Power of a Statistical Test

(8.5)

Type 2 error:
Probability of not rejecting H0 when H0 is false

Type I Error: Review

α

Type I Error: Reject H_0 when H_0 is true

Hypothesis testing:

Set maximum acceptable rate of Type I error:

α <- significance level

Choose a test with the most power to detect H_A.

Power

if $H_0: \mu = 100$

$H_A: \mu < 100$

e.g. rej if $\bar{x} < 80$

The power of a statistical test is the probability that the test correctly rejects H_0 (*when H_A is true*).

H_0 false

The **power** of a statistical test is related to Type II error.

- β is the probability of Type II error

$$\text{Power} = 1 - \underline{P[\text{Type II Error}]}$$
$$= 1 - \beta$$

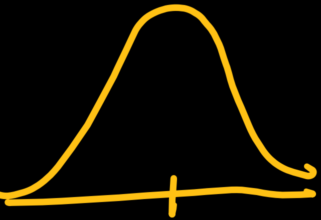
Type I and Type II Errors

Type 2 error:
Probability of not rejecting H0 when H0 is false

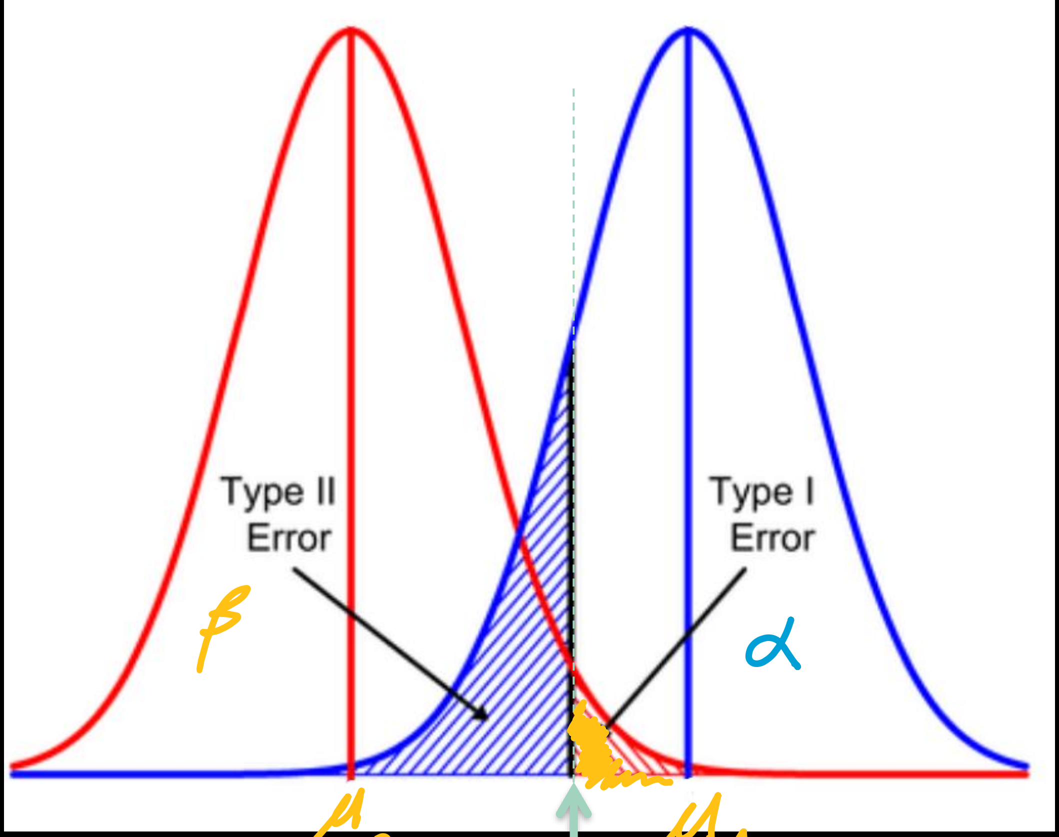
$H_0: \mu = \mu_0$

$H_A: \mu > \mu_0$

$\alpha = 0.05$



$z_\alpha = 1.645$



Rej H if $Z > 1.645$

Rej if $\bar{X} >$ ~

Type I and Type II Errors

Example:
H_0 : $\mu = 100$
H_A : $\mu > 100$

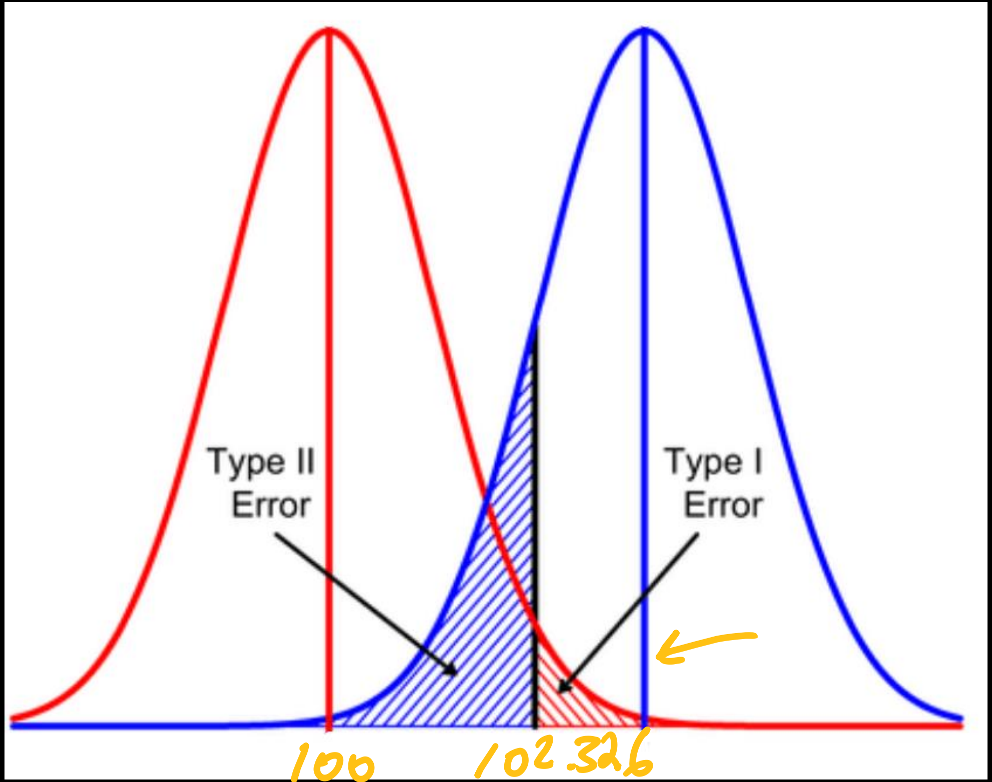
$\sigma = 10$, n = 100

