Bias and Variance of an Estimate

Mean Squared Error (MSE)

Method of Moments (MoM)

(6.4)

MLE Review

Let $X_1, X_2, \ldots, X_n$ be an iid sample from ~ Poisson(λ). Find an expression for the MLE of λ.

Bias

The **bias** of an estimator, $\hat{\theta}$, is defined as:

$$\boxed{Bias[\hat{\theta}] = E[\hat{\theta}] - \theta}$$

If the bias of an estimator equals 0, then it is an **unbiased estimator.**

i.e. if **E**$[\hat{\theta}]$ = θ **or** **E** $[\hat{p}]$ = p the estimator is unbiased.

Unbiased Estimation

If $\mathrm{E}[u(X_1, X_2, \ldots, X_n)] = \theta$,

Then $u(X_1, X_2, \ldots, X_n)$ is an **unbiased estimator** of θ.

Otherwise, $u(X_1, X_2, \ldots, X_n)$ is biased for θ.

Example of an Unbiased Estimator

MLE of p from a Bernoulli sample of size n:

$$\hat{p} = \frac{1}{n}\sum_i X_i$$

If an estimator is unbiased, $\mathrm{E}[u(X_1, X_2, \dots X_n)] = \theta$.

Here, $u(X_1, X_2, \dots, X_n)$ is $\hat{p}$.

If $X \sim Bern(p)$, E[X] = E[$\bar{X}$]=

Bernoulli example

$$E[\hat{p}] = E[\frac{1}{n}\sum X_i] = \frac{1}{n}E[\sum X_i]$$

$$= \frac{1}{n}E[X_1 + X_2 + \ \ldots + X_n]$$

$$= \frac{1}{n}(E[X_1] + E[X_2] + \ \ldots + E[X_n])$$

$$= \frac{1}{n}(\mathrm{p} + \mathrm{p} + \ldots + \mathrm{p}) = \mathrm{p}$$

A Simple Example of a Biased Estimator

Take random Bernoulli samples:

$X_1, X_2, \ldots, X_n \sim \text{Bern}(p)$, where p is unknown.

Instead of using $\bar{X}$ as my **estimator** for p, what if I don't care about likelihood and decide to blindly use $\hat{p}$ = ½ as my estimate? (Is that a good idea?)

What is the bias of this estimate?

Bias

In previous example, if $\hat{p}$ = ½

Bias = E[$\hat{p}$] $- p$ = ½ $- p \neq 0$

Let $X_1, X_2, \ldots, X_n \sim \mathrm{Bern}(p)$, where p is unknown.

What if we use $\hat{p} = X_1$ instead?

$E[\hat{p}] =$

$\mathrm{Var}[\hat{p}] =$

notes

Mean Squared Error (MSE)

$$\mathrm{MSE}(\hat{\theta}) = \mathrm{E}_\theta\left[(\hat{\theta} - \theta)^2\right].$$

$$\begin{aligned}
\mathrm{MSE}(\hat{\theta}) &= \mathbb{E}[(\hat{\theta} - \theta)^2] \\
&= \mathbb{E}(\hat{\theta}^2) + \mathbb{E}(\theta^2) - 2\theta\mathbb{E}(\hat{\theta}) \\
&= \mathrm{Var}(\hat{\theta}) + (\mathbb{E}\hat{\theta})^2 + \theta^2 - 2\theta\mathbb{E}(\hat{\theta}) \\
&= \mathrm{Var}(\hat{\theta}) + (\mathbb{E}\hat{\theta} - \theta)^2 \\
&= \mathrm{Var}(\hat{\theta}) + \mathrm{Bias}^2(\hat{\theta})
\end{aligned}$$

Mean Squared Error (MSE)

$$\mathrm{MSE}(\hat{\theta}) = \mathrm{E}_{\theta}\left[(\hat{\theta} - \theta)^2\right].$$

Alternative proof: Let X = $\hat{\theta} - \theta$.
E[X^2] = Var[X] + $(E[X])^2$.

$$\begin{aligned}\mathrm{MSE}(\hat{\theta}) &= \mathbb{E}[(\hat{\theta} - \theta)^2] \\ &= \mathrm{Var}(\hat{\theta} - \theta) + (\mathbb{E}[\hat{\theta} - \theta])^2 \\ &= \mathrm{Var}(\hat{\theta}) + \mathrm{Bias}^2(\hat{\theta})\end{aligned}$$

Moments (Review)

Given a random variable, X,

$E[X^k]$ is its k^{th} **raw moment** ("k^{th} **moment**").

$E[(X-\mu)^k]$ is its k^{th} **central moment.**

These are known as **theoretical** moments.

Sample Moment

The k^{th} **sample moment** is defined:

$$\frac{1}{n}\sum {X_i}^k$$

E.g. $\bar{X} = \frac{1}{n}\sum X_i$ is the 1^{st} sample moment.

Method of Moments

Method of Moments

- One of the oldest methods to obtaining parameter estimates.
- Starting with the first moment, set each sample moment equal to the corresponding theoretical moment:
  - *Set the second sample moment equal to the second theoretical moment. (If necessary)*
  - *Continue setting the third, fourth, etc. sample moments equal to the theoretical moments until the # of equations equals the # of parameters.*
- Solve for the parameters.
- For as many parameters are you are solving for, you will need to match that many moments.

MOM Steps (in Stat 400)

Step 1: Find E[X], the population mean.

- This will be a function of θ. We will call it g(θ)

Step 2: Set the population mean equal to the sample mean. $g(\theta) = E[X] = \bar{X}$.

Step 3: Solve for θ.

Step 4: Put a tilde over θ to signify that it is an estimator!

Note: you will only be expected to solve 1 parameter

Example

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = \frac{2(\theta - x)}{\theta^2} \qquad 0 < x < \theta \qquad \theta > 0.$$

a) Obtain the method of moments estimator of θ, $\tilde{\theta}$.

b) Is $\tilde{\theta}$ an unbiased estimator for θ?

c) Find $\mathrm{Var}(\tilde{\theta})$.

d) Find the MSE of $(\tilde{\theta})$.

notes

Find $\operatorname{Var}(\tilde{\theta})$.

Let $X_1, X_2, \ldots, X_n$ be a random sample from the distribution with probability density function

$$f(x) = \frac{2(\theta - x)}{\theta^2} \qquad 0 < x < \theta \qquad \theta > 0.$$

$$f(x;\beta,\delta) = \beta\,\delta\,x^{\delta-1}\,e^{-\beta x^{\delta}}, \qquad x > 0, \qquad \text{zero otherwise.}$$

$$\mathrm{E}(\mathrm{X}) = \frac{1}{\beta^{1/\delta}}\,\Gamma\left(\frac{1}{\delta}+1\right). \qquad \overline{\mathrm{X}} = \frac{1}{\tilde{\beta}^{1/\delta}}\,\Gamma\left(\frac{1}{\delta}+1\right).$$

$$\tilde{\beta} = \left(\frac{\Gamma\left(\frac{1}{\delta}+1\right)}{\overline{\mathrm{X}}}\right)^{\delta}.$$

Gamma example

Review: Conditional Distribution Examples