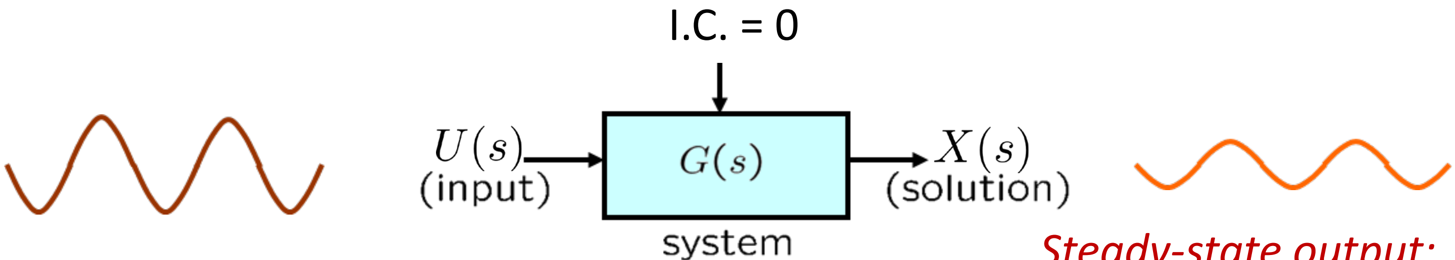
ME 340 Dynamics of Mechanical Systems

# Frequency Response and Bode Plot
# Part 5

# Frequency response:




*Input:* $u(t) = U_0 \sin(\omega t)$

*Steady-state output:*
$x_{ss}(t) = U_0|G(j\omega)| \sin\big(\omega t + \angle G(j\omega)\big)$

- For a stable LTI system, for a sinusoidal input,
  - The steady-state output, i.e., frequency response, is sinusoidal.
  - The frequency of the output is the same as the input.
  - The magnitude is amplified by $M(\omega) = |G(j\omega)|$
  - There is a shift in phase of $\varphi(\omega) = \angle G(j\omega)$

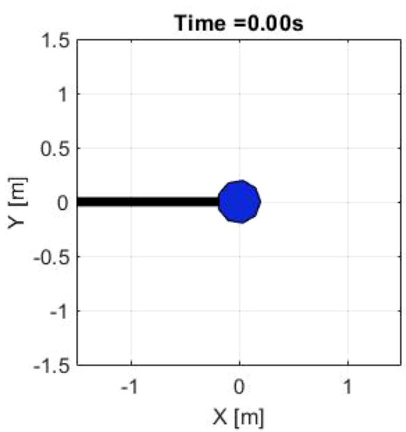
Bode Plot

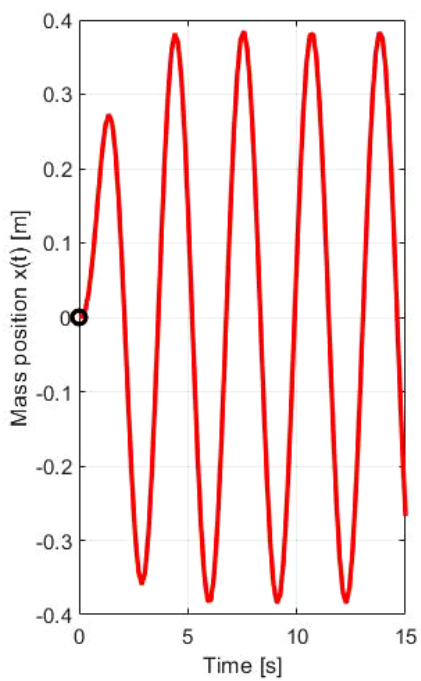
- Note that $G(j\omega) = |G(j\omega)|e^{\angle G(j\omega)} = M(\omega)e^{\varphi(\omega)}$

# Frequency response: 2$^{nd}$ order systems

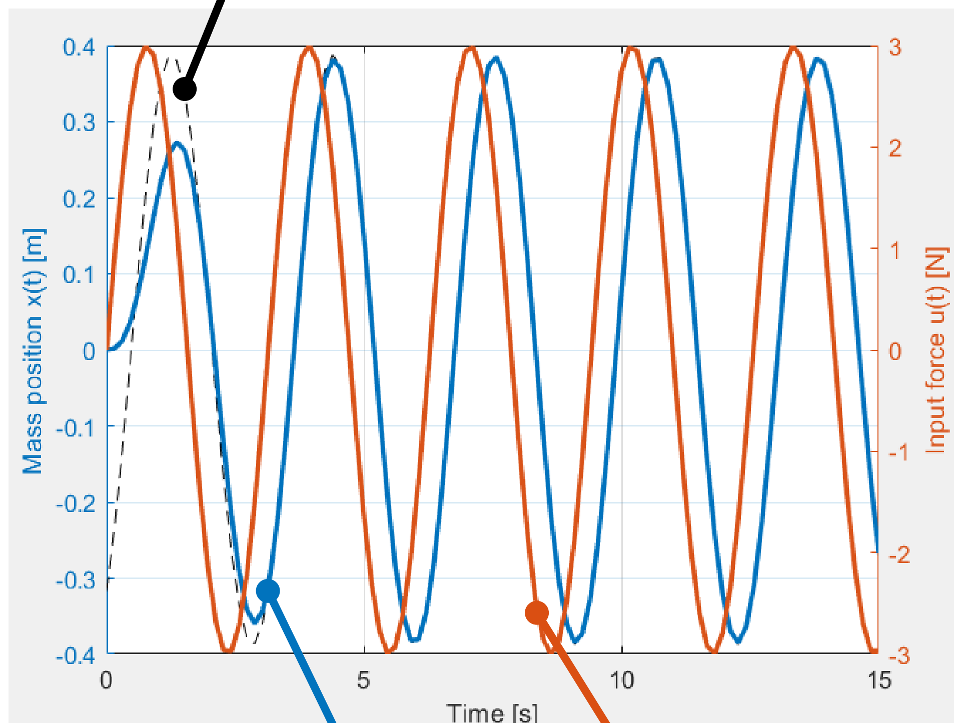
- Frequency response:

Lecture25_FreqResp_Mass.m




$$x_{ss}(t) = U_0|G(j\omega)|\sin\big(\omega t + \angle G(j\omega)\big)$$




$x(t)$

$u(t) = U_0 \sin(\omega t)$

# Bode plots

- Graphical representation of the frequency response, $G(j\omega)$ vs. $\omega$
  - Bode plots consist of two plots: $20 \log M(\omega)$ vs. $\omega$ and $\phi(\omega)$ vs. $\omega$
  - Usually, logarithmic scale is used for the frequency axis
  - $M(\omega)$ is plotted in decibels (dB)

- ODEs of LTI systems lead to TF's of the form poly/poly
  - Further any polynomial can be factored as products of terms of the form $(Ts + 1)$ and $(s^2 + 2\zeta\omega_n s + \omega_n^2)$
  - We can generate all others by knowing these and some simple rules

