ME 340 Dynamics of Mechanical Systems

Lagrangian Dynamics
Part 1

Recap

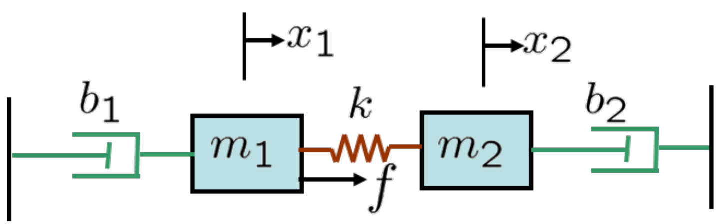
- To analyze the behavior of a mechanical system, we need to derive the *equation(s) of motion*.
- So far, our tools include
  - 牛二定律 Newton's 2nd law + free body diagram: planar systems

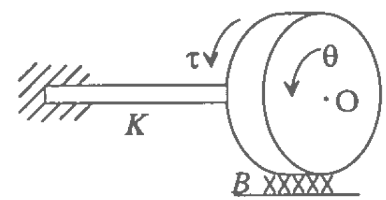
$$\sum F = m\ddot{x} \qquad \sum \tau = I\ddot{\theta}$$

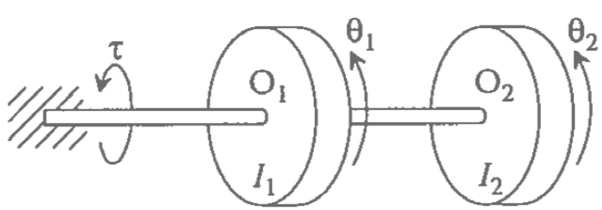
  - 能量守恒 Energy method: 1DoF conservative systems

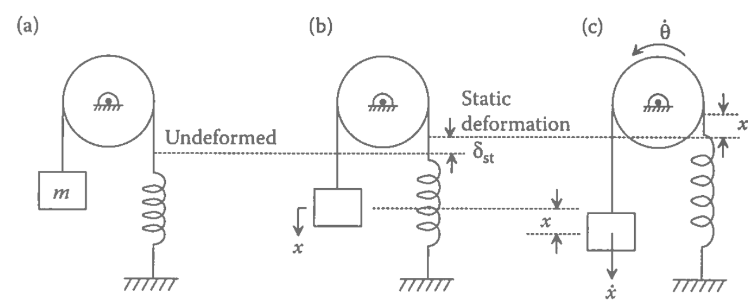
$$T + V = \text{constant} \Leftrightarrow \frac{d}{dt}(T + V) = 0$$

- What we have seen...







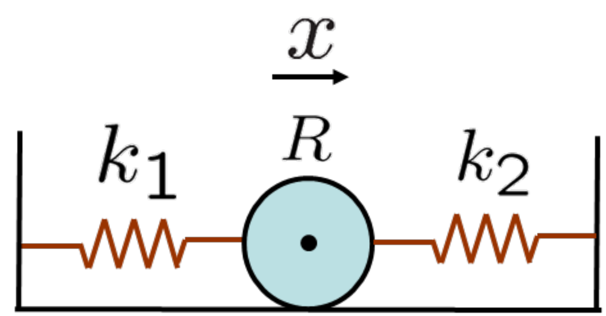


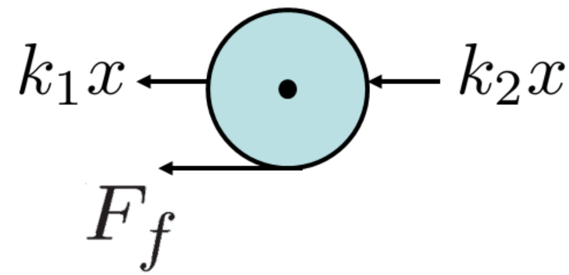
Example: rotational + translational motion