Define a rejection region at $\alpha = 0.01$.

$Z_{0.01}$

Rej H_0 if $z > 2.326$

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$



$$2.326 = \frac{\bar{X} - 100}{10/\sqrt{100}}$$

$\bar{X} = 102.326$

Rej $\bar{X} > 102.326$

What if H_A is true? $\quad H_A: \mu > 100$

e.g. 103 $\quad$ Rej H_0 if $\bar{X} > 102.32$

What is the power of this test at $\mu = 103$

$$\text{Power} = P[\text{reject } H_0 \mid H_A \text{ true}]$$

$$= P[\bar{X} > 102.32 \mid \mu = 103]$$

$$= P\left[\frac{\bar{X} - 103}{10/\sqrt{100}} > \frac{102.32 - 103}{10/\sqrt{100}}\right]$$

β @ 103

$= 1 - \text{power}$

$= .25$

$$= P[Z > -0.674] = 0.75$$

Power Function

The power function shows the probability of rejecting H_0 at different values of θ.

Note: The power function is sometimes denoted differently by different people. E.g.: $\beta(\theta)$, $B(\theta)$, or $\boldsymbol{K(\theta)}$.

- The β in this power function above is **<u>not</u>** the same as the β for type II error.
    - β is the name (letter) of the power function.
    - Need to look at context
- Textbook sometimes calls $\beta(\theta)$ as "K(μ)" when dealing with the mean.

