Homework 6

1. Compute the Laplace transform of $g(t) = \begin{cases} t^3 & 0 \le t \le 1 \\ 0 & otherwise \end{cases}$

   [Hint. Note the identity $t^3 = (t-1)^3 + 3(t-1)^2 + 3(t-1) + 1$].

$$g(t) = \begin{cases} t^3, & 0 \le t \le 1 \\ 0, & \text{else} \end{cases} \Rightarrow g(t) = u_0 \cdot t^3 - u_1 t^3 \quad (\text{here } u_c \text{ stands for } u_s(t-c))$$

$$\therefore G(s) = \mathcal{L}\{u_0 t^3 - u_1 t^3\} = \frac{3!}{s^4} - \mathcal{L}\{u_1 t^3\} = \frac{3!}{s^4} - \mathcal{L}\{u_1[(t-1)^3 + 3(t-1)^2 + 3(t-1) + 1]\}$$

$$= \boxed{\frac{3!}{s^4} - e^{-s}\left[\frac{3!}{s^4} + 3 \cdot \frac{2!}{s^3} + 3 \cdot \frac{1!}{s^2} + \frac{1}{s}\right]}$$

2. Find the inverse Laplace transform of $X(s) = \frac{2e^{-\theta s}(s+1)}{((s+1)^2+1)^2}$.

$$\because X(s) = \frac{2e^{-\theta \cdot s}(s+1)}{((s+1)^2+1)^2} = 2 \cdot e^{-\theta s} \cdot \frac{(s+1)}{((s+1)^2+1)^2} = 2e^{-\theta \cdot s} \cdot F(t)$$

$$\therefore \mathcal{L}\{X(s)\} = u_\theta \cdot f(t-\theta) \quad \text{where } \mathcal{L}\{f(t)\} = F(t) = \frac{s+1}{((s+1)^2+1)^2} = -\frac{1}{2} \cdot \left(\frac{1}{(s+1)^2+1}\right)'$$

$$X(s) = \frac{2e^{-\theta s}(s+1)}{((s+1)^2+1)^2} = -e^{-\theta s} \cdot \left(\frac{1}{(s+1)^2+1}\right)'$$

$$\mathcal{L}\{X(s)\} = -u_s(t-\theta) \cdot f(t) \quad \text{where } \mathcal{L}\{f(t)\} = \left[\frac{1}{(s+1)^2+1}\right]'$$

$$\begin{cases} \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\} = e^{-t} \cdot u_s(t) \cdot \sin t \\ \mathcal{L}^{-1}\{-G'(s)\} = t \cdot g(t) \\ G'(s) = \left[\frac{1}{(s+1)^2+1}\right]' \end{cases} \Rightarrow \mathcal{L}^{-1}\left\{\left[\frac{1}{(s+1)^2+1}\right]'\right\} = t \cdot e^{-t} \cdot u_s(t) \cdot \sin t$$

$$\Rightarrow x(t) = u_s(t-\theta) \cdot u_s(t-\theta) \cdot e^{-(t-\theta)} \cdot \sin(t-\theta) \cdot (t-\theta)$$

$$= u_s(t-\theta) \cdot (t-\theta) \cdot \sin(t-\theta) \cdot e^{-(t-\theta)}$$

3. Find the inverse Laplace transform of $F(s) = \frac{4}{s^4 - s^2}$.

$$\because \frac{4}{s^4 - s^2} = \frac{4}{s^2(s^2-1)} = \frac{4}{s^2(s+1)(s-1)} = 4\left[\frac{1}{s(s-1)} \cdot \frac{1}{s(s+1)}\right] = 4\left[\left(\frac{1}{s-1} - \frac{1}{s}\right)\left(\frac{1}{s} - \frac{1}{s+1}\right)\right]$$

$$= 4\left[\left(\frac{1}{s-1}\cdot\frac{1}{s}\right) - \frac{1}{(s-1)}\cdot\frac{1}{(s+1)} - \frac{1}{s^2} + \frac{1}{s}\cdot\frac{1}{s+1}\right]$$

$$= 4\left[\frac{1}{s-1} - \cancel{\frac{1}{s}} - \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) - \frac{1}{s^2} + \cancel{\frac{1}{s}} - \frac{1}{s+1}\right]$$

$$= 4\left[\frac{1}{2}\cdot\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} - \frac{1}{s^2}\right] = 2\cdot\frac{1}{s-1} - 2\cdot\frac{1}{s+1} - 4\frac{1}{s^2}$$

$$\therefore \mathcal{L}^{-1}\left\{\frac{4}{s^4 - s^2}\right\} = \left[2\cdot e^{t} - 2\cdot e^{-t} - 4t\right] u_s(t)$$

