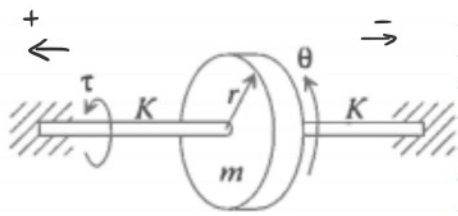
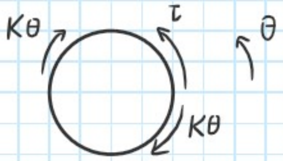
Homework 10

1. Consider the system shown by the figure on the right.
   (a) Draw the free-body diagram. (5 points)
   (b) Derive the equation of motion. (10 points)
   (c) Determine the transfer function (assuming all initial conditions are zero). (5 points)



a)



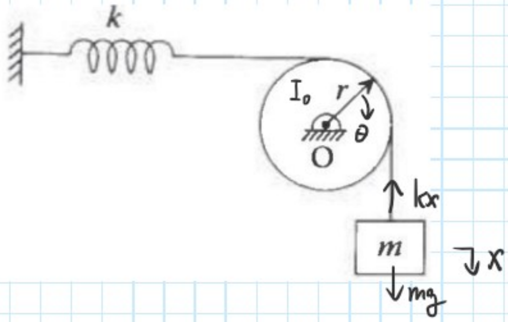
b).

$$I\ddot{\theta} = \tau - 2K\theta$$

c). $$\ddot{\theta} + \frac{2K}{I}\theta = \frac{\tau}{I} \Rightarrow \left(Is^2 + 2K\right)X(s) = U(s)$$

$$\therefore\ G(s) = \frac{X(s)}{U(s)} = \frac{1}{Is^2+2K}$$

2. Consider the pulley system as shown on the right. A block of mass m is connected to a translational spring of stiffness k through a cable, which passes by a pulley. The pulley rotates about a fixed mass center O. The moment of inertia of the pulley about its mass center is I_o. Determine the equation of motion using the energy method. (20 points)



$T: \frac{1}{2}m\cdot\dot{x}^2 + \frac{1}{2}I_0\dot{\theta}^2$

$r\theta = x$

$r\dot{\theta} = \dot{x}$

$V: -mgx + \frac{1}{2}kx^2$

$r\ddot{\theta} = \ddot{x}$

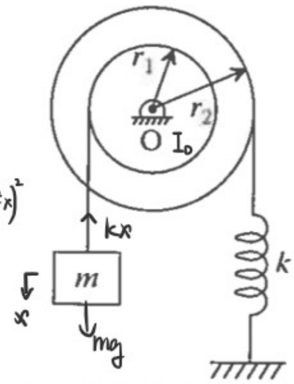
$\therefore$ Total energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_o\dot{\theta}^2 - mgx + \frac{1}{2}kx^2$

$$\therefore \frac{dE}{dt} = m\dot{x}\ddot{x} + I_o\dot{\theta}\ddot{\theta} - mg\dot{x} + kx\dot{x} = 0$$

$$m\dot{x}\ddot{x} + \frac{I_o}{r^2}\dot{x}\ddot{x} - mg\dot{x} + kx\dot{x} = 0$$

$$\left(m + \frac{I_o}{r^2}\right)\ddot{x} - mg + kx = 0$$

3. The double pulley system below has an inner radius of r_1 and an outer radius of r_2. The mass moment of inertia of the pulley about the point O is I_o. A translational spring of stiffness k and a block of mass m are suspended by cables wrapped around the pulley as shown. Determine the equation of motion using the energy method. (30 points)



$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I_0\cdot\dot{\theta}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{I_o}{r_1^2}\dot{x}^2$

