Lec03: Heat (Diffusion) Equation

Chapter Two

Section 2.3-2.5

Announcement

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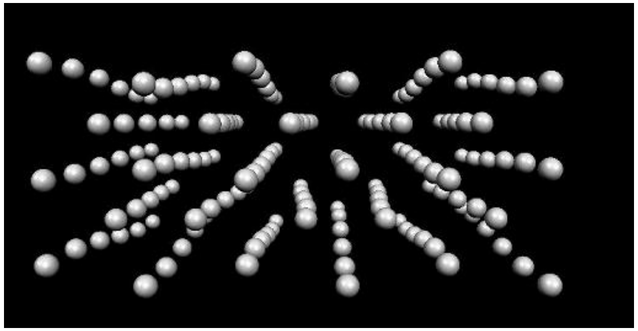
- HW 1 due today
  - Upload to BB
- Quiz02 due today

Recap:

- Fourier Law (3 different coordinate systems)

$$q = -kA\nabla T \quad \text{or} \quad q'' = -k\nabla T$$



Collective atomic vibration - Phonon

- Thermal conductivity
  - Solids (Phonons + Electrons + others)

$$k = \frac{1}{3}\underbrace{C}_{\text{energy carrier specific heat per unit volume.}}\underbrace{\overline{v}}_{\text{average energy carrier velocity, } \overline{v} < \infty.}\underbrace{\lambda_{\text{mfp}}}_{\text{mean free path} \rightarrow \text{average distance traveled by an energy carrier before a collision with something in solid.}} \qquad (2.7)$$

  - Fluids
  - Mixtures (Effective medium)
  - Nanoscale effects

- Thermal Diffusivity
  - $\alpha = \dfrac{k}{\rho c_p}$

Content and Quiz 03

1. Write down the basic heat diffusion equations in 3 different coordinate systems.

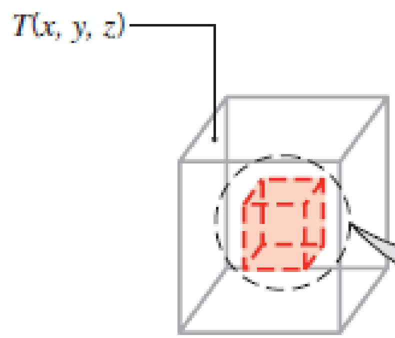
2. List the 3 common boundary conditions and the associated expressions

3. Write down the equation for 1D heat diffusion equation across a wall of thickness x

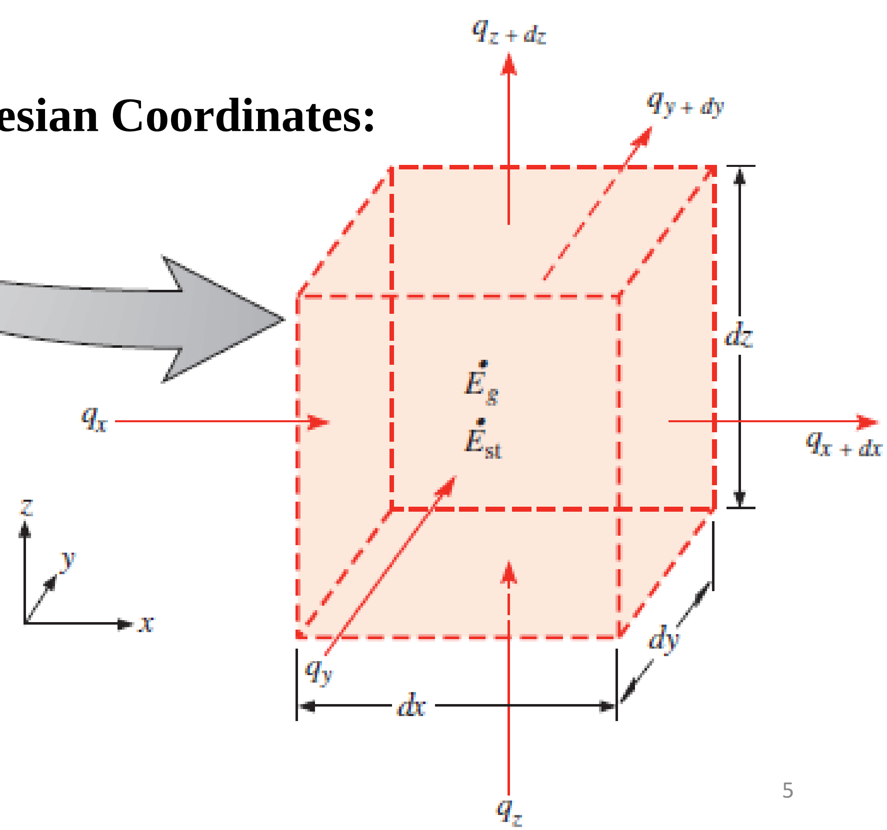
4. What is the equivalent thermal resistance in this wall with surface temperatures T_1 and T_2?

5. What is the expression for the overall heat transfer coefficient U for a composite wall?

- To calculate heat flux at any point, need Temperature Distribution =>
- Temperature distribution from Heat Diffusion Equation =>
- Heat Diffusion Equation from Conservation of energy applied to a differential control volume

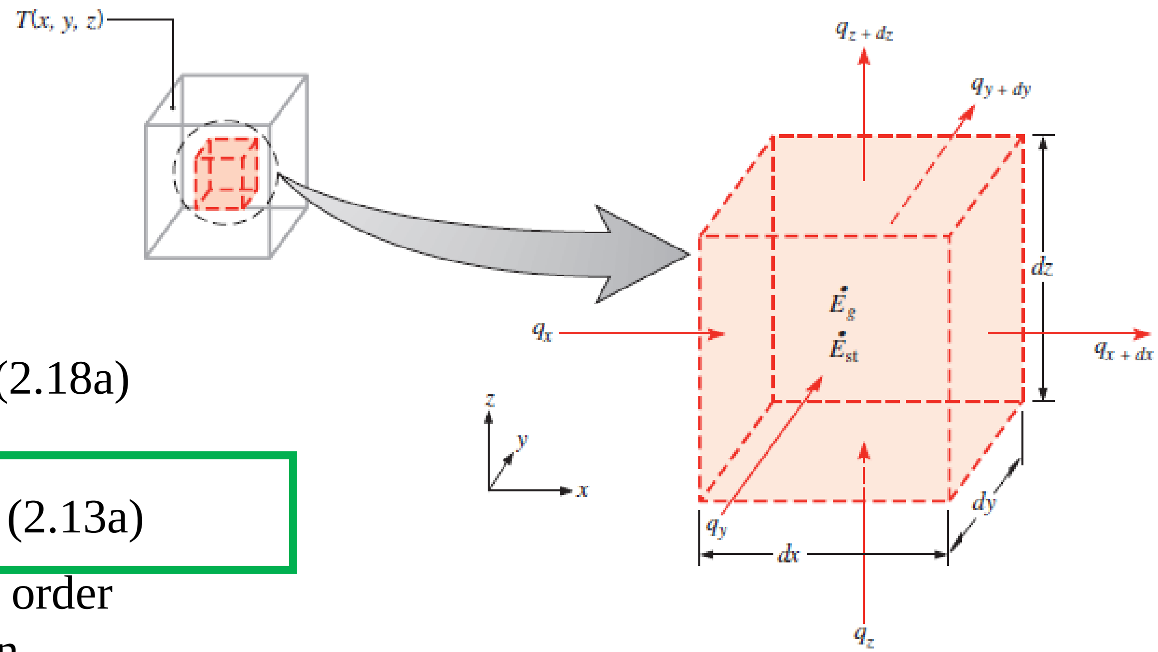


Cartesian Coordinates:



From this picture, can you identify the terms for the 1st Law:

- $\dot{E}_{in}$ (surface)
- $\dot{E}_{out}$ (surface)
- $\dot{E}_g$ (volume)
- $\dot{E}_{st}$ (volume)



Heat Conduction:

$$q_x = -k dydz \frac{\partial T}{\partial x} \qquad \text{(2.18a)}$$

$$q_{x+dx} = \boxed{q_x + \frac{\partial q_x}{\partial x} dx \qquad \text{(2.13a)}}$$

- Taylor expansion to the 1st order
- Same for y- and z-direction

Heat Generation:

$$\dot{E}_g = \boxed{\dot{q}_g dxdydz \quad \text{(2.14)}}$$

- +ve if heat generated

Heat Storage:

$$\dot{E}_{st} = \boxed{\frac{\partial U_{sens}}{\partial t} = \rho c_v \frac{\partial T}{\partial t} dxdydz \quad \text{(2.15)}}$$

