Homework 5

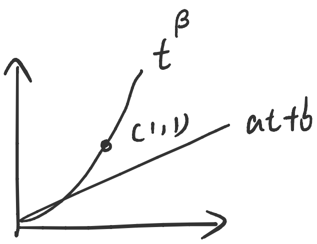
1. The following problem relates to the existence theorem of Laplace Transforms.
   (a) Consider the function $h(t) = t^\beta - (at + \log_e M)$, where $\beta \geq 2$, a, and $M \geq 1$ are positive real numbers.
   ≥ 0 ≥ 0
      i. Show that $h(t) > 0$ for all $t > \max\{1, a + \log_e M\}$. $\{1, a+b\}$
      ii. Deduce that $e^{t^{10}} > Me^{at}$ for all $t > \max\{1, a + \log_e M\}$.

   (b) Consider the function $f(t) = e^{t^{10}} u_s(t)$, where $u_s(t)$ is a unit-step function. Does $\mathcal{L}\{f(t)\}$ exist? Justify your response.

   (c) Consider the function

$$f(t) = \begin{cases} e^{t^{10}} & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Does $\mathcal{L}\{f(t)\}$ exist? Justify your response.



a). i) let $\ln M = b \geq 0$

To show $h(t) > 0$ is the same to show $t^\beta > at+b$

Let $f(t) = t^\beta$ $g(t) = at+b$

① $a+b \leq 1$

$f(1) = 1^\beta = 1$

$g(1) = a+b \leq f(1)$

$\because f'(1) = \beta \geq 2$

$g'(1) = a \leq 1 < f'(1)$

$\therefore f(1) > g(1)$

$\therefore h(t) > 0$

② $a+b \geq 1$

$f(a+b) = (a+b)^\beta \geq (a+b)^2 = a^2 + 2ab + b^2$

$g(a+b) = a(a+b) + b = a^2 + ab + b$

$\therefore f(a+b) - g(a+b) = ab + b^2 - b = b(a+b-1) > 0$

$f'(a+b) = \beta(a+b)^{\beta-1} \geq 2(a+b)$

$g'(a+b) = a$

$\therefore f'(a+b) - g'(a+b) = a + 2b > 0$

$\therefore f(a+b) > g(a+b)$

$\therefore h(t) > 0$

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a) ii). To prove $e^{t^{10}} > Me^{at}$ equals to prove $t^{10} > \ln M + at = at + b$

From a) i). we know $t^\beta > at+b$ for $t > \max\{1, a+b\}$

$\therefore t^{10} > at + b$

$\therefore e^{t^{10}} > M \cdot e^{at}$

(b) Consider the function $f(t) = e^{t^{10}} u_s(t)$, where $u_s(t)$ is a unit-step function. Does $\mathcal{L}\{f(t)\}$ exist? Justify your response.

(c) Consider the function

$$f(t) = \begin{cases} e^{t^{10}} & \text{if } 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Does $\mathcal{L}\{f(t)\}$ exist? Justify your response.

B

b). $f(t) = e^{t^{10}} \cdot u_0(t)$ Let $m(t) = e^{t^{10}}$

$\mathcal{L}\{f(t)\} = \mathcal{L}\{m(t) \cdot u_0(t)\} = e^{-0} \cdot M(s) = \mathcal{L}\{e^{t^{10}}\} = \int_0^\infty e^{t^{10} - st} dt$

$\because \lim_{t \to \infty} e^{t^{10} - st} = \infty$ $\therefore$ DNE

c). Yes. Because $\int_0^\infty f(t) e^{-st} = \int_0^1 e^{t^{10}} \cdot e^{-st} dt + \int_1^\infty 0 e^{-st} dt = \int_0^1 e^{t^{10} - st} dt$ which is finite

2. Compute $\int_0^\infty t^6 e^{-3t} dt$. [Hint use the definition of Laplace transform and the Laplace Transform tables].

$\because F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} \cdot f(t)\, dt$

$\therefore$ in $\int_0^\infty e^{-3t} t^6 dt$, $s = 3$, $f(t) = t^6$

From the transformation table

we know $\mathcal{L}\{t^6\} = \frac{6!}{s^7}$

$\therefore \int_0^\infty e^{-3t} t^6 dt = \frac{6!}{3^7}$

3. Given the trigonometric identity $\sin 3t = 3 \sin t - 4 \sin^3 t$, determine $\mathcal{L}\{\sin^3 t\, u_s(t)\}$ [Use Laplace Transform Tables].

三倍角公式

$\mathcal{L}\{\sin^3 t \cdot u_0(t)\} = \mathcal{L}\{u_0(t) \cdot \frac{3\sin t - \sin 3t}{4}\} = \mathcal{L}\{u_0(t) \cdot \frac{3}{4}\sin t - u_0(t) \cdot \frac{1}{4} \sin 3t\}$

$= \frac{3}{4} \cdot \frac{1}{s^2+1} - \frac{1}{4} \cdot \frac{3}{s^2+9} = \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$

4. Given that $\mathcal{L}\{\cos^2 t\} = \frac{s^2+2}{s(s^2+4)}$, determine Laplace transform of $f(t) = 2 \cos t \sin t$ where $t \geq 0$.

$\boxed{\mathcal{L}\{f'(t)\} = sF(s) - f(0)}$

$\mathcal{L}\{\cos^2 t\} = \frac{s^2+2}{s(s^2+4)}$

$\mathcal{L}\{(\cos^2 t)'\} = \mathcal{L}\{2\cos t \cdot -\sin t\} = \frac{s^2+2}{s(s^2+4)} \cdot s - \cos^2(0) = \frac{s^2+2}{s^2+4} - 1 = -\frac{2}{s^2+4}$

$\therefore \mathcal{L}\{2\cos t \cdot \sin t\} = \frac{2}{s^2+4}$

5.

(a) Show that $(-1)^k \sin(t - k\pi) = \sin(t)$ and $(-1)^k \cos(t - k\pi) = \cos(t)$ for any positive integer k [Use appropriate trigonometric identities].

(b) Determine the Laplace transform of $f(t) = \sin(t)\, u_s(t - k\pi)$, where k is an integer.

a). ① $k \in$ odd $\quad (-1)^k \sin(t-k\pi) = -\sin(t-\pi) = \sin(t)$

② $k \in$ even $\quad (-1)^k \sin(t-k\pi) = \sin(t-0) = \sin(t)$

① $k \in$ odd $\quad (-1)^k \cdot \cos(t-k\pi) = -\cos(t-\pi) = \cos t$

② $k \in$ even $\quad (-1)^k \cdot \cos(t-k\pi) = \cos(t-2\pi) = \cos t \qquad$ q.e.d.

b) $\mathcal{L}\{\sin(t)\, u_s(t-k\pi)\} = e^{-k\pi s} \cdot \mathcal{L}\{\sin(t+k\pi)\} = e^{-k\pi s} \mathcal{L}\{(-1)^k \cdot \sin t\}$

$= (-1)^k \cdot e^{-k\pi s} \cdot \dfrac{1}{s^2+1}$