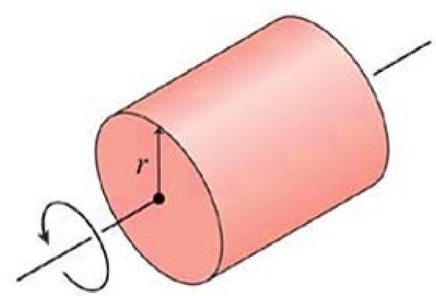
- The system:
  - A cylinder is connected with two springs.
  - Friction exists between the cylinder and ground.
  - Rolling without slipping.
- Free-body diagram
  - Translational: $m\ddot{x} = -F_f - k_1 x - k_2 x$
  - Rotational: $I\ddot{\theta} = R \cdot F_f$
- Geometric constraint: $x = R \cdot \theta$
- Equation of motion

$$\left(\frac{3m}{2}\right)\ddot{x} + (k_1 + k_2)x = 0$$

Is energy method applicable?







Disk or solid cylinder about its axis
$I = \frac{1}{2}MR^2$

Example: rotational + translational motion

$Translation: m\ddot{x} = -F_f - k_1 x - k_2 x$

$Rotation: I\ddot{\theta} = RF_f$

$Geometric\ constraint: x = R\theta$

$$\frac{mR^2}{2}\ddot{\theta} = RF_f \therefore F_f = \frac{m}{2}\ddot{x}$$

$$m\ddot{x} = -\frac{m}{2}\ddot{x} - k_1 x - k_2 x$$

$$\boxed{\frac{3}{2}m\ddot{x} + (k_1 + k_2)x = 0}$$

$$T + V = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}k_1 x^2 + \frac{1}{2}k_2 x^2$$

$$T + V = \frac{1}{2}m\dot{x}^2 + \frac{mR^2}{4}\left(\frac{\dot{x}}{R}\right)^2 + \frac{1}{2}(k_1 + k_2)x^2 = \frac{3}{4}m\dot{x}^2 + \frac{1}{2}(k_1 + k_2)x^2$$

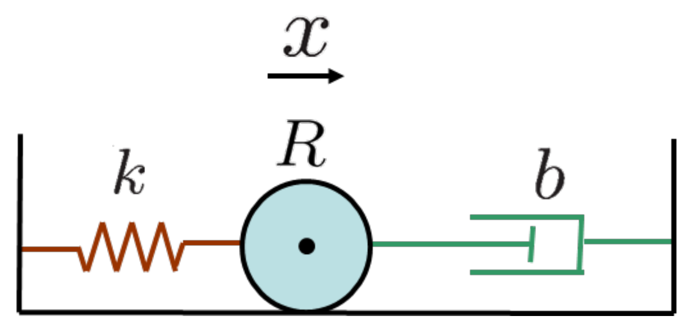
$$\frac{d}{dt}(T + V) = 0 = \frac{3}{2}m\dot{x}\ddot{x} + (k_1 + k_2)x\dot{x}$$

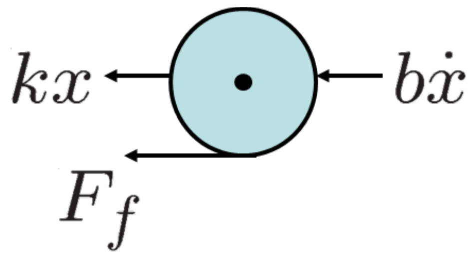
$$\boxed{\frac{3}{2}m\ddot{x} + (k_1 + k_2)x = 0}$$

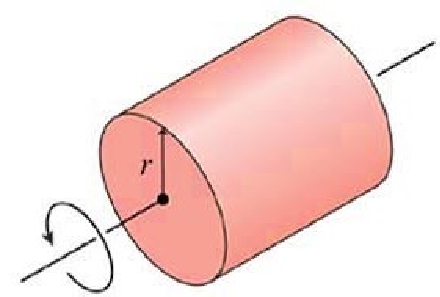
Example: rotational + translational motion

