2.1 Discrete Random Variables

Terms:

- Random Variable
- Support
- Discrete Random Variable
- Probability Mass Function
- Cumulative Distribution Function
- Expected Value (Expectation)

Random Variables

Definition 2.1-1

Given a random experiment with an outcome space S, a function X that assigns one and only one real number $X(s) = x$ to each element s in S is called a **random variable**. The **space** of X is the set of real numbers $\{x : X(s) = x, s \in S\}$, where $s \in S$ means that the element s belongs to the set S.

A random variable associates a numerical value to each outcome of a random experiment.

Used to describe the elements of S numerically.

Random variables – Coin Example

A fair coin is flipped. The set of possible outcomes is Heads and Tails. $S = \{H, T\}$

Suppose we are interested in whether the outcome was Tails or not.

Let X be a real valued function defined on S such that $X(H) = 0$ and $X(T) = 1$.

- Domain of X: Outcome space, S
- Range of X: real numbers $\{x: x = 0,1\}$

X is a random variable, and the space of X is the set of numbers $\{0,1\}$.

Types of Random Variables

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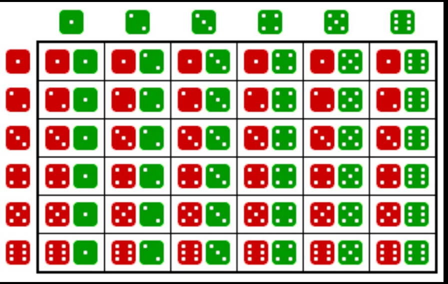
- A random variable is **discrete** if it has a countable number of values. (can be countably infinite)

连续

- A random variable is **continuous** if it is uncountable. (represents something on a continuous scale and can take any values in an interval)

Example: Defining a Random Variable

Consider the sample space for rolling two die: 36种可能

$S =$ 

Support ?

Say we are interested <u>only</u> in the <u>sum</u> of the number of spots.

We can define a random variable, X, by assigning a real number to each outcome in S.

$X = \{2, 3, 4 \dots 12\}$

X({}) = 2, X({}) = 3, ...
X({}) = 12

What is the space of X?

Example: Defining a Random Variable (continued)

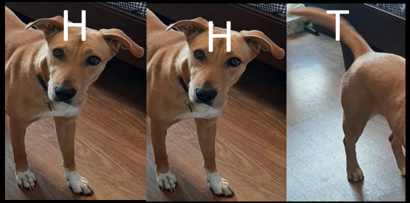
The space of X is $\mathcal{X} = \{2,3,4,5,6,7,8,9,10,11,12\}$

For convenience, we don't need the original sample space anymore. We can use the space of X instead.

Discrete random variable

If a sample space, S, contains a **countable** number of points, we call S a **discrete sample space**.

Any random variable X (with space, $\mathcal{X}$) arising from a discrete sample space, S, is called a **discrete random variable**.



Discrete random variable (examples)

- Number of "heads" in 3 flips of a coin

  S = {HHH, HHT, …, TTH ,TTT} $\rightarrow$ $\mathcal{X}$ = {0,1,2,3}

- Number of diaper changes I need to make per night:
  S = {0,1,2,…}

- Hulk start with 15mL saliva for Covid testing but accidentally spill random amount. Want to know **exactly** how much saliva have left in tube?!?! S = [0, 15)

- "Hulk start with 15mL saliva... **rounded to nearest mL**?"

  S = {0,1,2,…14,15}

Probability Mass Function

For a random variable X, the probability that the random variable takes a value, x, is denoted $P[X = x]$.

For a discrete random variable, this is also denoted by $f(x)$.

$f(x)$ is called the **probability mass function.**

Properties of a pmf

Definition 2.1-2

The pmf $f(x)$ of a discrete random variable X is a function that satisfies the following properties:

(a) $f(x) > 0, \qquad x \in S;$

(b) $\displaystyle\sum_{x \in S} f(x) = 1;$

(c) $\displaystyle P(X \in A) = \sum_{x \in A} f(x), \qquad \text{where } A \subset S.$

If a pmf satisfies all 3 properties, then it is a valid discrete probability distribution.

Probability mass function

x	$f(x)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$
$\vdots$	$\vdots$
x_n	$f(x_n)$

If the pmf is simple, it may be written as a table or list.

- Example: Flip a fair coin twice. let X be defined as the number of heads observed. What is the pmf of X?

Most of the time, it is written as a formula.

