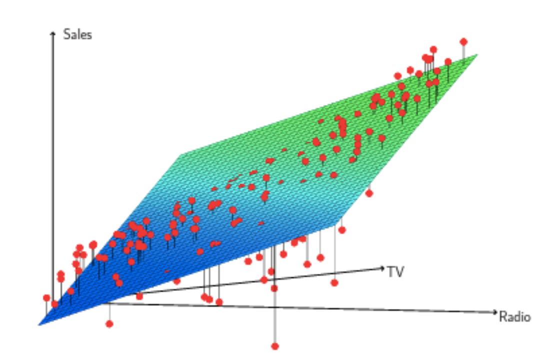
Least-squares estimation

- In the MLR model, for given a observed sample of size n such that $\{(x_{1i}, \ldots, x_{pi}, y_i)\}_{i=1}^{n}$, the method of least-squares estimates the regression coefficients which **minimizes the residual sum-of-sum squares (RSS)**:

$$RSS(\hat{\beta}) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - (\hat{\beta}_0 + \hat{\beta}_1 * x_{i1} + \ldots + \hat{\beta}_p * x_{ip})\right)^2.$$

Least-squares estimation

- For example, for advertising data with two predictors, we are trying to fit a **regression plane**.
- This plane is chosen to **minimize the sum of the squared vertical distances** between each observation (shown in red) and the plane.



Finding least-squares estimators

- Let $\mathcal{J}(\beta) = \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 * x_{i1} + ... + \beta_p * x_{ip})\right)^2$ be the **objective function**.
- To minimize $\mathcal{J}(\beta)$, we take partial derivatives with respect to β_j ($j = 0, 1, 2, ..., p$), set them to zero, and solve the resulting system of equations simultaneously, such that.

$$\left.\frac{\partial \mathcal{J}(\beta)}{\partial \beta_0}\right|_{\hat{\beta}} = -2\sum_{i=1}^{n} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 * x_{i1} + ... + \hat{\beta}_p * x_{ip})) = 0 \quad (1)$$

$$\left.\frac{\partial \mathcal{J}(\beta)}{\partial \beta_j}\right|_{\hat{\beta}} = -2\sum_{i=1}^{n} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 * x_{i1} + ... + \hat{\beta}_p * x_{ip}))x_{ip} = 0$$

- for $j = 1, 2, ..., p$.

Finding least-squares estimators

- However, solving the resulting **homogeneous system of (p+1) linear equations in (p+1) unknowns** simultaneously is not algebraically easy.

Matrix notation

- Then, denote $n \times 1$ response vector $\mathbf{Y}$, $n \times (p+1)$ design matrix $\mathbf{X}$, $(p+1) \times 1$ regression vector β, and $n \times 1$ error vector ϵ, respectively, as:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ . \\ . \\ Y_n \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & X_{11} & ... & X_{1p} \\ 1 & X_{21} & ... & X_{2p} \\ . & . & ... & . \\ . & . & ... & . \\ 1 & X_{n1} & ... & X_{np} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_2 \\ . \\ . \\ \beta_p \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ . \\ . \\ \epsilon_n \end{pmatrix}.$$

Matrix notation

- Then, the MLR model can be written in the **matrix form**:

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon.$$

- The **least-squares estimates** of β is the solution to the following **optimization** problem.

$$\hat{\beta}_{OLS} = \underset{\beta}{argmin} \left\|\epsilon\right\|_2^2 = \underset{\beta}{argmin} \left\|\mathbf{y} - \mathbf{x}\beta\right\|_2^2 = \underset{\beta}{argmin} (\mathbf{y}-\mathbf{x}\beta)^T(\mathbf{y}-\mathbf{x}\beta),$$

- since $\sum_{i=1}^{n} \epsilon_i^2 = \epsilon^T\epsilon = \left\|\epsilon\right\|_2^2$.