- The system:
  - A cylinder is connected with a spring and a damper.
  - Friction exists between the cylinder and ground.
  - Rolling without slipping.
- Free body diagram
  - Translational: $m\ddot{x} = -F_f - kx - b\dot{x}$
  - Rotational: $I\ddot{\theta} = RF_f$
- Geometric constraint: $x = R\theta$
- Equation of motion

$$\left(\frac{3m}{2}\right)\ddot{x} + b\dot{x} + kx = 0$$







Disk or solid cylinder about its axis
$I = \frac{1}{2}MR^2$

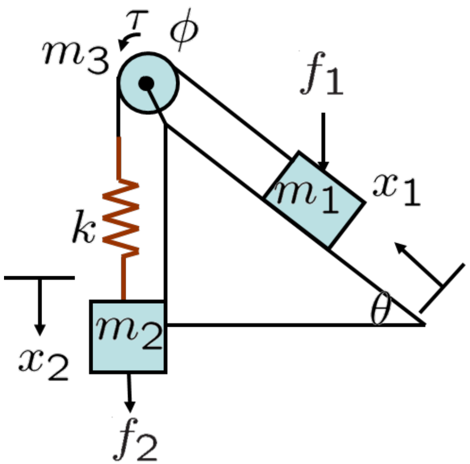
Is energy method applicable? X 阻尼做负功

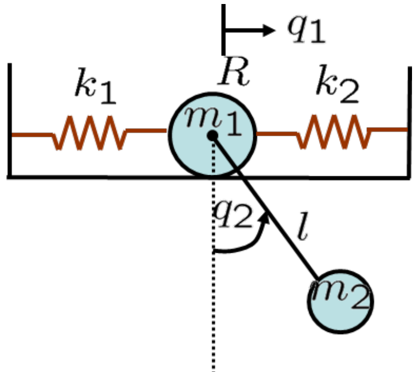
Example: rotational + translational motion

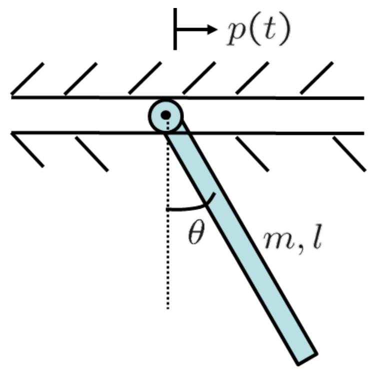
$$Translation: m\ddot{x} = -F_f - kx - bx$$
$$Rotation: I\ddot{\theta} = RF_f$$
$$Geometric\ constraint: x = R\theta$$

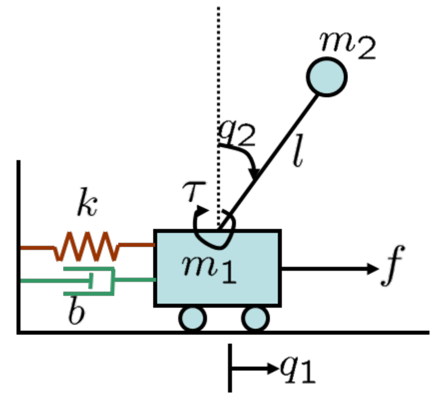
$$\boxed{\frac{3}{2}m\ddot{x} + b\dot{x} + kx = 0}$$

What if we have the systems below…









- When dealing with complex mechanical systems with high number of degrees of freedom, 高自由度
  - Using Newton's laws requires free body diagrams for all elements
  - Energy method only applies to scenarios where the system is 1DoF and conservative
- We need a more powerful tool to analyze such systems.

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Lagrangian dynamics

(旧) 能量守恒法

- To apply energy approach:
  - All components have to be energy storing
  - No work done on the system
  - We only get one equation – only good for 1 degree-of-freedom (DOF) systems
  - Extremely limiting

(新) 拉格朗日动学

- Lagrangian dynamics:
  - Based on both Energy and Work, e.g., work done by dampers
  - Can handle many generalized coordinates – one equation for each DOF
  - Simpler in some cases over Free Body Diagram method

Lagrangian dynamics

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

- q_i: generalized coordinates, $1 \leqslant i \leqslant N$
- $\dot{q}_i$: generalized velocities
- T is the total Kinetic Energy in the system
- V is the total Potential Energy
- Q_i: generalized non-conservative forces

q_i describe the configuration of the system relative to some reference configuration (position, angle, etc)

