Chi Squared Tests:

Goodness of fit (9.1)

Independence (9.2)

Last Class!

χ^2 Goodness of Fit Test - Background

Developed by Pearson in 1900

Tests the appropriateness of different models

Approximate test for use with large samples (large n).

Only appropriate when all cell counts are greater than 5
(*Otherwise, consider calculating exact p-value*)

χ^2 Goodness of Fit Test

Random sample of size n classified into k categories or cells.

- Let $Y_1, Y_2, \ldots, Y_k$ denote the respective cell frequencies.

$$\sum_{i=1}^{k} Y_i = n$$

- Denote cell probabilities $p_1, p_2, \ldots, p_k$.
- H_0: $p_1 = p_{10}$, and $p_2 = p_{20}$, … , and $p_k = p_{k0}$.
- H_A: H_0 not true

χ^2 Goodness of Fit Test

	Group 1	Group 2	…	Group k	Total
Observed Freq (O_i)	Y_1	Y_2	…	Y_k	n
Probability | H_0 (p_{i0})	p_{10}	p_{20}	…	p_{k0}	1
Expected Freq (E_i)	np_{10}	np_{20}	…	np_{k0}	n

χ^2 Goodness of Fit Test

Test statistic:

$$X^2 = \sum_{i=1}^{k} \frac{(Y_i - np_{i0})^2}{np_{i0}} = \sum_{i=1}^{k} \frac{(Obs_i - Exp_i)^2}{Exp_i} \sim \chi^2_{(k-1)}$$

Reject H_0 if $X^2 \geq \chi^2_{(k-1),\alpha}$

Chi Squared Goodness of Fit Example

Suppose Bertie Botts Every Flavor Beans has a Holiday sampler with 5 flavors. They claim that there is an even distribution (p=0.2 for each flavor).
David goes through a box of 1000 candies, and counts the following:

Flavor	Number of Pieces of Candy	Expected Number of Pieces of Candy
Lemon	180	200
Green Apple	250	200
Vomit	120	200
Dirt	225	200
Soap	225	200

Suppose Bertie Botts Every Flavor Beans has a Holiday sampler with 5 flavors. They claim that there is an even distribution of flavors (p=0.2 for each flavor). David goes through a box of 1000 candies, and counts the following:

Test whether the claim is true at α = 0.05.

H_0:

H_A:

Flavor	Observed	Expected	Obs - Exp
Lemon	180	200	180-200 = -20
Green Apple	250	200	250-200 = 50
Vomit	120	200	120-200 = -80
Dirt	225	200	225-200 = 25
Soap	225	200	225-200 = 25

2 + 12.5 + 32 + 3.125 + 3.125 = 52.75

p-value:

Decision: “ “

Random Digit Example (Goodness of Fit)

When making random numbers, people are usually reluctant to record the same or consecutive numbers in adjacent positions. Even though these true probabilities should be p_{10} = 1/10 and p_{20} = 2/10, respectively. Albert tests a friend's concept of a random sequence by asking them to generate a sequence of 51 *random* digits.

Suppose they generate the following sequence: (Ex 9.1-1)

5	8	3	1	9	4	6	7	9	2	6	3	0
8	7	5	1	3	6	2	1	9	5	4	8	0
3	7	1	4	6	0	4	3	8	2	7	3	9
8	5	6	1	8	7	0	3	5	2	5	2	

Use a goodness of fit test to make a statistical decision about whether this sequence seems to be truly random or not (at α = 0.05)

5	8	3	1	9	4	6	7	9	2	6	3	0
8	7	5	1	3	6	2	1	9	5	4	8	0
3	7	1	4	6	0	4	3	8	2	7	3	9
8	5	6	1	8	7	0	3	5	2	5	2	

Random Digit Example (Goodness of Fit)

$H_0 : p_1 = p_{10} = 1/10, \quad p_2 = p_{20} = 2/10, \quad p_3 = p_{30} = 7/10$

$H_A :$

	Group 1	Group 2	…	Group k	Total
Observed Freq (O_i)	Y_1 0	Y_2 8	…	Y_3 42	n 50
Probability | H_0 (p_{i0})	p_{10}	p_{20}	…	p_{k0}	1
Expected Freq (E_i)	np_{10}	np_{20}	…	np_{k0}	n

$$\frac{(0-5)^2}{5} + \frac{(8-10)^2}{10} + \frac{(42-35)^2}{35} = 6.8$$

$$6.8 > 5.991 = \chi^2_{0.05}(2).$$

Definition

Contingency table

- Summarizes relationship between categorical variables by displaying frequency distribution.

e.g.

