Homework 3

Exercise 1

泊松过程 $\lambda = 0.5$

While taking bike rides with Chloe, we encounter squirrels according to a Poisson process with a rate of 0.5 squirrels per mile.

a) (0.5) What is the probability of not encountering any squirrels in 1 mile?

b) (0.5) What is the probability of not encountering any squirrels in 6 miles?

c) (0.5) What is the probability of encountering exactly 4 squirrels in 10 miles?

d) (0.5) What is the probability of encountering at least 2 squirrels in 10 miles?

e) (0.5) On a 10 mile bike ride, broken into 10 mile-long increments, what is the probability that we encounter 0 squirrels on exactly 4 of these increments (and do encounter squirrels on the other 6 mile-long increments)?

a). $\lambda = 0.5$

$$f(0) = \frac{\lambda^0 e^{-\lambda}}{0!} = \boxed{\frac{1}{\sqrt{e}}}$$

b). $\lambda_2 = 6 \times \lambda = 3$

$$f(0) = \frac{\lambda_2^0 \cdot e^{-\lambda_2}}{0!} = \frac{1 \times e^{-3}}{1} = \boxed{\frac{1}{e^3}}$$

c). $\lambda_3 = 10 \times \lambda = 5$

$$f(4) = \frac{\lambda_3^4 \times e^{-\lambda_3}}{4!} = \frac{5^4 \cdot e^{-5}}{4!} = \boxed{\frac{625}{24 \cdot e^5} \approx 17.56}$$

d). $\lambda_4 = 10 \times \lambda = 5$

$$P = 1 - f(0) - f(1) = 1 - \frac{5^0 e^{-5}}{0!} - \frac{5^1 e^{-5}}{1!} = \boxed{1 - 6e^{-5} \approx 95.96}$$

e). From a) we know $P(X=0) = \frac{1}{\sqrt{e}}$

$$P = C_{10}^4 \cdot P(X=0)^4 \cdot (1 - P(X=0))^6 = 210 \cdot \left(\frac{1}{\sqrt{e}}\right)^4 \cdot \left(1 - \frac{1}{\sqrt{e}}\right)^6 = \boxed{\frac{210}{e^2} \cdot \left(1 - \frac{1}{\sqrt{e}}\right)^6}$$

Exercise 2

While wandering around the halls of North Shore High School after hours, Regina George encounters lots of minions. Suppose minions appear one-at-a-time. You may assume that these encounters are independent and that these are there are only 4 types of minions she may run into:

Minion	Probability
Freshman	0.4
Sophomore	0.25
Junior	0.2
Senior	0.15

a) (0.5 pt) What is the probability that she sees her first Junior **after** her 4th encounter?

b) (0.5 pt) What is the probability that she sees her third Senior **on** (exactly) her 15th encounter?

c) (0.5 pt) What is the probability that she sees exactly 3 Freshmen and exactly 5 Sophomores in 11 encounters? 0.4 0.25

d) (0.5 pt) Suppose she encounters 8 minions. What is the probability that **at least** 2 are Sophomores?

e) (0.5 pt) Suppose she encounters 10 minions. What is the probability that **exactly** 3 of them are Seniors?

a). $\underline{\times}\ \underline{\times}\ \underline{\times}\ \underline{\times}\ \underline{\triangle}$ $\quad P = 0.8^4 \times 0.2 = 8.19\%$

b). $P = C_{14}^{2} \times (0.15)^3 \times (0.85)^{12} = 4.36\%$

c). $P = C_{11}^{3} \times C_{8}^{5} \times (0.4)^3 \times (0.25)^5 \times (0.35)^3 = 2.48\%$

d). $P = 1 - C_8^0 \cdot (0.25)^0 \cdot (0.75)^8 - C_8^1 \cdot (0.25)^1 \cdot (0.75)^7 = 63.29\%$

e). $P = C_{10}^{3} \cdot (0.15)^3 \cdot (0.85)^7 = 12.98\%$

Exercise 3 Use R to find the following.

Show your commands and output for credit! (no handwritten code)

Use the same information from the previous exercise.

a) (0.5 pt) Find the probability that it takes **fewer than** 60 trials to find the 20th Sophomore.

b) (0.5 pt) What is the probability that she sees her 40th Senior **by** (including) her 200th encounter?

c) (0.5 pt) Suppose she encounters 1000 minions. What is the probability that **fewer than** 220 are Juniors?

d) (0.5 pt) Suppose she encounters 1000 minions. What is the probability that **at least** 420 are Freshmen?

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rm(list=ls()) #remove all da

#Question A
P1=pnbinom(39,20,0.25)

#Question B
P2=dbinom(39,199,0.15)*0.15


#Question C
P3=pbinom(219,1000,0.2)

#Question D
P4=1-pbinom(419,1000,0.4)
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P1	0.0797041597702…
P2	0.0023093821430…
P3	0.9371585047183…
P4	0.1042791471109…

Exercise 4

(a) (1 pt) Let X denote a discrete random variable that takes values $1, 2, 3, ..., n$. Show

$$\frac{d}{dt}M_X(t) = E[Xe^{tX}],$$

where $M_X(t)$ denotes the moment generating function.

(b) (0.5 pt) Suppose a discrete random variable X has mgf given by

$$\frac{\frac{2}{11}e^t}{1-\frac{9}{11}e^t}$$

Find $P(X = 2)$.

(c) (0.5 pt) For the mgf in the part (b), find $E[X^2]$

Hint: There are multiple correct approaches to finding this answer

(d) (1 pt) Let X be $Poisson(\lambda)$. Show the MGF of X is given by

$$M_X(t) = \exp\{\lambda(e^t - 1)\}.$$

a). $\because \frac{d}{dt}M_X(t) = E[Xe^{tX}] = \sum_X^n X \cdot e^{tX} \cdot P(X)$

$\therefore M_X(t) = \sum_X^n e^{tX} \cdot P(X) = E[e^{tX}]$ $\quad\therefore M_X(t)$ is m.g.f.

b). $M_X(t) = \dfrac{\frac{2}{11}e^t}{1-\frac{9}{11}e^t} = \frac{2}{11}e^t \cdot \sum_{n=0}^{\infty}\left(\frac{9}{11}e^t\right)^n = \sum_{n=0}^{\infty} \frac{2}{11}\cdot\left(\frac{9}{11}\right)^n \cdot e^{(n+1)t}$

$\because M_X(t) = E[e^{tX}] = \sum_X^n e^{tX} \cdot P(X)$

$\therefore$ let $n=1 \Rightarrow P(X=2) = \frac{2}{11}\cdot\left(\frac{9}{11}\right)^1 = \boxed{\frac{18}{121}}$

c). $\because M_X(t) = \dfrac{\frac{2}{11}e^t}{1-\frac{9}{11}e^t}$ $\quad\therefore E[X^2] = M_X''(t=0) = \boxed{55}$

d). $\because X \sim Posi(\lambda)$

$\therefore P(X=x) = \dfrac{\lambda^x e^{-\lambda}}{x!}$

$\therefore M_t(x) = E(e^{tx}) = \sum_X e^{tX} \cdot P(x) = \sum_x^n e^{tx} \cdot \dfrac{\lambda^x e^{-\lambda}}{x!} = e^{-\lambda} \cdot \sum_x^n \dfrac{(\lambda \cdot e^t)^x}{x!} = e^{-\lambda} \cdot e^{\lambda e^t}$

$= e^{\lambda(e^t-1)}$