2.4 – 2.6

Bernoulli, Binomial, Geometric, Negative Binomial, Hypergeometric, Multinomial, and Poisson Distributions

Bernoulli Experiment

A Bernoulli experiment is a random experiment where the outcome can be classified as one of two mutually exclusive ways (Heads/Tails, Pass/Fail)

A sequence of Bernoulli trials occurs when a Bernoulli experiment is performed several **independent** times, and the success probability, p, remains the same.

Bernoulli experiment examples

Flip a fair coin where “heads” is a success. This is a Bernoulli experiment with $p=0.5$.

Roll a die. “6” is success, everything else is a failure.

Did a randomly selected student read the entire syllabus? (yes/no)

Bernoulli Distribution $X \sim Bernoulli(p)$

If random variable, X, has a Bernoulli distribution:

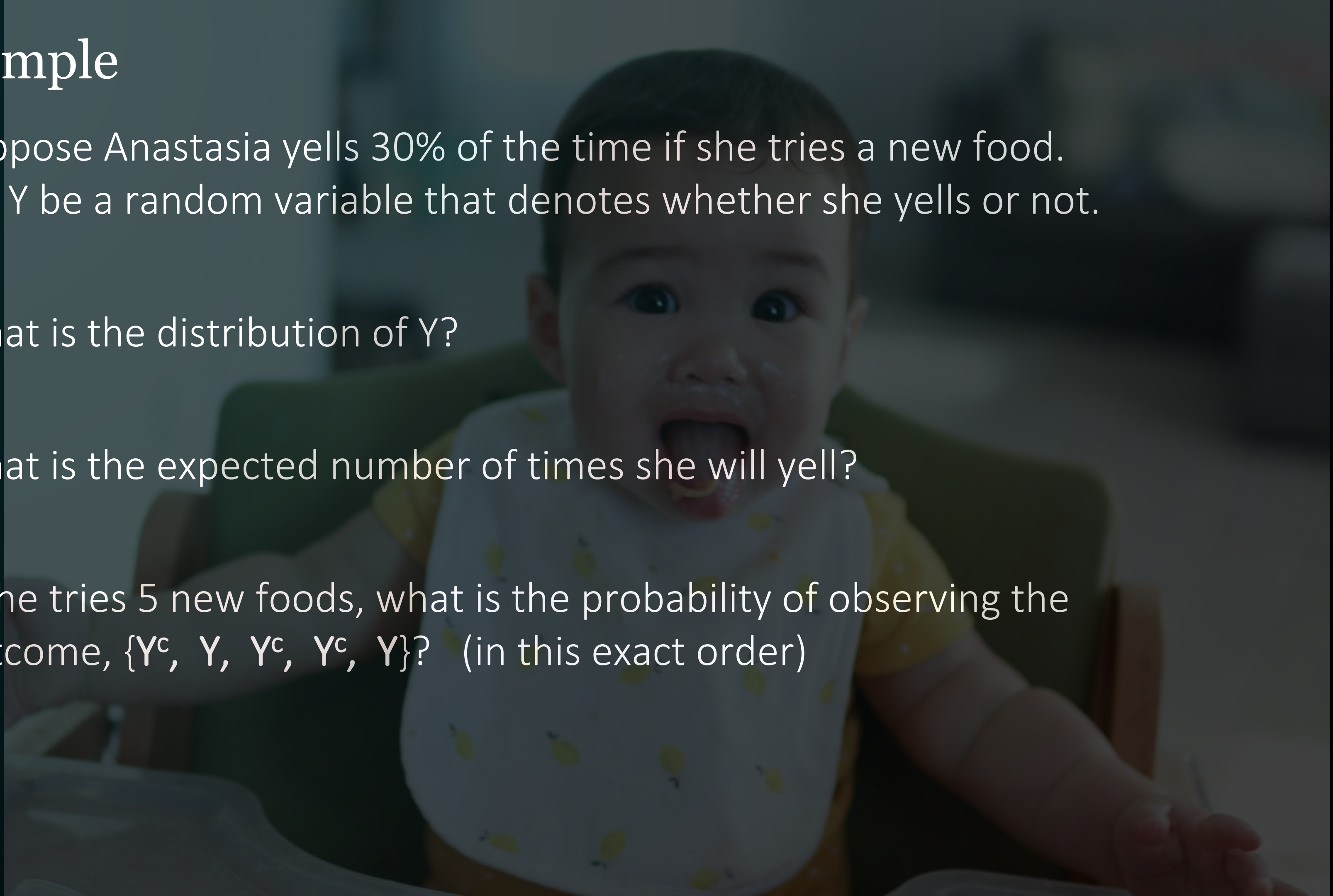
$$f(x) = p^x (1-p)^{1-x}, \ x = \{0,1\}$$

