ME 340 Dynamics of Mechanical Systems

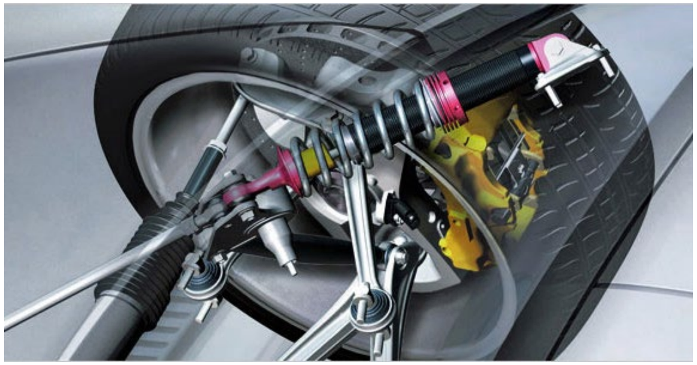
Mechanical Systems
Part 1

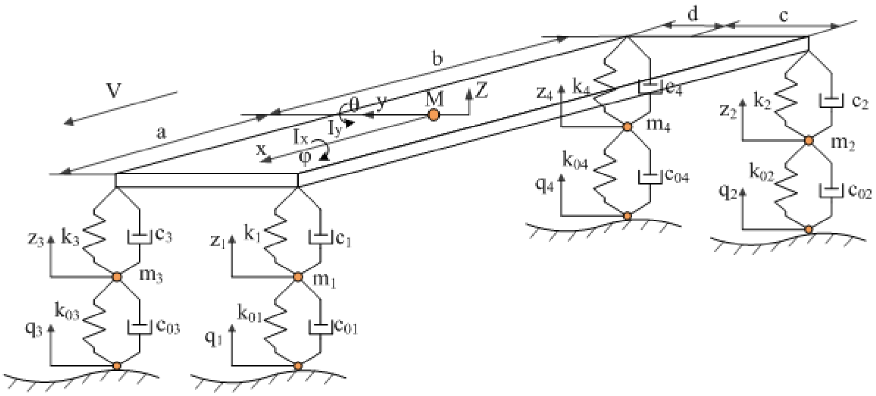
Mechanical systems

- We will focus on the modeling of mechanical systems.
- In the real world, mechanical systems could be extremely complex.
- Good news is that a complex mechanical system can be modeled as a system of simple components.
- Topics to be discussed:
  - Mechanical elements
  - Translational systems
  - Rotational systems
  - Energy method

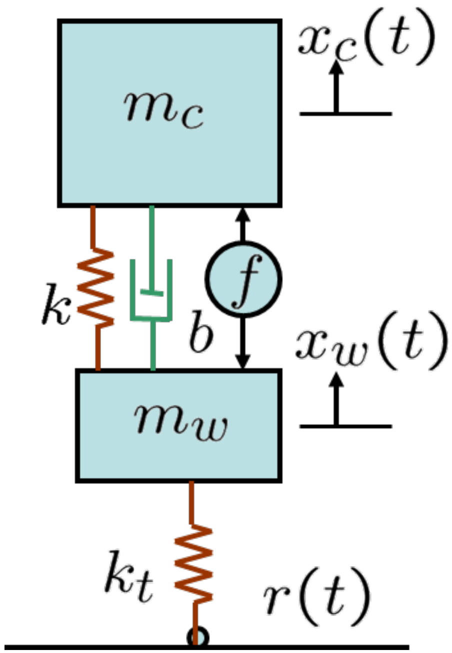
Example: car suspension

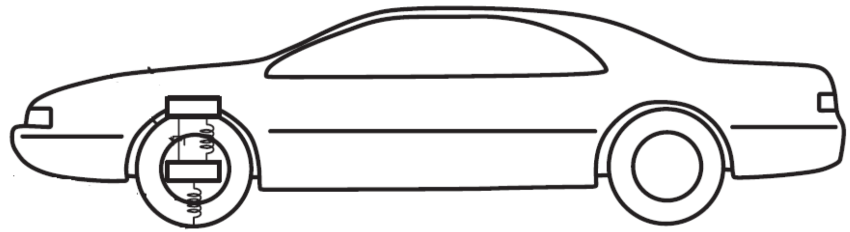
- Complex system, but we can see many common key elements:
  - Masses, springs, dampers, etc.





