Bivariate Distributions

Discrete (4.1)
Continuous (4.4)

Bivariate Distributions

Ch 2-3: Univariate. Only one measurement for observations. For example:

- Waiting time
- Number of successes in *n* trials
- Number of occurrences in a unit time, etc.

Ch 4: Use multiple variables to predict an outcome

Discrete Bivariate Distributions

Definition 4.1-1

Let X and Y be two random variables defined on a discrete space. Let S denote the corresponding two-dimensional space of X and Y, the two random variables of the discrete type. The probability that $X = x$ and $Y = y$ is denoted by $f(x, y) = P(X = x, Y = y)$. The function $f(x, y)$ is called the **joint probability mass function** (joint pmf) of X and Y and has the following properties:

(a) $0 \le f(x, y) \le 1$.

(b) $\displaystyle\sum\sum_{(x,y)\in S} f(x, y) = 1$.

(c) $\displaystyle P[(X, Y) \in A] = \sum\sum_{(x,y)\in A} f(x, y)$, where A is a subset of the space S.

Discrete Bivariate Example

Let $f(x,y) = \frac{xy^2}{30}$, $x = 1,2,3$ $y = 1,2.$

(a) $0 \le f(x,y) \le 1$.

(b) $\sum\sum_{(x,y)\in S} f(x,y) = 1$.

(c) $P[(X,Y) \in A] = \sum\sum_{(x,y)\in A} f(x,y)$, where A is a subset of the space S.

Discrete Bivariate Example

Let X and Y be two discrete random variables with the following joint pmf, *f(x,y)*:

e.g. f(3,0) = 0.31

		X			
		3	4	5	
Y	0	0.31	0.21	0.21	0.73
	1	0.03	0.04	0.05	0.12
	2	0.02	0.03	0.04	0.09
	3	0.01	0.02	0.03	0.06
		0.37	0.30	0.33	1.0

Marginal pmf

Definition 4.1-2

Let X and Y have the joint probability mass function $f(x, y)$ with space S. The probability mass function of X alone, which is called the **marginal probability mass function of** X, is defined by

$$f_X(x) = \sum_y f(x, y) = P(X = x), \qquad x \in S_X,$$

where the summation is taken over all possible y values for each given x in the x space S_X. That is, the summation is over all (x, y) in S with a given x value. Similarly, the **marginal probability mass function of** Y is defined by

$$f_Y(y) = \sum_x f(x, y) = P(Y = y), \qquad y \in S_Y,$$

Marginal probability

- $$f(y) = \begin{cases} 0.73, & y = 0 \\ 0.12, & y = 1 \\ 0.09, & y = 2 \\ 0.06, & y = 3 \end{cases}$$

		X			
		3	4	5	
	0	0.31	0.21	0.21	0.73
Y	1	0.03	0.04	0.05	0.12
	2	0.02	0.03	0.04	0.09
	3	0.01	0.02	0.03	0.06
		0.37	0.30	0.33	1.0

Independence of X and Y

X and Y are independent **iff:**

- for every x ∈ S_x and y ∈ S_y,

$$\mathrm{P}[\mathrm{X} = \mathrm{x}, \mathrm{Y} = \mathrm{y}] = \mathrm{P}[\mathrm{X} = \mathrm{x}]\mathrm{P}[\mathrm{Y} = \mathrm{y}]$$

i.e.,

$$\boxed{f_{XY}(x, y) = f_X(x) f_Y(y)}$$

Examples

Bivariate Discrete

1

Let $f(x, y) = \frac{xy^2}{30}, \quad x = 1,2,3 \quad y = 1,2.$

A) Find the marginal pmf of X: $f_X(x) = \frac{x}{6}, \quad x = 1,2,3.$

B) Find the marginal pmf of Y: $f_Y(y) = \frac{y^2}{5}, \quad y = 1,2.$

C) Find P[X=Y] : 9/30

D) Are X and Y independent? (Yes)

