Lec04: One-Dimensional, Steady-State Conduction without Thermal Energy Generation

Chapter Three

Section 3.2 – 3.4

Announcement

ZJUI

- HW2 due Monday after the Oct Holidays
  - HW will be up by Thurs

- Who is still missing in the Dingtalk grp?

Content and Quiz 04

ZJUI

1. What causes contact resistance?

2. Write down the equation for 1D heat diffusion for Cylindrical and Spherical coordinate system.

3. How does q vary with r in both the Cylindrical and Spherical coordinate system when heat is conducted along the r-direction?

4. What is the thermal resistance for both the Cylindrical and Spherical coordinate system?

5. What is the concept of critical thermal insulation thickness?

Recap

1. 3D heat diffusion equation in three coordinate systems:

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

2. Three boundary conditions:

   A. Constant temperature, $T(x,t) = T_s$

   B. Constant heat flux

   - Insulated, $\frac{dT}{dx}|_{x=x0} = 0$
   - Constant heat flux, $-k\frac{dT}{dx}|_{x=x0} = q“$

   C. Convective heat flux, $-k\frac{dT}{dx}|_{x=x0} = h[T_\infty - T(x,t)]$

Thermal Resistances $\left(R_t = \dfrac{\Delta T}{q}\right)$ and Thermal Circuits:

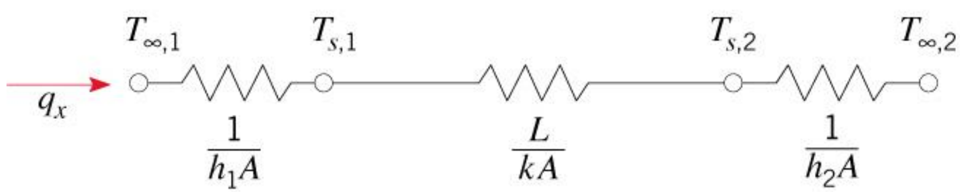
Conduction in a plane wall: $R_{t,\text{cond}} = \dfrac{L}{kA}$ (3.6)

Convection: $R_{t,\text{conv}} = \dfrac{1}{hA}$ (3.9)

$$R_{t,\text{rad}} = \frac{1}{h_r A} \qquad R''_{t,\text{rad}} = \frac{1}{h_r}$$

$$h_r = \varepsilon\sigma\left(T_s + T_{\text{sur}}\right)\left(T_s^2 + T_{\text{sur}}^2\right) \quad (1.9)$$

Thermal circuit for plane wall with adjoining fluids:



$$R_{\text{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A} \quad (3.12)$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{\text{tot}}} \quad (3.11)$$

Consider: Thermal resistance per unit surface area:

$$R''_{t,\mathrm{cond}} = \frac{L}{k} \qquad R''_{t,\mathrm{conv}} = \frac{1}{h}$$

Units: $R_t \leftrightarrow \mathrm{K/W}$

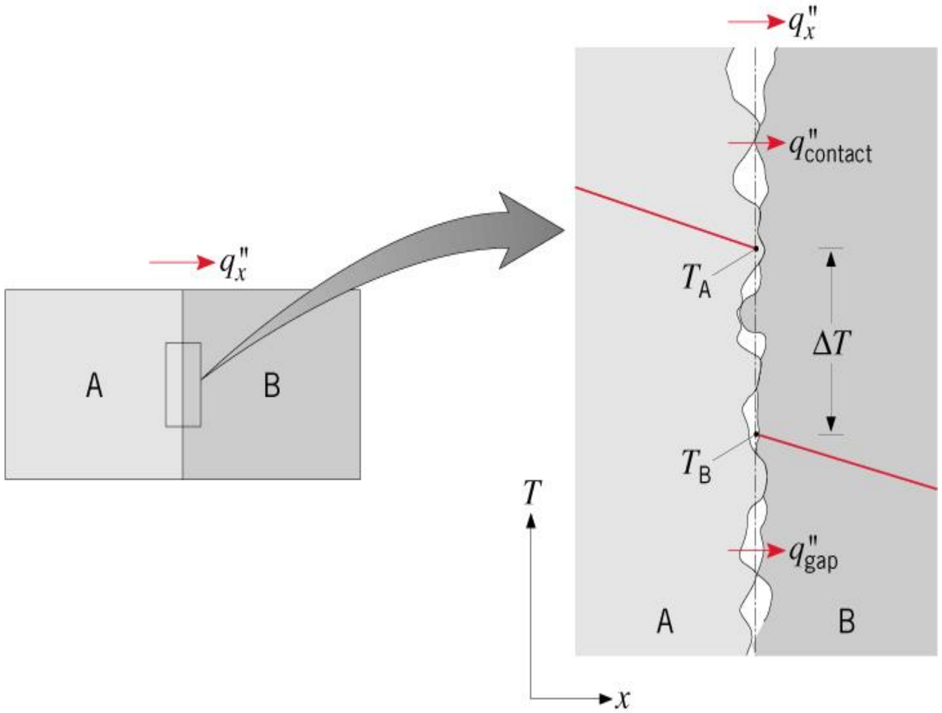
$R''_t \leftrightarrow \mathrm{m^2 \cdot K/W}$

Given that: $R_{t,\mathrm{cond}} = \dfrac{L}{kA}$

how to get $R''_{t,cond}$?

Contact Resistance

Contact Resistance:



$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$

$$R_{t,c} = \frac{R''_{t,c}}{A_c}$$

Units:

R_t: K/W

R''_t: $m^2 \cdot K/W$

The exact values depend on: Materials A and B, surface finishes, interstitial conditions, and contact pressure (Tables 3.1 and 3.2 of text)

Contact resistance

- Mainly due to surface roughness

- Contact spots with gaps. (Gaps filled with fluid can reduce resistance.)
  - Simply represent a rough surface as two parallel resistors, one for air gap, another for points of contact

- Contact resistance can be decreased by
  - Increasing the area of contacts. (How to do that?)
  - Contact resistance decreases if air gap is replaced by a higher *k* fluid/solid. What kind of solid can replace the fluid?

- How can you include this resistance into the composite wall analysis?

Example 1

"HW" company invented a top-secret high-performance chip. However, it is worried by the chip's heat transfer characteristics and decided to give you (a heat transfer expert) a 1 million USD contract to test if the chip will operate under the maximum allowed 85°C.

This thin silicon chip is attached on an 8-mm thick aluminum substrate using a 0.02-mm thick epoxy. The whole setup is insulated only on its sides, exposing the top and bottom. The chip and substrate are both 10 mm x 10 mm . Their exposed surfaces are cooled by air (T = 25 °C and h = 100 W/m^{2}K). This chip gives out a heat flux of 10^4 W/m^2 under normal operation. Will this chip overheat? Treat the epoxy joint as a thermal resistance, R_t'' = 0.9 x 10^{-4} m^{2}K/W.

Draw the thermal resistance circuit to solve the above problem.