Example: car suspension



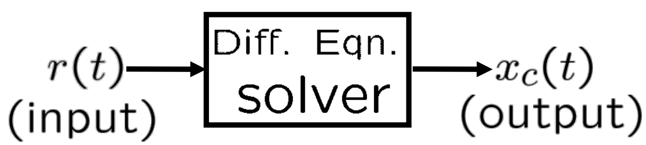


- m_c: chassis mass (car assembly)
- m_w: wheel mass (wheel assembly)
- k, b: passive shock absorbers
- k_t pneumatic tire stiffness
- f: active suspension
- r: road disturbance

$$m_c\ddot{x}_c + k(x_c - x_w) + b(\dot{x}_c - \dot{x}_w) = f$$
$$m_w\ddot{x}_w + k(x_w - x_c) + k_t(x_w - r) + b(\dot{x}_w - \dot{x}_c) = -f$$

differential equation

- It is desired that chassis position $x_c(t)$ is insensitive to road disturbances $r(t)$.



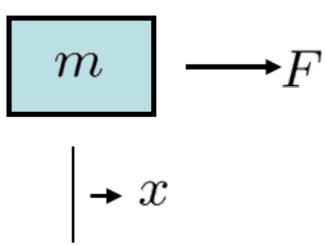
Recap: Newton's 2$^{\text{nd}}$ Law

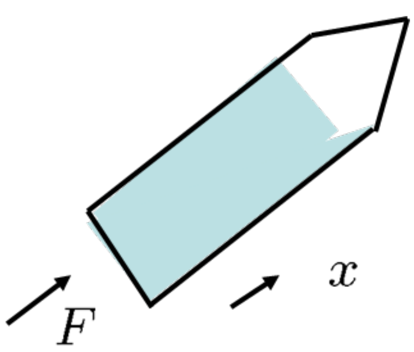
- The time-rate-of-change of momentum is proportional to the force.

$$F = \frac{d}{dt}(mv) = \frac{d}{dt}(m\dot{x})$$

  - More generally: $\frac{d}{dt}(m\dot{r})$, where r is the position vector measured from an *inertial frame*.
  - Note that $F = m\frac{d^2x}{dt^2}$ only when the mass m is a constant, i.e., $F = ma$ is true only when m is a constant.
  - For example, in a rocket, the fuel, which forms a significant mass of the whole rocket, gets used up as the rocket flies, namely, m is NOT a constant.

$$F = \frac{d}{dt}(m\dot{x}) = \dot{m}\dot{x} + m\ddot{x} = \dot{m}v + ma$$ *Product rule for derivatives*





Springs

- Have a free length or natural length (say L).
- They always "want" to be free or natural.
- They always exert a restoring force when moved from natural/free state.
    - When *stretched* or *compressed* (say by a length x), they exert a restoring force kx in a direction so as to restore to the "natural state."
    - Whether stretched or compressed, force exerted by the spring on the mass is always opposite to the direction of its displacement, i.e.,

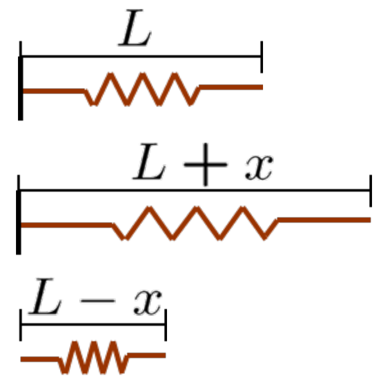
$$F = -kx$$

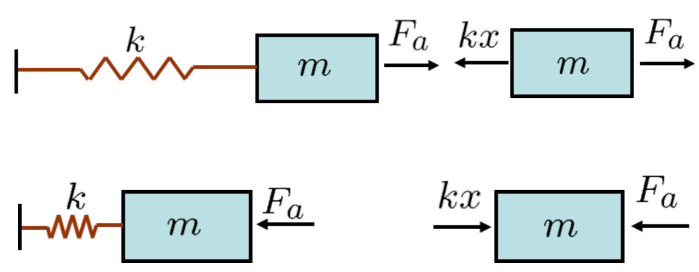
k: spring constant or stiffness
x: displacement from the natural state

