Laboratory handout 1 – Mathematical preliminaries

In this handout, an expression on the left of the symbol := is defined in terms of the expression on the right. In contrast, an expression on the left of the symbol = is either equal to the expression on the right as a result of a computation, or is assumed to equal the expression on the right, provided that certain variables are chosen appropriately. Thus,

$$(a+b)(a-b) = a^2 - b^2 \tag{1}$$

since the right-hand side is obtained by computation applied to the left-hand side. In contrast,

$$p(u) := (a+u)(b-u) \tag{2}$$

defines the quadratic polynomial function p in terms of the coefficients a and b. In this case,

$$p(u) = 0 \tag{3}$$

is an equation in the variable u.

In this handout, **bold-faced** words denote terminology that should be remembered and understood in terms of the associated definition in words or mathematical notation. Please use the margin for personal notes, examples, diagrams, or questions.

In this handout, MATLAB commands are preceded by >> to indicate that they should be entered on the command-line in the MATLAB command window. When commands are broken across multiple lines, an ellipsis `...` must terminate each intermediate line. Displayed output is suppressed if a command ends with a semicolon. MATLAB commands or input is distinguished from the surrounding text by a `special font`. Help with a MATLAB command may be obtained by typing `doc` followed by the command name on the command line. All examples should be executed and the observations verified.

In this handout, numerical examples are chosen to result in simple arithmetic with rational numbers. This contrasts with real applications, where decimal numbers are the rule. A less purposeful choice of numerical examples will invariable require the use of a calculator.

A **complex number** is a pair of real numbers α and β arranged in the **rectangular form** $\alpha + \mathrm{j}\beta$, in terms of the **imaginary unit** j. We refer to α as the **real part** and β as the **imaginary part**, and write

$$\Re(\alpha + \mathrm{j}\beta) := \alpha,\ \Im(\alpha + \mathrm{j}\beta) := \beta. \tag{4}$$

A complex number is **purely real** if its imaginary part is zero, and **purely imaginary** if its real part is zero. A complex number equals zero if the real and imaginary parts both equal zero.

Complex numbers may be added by adding the real and imaginary parts separately, e.g.,

$$(\alpha + \mathrm{j}\beta) + (\gamma + \mathrm{j}\delta) := (\alpha + \gamma) + \mathrm{j}(\beta + \delta). \tag{5}$$

The order of addition does not matter. We write

$$(\alpha + \mathrm{j}\beta) - (\gamma + \mathrm{j}\delta) := (\alpha + \mathrm{j}\beta) + \big((-\gamma) + \mathrm{j}(-\delta)\big). \tag{6}$$

In particular, $\alpha - \mathrm{j}\beta = \alpha + \mathrm{j}(-\beta)$ is the **complex conjugate** $\overline{\alpha + \mathrm{j}\beta}$. The sum of a complex number and its conjugate equals twice its real part, since

$$(\alpha + \mathrm{j}\beta) + (\alpha - \mathrm{j}\beta) = 2\alpha. \tag{7}$$

The difference between a complex number and its conjugate equals j times twice its imaginary part, since

$$(\alpha + \mathrm{j}\beta) - (\alpha - \mathrm{j}\beta) = \mathrm{j}2\beta. \tag{8}$$

Complex numbers may be expressed in **polar form** $\rho e^{\mathrm{j}\theta}$ with $\rho \geq 0$, in terms of the **complex exponential function**

$$e^{r+\mathrm{j}\theta} := e^{r}(\cos\theta + \mathrm{j}\sin\theta). \tag{9}$$

We refer to ρ as the **magnitude** and, provided that $\rho > 0$, to θ as the **phase**, and write

$$\left|\rho e^{\mathrm{j}\theta}\right| := \rho,\ \arg\left(\rho e^{\mathrm{j}\theta}\right) := \theta. \tag{10}$$

The corresponding real and imaginary parts equal $\rho\cos\theta$ and $\rho\sin\theta$. It follows that the complex conjugate $\overline{\rho e^{\mathrm{j}\theta}} = \rho e^{\mathrm{j}(-\theta)}$ and that the imaginary unit has magnitude 1 and phase $\pi/2$.