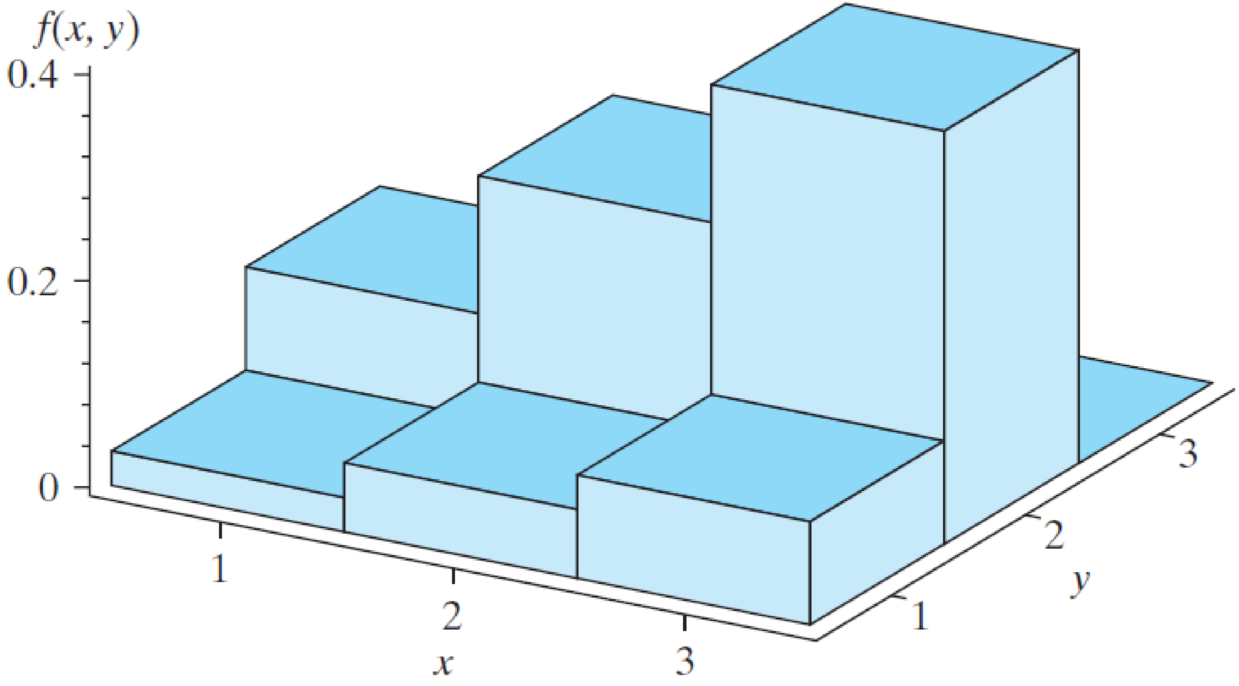


Figure 4.1-3 Joint pmf $f(x, y) = \dfrac{xy^2}{30}$, $x = 1, 2, 3$ and $y = 1, 2$

2

Let $f(x, y) = c(x + 2y),\ x = 1,2\quad y = 1,2,3$

A) What value must the constant c take, so that $f(x, y)$ is a valid joint pmf?

f(1,1) = c(1 + 2(1)) = 3c	f(2,1) = c(2+2(1)) = 4c
f(1,2) = c(1 + 2(2)) = 5c	f(2,2) = c(2+2(2)) = 6c
f(1,3) = c(1 + 2(3)) = 7c	f(2,3) = c(2+2(3)) = 8c

33c = 1 c = 1/33

2

Let $f(x, y) = \frac{1}{33}(x + 2y),\ x = 1,2 \quad y = 1,2,3$

B) Find P[Y > X].

C) Find the marginal pmf of X.

D) Find the marginal pmf of Y.

3

Let $f(x,y) = 6\left(\frac{1}{4}\right)^x \left(\frac{1}{3}\right)^y$, x = 1,2,3,... y = 1,2,3,...

A) Find an expression for the marginal pmf of x. $f_X(x) = 3\left(\frac{1}{4}\right)^x$

B) Show that the marginal pmf of x is a valid probability distribution.

3

Let $f(x,y) = 6\left(\frac{1}{4}\right)^x \left(\frac{1}{3}\right)^y$, x = 1,2,3,… y = 1,2,3,…

C) Evaluate P[Y > X].

3 A fair die is rolled. Then a coin with probability, p, of Heads is flipped as many times as the die roll says.

e.g., if the result of the die roll is a 3, then the coin is flipped 3 times.

Let X be the result of the die roll and Y be the number of times the coin lands Heads.

A) Find f(x,y).

B) Are X and Y independent?