- Assume incompressible => density unchanged => $c_v = c_p$
- No phase change => no latent heat

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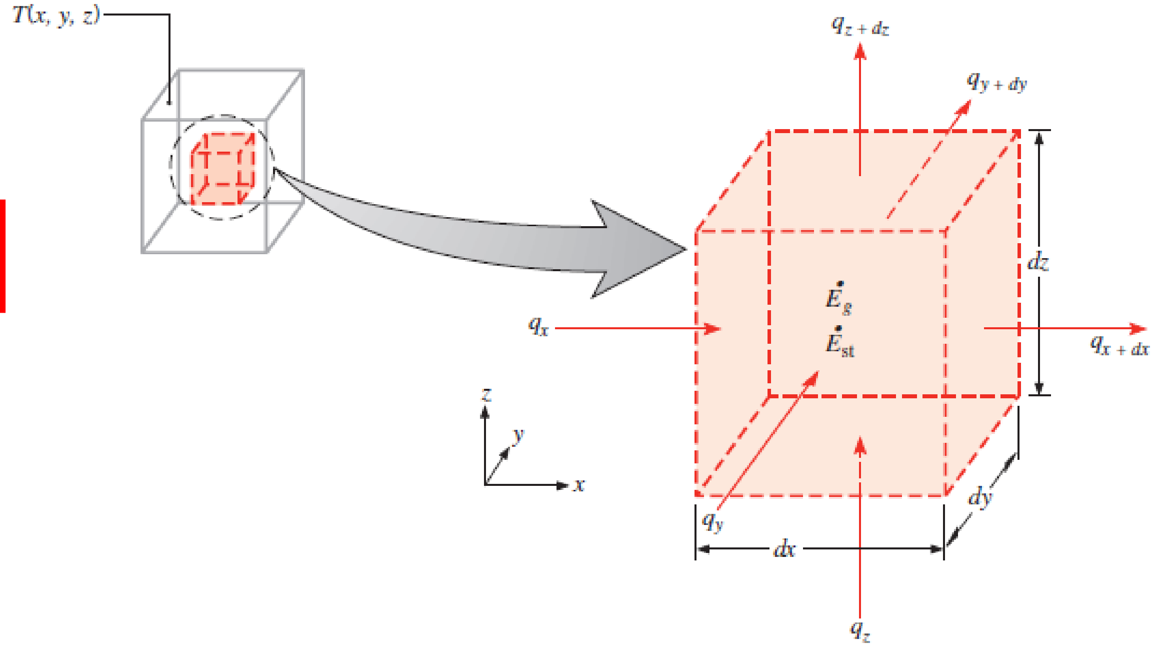
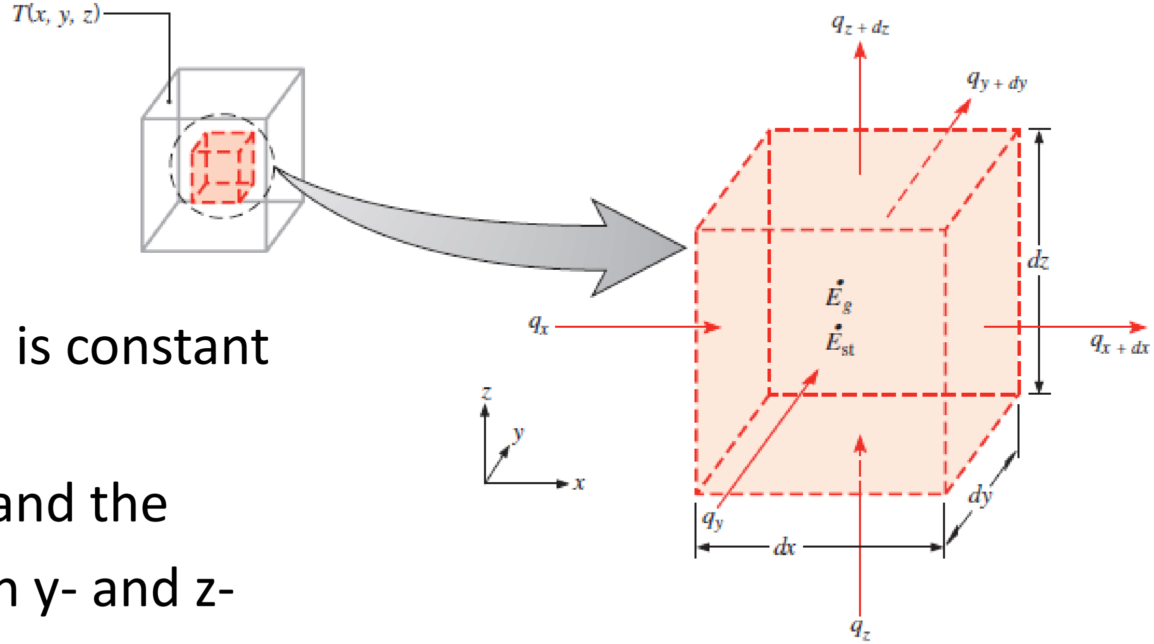
Applying 1st Law,

- $\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$



- $$q_x + q_y + q_z - \left(q_x + \frac{\partial q_x}{\partial x}dx\right) - \left(q_y + \frac{\partial q_y}{\partial y}dy\right) - \left(q_z + \frac{\partial q_z}{\partial z}dz\right) + \dot{q}_g dxdydz = \rho c_v \frac{\partial T}{\partial t}dxdydz$$



- Assume X-sectional area is constant
  => *dxdy = dydz = dxdz*
- Using $q_x = -kdydz\frac{dT}{dx}$ and the equivalent expressions in y- and z-direction,

$$\underbrace{\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)}_{\text{Net transfer of thermal energy into the control volume (inflow-outflow)}}+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.19)$$

Net transfer of thermal energy into the control volume (inflow-outflow)

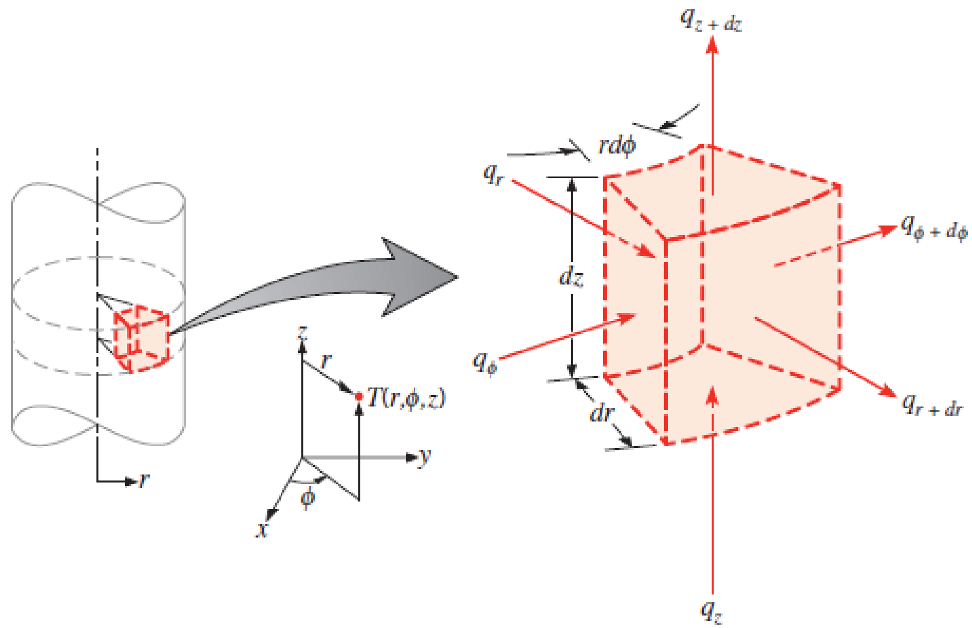
Thermal energy generation

Change in thermal energy storage

Heat Equation in other coordinate systems

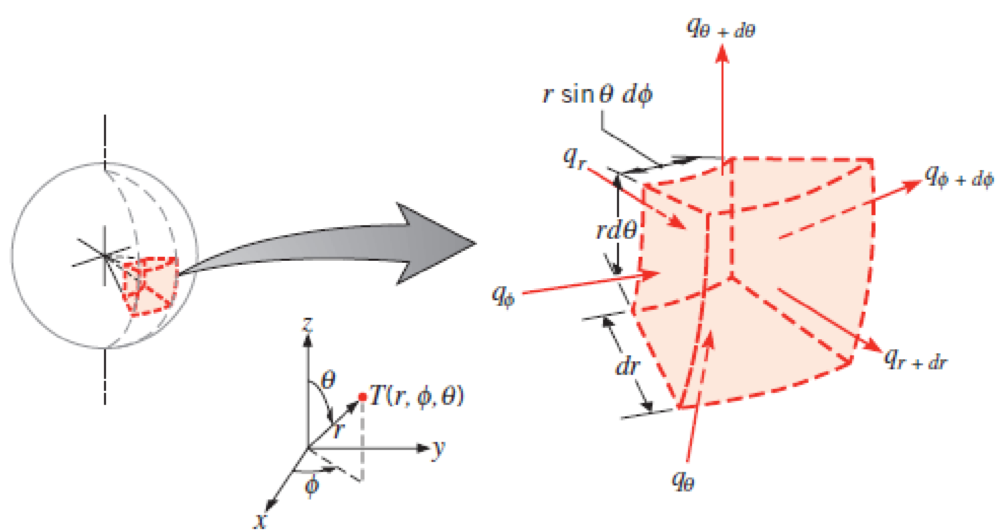
Cylindrical Coordinates:

$$\boxed{\frac{1}{r}\frac{\partial}{\partial r}\left(kr\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.26)}$$