4. A physical system is represented by an ordinary differential equation given by

$$\ddot{x} + 13\dot{x} + 42x = u(t), \; x(0) = x_0, \; \dot{x}(0) = \dot{x}_0.$$

(a) Determine $X(s)$ in terms of $U(s)$ and the initial conditions. Show that it is in the form given by

$$X(s) = G(s)U(s) + H(s, x_0, \dot{x}_0).$$

  i. Find the transfer function $G(s)$ for this system.

  ii. Show that $H(s, x_0, \dot{x}_0) = 0$ for all s when $x_0 = 0$ and $\dot{x}_0 = 0$.

(b) i. Find the state-space representation of the given system in the form $\dot{z} = Az + Bu$, where A and B are respectively 2×2 and 2×1 matrices.

  ii. Find and compare the roots of the characteristic equation, eigenvalues of the matrix A, and poles of the transfer function $G(s)$.

(c) Suppose a signal of interest $y(t) = 2x(t)$. Compute the transfer function $Q(s)$ such that $Y(s) = Q(s)U(s)$. (Assume $x_0 = 0$ and $\dot{x}_0 = 0$)
[Hint: $\frac{Y(s)}{U(s)} = \frac{Y(s)}{X(s)} \times \frac{X(s)}{U(s)}$].

(d) Suppose $v(t) = \dot{x} + x$. Compute the transfer function $R(s)$ such that $V(s) = R(s)U(s)$. (Assume $x_0 = 0$ and $\dot{x}_0 = 0$)

(e) If $u(t) = 8(1 - e^{-t})u_s(t)$. Find $x(t)$. (Assume $x_0 = 0$ and $\dot{x}_0 = 0$)

a). $\mathcal{L}\{\dot{x}\} = s\cdot X(s) - x_0$

$\mathcal{L}\{\ddot{x}\} = s\left(s\cdot X(s) - x_0\right) - \dot{x}_0 = s^2 X(s) - s x_0 - \dot{x}_0$

$\therefore s^2 X(s) - s x_0 - \dot{x}_0 + 13\left[s\cdot X(s) - x_0\right] + 42\, X(s) = U(t)$

$(s+6)(s+7)X(s) - (13+s)x_0 - \dot{x}_0 = U(t)$

$$X(s) = \frac{1}{(s+6)(s+7)}\cdot U(t) + \frac{(13+s)x_0 + \dot{x}_0}{(s+6)(s+7)}$$

$\therefore G(s) = \frac{1}{(s+6)(s+7)}$ i

$H(s, x_0, \dot{x}_0) = \frac{(13+s)x_0 + \dot{x}_0}{(s+6)(s+7)}$ ii

$\therefore H(s, 0, 0) = 0$

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[Hint: $\frac{Y(s)}{U(s)} = \frac{Y(s)}{X(s)} \times \frac{X(s)}{U(s)}$].

(d) Suppose $v(t) = \dot{x} + x$. Compute the transfer function $R(s)$ such that $V(s) = R(s)U(s)$. (Assume $x_0 = 0$ and $\dot{x}_0 = 0$)

(e) If $u(t) = 8(1 - e^{-t})u_s(t)$. Find $x(t)$. (Assume $x_0 = 0$ and $\dot{x}_0 = 0$)

b). $z = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{cases} x_1 = x \\ x_2 = \dot{x} \end{cases} \Rightarrow \dot{z} = \begin{bmatrix} \dot{x} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} x_2 \\ u(t) - 13x_2 - 42x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -42 & -13 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u(t)$

$\therefore \dot{z} = \begin{bmatrix} 0 & 1 \\ -42 & -13 \end{bmatrix} \cdot z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot u$ i

For characteristic equation: $\lambda^2 + 13\lambda + 42 = 0 \Rightarrow \lambda_1 = -6, \lambda_2 = -7$

For eigenvalue of A : $\det(A - \lambda I) = 0 \Rightarrow \lambda_1 = -6, \lambda_2 = -7$