Continuous Bivariate Distributions (4.4)

Continuous Bivariate Distributions

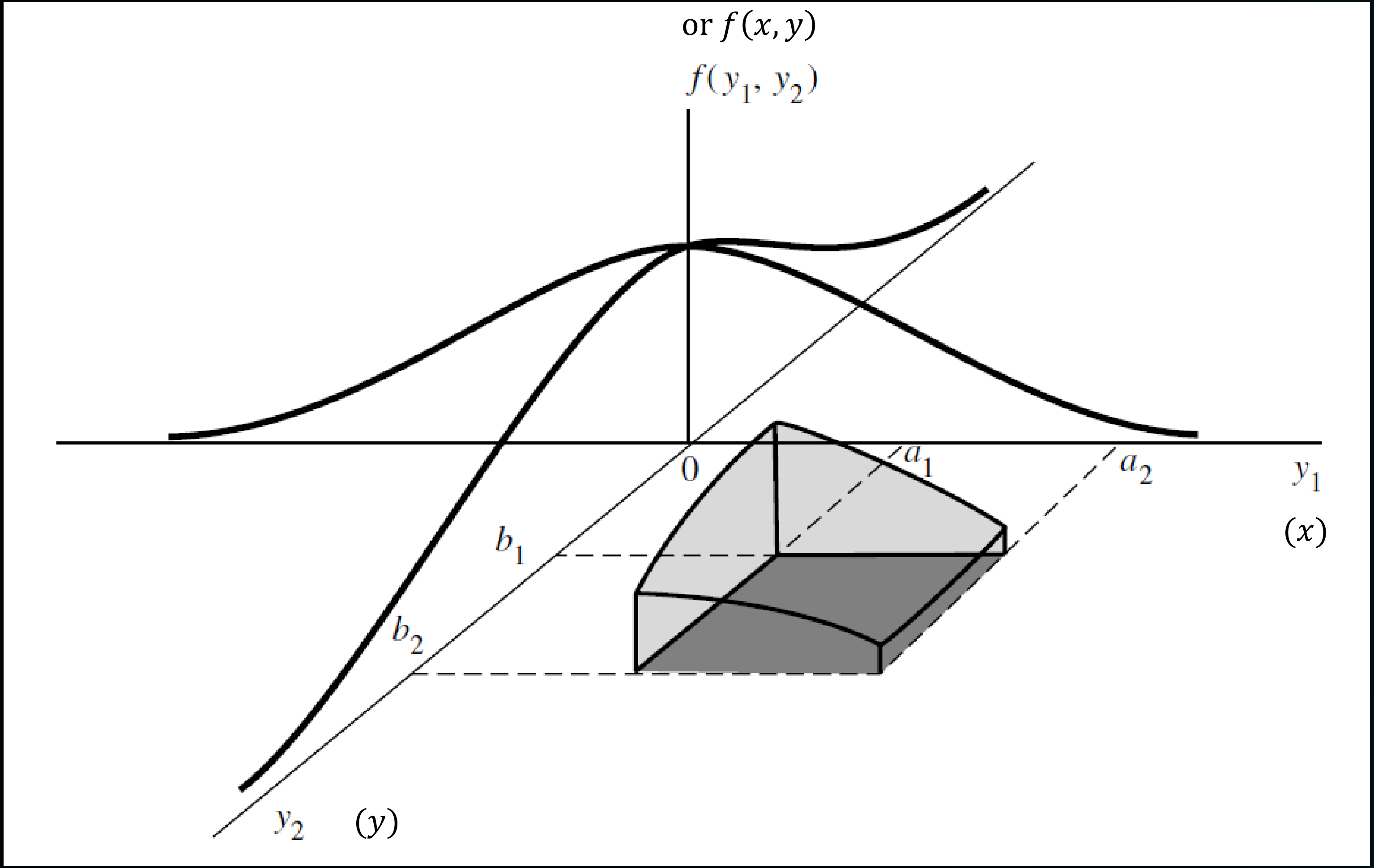
If X and Y are two continuous random variables, their **joint probability density function,** $f(x, y)$ represents the density at the point (x, y).

The joint pdf satisfies 3 properties:

(a) $f(x, y) \geq 0$, where $f(x, y) = 0$ when (x, y) is not in the support (space) S of X and Y.

(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$.

