$$P[A|B] = \frac{P[A \cap B]}{P[B]}$$ "Bayes' Rule"

1.5 Law of Total Probability

Note: The equation for the law of conditional probability (1.3) is also known as Bayes Rule.
Section (1.5) in the textbook is called "Bayes Rule", but the only new thing introduced in this section is the "Law of Total Probability".

A : vendor A $\qquad$ D : defective

Introductory Example

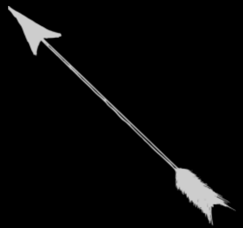
Hawkeye purchases arrows from 3 vendors: A, B and C. Out of his entire collection, 25% of his arrows are from **vendor A**, 35% from **vendor B**, and the rest are from **vendor C**. Assume we know:

$P[A] = 0.25$

$P[D \mid A] =$ 5% of the arrows from Vendor A are defective

4% from vendor B are defective

2% from vendor C are defective.



After a battle, he picks up an arrow at random and finds that it is defective. What is the probability that it came from vendor A?

$P[A \mid D]$

Law of Total Probability

Let $B_1, B_2, \ldots, B_m$ be a **partition** of the sample space, *S.*

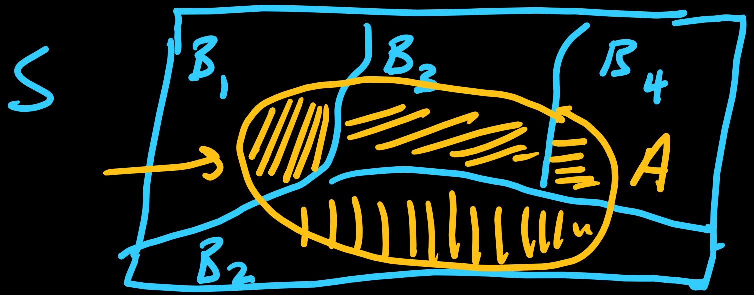
$$S = B_1 \cup B_2 \cup \cdots \cup B_m \text{ and } B_i \cap B_j = \emptyset, i \neq j.$$

Now, let A be an event.

We can write A as the union of *m* mutually exclusive events:

$$A = (B_1 \cap A) \cup (B_2 \cap A) \cup \cdots \cup (B_m \cap A).$$

$A \cap B_1$



Bayes Rule

Let $B_1, B_2, \ldots, B_m$ be a **partition** of the sample space, *S.*

P[A] = $\sum_{i=1}^{m} P(B_i \cap A)$ (using Multiplication Rule)

P[A] = $\sum_{i=1}^{m} P(B_i)P(A|B_i)$ Law of Total Probability

$$P[B_i \mid A] = \frac{P[B_i \cap A]}{P[A]} = \frac{P[A \mid B_i]P[B_i]}{\sum_{i=1}^{m} P(B_i)P(A|B_i)}$$

Conditional Probability (Bayes Theorem)

$$P(B_k \mid A) = \frac{P(B_k \cap A)}{P(A)}, \qquad k = 1, 2, \ldots, m.$$

Using the Law of Total Probability to re-write P[A], we can re-write **Bayes Theorem:**

$$P(B_k \mid A) = \frac{P(B_k)P(A \mid B_k)}{\sum_{i=1}^{m} P(B_i)P(A \mid B_i)}, \qquad k = 1, 2, \ldots, m.$$

Shortcut for 2 cases

If we are only considering B and B^c, we can write Bayes rule as follows:

$$\boldsymbol{P[B|A]} = \frac{P(B)P(A|B)}{P(B)P(A|B)+P(B^C)P(A|B^C)}$$

$P[A\cap B] \quad + \quad P(A\cap B^c]$

$P[A]$

25% of arrows come from **vendor A**, 35% from **vendor B**, and the rest are from **vendor C**.

Assume we know:

5% of the arrows from Vendor A are defective
4% from vendor B are defective
2% from vendor C are defective.