$$E[X] = \sum_{x=0}^{1} x\, p^x (1-p)^{1-x}$$
$$= 0(1-p) + 1(p) \quad = \boxed{p}$$

$$Var[X] = \sum_{x=0}^{1} (x-p)^2\, p^x (1-p)^{1-x} = \boxed{p(1-p)}$$

$$SD[X] = \sqrt{p(1-p)}$$

Example



Suppose Anastasia yells 30% of the time if she tries a new food.
Let Y be a random variable that denotes whether she yells or not.

What is the distribution of Y?

What is the expected number of times she will yell?

If she tries 5 new foods, what is the probability of observing the outcome, $\{Y^c, Y, Y^c, Y^c, Y\}$? (in this exact order)

Binomial Distribution

Often, we are interested only in the *total number of successes*, but **not** the actual order of occurrence.

If X = the total # of observed successes in n Bernoulli trials, then X has a **binomial distribution.**

- For x successes, there are $n - x$ failures.
- The Space of X is $0,1,2, \dots n$.

Binomial Distribution

$Notation$: $X \sim Binomial(n, p)$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \qquad x = 0,1,2,\dots,n$$

$$E[X] = np$$

$$Var[X] = np(1-p)$$

https://homepage.divms.uiowa.edu/~mbognar/applets/bin.html

Binomial Distribution (Summary)

X is a binomial random variable if the following are all true

1. A Bernoulli (success/fail) experiment is performed a constant number of times, n.
2. The random variable, X, is the (total) number of successes in n trials.
3. All trials are independent
4. The success probability, p, for every trial is constant.
    - *(The failure probability,* $1 - p$*, is also constant).*

$$
\begin{aligned}
E(X) &= \sum_{x=0}^{n} x \binom{n}{x} p^x (1-p)^{n-x} \\
&= \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x} \\
&= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} p^x (1-p)^{n-x}
\end{aligned}
$$

$x = y+1$ and $n = m+1$

$$
\begin{aligned}
E(X) &= \sum_{y=0}^{m} \frac{(m+1)!}{y!(m-y)!} p^{y+1} (1-p)^{m-y} \\
&= (m+1)p \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y} \\
&= np \sum_{y=0}^{m} \frac{m!}{y!(m-y)!} p^y (1-p)^{m-y}
\end{aligned}
$$

Binomial Example

Suppose Anastasia yells 30% of the time if she tries a new food. Let Y be a random variable that denotes the total number of times she yells if she tries 5 new foods.

What is the distribution of Y?

What is the expected number of times she will yell?

If Anastasia tries 5 new foods, what is the probability that she yells exactly twice?

$$f(2) = \binom{5}{2} 0.3^2 (1 - 0.3)^{5-2}$$

Binomial / Bernoulli Relationship

The Binomial distribution is a more *general* case of the Bernoulli distribution.

The Bernoulli Distribution is a more *specific* case of the Binomial distribution.   (specifically, when n = 1)

The sum of n independent Bernoulli Random Variables with the same parameter p is $\sim Binomial(n, p)$

2.5 Negative Binomial & Geometric Distribution

Geometric Distribution

Say we observe a sequence of independent Bernoulli trials until the first success occurs.

If X is the **number of trials** needed to observe the 1^{st} success, then X follows a **Geometric Distribution** with parameter, p.

Geometric Distribution $X \sim \text{Geom}(p)$

$f(x) = p(1\text{-}p)^{x\text{-}1}, \qquad x = 1,2,3, \ldots$

$E[X] = 1/p$

$$Var[X] = \frac{1-p}{p^2}$$

Question for home:

Show that the geometric distribution is a valid pmf regardless of p.