Complex numbers may be multiplied by multiplying the magnitudes and adding the phases, e.g.,

$$\rho_1 e^{\mathrm{j}\theta_1} \cdot \rho_2 e^{\mathrm{j}\theta_2} := \rho_1\rho_2 e^{\mathrm{j}(\theta_1+\theta_2)}. \tag{11}$$

Provided that $\rho_2 > 0$, we write

$$\rho_1 e^{\mathrm{j}\theta_1} \div \rho_2 e^{\mathrm{j}\theta_2} := \rho_1 e^{\mathrm{j}\theta_1} \cdot \frac{1}{\rho_2} e^{\mathrm{j}(-\theta_2)}. \tag{12}$$

In particular, $1 \div \rho e^{\mathrm{j}\theta} := \frac{1}{\rho} e^{\mathrm{j}(-\theta)}$ is the **inverse** $\left(\rho e^{\mathrm{j}\theta}\right)^{-1}$ provided that $\rho > 0$. The product of a nonzero complex number with its inverse equals 1, since

$$\rho e^{\mathrm{j}\theta} \cdot \frac{1}{\rho} e^{\mathrm{j}(-\theta)} = 1. \tag{13}$$

The product of a complex number with its conjugate equals the square of its magnitude, since

$$\rho e^{\mathrm{j}\theta} \cdot \rho e^{\mathrm{j}(-\theta)} = \rho^2. \tag{14}$$

The product of the imaginary unit j with itself equals $e^{\mathrm{j}\pi} = -1$.

Complex numbers may be represented graphically as points in a two-dimensional coordinate system, known as the **complex plane**, in terms of rectangular coordinates (α, β) given by the real and imaginary parts, or in terms of polar coordinates (ρ, θ) given by the magnitude and phase. It follows that $\rho = \sqrt{\alpha^2 + \beta^2}$ and $\theta \in (-\pi, \pi]$ satisfies

$$\cos\theta = \frac{\alpha}{\sqrt{\alpha^2 + \beta^2}}, \sin\theta = \frac{\beta}{\sqrt{\alpha^2 + \beta^2}}. \tag{15}$$

In MATLAB, the command

```
>> a = complex(1,2)
```

assigns the complex number $1 + j2$ to the variable a. Its real and imaginary parts are returned by the commands

```
>> real(a)
>> imag(a)
```

Its magnitude and phase (in the interval $(-\pi, \pi]$) are returned by the commands

```
>> abs(a)
>> angle(a)
```

The command

```
>> conj(a)
```

returns the complex conjugate $1 - j2$.

Complex numbers may be added, subtracted, multiplied, and divided in MATLAB by using the `+`, `-`, `*`, and `/` binary operators. Complex exponentials may be computed using the `exp` operator. As a special case, the command

```
>> abs(a)*exp(complex(0,1)*angle(a))
```

returns the content of a (within round-off error).

If $P(s)$ is a polynomial of degree m in the variable s, then there exist m (not necessarily distinct) complex numbers $\sigma_1, \ldots, \sigma_m$, such that

$$P(s) = (s - \sigma_1)(s - \sigma_2) \cdots (s - \sigma_m) \tag{16}$$

The numbers $\sigma_1, \ldots, \sigma_m$ are the **roots** of the polynomial. If there are only $n \leq m$ distinct roots $\sigma_1, \ldots, \sigma_n$, then

$$P(s) = (s - \sigma_1)^{k_1}(s - \sigma_2)^{k_2} \cdots (s - \sigma_n)^{k_n}, \tag{17}$$

where the **multiplicities** $k_1, \ldots, k_n$ satisfy $k_1 + k_2 + \cdots + k_n = m$. In this case, for every polynomial $Q(s)$ of degree $< m$, there exist constants $a_{1,1}, a_{1,2}, \ldots, a_{1,k_1}, a_{2,1}, a_{2,2}, \ldots, a_{2,k_2}, \ldots\ldots, a_{n,1}, a_{n,2}, \ldots, a_{n,k_n}$, such that

$$\begin{aligned} \frac{Q(s)}{P(s)} &= \frac{a_{1,1}}{s-\sigma_1} + \frac{a_{1,2}}{(s-\sigma_1)^2} + \ldots + \frac{a_{1,k_1}}{(s-\sigma_1)^{k_1}} \\ &\quad + \frac{a_{2,1}}{s-\sigma_2} + \frac{a_{2,2}}{(s-\sigma_2)^2} + \ldots + \frac{a_{2,k_2}}{(s-\sigma_1)^{k_2}} \\ &\qquad\qquad \vdots \\ &\quad + \frac{a_{n,1}}{s-\sigma_n} + \frac{a_{n,2}}{(s-\sigma_n)^2} + \ldots + \frac{a_{n,k_n}}{(s-\sigma_n)^{k_n}} \end{aligned} \tag{18}$$