(c) $P[(X, Y) \in A] = \iint_{A} f(x, y)\, dx\, dy$, where $\{(X, Y) \in A\}$ is an event defined in the plane.



Source: Wackerly, Mendenhall, Scheaffer – Mathematical Statistics

Marginal pdf

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy, \qquad x \in S_X,$$

integrate over the range of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx, \qquad y \in S_Y,$$

integrate over the range of X

Independence

X and Y are independent **iff:**

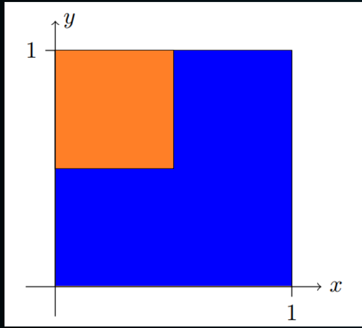
$$f(x, y) = f_X(x) f_Y(y),\ x \in S_X,\ y \in S_Y$$

1

$$P[(X,Y) \in A] = \iint_A f(x,y)\, dx\, dy$$

Suppose X and Y both have support (space): [0,1], with joint pdf,

$$f(x, y) = 4xy.$$



A) Find P[X < 0.5, Y > 0.5].

3/16

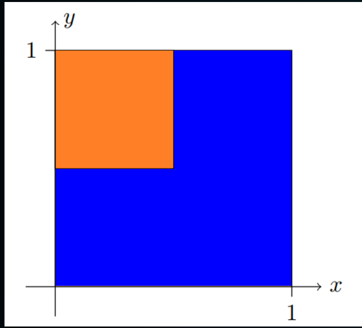
1

$$P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy$$

Suppose X and Y both have support (space): [0,1], with joint pdf,

$$f(x, y) = 4xy.$$

B) Are X and Y independent?



$$f_X(x) = 2x,\ 0 \le x \le 1$$

2

Suppose that the random variables X and Y have joint pdf,

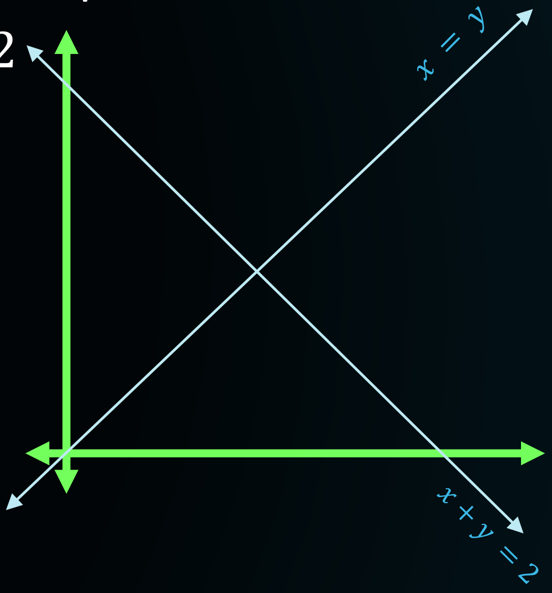
$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x + y \le 2 \\ 0, & elsewhere \end{cases}$$

A) Verify that this is a valid joint pdf.

2

Suppose that the random variables X and Y have joint pdf,

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x+y \le 2 \\ 0, & elsewhere \end{cases}$$



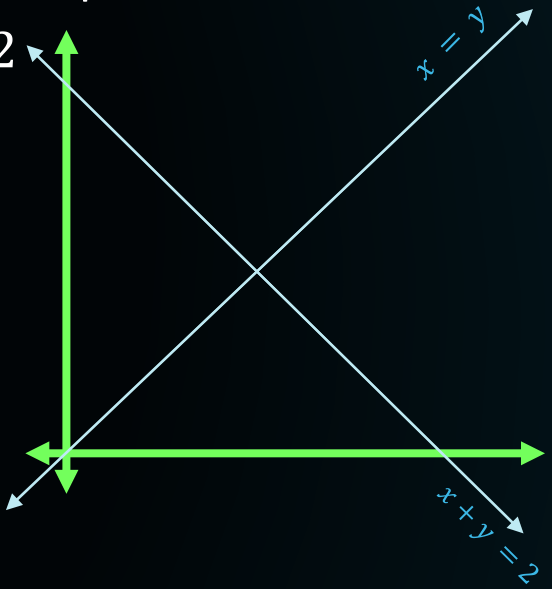
A) Verify that this is a valid joint pdf.