# Example: 1st order system

- Consider a transfer function: $G(s) = \frac{10}{0.5s+1}$

$$\begin{aligned}
G(s) &= \frac{10}{0.5s+1} \\
\Rightarrow G(j\omega) &= \frac{10}{0.5j\omega+1} \\
\Rightarrow M(\omega) &= \left|\frac{10}{0.5j\omega+1}\right| = \frac{10}{\sqrt{0.25\omega^2+1}} \\
\Rightarrow 20\log_{10} M(\omega) &= 20\log_{10} 10 - 10\log_{10}(0.25\omega^2+1) = 20 - 10\log_{10}(0.25\omega^2+1) \\
\text{and } \phi(\omega) &= -\tan^{-1} 0.5\omega
\end{aligned}$$

# Example: straight line approximations

- Consider a transfer function: $G(s) = \frac{10}{0.5s+1}$

$$20\log_{10} M(\omega) \quad = \quad 20\log_{10} 10 - 10\log_{10}(0.25\omega^2 + 1)$$

- Magnitude:
  - When $\omega$ is small ($\omega \ll 1/0.5$)

*DC gain (low-frequency magnitude): $M(\omega)$ when $\omega$ is very small*

$$20\log_{10} M(\omega) \approx 20\log_{10} 10 - \overbrace{10\log_{10}(0.25\omega^2 + 1)}^{\approx 0} = 20dB$$

  - When $\omega$ is large ($\omega \gg 1/0.5$)

*High-frequency slope (roll-off): coefficient of $\log_{10}\omega$*

$$\begin{aligned} 20\log_{10} M(\omega) &\approx 20\log_{10} 10 - 10\log_{10}(0.25\omega^2) \\ &= 20\log_{10} 10 - 20\log_{10}(0.5) - 20\log_{10}\omega \end{aligned}$$

  - The asymptotic approximations of magnitude for low- and high-frequency ranges intersect when $T\omega = 1$. $\omega = 1/T$ is called the *corner frequency* (also called break frequency).

- Phase:

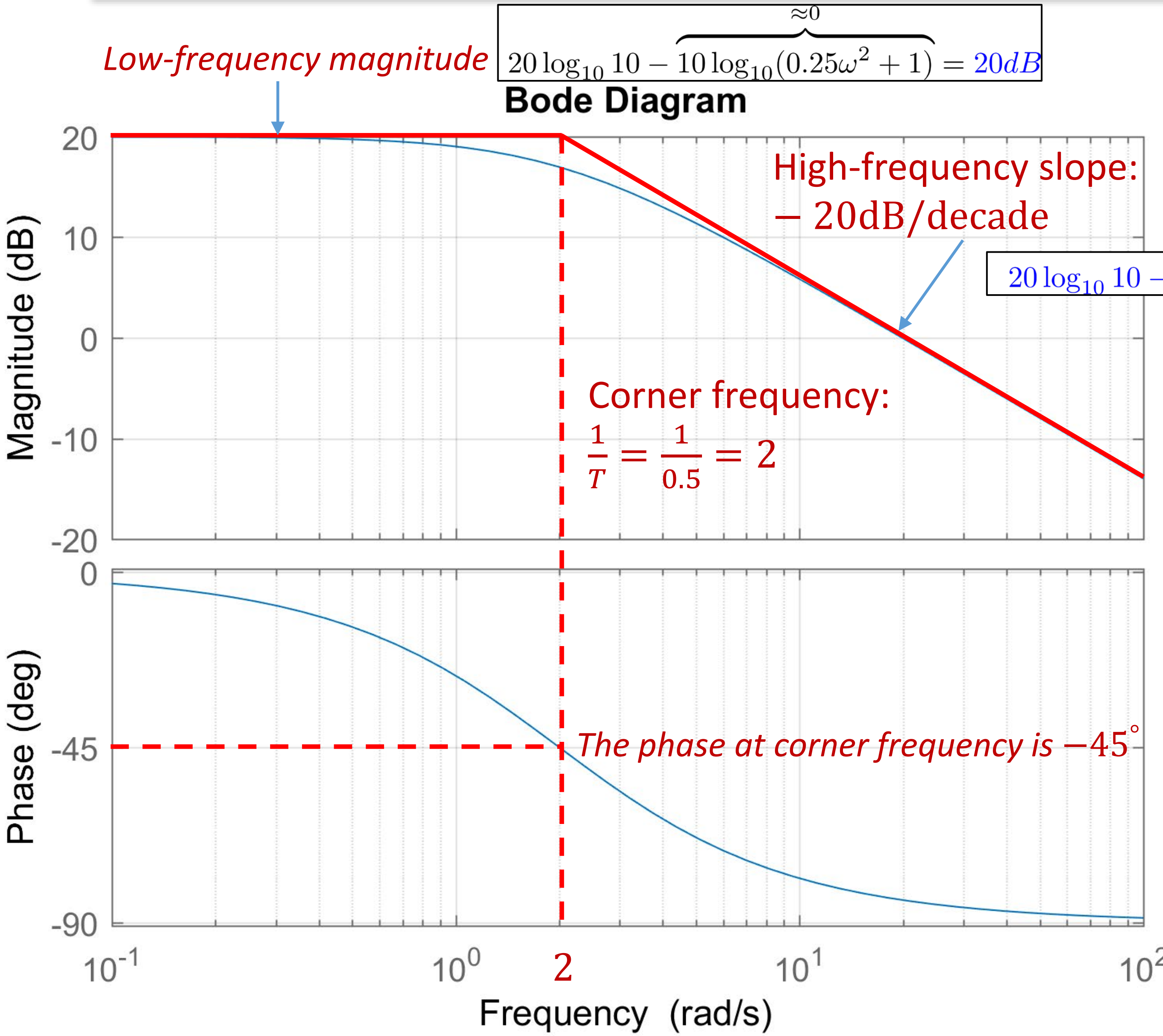
$$\phi(\omega) = -\tan^{-1} 0.5\omega$$

$$\tan^{-1}(0) = 0$$
$$\tan^{-1}(-\infty) = -90°$$

# Example: approximate (red) and exact (blue)

$$G(s) = \frac{10}{0.5s+1}$$

```
G = tf(10,[0.5 1]);
bode(G);
```



# Example: 1st order system

Ex.: Consider a transfer function: $G(s) = \frac{1/5}{\frac{3}{2}s+1}$ Corner frequency: $\omega = \frac{2}{3}$

$$G(j\omega) = \frac{1}{5}\frac{1}{\frac{3}{2}j\omega + 1}$$

$$|G(j\omega)| = \frac{1}{5}\frac{1}{\sqrt{\frac{9}{4}\omega^2 + 1}}$$

$$20\log|G(j\omega)| = 20\log\frac{1}{5} - 10\log\left(\frac{9}{4}\omega^2 + 1\right)$$

*For small frequency:*

$$20\log|G(j\omega)| = 20\log\frac{1}{5} - 10\log(1) = 20\log\frac{1}{5} \approx -14dB$$