Example 1

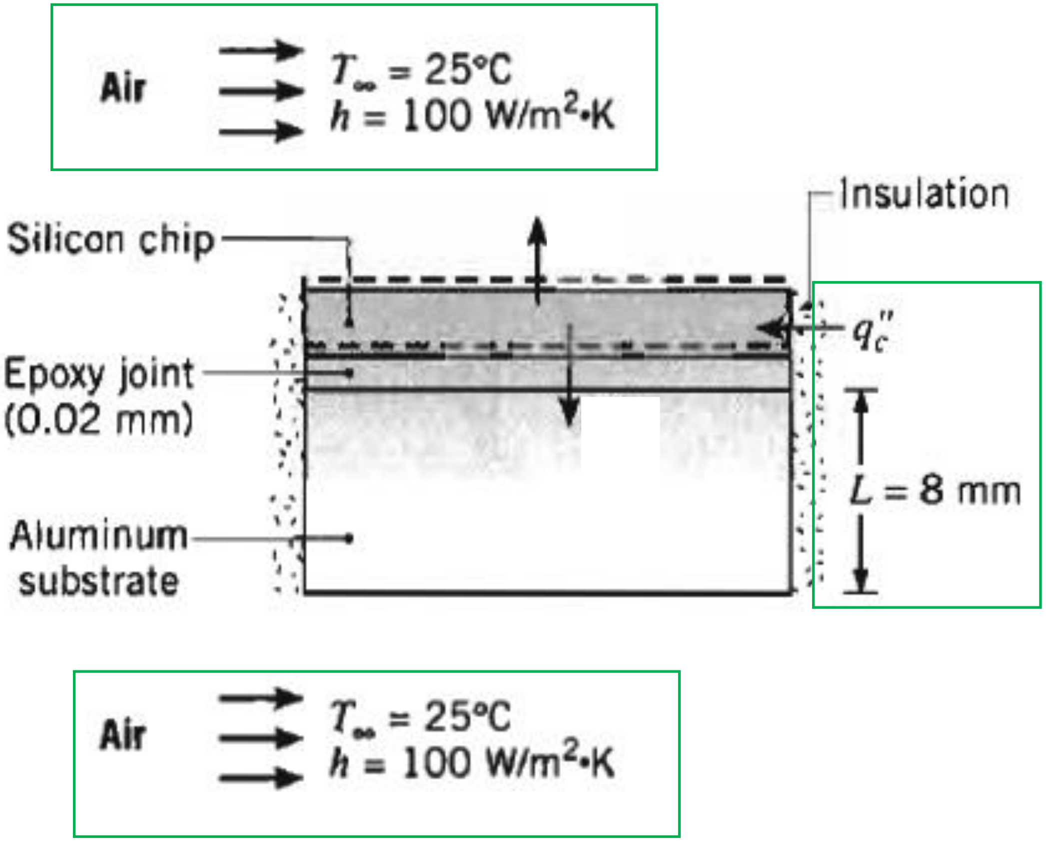
Known: Dimensions, heat dissipation, and maximum allowable temperature of a silicon chip. Thickness of aluminum substrate and epoxy joint. Convection conditions at exposed chip and substrate surfaces.

Find: Whether maximum allowable temperature is exceeded.

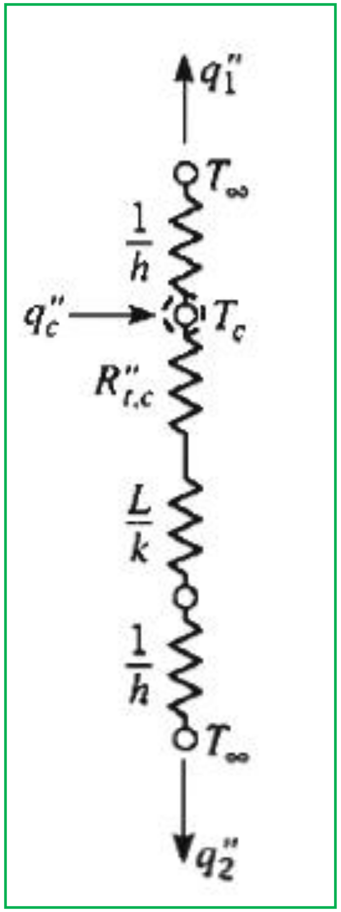
Assumptions:

1. Steady-state conditions.
2. One-dimensional conduction (negligible heat transfer from sides of composite).
3. Negligible chip thermal resistance (an isothermal chip).
4. Constant properties.
5. Negligible radiation exchange with surroundings.

Schematic:



Air
T_∞ = 25°C
h = 100 W/m²·K
Silicon chip
Epoxy joint
(0.02 mm)
Aluminum
substrate
Insulation
q_c''
L = 8 mm
Air
T_∞ = 25°C
h = 100 W/m²·K



q_1''
T_∞
$\frac{1}{h}$
q_c''
T_c
$R''_{t,c}$
$\frac{L}{k}$
$\frac{1}{h}$
T_∞
q_2''

Thermal resistance for 1D radial heat flow in Cylindrical Shell

ZJUI

1. **Specify appropriate form of the heat diffusion equation (or heat equation).**
2. **Solve for the temperature distribution.**
3. **Apply Fourier's law to determine the heat flux.**

Simplest Case: **One-Dimensional, Steady-State Conduction with No Thermal Energy Generation.**

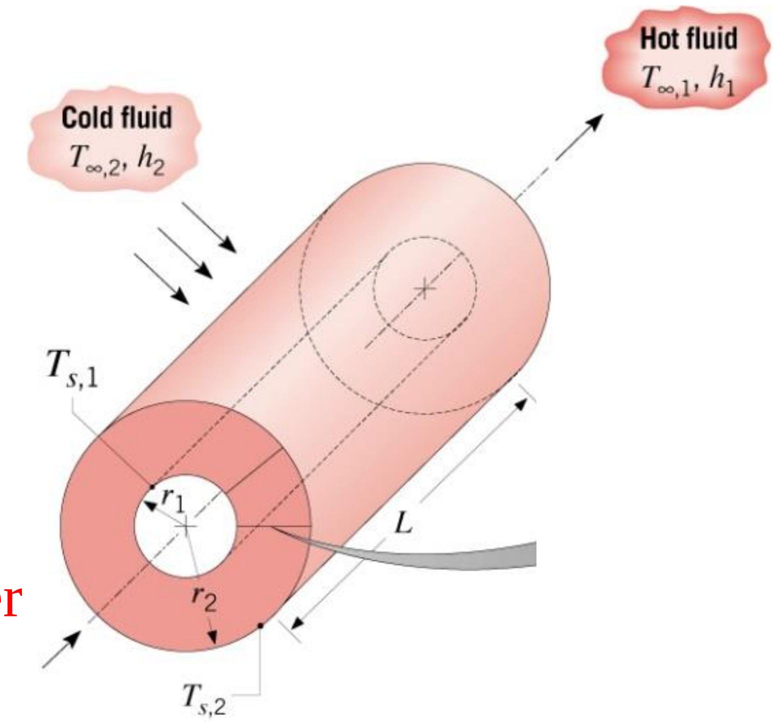
- Common Geometries:
  - The Plane Wall: Described in rectangular (x) coordinate. Area perpendicular to direction of heat transfer is constant (independent of x).
  - The Tube Wall: Radial conduction through tube wall.
  - The Spherical Shell: Radial conduction through shell wall.

Tube radial resistance:

Fourier Law:

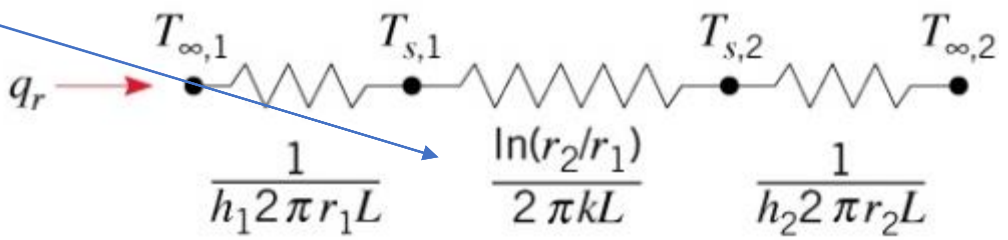
$$q_r = -k2\pi rL\frac{dT}{dr} \qquad (3.29)$$

$$\frac{q_r}{k2\pi rL}dr = -dT$$



Integrate and apply the BC at inner and outer surface:

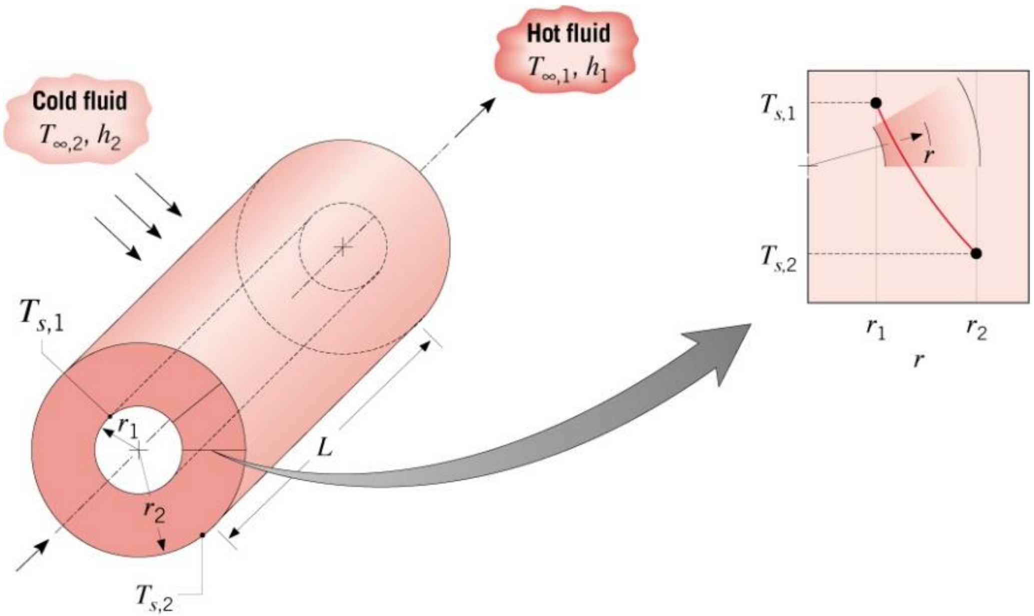
$$q_r\boxed{\frac{1}{k2\pi L}\ln\left(\frac{r_2}{r_1}\right)} = T_{s,1} - T_{s,2}$$