Spherical Coordinates:

$$\boxed{\frac{1}{r^2}\frac{\partial}{\partial r}\left(kr^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial \phi}\left(k\frac{\partial T}{\partial \phi}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(k\sin\theta\frac{\partial T}{\partial \theta}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t} \qquad (2.29)}$$



Simplifications

- Steady-State => $\dot{E}_{st} = 0$
- No generation => $\dot{E}_g = 0$

Example: One-Dimensional Conduction in the x-dir in a Planar Medium with Constant *k* and area and No Heat Generation. Simplifying from (2.19),

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

where $\alpha \equiv \frac{k}{\rho c_p} \rightarrow$ thermal diffusivity of the medium $[\mathrm{m}^2/\mathrm{s}]$

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Complications

- If k is not constant => k has to remain inside the derivative operator
- If area changes with direction => area cannot be canceled away but kept inside the derivative operator

Example 1

The temperature distribution across a wall 1 m thick at a certain instant of time is given as

$$T(x) = a + bx + cx^2$$

where T is in degrees Celsius and x is in meters, while $a = 900°\mathrm{C}$, $b = -300°\mathrm{C/m}$, and $c = -50°\mathrm{C/m^2}$. A uniform heat generation, $\dot{q} = 1000\ \mathrm{W/m^3}$, is present in the wall of area $10\ \mathrm{m^2}$ having the properties $\rho = 1600\ \mathrm{kg/m^3}$, $k = 40\ \mathrm{W/m \cdot K}$, and $c_p = 4\ \mathrm{kJ/kg \cdot K}$.

1. Determine the rate of heat transfer entering the wall ($x = 0$) and leaving the wall ($x = 1$ m).
2. Determine the rate of change of energy storage in the wall.
3. Determine the time rate of temperature change at $x = 0$, 0.25, and 0.5 m.

Basically, find $\dot{E}_{\text{in}}$ and $\dot{E}_{\text{out}}$ as there is no work and no mass flow. These reduce to q_{in} and q_{out}

Basically, find $\dot{E}_{\text{st}}$

Find $\frac{dT}{\mathrm{dt}}$

Example 1 - Solution

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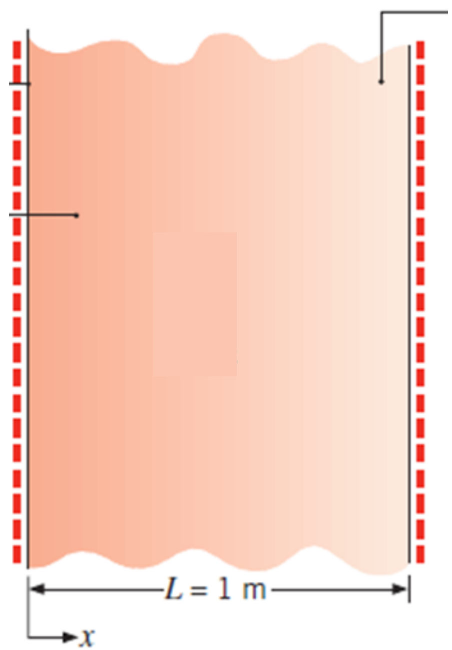
Known: Temperature distribution $T(x)$ at an instant of time t in a one-dimensional wall with uniform heat generation.

Find:

1. Heat rates entering, q_{in} $(x = 0)$, and leaving, q_{out} $(x = 1\text{ m})$, the wall.
2. Rate of change of energy storage in the wall, $\dot{E}_{st}$.
3. Time rate of temperature change at $x = 0$, 0.25, and 0.5 m.

Schematic:



Example 1 - Solution

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Assumptions:

1. One-dimensional conduction in the x-direction.
2. Isotropic medium with constant properties.
3. Uniform internal heat generation, $\dot{q}$ ($\mathrm{W/m^3}$).

Analysis:

1. Recall that once the temperature distribution is known for a medium, it is a simple matter to determine the conduction heat transfer rate at any point in the medium or at its surfaces by using Fourier's law. Hence the desired heat rates may be determined by using the prescribed temperature distribution with Equation 2.1. Accordingly,

$$q_{\text{in}} = q_x(0) = -kA\left.\frac{\partial T}{\partial x}\right|_{x=0} = -kA(b+2cx)_{x=0}$$

$$q_{\text{in}} = -bkA = 300°\mathrm{C/m} \times 40\ \mathrm{W/m\cdot K} \times 10\ \mathrm{m^2} = 120\ \mathrm{kW} \quad \triangleleft$$

$$q_{\text{out}} = q_x(L) = -kA\left.\frac{\partial T}{\partial x}\right|_{x=L} = -kA(b+2cx)_{x=L}$$

$$q_{\text{out}} = -(b+2cL)kA = -[-300°\mathrm{C/m} + 2(-50°\mathrm{C/m^2}) \times 1\ \mathrm{m}] \times 40\ \mathrm{W/m\cdot K} \times 10\ \mathrm{m^2} = 160\ \mathrm{kW} \quad \triangleleft$$

Example 1 - Solution

2. The rate of change of energy storage in the wall $\dot{E}_{st}$ may be determined by applying an overall energy balance to the wall. Using Equation 1.12c for a control volume about the wall,

$$\dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = \dot{E}_{st}$$

where $E_g = \dot{q}AL$, it follows that

$$\dot{E}_{st} = \dot{E}_{in} + \dot{E}_g - \dot{E}_{out} = q_{in} + \dot{q}AL - q_{out}$$

$$\dot{E}_{st} = 120\ \text{kW} + 1000\ \text{W/m}^3 \times 10\ \text{m}^2 \times 1\ \text{m} - 160\ \text{kW}$$

$$\dot{E}_{st} = -30\ \text{kW} \qquad \triangleleft$$

Example 1 - Solution

3. The time rate of change of the temperature at any point in the medium may be determined from the heat equation, Equation 2.21, rewritten as

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_p}\frac{\partial^2 T}{\partial x^2} + \frac{\dot{q}}{\rho c_p}$$

From the prescribed temperature distribution, it follows that

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}(b + 2cx) = 2c = 2(-50°\text{C/m}^2) = -100°\text{C/m}^2$$

Note that this derivative is independent of position in the medium. Hence the time rate of temperature change is also independent of position and is given by

$$\frac{\partial T}{\partial t} = \frac{40\ \text{W/m}\cdot\text{K}}{1600\ \text{kg/m}^3 \times 4\ \text{kJ/kg}\cdot\text{K}} \times (-100°\text{C/m}^2)$$

$$+ \frac{1000\ \text{W/m}^3}{1600\ \text{kg/m}^3 \times 4\ \text{kJ/kg}\cdot\text{K}}$$

$$\frac{\partial T}{\partial t} = -6.25 \times 10^{-4}\,°\text{C/s} + 1.56 \times 10^{-4}\,°\text{C/s}$$

$$= -4.69 \times 10^{-4}\,°\text{C/s}$$

Boundary and Initial Conditions

Initial and Boundary Conditions

- Heat Diffusion Equation is a differential equation
- Depends on the **Initial and Boundary conditions**

So, the question is how many do we need?

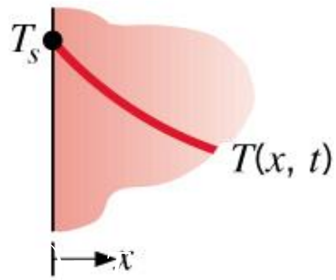
$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)+\dot{q}=\rho c_p\frac{\partial T}{\partial t}$$

Since heat equation is **first order in time** and **second order in space**,

- **One initial condition** or **one time-dependent condition** is needed if it is not steady state
- **Two boundary conditions** must be specified for **each coordinate direction**.

