MAT555E: Statistical Data Analysis for Computational Sciences

Fall22-Lecture 03: Multiple Linear Regression

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Learning Objectives

- Introduction to multiple linear regression
- Least-squares estimation
- Statistical tests
- Model selection criteria
- Implementing a multiple linear regression model with Python statsmodels library

Multiple Linear Regression Model

Multiple linear regression model set-up

- A **regression model** that involves **more than one independent variable** is called a **multiple linear regression (MLR) model**.
- Consider a MLR model with **p independent** variables:

$$Y = \beta_0 + \beta_1 * X_1 + ... + \beta_j * X_j + ... + \beta_p * X_p + \epsilon,$$

- where Y represents the response variable, X_j's $(j = 1, 2, ..., p)$ are independent variables, and with all the assumptions imposed on ϵ in SLR.
- Alternatively,

$$E(Y) = \beta_0 + \beta_1 * X_1 + ... + \beta_p * X_p.$$

Interpretation of regression coefficients

- The parameters β_j $(j = 0, 1, 2, ..., p)$ are called the **regression coefficients**.
- If the **range of the data** includes $X_1 = X_2 = ... = X_p = 0$, then β_0 is the expected value of Y when $X_1 = X_2 = ... = X_p = 0$. Otherwise, β_0 has **no physical interpretation**.
- The parameter β_j $(j = 1, 2, ..., p)$ represents the change expected in Y per unit change in X_j **when all of the remaining independent variables are held constant** $(\beta_j = \frac{\partial E(Y_i)}{\partial X_j}, j = 1, 2, ..., p)$.
- For this reason, the parameters β_j $(j = 1, 2, ..., p)$ are often called **partial regression coefficients**.

Marginal and partial effects of regression coefficients

- The effect of β in the linear model with a single predictor X is **usually not the same** as the effect of β of the same variable X in a model with multiple independent variables.
- The effect β is a marginal effect, **ignoring all other potential independent variables**, whereas β is a partial effect, **conditioning on the other independent variables**.

Multiple linear regression model set-up

- Suppose we have given a random sample of size n such that $\{(X_{i1}, \dots, X_{ip}, Y_i)\}_{i=1}^{n}$.
- Then, at individual data point level, we can write down the MLR model as:

$$Y_i = \beta_0 + \beta_1 * X_{i1} + \dots + \beta_p * X_{ip} + \epsilon_i, \quad i = 1, \dots, n.$$

- Alternatively,

$$E(Y_i) = \beta_0 + \beta_1 * X_{i1} + \dots + \beta_p * X_{ip}, \quad i = 1, \dots, n.$$

Model geometric interpretation

- In MLR, the equation $E(Y_i) = \beta_0 + \beta_1 * X_{i1} + ... + \beta_p * X_{ip}$ describes a **p-dimensional regression hyperplane** in the (p+1)-dimensional space of $Y, X_1, \dots, X_p$.
- The parameter β_0 is the **intercept** of the **p-dimensional hyperplane**.

> In linear algebra: the equation above defines a **p-dimensional hyperplane** where β_0 is the off-set from the origin and $\langle \beta_1, \dots, \beta_p \rangle$ is the vector normal to the hyperplane.

Recall

- In **SLR model**, the equation $E(Y_i) = \beta_0 + \beta_1 * X_{i1}$ describes a **regression line** in the **X-Y plane** and the parameter β_0 is the **y-intercept**.
- But, we never know the true values of β_0 and β_1.
- So, in SLR, actually, we are trying to **estimate the regression line** equation $\widehat{E(Y_i)} = \widehat{Y_i} = \hat{\beta}_0 + \hat{\beta}_1 * X_{i1}$ or
- In other words, we are trying to **fit the equation** $\widehat{Y_i} = \hat{\beta}_0 + \hat{\beta}_1 * X_{i1}$ to the data, which is called as **fitted regression line**, **least-squares line** etc.