Negative Binomial Distribution

More generally, suppose we observe a sequence of independent Bernoulli trials until the $\boldsymbol{r}^{\text{th}}$ success occurs.

If X is the number of trials needed to observe the r^{th} success, then X follows a **Negative Binomial** distribution with parameters $\boldsymbol{r}, \boldsymbol{p}$.

Negative Binomial Distribution

$$X \sim NB(r, p)$$

$$f(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r}, \qquad x = r,\ r+1,\ r+2, \ldots$$

$$E[X] = r/p$$

$$Var[X] = \frac{r(1-p)}{p^2}$$

Examples

2.4 - 2.5

1 A magical beer machine vending machine gives a random beer to the customer. It gives a stout 30% of the time and an IPA 70% of the time. Let X be the number of stouts you get out of 20 beers.

What is the distribution of X? What is its pmf?

What is the probability of getting fewer than 7 stouts?

pbinom(q, size, prob, lower.tail = TRUE, log.p = FALSE)

2 A magical beer machine vending machine randomly gives you a stout 30% of the time and an IPA 70% of the time. Thor gently smashes the machine until it gives him a stout. Let X represent the number of trials required for Thor to get his first stout.

What is the distribution of X? What is its pmf?

What is the probability of getting a stout on the 5th trial?

What is the probability of getting a stout within the first 5 trials?

3 A random variable $X \sim$ has a binomial distribution with $\mu = 6, \;\; \sigma^2 = 3.6$.

What is the distribution of X?

Find $P(X = 4)$.

Find $F(2)$.

4 Jacqueline hits her free throws with $p = 0.9$.

What is the probability that she has her first miss on the 7^{th} free throw?

What is the probability that she has her first miss on the 12^{th} attempt or later?

What is the probability that she has her 3^{rd} miss on the 30^{th} free throw?

The Hypergeometric & Multinomial Distributions

Hypergeometric Distribution

Out of a population of size N, suppose we have N_1 successes and N_2 failures.

(note, $N_1 + N_2 = N$, the probability of a success, $p = N_1 / N$)

Define a random variable X:

the number of successes in a random sample of size n.

If sampling is done without replacement, X follows a **hypergeometric distribution.**

Hypergeometric Distribution

$$X \sim HG(N, N_1, n)$$

(remember: $N = N_1 + N_2$*)*

$$f(x) = \frac{\binom{N_1}{x}\binom{N_2}{n-x}}{\binom{N}{n}},$$

$$x \le n, \quad x \le N_1, \quad n - x \le N_2$$

$$E[X] = n\frac{N_1}{N}$$

$$Var[X] = n\frac{N_1}{N}\frac{N_2}{N}\frac{N-n}{N-1}$$

Hypergeometric vs Binomial

If instead, sampling is done one at a time with replacement, $X \sim Binomial(n, p)$ e.g.

- <u>Binomial</u>: A magical beer machine gives the user a stout 30% of the time and an IPA 70% of the time. Let X be the number of stouts you get out of 20 beers.
- <u>Hypergeometric</u>: A nice minibar has 9 stouts and 21 IPAs. Let X be the number of stouts you get if you randomly select 20 beers.
    - What is the pmf of X? $f(x) = \frac{\binom{9}{x}\binom{21}{20-x}}{\binom{30}{20}}, \;\; x \leq 9$

Multinomial Distribution

Similar to binomial distribution, but for more than 2 groups. E.g.

- Color – Red/Green/Blue
- Your Major – Stats/Math/Engineering/Other

Multinomial Distribution

$$X = (X_1, X_2, \ldots X_k) \sim Multinomial(n, p_1, p_2, \ldots, pk)$$

$$f(x_1, x_2, \ldots x_k) = \frac{n!}{x_1! x_2! \ldots x_k!} p_1^{x_1} \, p_2^{x_2} \ldots p_k^{x_k}$$

- $E[X_i] = np_i$
- $Var[X_i] = np_i(1\text{-}p_i)$

Examples

2.4

1 A deck of cards contains 20 cards: 6 red cards and 14 black cards. 5 cards are drawn randomly *without replacement*.

What is the probability that exactly 4 red cards are drawn?

What is the probability that at least 2 black cards are drawn?