Finding least-squares estimators

- Denote the objective function $\mathcal{J}(\beta) = (\mathbf{y} - \mathbf{x}\beta)^T(\mathbf{y} - \mathbf{x}\beta)$ which is scalar, find the **gradient vector** of $\mathcal{J}(\beta)$ with respect to β such that:

$$\nabla \mathcal{J}(\beta) = \frac{\partial \mathcal{J}(\beta)}{\partial \beta} = \begin{pmatrix} \frac{\partial \mathcal{J}(\beta)}{\partial \beta_0} \\ \frac{\partial \mathcal{J}(\beta)}{\partial \beta_1}. \\ . \\ \frac{\partial \mathcal{J}(\beta)}{\partial \beta_p} \end{pmatrix},$$

- and then set the **gradient to zero** $\nabla \mathcal{J}(\beta) = 0$.

Finding least-squares estimators

- Let's expand $\mathcal{J}(\beta)$ first:

$$\begin{aligned}\mathcal{J}(\beta) &= \big((\mathbf{y}-\mathbf{x}\beta)^T(\mathbf{y}-\mathbf{x}\beta)\big) = \big((\mathbf{y}^T-\beta^T\mathbf{x}^T)(\mathbf{y}-\mathbf{x}\beta)\big)\\ &= \mathbf{y}^T\mathbf{y} - 2\mathbf{y}^T\mathbf{x}\beta + \beta^T\mathbf{x}^T\mathbf{x}\beta,\end{aligned}$$

- since the dot product $\mathbf{y}^T\mathbf{x}\beta$ is scalar, $(\beta^T\mathbf{x}^T y)^T = \mathbf{y}^T\mathbf{x}\beta$.

Finding least-squares estimators

- Applying the rules for **differentiation of a scalar with respect to a vector**, we get:

$$\nabla \mathcal{J}(\beta) = \frac{\partial \mathcal{J}(\beta)}{\partial \beta} = -2\mathbf{x}^T\mathbf{y} + 2\mathbf{x}^T\mathbf{x}\beta,$$

- since $\frac{\partial a^T \beta}{\partial \beta} = a$ and $\frac{\partial \beta^T \mathbf{S} \beta}{\partial \beta} = 2\mathbf{S}\beta$, where $\mathbf{S}$ is a symmetric matrix.

Finding least-squares estimators

- Setting the gradient vector $\nabla \mathcal{J}(\beta) = 0$, we get the **normal equations**:

$$\mathbf{x}^T \mathbf{y} = \mathbf{x}^T \mathbf{x} \beta.$$

- Here we assume that **the number of columns in $\mathbf{x}$ is less than the number of rows** such as $(p + 1) < n$ and the rank of $\mathbf{x}$ is $(p + 1)$, hence $\mathbf{x}$ has a **full column rank** $(p + 1)$.

Least-squares estimators

- Since $rank(\mathbf{x}^T\mathbf{x}) = rank(\mathbf{x}) = (p+1)$, this leads the symmetric matrix $\mathbf{x}^T\mathbf{x}$ to has a **full rank** of $(p+1)$.
- In this case $\det(\mathbf{x}^T\mathbf{x}) \neq 0$, that the square matrix $\mathbf{x}^T\mathbf{x}$ is non-singular and, in turn, is **invertible**.
- Then the normal equations yield the **least-squares estimator** of β as:

$$\hat{\beta}_{OLS} = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y},$$

- where $\mathbf{x}^T\mathbf{x}$ is a $(p+1)\times(p+1)$ **symmetric matrix** and $\mathbf{x}^T\mathbf{y}$ is a $(p+1)\times 1$ vector.
- Note that $\mathbf{x}^T\mathbf{x}$ is also known as Gram matrix of $\mathbf{x}$.

