2.3 Moment Generating Function

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1st Moment $E[X^1]$

2nd Moment $E[X^2]$

What is a Moment?

The r^{th} **moment** of a random variable, X, is $\boxed{E[X^r].}$

- Also called: moment about the origin, raw moment

The r^{th} **central moment** of a random variable, X, is the expected value of the rth power of the deviation of a random variable from its mean: $\boxed{E[(X-\mu_X)^r]}$

2nd. $E[(X-\mu_X)^2] = Var$

Moment Generating Function

Definition 2.3-1

Let X be a random variable of the discrete type with pmf $f(x)$ and space S. If there is a positive number h such that

$$E(e^{tX}) = \sum_{x \in S} e^{tx} f(x)$$

exists and is finite for $-h < t < h$, then the function defined by

$$\boxed{M(t) = E(e^{tX})}$$ MGF

is called the **moment-generating function of** X (or of the distribution of X). This function is often abbreviated as mgf.

The above equation is the mgf if this expected value exists in some neighborhood of 0.

Properties of MGFs 性质

①
1. If two random variables have the same MGF, then they have the same distribution. i.e. if X and Y are random variables that have the same MGF: $M_X(t) = M_Y(t)$, then X and Y have the exact same distribution (pmf, cdf, etc)

②
2. For two independent random variables, X and Y, the MGF of their sum is the product of their MGFs:

$$M_{X+Y}(t) = M_X(t)M_Y(t)$$

(works for more than 2 as well)

Properties of MGFs

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3. The n^{th} derivative of $M_X(t)$ evaluated at t=0 is equal to the n^{th} moment, $E[X^n]$.

$$E[X] = M_X'(t)\Big|_{t=0}$$

$$E[X^2] = M_X''(t)\Big|_{t=0}$$

$$E[e^{tX}] = E\left[1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots\right]$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

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$$M_X'(t) = E\left[0 + X + \frac{2tX^2}{2!} + \frac{3(tX)^2 X}{3!} + \cdots\right]$$

$$M_X'(0) = E[0 + X + 0 + 0 \cdots]$$

$$M_X''(0) = E[0 + 0 + X^2 + 0 \cdots]$$

Examples

Moment Generating Function

Binomial mgf example

$E[X] = n \cdot p$

Let X ~ Binom(n, p). The mgf is:

$$M_x(t) = E[e^{tX}] = \sum_{X} e^{tX} \cdot f(x)$$

$$= \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

公式

$$\boxed{(a+b)^n = \sum_{x=0}^{n} C_n^x a^{n-x} b^x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

$$= \boxed{[pe^t + (1-p)]^n}$$

Binomial mgf example

1. Compute the first two moments of the Binomial Distribution using its mgf: $M_X(t) = [pe^t + (1-p)]^n$

$M'(t) = n[pe^t + (1-p)]^{n-1}pe^t.$

$M''(t) = n(n-1)[pe^t + (1-p)]^{n-2}p^2e^{2t} + n[pe^t + (1-p)]^{n-1}pe^t.$

$E[X] = M'(0) = np$

$E[X^2] = M''(0) = n(n-1)p^2 + np$

$Var[X] = E[X^2] - (E[X])^2 = n(n-1)p^2 + np - (np)^2$

$= np - np^2 = np(1-p)$

2. Suppose a random variable X has moment generating function: $M(t) = (\frac{2}{3} + \frac{1}{3}e^{t})^{10}$

What is E[X]?

Hint: This question may also be done via a shortcut method if you recognize the distribution

$$f(x) = \binom{10}{x}\left(\frac{1}{3}\right)^{x}\left(\frac{2}{3}\right)^{10-x}$$

3.

Suppose X has the MGF

$$m_X(t) = (1-2t)^{-\frac{1}{2}} \text{ for } t < \frac{1}{2}.$$

Find the first and second moments of X.

Solution: We have

$$m'_X(t) = -\frac{1}{2}(1-2t)^{-\frac{3}{2}}(-2) = (1-2t)^{-\frac{3}{2}},$$
$$m''_X(t) = -\frac{3}{2}(1-2t)^{-\frac{5}{2}}(-2) = 3(1-2t)^{-\frac{5}{2}}.$$

So that

$$\mathbb{E}X = m'_X(0) = (1-2\cdot 0)^{-\frac{3}{2}} = 1,$$
$$\mathbb{E}X^2 = m''_X(0) = 3(1-2\cdot 0)^{-\frac{5}{2}} = 3.$$

4. Suppose we have 3 independent random variables:

$W, X, Y \sim \text{Bernoulli}(p)$

Let Z be the sum of all three: $Z = W + X + Y$.

Show that $Z \sim \text{Binom}(n, p)$

Note: The mgf of a Bernoulli random variable is $M(t) = (1 - p + pe^t)$

Q: Can you show $\sum Geometric$ = NB?

5 (Example 2.3-5) Given the following mgf,

$$M(t) = e^t\left(\frac{3}{6}\right) + e^{2t}\left(\frac{2}{6}\right) + e^{3t}\left(\frac{1}{6}\right), \qquad -\infty < t < \infty,$$

What is E[X]?

$M'(t) =$

$M'(0) = 1 \cdot \frac{3}{6} + 2 \cdot \frac{2}{6} + 3 \cdot \frac{1}{6}$

What is f(x)?

the support of X is $S = \{1, 2, 3\}$ and the associated probabilities are

$$P(X = 1) = \frac{3}{6}, \qquad P(X = 2) = \frac{2}{6}, \qquad P(X = 3) = \frac{1}{6}.$$

Alternatively, f(x) = $\frac{4-x}{6}$, x = 1,2,3.

Moment Generating Function

Suppose the space of X is $\mathcal{X} = \{x_1, x_2, x_3, \ldots\}$

What is an expression for the mgf?

$M(t) = E[e^{tX}]$

$= \sum_{x} e^{tx} \cdot f(x)$

$$M(t) = e^{tx_1}f(x_1) + e^{tx_2}f(x_2) + e^{tx_3}f(x_3) + \ldots$$

The coefficient of each e^{tx_i} is the pmf, $f(x_i)$ or $P[X = x_i]$

6. Given the following mgf, what is f(x)?

$$M_X(t) = \frac{1}{7}e^{2t} + \frac{3}{7}e^{3t} + \frac{2}{7}e^{5t} + \frac{1}{7}e^{8t}$$

x	$f(x)$
2	$\frac{1}{7}$
3	$\frac{3}{7}$
5	$\frac{2}{7}$
8	$\frac{1}{7}$

7. Suppose X and Y are independent Poisson random variables with parameters λ_x, λ_y, respectively. Find the distribution of X + Y

$X \sim Pois(6) \Rightarrow M_X(t) = e^{6(e^t-1)}$

$M(t) = e^{\lambda(e^t-1)}$

$Y \sim Pois(7) \Rightarrow M_Y(t) = e^{7(e^t-1)}$

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$X+Y \sim Pois(13)$

$M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$ 证明很简单

$= e^{6(e^t-1)} \cdot e^{7(e^t-1)} = e^{13(e^t-1)}$