Examples:

$$f(x) = \frac{(|x|+1)^2}{9}, \qquad x = -1, 0, 1.$$

$$f(x) = \frac{5-x}{10}, \qquad x = 1, 2, 3, 4.$$

$$f(x) = \begin{cases} 0.9, & x = 0, \\ \dfrac{c}{x}, & x = 1, 2, 3, 4, 5, 6, \end{cases}$$

Cumulative Distribution Function

The function defined by

$$F(x) = P(X \leq x), -\infty < x < \infty$$

is called the **cumulative distribution function.**

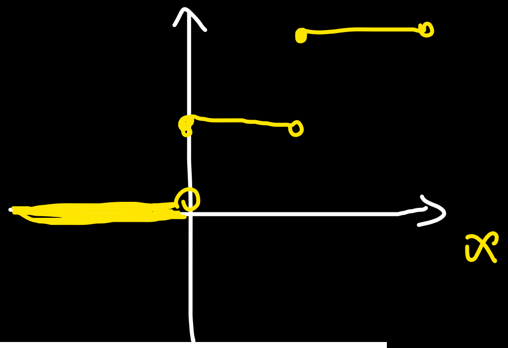
You may also see the cdf referred to as the **distribution function** of a random variable, X.

CDF examples

1. Flip a fair coin twice. let X be defined as the number of heads observed. What is the cdf of X?

$$F(x) = P[X \leq x]$$

$$F(3) = 1$$



2. Given the following pmf, what is F(x)?

$$f(x) = \frac{5-x}{10}, \qquad x = 1, 2, 3, 4.$$

$$F(1) = f(1)$$

$$F(2) = f(1) + f(2)$$

$$F(3) = f(1) + f(2) + f(3)$$

2.2 Expected Value

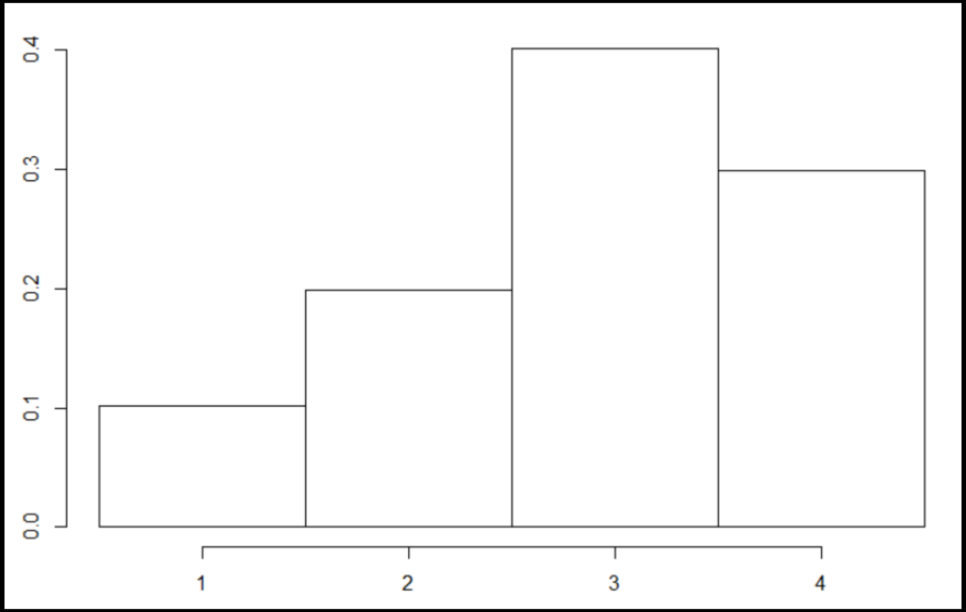
Expected Value of X (or 'expectation' of X)

Let X be a discrete random variable with probability mass function f(x). The **expected value** of X can be denoted $E[X]$, μ, or μ_X, and is given by:

$$\boxed{E[X] = \mu = \mu_X = \sum_{\{all\ x\}} x \cdot f(x)}$$

Expected Value of X

f(x)



x

x	f(x)	F(x)
1	0.1	
2	0.2	
3	0.4	
4	0.3	

E[X] =

Expectation of a function of X

Definition 2.2-1

If $f(x)$ is the pmf of the random variable X of the discrete type with space S, and if the summation

$$\sum_{x\in S} u(x)f(x), \qquad \text{which is sometimes written} \qquad \sum_{S} u(x)f(x),$$

exists, then the sum is called the **mathematical expectation** or the **expected value** of $u(X)$, and it is denoted by $E[u(X)]$. That is,

$$E[u(X)] = \sum_{x\in S} u(x)f(x).$$

Some properties of expectation

Theorem 2.2-1 When it exists, the mathematical expectation E satisfies the following properties:

(a) If c is a constant, then $E(c) = c$.

