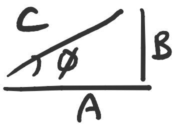
Homework 2

1. Show that $A\sin(\omega t) + B\cos(\omega t)$ can be written as $C\sin(\omega t + \phi)$ where $C = \sqrt{A^2 + B^2}$ and $\phi = \tan^{-1}\left(\frac{B}{A}\right)$.



$$C \cdot \sin(wt + \phi)$$

$$= C \cdot [\sin(wt) \cdot \cos\phi + \cos(wt) \cdot \sin(\phi)]$$

$$= C \cdot [\sin(wt) \cdot \frac{A}{C} + \cos(wt) \cdot \frac{B}{C}]$$

$$= A \cdot \sin(wt) + B\cos(wt)$$

2. State the order of the following ordinary differential equations and state if they are (i) time-invariant or not, and (ii) linear or not

   (a) $\ddot{x} + x^3 e^x \dot{x} + x = 0$
   (b) $e^t \ddot{x} + \dot{x} + x = 0$
   (c) $\ddot{x} + \sin(tx)\dot{x} + x = 0$

	Order	Time-invariant	Linear
a)	2	✓	✗
b)	2	✗	✓
c)	2	✗	✗

3. Determine if the following differential equations are linear or not by checking if they satisfy the super-position principle

   (a) $\dot{x} + t^2 x = 0$
   (b) $\dot{x} + x^2 t = 0$

Linear

a) $f(x) = \dot{x} + t^2 x$ $\quad \therefore f(x_1) + f(x_2) = \dot{x}_1 + \dot{x}_2 + t^2(x_1 + x_2) = (\dot{x}_1 + \dot{x}_2) + t^2(x_1 + x_2) = f(x_1 + x_2)$

non-Linear

b). $f(x) = \dot{x} + x^2 t$ $\quad \therefore f(x_1) + f(x_2) = \dot{x}_1 + \dot{x}_2 + t(x_1^2 + x_2^2)$ $\quad \therefore f(x_1 + x_2) \neq f(x_1) + f(x_2)$

$f(x_1 + x_2) = \dot{x}_1 + \dot{x}_2 + t(x_1 + x_2)^2$

4. Let $x_1(t)$ and $x_2(t)$ be solutions to a linear ordinary differential equation (ODE): $a_2\ddot{x} + a_1\dot{x} + a_0 x = 0$.
Show that $y(t) = \alpha x_1(t) + \beta x_2(t)$ is also a solution to this ODE for any constants α and β.

$\because$ $a_2 \cdot \ddot{x}_1(t) + a_1 \dot{x}_1(t) + a_0 x_1(t) = 0$ ①

$a_2 \ddot{x}_2(t) + a_1 \dot{x}_2(t) + a_0 \cdot x_2(t) = 0$ ②

$\therefore$ α ① $+ \beta \cdot$ ② is as follows.

$$a_2(\alpha \cdot \ddot{x}_1(t) + \beta \ddot{x}_2(t)) + a_1(\alpha \cdot \dot{x}_1(t) + \beta \dot{x}_2(t)) + a_0(\alpha x_1(t) + \beta x_2(t)) = 0$$

$$a_2(\alpha \cdot x_1(t) + \beta x_2(t))'' + a_1(\alpha \cdot x_1(t) + \beta x_2(t))' + a_0(\alpha x_1(t) + \beta x_2(t)) = 0$$

$$a_2 \ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0$$

q.e.d.

5. Given the linear ODE: $x^{(3)} + 3\ddot{x} + 4\dot{x} + 2x = u(t)$ 线性高阶 ⇒ 矩阵 ODE

(a) Convert this third order ODE into first-order form $\dot{z} = f(z, u)$ where z is a 3×1 vector
(b) With $u = 0$ and the initial conditions $x(0) = 1$, $\dot{x}(0) = -1$, $\ddot{x}(0) = 1$ determine the value of the vector $\dot{z}(0)$.

a). let $z = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ $\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = \ddot{x} \end{cases}$

$\because$ $x''' + 3x'' + 4x' + 2x = u$

$\therefore$ $x''' = u - 3x'' - 4x' - 2x$

$\therefore$ $\begin{cases} x_1' = x_2 & = 0x_1 + 1x_2 + 0x_3 \\ x_2' = x_3 & = 0x_1 + 0x_2 + 1x_3 \\ x_3' = u - 3x_3 - 4x_2 - 2x_1 & = -2x_1 - 4x_2 - 3x_3 + u \end{cases}$

$\therefore$ $\boxed{\dot{\vec{z}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \cdot \vec{z} + \begin{bmatrix} 0 \\ 0 \\ u \end{bmatrix}}$

b). $\because$ $u = 0$

$\therefore$ $\begin{cases} \dot{\vec{z}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \cdot \vec{z} \\ \vec{z}(0) = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \end{cases}$

$\therefore$ $\dot{\vec{z}}(0) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

