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ME 340 Dynamics of Mechanical Systems

Lagrangian Dynamics
Part 3

Non-conservative forces

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

- The examples that we have seen do not involve non-conservative forces. 前几个例子没有非保守力
- Example: damping force, motor torque, friction, external forces
- Generalized *non-conservative forces*
  - Work done by force depends on the path (NOT only on the end points)
- When applying Lagrangian equations, an essential step (Step 4) is to derive generalized non-conservative forces.

Non-conservative forces 非保守力

- Let N non-conservative forces $\{F_j\}$ act respectively at points $r_j(q_1, \dots, q_L)$, $1 \le j \le N$
- Then work done dW_{nc} in moving each r_j to $r_j + dr_j$

$$dW_{nc} = \sum_{j=1}^{N} F_j \, dr_j$$

Represent dr_j using dq_i

$$\boxed{dr_j = \sum_{i=1}^{L} \frac{\partial r_j}{\partial q_i} dq_i}$$

$$dW_{nc} = \sum_{i=1}^{L} \left(\sum_{j=1}^{N} F_j \frac{\partial r_j}{\partial q_i} \right) dq_i$$

Chain rule

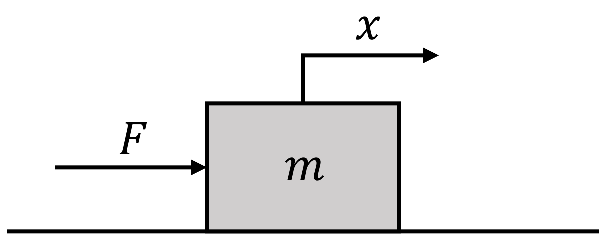
$$dW_{nc} = \sum_{i=1}^{L} Q_i dq_i$$

推导过程

Express work increment as a function of generalized coordinates.

Non-conservative forces

- Example:



$r_1 = x$ $\quad\quad$ $F_1 = F$

$$\frac{\partial r}{\partial q_1} = \frac{\partial r}{\partial x} = 1$$

$$\boxed{Q_1 = F_1 . \frac{\partial r}{\partial x_1} = F . 1 = F}$$

$$m\ddot{x} = F$$

Non-conservative torques

- Let M non-conservative torques $\{\tau_k\}$ cause respective rotation at angles $\theta_k(q_1, \dots, q_L)$, $1 \leq k \leq M$.
- Then work done dW_{nc} in moving each θ_k to $\theta_k + d\theta_k$ is given by

$$dW_{nc} = \sum_{k=1}^{M} \tau_k \, d\theta_k$$

Represent $d\theta_j$ using dq_i

$$\boxed{d\theta_k = \sum_{i=1}^{L} \frac{\partial \theta_k}{\partial q_i} dq_i}$$

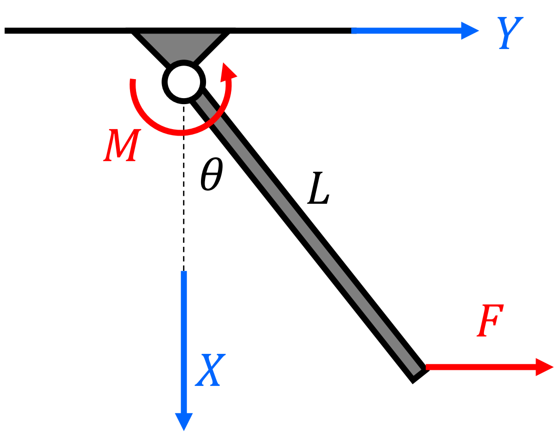
$$dW_{nc} = \sum_{i=1}^{L} \left(\sum_{k=1}^{M} \tau_k \frac{\partial \theta_k}{\partial q_i} \right) dq_i$$

$$dW_{nc} = \sum_{i=1}^{L} Q_i dq_i$$

Express work increment as a function of generalized coordinates.