Table 9.2-1 Undergraduates at the University of Iowa

	College					
Gender	Business	Engineering	Liberal Arts	Nursing	Pharmacy	Totals
Male	21	16	145	2	6	190
	(16.625)	(9.5)	(152)	(7.125)	(4.75)	
Female	14	4	175	13	4	210
	(18.375)	(10.5)	(168)	(7.875)	(5.25)	
Totals	35	20	320	15	10	400

χ^2 Test for Homogeneity & Independence (9.2)

Tests relationship between two categorical variables

The process of testing is the same. Only the name is different

Difference in names depends on how data is collected.

χ^2 Test for **Homogeneity** (9.2)

χ^2 Test for Homogeneity: (**one** margin fixed)

Tests whether two or more sub-groups of a population share the same distribution of a single categorical variable. E.g., do different age groups have the same proportion of people who prefer Twitch, YouTube Live, or Zoom?

Independent random samples from r populations.

Each sample is classified into c response categories.

H_0 : In each category, the probabilities are equal for all r populations.

χ^2 Test for **Independence** (9.2)

χ^2 Test for Independence: (**no** margins fixed)

Tests whether two categorical variables are associated with one another in the population, e.g. age group vs video streaming platform preference.

A random sample of size n is simultaneously classified with respect to two characteristics, one has r categories and the other c categories.

H_0 : The two classifications are independent; i.e., each cell probability is the product of the row and column marginal probabilities

χ^2 Test for Homogeneity & Independence (9.2)

Test statistic (both test):

$$X^2 = \sum \frac{(O-E)^2}{E} \sim \chi^2_{(r-1)*(c-1)}$$

O = observed cell frequency

$$E = \frac{\text{row total} * \text{column total}}{\text{overall total}}$$

(more formally): $$X^2 = \sum_{j=1}^{h} \sum_{i=1}^{k} \frac{(Y_{ij} - n_j p_{ij})^2}{n_j p_{ij}} \sim \chi^2_{(r-1)(c-1)}$$

χ^2 Test of homogeneity example:

Edward is wondering which instructor to select for Stat 420. Kayleigh doesn't know much about them, so she states

H_0: their grade distributions are the same.

We collect 2 separate random samples from each instructor and obtain the following data.

	Grade					
	A	B	C	D	F	Totals
Group I	8	13	16	10	3	50
Group II	4	9	14	16	7	50

Perform a χ^2 test of homogeneity at significance level 0.05 to determine if these grade distributions are similar.

	Grade					
	A	B	C	D	F	Totals
Group I	8	13	16	10	3	50
Group II	4	9	14	16	7	50

$$q = \frac{(8-6)^2}{6} + \frac{(13-11)^2}{11} + \frac{(16-15)^2}{15} + \frac{(10-13)^2}{13} + \frac{(3-5)^2}{5}$$

$$+ \frac{(4-6)^2}{6} + \frac{(9-11)^2}{11} + \frac{(14-15)^2}{15} + \frac{(16-13)^2}{13} + \frac{(7-5)^2}{5}$$

$$= \frac{4}{6} + \frac{4}{11} + \frac{1}{15} + \frac{9}{13} + \frac{4}{5} + \frac{4}{6} + \frac{4}{11} + \frac{1}{15} + \frac{9}{13} + \frac{4}{5} = 5.18.$$

Example

A random sample of 400 undergraduate students at the University of Iowa was selected, then classified by college and gender. Are these variables independent at α=0.01?

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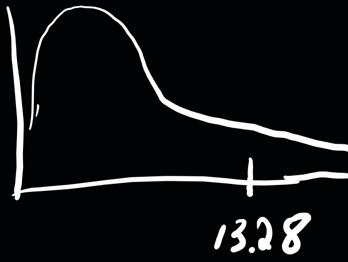

$$\frac{(21-16.625)^2}{16.625}+\frac{(14-18.375)^2}{18.375}+\cdots+\frac{(4-5.25)^2}{5.25}$$

$$1.15+1.04+4.45+4.02+0.32+0.29+3.69$$

$$+3.34+0.33+0.30 \;=\; 18.93.$$

Critical value: 13.28