Hooke's Law **This is called a *linear* spring.**

- If some external force F_a is applied and m is constant,

$$\frac{d}{dt}(m\dot{x}) = -kx + F_a \Leftrightarrow m\ddot{x} + kx = F_a$$





Springs

- Have a free length or natural length (say L).
- They always "want" to be free or natural.
- They store *potential energy* when moved from natural/free state.
- When stretched or compressed by a length x, the potential energy is given by

$$V = \frac{1}{2}kx^2$$

- Kinetic energy of the mass:

$$T = \frac{1}{2}mv^2$$

- Therefore, the total energy

$$T.E. = T + V$$

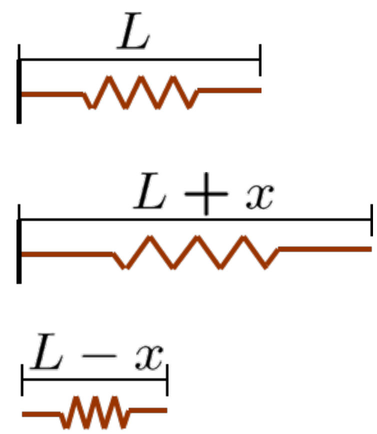
- When no external force is acting, T.E. is constant

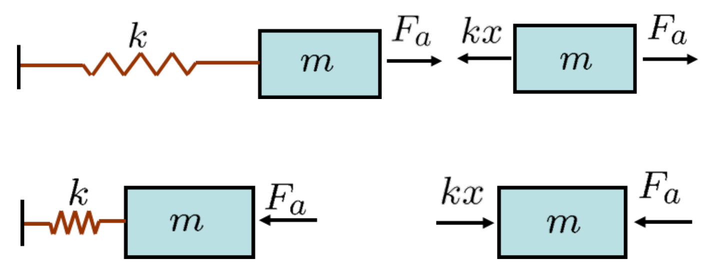
$$\frac{d}{dt}(T.E.) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = 0 \Rightarrow \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

$$\Rightarrow m\dot{x}\ddot{x} + k\dot{x}x = 0 \Rightarrow m\ddot{x} + kx = 0$$

Same as what we got in the last slide with $F_a = 0$





Springs in parallel

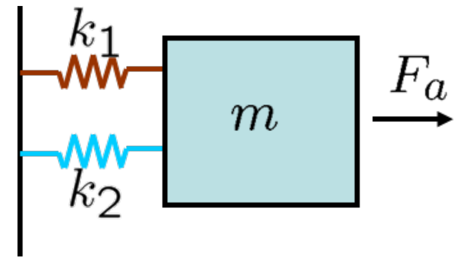
- Spring-mass systems
  - Very commonly seen as components of bigger systems.
- Free body diagrams
  - Springs in parallel
    - Each spring compresses or stretches by the same amount x
    - Springs will exert force $k_1 x$ and $k_2 x$ opposite to the acceleration direction
    - Apply Newton's 2nd law

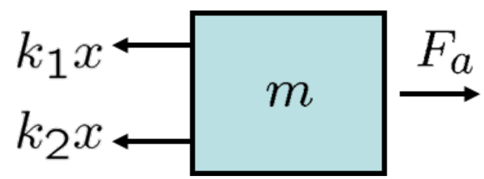
$$m\ddot{x} = F_a - k_1 x - k_2 x$$

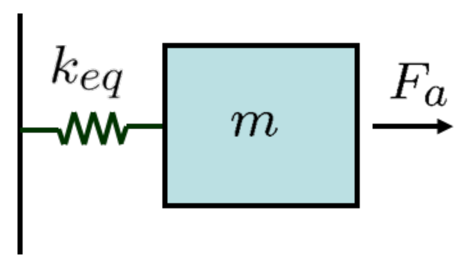
$$\Rightarrow \quad m\ddot{x} + \underbrace{(k_1 + k_2)}_{k_{eq}} x = F_a$$

