Homework 3

1. Express the following fractions as a sum of their partial fractions

   (a) $\frac{3s+4}{s^2+3s+2}$

   (b) $\frac{3s^2+8s+6}{(s+1)^2(s+2)}$

a). $\frac{3s+4}{s^2+3s+2} = \frac{3s+4}{(s+2)(s+1)} = \frac{2}{s+2} + \frac{1}{s+1}$

b). $\frac{3s^2+8s+6}{(s+1)^2(s+2)} = \frac{2}{s+2} + \frac{1}{s+1} + \frac{1}{(s+1)^2}$

2. Consider the ordinary differential equation

$$\ddot{x} + 5\dot{x} + 6x = u.$$

   (a) Determine the state-space representation $\dot{z} = f(z, u)$ for this ordinary differential equation.

   (b) Show that $f(z)$ can be written in the form $Az + Bu$, where A is a 2×2 matrix and B is a 2×1 vector. Find A and B.

a). $\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -5x_2 - 6x_1 + u \end{cases} \Rightarrow \dot{z} = \begin{bmatrix} x_2 \\ u - 5x_2 - 6x_1 \end{bmatrix}$

b). $\dot{z} = \underbrace{\begin{bmatrix} 0 & 1 \\ -6 & -5 \end{bmatrix}}_{A_{2\times 2}} \cdot z + \underbrace{\begin{bmatrix} 0 \\ u \end{bmatrix}}_{B_{2\times 1}}$

3. Consider the time-invariant ordinary differential equation

$$\ddot{x} + 10\dot{x} + 9x = 0$$

(a) Determine its characteristic polynomial.

(b) Determine the solution $x(t)$ for this ordinary differential equation, given that $x(0) = 0$ and $\dot{x}(0) = 8$.

(c) Determine a state-space representation of the form $\dot{z} = Az$ for this ordinary differential equation, where A is a 2×2 matrix; that is, find A.

(d) Determine the polynomial in λ by computing determinant$(A - \lambda I)$; here $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an identity matrix. How does this polynomial compare with the characteristic polynomial obtained in (a)?

a). let $\left.\begin{array}{l} x = e^{\lambda t} \\ \therefore\ \dot{x} = \lambda e^{\lambda t} \\ \ddot{x} = \lambda^2 e^{\lambda t} \end{array}\right\} \Rightarrow \lambda^2 e^{\lambda t} + 10\lambda e^{\lambda t} + 9e^{\lambda t} = 0$

$\therefore \boxed{\lambda^2 + 10\lambda + 9 = (\lambda+9)(\lambda+1) = 0}$

b). From a) we know $\lambda_1 = -9,\ \lambda_2 = -1$

$\therefore x_1 = e^{-9t},\ x_2 = e^{-t}$

$x = C_1 e^{-9t} + C_2 e^{-t}$

$\because \begin{cases} x(0) = 0 \\ \dot{x}(0) = 8 \end{cases} \therefore \begin{cases} C_1 + C_2 = 0 \\ -9C_1 - C_2 = 8 \end{cases} \therefore \begin{cases} C_1 = -1 \\ C_2 = 1 \end{cases} \therefore \boxed{x = e^{-t} - e^{-9t}}$

c). $\begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -10x_2 - 9x_1 \end{cases} \Rightarrow \dot{z} = \begin{bmatrix} 0 & 1 \\ -9 & -10 \end{bmatrix} \cdot z,\ \boxed{A = \begin{bmatrix} 0 & 1 \\ -9 & -10 \end{bmatrix}}$

d). $A - \lambda_1 I = \begin{bmatrix} 0 & 1 \\ -9 & -10 \end{bmatrix} - \begin{bmatrix} -9 & 0 \\ 0 & -9 \end{bmatrix} = \begin{bmatrix} 9 & 1 \\ -9 & -1 \end{bmatrix}$ $\therefore \det = 0$

$A - \lambda_2 I = \begin{bmatrix} 0 & 1 \\ -9 & -10 \end{bmatrix} - \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -9 & -9 \end{bmatrix}$ $\therefore \det = 0$

$\therefore$ computing $\det(A - \lambda I) = 0$ we get the same λ in a).

4. Consider a first-order system

$$\tau \dot{y} + y = ku(t), \qquad y(0) = y_0$$

where $\tau > 0$ and k are constants.