After a battle, an arrow is picked at random and found to be defective. What is the probability that it came from vendor A?

$$P[A \mid D] = \frac{P[A \cap D]}{P[D]}$$

$$\begin{aligned}
\mathrm{P}(A|D) &= \frac{\mathrm{P}(D|A)\mathrm{P}(A)}{\mathrm{P}(D|A)\mathrm{P}(A) + \mathrm{P}(D|B)\mathrm{P}(B) + \mathrm{P}(D|C)\mathrm{P}(C)} \\
&= \frac{0.05 \times 0.25}{0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.4} \\
&= 0.362
\end{aligned}$$

1.5 Bayes' Rule

Examples

1. Jack Sparrow believes he has a

A_1 : GPA 3.5-4.0

25% chance of ending up with a GPA between 3.5 - 4.0,

$P[A_1] = 0.25$

35% chance of ending with a 3.0 – 3.5, A_2

40% chance of ending up with a GPA less than 3.0. A_3

His advisor, Captain Hook, tells him that Jack has the following chances of getting into grad school for each GPA bracket:

G = grad school

$P[G \mid A_1]$	3.5 – 4.0 GPA:	$p = 0.8$
	3.0 – 3.5 GPA:	$p = 0.5$
	Below 3.0 GPA:	$p = 0.1$

1. Based on this information, what is the probability that Jack gets into grad school?

$$P[G] = P[G \mid A_1] \cdot P[A_1] + P[G \mid A_2] \cdot P[A_2] + P[G \mid A_3] \cdot P[A_3] = \boxed{0.415}$$

2. Suppose Jack has been accepted into grad school. What is the probability that Jack ended up with a GPA between 3.0 and 3.5?

A_1 3.5 - 4.0: p = 0.25

A_2 3.0 – 3.5: p = 0.35

A_3 < 3.0: p = 0.4

If 3.5 – 4.0 GPA: *p = 0.8*

If 3.0 - 3.5 GPA: *p = 0.5*

If < 3.0 GPA: *p = 0.1*

$$P[A_2 \mid G] = \frac{P[A_2 \cap G]}{P[G]}$$

$$= \frac{P[G \mid A_2] \cdot P[A_2]}{P[G]}$$

$$= \frac{0.5 \cdot 0.35}{0.415} = 0.42$$

A B C

3. Albus Dumbledore, Bill Weasley, and Cornelius Fudge are about to be executed. A crystal ball reveals that one will be randomly spared but won't show who is spared.

Albus convinces it to show him one of the people that will be executed.

- "If Bill is going to live, show me Cornelius' execution.
- If Cornelius is going to live, show me Bill's execution.
- If I am going to live, open **RStudio** and use `sample(1:2,1)` to decide which execution to show me"

multiple choice

a. 0

b. 1/3

c. 1/2

d. 2/3

e. depends

→ The ball shows him Bill's execution. Given this new information, what is the probability that Albus will live?

$P[A] = 1/3 \qquad P[B] = 1/3 \qquad P[C] = 1/3$

A = Albus lives

B = Bill lives

C = Cornelius Lives

→ b = sees Bill execution

c = sees Cornelius execution

$P[b]$

$$P(A|b) = \frac{P(b|A)P(A)}{P(b|A)P(A) + P(b|B)P(B) + P(b|C)P(C)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 \times \frac{1}{3} + 1 \times \frac{1}{3}} = \frac{1}{3}.$$

1

A = Albus lives

B = Bill lives

C = Cornelius Lives

b = sees Bill execution

c = sees Cornelius execution

$$P(C|b) = \frac{P(b|C)P(C)}{P(b|A)P(A) + P(b|B)P(B) + P(b|C)P(C)}$$

$$= \frac{2}{3}$$

When someone comes to the house, Chloe barks **90%** of the time if they ring the doorbell. When they do **not** ring the doorbell, she barks 30% of the time anyway.

Assume people ring the doorbell **80%** of the time.

D : doorbell

B : bark

4. Find the overall probability that Chloe barks ~~when someone comes to the door~~.

$$P[B] = P[B \cap D] + P[B \cap D^c]$$

$$= \underline{P[B|D]} \cdot P[D] + P[B|D^c]\, P[D^c]$$

$$= (0.9)(0.8) + (0.3)(0.2)$$

$$= \boxed{0.78}$$

5. Given that Chloe barked when someone came to the door, what is the probability that they rang the bell?

$$P[D \mid B] = \frac{P[D \cap B]}{P[B]} = \frac{P[B \mid D]\,P[D]}{P[B]}$$

$$= \frac{(0.9)(0.8)}{0.78} = 0.923 \qquad {A^{c}}^{c} = A$$

$$[B^c \mid D^c]^c = B \mid D^c$$