In MATLAB, the command

```
>> [a,s] = residue([1, 0], [1, 2, 1])
```

returns the coefficients $a_{1,1} = 1$ and $a_{1,2} = -1$ and the root $\sigma_1 = -1$ (with multiplicity) corresponding to the ratio of the two polynomials $P(s) := (s+1)^2 = s^2 + 2s + 1$ and $Q(s) := s$, i.e.,

$$\frac{s}{s^2 + 2s + 1} = \frac{1}{s+1} - \frac{1}{(s+1)^2}. \tag{19}$$

As the `residue` algorithm is numerical and very sensitive to round-off errors, care should be taken in relying on its output.

If the **determinant**

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} := a_1 b_2 - a_2 b_1 = 0, \tag{20}$$

then the **matrix product**

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := \begin{pmatrix} a_1 v_1 + b_1 v_2 \\ a_2 v_1 + b_2 v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{21}$$

provided that $a_1 v_1 + b_1 v_2 = 0$. In this case, the column matrix

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{22}$$

is a **nullvector** of the **singular** matrix

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{23}$$

A number λ, such that the determinant

$$\begin{vmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{vmatrix} = 0, \tag{24}$$

is an **eigenvalue** of the matrix

$$A := \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}. \tag{25}$$

Any nonzero nullvector

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{26}$$

of the matrix

$$\begin{pmatrix} a_1 - \lambda & b_1 \\ a_2 & b_2 - \lambda \end{pmatrix} \tag{27}$$

is a **right eigenvector** of the matrix A corresponding to the eigenvalue λ. In this case, the matrix product

$$\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \end{pmatrix}. \tag{28}$$

Any nonzero nullvector of the matrix

$$\begin{pmatrix} a_1 - \lambda & a_2 \\ b_1 & b_2 - \lambda \end{pmatrix} \tag{29}$$

is a **left eigenvector** of the matrix A corresponding to the eigenvalue λ and, equivalently, a right eigenvector of the **transpose**

$$A^T := \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} \tag{30}$$

corresponding to the same eigenvalue λ.

In MATLAB, the command

```
>> A = [1, 2; 5, 4]
```

assigns the matrix

$$\begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} \tag{31}$$

to the variable `A`. Its determinant is returned by the command

```
>> det(A)
```

Its eigenvalues are assigned to the variable `lambda` by the command

```
>> lambda = eig(A)
```

and a pair of corresponding right eigenvectors are returned by the commands

```
>> null([1-lambda(1), 2; 5, 4-lambda(1)])
>> null([1-lambda(2), 2; 5, 4-lambda(2)])
```

or as columns of the matrix `v` returned by the command

```
>> [v, ~] = eig(A)
```

Similarly, a pair of corresponding left eigenvectors are given by the columns of the matrix `w` returned by the command

```
>> [w, ~] = eig(A')
```

where `A'` represents the matrix transpose.

The matrix product of `A` with its first right eigenvector is then computed with the command

```
>> A * v(:,1)
```

and should be compared to the content of `v(:,1)`. Similarly, the product of the transpose of the first left eigenvector with `A` is computed with the command

```
>> w(:,1)' * A
```

and should be compared to the content of `w(:,1)'`.

It is very often helpful to visualize mathematical results through suitable graphical representations. The **graph** of a function $f(t)$ of a single variable is a collection of points with coordinates $(t, f(t))$ in a suitable coordinate system.

A complex-valued function $H(s)$ of a complex variable s may be visualized in terms of two two-dimensional surfaces of points $\left(\Re(s), \Im(s), |H(s)|\right)$ and $\left(\Re(s), \Im(s), \arg(H(s))\right)$ in a three-dimensional coordinate system. An alternative graphical representation is in terms of a **contour plot** consisting of a collection of one-dimensional contours in the complex plane for which $|H(s)|$ and $\arg(H(s))$, respectively, are constant.