*For high frequency:*

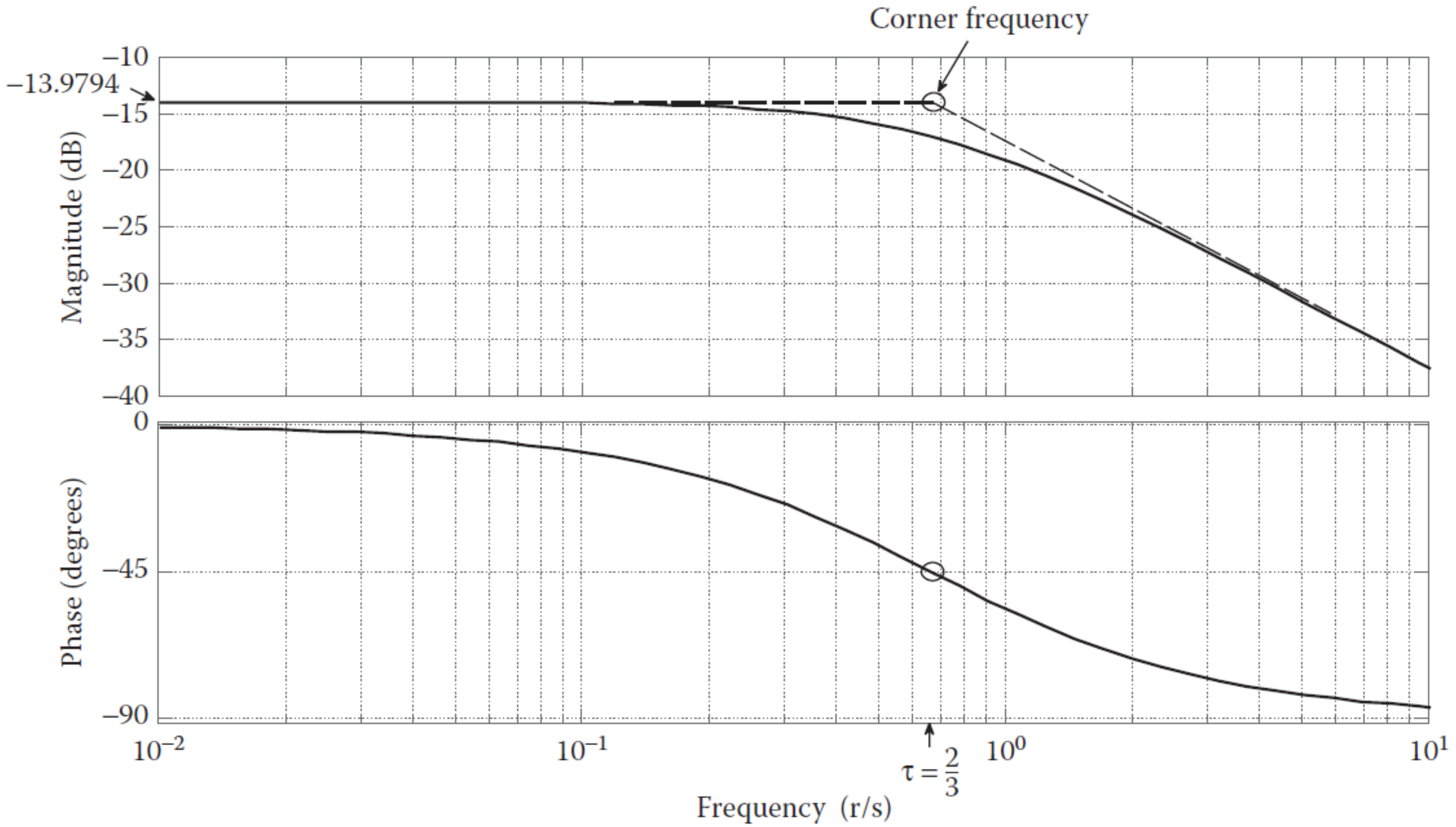
$$20\log|G(j\omega)| = 20\log\frac{1}{5} - 10\log\left(\frac{9}{4}\omega^2\right)$$

$$20\log|G(j\omega)| = 20\log\frac{1}{5} - 10\log\left(\frac{9}{4}\right) - 20\log(\omega)$$

# Example: 1$^{st}$ order system

Ex.: Consider a transfer function: $G(s) = \frac{1/5}{\frac{3}{2}s+1}$ Corner frequency: $\omega = \frac{2}{3}$

$$20\log|G(j\omega)| = 20\log\frac{1}{5} - 10\log\left(\frac{9}{4}\omega^2+1\right)$$



# Bode plots of 2nd order systems ($\zeta < 1$)

- Transfer function: $G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$$\Rightarrow G(j\omega) = \frac{\omega_n^2}{(\omega_n^2 - \omega^2) + 2j\zeta\omega_n\omega}$$

$$\Rightarrow M(\omega) = \left| \frac{1}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right) + 2j\zeta\left(\frac{\omega}{\omega_n}\right)} \right| = \frac{1}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}}$$

$$\Rightarrow 20\log_{10} M(\omega) = -10\log_{10}\left(\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2\right)$$

$$\text{and } \phi(\omega) = -\tan^{-1}\frac{2\zeta\left(\frac{\omega}{\omega_n}\right)}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

*DC gain (low-frequency magnitude)?*
*High-frequency slope?*

- Straight line approximations
  - When $\omega$ is small ($\omega \ll \omega_n$) Magnitude: $20\log_{10} M(\omega) \approx 0$, Phase: $\phi(\omega) \approx 0$
  - When $\omega \approx \omega_n$ Magnitude: $20\log_{10} M(\omega) \approx -20\log_{10}(2\zeta)$, Phase: $\phi(\omega) \approx -90°$
  - When $\omega$ is large ($\omega \gg \omega_n$) Magnitude: $20\log_{10} M(\omega) \approx -40\log_{10}\left(\frac{\omega}{\omega_n}\right)$, Phase: $\phi(\omega) \approx -180°$
  - Corner frequency at $\omega = \omega_n$

# Bode plots of 2nd order systems ($\zeta < 1$)

$$M(\omega) = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}}$$

- $M(\omega)$ is maximum when the denominator is minimum.