The form of $\mathrm{x}^T\mathrm{x}$ matrix and $\mathrm{x}^T\mathrm{y}$ vector

$$\mathbf{x}^T\mathbf{x} = \begin{pmatrix} n & \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i2} & \dots & \sum_{i=1}^n x_{ip} \\ \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i1}^2 & \sum_{i=1}^n x_{i1}x_{i2} & \dots & \sum_{i=1}^n x_{i1}x_{ip} \\ . & . & . & . & . \\ . & . & . & . & . \\ \sum_{i=1}^n x_{ip} & \sum_{i=1}^n x_{i1}x_{ip} & \sum_{i=1}^n x_{i2}x_{ip} & \dots & \sum_{i=1}^n x_{ip}^2 \end{pmatrix} \quad \text{and}$$

$$\mathbf{x}^T\mathbf{y} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_{i1}y_i \\ . \\ . \\ \sum_{i=1}^n x_{ip}y_i \end{pmatrix}.$$

Issues with Inverse of $\mathrm{x}^T\mathrm{x}$ Matrix

Linearly independent

A collection of (p+1) elements in a vector space are linearly dependent if at least one element in this collection can be expressed as a linear combination of the remaining p elements. If no element, however, can be expressed in this fashion, then the (p+1) elements are linearly independent.

Linearly independent columns and full column rank

- Here we assumed that our design matrix $\mathbf{x}$ has a **full column rank** $(p+1)$ (for $(p+1) < n$) which implies that $(p+1)$ **columns are linearly independent of each other**.
- Statistically, this further implies that the explanatory variables, are **linearly independent of each other**!!!
- If the columns of $\mathbf{x}$ are **linearly related**, then $\mathbf{x}$ has a column rank r which is **less than** (p+1).

Linearly dependent columns, multicollinearity problem, and OLS

- This issue in statistics leads to **multicollinearity** problem.
- In statistics, the concept of **multicollinearity** refers to a situation in which **more than two explanatory variables in a multiple regression model are highly linearly related**.
- In case of multicollinearity, the design matrix $\mathbf{x}$ has less than full column rank and some of the singular values of $\mathbf{x}$ will be **zero**, this implies that at least one of the eigenvalues of $\mathbf{x}^T\mathbf{x}$ is equal to zero, then the square matrix $\mathbf{x}^T\mathbf{x}$ is singular, and therefore the square matrix $\mathbf{x}^T\mathbf{x}$ cannot be inverted.
- Note: The singular values of $\mathbf{x}$ are the positive square roots of the eigenvalues of $\mathbf{x}^T\mathbf{x}$.

Multicollinearity problem and OLS

- Under these circumstances, the ordinary least squares estimator $\hat{\beta}_{OLS} = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y}$ does not exist.
- In practice, such **perfect multicollinearities rarely occur** in **statistical applications**.
- Rather, the columns of $\mathbf{x}$ may be **nearly** linearly related.
- In this case, the rank of $\mathbf{x}$ is (p+1), but some of the singular values of $\mathbf{x}$ will be **near zero**.

Generalized inverse

- The literature shows that any solution of $\mathbf{x}^T\mathbf{y} = \mathbf{x}^T\mathbf{x}\beta$ is of the form $\mathbf{G}\mathbf{x}^T\mathbf{y}$ with $\mathbf{G}$ a generalized inverse of $\mathbf{x}^T\mathbf{x}$.
- Moore Penrose inverse is the most commonly used generalized matrix inversion approach.

Generalized inverse with numpy and statsmodels

- For data analysis problems when the code is written from scratch, to avoid matrix inversion failure problems, you can use numpy.linalg.pinv() which computes the (Moore-Penrose) pseudo-inverse of a matrix rather than numpy.linalg.inv().
- For example, OLS.fit() in `statsmodels` library uses `Moore Penrose inverse` as a **default method** to solve the least squares problem.

Generalized inverse with scikit-learn and scipy

- Interestingly, at first sight, LinearRegression in `scikit-learn` seems that it does not use any generalized matrix inversion approach.
- However, in the notes, to my understanding, it says that it depends on scipy.linalg.lstsq and scipy.optimize.nnls.

Multicollinearity problem and OLS

- Nevertheless, the presence of multicollineartiy in $\mathbf{x}$ has adverse effects on the least-squares estimate $\hat{\beta}_{OLS}$.
- When the determinant of $(\mathbf{x}^T\mathbf{x})^{-1}$ is close to zero, the matrix elements get very large in magnitude.
- Since $\hat{\beta}_{OLS} = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y}$ and $Var(\hat{\beta}) = \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}$ depends on $(\mathbf{x}^T\mathbf{x})^{-1}$,
- Large variance associated with the elements of $\hat{\beta}_{OLS}$ can therefore to expected and this causes $\hat{\beta}_{OLS}$ to become an unreliable estimate for β.
- This problem will take us to the **ridge regression** around Week 7.

