Lab 4 Pre-Lab

1. Express the signal $x(t) = 3e^{-t}\sin 2t$ as an exponentially decaying harmonic signal.

$$x(t) = 3e^{-t}\cdot \sin 2t = 3e^{-t}\cdot \cos\left(2t - \frac{\pi}{2}\right) \qquad x(t) = Ae^{-\gamma t}\cos(\omega t - \theta)$$

2. Find all local extrema of the exponentially decaying harmonic signal $x(t) = 3e^{-t/2}\cos(3t-1)$ and determine the elapsed time between consecutive minima.

$$x(t) = 3e^{-\frac{t}{2}}\cos(3t-1)$$

$$\dot{x}(t) = -\frac{3}{2}e^{-\frac{t}{2}}\cos(3t-1) - 9e^{-\frac{t}{2}}\cdot\sin(3t-1) = -3e^{-\frac{t}{2}}\left[\frac{1}{2}\cos(3t-1) + 3\sin(3t-1)\right]$$

$$= -3e^{-\frac{t}{2}}\cdot\cos\left(3t-1+\tan^{-1}(-6)\right)\cdot\frac{\sqrt{37}}{2}$$

$$\ddot{x}(t) = \frac{37}{4}\cdot 3e^{-\frac{t}{2}}\cdot\cos\left(3t-1+2\tan^{-1}(-6)\right)$$

Let $\dot{x}(t) = 0 \Rightarrow t = \dfrac{1+\tan^{-1}(6)}{3} + \dfrac{2k\pi}{3}$ (local minimum)

$t = \dfrac{1+\tan^{-1}(6)}{3} + \dfrac{\pi + 2k\pi}{3}$ (Local maximum)

$\therefore$ elapsed time $= \dfrac{2\pi}{3}$

3. Determine the logarithmic decrement for the exponentially decaying harmonic signal $x(t) = e^{-3t}\cos(t+\pi/2)$ and relate this to the ratio between successive minima.

$$\dot{x}(t) = -3e^{-3t}\cos\left(t+\frac{\pi}{2}\right) - e^{-3t}\sin\left(t+\frac{\pi}{2}\right)$$

$$= -e^{-3t}\left[3\cos\left(t+\frac{\pi}{2}\right) + \sin\left(t+\frac{\pi}{2}\right)\right] = -e^{3t}\left[\cos\left(t+\frac{\pi}{2} - \tan^{-1}\frac{1}{3}\right)\right]$$

$$\therefore t_n = \tan^{-1}\frac{1}{3} - \frac{\pi}{2} + k\pi$$

$$\therefore \frac{x(t_{n+1})}{x(t_n)} = e^{3\times\pi}\times -1 = -e^{3\pi}$$

4. Measurements on a mechanical system result in a signal that is dominated by the solution to the differential equation

$$2\frac{d^2x}{dt^2}(t) + 24\frac{dx}{dt}(t) + 18x(t) = 2.$$

Determine whether the system is overdamped or underdamped.

$$2\ddot{x}(t) + 24\dot{x}(t) + 18x(t) = 2$$

$$\ddot{x}(t) + 12\dot{x}(t) + 9x(t) = 1$$

$$\therefore \begin{cases} \omega_n^2 = 9 \\ 2\xi\omega_n = 12 \end{cases} \Rightarrow \begin{cases} \omega_n = 3 \\ \xi = 2 > 1 \end{cases} \quad \therefore \text{Overdamped}$$

5. For what value of k is the mass-spring-damper system, governed by the differential equation

$$m\frac{d^2x}{dt^2}(t) + c\frac{dx}{dt}(t) + kx(t) = f(t),$$

critically damped?

$$\ddot{x}(t) + \frac{c}{m}\dot{x}(t) + \frac{k}{m}x(t) = \frac{1}{m}f(t)$$

$$\begin{cases} 2\xi\omega_n = \frac{c}{m} \\ \omega_n^2 = \frac{k}{m} \end{cases} \Rightarrow \xi = \frac{c}{2\sqrt{mk}} = 1 \quad \therefore k = \frac{c^2}{4m}$$

6. Graph the unit step response of a linear, second-order, time-invariant system with natural frequency $\omega_n = 1/2$ and damping ratio $\zeta = 1/\sqrt{2}$.

$$\ddot{x} + \frac{1}{\sqrt{2}}\dot{x} + \frac{1}{4}x = 0$$

For unit step response

$$x(t) = \frac{A}{\omega_n^2}\left[1 - e^{\xi\omega_n t} \cdot \left(\cos\omega_d t + \frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right)\right.$$

$$= 4\left[1 - e^{\frac{t}{2\sqrt{2}}} \cdot \left(\cos\frac{1}{2\sqrt{2}}t + \sin\frac{1}{2\sqrt{2}}t\right)\right.$$