Homework 7

1. Consider the physical system represented by an ordinary differential equation
$$\ddot{x} + 5\dot{x} + 4x = u(t).$$
   (a) Find the state-space representation for this system.
   (b) Find the transfer function that represents this system. (assuming zero initial conditions)
   (c) Find the *impulse response* for this system; that is, determine the solution $x(t)$ when the input $u(t)$ is unit impulse function $\delta(t)$, and *zero initial conditions*, that is $x(0) = 0$ and $\dot{x}(0) = 0$.
   (d) Find the *step response* for this system; that is, determine the solution $x(t)$ when the input $u(t)$ is unit step function $u_s(t)$, with zero initial conditions.
   (e) Find the *free response* for this system when initial conditions are given by $x(0) = 2$ and $\dot{x}(0) = -5$.

(a). $z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases}$ $\Rightarrow \dot{z} = \begin{pmatrix} x_2 \\ u(t) - 5x_2 - 4x_1 \end{pmatrix} \Rightarrow \boxed{\dot{z} = \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix} \cdot z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot u(t)}$

(b). $s^2 X(s) + 5s X(s) + 4X(s) = U(s)$

$$X(s) = \frac{1}{(s+4)(s+1)} U(s) \Rightarrow \boxed{G(s) = \frac{1}{(s+1)(s+4)}}$$

(c). $X(s) = G(s) \cdot 1 = \frac{1}{3}\left[\frac{1}{s+1} - \frac{1}{s+4}\right] \Rightarrow \boxed{x(t) = \frac{1}{3}\left[e^{-t} - e^{-4t}\right] \cdot u_s(t)}$

(d). $X(s) = G(s) \cdot \frac{1}{s} = \frac{1}{3}\left[\frac{1}{s+1} \cdot \frac{1}{s} - \frac{1}{s} \cdot \frac{1}{s+4}\right] = \frac{1}{3}\left[\frac{1}{s} - \frac{1}{s+1} - \frac{1}{4}\left(\frac{1}{s} - \frac{1}{s+4}\right)\right]$

$$= \frac{1}{4} \cdot \frac{1}{s} - \frac{1}{3} \cdot \frac{1}{s+1} + \frac{1}{12} \cdot \frac{1}{s+4} = \boxed{\left(\frac{1}{4} - \frac{1}{3}e^{-t} + \frac{1}{12}e^{-4t}\right) \cdot u_s(t)}$$

(e). $\ddot{x} + 5\dot{x} + 4x = u(t)$, $x_0 = 2$, $\dot{x}_0 = -5$

$\mathcal{L}\{\dot{x}\} = sX(s) - 2$

$\mathcal{L}\{\ddot{x}\} = s\left[sX(s) - 2\right] - 5 = s^2 X(s) - 2s - 5$

$\therefore X(s) = \frac{1}{(s+4)(s+1)} \cdot U(s) + \frac{2s+5}{(s+1)(s+4)}$

$\because$ Free response

$\therefore X(s) = \frac{2s+5}{(s+1)(s+4)} \Rightarrow x(t) = \frac{1}{s+1} + \frac{1}{s+4} = \boxed{\left[e^{-t} + e^{-4t}\right] \cdot u_s(t)}$

2. Consider a linear time-invariant system whose step response is given by

$$x(t) = \left(e^{-3t}u_s(t)\right) \star \left(e^{-5t}u_s(t)\right) \star u_s(t).$$

(a) Find the transfer function that represents this system.
(b) Find the ordinary differential equation that represents this system. ODE
(c) Find the impulse response for this system.
(d) Find the step-response for this system.
(e) Find the state-space representation of this system.

a). $X(s) = \mathcal{L}\{e^{-3t}\cdot u_s(t)\}\cdot\mathcal{L}\{e^{-5t}\cdot u_s(t)\}\cdot\mathcal{L}\{u_s(t)\cdot 1\}$

$= \frac{1}{s+3}\cdot\frac{1}{s+5}\cdot\frac{1}{s} = \frac{1}{(s+3)(s+5)}\cdot U(s) \Rightarrow \boxed{G(s) = \frac{1}{(s+3)(s+5)}}$

b). $\because G(s) = \frac{1}{(s+3)(s+5)} \quad \therefore \ddot{x} + 8\dot{x} + 15x = u(t)$

c). $X(s) = G(s)\cdot 1 \Rightarrow \boxed{x(t) = \frac{1}{2}\left[e^{-3t} - e^{-5t}\right]\cdot u_s(t)}$

d). $X(s) = G(s)\cdot\frac{1}{s} = \frac{1}{2}\left[\frac{1}{s+3} - \frac{1}{s+5}\right]\cdot\frac{1}{s} = \frac{1}{6}\left[\frac{1}{s} - \frac{1}{s+3}\right] - \frac{1}{10}\left[\frac{1}{s} - \frac{1}{s+5}\right]$

