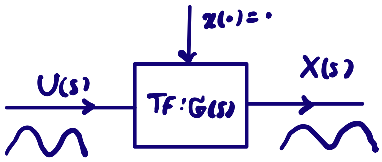


For a stable 1^{st} order LTI system, if a sinusoidal input $u(t)$ is applied, the steady state response is also sinusoidal with same frequency!

对于1阶时不变系统，若 $u(t)$ 是正弦 ⇒ 稳态响应也为正弦
sinusoild　steady state

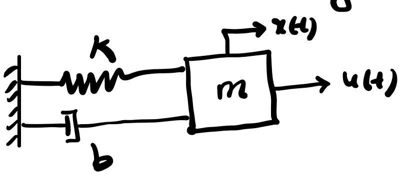
- The amplitude is scaled with $|G(jw)|$
- The phase is also shifted with $\angle G(jw)$

Input: $u(t) = U_0 \sin(wt)$

Output (steady state): $x_{ss}(t) = U_0 \, |G(jw)| \sin(wt + \angle G(jw))$

The transfer function determines how the system behaves in the long run!

For second order system:



$$m\ddot{x} + b\dot{x} + Kx = u(t)$$

$$\ddot{x} + 2\zeta w_n \dot{x} + w_n^2 x = \frac{w_n^2}{K} u(t)$$

$$w_n = \sqrt{K/m}, \quad \zeta = \frac{b}{2\sqrt{mK}}$$

$$G(s) = \frac{X(s)}{U(s)} = \frac{w_n^2}{K} \times \frac{1}{s^2 + 2\zeta w_n s + w_n^2}$$

$$s = j\omega \rightsquigarrow \quad G(j\omega) = \frac{\omega_n^2}{k} \frac{1}{(j\omega)^2 + 2\zeta\omega_n (j\omega) + \omega_n^2}$$

$$= \frac{1}{k} \frac{1}{\frac{1}{\omega_n^2}\left[-\omega^2 + j\, 2\zeta\omega_n\omega + \omega_n^2\right]}$$

$$= \frac{1}{k} \frac{1}{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right) + j\, \frac{2\zeta\omega}{\omega_n}} = r e^{j\theta}$$

$$|G(j\omega)| = \frac{1}{k} \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2}}, \quad \angle G(j\omega) = 0 - \tan^{-1}\left(\frac{2\zeta\omega/\omega_n}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right)$$

$$u(t) = U_0 \sin(\omega t)$$

$$x_{ss}(t) = U_0 \, |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

For a general stable LTI system:

$$u(t) = \sum_i A_i \sin(\omega_i t + \theta_i) \longrightarrow \boxed{G(s)} \longrightarrow x_{ss} = \sum_i A_i \, |G(j\omega_i)| \sin(\omega_i t + \theta_i + \angle G(j\omega_i))$$

IC = 0

System

For each input component:

- we will get a sinusoidal output
- Frequency is the same
- Magnitude is amplified by $|G(j\omega_i)|$
- phase is shifted by $\angle G(j\omega_i)$

Ex1: $\ddot{x} + 8\dot{x} + 15x = \underbrace{\sqrt{41}\ \sin(4t)}_{u(t)}$, x_{ss} ?

a) what are the input components?

$u(t) = \sqrt{41}\ \sin(4t) \rightarrow \omega = 4\ rad/s$

$U_0 = \sqrt{41}$

b) Transfer function ?

$$s^2 X(s) + 8s X(s) + 15 X(s) = U(s)$$

$$G(s) = \frac{X(s)}{U(s)} = \frac{1}{s^2 + 8s + 15}$$

c) what are $M(\omega)$ and $\phi(\omega)$?

$|G(j\omega)| = M(\omega)$

$\angle G(j\omega) = \phi(\omega)$

$$G(j\omega) = \frac{1}{(j\omega)^2 + 8(j\omega) + 15} = \frac{1}{(15 - \omega^2) + j\,8\omega}$$

$$M(\omega) = \left.\frac{1}{\sqrt{(15-\omega^2)^2 + (8\omega)^2}}\right|_{\omega=4} \qquad \phi(\omega) = 0 - \tan^{-1}\left(\frac{8\omega}{15-\omega^2}\right)\Bigg|_{\omega=4}$$

d) what is the steady state response?

$$x_{ss}(t) = \sqrt{41} \times M(4) \times \sin(4t + \phi(4))$$

what if $\ddot{x} + 8\dot{x} + 15x = \sqrt{41}\ \underbrace{\cos(4t)}_{\hookrightarrow \sin(4t + \pi/2)}$?

$$x_{ss}(t) = \sqrt{41} \times M(4) \times \sin(4t + \pi/2 + \phi(4))$$

what if $\ddot{x} + 8\dot{x} + 15x = \underbrace{\sqrt{41}\,\sin(4t)}_{u_1(t)} + \underbrace{2\,u_s(t)}_{u_2(t)}$

$u_1(t) = \sqrt{41}\,\sin(4t) \rightsquigarrow A_1 = \sqrt{41},\ \omega_1 = 4,\ \theta_1 = 0$

$u_2(t) = 2\,u_s(t) \rightsquigarrow A_2 = 2,\ \omega_2 = 0,\ \theta_2 = \pi/2$

$x_{ss}(t) = x_{ss_1}(t) + x_{ss_2}(t)$

w/ $x_{ss_1}(t) = \sqrt{41} \times M(4) \times \sin(4t + \phi(4))$

$x_{ss_2}(t) = 2 \times M(0) \times \sin(\pi/2 + \phi(0))$