Some common boundary/surface conditions cases (eqn 2.31 – 2.34):

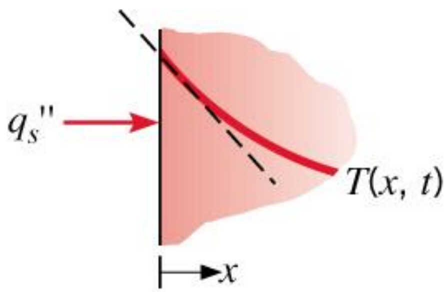
1. **Constant Surface Temperature (Dirichlet condition):** (example, in contact with boiling/melting materials)



$$T(0,t) = T_s$$

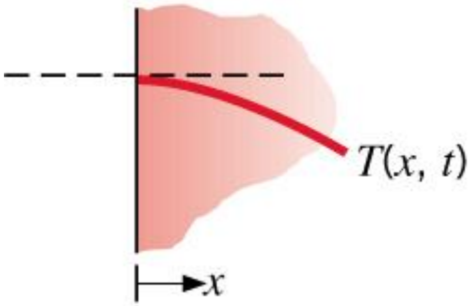
2. **Constant Heat Flux (Neumann condition):** (example, attached to a constant heater)

Applied Flux



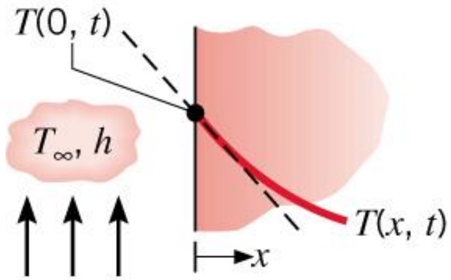
$$-k\frac{\partial T}{\partial x}|_{x=0} = q_s''$$

Insulated Surface



$$\frac{\partial T}{\partial x}|_{x=0} = 0$$

3. **Convection:**



$$-k\frac{\partial T}{\partial x}|_{x=0} = h[T_\infty - T(0,t)]$$

- Can you write **one with radiation** at the surface?

Example 2

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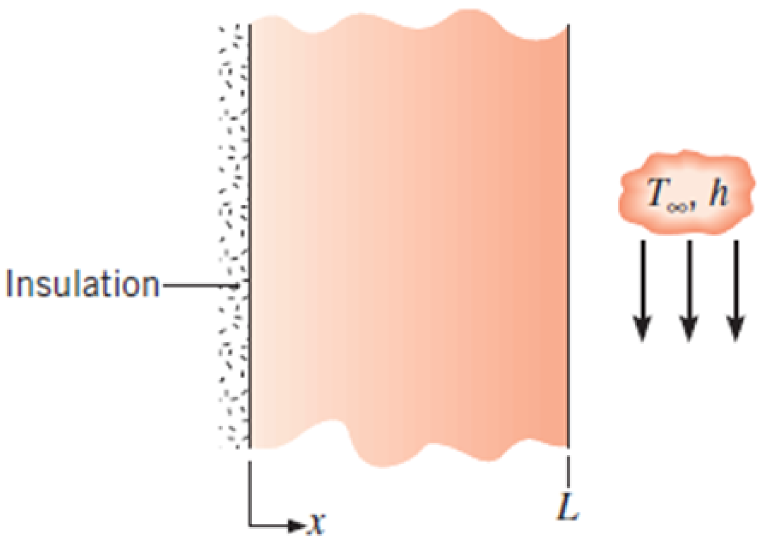
The plane wall with constant properties and no internal heat generation shown in the figure is initially at a uniform temperature T_i. Suddenly the surface at $x = L$ is heated by a fluid at T_∞ having a convection heat transfer coefficient h. The boundary at $x = 0$ is perfectly insulated.



(a) Write the differential equation, and identify the boundary and initial conditions that could be used to determine the temperature as a function of position and time in the wall.

(b) On $T - x$ coordinates, sketch the temperature distributions for the following conditions: initial condition ($t \le 0$), steady-state condition ($t \to \infty$), and two intermediate times.

(c) On $q''_x - t$ coordinates, sketch the heat flux at the locations $x = 0$, $x = L$. That is, show qualitatively how $q''_x(0, t)$ and $q''_x(L, t)$ vary with time.

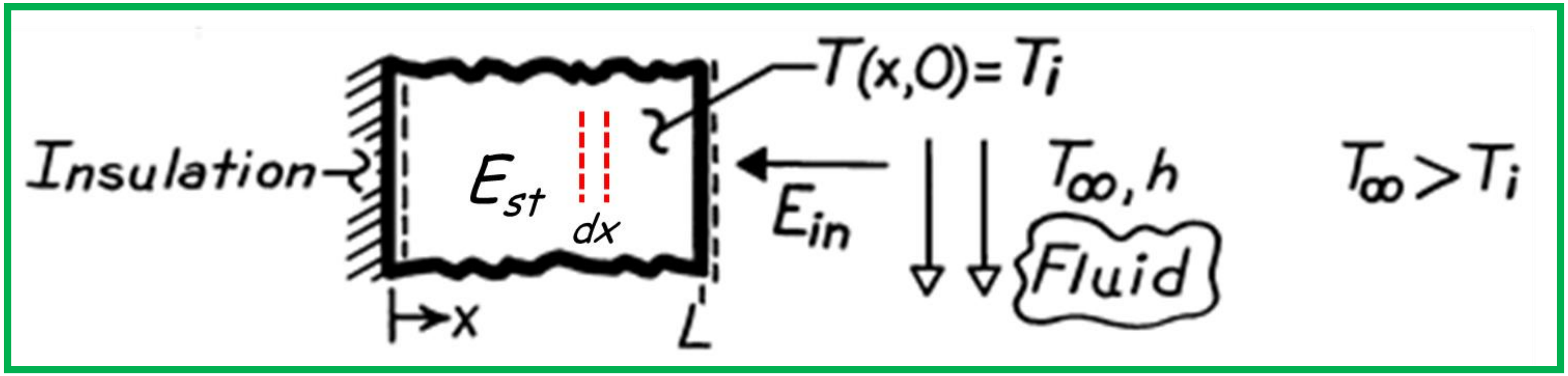
(d) Write an expression for the total energy transferred to the wall per unit volume of the wall (J/m^3).

Example 2 - Solution

KNOWN: Plane wall, initially at a uniform temperature, is suddenly exposed to convective heating.

FIND:
(a) Differential equation and initial and boundary conditions which may be used to find the temperature distribution, $T(x,t)$;
(b) Sketch $T(x,t)$ for the following conditions: initial ($t \leq 0$), steady-state ($t \to \infty$), and two intermediate times;
(c) Sketch heat fluxes as a function of time at the two surfaces;
(d) Expression for total energy transferred to wall per unit volume [J/m^3].

SCHEMATIC:



Example 2 - Solution

ASSUMPTIONS: (1) One-dimensional conduction, (2) Constant properties, (3) No internal heat generation.

(a) Differential equation and initial and boundary conditions which may be used to find the temperature distribution, $T(x,t)$;

ANALYSIS: (a) For one-dimensional conduction with constant properties, the heat equation and boundary condition have the form,

$$\dot{E}_{in} - \dot{E}_{out} + \dot{E}_g = \dot{E}_{st}$$

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$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

Boundary Conditions :

$$\frac{\partial T}{\partial x}|_{x=0} = 0$$

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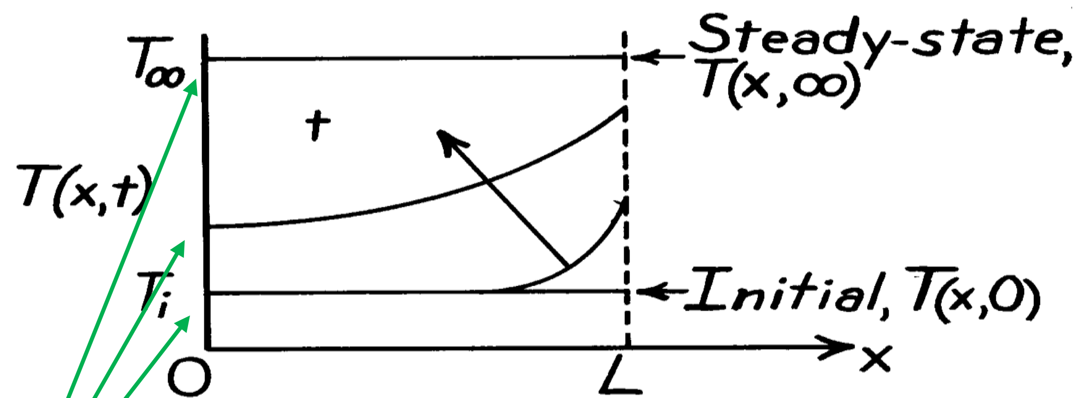
$$-k\frac{\partial T}{\partial x}|_{x=L} = h[T_\infty - T(L,t)]$$

(b) The temperature distributions are shown on the sketch.