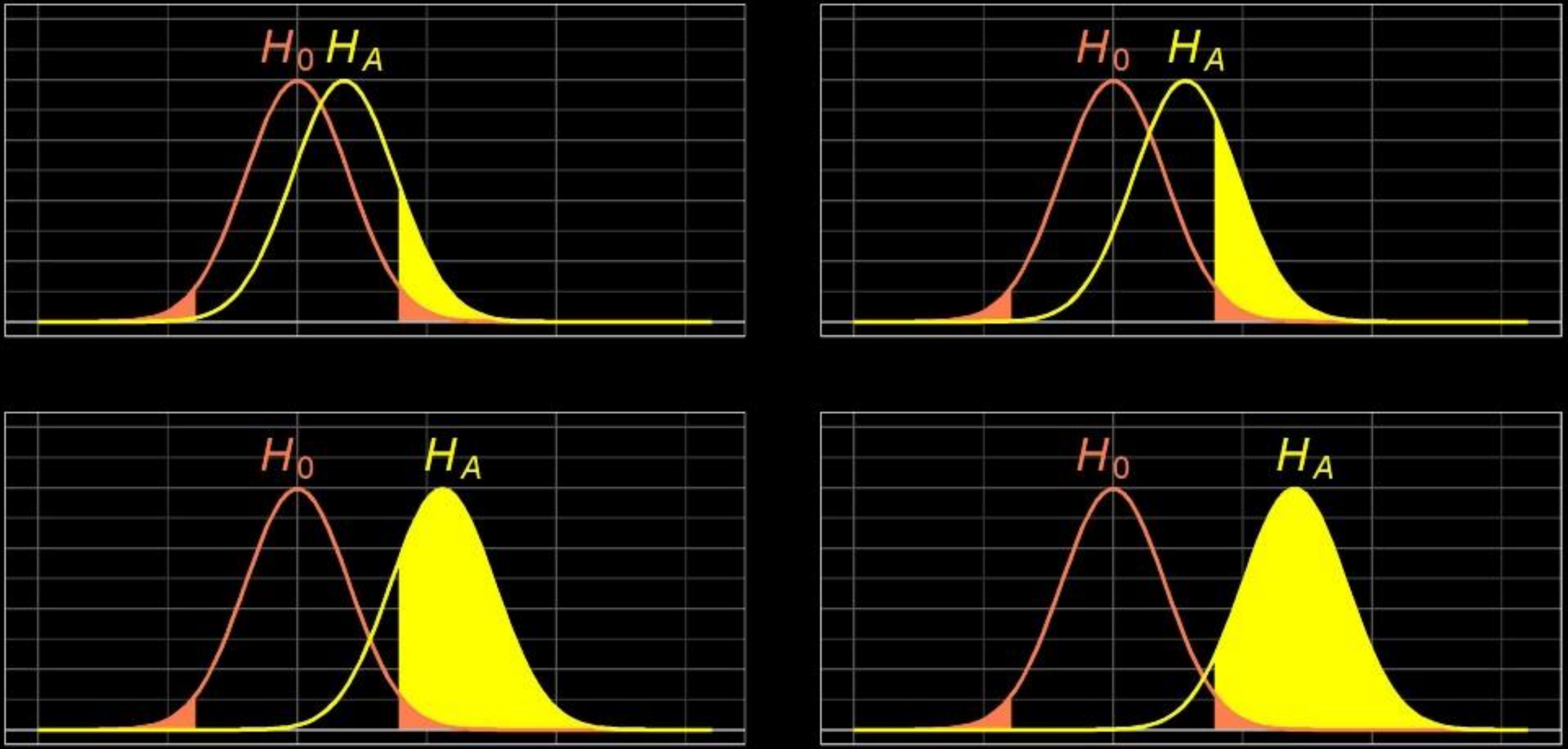


Image Source: https://towardsdatascience.com/power-analysis-made-easy-dfee1eb813a

Example 8.5-2

$P[\text{rej } H_0 \mid H_0 \text{ true}] =$

Let $X_1, X_2, \ldots, X_n$ be a random sample $\sim N(\mu, 100)$. Let $n = 25$.

Suppose we want to test whether the true mean is 60 (H_0) versus if it is greater than 60 (H_A).

H_0: $\mu = 60$

H_A: $\mu > 60$

Test statistic:

$\alpha = P[\bar{X} > 62 \mid \mu = 60]$

$= P\left[Z > \frac{62 - 60}{10/\sqrt{25}}\right] = P[Z \geq 1]$

$= .1587$

Suppose we choose a test that rejects H_0 if and only if $\boxed{\bar{x} \geq 62}$.

What are the consequences of this test *(what does the power curve look like)?*

if $\mu = 62$ Power $= 0.5$ $\Big\}$ if $\mu = 63$,

Power Function

If the true mean under H_A is μ, and X ~ N(μ,100),

then $\bar{X}$ ~ N(μ, 100/n) = N(μ, 4)

The probability of rejecting H_0 is given by under H_A

$K(\mu) = P[\bar{X} \geq 62 \ ; \mu]$

$= P\left[\frac{\bar{X} - \mu}{2} \geq \frac{62 - \mu}{2} \ ; \mu\right] \qquad = P\left[Z \geq \frac{62 - \mu}{2} \ ; \mu\right]$