(a) *free response:* determine the free response, that is find $y(t)$ when $u(t) = 0$ and $y_0 \neq 0$. Show that the *steady state* response $y_{ss}(t) = \lim_{t\to\infty} y(t) = 0$. 通解

(b) *sinusoidal response:*

i. determine the *sinusoidal* response, that is find $y(t)$ when $u(t) = A\sin\omega t$ and $y_0 \neq 0$.

ii. let $G(s) = \frac{k}{\tau s+1}$. Show that the *steady-state* sinusoidal response

$$y_{ss}(t) = \lim_{t\to\infty} y(t) = A|G(j\omega)|\sin(\omega t + \phi), \text{ where } \phi = \angle G(j\omega)$$

[Hint: It is enough to show that $\lim_{t\to\infty}\{y(t) - A|G(j\omega)|\sin(\omega t + \phi)\} = 0$]

a). $\because \tau \cdot y' + y = 0$

$\therefore \tau \cdot \frac{dy}{dt} = -y$

$\int \tau \frac{dy}{y} = \int \frac{-1}{dt}$

$\tau \ln y = -t + C$

$y = e^{-\frac{t}{\tau}} + C$

$\because y(0) = y_0 \quad \therefore y = e^{-\frac{t}{\tau}} + y_0 - 1$

$\therefore y_{ss}(t) = e^{-\frac{t}{\tau}} \Rightarrow \lim_{t\to\infty} y_{ss}(t) = 0$

b). i). $\tau y' + y = k \cdot A \cdot \sin(wt)$

From a) we know $y_H = C_1 e^{-\frac{t}{\tau}}$, $\lambda = -\frac{1}{\tau}$

For the y_p of $u(t) = A \cdot \sin(wt)$

$y_p = C_2 \cdot \sin(wt) + C_3 \cdot \cos(wt)$

$\therefore y_p' = wC_2 \cdot \cos(wt) - w \cdot C_3 \cdot \sin(wt)$

$\therefore \begin{cases} C_3 + w\tau C_2 = 0 \\ C_2 - w\tau C_3 = k \cdot A \end{cases} \Rightarrow \begin{cases} C_2 = \frac{kA}{1+w^2\tau^2} \\ C_3 = \frac{-kAw\tau}{1+w^2\tau^2} \end{cases}$

$\therefore y = y_H + y_p = C_1 \cdot e^{-\frac{t}{\tau}} + C_2 \cdot \sin(wt) + C_3 \cdot \cos(wt)$

b ii).

$$|G(jw)| = \left|\frac{k}{\tau jw + 1}\right| = \left|\frac{k(1 - j\tau w)}{1 + \tau^2 w^2}\right| = \frac{k}{\sqrt{1+\tau^2 w^2}}$$

$$\lim_{t\to\infty} y(t) = C_2 \sin(wt) + C_3 \cdot \cos(wt)$$

$$= \frac{kA}{1+w^2\tau^2}\sin(wt) - \frac{kAw\tau}{1+w^2\tau^2}\cos(wt)$$

$$= \frac{k \cdot A}{\sqrt{1+w^2\tau^2}}\left[\frac{1}{\sqrt{1+w^2\tau^2}}\sin(wt) - \frac{w\tau}{\sqrt{1+w^2\tau^2}} \cdot \cos(wt)\right]$$

$$= A \cdot |G(jw)| \cdot \sin(wt + \emptyset)$$

q.e.d

5. Solve the following ordinary differential equations with given initial conditions:

✓(a) $\dot{x} + x = \cos t\,,\quad x(0) = 1$

✓(b) $\ddot{y} + 4\dot{y} + 3y = 2e^{-t},\quad y(0) = 0;\ \dot{y}(0) = 0.5$

✓(c) $\ddot{x} + x = e^{-t},\quad x(0) = 1;\ \dot{x}(0) = \frac{1}{2}$

✓(d) $\frac{d^4x}{dt} + 2\ddot{x} + x = 0,\quad x(0) = 1;\ \dot{x}(0) = 1;\ \ddot{x}(0) = -1;\ \dddot{x}(0) = -3$

a). $x' + x = \cos t$

① For x_H

$\left.\begin{array}{l} x' + x = 0 \\ x(0) = 1 \end{array}\right\} \Rightarrow x_H = e^{-t}$

② For x_p

$x_p = C_1 \cdot \sin t + C_2 \cdot \cos t$

$\because x(0) = 1$

$\therefore C_1 \cdot \cos t - C_2 \cdot \sin t + C_1 \sin t + C_2 \cdot \cos t = \cos t$

$\begin{cases} C_1 + C_2 = 1 \\ C_1 - C_2 = 0 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{1}{2} \\ C_2 = \frac{1}{2} \end{cases} \Rightarrow x_p = \frac{1}{2}\sin t + \frac{1}{2}\cos t$

$\therefore \boxed{x = x_H + x_p = \frac{1}{2}e^{-t} + \frac{1}{2}\sin t + \frac{1}{2}\cos t}$

b). ① For y_H. $y'' + 4y' + 3y = 0$

$\therefore \lambda_1 = -3,\ \lambda_2 = -1$

$y_1 = e^{-3t},\ y_2 = e^{-t}$

$\therefore y_H = C_1 e^{-3t} + C_2 e^{-t}$

② For y_p.