Revisiting Advertising Data

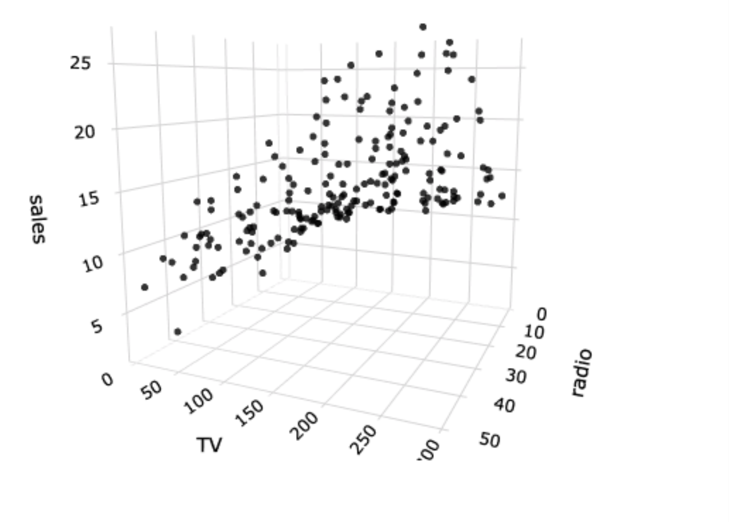
Revisiting advertising data

- Consider the advertising data which consists of `sales` of a product in 200 different markets, along with advertising budgets for the product in each of those markets for three different media: `TV`, `radio`, and `newspaper`.
- Now assume a **multiple linear regression model** for sales with **two predictors**: `TV` and `radio` such that:

$$sales_i = \beta_0 + \beta_1 * TV_i + \beta_2 * radio_i + \epsilon_i$$

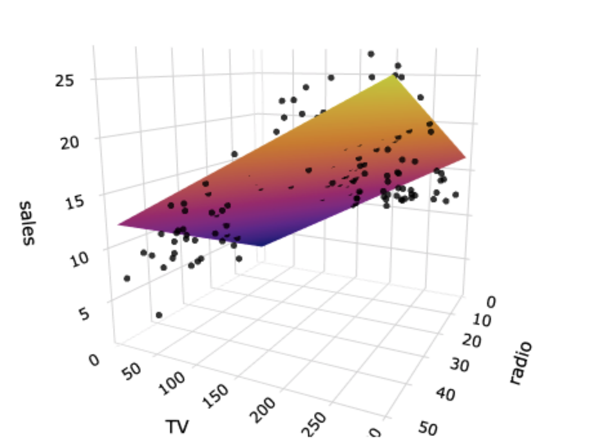
- for $i = 1, 2, \ldots, 200$ and $\epsilon_i \sim N(0, \sigma^2)$.

3D-scatter plot of advertising data



Fitting a regression plane

- Since we have **two predictors**, now, we are trying to **fit a regression plane**, $\widehat{sales}_i = \hat{\beta}_0 + \hat{\beta}_1 * TV_i + \hat{\beta}_2 * radio_i$ to the advertising data.



Multiple Linear Regression Model Continued

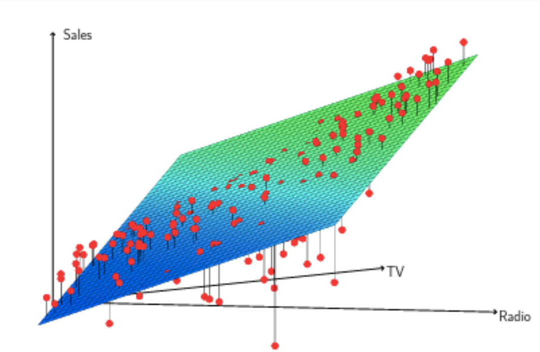
Least-squares estimation

- In the MLR model, for given a observed sample of size n such that $\{(x_{i1}, \dots, x_{ip}, y_i)\}_{i=1}^{n}$, the method of least-squares estimates the regression coefficients which **minimizes the residual sum-of-sum squares (RSS)**:

$$RSS(\hat{\beta}) = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(y_i - (\hat{\beta}_0 + \hat{\beta}_1 * x_{i1} + \dots + \hat{\beta}_p * x_{ip})\right)^2.$$

Least-squares estimation

- For example, for advertising data with two predictors, we are trying to fit a **regression plane**.
- This plane is chosen to **minimize the sum of the squared vertical distances** between each observation (shown in red) and the plane.