$H_0: \mu = 60$

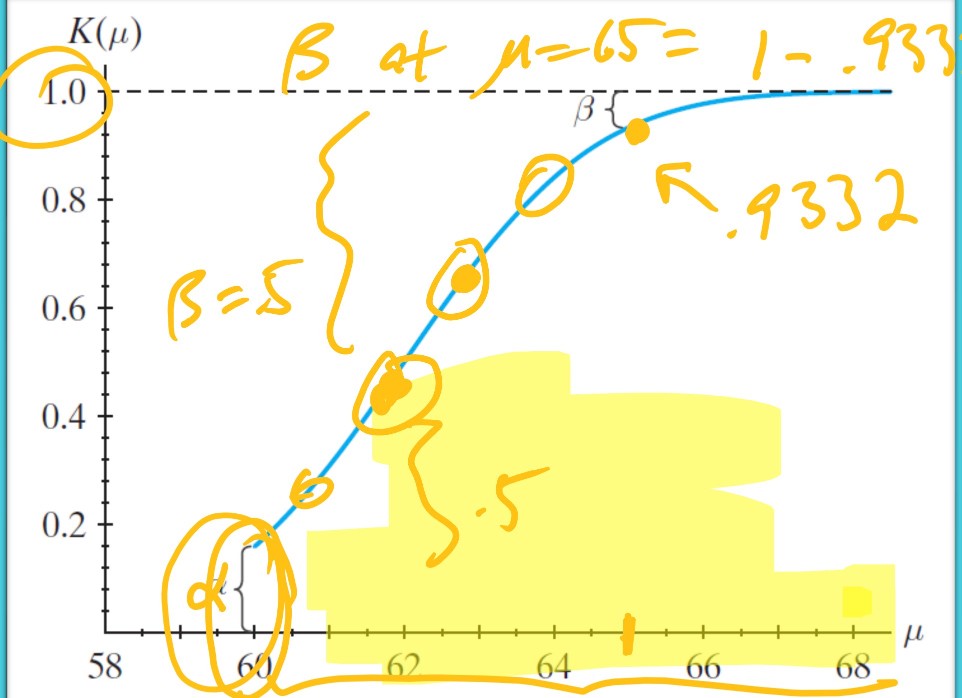
P[rej]

