Homework 1 条件概率

Exercise 1

The probability that Chloe acts hungry at 7pm given that she has already eaten dinner is 0.4. The probability that Chloe acts hungry at 7pm given that she has not had dinner is 0.98. Assume that there is probability 0.9 that Chloe has eaten dinner by 7pm on a given day.

a) (1 pt) If Chloe is acting hungry at 7pm, find the probability that she has not had dinner yet.

b) (1 pt) If Chloe is not acting hungry at 7pm, find the probability that she has already had dinner.

Let A be the event of hungry, B be the event of having dinner

$P(A|B) = 0.4$
$P(A|B^c) = 0.98$
$P(B) = 0.9$

a). Find $P[B^c|A]$

$$P[B^c|A] = \frac{P[B^c \cap A]}{P[A]} \underset{\text{全概率}}{=} \frac{P[A\cap B^c]}{P[A\cap B] + P[A\cap B^c]} \overset{\text{概率乘法}}{\underset{\text{概率乘法}}{=}} \frac{P[A|B^c]\cdot P[B^c]}{P[A|B]\,P[B] + P[A|B^c]\cdot P[B^c]} = \frac{0.98\times 0.1}{0.4\times 0.9 + 0.98\times 0.1}$$

$$\boxed{\approx 0.214}$$

b). Find $P[B|A^c]$

$$P[A] = P[A|B]\cdot P[B] + P[A|B^c]\cdot P[B^c] = 0.458 \Rightarrow P[A^c] = 0.542$$

$$P[B|A^c] = \frac{P[B\cap A^c]}{P[A^c]} = \frac{P[A^c|B]\cdot P[B]}{1-P[A]} = \frac{(1-P[A|B])\ P[B]}{1-P[A]} = \boxed{0.996}$$

Exercise 2

Draw one card at random from a standard deck of cards. The sample space S is the collection of the 52 cards. Assume that the probability set function assigns 1/52 to each of the 52 outcomes. Let the following events be defined:

- A = {Queen or King}
- B = {Spade}
- C = {Black Jack or Queen}.

Find:

a) (0.5 pt) $P[A \cup B]$
b) (0.5 pt) $P[A \cap B]$
c) (0.5 pt) $P[A \cup (B \cap C)]$
d) (0.5 pt) $P[A^C \cup B]$

a). $P[A \cup B] = P[A] + P[B] - P[A \cap B] = \frac{4+4}{52} + \frac{13}{52} - \frac{2}{52} = \boxed{\frac{27}{52}}$

b). $P[A \cap B] = \frac{2}{52} = \boxed{\frac{1}{26}}$

c) $P[A \cup (B \cap C)] = P[A] + P[B \cap C] - P[A \cap (B \cap C)] = \frac{4+4}{52} + \frac{2}{52} - 0 = \frac{10}{52} = \boxed{\frac{5}{26}}$

d). $P[A^c \cup B] = P[A^c] + P[B] - P[A^c \cap B] = \left(1 - \frac{8}{52}\right) + \frac{13}{52} - \frac{12}{52} = \boxed{\frac{45}{52}}$

Exercise 3

Suppose $S = \{1, 2, 3, 4, 5, ...\}$ and for any element of S,

$$P[k] = c\frac{5^k}{k!}$$

Find the value of c that makes this a valid probability distribution. (2 pts)

$P[1] + P[2] + \cdots = 1$

$c\left[\frac{5^1}{1!} + \frac{5^2}{2!} + \cdots \frac{5^n}{n!}\right]_{n \to \infty} = 1$

$c \cdot e^5 = 1$

$\boxed{c = \frac{1}{e^5}}$

Exercise 4

Suppose $S = \{2, 3, 4...\}$ and

$$P[k] = \frac{c}{(5/2)^k}$$

a) (1.5 pts) Find a value of c that makes this a valid probability distribution.

b) (1 pt) Find P[Outcome is *greater* than 3].

a). $1 = P[2] + P[3] + \cdots$

$$= c\left[\left(\tfrac{2}{5}\right)^2 + \left(\tfrac{2}{5}\right)^3 + \cdots \left(\tfrac{2}{5}\right)^n \right]_{n\to\infty}$$

$$= c\left[\left(\tfrac{2}{5}\right) + \cdots \left(\tfrac{2}{5}\right)^n \cdots \right] - c\cdot\tfrac{2}{5}$$

$$= c\cdot\left[\frac{\frac{2}{5}}{1-\frac{2}{5}} \right] - c\cdot\tfrac{2}{5} = \left(\tfrac{2}{3} - \tfrac{2}{5}\right)c \Rightarrow \boxed{c = \frac{15}{4}}$$

b) $P[n] = \frac{15}{4}\cdot\left(\frac{2}{5}\right)^n \quad n \geq 2$

$$P[4] + P[5] + \cdots = 1 - P[2] - P[3] = 1 - \frac{15}{4}\left(\frac{2}{5}\right)^2 - \frac{15}{4}\cdot\left(\frac{2}{5}\right)^3 = \boxed{\frac{4}{25}}$$

Exercise 5

Suppose $P[A] = 0.5$, $P[B^C] = 0.3$, and $P[A \cap B] = 0.2$

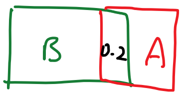
a) (0.5 pts) Find $P[B|A]$

b) (0.5 pts) Find $P[B^C|A^C]$

c) (0.5 pts) Find $P[A^C|B]$

$P[A] = 0.5$ $\quad P[A \cap B] = 0.2$

$P[B] = 0.7$



a). $$P[B|A] = \frac{P[B \cap A]}{P[A]} = \frac{0.2}{0.5} = \boxed{\frac{2}{5}}$$

b). $$P[B^c|A^c] = \frac{P[B^c \cap A^c]}{P[A^c]} = \frac{P[A \cup B]^c}{1 - P[A]} = \frac{\left[P[A] + P[B] - P[A \cap B]\right]^c}{1 - P[A]} = \frac{1 - [0.5 + 0.7 - 0.2]}{1 - 0.5} = \boxed{0}$$

c). $$P[A^c|B] = \frac{P[A^c \cap B]}{P[B]} = \frac{1 - P[A]}{P[B]} = \frac{1 - 0.5}{0.7} = \boxed{\frac{5}{7}}$$