    - Equivalent spring constant or stiffness:

$$k_{eq} = k_1 + k_2$$



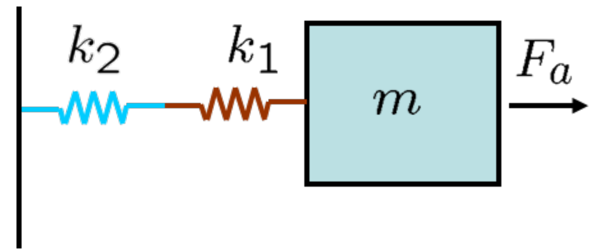


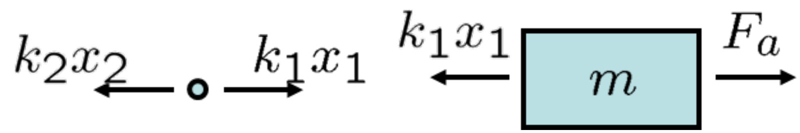


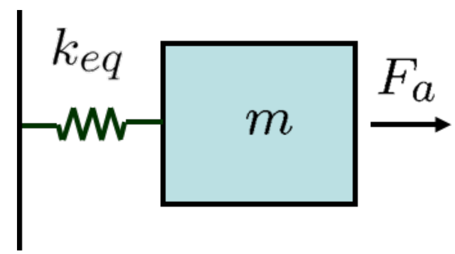
Springs in series

- Springs in series
  - Each spring moves by different amounts x_1 and x_2
  - The displacement of the mass is
    $$x_1 + x_1 \triangleq x$$
  - Free-body diagram for the mass
    $$m\ddot{x} = F_a - k_1 x_1$$
  - Free-body diagram for the massless junction, where two springs meet
    $$0 = k_1 x_1 - k_2 x_2$$
  - With $x = x_1 + x_2$, we get

$$\begin{aligned} m\ddot{x} + k_1 x_1 &= F_a \\ m\ddot{x} + \underbrace{\frac{k_1 k_2}{k_1 + k_2}}_{k_{eq}} x &= F_a \end{aligned}$$