rej if $\bar{X} > 62$

Table 8.5-1 Values of the power function

μ	$K(\mu)$
60	0.1587
61	0.3085
62	0.5000
63	0.6915
64	0.8413
65	0.9332
66	0.9772

β at $\mu = 65 = 1 - .9332$



.9332

$\beta = .5$

.5

H_0

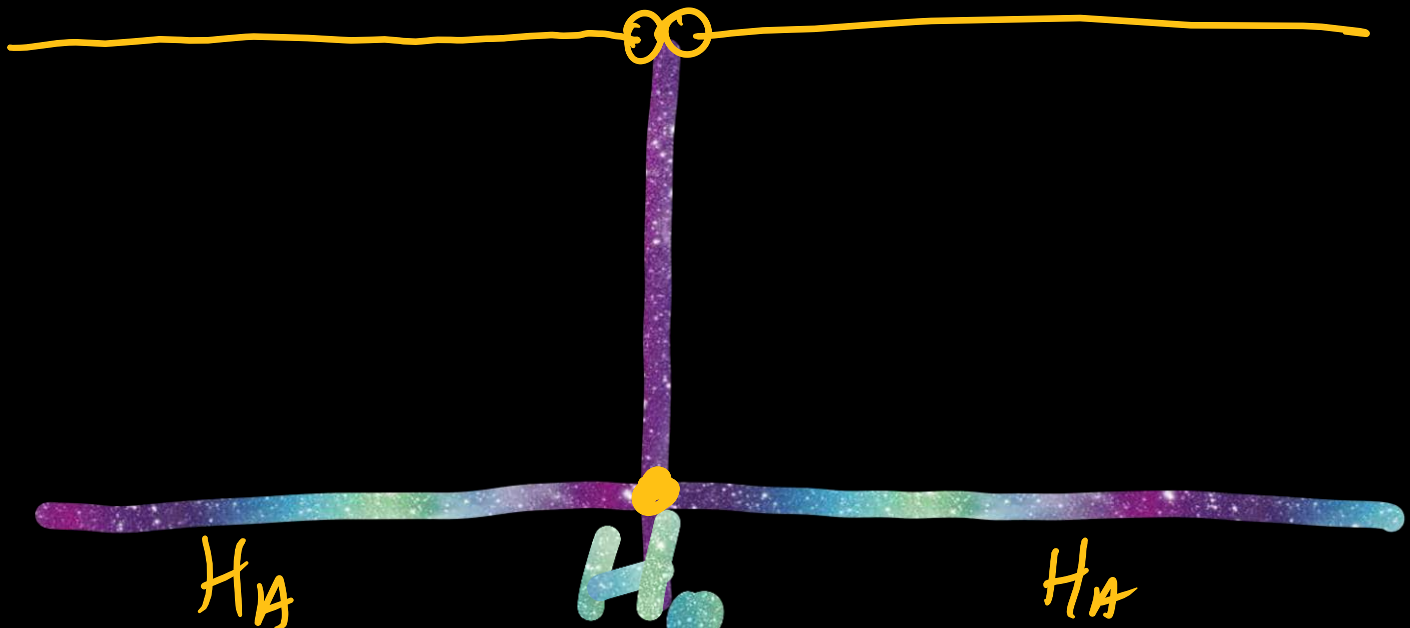
H_A

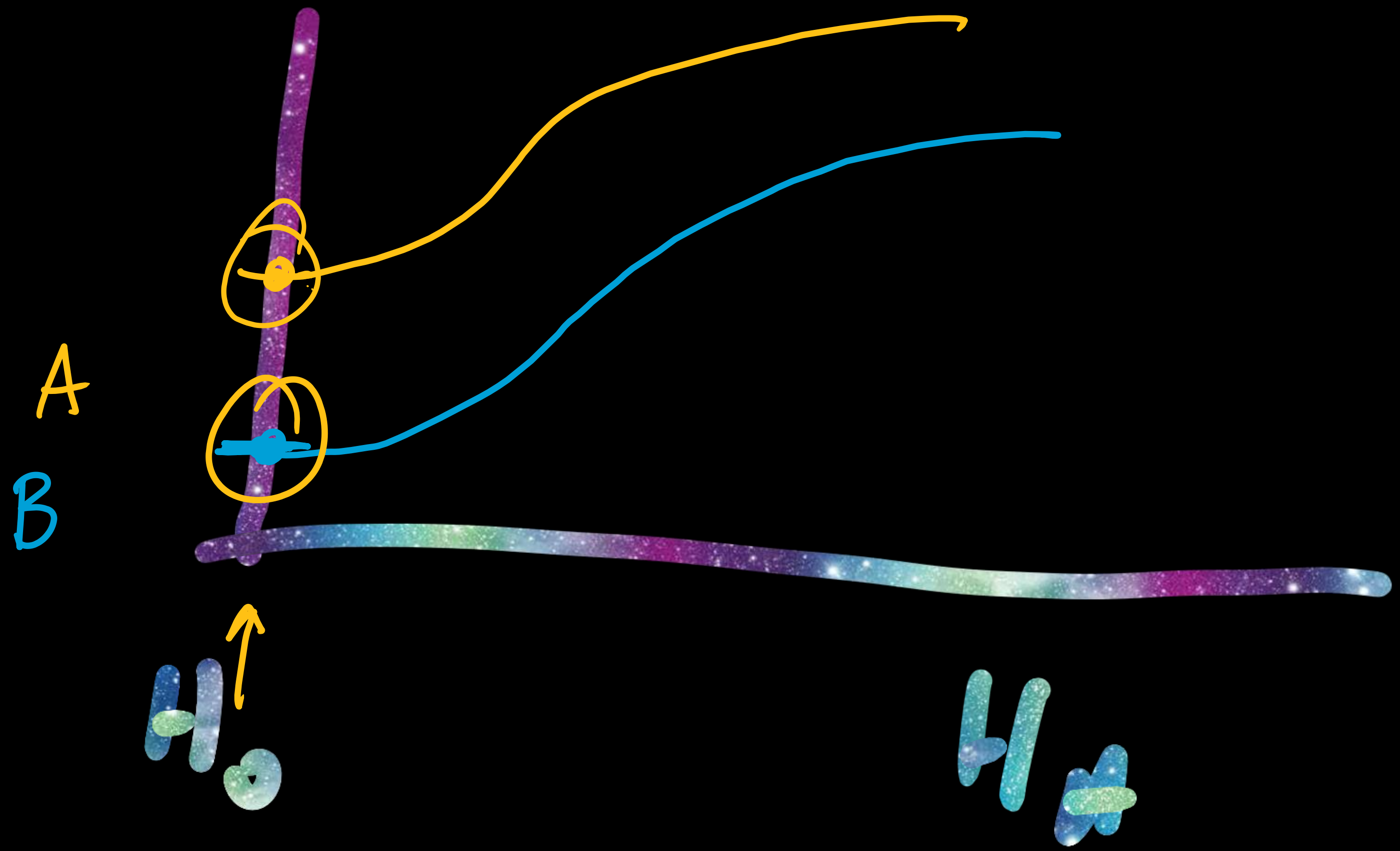
Figure 8.5-2 Power function $K(\mu) = 1 - \Phi([62 - \mu]/2)$

Ideal power function?

$H_A: \mu \neq \mu_0$

- In the previous example, what would an ideal power function look like?




A
B
H_0
H_A

power = P[rej H_0 | H_A true]

Example 2

α = P[rej H_0 | H_0 true]

Assume that the number of grams of caffeine that Albert ingests every day follows an approximately normal distribution with unknown mean and standard deviation 16.

Let $n = 16$, $\alpha = 0.01$

Test H_0: $\mu = 100$ vs H_A: $\mu > 100$.

Define a rejection region for H_0.

$z > 2.326$

Rej H_0 if $\bar{X} > 109.304$

Example 2

Type 1 ↑, Power ↑, Type 2 ↓

What is the power at μ = 108?