Non-conservative torques

- Example:



$$\theta_1 = \theta \qquad \frac{\partial \theta_1}{\partial q_1} = \frac{\partial \theta_1}{\partial \theta} = 1$$

$$r_1 = \begin{bmatrix} L\cos(\theta) \\ L\sin(\theta) \end{bmatrix} \qquad \frac{\partial r_1}{\partial q_1} = \frac{\partial r_1}{\partial \theta} = \begin{bmatrix} -L\sin(\theta) \\ L\cos(\theta) \end{bmatrix}$$

$$Q_1 = M_1 . \frac{\partial \theta_1}{\partial \theta} + F_1 . \frac{\partial r_1}{\partial \theta} = M.1 + \begin{bmatrix} 0 \\ F \end{bmatrix} . \begin{bmatrix} -L\sin(\theta) \\ L\cos(\theta) \end{bmatrix}$$

$$\boxed{Q_1 = M + FL\cos(\theta)}$$

Non-conservative forces

- To determine non-conservative generalized force Q_i
  - For each r_j compute $\frac{\partial r_j}{\partial q_i}$, and for each θ_k compute $\frac{\partial \theta_k}{\partial q_i}$

$$Q_i = \sum_{j=1}^{N} F_j \frac{\partial r_j}{\partial q_i} + \sum_{k=1}^{M} \tau_k \frac{\partial \theta_k}{\partial q_i}$$

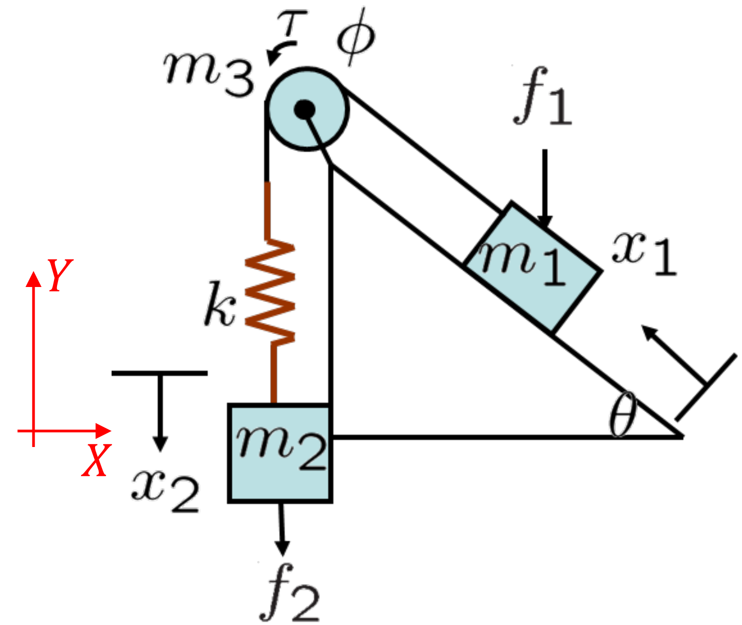
- Or
  - For each r_j compute $dr_j = \sum_{i=1}^{L} \frac{\partial r_j}{\partial q_i} dq_i$, and for each θ_k compute $d\theta_k = \sum_{i=1}^{L} \frac{\partial \theta_k}{\partial q_i} dq_i$
  - Compute

$$dW_{nc} = \sum_{j=1}^{N} F_j dr_j + \sum_{k=1}^{M} \tau_k d\theta_k$$

  - Q_i is the coefficient of dq_i in the expression of dW_{nc}.

