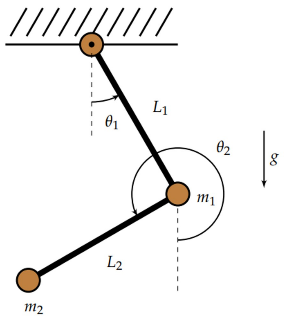
Lab 7 Pre-Lab

1. Consider a double pendulum consisting of two particles swinging in a vertical plane under the influence of gravity, such that the first particle of mass m_1 is located at a distance L_1 from a stationary suspension, and the second particle of mass m_2 is located at a distance L_2 from the first particle, as shown in the figure.



(a) Show that the kinetic energy of the mechanism equals

$$T = \frac{m_1 + m_2}{2} L_1^2 \dot{\theta}_1^2 + \frac{m_2}{2} L_2^2 \dot{\theta}_2^2 + m_2 L_1 L_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 \dot{\theta}_2$$

in terms of the angles θ_1 and θ_2.

(b) Show that the potential energy of the mechanism equals

$$V = -(m_1 + m_2) L_1 g \cos\theta_1 - m_2 L_2 g \cos\theta_2$$

(c) Show that Lagrange's equations equal

$$(m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 L_1 L_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (m_1 + m_2) L_1 g \sin\theta_1 = 0$$

and

$$m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 - m_2 L_1 L_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 + m_2 L_2 g \sin\theta_2 = 0$$

a). $|V_1| = L_1 \cdot \dot{\theta}_1$

$$\vec{R}_2 = L_1 \cdot e^{j\theta_1} + L_2 \cdot e^{j\theta_2} \Rightarrow \vec{V}_2 = j\dot{\theta}_1 L_1 \cos\theta_1 + j\dot{\theta}_2 L_2 \cdot \cos\theta_2 - \dot{\theta}_1 L_1 \sin\theta_1 - \dot{\theta}_2 L_2 \sin\theta_2$$

$$\Rightarrow |V_2| = \left[(\dot{\theta}_1 L_1 \cos\theta_1 + \dot{\theta}_2 L_2 \cos\theta_2)^2 + (\dot{\theta}_1 L_1 \sin\theta_1 + \dot{\theta}_2 L_2 \sin\theta_2)^2\right]^{\frac{1}{2}}$$

$$= \left[(\dot{\theta}_1 L_1)^2 + (\dot{\theta}_2 L_2)^2 + 2\dot{\theta}_1 \dot{\theta}_2 L_1 L_2 \cos(\theta_1 - \theta_2)\right]^{\frac{1}{2}}$$

$\therefore$ Kinetic energy: $\frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2$

$$= \frac{1}{2} m_1 \cdot L_1^2 \cdot \dot{\theta}_1^2 + \frac{1}{2} m_2 \cdot \left[(\dot{\theta}_1 L_1)^2 + (\dot{\theta}_2 L_2)^2 + 2\dot{\theta}_1 \dot{\theta}_2 L_1 L_2 \cos(\theta_1 - \theta_2)\right]$$

$$= \frac{m_1 + m_2}{2} L_1^2 \dot{\theta}_1^2 + m_2 \cdot \dot{\theta}_2^2 L_2^2 + m_2 \dot{\theta}_1 \dot{\theta}_2 L_1 \cdot L_2 \cos(\theta_1 - \theta_2)$$

b). Potential energy: $-m_1 g \cdot L_1 \cos\theta_1 - m_2 g \cdot [L_1 \cos\theta_1 + L_2 \cos\theta_2]$

$$= -(m_1 + m_2) g L_1 \cos\theta_1 - m_2 g L_2 \cos\theta_2$$

c). $\frac{\partial T}{\partial \dot{\theta}_1} = (m_1 + m_2) L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_2 - \theta_1)$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 \cos(\theta_2 - \theta_1) - m_2 L_1 L_2 \dot{\theta}_2 \sin(\theta_2 - \theta_1)(\dot{\theta}_2 - \dot{\theta}_1)$$

$$\frac{\partial T}{\partial \theta_1} = m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_2 - \theta_1)$$

$$\frac{\partial V}{\partial \theta_1} = (m_1 + m_2) g L_1 \cdot \sin\theta_1$$

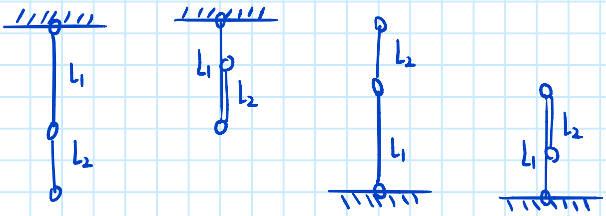
$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = Q_1 = 0 \Rightarrow (m_1 + m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 L_1 L_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 + (m_1 + m_2) L_1 g \sin\theta_1 = 0$$

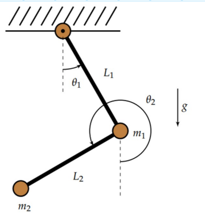
And same for $i=2$

2. Determine the four equilibrium configurations of the equations of motion shown in Prob. 1 by letting $\dot{\theta}_1 = 0$, $\dot{\theta}_2 = 0$, $\ddot{\theta}_1 = 0$, and $\ddot{\theta}_2 = 0$ and solving for all possible values of θ_1 and θ_2.

$\because \dot{\theta}_1 = 0, \ \dot{\theta}_2 = 0, \ \ddot{\theta}_1 = 0, \ \ddot{\theta}_2 = 0$

$$\therefore \begin{cases} (m_1+m_2) L_1 g \sin\theta_1 = 0 \\ m_2 L_2 g \sin\theta_2 = 0 \end{cases} \Rightarrow \begin{cases} \theta_1 = k\pi \\ \theta_2 = k\pi \end{cases}$$





线性化

3. Show that the linearized equations of motion about the equilibrium configuration in which both particles are located below the stationary suspension are given by

$$(m_1+m_2)L_1^2\ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 + (m_1+m_2) g L_1 \theta_1 = 0$$

and

$$m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 + m_2 g L_2 \theta_2 = 0$$

Original Equation:

1. $(m_1+m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \overset{=1}{\cos(\theta_1-\theta_2)} \ddot{\theta}_2 + m_2 L_1 L_2 \overset{=0}{\sin(\theta_1-\theta_2)} \dot{\theta}_2^2 + (m_1+m_2) L_1 g \sin\theta_1 = 0$

2. $m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \underset{=1}{\cos(\theta_1-\theta_2)} \ddot{\theta}_1 - m_2 L_1 L_2 \underset{=0}{\sin(\theta_1-\theta_2)} \dot{\theta}_1^2 + m_2 L_2 g \sin\theta_2 = 0$

Linearization:

Taylor expansion: $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{3!} + \cdots$

Because here $\theta_{1_0} = \theta_{2_0} = 0$ Therefore, we expand the sin at $\theta = 0$

$\therefore \sin\theta_1 \approx \theta_1 \quad \sin\theta_2 \approx \theta_2 \quad \cos(\theta_1-\theta_2) \approx 1 \quad \sin(\theta_1-\theta_2) \approx 0$

$\therefore$ 1. $(m_1+m_2) L_1^2 \ddot{\theta}_1 + m_2 L_1 L_2 \ddot{\theta}_2 + (m_1+m_2) g L_1 \theta_1 = 0$

2. $m_2 L_2^2 \ddot{\theta}_2 + m_2 L_1 L_2 \ddot{\theta}_1 + m_2 L_2 g \theta_2 = 0$