Power = P[$\bar{X} \geq 109.304 \mid \mu = 108$]

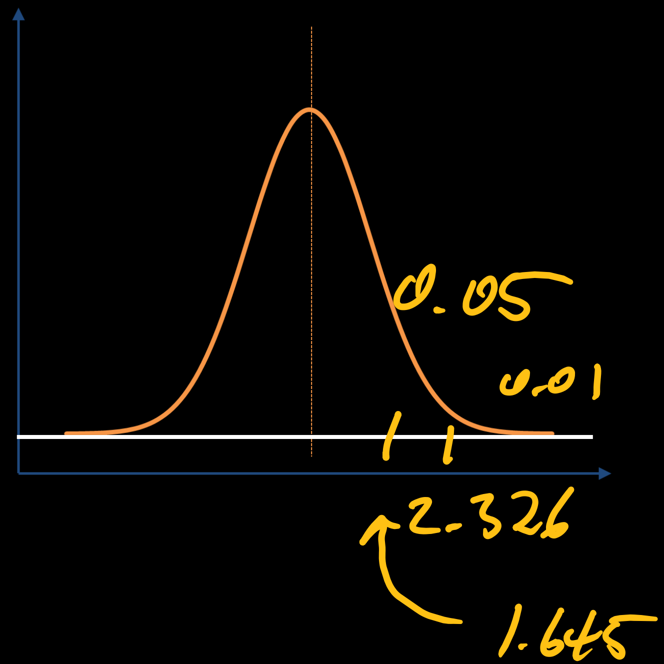
$= P\left[Z \geq \frac{109.304 - 108}{16/\sqrt{16}}\right] = P[Z \geq 0.326]$