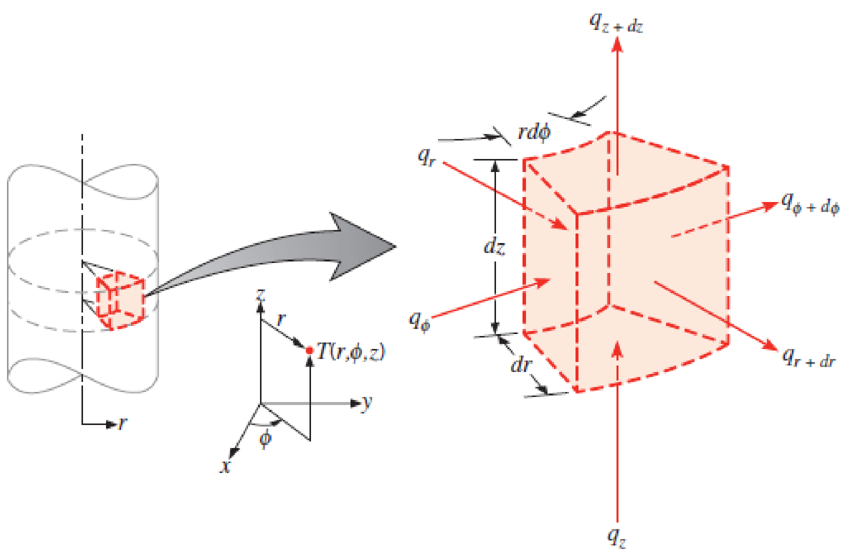
Heat Equation

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

$$\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right) + \dot{q} = \rho c_p \frac{\partial T}{\partial t}$$



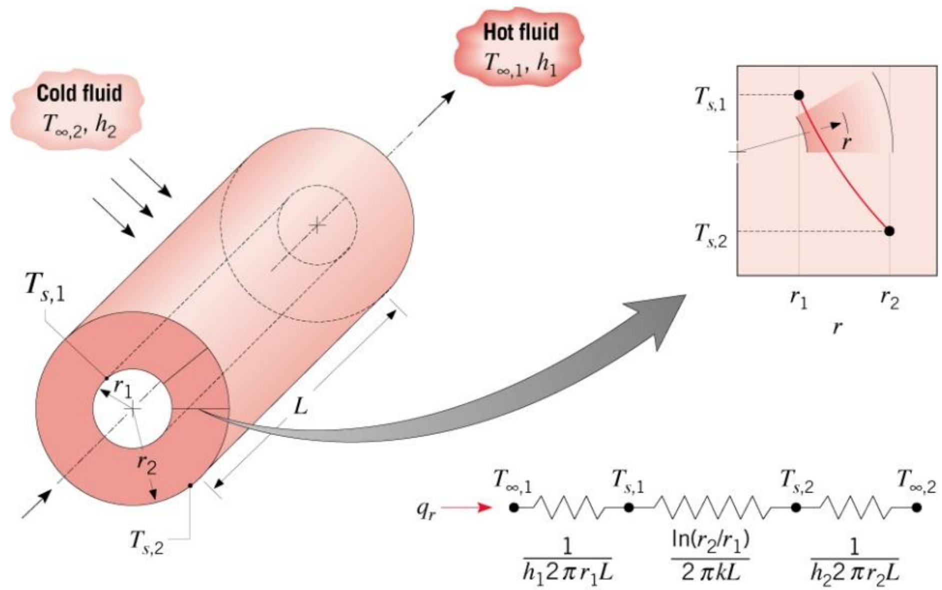
- Steady-State (SS) Heat transfer + no Heat Generation + HT in *r*-dir
- The above equation simplify to?



$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0 \qquad (3.28)$$

Heat Equation for cylindrical coordinates and constant k gives Temperature Distribution :

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0 \qquad (3.28)$$



$$\boxed{T(r)=\frac{T_{s,1}-T_{s,2}}{\ln(r_1/r_2)}\ln\left(\frac{r}{r_2}\right)+T_{s,2} \qquad (3.31)}$$

Thermal Resistance for cylindrical shell/tube

Differentiate and put (3.31) into Fourier Law for cylinder:

$$q_r=-k2\pi rL\frac{dT}{dr} \qquad (3.29)$$

$$q_r\boxed{\frac{1}{k2\pi L}\ln\left(\frac{r_2}{r_1}\right)}=T_{s,1}-T_{s,2}$$

- Heat Rate and Heat Flux:

$$q_r = 2\pi r L q_r'' = \frac{2\pi L k}{\ln(r_2/r_1)}\left(T_{s,1} - T_{s,2}\right)$$ [W] – heat rate (3.32)

$$q_r'' = -k\frac{dT}{dr} = \frac{k}{r\ln(r_2/r_1)}\left(T_{s,1} - T_{s,2}\right)$$ [W/m²] –per unit area (heat flux)

$$q_r' = 2\pi r q_r'' = \frac{2\pi k}{\ln(r_2/r_1)}\left(T_{s,1} - T_{s,2}\right)$$ [W/m] – heat per unit length

- Conduction Resistance:

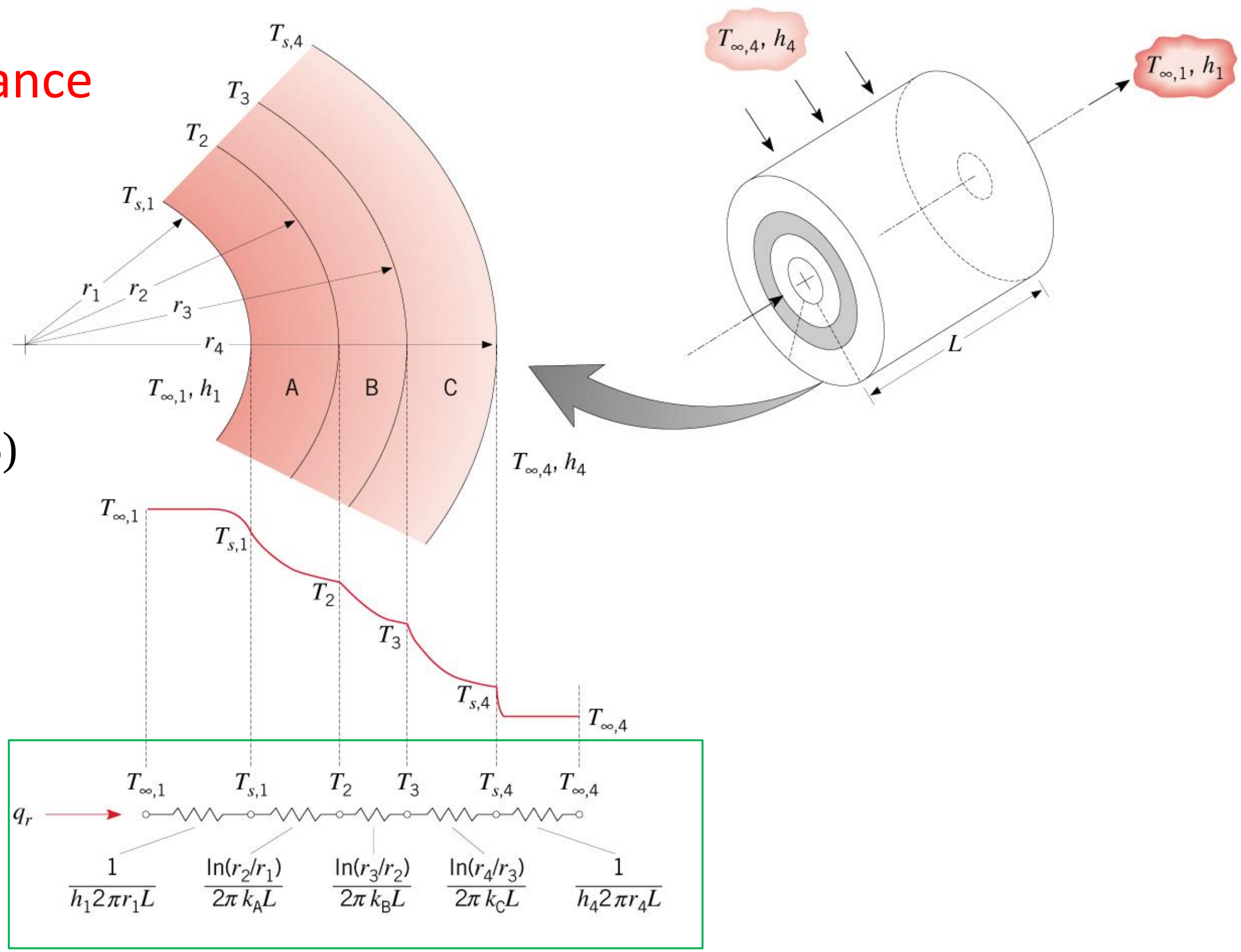
$$R_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi L k}$$ [K/W] (3.33)