Boundary Conditions

$$\frac{\partial T}{\partial x}|_{x=0} = 0 \text{ at any time t}$$

$$-k\frac{\partial T}{\partial x}|_{x=0} = h[T_\infty - T(L,t)]$$

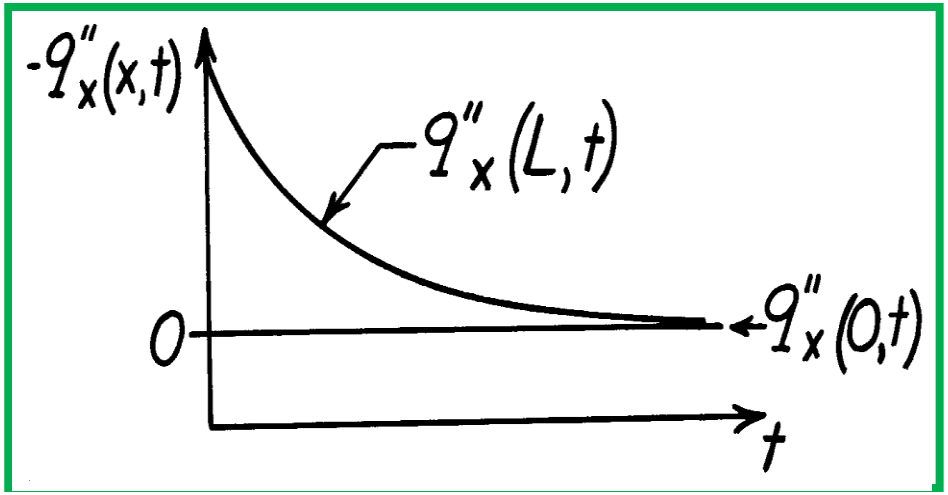
- **The gradient at x = 0 is always zero, since this boundary is adiabatic.**
- **The gradient at x = L decreases with time. Why?**

Example 2 - Solution

c) The heat flux, $q''_x(x,t)$, as a function of time, is shown on the sketch for the surface

$x = 0$ and

$x = L$.



$$q''_x(x,t) = -k\frac{\partial[T(t)]}{\partial x}$$

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d) The total energy transferred to the wall may be expressed as

$$E_{in} = \int_0^\infty q''_{\text{conv}} A_s dt$$

$$E_{in} = hA_s \int_0^\infty (T_\infty - T(L,t))dt$$

Dividing both sides by A_sL, the energy transferred per unit volume is

$$\frac{E_{in}}{V} = \frac{h}{L}\int_0^\infty [T_\infty - T(L,t)]dt \qquad [\text{J/m}^3]$$

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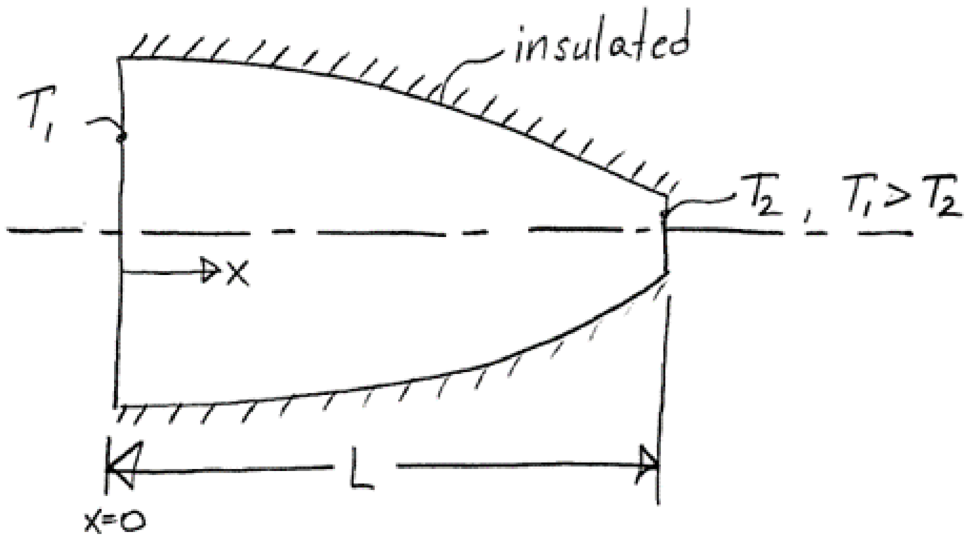
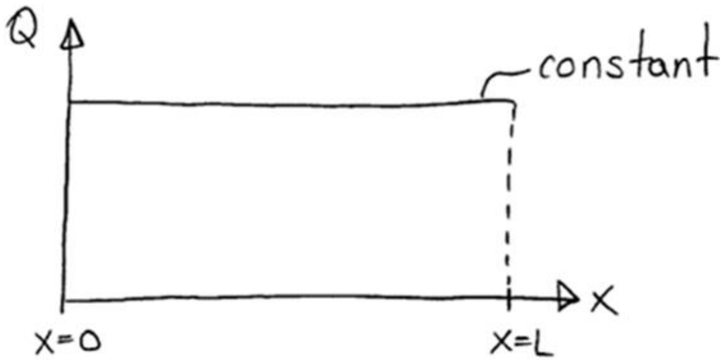
Example | Assume 1-D, steady-state, heat conduction through the axisymmetric shape below. Assume no heat generation & that it is well insulated on the outsides. Assume constant properties.



Sketch the heat flux distribution and temperature distribution as a function of x. Estimate the general trend, no need detailed calculation.

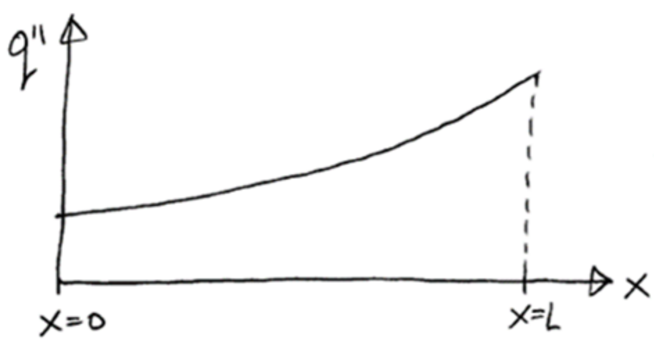
Example 3 - solution

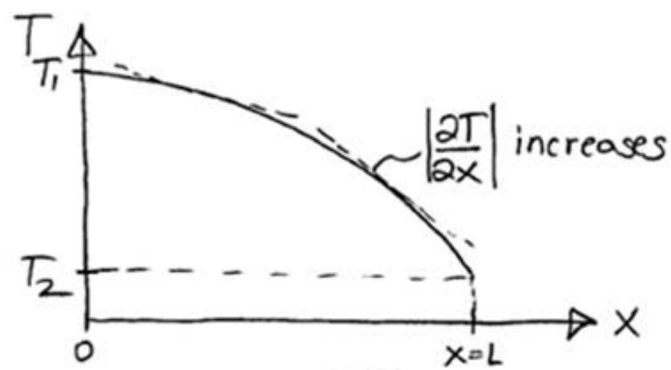
Conservation of energy



$$\Rightarrow Q = -kA_x\frac{\partial T}{\partial x} = \text{constant}$$

A_x is decreasing so $\frac{\partial T}{\partial x}\uparrow$





Short Summary

- Heat Diffusion Equations
  - 3 different coordinate systems
  - Simplications
  - Complications
- Boundary Conditions