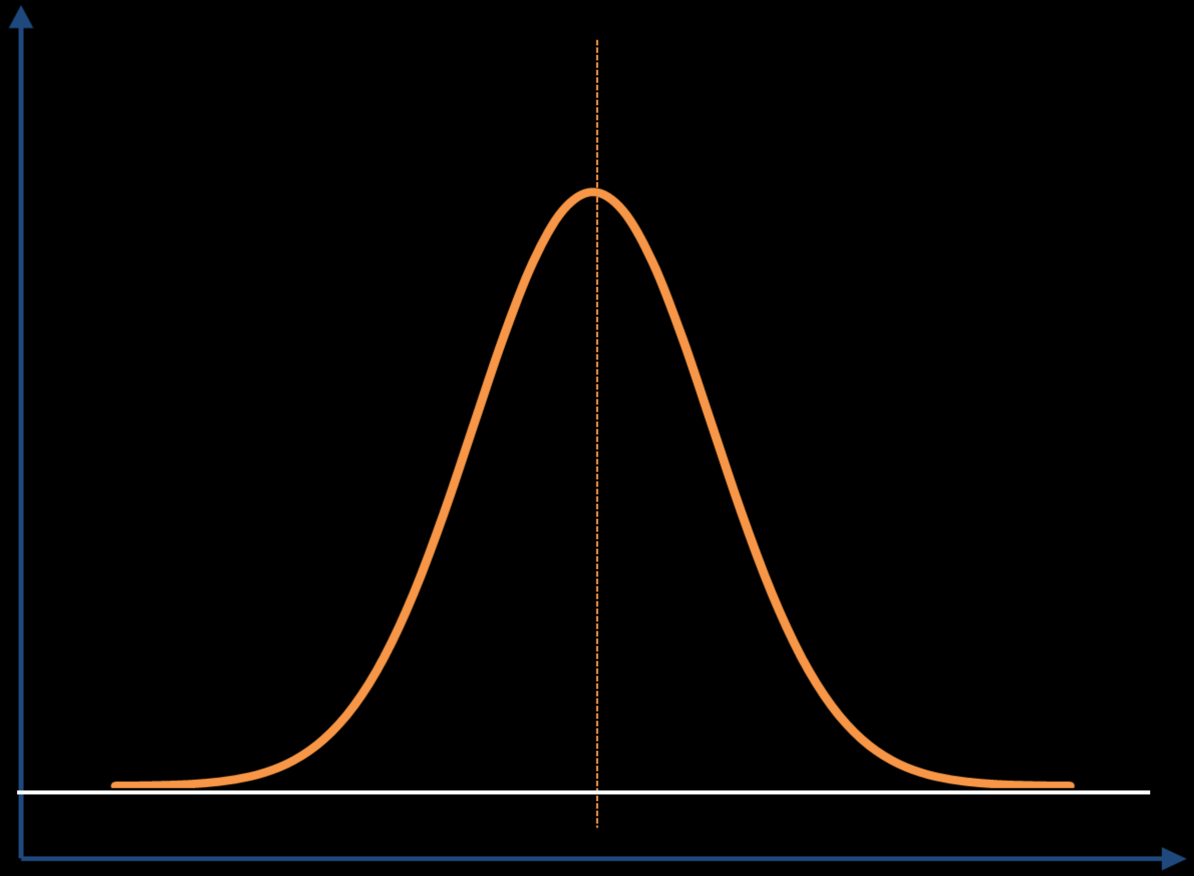
= 0.3722

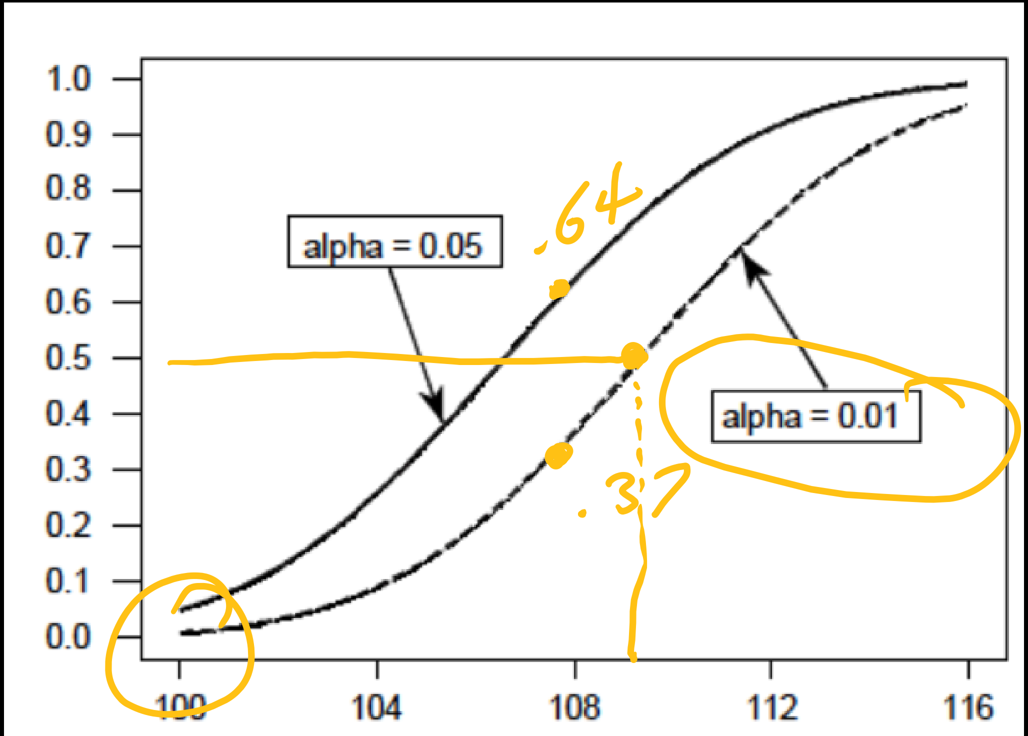
what if we used $\alpha = 0.05$

cut-off $\bar{X} > 106.58$

power = 0.6404






1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
100
104
108
112
116
alpha = 0.05
alpha = 0.01
.64
.37

qt(.975, 15)

$\alpha/2$

Example 3

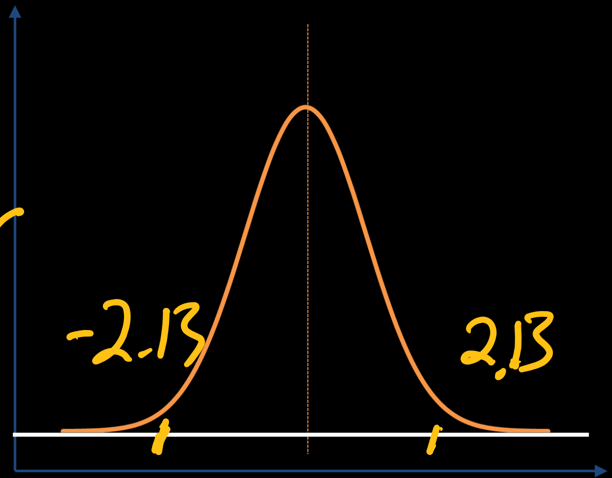
Suppose we have a normally distributed random sample with $n = 16$, $s = 8$.

We wish to test $H_0: \mu = 50$ vs $H_A: \mu \neq 50$.

What is the power of a level $\alpha = 0.05$ test?

t_{15}

Rej H_0 if $t_{15} > 2.13$ or $t_{15} < -2.13$



$t_{15,\, 0.025}$

$$2.13 = \frac{\bar{X} - 50}{8/\sqrt{16}}$$

Rej if $\bar{X} > 54.26$ or $\bar{X} < 45.74$

$P[t_{15} > 0] = 1/2$ 51 power at 54.26

Example 3

Suppose we have a normally distributed random sample with $n = 16$.

We wish to test $H_0: \mu = 50$ vs $H_A: \mu \neq 50$

curve

What is the power of a level $\alpha = 0.05$ test?

