Homework 4 指数

Exercise 1

Let random variable X have probability density function

$$f(x) = -ln\,(0.5)\,(.5)^x, \qquad x \geq 0$$

a) (1 pt) Find an expression for $F(x)$ by hand. Show all your work and calculus steps!

Hint: $\int a^x = \frac{a^x}{ln(a)} + c$

Hint: Be sure that F(0) = 0

b) (1 pt) Plot the graph of $f(x)$ **and** $F(x)$ for x values: $0 < \text{x} < 10$ **using R**. Show code and plots.

You should have two graphs: 1) x on the horizontal axis, and the pdf on the vertical. 2) x on the horizontal, and the cdf on the vertical.

Parts (c) and (d): You may use an online integral calculator such as **Symbolab** *and do not need to show work for the calculus portions. Still set up your equations so it is clear what it is you solved.*

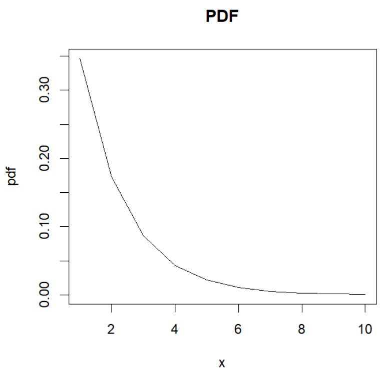
c) (0.5 pt) Evaluate $E[X]$.

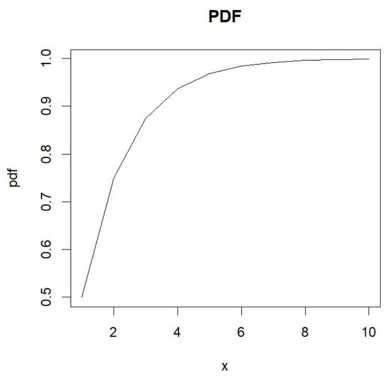
d) (0.5 pt) Evaluate $Var[X]$.

a). $F(x) = \int_0^x f(t)\,dt = -\int_0^x \ln\frac{1}{2}\cdot\left(\frac{1}{2}\right)^t dt = -\left(\frac{1}{2}\right)^t\Big|_0^x = \boxed{1-\left(\frac{1}{2}\right)^x}$

b).

```r
rm(list=ls()) #remove all data

fx=function(x){
  -log(0.5)*(0.5)^x
}

Fx=function(x){
  1-(0.5)^x
}

x=1:0.01:10

plot(fx(x),type="l",main='PDF',xlab='x',ylab='pdf')

plot(Fx(x),type="l",main='PDF',xlab='x',ylab='pdf')
```





c). $E[X] = \int_0^\infty x\cdot f(x) = \int_0^\infty -x\ln\frac{1}{2}\cdot\left(\frac{1}{2}\right)^x dx = -\left[x\cdot\left(\frac{1}{2}\right)^x\Big|_0^\infty - \int_0^\infty \left(\frac{1}{2}\right)^x dx\right]$

$= -\left[0-\left(0-\frac{1}{\ln\frac{1}{2}}\right)\right] = \boxed{\frac{1}{\ln 2}}$

d). $Var[x] = E[(x-\mu)^2] = \int_0^\infty \left(x-\frac{1}{\ln 2}\right)^2\cdot f(x)\,dx = \boxed{2.08137}$

From Wolframe Alpha

Exercise 2

Let T denote the time it takes for a computer to shut down. Suppose T follows an Exponential distribution with mean 15 seconds. A computer lab has 10 independent computers that must all be shut down at the end of the day. $\to$ 时间 $\to \theta$

a) (0.5 pt) What is the probability that it takes a randomly selected computer **at least** 10 seconds to shut down?

b) (0.5 pt) What is the probability that it takes a randomly selected computer **at least** 1 minute to shut down?

c) (0.5 pt) What is the probability that exactly 7 of the 10 computers will be successfully shut down **within** the first 30 seconds?

$\lambda = \frac{1}{15} \qquad n=10$

a). $P(0 \leqslant X \leqslant 10) = \int_0^{10} \lambda e^{-\lambda x} dx = -e^{-\lambda x}\Big|_0^{10} = -e^{-\frac{1}{15}\cdot 10} - (-e^{0}) = \boxed{1-e^{-\frac{2}{3}}}$

$P(X>10) = 1 - P(0 \leqslant X \leqslant 10) = \boxed{e^{-\frac{2}{3}}}$

b). $P(0 \leqslant X \leqslant 60) = \int_0^{60} \lambda e^{-\lambda x} dx = 1-e^{-4}$

$P(X>60) = 1 - P(0 \leqslant X \leqslant 60) = \boxed{e^{-4}}$

c). $P(0 \leqslant X \leqslant 30) = \int_0^{30} \lambda e^{-\lambda x} dx = 1-e^{-2}$

$P = C_{10}^{3} \times (1-e^{-2})^7 (e^{-2})^3 = 120 \times (1-e^{-2})^7 e^{-6} \approx \boxed{0.10749}$

Exercise 3

Given the following pdf for a random variable, T,

$$f(t) = c \cdot (3t - t^2),\ 0 \le t \le 3.$$

a) (0.5 pt) Find a value of c that makes this a valid pdf.

b) (0.5) What is $P[1 < T < 2]$?

c) (0.5 pt) Evaluate $E[T]$.

d) (0.5 pt) Find (evaluate) the 30th percentile of T.