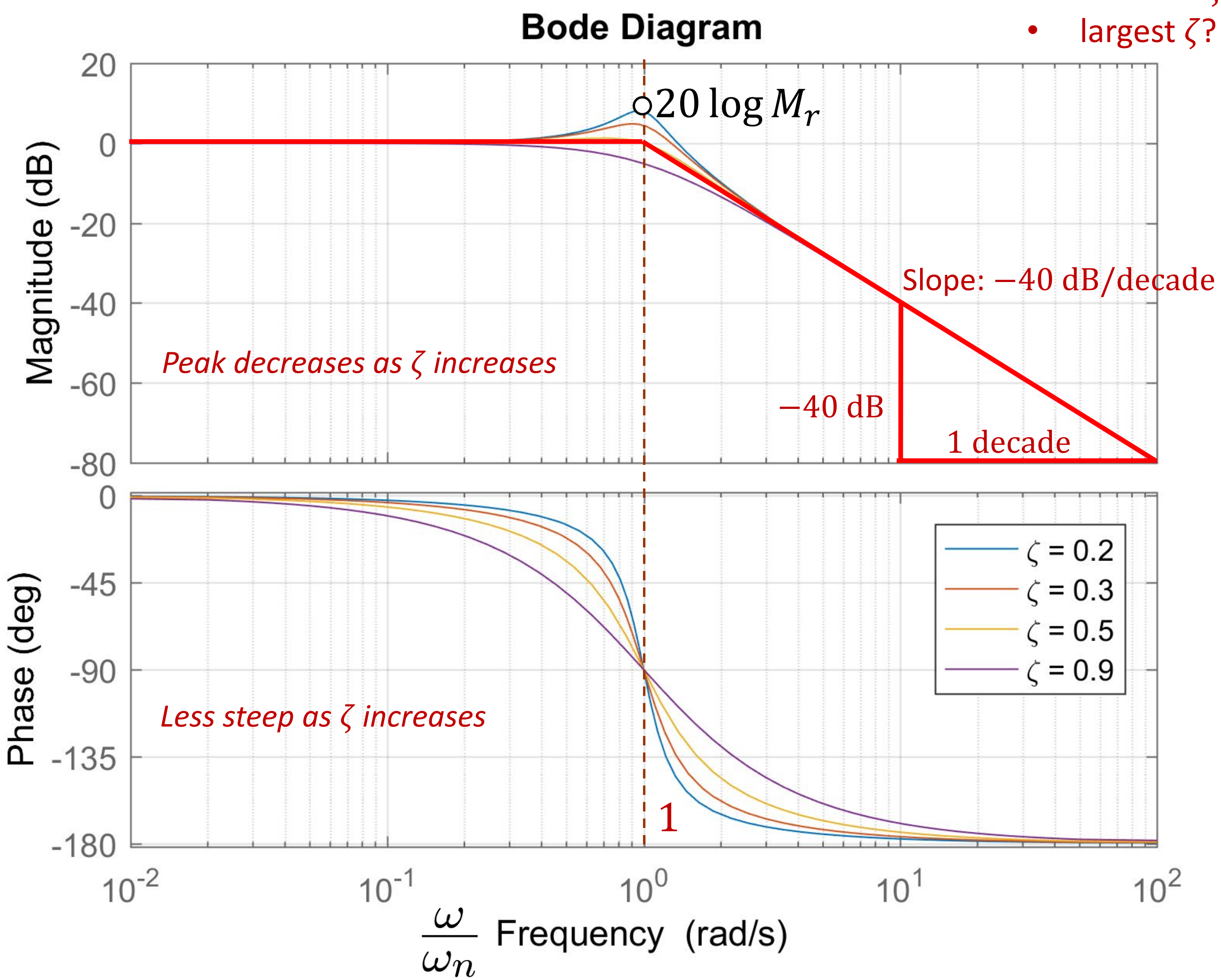
$$\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2+\left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2 = \left(\left(\frac{\omega}{\omega_n}\right)^2-(1-2\zeta^2)\right)^2+4\zeta^2(1-\zeta^2)$$

- This implies that at *resonance* (where $M(\omega)$ is maximum)
  - *Resonant frequency:* $\omega_r = \omega_n\sqrt{1-2\zeta^2}$
  - Peak value is given by
  $$M_r = M(\omega_r) = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$
    - When $\zeta = \frac{1}{\sqrt{2}} = 0.707, M(\omega) = 1$. There's no peak when $\zeta > 0.707$
    - When $\zeta$ gets larger, $M_r$ increases
    - $M_r$ goes to infinity as $\zeta$ goes to zero.

# Bode plots of 2nd order systems ($\zeta < 1$)

Which plot corresponds to:

- smallest $\zeta$?
- largest $\zeta$?



# Bode plots of 2nd order systems ($\zeta < 1$):

Ex.: $G(s) = \frac{8}{9} \frac{2.25}{s^2 + 0.9s + 2.25} = \frac{8}{9} \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$\omega_n = 1.5rad/s$

$\zeta = 0.3$

$$G(j\omega) = \frac{8}{9} \frac{2.25}{\sqrt{(2.25 - \omega^2) + 0.9\omega j}}$$

$$|G(j\omega)| = \frac{8}{9} \frac{2.25}{\sqrt{(2.25 - \omega^2)^2 + (0.9\omega)^2}}$$

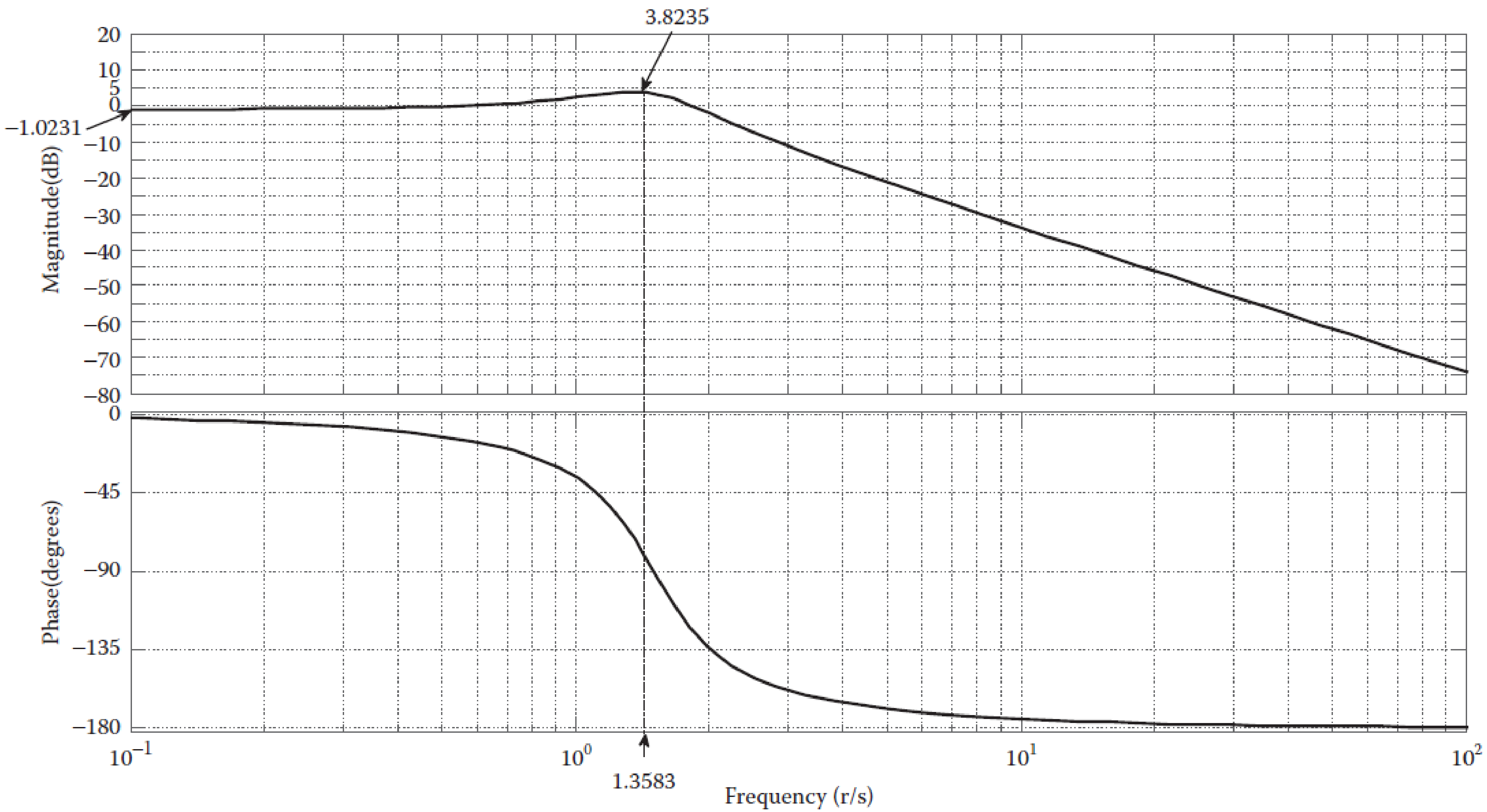
$$20 \log|G(j\omega)| = 20 \log \frac{8}{9} + 20 \log \left( \frac{2.25}{\sqrt{(2.25 - \omega^2)^2 + (0.9\omega)^2}} \right)$$