For purely imaginary $s = \mathrm{j}\omega$, we may visualize the function $H(\mathrm{j}\omega)$ in terms of two one-dimensional curves of points $\left(\omega, |H(\mathrm{j}\omega)|\right)$ and $\left(\omega, \arg(H(\mathrm{j}\omega))\right)$ in a two-dimensional coordinate system. For the function

$$H(s) := \frac{1}{s+1} \tag{32}$$

the graph of $|H(\mathrm{j}\omega)|$ is the collection of points with coordinates

$$\left(\omega, \frac{1}{\sqrt{\omega^2+1}}\right) \tag{33}$$

and the graph of $\arg(H(\mathrm{j}\omega))$ is the collection of points with coordinates

$$\left(\omega, -\arctan\omega\right) \tag{34}$$

A complex-valued function $f(t)$ of a real variable t may be visualized in terms of a one-dimensional curve in the complex plane with coordinates $\left(\Re\left(f(t)\right), \Im\left(f(t)\right)\right)$. For the function

$$f(t) := \sin(4t)e^{\mathrm{j}t} \tag{35}$$

points on the corresponding curve have coordinates

$$\left(\sin 4t\cos t, \sin 4t\sin t\right). \tag{36}$$

Alternatively, a complex-valued function $f(t)$ may be visualized in terms of two one-dimensional curves with coordinates $\left(t, \Re\left(f(t)\right)\right)$ and $\left(t, \Im\left(f(t)\right)\right)$ in a two-dimensional coordinate system. For the function

$$f(t) := e^{(-1/2+\mathrm{j}2)t} \tag{37}$$

points on the corresponding curves have coordinates $\left(t, e^{-t/2}\cos(2t)\right)$ and $\left(t, e^{-t/2}\sin(2t)\right)$, respectively.

In MATLAB, the command

```
>> t = 0:2*pi/100:2*pi;
```

assigns the **array** $[0, 2\pi/100, 4\pi/100, \ldots, 2\pi]$ to the variable `t`. The commands

```
>> t.^2
>> sin(t)
```

return arrays with each element of `t` squared and substituted into `sin`, respectively. The period in the `.^` operator indicates that the operator `^` should be applied to each element of `t` separately.

The commands

```
>> plot(real(sin(4*t).*exp(complex(0,t))), ...
        imag(sin(4*t).*exp(complex(0,t))), 'r')
>> grid on
```

```
>> title('Plot of {f(t)=sin(4t)e^{jt}}')
>> xlabel('\Re')
>> ylabel('\Im')
```

graph a curve through points in the complex plane with coordinates $\big(\sin 4t \cos t, \sin 4t \sin t\big)$ for each t in the variable `t`. Similarly, the commands

```
>> plot(t, real(exp(complex(-1/2,2)*t)), 'r', ...
        t, imag(exp(complex(-1/2,2)*t)), 'b')
>> grid on
>> axis([0, 2*pi, -inf, inf])
>> title('Plot of {e^{(-1/2+j2)t}}')
>> xlabel('t')
```

graph two curves through points with coordinates $(t, e^{-t/2}\cos 2t)$ and $(t, e^{-t/2}\sin 2t)$, respectively, for each t in the variable `t`.

The commands

```
>> om = 10.^(-4:.1:4);
>> semilogx(om, 1./sqrt(om.^2+1), 'r')
>> grid on
>> axis([10^(-4), 10^4, 0, 1.2])
>> title('Plot of magnitude of {H(j\omega)=1/(j\omega +1)}')
>> xlabel('\omega')
```

assign the array $[10^{-4}, 10^{-3.9}, 10^{-3.8}, \ldots, 10^{4}]$ to the variable `om`, and graph a curve through points with coordinates $(\omega, 1/\sqrt{\omega^2+1})$ for each ω in `om`, with a logarithmic scale on the horizontal axis.

The commands

```
>> [re, im] = meshgrid(-0.99:0.01:-0.5,-0.5:0.01:0.5);
>> mag = 1./abs(complex(re,im)+1);
```

assign arrays of real and imaginary parts of a **point grid** on the rectangular domain $\{s \mid -0.99 \le \Re(s) \le -0.05, -0.5 \le \Im(s) \le 0.5\}$ to the variables `re` and `im`, respectively, and an array of magnitudes $1/|s+1|$, evaluated on the grid, to the variable `mag`. The commands

```
>> contour(re, im, mag, 100)
>> grid on
>> title('Contour plot of {1/|s+1|}')
>> xlabel('\Re(s)')
>> ylabel('\Im(s)')
```

graphs 100 contours of constant $1/|s+1|$ in the complex plane. Similarly, the commands

```
>> surface(re, im, mag, 'EdgeColor', [0.8 0.8 0.8], ...
                        'FaceColor', 'r')
>> title('Surface plot of {1/|s+1|}')
>> view(-45,45)
>> xlabel('\Re(s)')
>> ylabel('\Im(s)')
```

adds the graph of a two-dimensional surface connecting points with coordinates $\left(\Re(s), \Im(s), 1/|s+1|\right)$ in a three-dimensional coordinate system for each s in the grid.