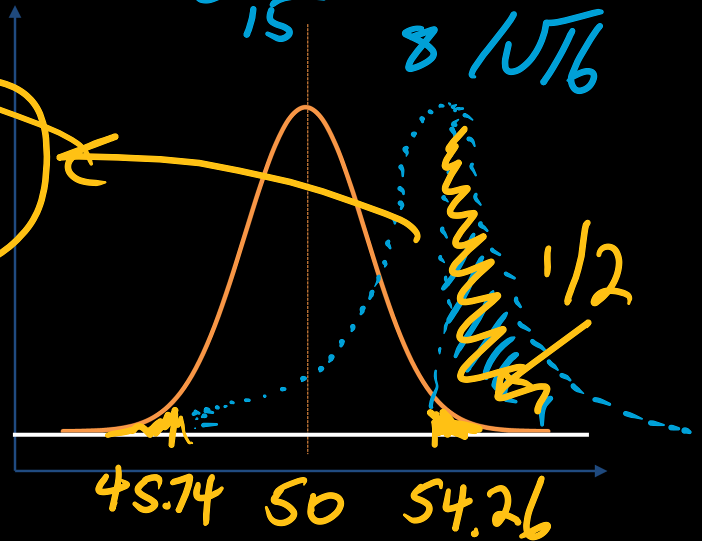
$$t_{15} = \frac{45.74 - 54.26}{8/\sqrt{16}}$$

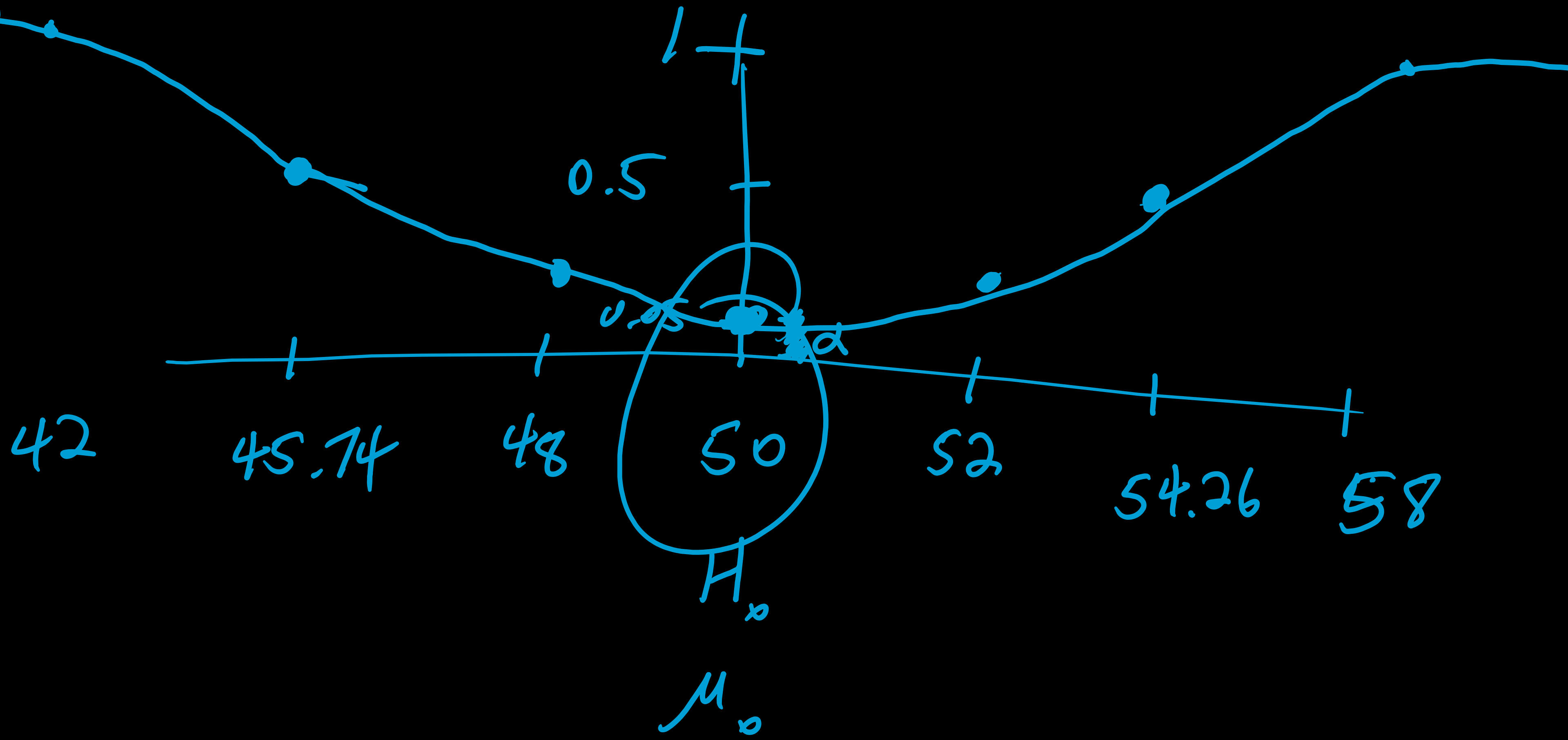
Power $= P[\bar{X} > 54.26 \mid \mu = 54.26] +$

$P[\bar{X} < 45.74 \mid \mu = 54.26]$

$= \frac{1}{2} + P[t_{15} < -4.26]$

≈ 0




1
0.5
0.05
α
42
45.74
48
50
52
54.26
58
H_0
μ_0