百分位数

a). $\because \int_0^3 f(t)\,dt = 1$

$\therefore c\int_0^3 3t - t^2\,dt = c\cdot\left(\frac{3}{2}t^2 - \frac{1}{3}t^3\right)\Big|_0^3 = c\cdot\left[\frac{27}{2} - 9\right] = c\cdot\frac{9}{2} = 1$

$\therefore \boxed{c = \frac{2}{9}}$ $\quad f(t) = \frac{2}{9}(3t - t^2)$

b). $P[1<T<2] = \int_1^2 f(t)\,dt = \frac{2}{9}\left(\frac{3}{2}t^2 - \frac{1}{3}t^3\right)\Big|_1^2 = \frac{7}{21} = \boxed{\frac{13}{27}}$

c). $E(T) = \int_0^3 t\cdot f(t)\,dt = \int_0^3 \frac{2}{9}\left[3t^2 - t^3\right]dt = \frac{2}{9}\left[t^3 - \frac{1}{4}t^4\right]\Big|_0^3 = \boxed{1.5}$

d). $\because 0.3 = \int_0^{\pi_{0.3}} f(t)\,dt$

$\therefore 0.3 = \frac{2}{9}\left[\frac{3}{2}t^2 - \frac{1}{3}t^3\right]\Big|_0^{\pi_{0.3}}$

$\therefore \pi_{0.3} = \boxed{1.08977}$

Exercise 4

Suppose Tony Montana runs into Hector Salamanca according to a Poisson process with an average of 1 run-in per day. Assume that the week starts on Sunday at 12:00am.

泊松过程 $\lambda_{天} = 1$

Hint: Sunday is equivalent to time, t, $0 < t < 1$

a) (0.5 pt) Walter is trying to avoid Tuco. What is the probability that he does not run into Tuco in any given day?

b) (0.5 pt) What is the probability that he runs into Tuco **before** (not including) Wednesday? (*midnight Sunday through all of Tuesday*) 负 = 项

c) (0.5 pt) What is the probability that the 5th run-in occurs within the first week?

d) (0.5 pt) What is the probability that Walter has his second run-in with into Tuco on either Monday or Tuesday? ($Monday \cup Tuesday$)

a). $\lambda_{day} = 1$

$$P(x=0) = \frac{e^{-\lambda} \cdot \lambda^{x}}{0!} = \boxed{\frac{1}{e}}$$

b). $\downarrow \downarrow \downarrow$ _ _ _ W _ _

$$P_1 = 1 - P(X=0) = 1 - \frac{1}{e}$$

$$P_2 = \frac{1}{e}\left(1 - \frac{1}{e}\right)$$

$$P_3 = \frac{1}{e} \cdot \frac{1}{e}\left(1 - \frac{1}{e}\right) = \frac{1}{e^2} - \frac{1}{e^3}$$

$$P = P_1 + P_2 = 1 - \frac{1}{e} + \frac{1}{e} - \frac{1}{e^2} + \frac{1}{e^2} - \frac{1}{e^3} = \boxed{1 - \frac{1}{e^3}}$$

c). $\lambda_{week} = 7 \times \lambda_{day} = 7$

$$P(X \leq 4) = P(0) + P(1) + P(2) + P(3) + P(4) = \frac{e^{-7}}{1} + \frac{7e^{-7}}{1} + \frac{7^2 e^{-7}}{2} + \frac{7^3 e^{-7}}{6} + \frac{7^4 e^{-7}}{24} = 0.17299$$

$$P(X \geq 5) = 1 - P(X \leq 4) = \boxed{0.82701}$$

d) $\lambda = 1$

$$P = \int_0^3 \frac{1}{\Gamma(2) \cdot 1} \cdot x^{2-1} \cdot e^{-\frac{x}{1}} dx = \int_0^3 \frac{1}{\Gamma(2)} x e^{-x} dx = \int_0^3 x e^{-x} dx = \boxed{1 - \frac{4}{e^3}} \approx 0.8$$

Exercise 5

Let X ~ $N(10, 5^2)$ 正态分布

a) (0.25 pt) Evaluate $P[X > 12]$.

b) (0.25 pt) Evaluate $P[7 < X < 10]$.

c) (0.5 pt) Evaluate $P[3X + 5 > 45]$.

d) (0.5 pt) What value is the 20th percentile of X?

a). $\mu = 10 \quad \sigma^2 = 5$

$P[X > 12]$

$= P[\frac{X-10}{5} > \frac{12-10}{5} = 0.4]$

$= P[Z > 0.4] = 1 - \text{pnorm}(0.4) = \boxed{0.3446}$

b) $P[7 < X < 10]$

$= P[\frac{7-10}{5} < \frac{X-10}{5} < \frac{10-10}{5}]$

$= P[-0.6 < Z < 0]$

$= \text{pnorm}(0) - \text{pnorm}(-0.6) = \boxed{0.2257}$

c). $P[3X + 5 > 45]$

$= P[X > \frac{40}{3}]$

$= P[\frac{X-10}{5} > \frac{\frac{40}{3}-10}{5} = \frac{2}{3}]$

$= P[Z > \frac{2}{3}] = 1 - \text{pnorm}[\frac{2}{3}] = \boxed{0.2525}$

d). $Z = CDF^{-1}(20\%)$

$= \text{qnorm}(0.2)$

$= -0.8416$

$\because Z = \frac{X-10}{5}$

$\therefore X = Z \cdot 5 + 10 = \boxed{5.792}$