$$R'_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi k}$$ $[\mathrm{m}\cdot\mathrm{K/W}]$

Composite Cylindrical Shell

Composite Wall with Negligible Contact Resistance

$$
q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{tot}}} = UA\left(T_{\infty,1} - T_{\infty,4}\right) \qquad (3.35)
$$



Note:

- R_{tot} *is a constant independent of radius*
- $R_{\text{tot}} = \frac{1}{UA}$
- Is U independent of radius?

  A changes with r, so U_i changes as well => $U_i = (A_i R_{\text{tot}})^{-1}$

Is thicker insulation better for insulating heat loss?

Example 1: Critical Insulation thickness

Problem:

A thin-walled copper tube of radius r_i is used to transport a low-temperature refrigerant and is at a temperature T_i that is less than that of the ambient air at T_∞ around the tube. Is there an optimum thickness associated with application of insulation to the tube?

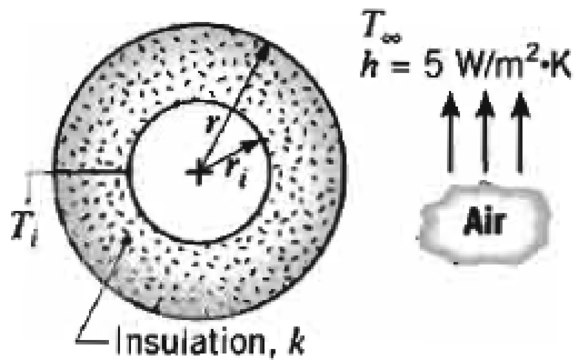
Known: Radius r_i and temperature T_i of a thin-walled copper tube to be insulated from the ambient air.

Find:

1. Whether there exists an optimum insulation thickness that minimizes the heat transfer rate.
2. Thermal resistance associated with using cellular glass insulation of varying thickness.

Example 1: Critical Insulation thickness

Schematic:



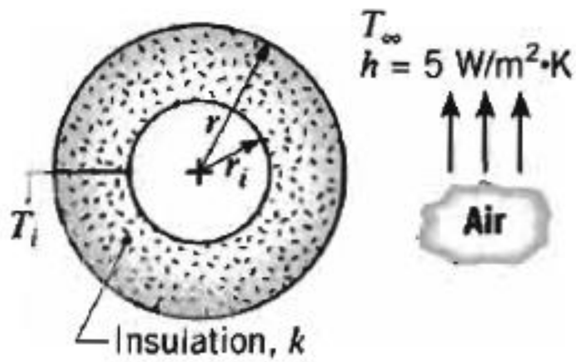
Assumptions:

1. Steady-state conditions.
2. One-dimensional heat transfer in the radial (cylindrical) direction.
3. Negligible tube wall thermal resistance.
4. Constant properties for insulation.
5. Negligible radiation exchange between insulation outer surface and surroundings.
6. **No Heat Generation**

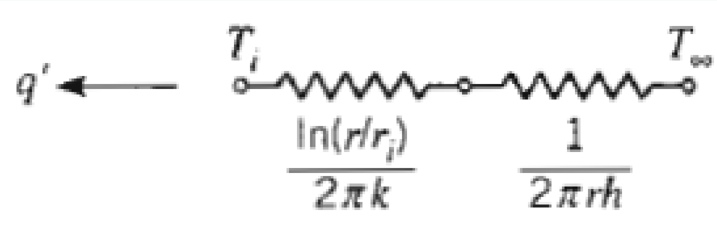
Solution (Method A – Conversation of energy):

$$\dot{E}_{in} - \dot{E}_{\text{out}} + \dot{E}_{\text{g}} = \dot{E}_{st}$$ **??? Not straightforward!!!**

Schematic:



Solution (Method 2 – thermal resistance):



q'' – heat rate per unit area (heat flux) [W/m^2]
q' – heat per unit length [W/m]
q – heat rate [W]

$$R'_{\mathrm{tot}} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h}$$

ZJUI

Solution:

$$R'_{tot} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h}$$

Why is there an optimal insulation thickness?

Convection resistance decreases with *r* !

$$R'_{tot} = \frac{\ln(r/r_i)}{2\pi k} + \frac{1}{2\pi r h}$$

Conduction resistance increases with *r*!

- Is there an optimal point? If yes, is it a minimum or maximum?
- How to find it?
  - We can dR/dr = 0 to find this point!

ZJUI

Solution:

$$\frac{dR'_{\text{tot}}}{dr} = 0$$

Hence

$$\frac{1}{2\pi kr} - \frac{1}{2\pi r^2 h} = 0$$

or

$$\boxed{r = \frac{k}{h}}$$

To determine whether the foregoing result maximizes or minimizes the total resistance, the second derivative must be evaluated. Hence

$$\frac{d^2R'_{\text{tot}}}{dr^2} = -\frac{1}{2\pi kr^2} + \frac{1}{\pi r^3 h}$$

or, at $r = k/h$,

$$\frac{d^2R'_{\text{tot}}}{dr^2} = \frac{1}{\pi(k/h)^2}\left(\frac{1}{k} - \frac{1}{2k}\right) = \frac{1}{2\pi k^3/h^2} > 0$$

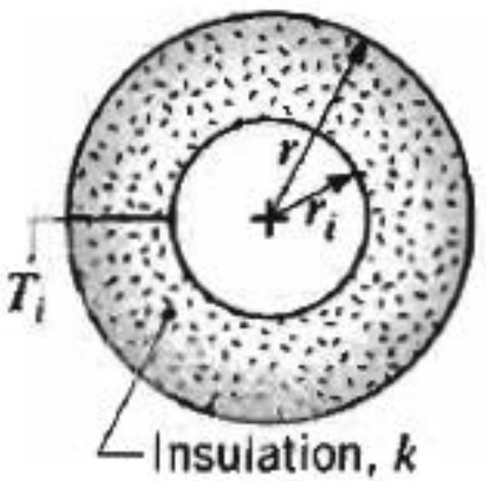
Example 1: Critical Insulation thickness