Finding least-squares estimators

- Let $\mathcal{J}(\beta) = \sum_{i=1}^{n} \left(y_i - (\beta_0 + \beta_1 * x_{i1} + ... + \beta_p * x_{ip})\right)^2$ be the **objective function**.
- To minimize $\mathcal{J}(\beta)$, we take partial derivatives with respect to β_j ($j = 0, 1, 2, ..., p$), set them to zero, and solve the resulting system of equations simultaneously, such that.

$$\left.\frac{\partial \mathcal{J}(\beta)}{\partial \beta_0}\right|_{\hat{\beta}} = -2 \sum_{i=1}^{n} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 * x_{i1} + ... + \hat{\beta}_p * x_{ip})) = 0 \quad (1)$$

$$\left.\frac{\partial \mathcal{J}(\beta)}{\partial \beta_j}\right|_{\hat{\beta}} = -2 \sum_{i=1}^{n} (y_i - (\hat{\beta}_0 + \hat{\beta}_1 * x_{i1} + ... + \hat{\beta}_p * x_{ip}))x_{ij} = 0$$

- for $j = 1, 2, ..., p$.

Finding least-squares estimators

- However, solving the resulting **homogeneous system of (p+1) linear equations in (p+1) unknowns** simultaneously is not algebraically easy.

Matrix notation

- Then, denote $n \times 1$ response vector $\mathbf{Y}$, $n \times (p+1)$ design matrix $\mathbf{X}$, $(p+1) \times 1$ regression vector β, and $n \times 1$ error vector ϵ, respectively, as:

$$\mathbf{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ . \\ . \\ Y_n \end{pmatrix}, \quad \mathbf{X} = \begin{pmatrix} 1 & X_{11} & ... & X_{1p} \\ 1 & X_{21} & ... & X_{2p} \\ . & . & ... & . \\ . & . & ... & . \\ 1 & X_{n1} & ... & X_{np} \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ . \\ . \\ \beta_p \end{pmatrix}, \quad \epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ . \\ . \\ \epsilon_n \end{pmatrix}.$$

Matrix notation

- Then, the MLR model can be written in the **matrix form**:

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon.$$

- The **least-squares estimates** of β is the solution to the following **optimization** problem.

$$\hat{\beta}_{OLS} = \underset{\beta}{argmin} \left\| \epsilon \right\|_2^2 = \underset{\beta}{argmin} \left\| \mathbf{y} - \mathbf{x}\beta \right\|_2^2 = \underset{\beta}{argmin} (\mathbf{y} - \mathbf{x}\beta)^T (\mathbf{y} - \mathbf{x}\beta),$$

- since $\sum_{i=1}^{n} \epsilon_i^2 = \epsilon^T \epsilon = \left\| \epsilon \right\|_2^2$.

Finding least-squares estimators

- Denote the objective function $\mathcal{J}(\beta) = (\mathbf{y} - \mathbf{x}\beta)^T(\mathbf{y} - \mathbf{x}\beta)$ which is scalar, find the **gradient vector** of $\mathcal{J}(\beta)$ with respect to β such that:

$$\nabla \mathcal{J}(\beta) = \frac{\partial \mathcal{J}(\beta)}{\partial \beta} = \begin{pmatrix} \frac{\partial \mathcal{J}(\beta)}{\partial \beta_0} \\ \frac{\partial \mathcal{J}(\beta)}{\partial \beta_1}. \\ . \\ \frac{\partial \mathcal{J}(\beta)}{\partial \beta_p} \end{pmatrix},$$

- and then set the **gradient to zero** $\nabla \mathcal{J}(\beta) = 0$.

