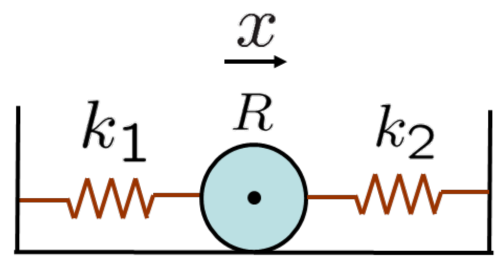
ME 340 Dynamics of Mechanical Systems

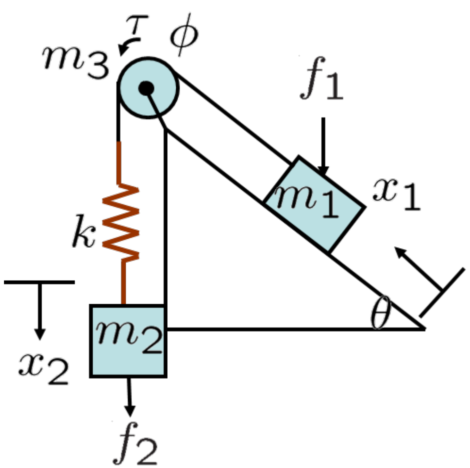
Lagrangian Dynamics
Part 2

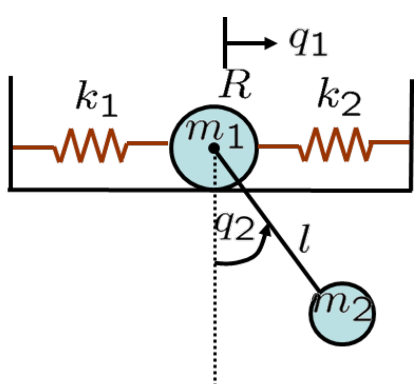
Energy vs Lagrangian methods:

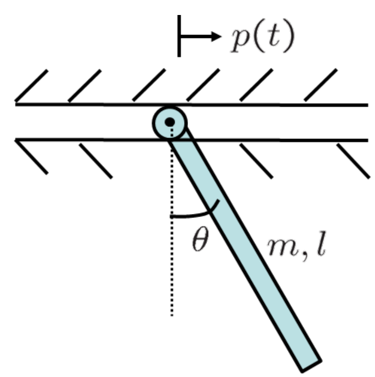
- Energy approach:
  - All components are energy storing
  - No work done on the system
  - We only get one equation – only good for 1 DOF

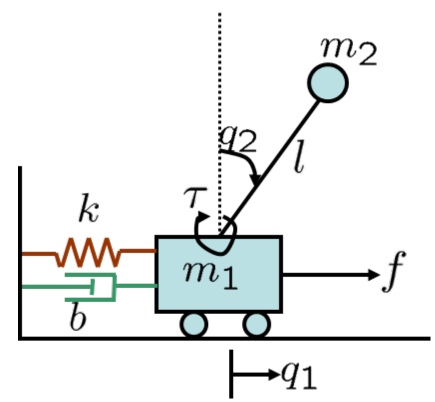


- Lagrangian dynamics:
  - Based on both Energy and Work, e.g., work done by dampers
  - Many generalized coordinates – one equation for each DOF
  - Simpler in some cases over Free Body Diagram method









Lagrangian dynamics

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

- q_i: generalized coordinates, $1 \leqslant i \leqslant N$
- $\dot{q}_i$: generalized velocities
- T is the total Kinetic Energy in the system
- V is the total Potential Energy
- Q_i: generalized non-conservative forces

- $L = T - V$ is called the Lagrangian of the system. Equivalently,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i$$

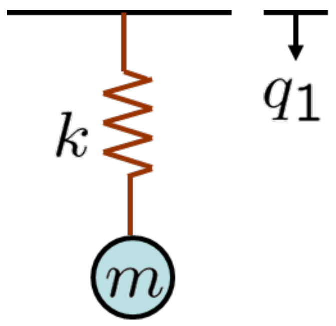
Lagrangian dynamics

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

General procedure

- Step 1: determine DOF and generalized coordinates;
- Step 2: write out the potential and kinetic energy;
- Step 3: calculate derivatives;
- Step 4: determine non-conservative generalized forces;
- Step 5: derive Lagrangian equations.