(b) If c is a constant and u is a function, then

$$E[c\,u(X)] = cE[u(X)].$$

(c) If c_1 and c_2 are constants and u_1 and u_2 are functions, then

$$E[c_1u_1(X) + c_2u_2(X)] = c_1E[u_1(X)] + c_2E[u_2(X)].$$

Examples:

(a) E[5] = 5, E[2] = 2

(b) E[2X] = 2E[X], $E[7X^2] = 7E[X^2]$

(c) E[2X + 5Y] = 2E[X] + 5E[Y] , $E[3X + 6X^2 – X^3] = 3E[X] + 6E[X^2] - E[X^3]$

Examples

X

1. Thanos has some flavored JUULs at home with the following proportion of flavors:

1/2 berry, 1/4 orange, 1/4 lemon.

If he picks up the following JUUL flavor, it will vaporize the following number of people:

Berry – 50, Orange – 30, Lemon - 40.

$X = \{30, 40, 50\}$

Let the random variable, X, represent the number of vaporized people if Thanos randomly selects a JUUL.

a) What is the pmf of X?

$E = \sum x \cdot f(x)$

x	$f(x)$
30	$\frac{1}{4}$
40	$\frac{1}{4}$
50	$\frac{1}{2}$

Vaping is bad for your health.

b) Does the expected value need to be an element of the sample space?

b) Let X be a random variable that represents the number of people vaporized if Thanos eats a random gummy. Find the expected value of X.

$$E[X] = (30)(1/4) + (40)(1/4) + (50)(1/2)$$

c) If Thanos selects a random JUUL, find the expected value of the number of eyeballs vaporized. $E[2X] = 2E[X]$

e) Find $E[X^2]$.

x	x^2	$f(x)$
30	900	$\frac{1}{4}$
40	1600	$\frac{1}{4}$
50	2500	$\frac{1}{2}$

$$E[X^2] = 900 \times \frac{1}{4} + 1600 \times \frac{1}{4} + 2500 \times \frac{1}{2}$$

Example 2

Suppose a discrete random variable X has the following probability distribution:

$$P(X=0) = 2-\sqrt{e}, \qquad P(X=k) = \frac{1}{2^k \cdot k!}, \quad k = 1, 2, 3, \ldots$$

a) Find E[X].

$$E[X] = 1 \cdot \frac{1}{2^1 \cdot 1!} + 2 \cdot \frac{1}{2^2 \cdot 2!} + 3 \cdot \frac{1}{2^3 \cdot 3!} + \cdots\cdots$$

知识点 $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ ☆

$$E(X) = \sum_{\text{all } x} x \cdot f(x) = 0 \cdot \left(2 - e^{1/2}\right) + \sum_{k=1}^{\infty} k \cdot \frac{1}{2^k \cdot k!} = \sum_{k=1}^{\infty} \frac{1}{2^k \cdot (k-1)!}$$

$$= \frac{1}{2} \cdot \sum_{k=1}^{\infty} \frac{1}{2^{k-1} \cdot (k-1)!} = \frac{1}{2} \cdot \sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!} = \frac{e^{1/2}}{2}.$$

Example

Suppose a discrete random variable X has the following probability distribution:

$$P(X=0) = 2-\sqrt{e}, \qquad P(X=k) = \frac{1}{2^k \cdot k!}, \quad k = 1, 2, 3, \ldots$$

b) Find E[X²].

$$= \sum_{x=1}^{\infty} x^2 \cdot \frac{1}{2^x \cdot x!} = \sum_{x=1}^{\infty} x^2 \frac{(\frac{1}{2})^x}{x!} = \sum_{x=1}^{\infty} \frac{(\frac{1}{2})^{x-1}}{(x-1)!} \cdot \frac{1}{2} \cdot (x-1+1) = \sum_{x=1}^{\infty} \left[\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{(\frac{1}{2})^{x-2}}{(x-2)!} + \frac{1}{2} \cdot \frac{(\frac{1}{2})^{x-1}}{(x-1)!} \right]$$

$$= \frac{1}{4} \cdot e^{\frac{1}{2}} + \frac{1}{2} \cdot e^{\frac{1}{2}} = \frac{3}{4}\sqrt{e}$$

$$E(X(X-1)) = \sum_{k=2}^{\infty} k \cdot (k-1) \cdot \frac{1}{2^k \cdot k!} = \sum_{k=2}^{\infty} \frac{1}{2^k \cdot (k-2)!}$$

$$= \frac{1}{4} \cdot \sum_{k=2}^{\infty} \frac{1}{2^{k-2} \cdot (k-2)!} = \frac{1}{4} \cdot \sum_{n=0}^{\infty} \frac{1}{2^n \cdot n!} = \frac{e^{1/2}}{4}.$$