Finding least-squares estimators

- Let's expand $\mathscr{J}(\beta)$ first:

$$\begin{aligned}\mathscr{J}(\beta) &= \big((\mathbf{y}-\mathbf{x}\beta)^T(\mathbf{y}-\mathbf{x}\beta)\big) = \big((\mathbf{y}^T-\beta^T\mathbf{x}^T)(\mathbf{y}-\mathbf{x}\beta)\big)\\ &= \mathbf{y}^T\mathbf{y} - 2\mathbf{y}^T\mathbf{x}\beta + \beta^T\mathbf{x}^T\mathbf{x}\beta,\end{aligned}$$

- since the dot product $\mathbf{y}^T\mathbf{x}\beta$ is scalar, $(\beta^T\mathbf{x}^T y)^T = \mathbf{y}^T\mathbf{x}\beta$.

Finding least-squares estimators

- Applying the rules for **differentiation of a scalar with respect to a vector**, we get:

$$\nabla \mathcal{J}(\beta) = \frac{\partial \mathcal{J}(\beta)}{\partial \beta} = -2\mathbf{x}^T\mathbf{y} + 2\mathbf{x}^T\mathbf{x}\beta,$$

- since $\frac{\partial a^T\beta}{\partial \beta} = a$ and $\frac{\partial \beta^T\mathbf{S}\beta}{\partial \beta} = 2\mathbf{S}\beta$, where $\mathbf{S}$ is a symmetric matrix.

Finding least-squares estimators

- Setting the gradient vector $\nabla \mathcal{J}(\beta) = 0$, we get the **normal equations**:

$$\mathbf{x}^T \mathbf{y} = \mathbf{x}^T \mathbf{x} \beta.$$

- Here we assume that **the number of columns in $\mathbf{x}$** is less than the number of rows such as $(p+1) < n$ and the rank of $\mathbf{x}$ is $(p+1)$, hence $\mathbf{x}$ has a **full column rank** $(p+1)$.

Least-squares estimators

- Since $rank(\mathbf{x}^T\mathbf{x}) = rank(\mathbf{x}) = (p+1)$, this leads the symmetric matrix $\mathbf{x}^T\mathbf{x}$ to has a **full rank** of $(p+1)$.
- In this case $\det(\mathbf{x}^T\mathbf{x}) \neq 0$, that the square matrix $\mathbf{x}^T\mathbf{x}$ is non-singular and, in turn, is **invertible**.
- Then the normal equations yield the **least-squares estimator** of β as:

$$\hat{\beta}_{OLS} = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y},$$

- where $\mathbf{x}^T\mathbf{x}$ is a $(p+1)\times(p+1)$ **symmetric matrix** and $\mathbf{x}^T\mathbf{y}$ is a $(p+1)\times 1$ vector.
- Note that $\mathbf{x}^T\mathbf{x}$ is also known as Gram matrix of $\mathbf{x}$.

The form of $\mathrm{x}^T\mathrm{x}$ matrix and $\mathrm{x}^T\mathrm{y}$ vector

$$\mathbf{x}^T\mathbf{x} = \begin{pmatrix} n & \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i2} & \dots & \sum_{i=1}^n x_{ip} \\ \sum_{i=1}^n x_{i1} & \sum_{i=1}^n x_{i1}^2 & \sum_{i=1}^n x_{i1}x_{i2} & \dots & \sum_{i=1}^n x_{i1}x_{ip} \\ . & . & . & . & . \\ . & . & . & . & . \\ \sum_{i=1}^n x_{ip} & \sum_{i=1}^n x_{i1}x_{ip} & \sum_{i=1}^n x_{i2}x_{ip} & \dots & \sum_{i=1}^n x_{ip}^2 \end{pmatrix} \quad \text{and}$$

$$\mathbf{x}^T\mathbf{y} = \begin{pmatrix} \sum_{i=1}^n y_i \\ \sum_{i=1}^n x_{i1}y_i \\ . \\ . \\ \sum_{i=1}^n x_{ip}y_i \end{pmatrix}.$$

Issues with Inverse of $\mathrm{x}^T\mathrm{x}$ Matrix

Linearly independent

A collection of (p+1) elements in a vector space are linearly dependent if at least one element in this collection can be expressed as a linear combination of the remaining p elements. If no element, however, can be expressed in this fashion, then the (p+1) elements are linearly independent.