$\therefore \boxed{x(t) = \left[\frac{1}{15} - \frac{1}{6}e^{-3t} + \frac{1}{10}e^{-5t}\right]\cdot u_s(t)}$

e). Let $z = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \quad \dot{z} = \begin{pmatrix} x_2 \\ -8x_2 - 15x_1 + u(t) \end{pmatrix} \Rightarrow \boxed{\dot{z} = \begin{pmatrix} 0 & 1 \\ -15 & -8 \end{pmatrix}\cdot z + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)}$

3. Transfer function of a system that relates input $u(t)$ to the output $y(t)$ is given by $G(s) = \frac{X(s)}{U(s)} = \frac{1}{s+14}$. This system is given an input $u(t) = 2u_s(t)$. If $x(0) = 0$, what is $\lim_{t\to\infty} x(t)$? Does the final value theorem apply to this problem? Why or why not?

$$\left.\begin{aligned} &x(0) = 0 \\ &X(s) = \frac{1}{s+14}\cdot U(s) \\ &u(t) = 2u_s(t) \Rightarrow U(s) = \frac{2}{s} \end{aligned}\right\} \Rightarrow X(s) = \frac{2}{s(s+14)} = \frac{1}{7}\left[\frac{1}{s} - \frac{1}{s+14}\right]$$

$$\Rightarrow x(t) = \frac{1}{7}\left[1 - e^{-14t}\right] u_s(t)$$

$$\Rightarrow \lim_{t\to\infty} x(t) = \boxed{\frac{1}{7}}$$

According to Final value Theorem:

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} s\cdot X(s) \quad \text{if} \quad \lim_{s\to\infty} sF(s) \text{ exsits}$$

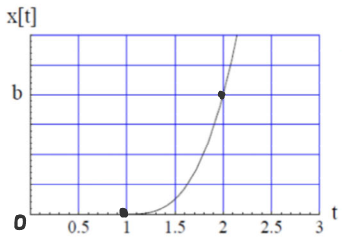
$$\because \lim_{s\to\infty} s\cdot F(s) = \lim_{s\to\infty} \frac{1}{s+14}\cdot\frac{2}{s}\cdot s = \lim_{s\to\infty} \frac{2}{s+14} = 0 \quad \text{which exsits}$$

$$\therefore \lim_{t\to\infty} x(t) = \lim_{s\to 0} sF(s) = \frac{2}{14} = \frac{1}{7}$$

4. A system is described by

$$a\frac{d^3x(t)}{dt^3} = u(t).$$

This system is initially at rest with $x(0) = 0$ is given an input $u(t) = \delta(t-1)$. The corresponding output $x(t)$ looks as shown in the adjoining figure. Find the constant a in terms of the constant b. ('b' is in the graph)



$$a\cdot x^{(3)}(t) = \delta(t-1)$$

$$a\cdot x^{(3)}(t) = \delta(t-1)\cdot u_s(t-1)$$

$$\because a\cdot x^{(3)}(t) = u(t)$$

$$\therefore a\left[s^3\cdot X(s) - s^2 x_0 - s\dot{x}_0 - \ddot{x}_0\right] = U(s)$$

$$a\left[s^3X(s) - s\dot{x}_0 - \ddot{x}_0\right] = U(s)$$

$\because$ system is at rest from $[0,1]$

$$\therefore X(s) = \frac{1}{a}\cdot\frac{1}{s^3}\cdot U(s)$$

$$U(s) = \mathcal{L}\{\delta(t-1)\} = \mathcal{L}\{\delta(t-1)\cdot u_s(t-1)\} = e^{-t}\cdot 1$$

$$\therefore X(s) = \frac{1}{2a}\cdot e^{-1}\cdot\frac{2}{s^3} \Rightarrow x(t) = (t-1)^2\cdot u_s(t-1)\cdot\frac{1}{2a}$$

From graph we see $x(2) = b$

$$\therefore \frac{1}{2a} = b \Rightarrow \boxed{a = \frac{1}{2b}}$$

5. Consider a linear time-invariant system whose state-space representation is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

Let $x(t) = z_1(t)$ be the output signal of interest.

(a) Find the transfer function $X(s)/U(s)$ that represents this system. (assuming zero initial conditions)

(b) Find an ordinary differential equation in terms of the signal $x(t)$ that represents this system.

(c) Find the impulse response for this system.

(d) Find the step-response for this system.

a). $\because \ \dot{z} = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \cdot z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$

$\therefore \ \ddot{x} + 3\dot{x} + 2x = u(t)$

$\therefore \ \boxed{G(s) = \dfrac{1}{(s+1)(s+2)}}$

b). $\boxed{\ddot{x}(t) + 3\dot{x}(t) + 2x(t) = u(t)}$

c). $X(s) = G(s) \cdot 1 = \frac{1}{s+1} - \frac{1}{s+2} \Rightarrow \boxed{x(t) = e^{-t} - e^{-2t}}$

d). $X(s) = G(s) \cdot \frac{1}{s} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{2}\left[\frac{1}{s} - \frac{1}{s+2}\right] \Rightarrow \boxed{x(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}}$