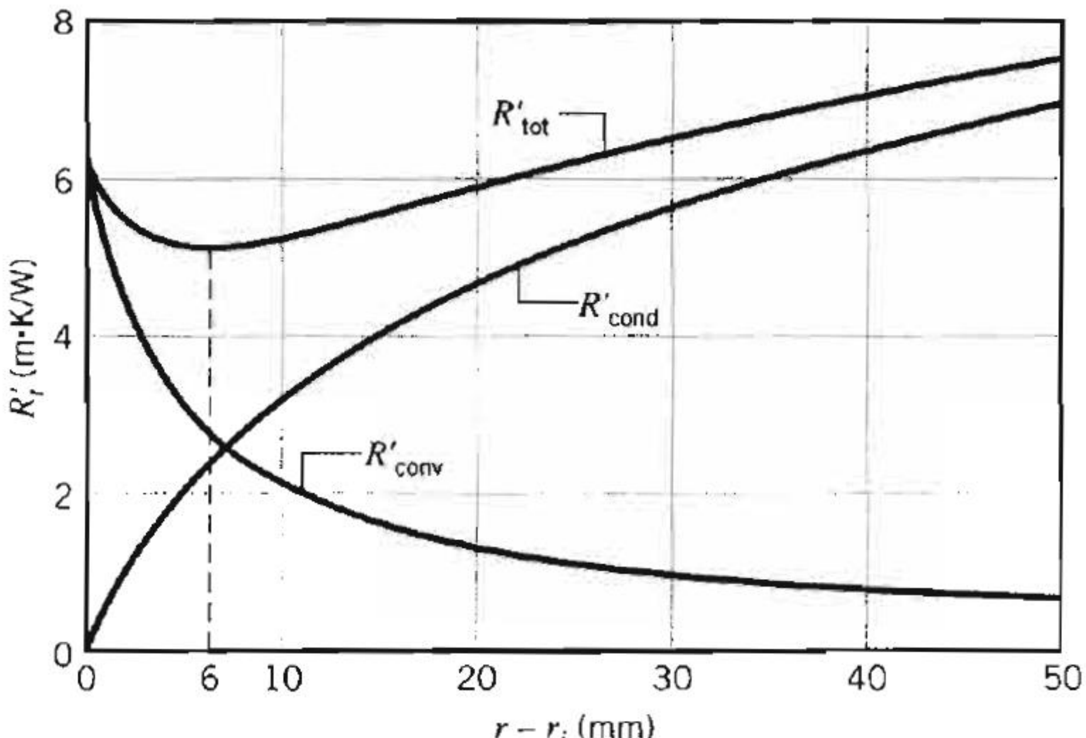
Solution:

From the above result it makes more sense to think in terms of a *critical insulation radius*

$$\boxed{r_{cr} \equiv \frac{k}{h}}$$

which maximizes heat transfer, that is, below which q' increases with increasing r and above which q' decreases with increasing r.

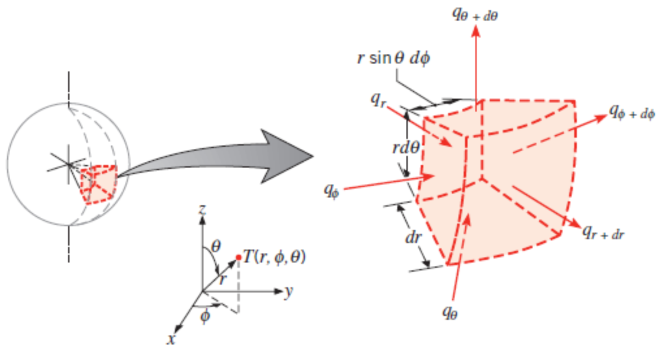


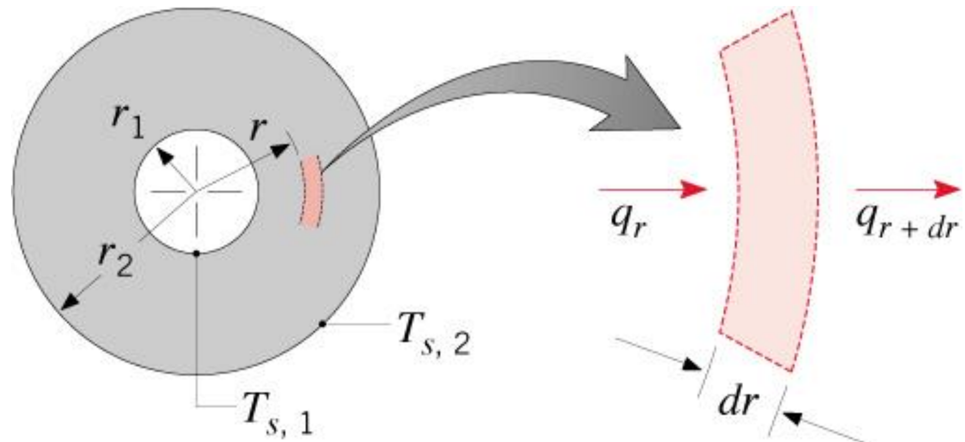


Thermal resistance for a 1D radial HT across a Spherical Shell

1D SS HT no Heat Gen : Sphere Shell

Consider only heat transfer in the r-direction:





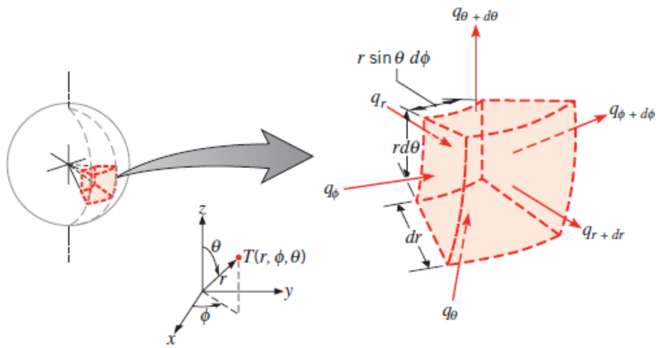
Fourier Law and Heat Equation :

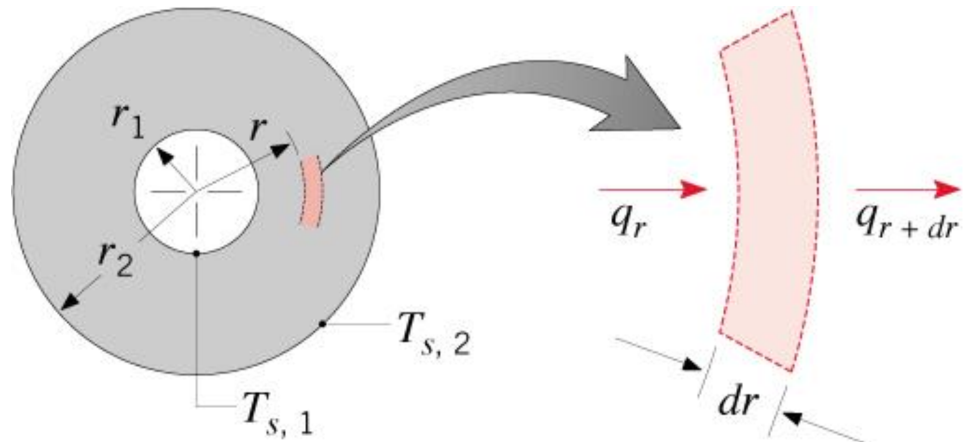
$$q_r = -k 4\pi r^2 \frac{dT}{dr} \qquad (3.38) \qquad \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dT}{dr}\right) = 0$$

What does the heat equation tell us about the variation of q_r with r ?