Linearly independent columns and full column rank

- Here we assumed that our design matrix $\mathbf{x}$ has a **full column rank** $(p+1)$ (for $(p+1) < n$) which implies that $(p+1)$ columns are linearly independent of each other.
- Statistically, this further implies that the explanatory variables, are **linearly independent of each other**!!!
- If the columns of $\mathbf{x}$ are **linearly related**, then $\mathbf{x}$ has a column rank r which is **less than** (p+1).

Linearly dependent columns, multicollinearity problem, and OLS

- This issue in statistics leads to **multicollinearity** problem.
- In statistics, the concept of **multicollinearity** refers to a situation in which **more than two explanatory variables in a multiple regression model are highly linearly related**.
- In case of multicollinearity, the design matrix $\mathbf{x}$ has less than full column rank and some of the singular values of $\mathbf{x}$ will be **zero**, this implies that at least one of the eigenvalues of $\mathbf{x}^T\mathbf{x}$ is equal to zero, then the square matrix $\mathbf{x}^T\mathbf{x}$ is singular, and therefore the square matrix $\mathbf{x}^T\mathbf{x}$ cannot be inverted.
- Note: The singular values of $\mathbf{x}$ are the positive square roots of the eigenvalues of $\mathbf{x}^T\mathbf{x}$.

Multicollinearity problem and OLS

- Under these circumstances, the ordinary least squares estimator $\hat{\beta}_{OLS} = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y}$ does not exist.
- In practice, such **perfect multicollinearities rarely occur** in **statistical applications**.
- Rather, the columns of $\mathbf{x}$ may be **nearly** linearly related.
- In this case, the rank of $\mathbf{x}$ is (p+1), but some of the singular values of $\mathbf{x}$ will be **near zero**.

Generalized inverse

- The literature shows that any solution of $\mathbf{x}^T\mathbf{y} = \mathbf{x}^T\mathbf{x}\beta$ is of the form $\mathbf{G}\mathbf{x}^T\mathbf{y}$ with $\mathbf{G}$ a generalized inverse of $\mathbf{x}^T\mathbf{x}$.
- Moore Penrose inverse is the most commonly used generalized matrix inversion approach.

Generalized inverse with numpy and statsmodels

- For data analysis problems when the code is written from scratch, to avoid matrix inversion failure problems, you can use numpy.linalg.pinv() which computes the (Moore-Penrose) pseudo-inverse of a matrix rather than numpy.linalg.inv().
- For example, OLS.fit() in `statsmodels` library uses `Moore Penrose inverse` as a **default method** to solve the least squares problem.

Generalized inverse with scikit-learn and scipy

- Interestingly, at first sight, LinearRegression in `scikit-learn` seems that it does not use any generalized matrix inversion approach.
- However, in the notes, to my understanding, it says that it depends on scipy.linalg.lstsq and scipy.optimize.nnls.

Multicollinearity problem and OLS

- Nevertheless, the presence of multicollineartiy in $\mathbf{x}$ has adverse effects on the least-squares estimate $\hat{\beta}_{OLS}$.
- When the determinant of $(\mathbf{x}^T\mathbf{x})^{-1}$ is close to zero, the matrix elements get very large in magnitude.
- Since $\hat{\beta}_{OLS} = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y}$ and $Var(\hat{\beta}) = \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}$ depends on $(\mathbf{x}^T\mathbf{x})^{-1}$,
- Large variance associated with the elements of $\hat{\beta}_{OLS}$ can therefore to expected and this causes $\hat{\beta}_{OLS}$ to become an unreliable estimate for β.
- This problem will take us to the **ridge regression** around Week 7.

How to check multicollinearity?

- One quick **mathematical way** to check for multicollinearity is to calculate:

$$\kappa(\mathbf{x}) = \sqrt{\frac{e_{max}(\mathbf{x}^T\mathbf{x})}{e_{min}(\mathbf{x}^T\mathbf{x})}},$$

- where e_{max} and e_{min} are the maximum and minimum eigenvalues of $\mathbf{x}^T\mathbf{x}$, respectively. - $\kappa(\mathbf{x})$ is less than 10, then there is **no serious problem with multicollinearity**. - Values of $\kappa(\mathbf{x})$ between 10 and 30 indicate **moderate to strong multicollinearity**, and if $\kappa(\mathbf{x})>$ 30, **severe multicollinearity** is implied.