2 Suppose the majors of students taking Stat 400 can be broken down as follows:

多项分布

Math	Statistics	Other
10%	20%	70%

$n=10$

Out of 10 randomly sampled students, calculate the probability that this group contains:

- A) 2 Math, 2 Stats, and 6 Other

$$\frac{10!}{2!\,2!\,6!} \times (0.1)^2 \times (0.2)^2 \times (0.7)^6 = C_{10}^{2} \cdot C_{8}^{2} \cdots$$

- B) At least one Stats student

= 1 – .8^10 = .8926

3 When Iron Man and Captain America play Connect 4 against each other, Iron Man wins 40% of the time, loses 35% of the time and draws 25% of the time. Assume results of games are independent.

If they play 12 games, what is the probability that Iron Man wins 7, loses 2, and draws 3 games?

$$f(x_1, x_2, \dots x_k) = \frac{n!}{x_1! x_2! \dots x_k!} p_1^{x_1}\, p_2^{x_2} \dots p_k^{x_k} = \frac{12!}{7!2!3!} \times 40^7 \times 35^2 \times 25^3 = 0.0248$$

If they play 12 games, what is the expected value of the number of games that they will tie?

$$E[X_3] = n \cdot P_3 = 12 \times 0.25 = 3$$

2.6 The Poisson Distribution

Poisson Process 泊松过程

Definition 2.6-1

Let the number of occurrences of some event in a given continuous interval be counted. Then we have an **approximate Poisson process** with parameter $\lambda > 0$ if the following conditions are satisfied:

(a) The numbers of occurrences in nonoverlapping subintervals are independent.

(b) The probability of exactly one occurrence in a sufficiently short subinterval of length h is approximately λh.

(c) The probability of two or more occurrences in a sufficiently short subinterval is essentially zero.

Poisson Process Examples

- # of cell phone calls passing through a relay tower between 9 and 11 a.m.
- Number of customers that show up to Oberweis between 5-6pm.
- Number of defects in a 100-foot roll of aluminum screen that is 2 feet wide.

Poisson Distribution

$$X \sim Poisson(\lambda)$$

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0,1,2,\ldots$$

- $E[X] = \lambda$
- $Var[X] = \lambda$

Note: λ is the Poisson rate.

Poisson Parameter Scaling

If events occur according to a Poisson process with rate λ, then the rate for a Poisson process in an interval of length t is λt.

Every minute, cars pull up to a drive-through according to a Poisson process with rate λ = 3.

- In an interval of length 1 hour, the rate is $\lambda = 180$.

3 * 60 (minutes in an hour) = 180

Examples

2.6

2 Suppose Albert receives spam mail every day according to a Poisson Process with an expected value of 10.

$X \sim Pois(10)$

What is the distribution of X?

$P[X=8]$

What is the probability that Albert receives 8 items of spam in a given day?

$$f(8) = \frac{e^{-10} \cdot 10^8}{8!} = 0.112$$

What is the probability that Albert receives 10 items of spam in a given day?