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} = 1.36rad/s$$

$$\max|G(j\omega)| = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} = 1.74 \therefore 20 \log \max|G(j\omega)| = 5.85dB$$

# Bode plots of 2nd order systems ($\zeta < 1$):

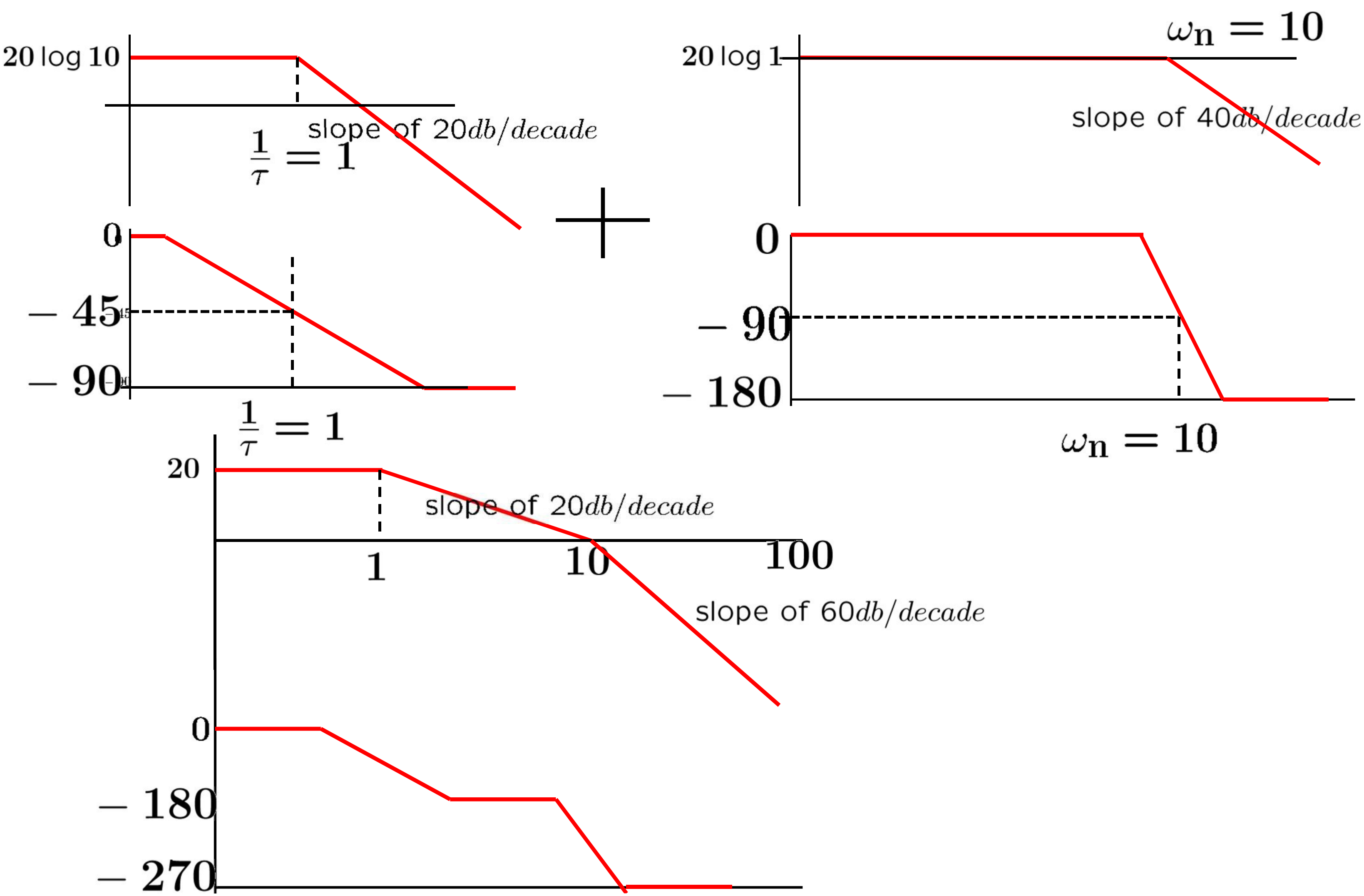
Ex.: $G(s) = \frac{8}{9}\frac{2.25}{s^2 + 0.9s + 2.25} = \frac{8}{9}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ $\omega_n = 1.5rad/s$ $\zeta = 0.3$