One-Dimensional, Steady-State Conduction without Thermal Energy Generation

Chapter Three

Section 3.1 Plane Wall

Plane Wall

1D Heat equation

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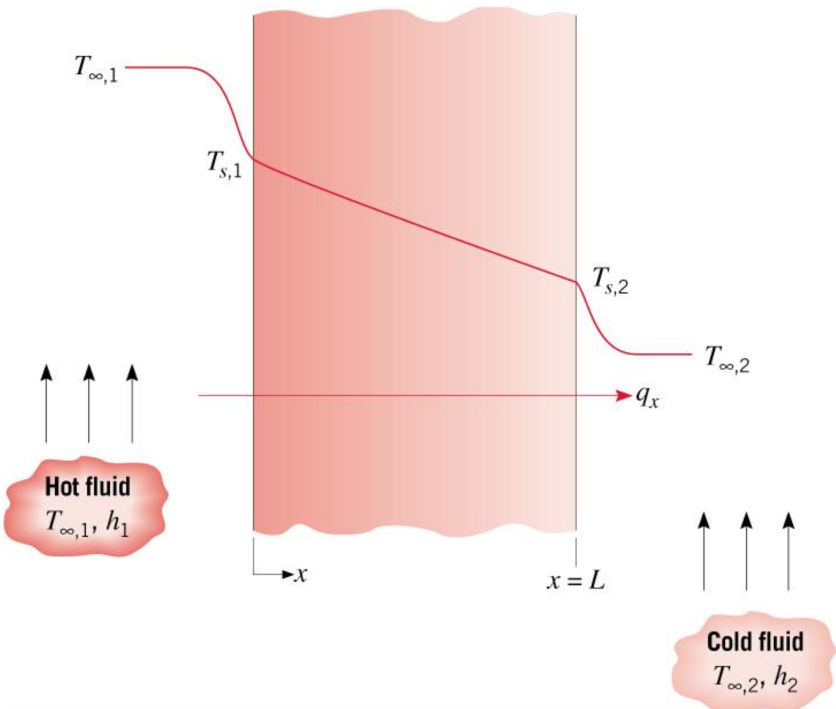
Consider a plane wall between two fluids of different temperature:



1. Heat Equation:

$$\dot{E}_{in} - \dot{E}_{\text{out}} + \dot{E}_{\text{g}} = \dot{E}_{st}$$

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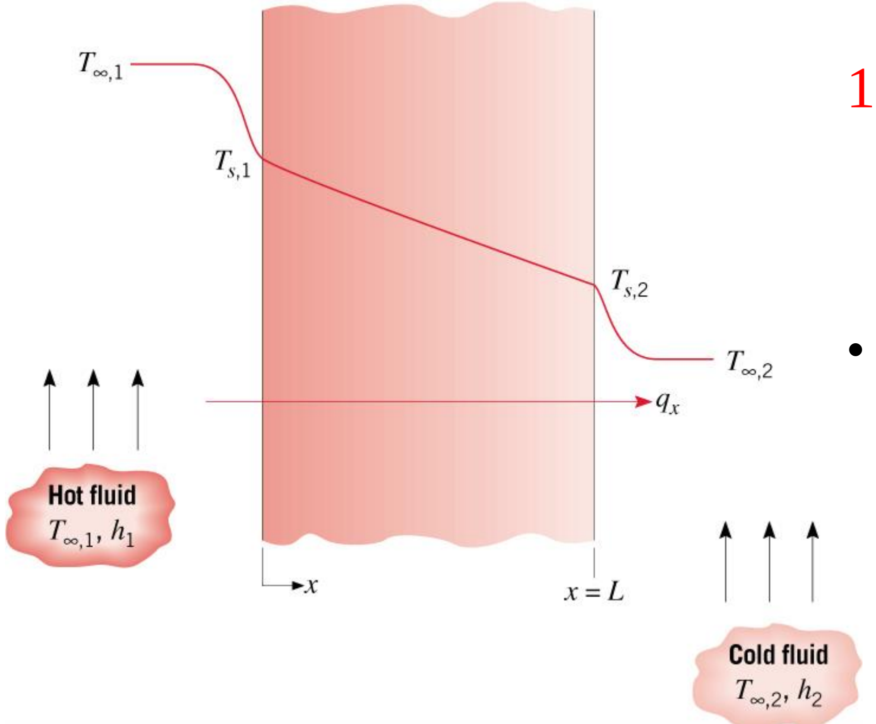
Consider a plane wall between two fluids of different temperature:



1. Heat Equation simplified to:

$$\boxed{\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \qquad (3.1)}$$

- Implications:

  What is $\left(k\frac{dT}{dx}\right)$?

  Heat flux $\left(q''_x\right)$ is independent of x.

  Heat rate $\left(q_x\right)$ is independent of x.

2. Boundary Conditions:

$$\boxed{T(0) = T_{s,1}, \quad T(L) = T_{s,2}}$$

- Temperature Distribution for constant k:

$$\boxed{T(x) = T_{s,1} + \left(T_{s,2} - T_{s,1}\right)\frac{x}{L}} \qquad (3.3)$$

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3. To find Heat Flux (apply Fourier Law and differentiate temperature):

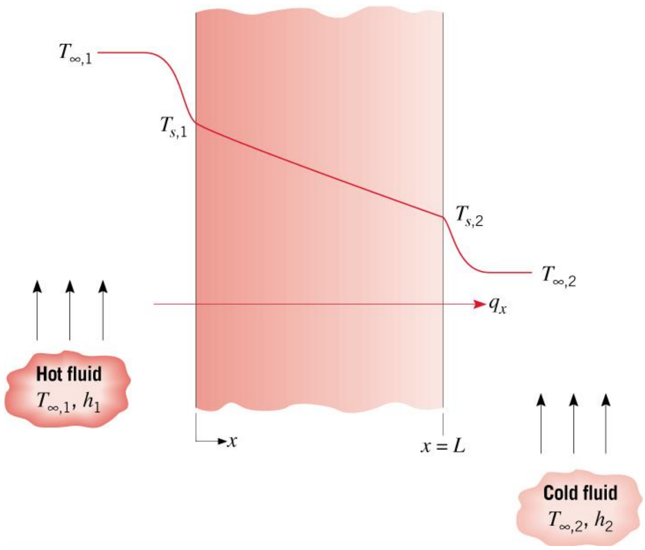
$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}\left(T_{s,1} - T_{s,2}\right) \tag{3.4}$$

$$q''_x = -k\frac{dT}{dx} = \frac{k}{L}\left(T_{s,1} - T_{s,2}\right) \tag{3.5}$$

Note:

- Difficult to measure the surface temperature T_s but easier to measure the fluid temperatures ($T_{\infty,1}$ and $T_{\infty,2}$).
- So, how to get the surface temperatures, T_s?
- Use surface balance to get $T_{s,1}$ and $T_{s,2}$:

$$-k\frac{\partial T}{\partial x}|_x = h\left[T_{\infty,x} - T_{s,x}\right]$$



Thermal Resistance

Applicable to 1D HT

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Thermal Resistances $\left(R_t = \dfrac{\Delta T}{q}\right)$ **and Thermal Circuits:**

Conduction in a plane wall:

$$R_{t,\text{cond}} = \frac{L}{kA} \quad (3.6)$$

Convection:

$$R_{t,\text{conv}} = \frac{1}{hA} \quad (3.9)$$

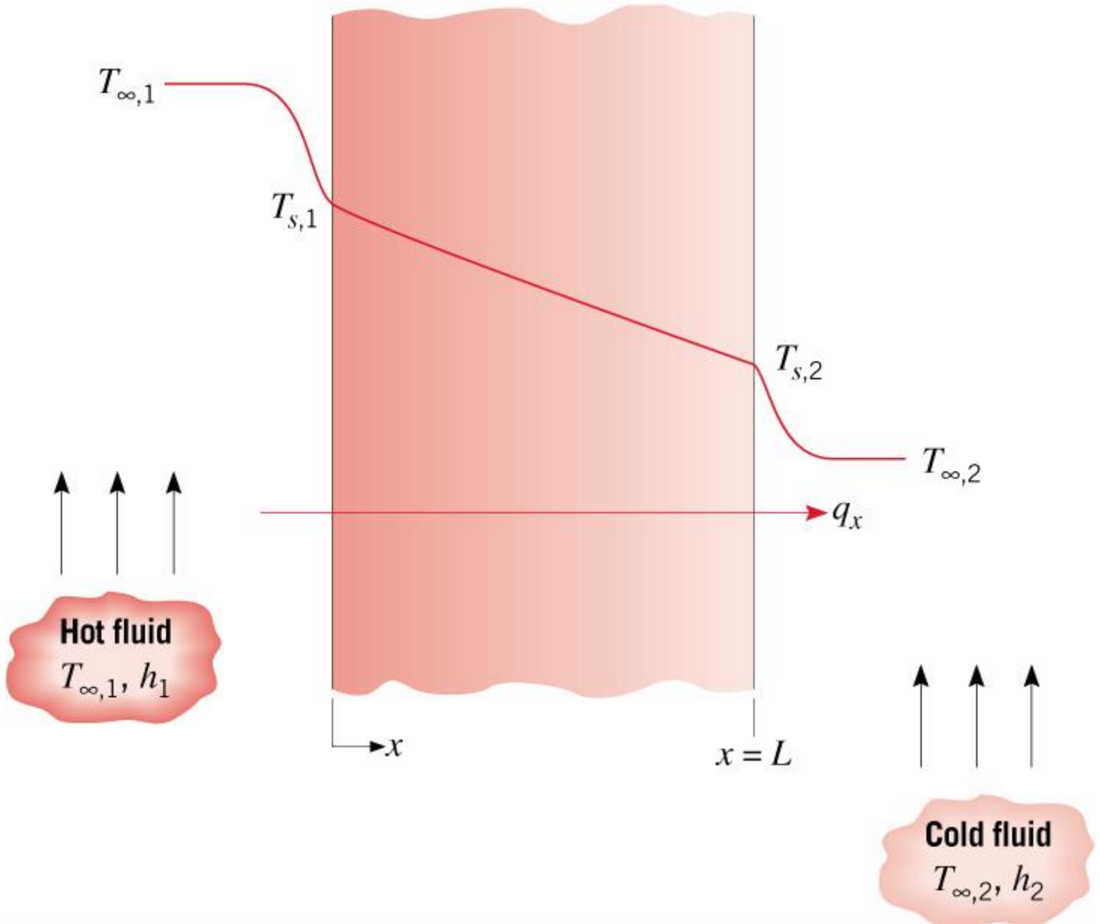
Radiation (can you derive?):