A trivial example: spring-mass system 例题



- ⋆ q_1: position of the mass, only 1 DOF
- ⋆ $\dot{q}_1$: velocity of the mass
- ⋆ $T = \frac{1}{2} m \dot{q}_1^2$ is the total Kinetic Energy
- ⋆ $V = \frac{1}{2} k q_1^2 - m g q_1$ is the total Potential Energy
- ⋆ $Q_1 = 0$ no non-conservative forces

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} = Q_1$$

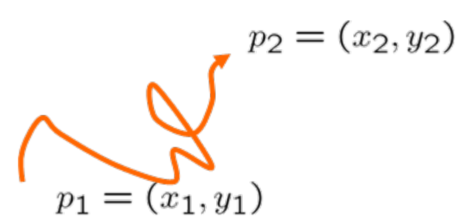
$$\Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial \dot{q}_1}\left(\frac{1}{2} m \dot{q}_1^2\right)\right) - \frac{\partial}{\partial q_1}\left(\frac{1}{2} m \dot{q}_1^2\right) + \frac{\partial}{\partial q_1}\left(\frac{1}{2} k q_1^2 - m g q_1\right) = 0$$

$$\Rightarrow \frac{d}{dt}(m \dot{q}_1) - \frac{\partial}{\partial q_1}\left(\frac{1}{2} m \dot{q}_1^2\right) + \frac{\partial}{\partial q_1}\left(\frac{1}{2} k q_1^2 - m g q_1\right) = 0$$

$$\Rightarrow m \ddot{q}_1 + 0 + k q_1 - m g = 0$$

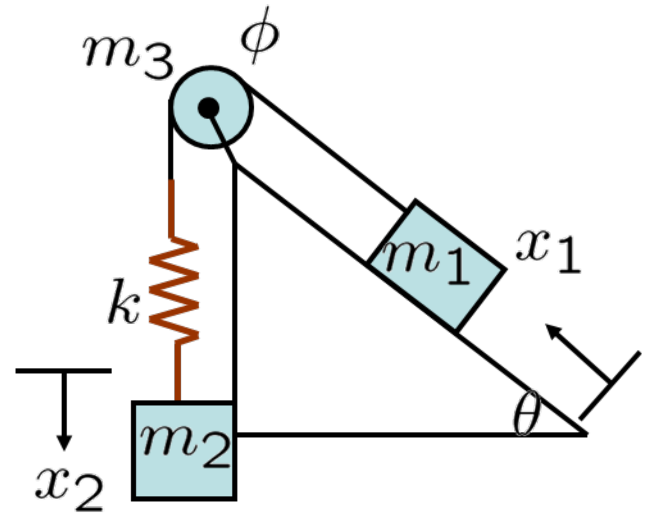
Kinetic and potential energy

- The *kinetic energy* of an object is the energy that it possesses due to its motion.
  - For example, $\frac{1}{2}m\dot{x}^2, \frac{1}{2}I_C\dot{\theta}^2$
- *Potential energy*: $V = V_{\text{elastic}} + V_{\text{gravity}}$, “stored” energy
  - V_{elastic}: stored in springs
    - For example, $\frac{1}{2}kx^2$ (linear springs), $\frac{1}{2}K\theta^2$ (torsional springs)
  - V_{gravity}: stored in mass with potential field
    - mgh, where h is taken w.r.t. some fixed point (datum)
    - V_{gravity} is “extra” potential energy from the gravity datum
  - Change in potential energy by taking the object from point A to point B is $V_B - V_A$
    - Does not depend on the path
  - Work done by conservative forces
    - Independent of the path
    - Depends only on the end points



Wedge example

例题2.



- Three masses
- Smooth wedge surface → no friction
- No external forces/torques
- DOF: 2
- Generalized coordinates

$$q_1 = x_1,\ q_2 = x_2,\ x_1 = R\phi \Rightarrow \dot{x}_1 = R\dot{\phi}$$

- Kinetic energy

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}\underbrace{\left(\frac{1}{2}m_3R^2\right)}_{\text{Solid cylinder}}\left(\frac{\dot{x}_1}{R}\right)^2$$