Exercises

1. Determine the real and imaginary parts of the complex number $2 - \mathrm{j}$.

2. Determine the complex conjugate of the complex number $\mathrm{j}2 + 2$.

3. Convert the complex number $\mathrm{j}2 - 1$ to polar form.

4. Write the complex number $\mathrm{j} \cdot e^{\mathrm{j}\pi/4}$ in rectangular form.

5. Compute the derivative of the complex exponential $e^{\mathrm{j}\theta}$ with respect to θ and compare this to $\mathrm{j}e^{\mathrm{j}\theta}$.

6. Determine the magnitude of the complex number $3 + \mathrm{j}4$.

7. Determine the phase of the complex number $(-1+\mathrm{j})/\sqrt{2}$.

8. Find a formula for the product of two complex numbers in rectangular form.

9. Find a formula for the ratio of two complex numbers in rectangular form.

10. Show that arithmetic with purely real complex numbers is identical to arithmetic with real numbers.

11. Identify the points in the complex plane corresponding to the complex numbers $\mathrm{j}3 - 2$ and $2e^{\mathrm{j}\pi/3}$.

12. Find constants $a_{1,1}$ and $a_{2,1}$ such that

$$\frac{2s-1}{s^2+3s+2} = \frac{a_{1,1}}{s+1} + \frac{a_{2,1}}{s+2}$$

13. Compute the determinant of the matrix

$$\begin{pmatrix} 3 & -1 \\ 2 & -1 \end{pmatrix}$$

14. Find a nullvector for the singular matrix

$$\begin{pmatrix} 4 & -1 \\ -4 & 1 \end{pmatrix}.$$

15. Determine all eigenvalues of the matrix

$$\begin{pmatrix} -9 & 4 \\ -24 & 11 \end{pmatrix}$$

16. Determine the left and right eigenvectors of the matrix

$$\begin{pmatrix} 9 & 5 \\ -5 & 15 \end{pmatrix}$$

Solutions

1. Here, $\Re(2-\mathrm{j}) = 2$ and $\Im(2-\mathrm{j}) = -1$.

2. Here, $\overline{\mathrm{j}2+2} = -\mathrm{j}2+2$.

3. Here, $\mathrm{j}\sqrt{3}-1 = 2e^{\mathrm{j}2\pi/3}$.

4. Since $\mathrm{j} = e^{\mathrm{j}\pi/2}$,

$$\mathrm{j}\cdot e^{\mathrm{j}\pi/4} = e^{\mathrm{j}3\pi/4} = -\frac{1}{\sqrt{2}} + \mathrm{j}\frac{1}{\sqrt{2}}.$$

5. Since $e^{\mathrm{j}\theta} = \cos\theta + \mathrm{j}\sin\theta$,

$$\frac{\mathrm{d}}{\mathrm{d}\theta}e^{\mathrm{j}\theta} = -\sin\theta + \mathrm{j}\cos\theta = \mathrm{j}(\cos\theta + \mathrm{j}\sin\theta) = \mathrm{j}e^{\mathrm{j}\theta}.$$

6. $3 + \mathrm{j}4 = \rho e^{\mathrm{j}\theta}$, where $\rho = \sqrt{3^2 + 4^2} = 5$.

7. $(-1 + \mathrm{j})/\sqrt{2} = \rho e^{\mathrm{j}\theta}$, where $\theta = 3\pi/4$ is the unique angle in $(-\pi, \pi]$ that satisfies

$$\cos\theta = -\frac{1}{\sqrt{2}}, \ \sin\theta = \frac{1}{\sqrt{2}}.$$

8. Since $\rho e^{\mathrm{j}\theta} = a + \mathrm{j}b$ implies that $a = \rho\cos\theta$ and $b = \rho\sin\theta$,

$$\begin{aligned}\rho_1 e^{\mathrm{j}\theta} \cdot \rho_2 e^{\mathrm{j}\theta_2} &= \rho_1\rho_2 e^{\mathrm{j}(\theta_1+\theta_2)} \\ &= \rho_1\rho_2(\cos(\theta_1+\theta_2) + \mathrm{j}\sin(\theta_1+\theta_2)) \\ &= \rho_1\cos\theta_1\rho_2\cos\theta_2 - \rho_1\sin\theta_1\rho_2\sin\theta_2 \\ &\qquad + \mathrm{j}\big(\rho_1\sin\theta_1\rho_2\cos\theta_2 + \rho_1\cos\theta_1\rho_2\sin\theta_2\big)\end{aligned}$$