How to remedy multicollinearity?

- One approach is to calculate the correlation coefficient between each pair of explanatory variables and omit the explanatory variable from the data analysis which is highly correlated with the rest.
- **Variance inflation factor** –> HW

Multiple Linear Regression Model Continued

The fitted values and residuals

- The **fitted value** vector $\mathbf{y}$ is:

$$\hat{\mathbf{y}} = \mathbf{x}\hat{\beta} = \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y},$$

- where the matrix $\mathbf{P} = \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T$ is called as **projection matrix**, which results in $\hat{\mathbf{y}}$ when right-multiplied with $\mathbf{y}$.
- The matrix $\mathbf{P}$ is also an idempotent matrix such that $\mathbf{P}^2 = \mathbf{P}$.
- The **residual** vector $\mathbf{e}$ is:

$$\mathbf{e} = \mathbf{y} - \hat{\mathbf{y}} = (\mathbf{I} - \mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T)\mathbf{y}.$$

The sampling distribution of $\hat{\beta}$

- The characteristics of $\hat{\beta}$ are:
  - Centered at β, i.e. $E(\hat{\beta}) = \beta$.
  - $Var(\hat{\beta}) = \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}$.
  - $\hat{\beta} \sim N(0, \sigma^2(\mathbf{x}^T\mathbf{x})^{-1})$.
- where the details are:
  $E(\hat{\beta}) = E((\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y}) = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{x}\beta = \beta$ and
- $Var(\hat{\beta}) = V((\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T\mathbf{y}) = (\mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T V(\mathbf{y})\mathbf{x}(\mathbf{x}^T\mathbf{x})^{-1} =$
  $\sigma^2(\mathbf{x}^T\mathbf{x})^{-1}$ where $V(\mathbf{y}) = \sigma^2\mathbf{I}$ and $\mathbf{x}$ is not random.

The form of $Var(\hat{\beta})$

- Note that $Var(\hat{\beta}) = \sigma^2(\mathbf{x}^T\mathbf{x})^{-1}$ is a $(p+1)\times(p+1)$ **symmetric matrix** called as the **variance-covariance matrix** of $\hat{\beta}$:

$$Var(\hat{\beta}) = \begin{pmatrix} Var(\hat{\beta}_0) & Cov(\hat{\beta}_0,\hat{\beta}_1) & Cov(\hat{\beta}_0,\hat{\beta}_2) & \dots & Cov\hat{\beta}_0,\hat{\beta}_p) \\ Cov(\hat{\beta}_1,\hat{\beta}_0) & Var(\hat{\beta}_1) & Cov(\hat{\beta}_1,\hat{\beta}_2) & \dots & Cov\hat{\beta}_1,\hat{\beta}_p) \\ Cov(\hat{\beta}_2,\hat{\beta}_0) & Cov(\hat{\beta}_2,\hat{\beta}_1) & Var(\hat{\beta}_2) & \dots & Cov\hat{\beta}_2,\hat{\beta}_p) \\ . & & & & \\ . & & & & \\ Cov(\hat{\beta}_p,\hat{\beta}_0) & Cov(\hat{\beta}_p,\hat{\beta}_1) & Cov(\hat{\beta}_p,\hat{\beta}_2) & \dots & Var(\hat{\beta}_p) \end{pmatrix}.$$

- The variance of an individual OLS estimator $\hat{\beta}_j$ $(j = 0, 1, \dots, p)$, $Var(\hat{\beta}_j)$, is the j-th **diagonal** element of the matrix $Var(\hat{\beta})$.