$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

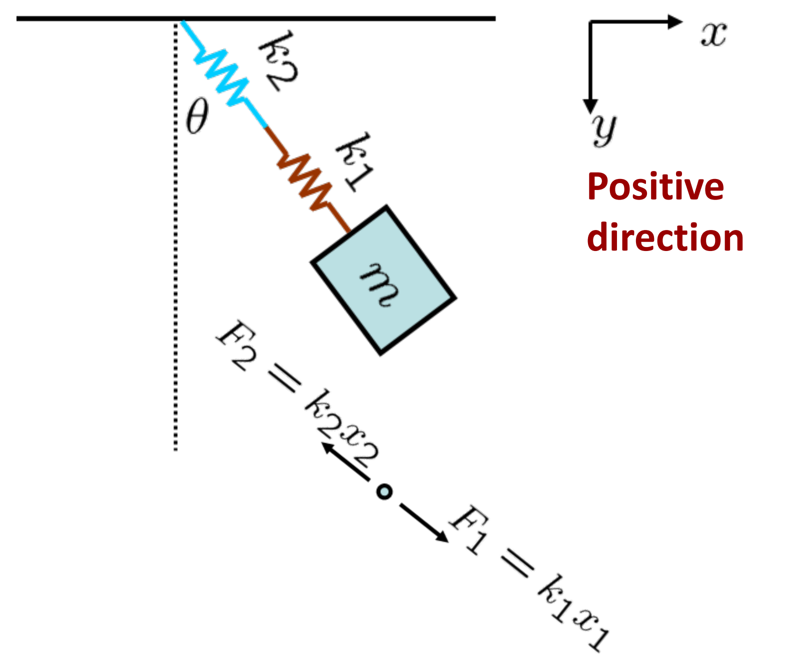
Example: pendulum with spring arm

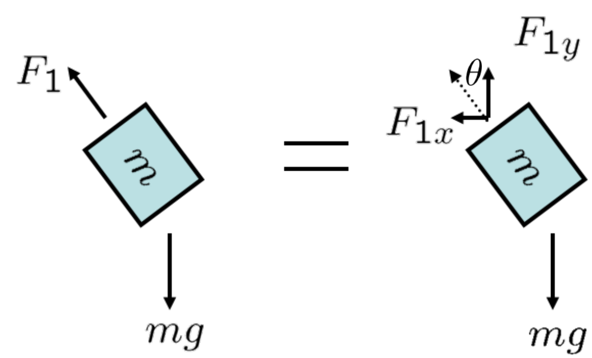
- The mass moves along *x* and *y* directions.
- Spring deforms by $l = \sqrt{x^2 + y^2}$
- Note that $\sin\theta = \frac{x}{\sqrt{x^2+y^2}}$, $\cos\theta = \frac{y}{\sqrt{x^2+y^2}}$
- Free body diagrams
  - $k_1 l_1 = k_2 l_2$ and $l_1 + l_2 = l = \sqrt{x^2 + y^2}$
  - Therefore, $F_1 = k_{eq} l = \frac{k_1 k_2}{k_1 + k_2}\sqrt{x^2 + y^2}$
  - Newton's 2nd Law

$$
\begin{aligned}
m\ddot{r} &= F \\
\Rightarrow m\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} &= \begin{pmatrix} -F_{1x} \\ mg - F_{1y} \end{pmatrix} \\
\Rightarrow \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} &= \begin{pmatrix} -k_{eq} l \sin\theta \\ mg - k_{eq} l \cos\theta \end{pmatrix} \\
\Rightarrow \begin{pmatrix} m\ddot{x} \\ m\ddot{y} \end{pmatrix} &= \begin{pmatrix} -k_{eq} x \\ mg - k_{eq} y \end{pmatrix}
\end{aligned}
$$

  - Equivalently,

$$
\begin{aligned}
m\ddot{x} + \frac{k_1 k_2}{k_1 + k_2} x &= 0 \\
m\ddot{y} + \frac{k_1 k_2}{k_1 + k_2} y &= mg
\end{aligned}
$$





Translational viscous damper

- Dampers (also called dashpots):
  - They exert a force which *always opposes motion*.
  - Resisting force ∝ *relative velocity*

$$F = b(\dot{y} - \dot{x})$$

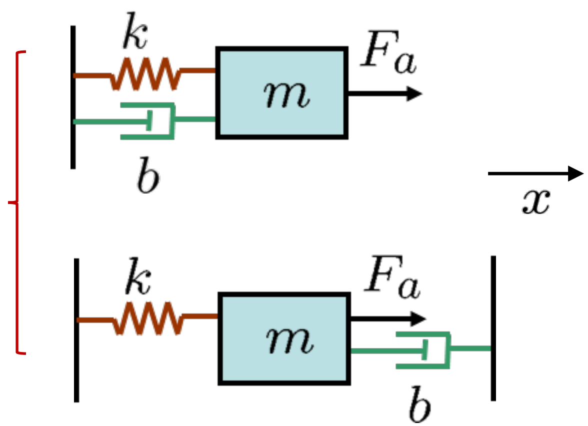
  - Free body diagram for *Mass-Spring-Damper system*

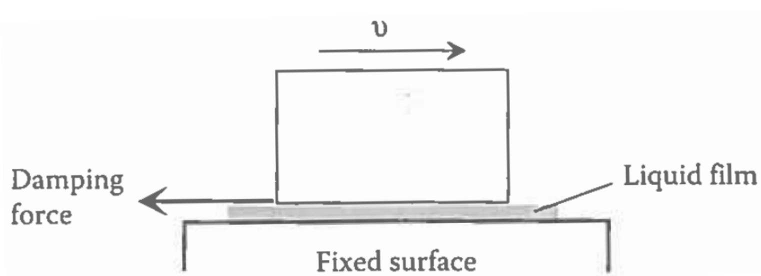
$$m\ddot{x} = -kx - b\dot{x} + F_a$$
$$\Leftrightarrow m\ddot{x} + b\dot{x} + kx = F_a$$

  - If some external force F_a is applied and m is constant.