implies that

$$(a_1 + \mathrm{j}b_1) \cdot (a_2 + \mathrm{j}b_2) = a_1a_2 - b_1b_2 + \mathrm{j}(a_1b_2 + a_2b_1)$$

9. Since

$$1 \div \rho e^{\mathrm{j}\theta} = \frac{1}{\rho}e^{\mathrm{j}(-\theta)} = \frac{\rho\cos\theta}{\rho^2} - \mathrm{j}\frac{\rho\sin\theta}{\rho^2}$$

implies that

$$1 \div (a + \mathrm{j}b) = \frac{a}{a^2+b^2} - \mathrm{j}\frac{b}{a^2+b^2}$$

it follows that

$$\begin{aligned}(a_1 + \mathrm{j}b_1) \div (a_2 + \mathrm{j}b_2) &:= (a_1 + \mathrm{j}b_1) \cdot 1 \div (a_2 + \mathrm{j}b_2) \\ &= \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + \mathrm{j}\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2}\end{aligned}$$

10. The conclusion follows from the observations

$$\begin{aligned}(a_1 + \mathrm{j}0) + (a_2 + \mathrm{j}0) &= (a_1 + a_2) + \mathrm{j}0 \\ (a_1 + \mathrm{j}0) - (a_2 + \mathrm{j}0) &= (a_1 - a_2) + \mathrm{j}0 \\ (a_1 + \mathrm{j}0) \cdot (a_2 + \mathrm{j}0) &= (a_1 \cdot a_2) + \mathrm{j}0 \\ (a_1 + \mathrm{j}0) \div (a_2 + \mathrm{j}0) &= (a_1 \div a_2) + \mathrm{j}0\end{aligned}$$

11. The complex numbers $j3-2$ and $2e^{j\pi/3}$ correspond to points in the complex plane, parameterized by the real and imaginary parts, with coordinates $(-2,3)$ and $(2\cos\pi/3, 2\sin\pi/3) = (1,\sqrt{3})$.

12. Multiply both sides of the equation by $s+1$, simplify, and set $s=-1$ to obtain $a_{1,1}=-3$. Similarly, multiply both sides of the equation by $s+2$, simply, and set $s=-2$ to obtained $a_{2,1}=5$.

13. Here,
$$\begin{vmatrix} 3 & -1 \\ 2 & -1 \end{vmatrix} = -1.$$

14. Since $4v_1 - v_2 = 0$ implies that $v_2 = 4v_1$, every nullvector is of the form
$$\begin{pmatrix} v \\ 4v \end{pmatrix}$$
for some v.

15. The condition
$$\begin{vmatrix} -9-\lambda & 4 \\ -24 & 11-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0$$
implies that λ equals 3 or -1.

16. The condition
$$\begin{vmatrix} 9-\lambda & 5 \\ -5 & 15-\lambda \end{vmatrix} = \lambda^2 - 24\lambda + 160 = 0$$
implies that λ equals $12+j4$ or $12-j4$. Since $(-3-j4)v_1 + 5v_2 = 0$ implies that $v_2 = (3+j4)v_1/5$, every right eigenvector corresponding to the eigenvalue $12+j4$ is of the form
$$\begin{pmatrix} 5v \\ (3+j4)v \end{pmatrix}$$
for some nonzero v. Similarly, since $(-3+j4)v_1 + 5v_2 = 0$ implies that $v_2 = (3-j4)v_1/5$, every right eigenvector corresponding to

the eigenvalue $12 - j4$ is of the form

$$\begin{pmatrix} 5v \\ (3 - j4)v \end{pmatrix}$$

for some nonzero v. From the equations $(-3 - j4)v_1 - 5v_2 = 0$ and $(-3 + j4)v_1 - 5v_2 = 0$, every left eigenvector corresponding to either of the eigenvalues $12 + j4$ and $12 - j4$ is of the form

$$\begin{pmatrix} 5v \\ (-3 - j4)v \end{pmatrix}$$

or

$$\begin{pmatrix} 5v \\ (-3 + j4)v \end{pmatrix},$$

respectively.