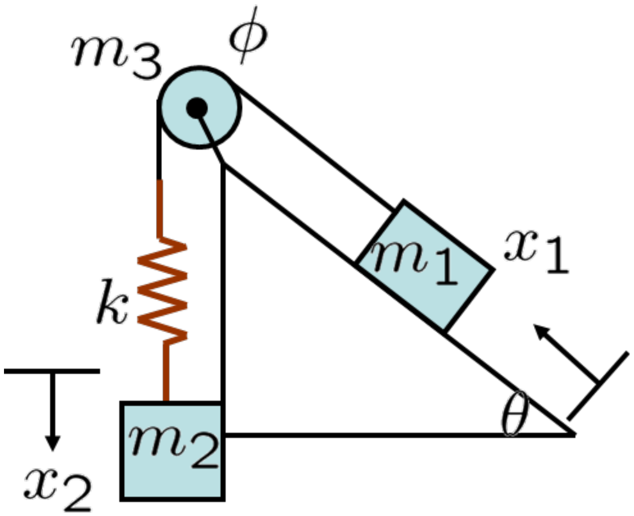
- Potential energy

$$V = m_1 g x_1 \sin\theta - m_2 g x_2 + \frac{1}{2}k(x_1 - x_2)^2$$

- Generalized forces

$$Q_1 = 0,\ Q_2 = 0$$

Wedge example



- Three masses
- Smooth wedge surface → no friction
- No external forces/torques

- Two equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) - \frac{\partial T}{\partial x_2} + \frac{\partial V}{\partial x_2} = 0$$

- Equations of motion

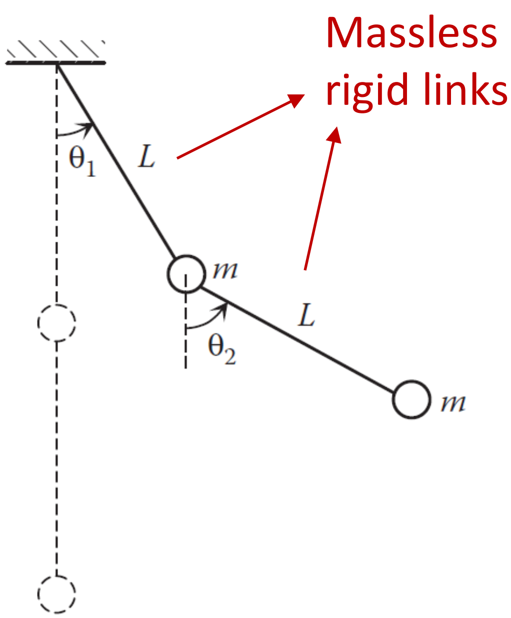
$$(m_1 + \frac{m_3}{2})\ddot{x}_1 + m_1 g \sin\theta + k(x_1 - x_2) = 0$$

$$m_2\ddot{x}_2 - m_2 g + k(x_2 - x_1) = 0$$

Example: double-pendulum system 双摆例题

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

- Two point masses
- DOF
- Generalized coordinates
- Kinetic energy
- Potential energy
- Generalized forces
- Equations of motion