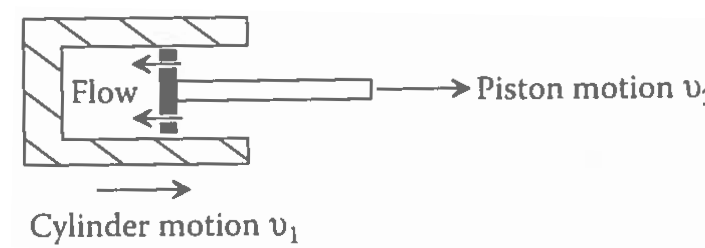
Two moving ends

Same equation of motion



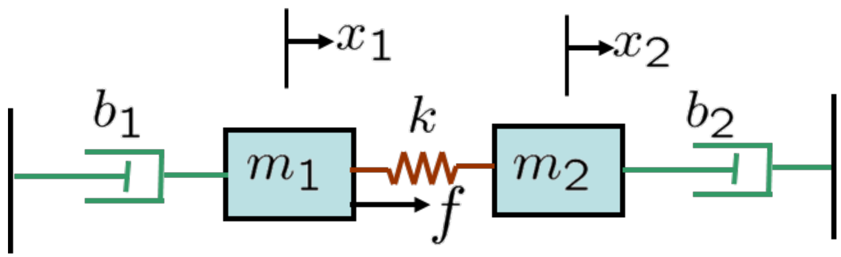


A mass sliding on a lubricated fixed surface



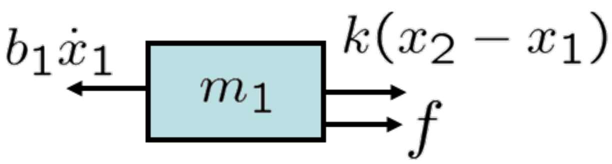
A piston-cylinder system

Example: mass-spring-damper system



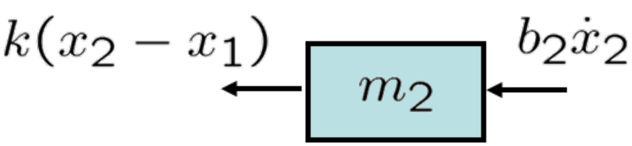
- Free body diagram
  - For mass 1:

$$m_1\ddot{x}_1 = -b_1\dot{x}_1 + k(x_2 - x_1) + f$$
$$\Leftrightarrow m_1\ddot{x}_1 + b_1\dot{x}_1 + kx_1 - kx_2 = f$$



  - For mass 2:

$$m_2\ddot{x}_2 = -b_2\dot{x}_2 - k(x_2 - x_1)$$
$$\Leftrightarrow m_2\ddot{x}_2 + b_2\dot{x}_2 + kx_2 - kx_1 = 0$$



  - Therefore,

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + kx_1 - kx_2 = f$$
$$m_2\ddot{x}_2 + b_2\dot{x}_2 + kx_2 - kx_1 = 0$$

Example: massless junction

- Free body diagram
  - For the massless point:

    $$0\ddot{x}_1 = b(\dot{x}_2 - \dot{x}_1) - kx_1 \Leftrightarrow b\dot{x}_1 + kx_1 = b\dot{x}_2$$

  - For the mass:

    $$m\ddot{x}_2 = -b(\dot{x}_2 - \dot{x}_1) \Leftrightarrow m\ddot{x}_2 + b\dot{x}_2 = b\dot{x}_1$$

  - Therefore,

    $$b\dot{x}_1 - b\dot{x}_2 + kx_1 = 0$$

    $$m\ddot{x}_2 + b\dot{x}_2 - b\dot{x}_1 = 0$$