Prelab Assignments

Complete these assignments before the lab. Show all work for credit.

1. By hand and, if necessary, using a calculator, determine the real and imaginary parts of the complex number $-j3+3$.

2. By hand and, if necessary, using a calculator, determine the complex conjugate of the complex number $4+j$.

3. By hand and, if necessary, using a calculator, convert the complex number $j/5-1/2$ to polar form.

4. By hand and, if necessary, using a calculator, write the complex number $-2j \cdot e^{j\pi/12}$ in rectangular form.

5. By hand and, if necessary, using a calculator, determine the magnitude of the complex number $12-j3$.

6. By hand and, if necessary, using a calculator, determine the phase of the complex number $(-2+j\sqrt{2})/\sqrt{3}$.

7. By hand and, if necessary, using a calculator, compute the inverse $(-3+2j)^{-1}$.

8. By hand and, if necessary, using a calculator, evaluate the ratio $(1+j) \div (2-j)$.

9. By hand and, if necessary, using a calculator, express

$$\frac{s-1}{s^2+3s+2}$$

   as a sum of terms of the form $a/(s-\sigma)$.

10. By hand and, if necessary, using a calculator, compute the determinant of the matrix

$$\begin{pmatrix} 6 & -1/2 \\ 1/2 & -1 \end{pmatrix}$$

11. By hand and, if necessary, using a calculator, find a nullvector for the singular matrix
$$\begin{pmatrix} 2 & 18 \\ -2/3 & -6 \end{pmatrix}.$$

12. By hand and, if necessary, using a calculator, determine all eigenvalues of the matrix
$$\begin{pmatrix} -9 & 5 \\ -24 & 11 \end{pmatrix}$$

13. By hand and, if necessary, using a calculator, determine the left and right eigenvectors of the matrix
$$\begin{pmatrix} 9 & 1 \\ -5 & 15 \end{pmatrix}$$

Report Assignments

Complete these assignments during the lab. *Show all work for credit.*

1. Use MATLAB to determine the real and imaginary parts of the complex number $-j3+3$.

2. Use MATLAB to determine the complex conjugate of the complex number $4+j$.

3. Use MATLAB to convert the complex number $j/5-1/2$ to polar form.

4. Use MATLAB to write the complex number $-2j \cdot e^{j\pi/12}$ in rectangular form.

5. Use MATLAB to determine the magnitude of the complex number $12-j3$.

6. Use MATLAB to determine the phase of the complex number $(-2+j\sqrt{2})/\sqrt{3}$.

7. Use MATLAB to compute the inverse $(-3+2j)^{-1}$.

8. Use MATLAB to evaluate the ratio $(1+j) \div (2-j)$.

9. Use MATLAB to express

$$\frac{s+1}{s^2+3s+2}$$

   as a sum of terms of the form $a/(s-\sigma)$.

10. Use MATLAB to compute the determinant of the matrix

$$\begin{pmatrix} 6 & -1/2 \\ 1/2 & -1 \end{pmatrix}$$

11. Use MATLAB to find a nullvector for the singular matrix

$$\begin{pmatrix} 2 & 18 \\ -2/3 & -6 \end{pmatrix}.$$

12. Use MATLAB to determine all eigenvalues of the matrix

$$\begin{pmatrix} -9 & 5 \\ -24 & 11 \end{pmatrix}$$

13. Use MATLAB to determine the left and right eigenvectors of the matrix

$$\begin{pmatrix} 9 & 1 \\ -5 & 15 \end{pmatrix}$$

14. Use MATLAB to graph a curve with coordinates given by the real and imaginary parts of $f(t) = e^{\cos 4t + \mathrm{j}t}$ for $t \in [0, 2\pi]$.

15. Use MATLAB to graph a curve with coordinates given by ω and $|H(\mathrm{j}\omega)|$ for $H(s) = 1/(s^2 + s + 3)$.

16. Use MATLAB to graph contour curves in the complex plane with coordinates $\big(\Re(s), \Im(s)\big)$ for the function $H(s) = (s + \bar{s})/e^{|s|}$.

17. Use MATLAB to graph a two-dimensional surface through a set of points with coordinates $\big(\Re(s), \Im(s), H(s)\big)$ for the function $H(s) = \mathrm{j}(s - \bar{s})/e^{|s|}$.