How to check multicollinearity?

- One quick **mathematical way** to check for multicollinearity is to calculate:

$$\kappa(\mathbf{x}) = \sqrt{\frac{e_{max}(\mathbf{x}^T\mathbf{x})}{e_{min}(\mathbf{x}^T\mathbf{x})}},$$

- where e_{max} and e_{min} are the maximum and minimum eigenvalues of $\mathbf{x}^T\mathbf{x}$, respectively.
- $\kappa(\mathbf{x})$ is less than 10, then there is **no serious problem with multicollinearity**.
- Values of $\kappa(\mathbf{x})$ between 10 and 30 indicate **moderate to strong multicollinearity**, and if $\kappa(\mathbf{x}) > 30$, **severe multicollinearity** is implied.

How to remedy multicollinearity?

- One approach is to calculate the correlation coefficient between each pair of explanatory variables and omit the explanatory variable from the data analysis which is highly correlated with the rest.
- **Variance inflation factor** –> HW

Multiple Linear Regression Model Continued

The fitted values and residuals

- The **fitted value** vector $\mathbf{y}$ is:

$$\hat{\mathbf{y}} = \mathbf{x}\hat{\beta} = \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y},$$

- where the matrix $\mathbf{P} = \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T$ is called as **projection matrix**, which results in $\hat{\mathbf{y}}$ when right-multiplied with $\mathbf{y}$.
- The matrix $\mathbf{P}$ is also an idempotent matrix such that $\mathbf{P}^2 = \mathbf{P}$.
- The **residual** vector $\mathbf{e}$ is:

$$\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}} = (\mathbf{I} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T)\mathbf{y}.$$

The sampling distribution of $\hat{\beta}$

- The characteristics of $\hat{\beta}$ are:
  - Centered at β, i.e. $E(\hat{\beta}) = \beta$.
  - $Var(\hat{\beta}) = \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}$.
  - $\hat{\beta} \sim N(0, \sigma^2(\mathbf{x}^T\mathbf{x})^{-1})$.
- where the details are:
  $E(\hat{\beta}) = E((\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y}) = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{x}\beta = \beta$ and
- $Var(\hat{\beta}) = V((\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y}) = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T V(\mathbf{y})\mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1} = \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}$ where $V(\mathbf{y}) = \sigma^2\mathbf{I}$ and $\mathbf{x}$ is not random.

The form of $Var(\hat{\beta})$

- Note that $Var(\hat{\beta}) = \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}$ is a $(p+1) \times (p+1)$ **symmetric matrix** called as the **variance-covariance matrix** of $\hat{\beta}$:

$$Var(\hat{\beta}) = \begin{pmatrix} Var(\hat{\beta}_0) & Cov(\hat{\beta}_0, \hat{\beta}_1) & ... & Cov\hat{\beta}_0, \hat{\beta}_p) \\ Cov(\hat{\beta}_1, \hat{\beta}_0) & Var(\hat{\beta}_0) & ... & Cov\hat{\beta}_1, \hat{\beta}_p) \\ Cov(\hat{\beta}_2, \hat{\beta}_0) & Cov(\hat{\beta}_2, \hat{\beta}_1) & ... & Cov\hat{\beta}_2, \hat{\beta}_p) \\ . & & & \\ . & & & \\ Cov(\hat{\beta}_p, \hat{\beta}_0) & Cov(\hat{\beta}_p, \hat{\beta}_1) & ... & Var(\hat{\beta}_p) \end{pmatrix}.$$

- The variance of an individual OLS estimator $\hat{\beta}_j$ $(j = 0, 1, ..., p)$, $Var(\hat{\beta}_j)$, is the j-th **diagonal** element of the matrix $Var(\hat{\beta})$.