$$R_{t,\text{rad}} = ?$$

What assumptions are needed to use Thermal Circuits?

- Steady-state (SS) (E_{st} = 0)
- No heat generation (E_g = 0)

Thermal Resistance - Plane Wall

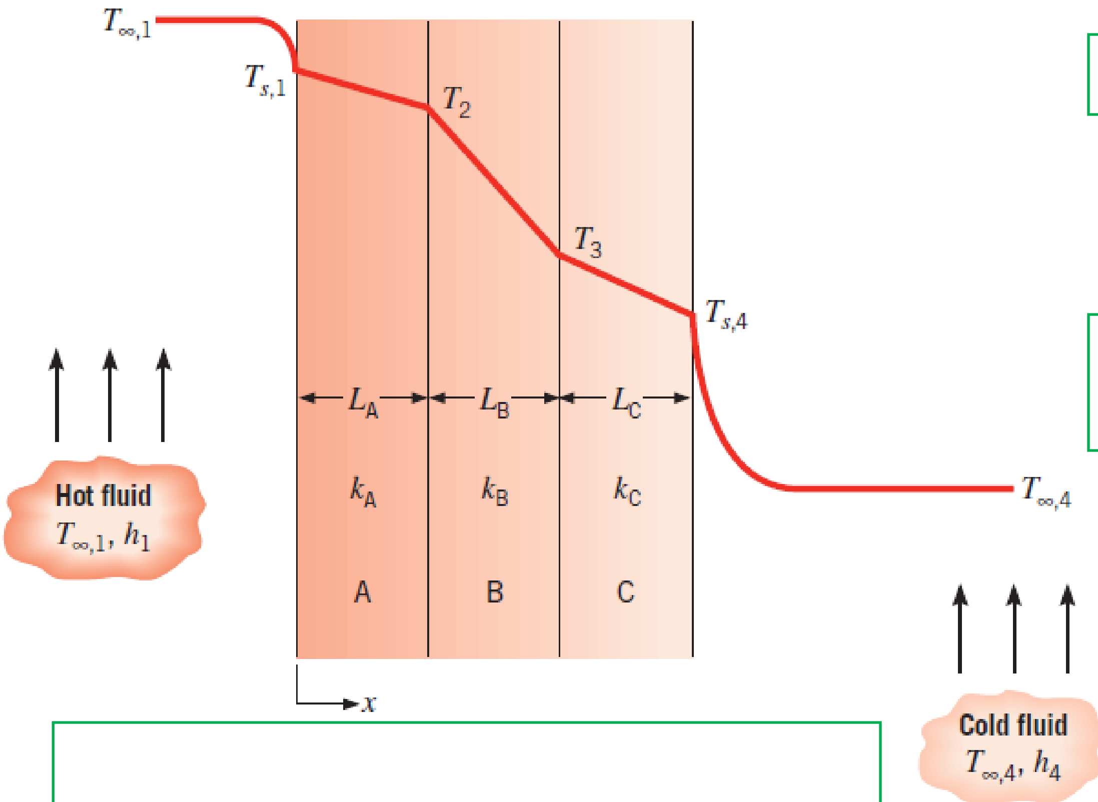


Thermal circuit for plane wall with adjoining fluids:

Composite Plane Wall

(with no contact resistance)

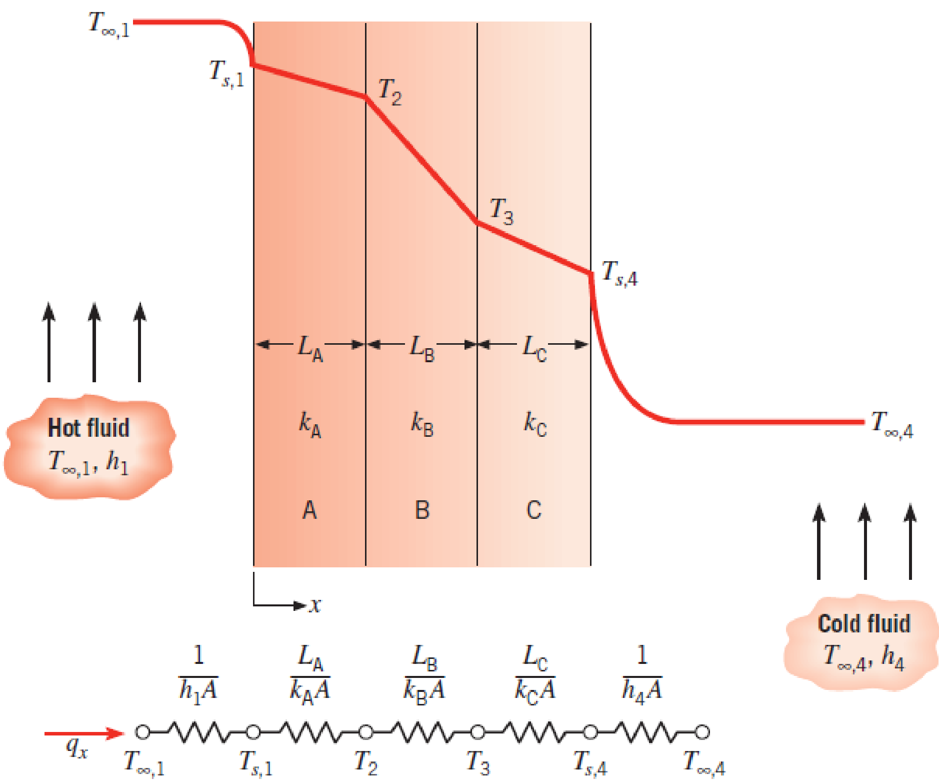
One solid wall == too simple => let us try many connected walls!



For the temperature gradients which wall has the highest k?

Are these resistances in series or parallel? Why?

$$\sum R_t = R_{\text{tot}}$$



$$R_{\text{tot}} = \sum R_t$$
$$= \frac{1}{A}\left[\frac{1}{h_1} + \frac{L_A}{k_A} + \frac{L_B}{k_B} + \frac{L_C}{k_C} + \frac{1}{h_4}\right]$$
$$= \frac{1}{AU}$$

So many resistances! Can we simplify?