$= \boxed{\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}}$

6. Given below is a system of two nonlinear differential equations.

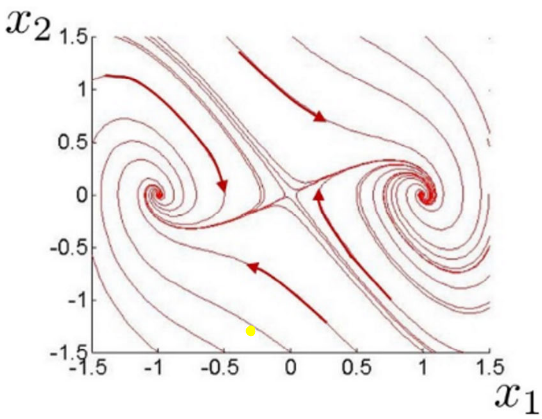
$$\ddot{x} + 2x\dot{y}e^{\dot{x}} + xy = 0$$
$$\ddot{y} + \dot{y}\dot{x} + 2y\sin(x) = 0.$$

Convert the nonlinear system to first-order form.

高阶非线性2元 ODE

$$\begin{cases} x_1 = x \\ x_2 = \dot{x} \\ x_3 = y \\ x_4 = \dot{y} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2 \cdot x_1 \cdot x_4 \cdot e^{x_2} - x_1 \cdot x_3 \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -x_4 \cdot x_2 - 2 \cdot x_3 \cdot \sin(x_1) \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 \cdot x_4 \cdot e^{x_2} - x_1 x_3 \\ x_4 \\ -x_4 \cdot x_2 - 2x_3 \cdot \sin(x) \end{bmatrix}$$

7. A phase portrait of a dynamical system $\dot{x} = f(x)$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is shown below.



(a) Determine the equilibrium points for this system using the displayed axis scales

(b) Estimate $\lim_{t\to\infty} x(t)$ when $x(0) = \begin{bmatrix} 0.5 \\ -1.5 \end{bmatrix}$.

a). From the graph we can see the equilibrium points are $(-1, 0)$ and $(1, 0)$

b). From the graph we know $\lim_{t\to\infty} x(t) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

8. The Lotka-Volterra model for predator-prey dynamics is:

$$\dot{x} = (a - by)x \qquad \begin{cases} \dot{x} = (40 - 2y)x \\ \dot{y} = (2x - 100)y \end{cases}$$
$$\dot{y} = (cx - d)y$$

Where x is the population of prey fish, y is the population of predator sharks, and we choose constants $a = 40$, $b = 2$, $c = 2$, and $d = 400$.

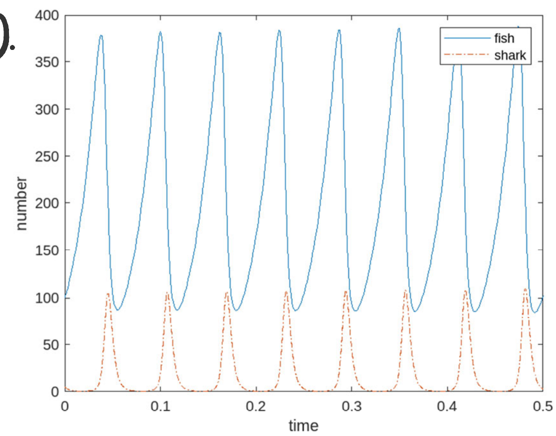
(a) Use MATLAB to simulate the evolution of the system starting from an initial condition of 100 fish, 5 sharks. Plot the population.of fish and sharks as a function of time. What is the minimum and maximum number of sharks that are ever present? Do these numbers suggest any problems with this model?
You can invoke ode45 by creating a new m-file called 'diffeqn' containing the following:

(b) Simulate with different initial conditions and plot the resulting trajectories in state space (the x-y plane) such that one or both of the species i) Die out ii) Grow infinitely large

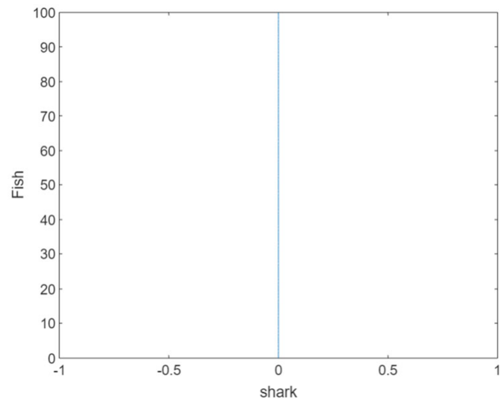
(c) Using MATLAB, construct a vector field plot for this differential equation. MATLAB provides a function called '`quiver`' which can be used to accomplish this.

- `quiver(x,y,u,v)` takes four arguments: the arguments x, y give the coordinates of each vector in the field, and u, v give the components of the vector at that point. It is possible to use explicit arrays u, v as arguments, or to express them as functions of x and y.
- `[X,Y] = meshgrid(a,b)` can be used to generate component matrices to use as arguments with `quiver`. The output matrices have constant rows (columns) with the value in the ith row (column) given by the ith entry of x (y).

a).



b).



Shark min = 109.4582

Shark max = 0.4720

∴ The maximum and minimum are not integer

c).