1D SS HT no Heat Gen : Sphere Shell

ZJUI





Heat Equation :

$$\boxed{\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0}$$

Temperature Distribution:

$$\boxed{T(r)=T_{s,1}-\left(T_{s,1}-T_{s,2}\right)\frac{1-\left(r_{1/}r\right)}{1-\left(r_1/r_2\right)}}$$

ZJUI

Heat Flux, Heat Rate and Thermal Resistance:

$$q_r'' = -k\frac{dT}{dr} = \frac{k}{r^2\left[(1/r_1)-(1/r_2)\right]}\left(T_{s,1}-T_{s,2}\right)$$

$$q_r = 4\pi r^2 q_r'' = \frac{4\pi k}{(1/r_1)-(1/r_2)}\left(T_{s,1}-T_{s,2}\right) \qquad (3.40)$$

$$R_{t,\text{cond}} = \frac{(1/r_1)-(1/r_2)}{4\pi k} \qquad (3.41)$$

Composite Shell:

$$q_r = \frac{\Delta T_{\text{overall}}}{R_{\text{tot}}} = UA\Delta T_{\text{overall}}$$

$$UA = R_{\text{tot}}^{-1} \leftrightarrow \text{Constant}$$

$$U_i = \left(A_i R_{\text{tot}}\right)^{-1} \leftrightarrow \text{ Depends on } A_i$$

Summary of thermal resistance

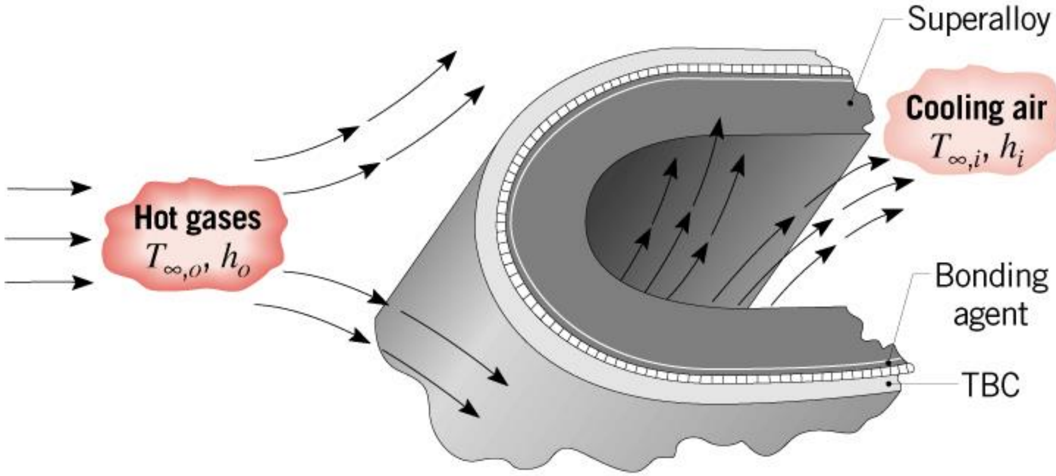
TABLE 3.3 One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	Cylindrical Wall[a]	Spherical Wall[a]
Heat equation	$\frac{d^2T}{dx^2}=0$	$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$	$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dT}{dr}\right)=0$
Temperature distribution	$T_{s,1}-\Delta T\frac{x}{L}$	$T_{s,2}+\Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1}-\Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\,\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\,\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\,\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\,\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\text{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$

[a]The critical radius of insulation is $r_{cr}=k/h$ for the cylinder and $r_{cr}=2k/h$ for the sphere.

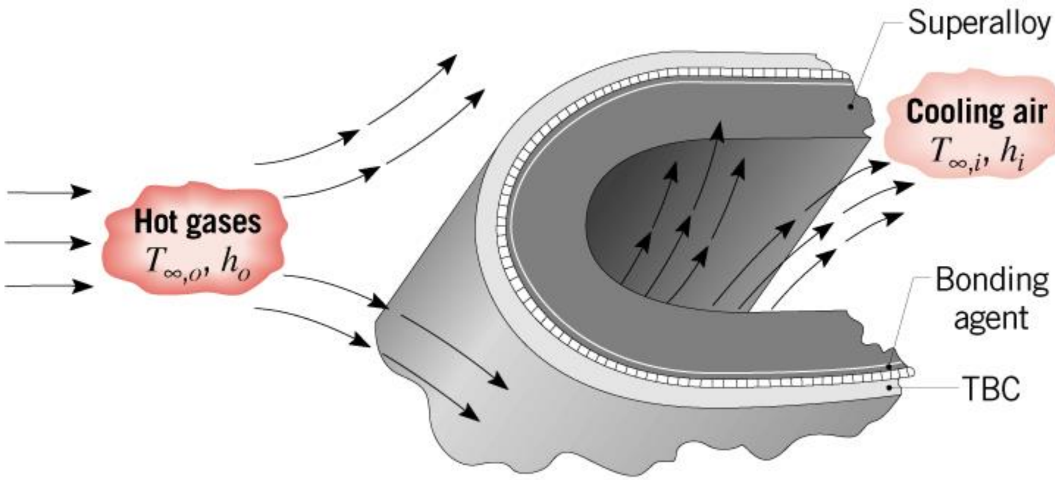
Example 2: TBC and contact resistance

Assessment of thermal barrier coating (TBC) for protection of turbine blades. Determine maximum blade temperature with and without TBC.

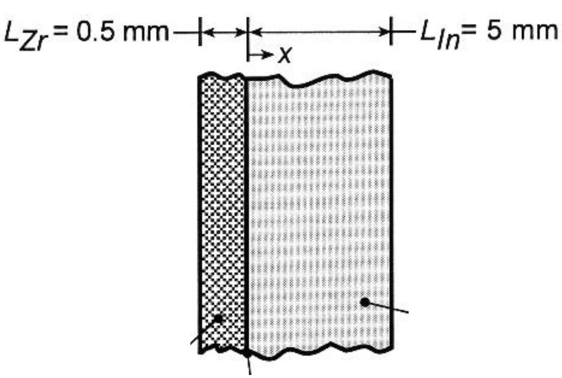


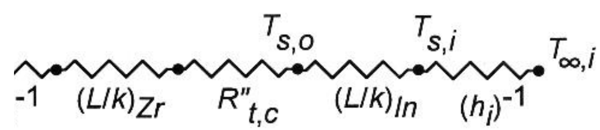
Draw a thermal resistance with and without the thermal contact resistance from the bonding agent between the TBC and blade. Is this thermal constant resistance bad?

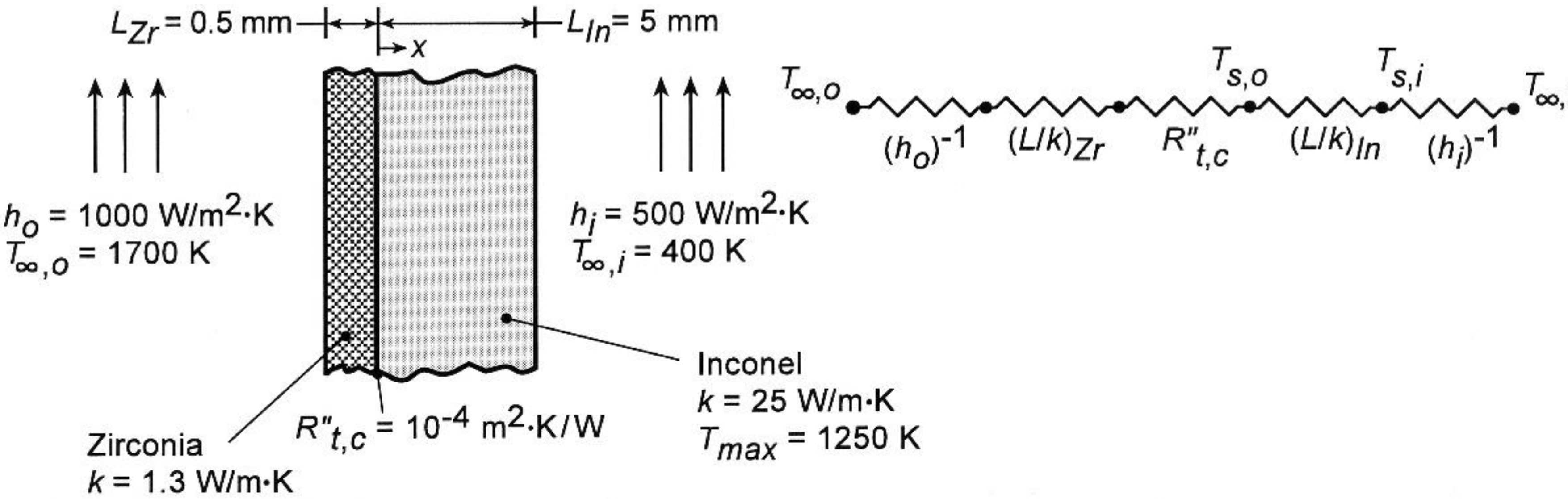
Example 2: TBC and contact resistance



ASSUMPTIONS: (1) One-dimensional, steady-state conduction in a composite plane wall, (2) Constant properties, (3) Negligible radiation.