Overall Heat Transfer Coefficient (U) :

Concept from Newton's law of cooling equation:

From:

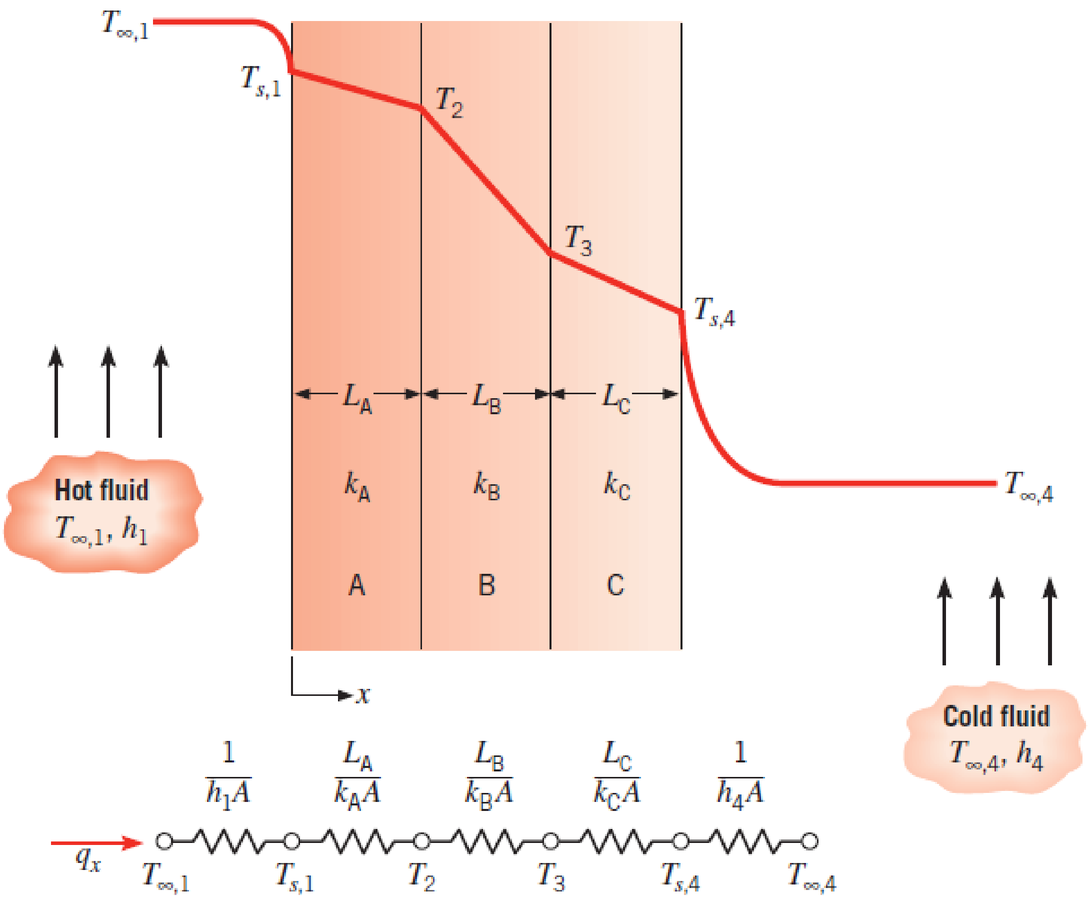
$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t} \quad (3.14)$$

Convert to:

$$q_x = UA\Delta T_{\text{overall}} \quad (3.17)$$

$$\frac{1}{UA} = \sum R_t = R_{tot} \quad (3.19)$$

Also, as q_x **is the same across all walls =>**



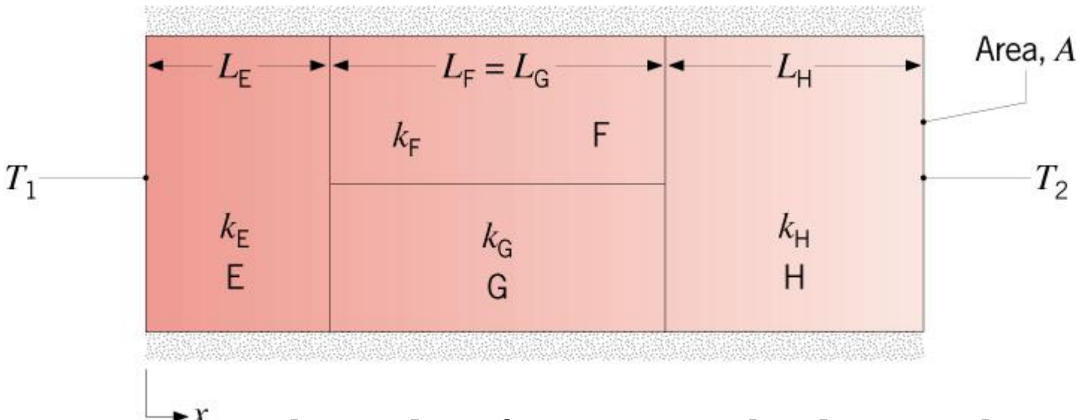
Hot Fluid to Wall A
= Inside Wall A
= Inside Wall B
= ….

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1 A)} \qquad (3.16)$$

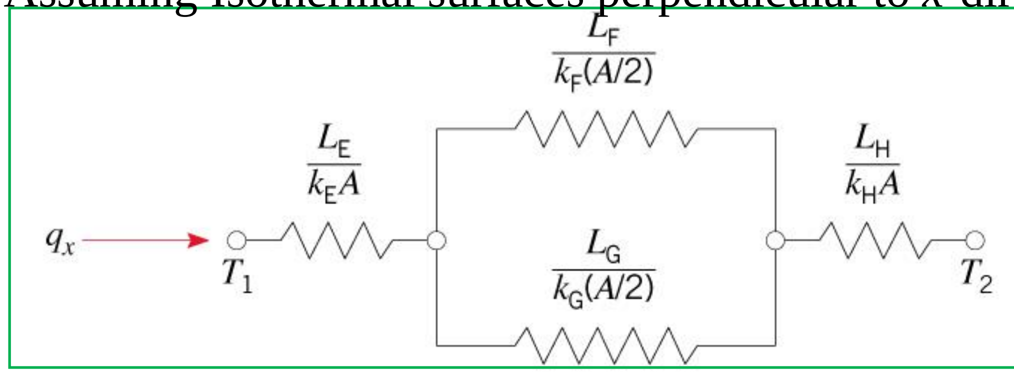
$$= \frac{T_{s,1} - T_{s,2}}{(L_A/k_A A)}$$

$$= \frac{T_2 - T_3}{(L_B/k_B A)}$$

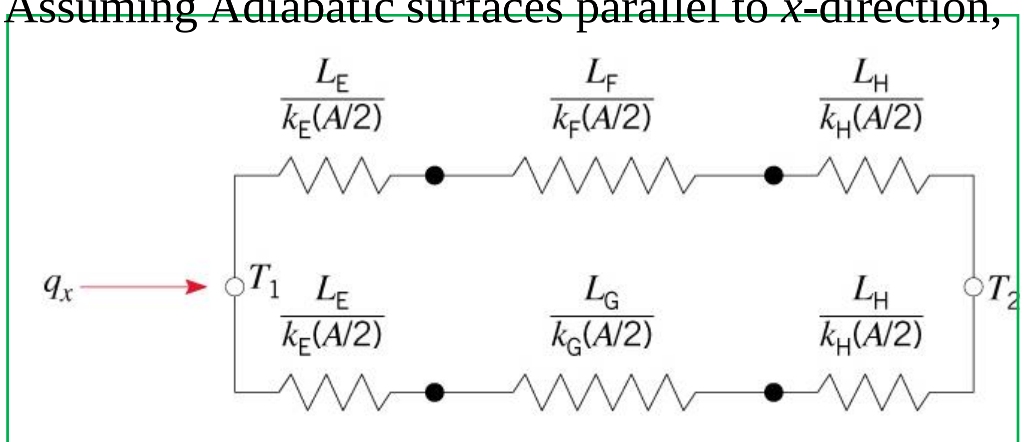
ZJUI



Assuming Isothermal surfaces perpendicular to x-direction,



Assuming Adiabatic surfaces parallel to x-direction,



Note:

- One-dimensional conditions **not valid** if $k_F \neq k_G$
- Circuits based on assumption of **isothermal surfaces normal to *x* direction** or **adiabatic surfaces parallel to *x* direction**

Example 4

You picked up scuba diving when you were at exchange in UIUC. Diving involves a greater heat loss than on land. The seawater is assumed to be at 10^{o}C. To reduce heat loss, you wear a special suit made from nanostructured silica aerogel insulation with extremely low k of 0.014 W/mK but with an outer surface emissivity of 0.95. What thickness is needed to reduce the heat loss to 100 W (typical of human metabolic heat generation)? What is the resulting skin temperature?

Draw the thermal circuit to solve the above question.

Example 4 - Solution

Known: Inner surface temperature of a skin/fat layer of known thickness, thermal conductivity, and surface area. Thermal conductivity and emissivity of snow and wet suits. Ambient conditions.

Find: Insulation thickness needed to reduce heat loss rate to 100 W and corresponding skin temperature.

Assumptions:

1. Steady-state conditions.
2. One-dimensional heat transfer by conduction through the skin/fat and insulation layers.
3. Contact resistance is negligible.
4. Thermal conductivities are uniform.
5. Radiation exchange between the skin surface and the surroundings is between a small surface and a large enclosure at the air temperature.
6. Liquid water is opaque to thermal radiation.
7. Solar radiation is negligible.
8. Body is completely immersed in water in part 2.

Example 4 - Solution