$$f(10) = \frac{e^{-10} \cdot 10^{10}}{10!} = 0.125$$

0 items of spam? $f(0) = e^{-10}$

2 Suppose Albert receives spam mail every day according to a Poisson Process with an expected value of 10.

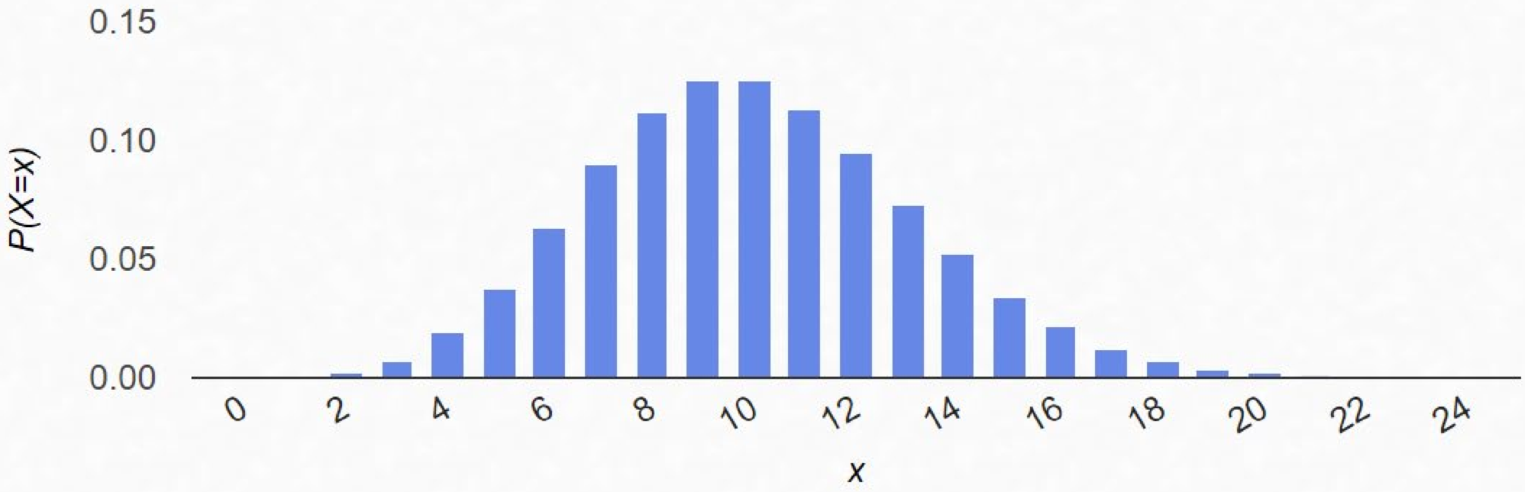
Find P[Albert receives 10 items of spam in a given day].

Find P[0 items of spam in a given day]?

$X \sim Pois\left(\frac{10}{24}\right)$

Find P[Albert receives 1 item of spam in a given hour]?

$$f(1) = \frac{e^{-\frac{10}{24}} \cdot \left(\frac{10}{24}\right)}{1!}$$



$$\mu = E(X) = 10 \quad \sigma = SD(X) = 3.162 \quad \sigma^2 = Var(X) = 10$$

R functions

Prefixes:

`d` to generate the probability mass or density function
Example: `dnorm(x,mean,sd)`

`p` to generate the probability $P(X \le x)$ (the cdf)
Example: `pnorm(q,mean,sd)`

`q` to generate the quantile $P(X \le x) > q$ (inverse cdf)
Example: `qnorm(p,mean,sd)`

`r` to generate a random variable having the specified distribution
Example: `rnorm(n,mean,sd)`

Distribution	Functions	Specifications
beta	`dbeta` `pbeta` `qbeta` `rbeta`	shape parameters α and β
binomial	`dbinom` `pbinom` `qbinom` `rbinom`	number of trials and probability of success on each trial
cauchy	`dcauchy` `pcauchy` `qcauchy` `rcauchy`	location α and scale β parameters
chi-square	`dchisq` `pchisq` `qchisq` `rchisq`	degrees of freedom
exponential	`dexp` `pexp` `qexp` `rexp`	rate λ
F	`df` `pf` `qf` `rf`	numerator and denominator degrees of freedom
gamma	`dgamma` `pgamma` `qgamma` `rgamma`	shape α and scale β parameters
geometric	`dgeom` `pgeom` `qgeom` `rgeom`	probability of success
hypergeometric	`dhyper` `phyper` `qhyper` `rhyper`	number in group 1 (n), number in group 2 (m), number drawn (k)
logistic	`dlogis` `plogis` `qlogis` `rlogis`	location and scale parameters
lognormal	`dlnorm` `plnorm` `qlnorm` `rlnorm`	mean μ and standard deviation σ on the log scale
negative binomial	`dnbinom` `pnbinom` `qnbinom` `rnbinom`	number of trials and probability of success on each trial
normal	`dnorm` `pnorm` `qnorm` `rnorm`	mean μ and standard deviation σ
poisson	`dpois` `ppois` `qpois` `rpois`	rate parameter λ
t	`dt` `pt` `qt` `rt`	degrees of freedom
uniform	`dunif` `punif` `qunif` `runif`	min and max (θ_1 and θ_2) of the distribution
weibull	`dweibull` `pweibull` `qweibull` `rweibull`	shape α and scale β parameters

R functions

Source: https://meredithfranklin.github.io/R-Probability-Distributions.html

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`r` to generate a random variable having the specified distribution

Example: `rnorm(n,mean,sd)`