$\because y_p = C_3 \cdot te^{-t}$

$\therefore y_p' = -C_3 te^{-t} + C_3 e^{-t}$

$y_p'' = C_3 te^{-t} - C_3 e^{-t} - C_3 e^{-t} = C_3 te^{-t} - 2C_3 e^{-t}$

$\therefore C_3 = 1 \quad \therefore y_p = te^{-t}$

$\because \begin{cases} x(0) = 1 \\ \dot{x}(0) = \frac{1}{2} \end{cases} \quad \therefore \begin{cases} C_1 = \frac{1}{4} \\ C_2 = -\frac{1}{4} \end{cases}$

$\therefore \boxed{y = y_p + y_H = \frac{1}{4}e^{-3t} - \frac{1}{4}e^{-t} + te^{-t}}$

c). For x_H. $\ddot{x} + x = 0$

$\because \lambda^2 + 1 = 0$

$\therefore \lambda = \pm i$

$\therefore x_1 = \cos t,\ x_2 = \sin t$

For x_p. e^{-t}

$\because x_p = C_3 \cdot e^{-t}$

$\therefore x_p'' = C_3 e^{-t}$

$\therefore C_3 = \frac{1}{2},\ x_p = \frac{1}{2}e^{-t}$

$\therefore x = C_1 \cos t + C_2 \sin t + \frac{1}{2}e^{-t}$

$\because \begin{cases} x(0) = 1 \\ \dot{x}(0) = \frac{1}{2} \end{cases} \quad \therefore \begin{cases} C_1 = \frac{1}{2} \\ C_2 = 1 \end{cases} \quad \therefore x = \boxed{\frac{1}{2}\cos t + \sin t + \frac{1}{2}e^{-t}}$

d). For x_H

$\because \lambda^4 + 2\lambda^2 + 1 = 0$

$\therefore (\lambda^2 + 1)^2 = 0$

$\therefore \lambda_1 = i,\ \lambda_2 = i,\ \lambda_3 = -i,\ \lambda_4 = -i$

$\therefore x_1 = \cos t,\ x_2 = t\cos t$

$x_3 = \sin t,\ x_4 = t \cdot \sin t$

$\therefore x = C_1 \cos t + C_2 t\cos t + C_3 \sin t + C_4 t \cdot \sin t$

$\because \begin{cases} x(0) = 1 \\ \dot{x}(0) = 1 \\ \ddot{x}(0) = -1 \\ \dddot{x}(0) = -3 \end{cases} \Rightarrow \begin{cases} C_1 = 1 \\ C_2 = 1 \\ C_3 = 0 \\ C_4 = 0 \end{cases} \Rightarrow \boxed{x = \cos t + t\cos t}$

6. For a third-order, linear, time-invariant, homogenous ordinary differential equation with certain initial conditions, the solution is given by

$$x(t) = e^{-2t} + e^{-3t}(\cos t + \sin t)$$

(a) Determine the roots of the characteristic equation for this ordinary differential equation.
(b) Determine the characteristic polynomial for this ordinary differential equation.
(c) Determine the ordinary differential equation.
(d) Determine the initial conditions $x(0)$, $\dot{x}(0)$, and $\ddot{x}(0)$.
(e) Determine the solution to this ordinary differential equation when $x(0) = 1$, $\dot{x}(0) = -2$, and $\ddot{x}(0) = -4$.

a). $\because x(t) = e^{-2t} + e^{-3t}(\cos t + \sin t)$

$\therefore \boxed{\lambda_1 = -2,\ \lambda_2 = -3+i,\ \lambda_3 = -3-i}$

b). $\boxed{(\lambda+2)\left[(\lambda+3)^2+1\right] = 0}$

c). $(\lambda+2)\left[(\lambda+3)^2+1\right] = 0$

$(\lambda+2)\left[\lambda^2+6\lambda+10\right] = 0$

$\lambda^3+8\lambda^2+22\lambda+20 = 0$

$\boxed{\dddot{x}+8\ddot{x}+22\dot{x}+20x = 0}$

e). Fundamental Solution

$x_1 = e^{-2t}$

$x_2 = e^{-3t}\cos t$

$x_3 = e^{-3t}\sin t$

$\therefore x = C_1 e^{-2t} + C_2 e^{-3t}\cos t + C_3 e^{-3t}\sin t$

$\because \begin{cases} x(0) = 1 \\ \dot{x}(0) = -2 \\ \ddot{x}(0) = -4 \end{cases} \Rightarrow \begin{cases} C_1 = -3 \\ C_2 = +4 \\ C_3 = +4 \end{cases}$

$\therefore \boxed{x = -3e^{-2t} + 4e^{-3t}(\cos t + \sin t)}$

d). $\because x(t) = e^{-2t} + e^{-3t}(\cos t + \sin t)$

$\therefore \dot{x}(t) = -2e^{-2t} - 3e^{-3t}(\cos t + \sin t) + e^{-3t}(\cos t - \sin t)$

$= -2e^{-2t} + e^{-3t}(-4\sin t - 2\cos t)$

$\ddot{x}(t) = 4e^{-2t} + 3e^{-3t}(4\sin t + 2\cos t) + e^{-3t}(-4\cos t + 2\sin t)$

$= 4e^{-2t} + e^{-3t}(14\sin t + 2\cos t)$

$\therefore \boxed{\begin{cases} x(0) = 2 \\ \dot{x}(0) = -4 \\ \ddot{x}(0) = 6 \end{cases}}$