# Some Properties of Bode Plots

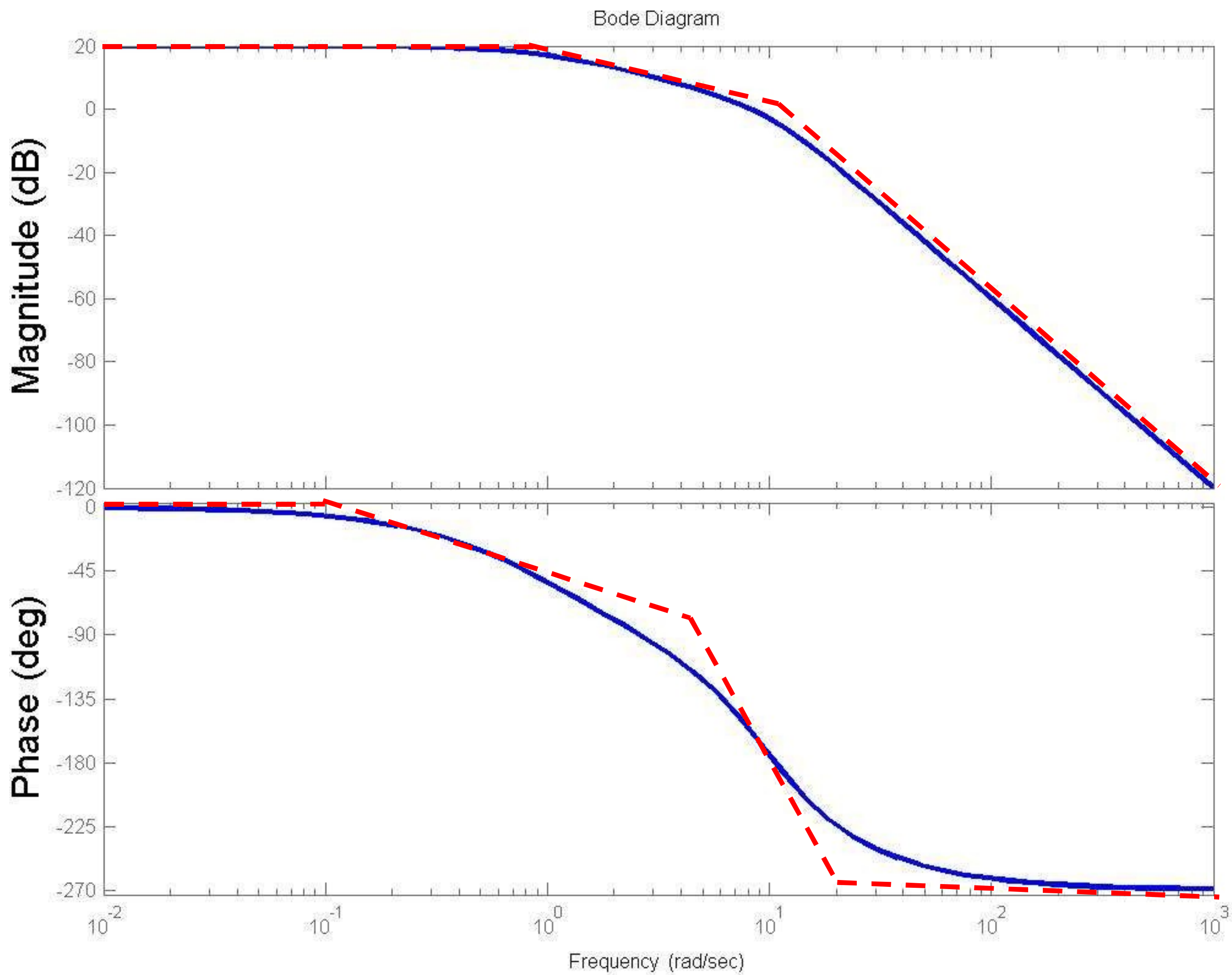
- The bode plot of a product of transfer functions is equal to the sum of the individual bode plots, For example if $G(s) = G_1(s)G_2(s)G_3(s)$ then Bode of $G(s)$ = Bode of $G_1(s)$ + Bode of $G_2(s)$ + Bode of $G_3(s)$
  - ⋆ obvious from $log|ab| = log|a| + log|b|$ and $\angle(ab) = \angle(a) + \angle(b)$ where $a$ and $b$ are complex numbers
- this implies bode plots of $\frac{1}{G(s)}$ is just the negative of bode plot of $G(s)$
- Example: $G(s) = \frac{1000}{(s+1)(s^2+14s+100)}$ Then the bode plot is equal to the bode plot of $\frac{10}{s+1}$ + the bode plot of $\frac{100}{s^2+14s+100}$
  - ⋆ makes life easy as we can easily plot these two standard plots and add them afterwards

# Examples



20 log 10
slope of 20db/decade
1/τ = 1
0
− 45
− 90
1/τ = 1
20 log 1
ωn = 10
slope of 40db/decade
0
− 90
− 180
ωn = 10
20
slope of 20db/decade
1
10
100
slope of 60db/decade
0
− 180
− 270

# Comparison of approximations with Matlab Plots

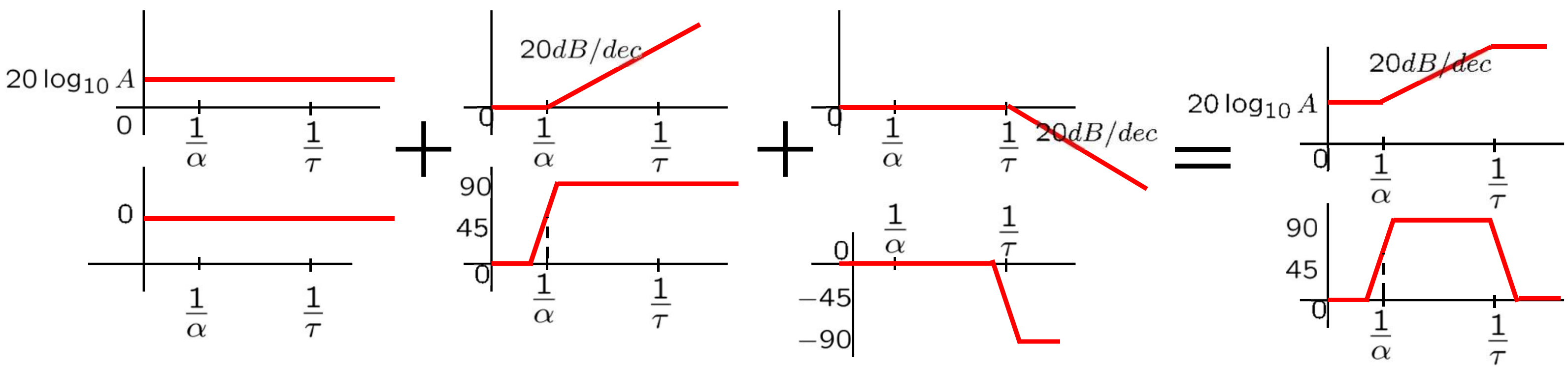


# Some more Examples

- Example: $G(s) = A\frac{\alpha s+1}{\tau s+1}$ $(\alpha > \tau > 0)$
  - $\star$

$$
\begin{aligned}
G(j\omega) &= A\frac{\alpha j\omega + 1}{\tau j\omega + 1} \\
\Rightarrow 20\log_{10}|G(j\omega)| &= 20\log_{10}|A| + 20\log_{10}|\alpha j\omega + 1| - 20\log_{10}|\tau j\omega + 1| \\
&= 20\log_{10}|A| + 20\log_{10}\sqrt{\alpha^2\omega^2+1} - 20\log_{10}\sqrt{\tau^2\omega^2+1} \\
\Rightarrow \phi(\omega) &= 0 + \tan^{-1}\alpha\omega - \tan^{-1}\tau\omega
\end{aligned}
$$

  - $\star$ drawing straight line approximations of each of these terms:



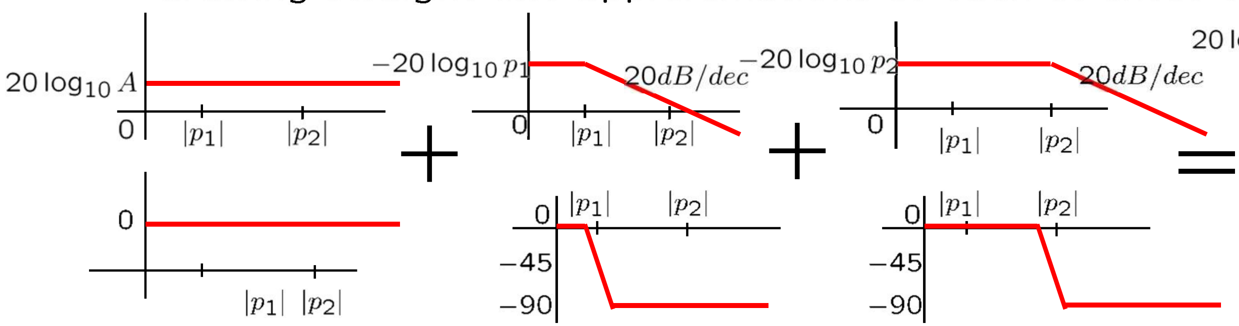

- Example: $G(s) = A\frac{\alpha s+1}{\tau s+1}$ $(0 < \alpha < \tau)$
  - $\star$ do it yourself as an exercise

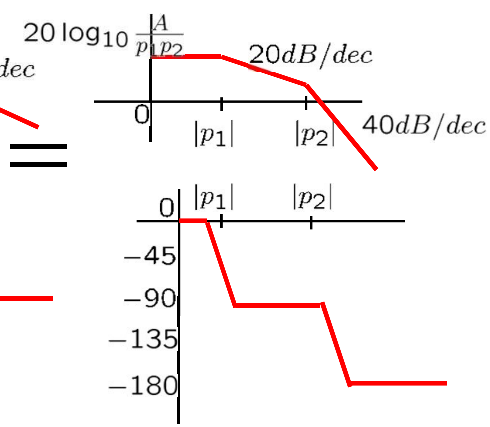
# Some more Examples

- Example: $G(s) = \frac{A}{s^2+2\zeta\omega_n s+\omega_n^2}$ $(\zeta > 1)$
  - ⋆ real poles: $p_1 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$, $p_2 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}$

$$
\begin{aligned}
G(s) &= \frac{A}{(s+p_1)(s+p_2)} \\
\Rightarrow G(j\omega) &= \frac{A}{(j\omega+p_1)(j\omega+p_2)} \\
\Rightarrow 20\log_{10}|G(j\omega)| &= 20\log_{10}|A| - 20\log_{10}(j\omega+p_1) - 20\log_{10}(j\omega+p_2) \\
&= 20\log_{10}|A| - 20\log_{10}\sqrt{(\omega^2+p_1^2)} - 20\log_{10}\sqrt{(\omega^2+p_2^2)} \\
\Rightarrow \phi(\omega) &= -\tan^{-1}\left(\frac{\omega}{p_1}\right) - \tan^{-1}\left(\frac{\omega}{p_2}\right)
\end{aligned}
$$

  - ⋆ drawing straight line approximations of each of these terms:




- Example: $G(s) = \frac{A}{s^2+2\zeta\omega_n s+\omega_n^2}$ $(\zeta = 1)$
  - ⋆ do it yourself

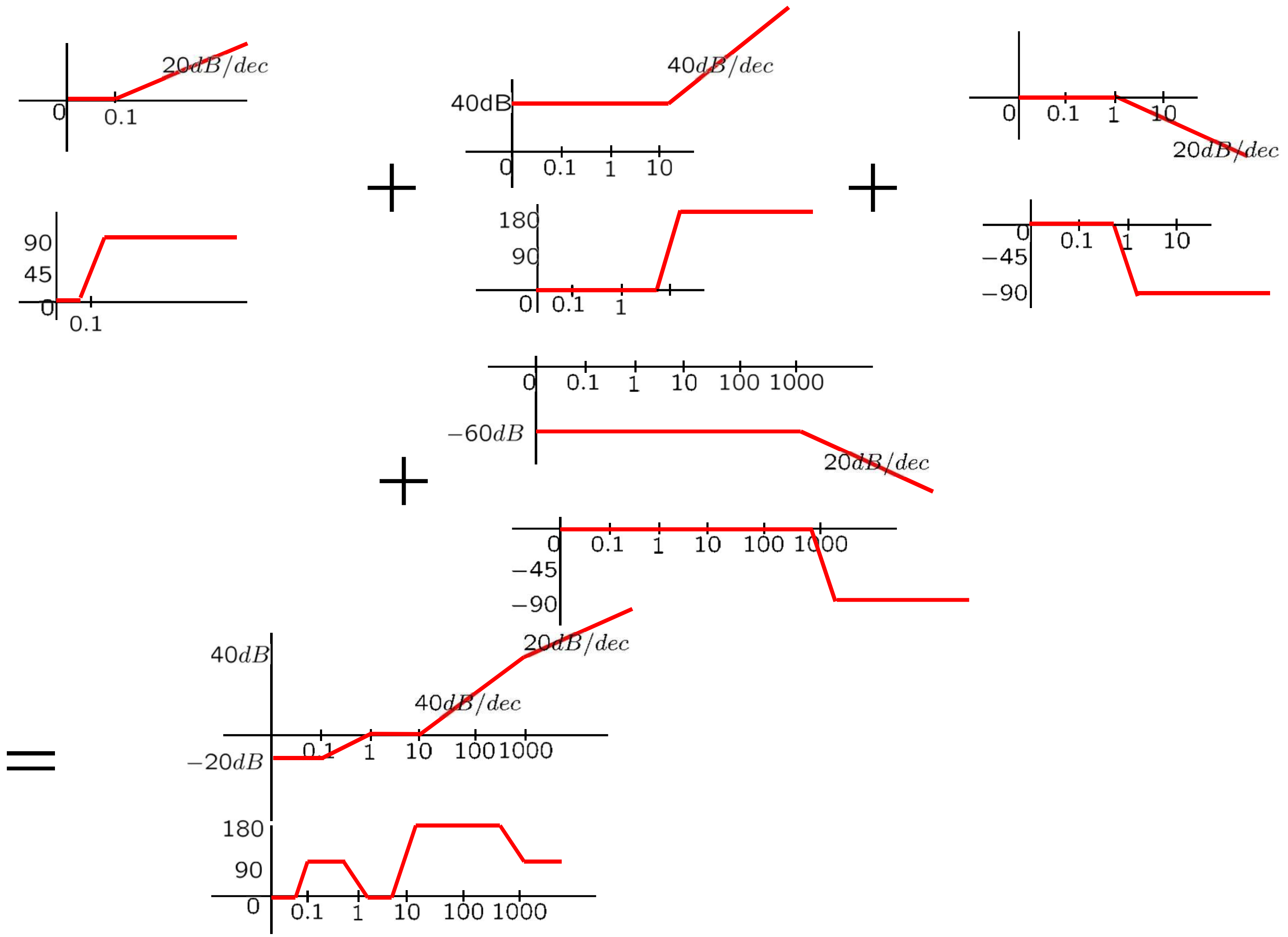
# Master Example

- Example: $G(s) = \frac{(10s+1)(s^2+20s+100)}{(s+1)(s+1000)}$
  - ⋆ $\zeta = 20/(2*10) = 1$

$$
\begin{aligned}
G(j\omega) &= \frac{(10j\omega+1)(j\omega+10)^2}{(j\omega+1)(j\omega+1000)} \\
\Rightarrow 20\log_{10}|G(j\omega)| &= 20\log_{10}|10j\omega+1| + 40\log_{10}|j\omega+10| \\
&\quad -20\log_{10}|j\omega+1| - 20\log_{10}|j\omega+1000| \\
&= 20\log_{10}\sqrt{100\omega^2+1} + 40\log_{10}\sqrt{\omega^2+100} \\
&\quad -20\log_{10}\sqrt{\omega^2+1} - 20\log_{10}\sqrt{\omega^2+10^6} \\
\Rightarrow \phi(\omega) &= \tan^{-1}(10\omega) + 2\tan^{-1}\left(\frac{\omega}{10}\right) - \tan^{-1}(1) - \tan^{-1}\left(\frac{\omega}{1000}\right)
\end{aligned}
$$