Wedge example with non-conservative forces

- Generalized non-conservative forces



$$r_1 = \begin{bmatrix} -x_1\cos\theta \\ x_1\sin\theta \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix};$$

$$r_2 = x_2, F_2 = f_2; \phi = \frac{x_1}{R}, \text{ torque } = \tau$$

- Method 1

$$\frac{\partial r_1}{\partial x_1} = \begin{bmatrix} -\cos\theta \\ \sin\theta \end{bmatrix}; \frac{\partial r_1}{\partial x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \frac{\partial r_2}{\partial x_1} = 0; \frac{\partial r_2}{\partial x_2} = 1; \frac{\partial \phi}{\partial x_1} = \frac{1}{R}; \frac{\partial \phi}{\partial x_2} = 0$$

$$Q_1 = F_1\frac{\partial r_1}{\partial x_1} + F_2\frac{\partial r_2}{\partial x_1} + \tau\frac{\partial \phi}{\partial x_1} = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix} \cdot \begin{bmatrix} -\cos\theta \\ \sin\theta \end{bmatrix} + f_2 \cdot 0 + \tau\frac{1}{R} = -f_1\sin\theta + \frac{\tau}{R}$$

$$Q_2 = F_1\frac{\partial r_1}{\partial x_2} + F_2\frac{\partial r_2}{\partial x_2} + \tau\frac{\partial \phi}{\partial x_2} = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} + f_2 + 0 = f_2$$

Wedge example with non-conservative forces

- Generalized non-conservative forces

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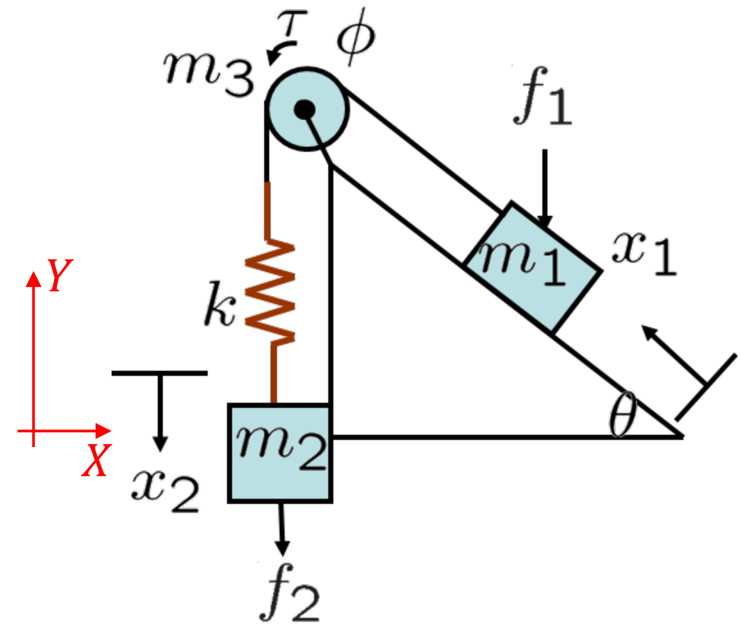

- Method 1

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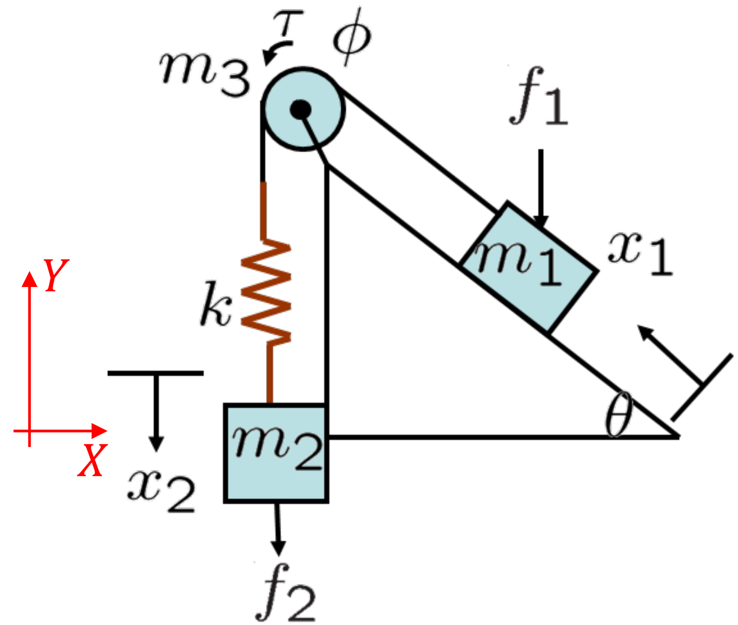
Wedge example with non-conservative forces