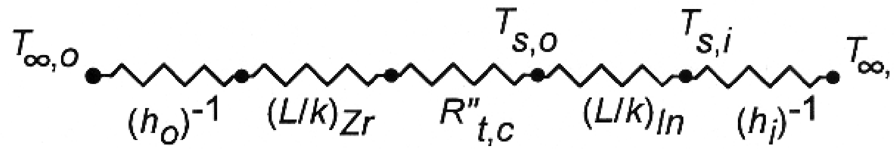


ANALYSIS: For a unit area, the total thermal resistance with the TBC is

$$R''_{tot,w} = h_o^{-1} + (L/k)_{Zr} + R''_{t,c} + (L/k)_{In} + h_i^{-1}$$

$$R''_{tot,w} = \left(10^{-3} + 3.85\times10^{-4} + 10^{-4} + 2\times10^{-4} + 2\times10^{-3}\right)\mathrm{m}^2\cdot\mathrm{K/W} = 3.69\times10^{-3}\ \mathrm{m}^2\cdot\mathrm{K/W}$$

Example 2: with Thermal Barrier Coating



With a heat flux of

$$q''_w = \frac{T_{\infty,o} - T_{\infty,i}}{R''_{tot,w}} = \frac{1300\,\mathrm{K}}{3.69\times10^{-3}\,\mathrm{m^2\cdot K/W}} = 3.52\times10^{5}\ \mathrm{W/m^2}$$

the inner and outer surface temperatures of the Inconel are

$$T_{s,i(\mathrm{w})} = \boxed{T_{\infty,i} + \left(\frac{q''_w}{h_i}\right) = 400K + \left(\frac{3.52\times10^5\,\frac{W}{m^2}}{500\,\frac{W}{m^2\cdot K}}\right) = 1104K}$$

$$T_{s,o(\mathrm{w})} = T_{\infty,i} + \left[\left(1/h_i\right) + \left(L/k\right)_{\mathrm{In}}\right] q''_w$$

$$= 400\,\mathrm{K} + \left(2\times10^{-3} + 2\times10^{-4}\right)\mathrm{m^2\cdot K/W}\left(3.52\times10^{5}\,\mathrm{W/m^2}\right) = 1174\,\mathrm{K}$$

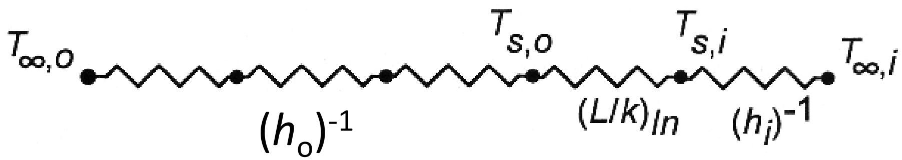
<

Example 2: without Thermal Barrier Coating

Without the TBC,

$$R''_{tot,wo} = h_o^{-1} + \left(L/k\right)_{In} + h_i^{-1} = 3.20 \times 10^{-3}\,\mathrm{m^2 \cdot K/W}$$

$$q''_{wo} = \frac{\left(T_{\infty,o} - T_{\infty,i}\right)}{R''_{tot,wo}} = 4.06 \times 10^5\ \mathrm{W/m^2}$$



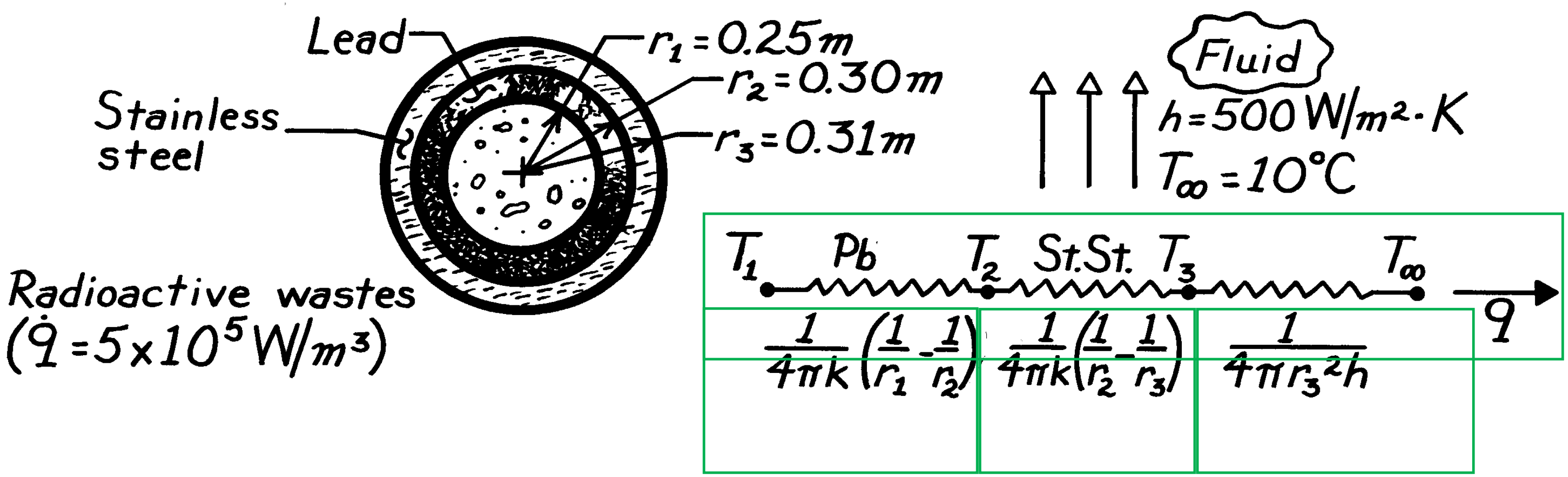
The inner and outer surface temperatures of the Inconel are then

$$T_{s,i(wo)} = \boxed{T_{\infty,i} + \left(q''_{wo}/h_i\right) = 1212\,\mathrm{K}}$$

$$T_{s,o(wo)} = \boxed{T_{\infty,i} + \left[\left(1/h_i\right) + \left(L/k\right)_{In}\right] q''_{wo} = 1293\,\mathrm{K}} \qquad <$$

Calculate the suitability of a composite spherical shell for storing radioactive wastes in oceanic waters from the thermal standpoint. (What do we want to test?) Draw a resistance circuit to help solve this question.

SCHEMATIC:



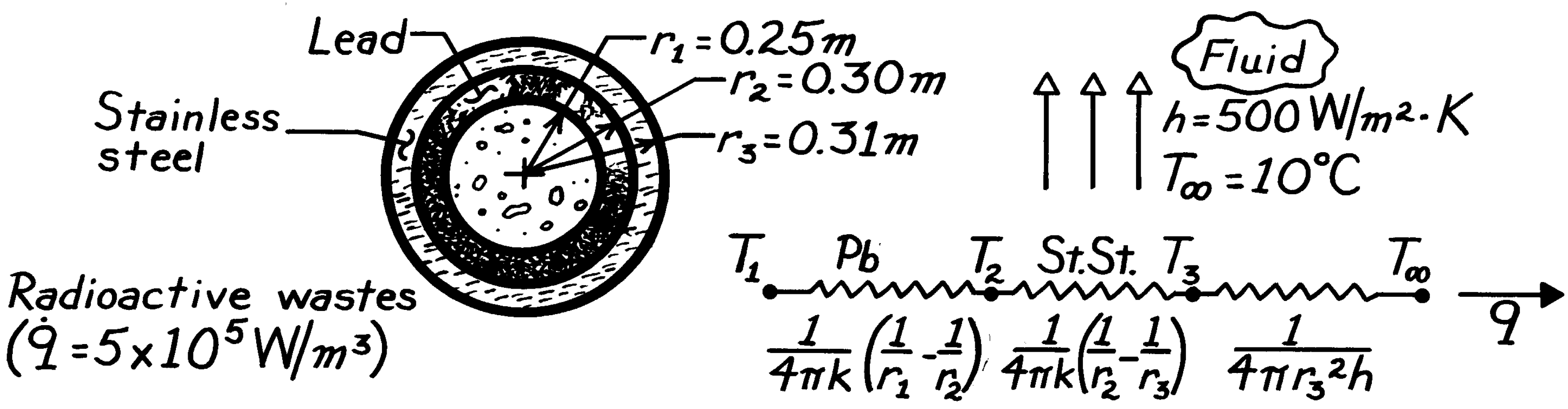
Example 3

ASSUMPTIONS: (1) One-dimensional conduction, (2) Steady-state conditions, (3) Constant properties at 300K, (4) Negligible contact resistance.