Q_i are forces (torque) that inject or remove energy

- $L = T - V$ is called the Lagrangian of the system. Equivalently,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

Review: partial derivative 偏微分复习

- A partial derivative of a function of multiple variables is its derivative with respect to one of those variables, with the others held constant.

$$(1)\quad z = f(x,y) = x^2 + xy + 2y^2$$

$$(2)\quad z = f(t) = x^2 + xy + 2y^2 \qquad \frac{\partial f}{\partial x}? \quad \frac{\partial f}{\partial y}? \quad \frac{df}{dt}?$$

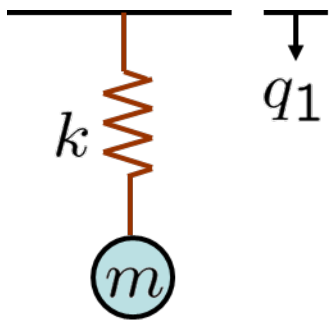
$$(3)\quad g(y,\theta) = mgx\sin\theta - mgy + \tfrac{1}{2}k(x+y)^2$$

$$(4)\quad T(x,\dot{x}) = \tfrac{1}{2}m\dot{x}^2 + mgx \qquad \frac{\partial T}{\partial \dot{x}}? \quad \frac{\partial T}{\partial x}? \quad \frac{dT}{dt}?$$

A simple example: spring-mass system

例题

- ⋆ q_1: position of the mass, only 1 DOF
- ⋆ $\dot{q}_1$: velocity of the mass
- ⋆ $T = \frac{1}{2} m \dot{q}_1^2$ is the total Kinetic Energy
- ⋆ $V = \frac{1}{2} k q_1^2 - m g q_1$ is the total Potential Energy
- ⋆ $Q_1 = 0$ no non-conservative forces



$$L = T - V = \frac{1}{2} m \dot{q}_1^2 - \frac{1}{2} k q_1^2 + m g q_1$$

$$\frac{\partial L}{\partial q_1} = -k q_1 + m g$$

$$\frac{\partial L}{\partial \dot{q}_1} = m \dot{q}_1$$

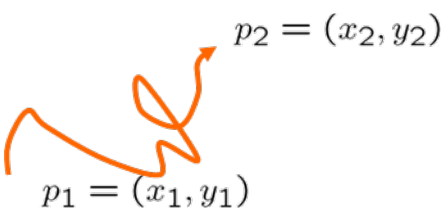
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = m \ddot{q}_1$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = Q_i = 0$$

$$\boxed{m \ddot{q}_1 + k q_1 - m g = 0}$$

Kinetic and potential energy

- The *kinetic energy* of an object is the energy associated with its translational or rotational motion.
  - For example, $\mathrm{T}_{\text{trans}} = \frac{1}{2}m\dot{x}^2$, $\mathrm{T}_{\text{rot}} = \frac{1}{2}I_C\dot{\theta}^2$
- *Potential energy*: $V = V_{\text{elastic}} + V_{\text{gravity}}$, is the "stored" energy
  - V_{elastic}: stored in springs
    - For example, $\mathrm{V}_{\text{elastic}} = \frac{1}{2}kx^2$ (linear springs),$\mathrm{V}_{\text{elastic}} = \frac{1}{2}K\theta^2$ (torsional springs)
  - V_{gravity}: stored in mass due to gravitational field
    - $V_{gravity} = mgy$, where y is taken w.r.t. some fixed point (datum)
    - V_{gravity} is "extra" potential energy from the gravity datum
  - Change in potential energy by taking the object from point A to point B is $V_B - V_A$
    - Does not depend on the path
  - Work done by conservative forces
    - Independent of the path
    - Depends only on the end points



Lagrangian dynamics

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

General procedure for Lagrangian method:

- Step 1: determine DOF, geometric constraints, and generalized coordinates q_i;
- Step 2: write out the potential and kinetic energies;
- Step 3: calculate partial and full derivatives;
- Step 4: determine non-conservative generalized forces Q_i;
- Step 5: assemble equations of motion.