Estimation of σ^2

- In many application, the parameter $Var(\epsilon_i) = \sigma^2$ is **often unknown**.
- As in SLR, we can **estimate it from the data** through the formula:

$$\hat{\sigma^2} = \frac{\sum_{i=1}^{n} e_i^2}{n-p-1} = \frac{\sum_{i=1}^{n} (y_i - \hat{y}_i)^2}{n-p-1} = \frac{(\mathbf{y} - \hat{\mathbf{y}})^T (\mathbf{y} - \hat{\mathbf{y}})}{n-p-1} = \frac{RSS}{n-p-1}.$$

Statistical tests

Hypothesis testing for β_j

- Suppose we wish to test the hypothesis that the **any regression parameter equals 0**, say, $\beta_j = 0$ $(j = 0, 1, \ldots, p)$.

- The appropriate **hypothesis** is:

$$\begin{aligned} H_0 &: \beta_j = 0 \\ H_1 &: \beta_j \neq 0 \end{aligned} \tag{2}$$

- Under H_0, the appropriate **test statistic** is:

$$t_j = \frac{\hat{\beta}_j}{se(\hat{\beta}_j)} = \frac{\hat{\beta}_j}{\sqrt{\widehat{Var}(\hat{\beta}_j)}} \sim t_{(n-p-1)},$$

- where $se(\hat{\beta}_j) = \sqrt{\hat{\sigma^2} diag_j((\mathbf{x}^T\mathbf{x})^{-1})}$.

Hypothesis testing for β_j

- **Decision rule:** We would reject $H_0 : \beta_j = 0$ at α significance level, if $t_j < -t_{1-\alpha/2,n-p-1}$ or $t_j > t_{1-\alpha/2,n-p-1}$, where $t_{1-\alpha/2,n-p-1}$ denotes the $100(1-\alpha/2)$ percentile of the t-distribution with $n-p-1$ degrees of freedom.
- The term α refers to the tolerance level for making a Type I error. (e.g., 10%, 5%, 1%).
- Alternatively, a P-value approach could also be used for decision making. We would reject $H_0 : \beta_j = 0$ at α significance level, if P-value $< \alpha$.

Hypothesis testing for a subset of q of β_j's

- Sometimes we want to test that a **particular subset of q of the coefficients** are zero.
- This corresponds to a null hypothesis:

$$H_0 : \beta_{p-q+1} = \beta_{p-q+2} = ... = \beta_p = 0$$
$$H_1 : \text{At least one of them is different from 0.}$$

- where for convenience we have put the **variables chosen for the omission** at the end of the list.
- Under such a null hypothesis, we can **reduce the full model to a smaller model** (the model that uses all the variables except those last q).

Hypothesis testing for a subset of q of β_j's

- Suppose that RSS and RSS_0 be the residual sum-of-squares based on the least-squares fit of the **full model** and the **reduced smaller model**, respectively.
- If the null hypothesis is true, then these two quantities should be similar.
- Under H_0, the appropriate **test statistic** is:

$$F = \frac{(RSS_0 - RSS)/q}{RSS/(n-p-1)} \sim F_{q,(n-p-1)}.$$

- **Decision rule:** We would reject H_0 at α significance level, if $F > F_{1-\alpha,q,(n-p-1)}$, where $F_{1-\alpha,q,(n-p-1)}$ denotes the $100(1-\alpha)$ percentile of the F-distribution with q and $(n-p-1)$ degrees of freedom.

Hypothesis testing for a subset of q of β_j's

- For linear regression models, an **individual t-test** is equivalent to an **F-test** for dropping a single coefficient β_j from the model.

Model Selection Criteria

Akaike Information Criterion

- Akaike Information Criterion (AIC) proposed a general measure of "model badness:"

$$AIC = -2log(\hat{\beta}) + 2m.$$

- where m is the number of parameters (specifically in regression models, it is the number of regression coefficients).
- The 'best' model can be chosen by seeing which has the lowest AIC. We must improve the log likelihood by one unit for every extra parameter.

Bayesian Information Criterion

- Bayesian Information Criterion (BIC) penalizes complex models more severely such as:

$$BIC = -2log(\hat{\beta}) + m * log(n).$$

- where m is the number of parameters (specifically in regression models, it is the number of regression coefficients) and n is the number of data points.
- Lowest BIC is taken to identify the 'best model', as before.
- BIC tends to favor simpler models than those chosen by AIC.

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