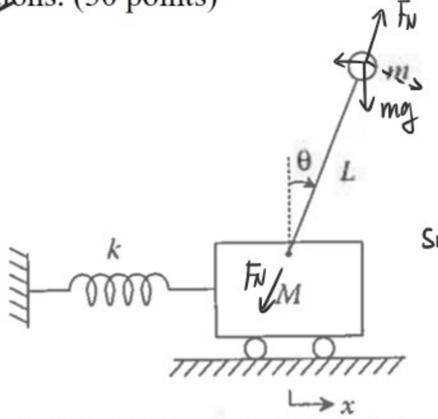
$V = -mgx + \frac{1}{2}k\left(\frac{r_2}{r_1}x\right)^2$

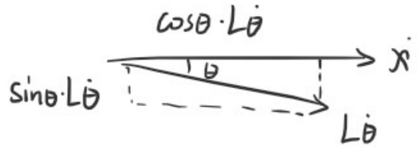
$E = T + V = \frac{1}{2}\left(m + \frac{I_o}{r_1^2}\right)\dot{x}^2 - mgx + \frac{1}{2}k\left(\frac{r_2}{r_1}x\right)^2$

$$\therefore \frac{dE}{dt} = \left(m + \frac{I_o}{r_1^2}\right)\dot{x}\ddot{x} - mg\dot{x} + \frac{r_2^2}{r_1^2}\cdot k\cdot x\dot{x} = 0$$

$$\therefore \left(m + \frac{I_o}{r_1^2}\right)\ddot{x} - mg + \frac{r_2^2}{r_1^2}kx = 0$$

4. Consider the following mechanical system, where a simple pendulum is pivoted on a cart of mass M. The pendulum consists of a point mass m concentrated at the tip of a massless rod of length L. The cart is connected to a translational spring of stiffness k. Denote the displacement of the cart as x and the angular displacement of the pendulum as θ. Derive the equations of motion using Lagrange's equations. (30 points)





$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

$q_1 = x \qquad q_2 = \theta \qquad Q_1 = Q_2 = 0$

$$T = \left(\tfrac{1}{2}M\dot{x}^2\right) + \left[\tfrac{1}{2}m(\cos\theta L\dot{\theta} + \dot{x})^2 + \tfrac{1}{2}m(\sin\theta L\dot{\theta})^2\right] = \tfrac{1}{2}(M+m)\dot{x}^2 + \tfrac{1}{2}mL^2\dot{\theta}^2 + m\cos\theta L\dot{\theta}\dot{x}$$

$$V = \tfrac{1}{2}kx^2 - mg \cdot L(1-\cos\theta) = \tfrac{1}{2}kx^2 + mgL\cos\theta - mgL$$

For q_1:
$$\begin{cases} \dfrac{\partial T}{\partial \dot{x}} = (M+m)\dot{x} + mL\cos\theta\cdot\dot{\theta} \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{x}}\right) = (M+m)\ddot{x} + \left[-mL\sin\theta\cdot\dot{\theta}^2 + mL\cos\theta\cdot\ddot{\theta}\right] \\ \dfrac{\partial T}{\partial x} = 0 \\ \dfrac{\partial V}{\partial x} = kx \end{cases}$$

$\therefore$ $(M+m)\ddot{x} - mL\sin\theta\cdot\dot{\theta}^2 + mL\cos\theta\cdot\ddot{\theta} + kx = 0$

For q_2:
$$\begin{cases} \dfrac{\partial T}{\partial \dot{\theta}} = mL^2\dot{\theta} + mL\cos\theta\cdot\dot{x} \Rightarrow \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\theta}}\right) = mL^2\cdot\ddot{\theta} - mL\sin\theta\cdot\dot{\theta}\dot{x} + mL\cos\theta\cdot\ddot{x} \\ \dfrac{\partial T}{\partial \theta} = 0 \\ \dfrac{\partial V}{\partial \theta} = -mg\sin\theta \end{cases}$$

$\therefore$ $mL^2\cdot\ddot{\theta} - mL\sin\theta\cdot\dot{\theta}\dot{x} + mL\cos\theta\cdot\ddot{x} - mg\sin\theta = 0$