  - ⋆ drawing straight line approximations of each of these terms:

# Master Example(cont'd.)



# Example

Match the following transfer functions

1. $G_A(s) = \frac{1}{(s+1)^2}$

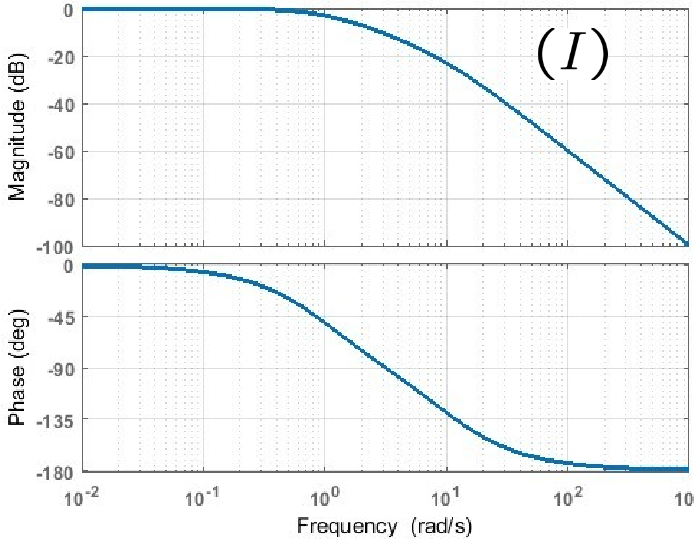
2. $G_B(s) = \frac{1}{(s+1)(s+10)}$

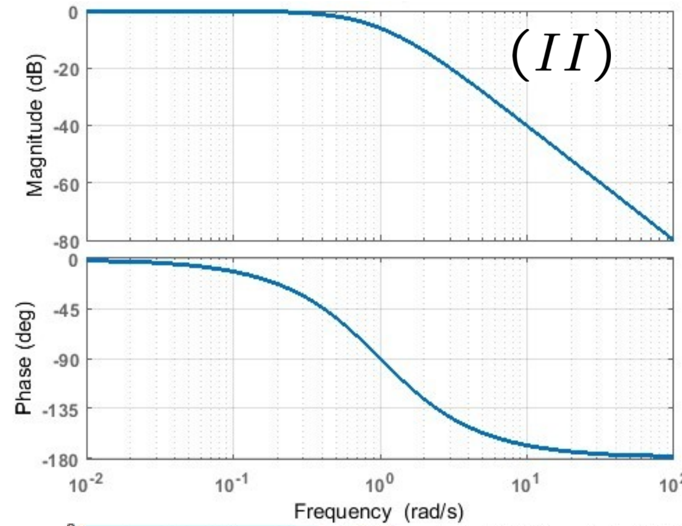
3. $G_C(s) = \frac{1}{s^2+0.001s+1}$

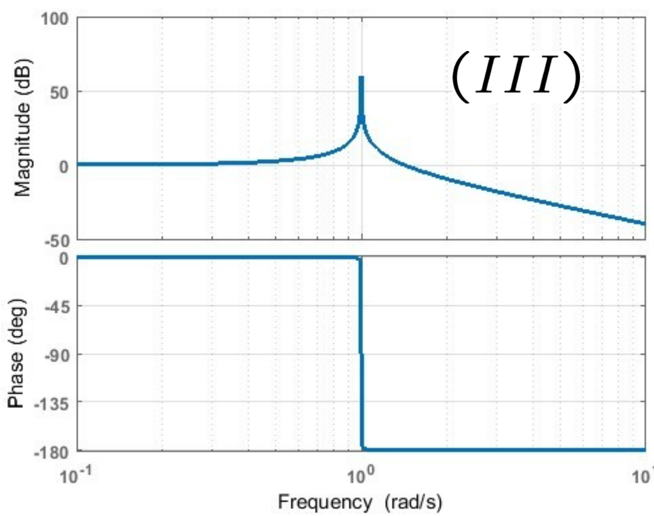
4. $G_D(s) = \frac{1}{10s+1}$

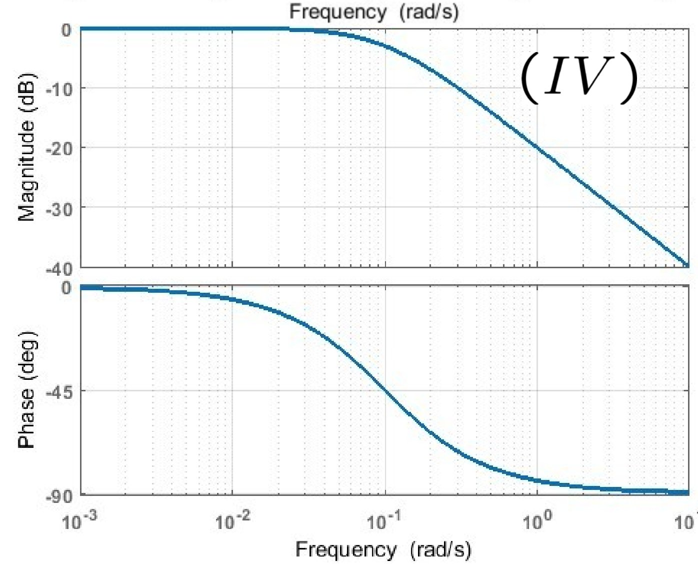
with the bode plots in the following page.

# Example(cont'd.)



# Solution

1. $G_A(s) = \frac{1}{(s+1)^2}$: 2nd order; corner frequencies (1 rad/s,1 rad/s); critically damped ($\zeta = 1$) which implies final slope is $-40 dB/decade$, and phase $-180$ degrees; Therefore (II)

2. $G_B(s) = \frac{1}{(s+1)(s+10)}$: 2nd order; overdamped ($\zeta > 1$) corner frequencies (1 rad/s,10 rad/s); which implies final slope is $-40 dB/decade$, and phase $-180$ degrees; Therefore (I)

3. $G_C(s) = \frac{1}{s^2+0.001s+1}$: 2nd order; corner frequencies (1 rad/s,); underdamped ($\zeta \ll 1$) which implies final slope is $-40 dB/decade$, and phase $-180$ degrees; Therefore (III)

4. $G_D(s) = \frac{1}{(10s+1)}$: 1st order; corner frequencies (1 rad/s which implies final slope is $-20 dB/decade$, and phase $-90$ degrees; Therefore (IV)