For poles of $G(s)$: $G(s) = \frac{1}{(s+7)(s+6)} \Rightarrow \lambda_1 = -6, \lambda_2 = -7$

$\therefore$ they are all the same ii

c). $y = 2x \Rightarrow \frac{1}{2}[\ddot{y} + 13\dot{y} + 42y] = u(t)$

$\Rightarrow \frac{1}{2}[s^2 Y(s) + 13 \cdot sY(s) + 42Y(s)] = U(s)$

$\Rightarrow (s+6)(s+7) \cdot Y(s) = 2U(s) \Rightarrow Y(s) = \frac{2}{(s+6)(s+7)} \cdot U(s) \Rightarrow Q(s) = \frac{2}{(s+6)(s+7)}$

d). $\left.\begin{array}{l} V(s) = s \cdot X(s) + X(s) \Rightarrow V(s) = (s+1) \cdot X(s) \\ X(s) = G(s) \cdot U(s) \end{array}\right\} \Rightarrow R(s) = \frac{s+1}{(s+6)(s+7)}$

e). $u(t) = 8(1 - e^{-t}) \cdot u_s(t) \Rightarrow U(s) = \frac{8}{s} - \frac{8}{s+1} = \frac{8}{s(s+1)}$

$\Rightarrow X(s) = G(s) \cdot U(s) = 8 \cdot \left[\frac{1}{s(s+1)(s+6)(s+7)}\right]$

$\Rightarrow x(t) = \left[\left(\frac{1}{6} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{6}\right) \cdot e^{-6t} + \left(\frac{1}{6} - \frac{1}{5}\right) \cdot e^{-t} + \left(\frac{1}{7} - \frac{1}{6}\right) e^{-7t}\right] \cdot 8$

$= \left[\frac{4}{21} + \frac{4}{15} e^{-6t} - \frac{4}{15} e^{-t} - \frac{4}{21} e^{-7t}\right] \cdot u_s(t)$

5. Show that for the given ordinary differential equation

$$\dot{x} = ax + bu,$$

the solution is given by

$$x(t) = e^{at}x_0 + \int_0^t e^{a(t-\tau)}bu(\tau)d\tau.$$

[Hint: Use Laplace transform of ODE and determine $X(s)$. Then find its inverse Laplace Transform $x(t)$ by using Tables and convolution theorem.]

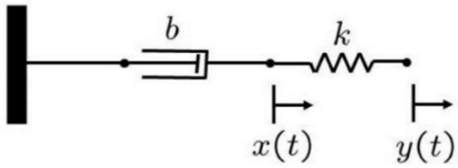
$\dot{x} = ax + bu \Rightarrow (s-a)X(s) - x_0 = b\,U(t)$

$$X(s) = \frac{b}{s-a}\cdot U(t) + \frac{x_0}{s-a}$$

$\Rightarrow x(t) = \mathcal{L}^{-1}\{X(s)\} = e^{at}\cdot x_0 + \int_0^t e^{a(t-\tau)}\cdot bu(\tau)d\tau$

6. Consider the system depicted here. If $x(t)$ denotes the displacement of a point to the left of the spring and $y(t)$ the displacement of a point to the right of the spring, then $x(t)$ and $y(t)$ are related by the differential equation

$$b\dot{x}(t) + kx(t) = ky(t).$$



Let $X(s)$ and $Y(s)$ denote the Laplace transforms of $x(t)$ and $y(t)$, respectively.

(a) Express $X(s)$ as a function of $x(0)$ and $Y(s)$.

(b) Let $b = k = 1$ and let $x(0) = 0$. Suppose you were to stretch the spring steadily from the right so that $y(t) = tu_s(t)$. Find $x(t)$ for $t \geq 0$. [Recall that Laplace Transform of $tu_s(t) = \frac{1}{s^2}$.]

a). $(bs+k)\cdot X(s) + b\cdot x_0 = k\cdot Y(s)$

$$X(s) = \frac{k}{bs+k}\cdot Y(s) - \frac{bx_0}{bs+k}$$

b). $X(s) = \frac{1}{s+1}\cdot Y(s) = \frac{1}{s+1}\cdot\frac{1}{s^2} = \frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1} \Rightarrow x(t) = \left[t - 1 + e^{-t}\right]\cdot u_s(t)$