(5) Volumetric heat generation throughout the waste

PROPERTIES: Table A-1, Lead: k = 35.3 W/m·K, MP = 601 K; Stainless.Steel.: k = 15.1 W/m·K.

SCHEMATIC:



ANALYSIS: From the thermal circuit, it follows that

$$q = \frac{T_1 - T_\infty}{R_{tot}} = \dot{q}\left[\frac{4}{3}\pi\, r_1^3\right]$$

The thermal resistances are:

$$R_{\mathrm{Pb}} = \left[1/\left(4\pi \times 35.3\ \mathrm{W/m\cdot K}\right)\right]\left[\frac{1}{0.25\mathrm{m}} - \frac{1}{0.30\mathrm{m}}\right] = 0.00150\ \mathrm{K/W}$$

$$R_{\mathrm{St.St.}} = \left[1/\left(4\pi \times 15.1\ \mathrm{W/m\cdot K}\right)\right]\left[\frac{1}{0.30\mathrm{m}} - \frac{1}{0.31\mathrm{m}}\right] = 0.000567\ \mathrm{K/W}$$

$$R_{\mathrm{conv}} = \left[1/\left(4\pi \times 0.31^2\,\mathrm{m}^2 \times 500\ \mathrm{W/m^2\cdot K}\right)\right] = 0.00166\ \mathrm{K/W}$$

$$R_{\mathrm{tot}} = 0.00372\ \mathrm{K/W}$$

The heat rate is then

$$q = 5\times10^5\ \mathrm{W/m^3}\left(4\pi/3\right)\left(0.25\mathrm{m}\right)^3 = 32,725\ \mathrm{W}$$

and the inner surface temperature is

$$T_1 = T_\infty + R_{\mathrm{tot}}q = 283\ \mathrm{K} + 0.00372\ \mathrm{K/W}\left(32{,}725\ \mathrm{W}\right)$$

$$= 405\ \mathrm{K} < \mathrm{MP} = 601\ \mathrm{K} \qquad <$$

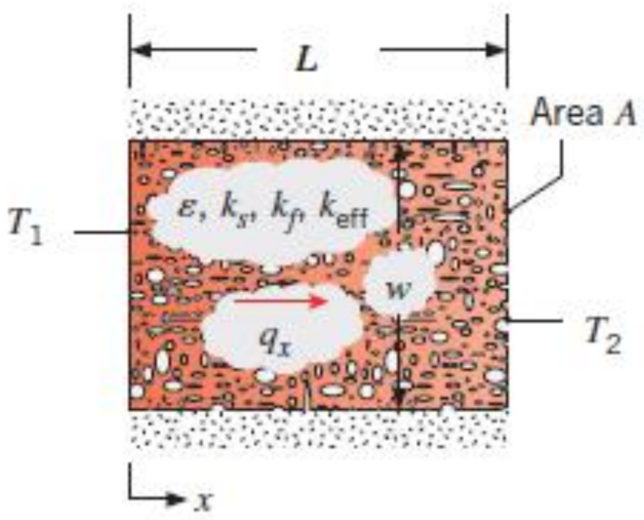
Hence, from the thermal standpoint, the proposal is adequate.

Multi-phase / Porous Media

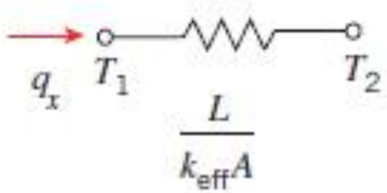
Optional

Media with holes!

- Saturated media consist of a solid phase (*s*) and a single fluid phase (*f*).
- Unsaturated media consist of solid, liquid, and gas phases.



Thermal resistor network

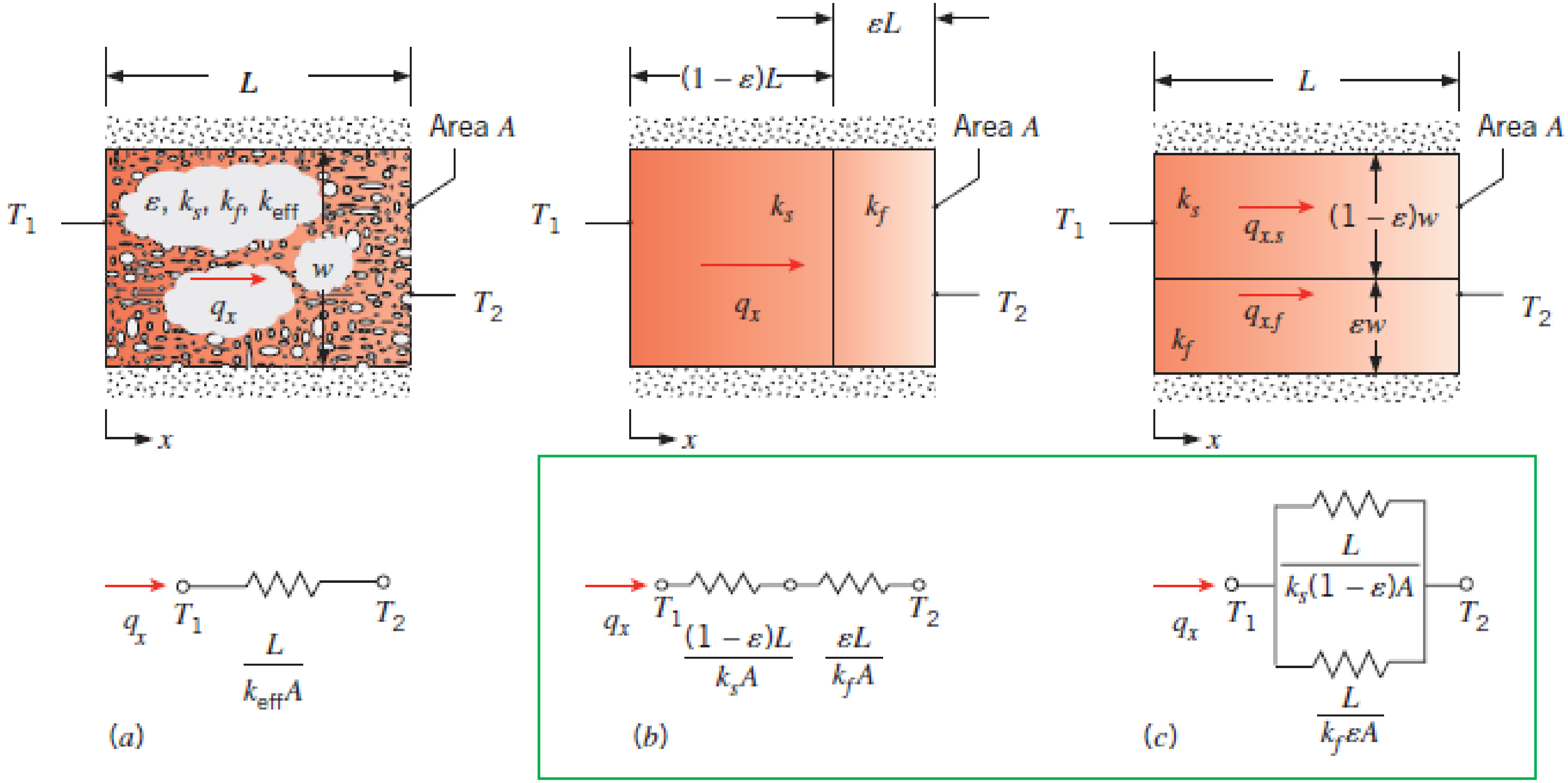


$$q_x = \frac{k_{eff}A}{L}(T_1 - T_2) \qquad (3.21)$$

The effective thermal conductivity (k_{eff}) of a saturated medium depends on:

- the k_s of solid (*s*) material
- its porosity ε
- its morphology (path/pattern)
- the k_f of interstitial fluid (*f*)

ZJUI



Estimate the upper and lower limits of k in this porous material:

- Fig b) treats the solid and gaps in series
- Fig c) treats them in parallel where the gaps occupy a certain fraction though the whole solid.