2

Suppose that the random variables X and Y have joint pdf,

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x + y \le 2 \\ 0, & elsewhere \end{cases}$$

B) Find $f_X(x)$.

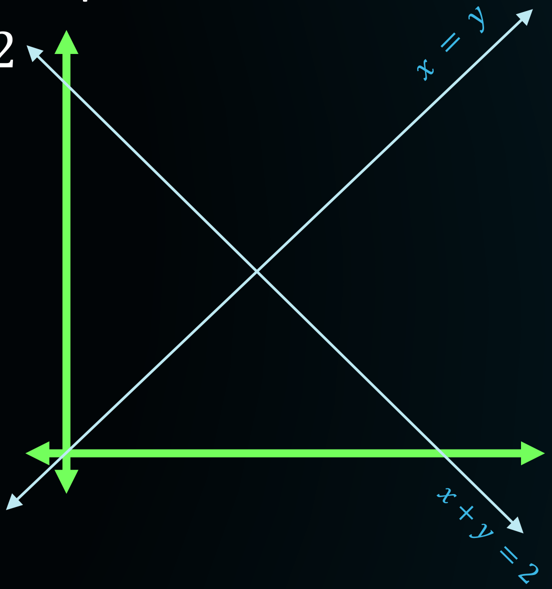


2

Suppose that the random variables X and Y have joint pdf,

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \ \ x+y \le 2 \\ 0, & elsewhere \end{cases}$$

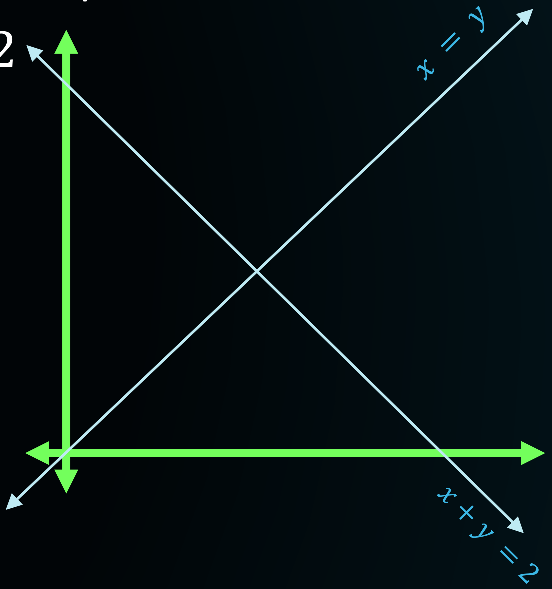
C) Find $f_Y(y)$.



2

Suppose that the random variables X and Y have joint pdf,

$$f(x, y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x + y \le 2 \\ 0, & elsewhere \end{cases}$$

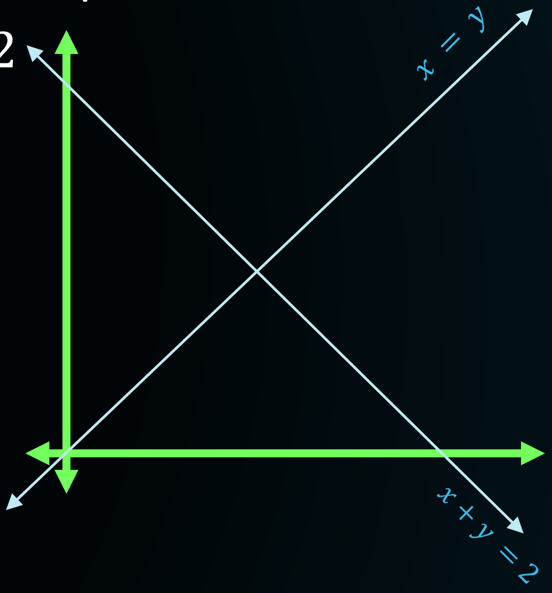


D) Evaluate E[X].

2

Suppose that the random variables X and Y have joint pdf,

$$f(x,y) = \begin{cases} 6x^2y, & 0 \le x \le y, \quad x+y \le 2 \\ 0, & elsewhere \end{cases}$$



E) Evaluate $P[X + Y < 1]$.