Estimation of σ^2

- In many application, the parameter $Var(\epsilon_i) = \sigma^2$ is **often unknown**.
- As in SLR, we can **estimate it from the data** through the formula:

$$\hat{\sigma^2} = \frac{\sum_{i=1}^{n} e_i^2}{n-p-1} = \frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{n-p-1} = \frac{(\mathbf{Y} - \hat{\mathbf{Y}})^T (\mathbf{Y} - \hat{\mathbf{Y}})}{n-p-1} = \frac{RSS}{n-p-1}.$$

Statistical tests

Hypothesis testing for β_j

- Suppose we wish to test the hypothesis that the **any regression parameter equals 0**, say, $\beta_j = 0$ $(j = 0, 1, \dots, p)$.

- The appropriate **hypothesis** is:

$$\begin{aligned} H_0 &: \beta_j = 0 \\ H_1 &: \beta_j \neq 0 \end{aligned} \tag{2}$$

- Under H_0, the appropriate **test statistic** is:

$$t_j = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)} = \frac{\hat{\beta}_j}{\sqrt{\widehat{Var}(\hat{\beta}_j)}} \sim t_{(n-p-1)},$$

- where $se(\hat{\beta}_j) = \sqrt{\hat{\sigma^2} diag((\mathbf{x}^T\mathbf{x})^{-1})}$.

Hypothesis testing for β_j

- **Decision rule:** We would reject $H_0 : \beta_j = 0$ at α significance level, if $t_j < -t_{1-\alpha/2,n-p-1}$ or $t_j > t_{1-\alpha/2,n-p-1}$, where $t_{1-\alpha/2,n-p-1}$ denotes the $100(1-\alpha/2)$ percentile of the t-distribution with $n-p-1$ degrees of freedom.
- The term α refers to the tolerance level for making a Type I error. (e.g., 10%, 5%, 1%).
- Alternatively, a P-value approach could also be used for decision making. We would reject $H_0 : \beta_j = 0$ at α significance level, if P-value $< \alpha$.

Hypothesis testing for a subset of q of β_j's

- Sometimes we want to test that a **particular subset of q of the coefficients** are zero.
- This corresponds to a null hypothesis:

$$H_0 : \beta_{p-q+1} = \beta_{p-q+2} = ... = \beta_p = 0$$
$$H_1 : \text{At least one of them is different from 0.}$$

- where for convenience we have put the **variables chosen for the omission** at the end of the list.
- Under such a null hypothesis, we can **reduce the full model to a smaller model** (the model that uses all the variables except those last q).

Hypothesis testing for a subset of q of β_j's

- Suppose that RSS and RSS_0 be the residual sum-of-squares based on the least-squares fit of the **full model** and the **reduced smaller model**, respectively.
- If the null hypothesis is true, then these two quantities should be similar.
- Under H_0, the appropriate **test statistic** is:

$$F = \frac{(RSS_0 - RSS)/q}{RSS/(n-p-1)} \sim F_{q,(n-p-1)}.$$

- **Decision rule:** We would reject H_0 at α significance level, if $F > F_{1-\alpha,q,(n-p-1)}$, where $F_{1-\alpha,q,(n-p-1)}$ denotes the $100(1-\alpha)$ percentile of the F-distribution with q and $(n-p-1)$ degrees of freedom.

Hypothesis testing for a subset of q of β_j's

- For linear regression models, an **individual t-test** is equivalent to an **F-test** for dropping a single coefficient β_j from the model.

Model Selection Criteria

Akaike Information Criterion

- Akaike Information Criterion (AIC) proposed a general measure of "model badness:"

$$AIC = -2log(\hat{\beta}) + 2m.$$

- where m is the number of parameters. The 'best' model can be chosen by seeing which has the lowest AIC. We must improve the log likelihood by one unit for every extra parameter.

Bayesian Information Criterion

- Bayesian Information Criterion (BIC) penalizes complex models more severely is:

$$BIC = -2log(\hat{\beta}) + m * log(n).$$

- where m is the number of parameters and n is the number of data points.
- Lowest BIC is taken to identify the ‘best model’, as before.
- BIC tends to favor simpler models than those chosen by AIC.

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