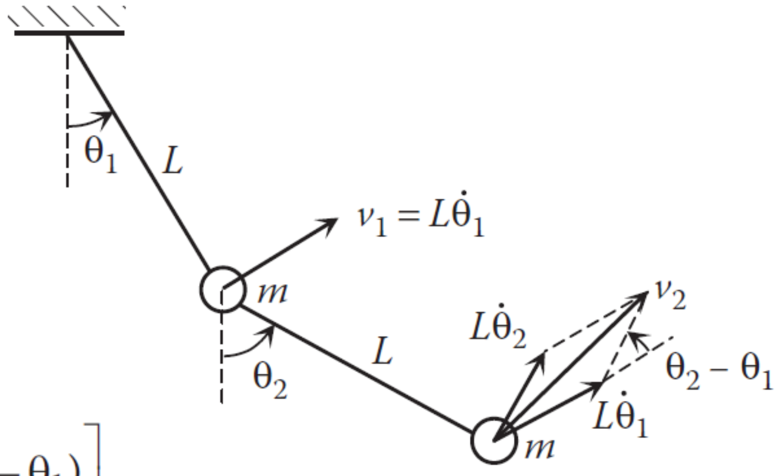
Example: double-pendulum system

$$T = \frac{1}{2} m v_1^2 + \frac{1}{2} m v_2^2$$

$$v_1 = L\dot{\theta}_1$$

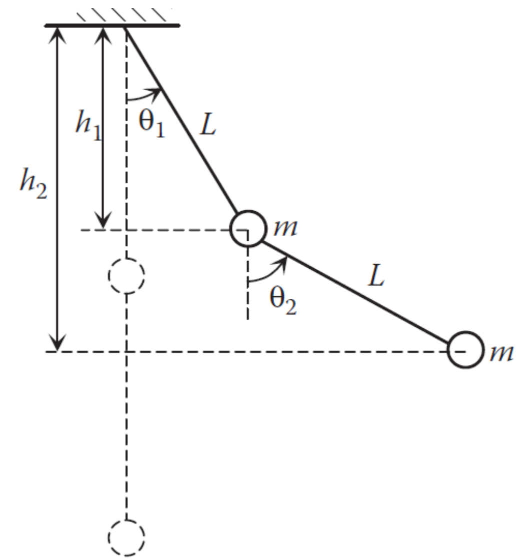
$$v_2^2 = \left(L\dot{\theta}_1\right)^2 + \left(L\dot{\theta}_2\right)^2 + 2L^2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_2 - \theta_1)$$

$$T = \frac{1}{2} m\left(L\dot{\theta}_1\right)^2 + \frac{1}{2} m\left[\left(L\dot{\theta}_1\right)^2 + \left(L\dot{\theta}_2\right)^2 + 2L^2\dot{\theta}_1\dot{\theta}_2 \cos(\theta_2 - \theta_1)\right]$$



$$V_g = -mgh_1 - mgh_2$$

$$V = -2mgL\cos\theta_1 - mgL\cos\theta_2$$



Example: double-pendulum system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=0, \quad i=1,2$$

$$\frac{\partial T}{\partial \dot{\theta}_1}=2mL^2\dot{\theta}_1+mL^2\dot{\theta}_2\cos(\theta_2-\theta_1)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)=2mL^2\ddot{\theta}_1+mL^2\ddot{\theta}_2\cos(\theta_2-\theta_1)-mL^2\dot{\theta}_2\sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)$$

$$\frac{\partial T}{\partial \theta_1}=-mL^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)(-1)$$

$$\frac{\partial V}{\partial \theta_1}=2mgL\sin\theta_1$$

$$\frac{\partial T}{\partial \dot{\theta}_2}=mL^2\dot{\theta}_2+mL^2\dot{\theta}_1\cos(\theta_2-\theta_1)$$

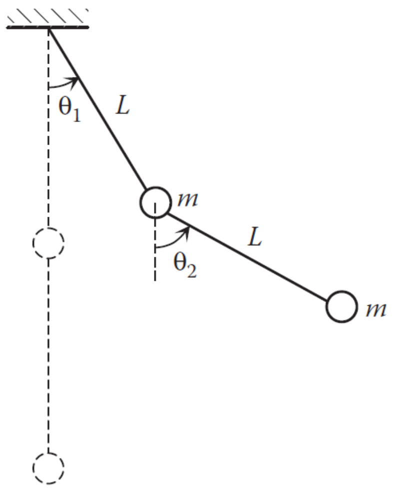
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right)=mL^2\ddot{\theta}_2+mL^2\ddot{\theta}_1\cos(\theta_2-\theta_1)-mL^2\dot{\theta}_1\sin(\theta_2-\theta_1)(\dot{\theta}_2-\dot{\theta}_1)$$

$$\frac{\partial T}{\partial \theta_2}=-mL^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)$$

$$\frac{\partial V}{\partial \theta_2}=mgL\sin\theta_2$$

Example: double-pendulum system

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=0, \quad i=1,2$$



$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right)-\frac{\partial T}{\partial \theta_1}+\frac{\partial V}{\partial \theta_1}=2mL^2\ddot{\theta}_1+mL^2\ddot{\theta}_2\cos(\theta_2-\theta_1)-mL^2\dot{\theta}_2^2\sin(\theta_2-\theta_1)+2mgL\sin\theta_1=0}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right)-\frac{\partial T}{\partial \theta_2}+\frac{\partial V}{\partial \theta_2}=mL^2\ddot{\theta}_2+mL^2\ddot{\theta}_1\cos(\theta_2-\theta_1)+mL^2\dot{\theta}_1^2\sin(\theta_2-\theta_1)+mgL\sin\theta_2=0}$$