- Generalized non-conservative forces

$$r_1 = \begin{bmatrix} -x_1 \cos\theta \\ x_1 \sin\theta \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix};$$

$$r_2 = x_2, F_2 = f_2; \phi = \frac{x_1}{R}, \text{ torque } = \tau$$



- Method 2

$$dr_1 = \begin{bmatrix} -dx_1 \cos\theta \\ dx_1 \sin\theta \end{bmatrix}; dr_2 = dx_2; d\phi = \frac{1}{R} dx_1$$

$$\begin{aligned} dW_{nc} &= F_1 dr_1 + F_2 dr_2 + \tau d\phi = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix} \cdot \begin{bmatrix} -dx_1 \cos\theta \\ dx_1 \sin\theta \end{bmatrix} + f_2 dx_2 + \frac{1}{R} \tau dx_1 \\ &= -f_1 \sin\theta dx_1 + f_2 dx_2 + \frac{1}{R} \tau dx_1 = \left(-f_1 \sin\theta + \frac{\tau}{R} \right) dx_1 + f_2 dx_2 \end{aligned}$$

$$\Rightarrow Q_1 = -f_1 \sin\theta + \frac{\tau}{R}, \; Q_2 = f_2$$

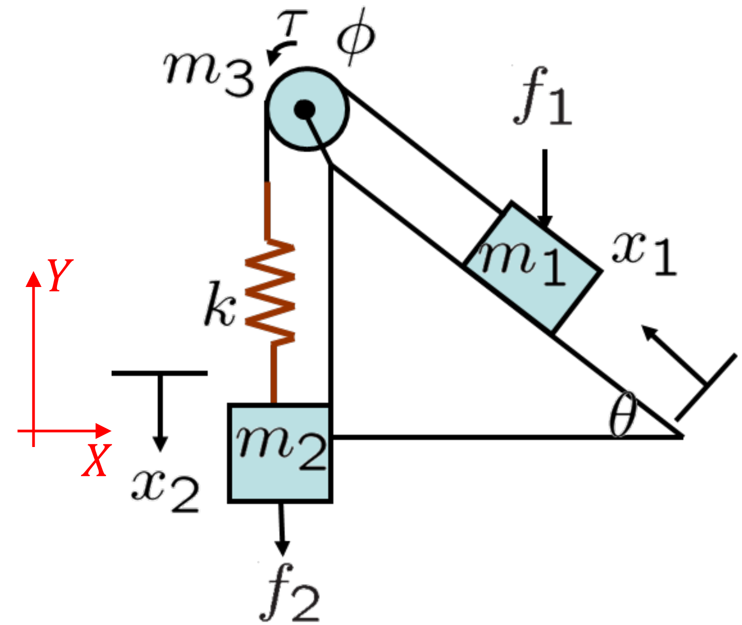
Wedge example with non-conservative forces

- Generalized non-conservative forces

$$r_1 = \begin{bmatrix} -x_1 \cos\theta \\ x_1 \sin\theta \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix};$$

$$r_2 = x_2, F_2 = f_2; \phi = \frac{x_1}{R}, \text{ torque } = \tau$$

- $Q_1 = -f_1 \sin\theta + \frac{\tau}{R},\ Q_2 = f_2$



- The equations of motion are

$$\left(m_1 + \frac{m_3}{2}\right)\ddot{x}_1 + m_1 g \sin\theta + k(x_1 - x_2) = -f_1 \sin\theta + \frac{\tau}{R}$$

$$m_2 \ddot{x}_2 - m_2 g + k(x_2 - x_1) = f_2$$