Fourier Transform

非周期	周期
CTFT $X(V) = \int_{-\infty}^{+\infty} x(t) \cdot e^{-j(2\pi V)t} dt$	**CTFS** $c_k = \frac{1}{T}\int_0^T x(t) \cdot e^{-j(2\pi v_0)kt}\ dt$ [**傅里叶系数**] $v_0 = \frac{1}{T}$ [基频率] $x(t) = \sum_{k=-\infty}^{+\infty} c_k \cdot e^{j(2\pi v_0)kt}$
DTFT $X_p(V) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi nV}$ $x[n] = \int_{-1/2}^{+1/2} X_p(V)e^{j2\pi nV}$	**DTFS** $c_k = \sum_{n=0}^{N-1} x[n] \cdot e^{-j2\pi\left(\frac{k}{N}n\right)}$ $k = 0,1,2 \ldots N-1$ $x[n] = \sum_{k=0}^{N-1} c_k \cdot e^{j\left(2\pi\frac{1}{N}k\right)\cdot n}$

DFT $X_{DFT}[k] = \sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$ $x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X_{DFT}[k]e^{j2\pi nk/N}$	**DFS** $X_{DFS}[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N}$ $x[n] = \sum_{k=0}^{N-1} X_{DFS}[k]e^{j2\pi nk/N}$

CTFT 公式

Table 9.1 Some Useful Fourier Transform Pairs

Entry	$x(t)$	$X(f)$	$X(\omega)$
1	$\delta(t)$	1	1
2	$\mathrm{rect}(t)$	$\mathrm{sinc}(f)$	$\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
3	$\mathrm{tri}(t)$	$\mathrm{sinc}^2(f)$	$\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$
4	$\mathrm{sinc}(t)$	$\mathrm{rect}(f)$	$\mathrm{rect}\left(\frac{\omega}{2\pi}\right)$
5	$\cos(2\pi\alpha t)$	$0.5[\delta(f+\alpha)+\delta(f-\alpha)]$	$\pi[\delta(\omega+2\pi\alpha)+\delta(\omega-2\pi\alpha)]$
6	$\sin(2\pi\alpha t)$	$j0.5[\delta(f+\alpha)-\delta(f-\alpha)]$	$j\pi[\delta(\omega+2\pi\alpha)-\delta(\omega-2\pi\alpha)]$
7	$e^{-\alpha t}u(t)$	$\dfrac{1}{\alpha+j2\pi f}$	$\dfrac{1}{\alpha+j\omega}$
8	$te^{-\alpha t}u(t)$	$\dfrac{1}{(\alpha+j2\pi f)^2}$	$\dfrac{1}{(\alpha+j\omega)^2}$
9	$e^{-\alpha\lvert t\rvert}$	$\dfrac{2\alpha}{\alpha^2+4\pi^2f^2}$	$\dfrac{2\alpha}{\alpha^2+\omega^2}$
10	$e^{-\pi t^2}$	$e^{-\pi f^2}$	$e^{-\omega^2/4\pi}$
11	$\mathrm{sgn}(t)$	$\dfrac{1}{j\pi f}$	$\dfrac{2}{j\omega}$
12	$u(t)$	$0.5\delta(f)+\dfrac{1}{j2\pi f}$	$\pi\delta(\omega)+\dfrac{1}{j\omega}$
13	$e^{-\alpha t}\cos(2\pi\beta t)u(t)$	$\dfrac{\alpha+j2\pi f}{(\alpha+j2\pi f)^2+(2\pi\beta)^2}$	$\dfrac{\alpha+j\omega}{(\alpha+j\omega)^2+(2\pi\beta)^2}$
14	$e^{-\alpha t}\sin(2\pi\beta t)u(t)$	$\dfrac{2\pi\beta}{(\alpha+j2\pi f)^2+(2\pi\beta)^2}$	$\dfrac{2\pi\beta}{(\alpha+j\omega)^2+(2\pi\beta)^2}$
15	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$	$\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi k}{T}\right)$
16	$x_p(t)=\sum_{k=-\infty}^{\infty}X[k]e^{j2\pi kf_0t}$	$\sum_{k=-\infty}^{\infty}X[k]\delta(f-kf_0)$	$\sum_{k=-\infty}^{\infty}2\pi X[k]\delta(\omega-k\omega_0)$

CTFT 性质

Table 9.2 Operational Properties of the Fourier Transform

Property	$x(t)$	$X(f)$	$X(\omega)$
Similarity	$X(t)$	$x(-f)$	$2\pi x(-\omega)$
Time Scaling	$x(\alpha t)$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{f}{\alpha}\right)$	$\frac{1}{\lvert\alpha\rvert}X\left(\frac{\omega}{\alpha}\right)$
Folding	$x(-t)$	$X(-f)$	$X(-\omega)$
Time Shift	$x(t-\alpha)$	$e^{-j2\pi f\alpha}X(f)$	$e^{-j\omega\alpha}X(\omega)$
Frequency Shift	$e^{j2\pi\alpha t}x(t)$	$X(f-\alpha)$	$X(\omega-2\pi\alpha)$
Convolution	$x(t)\star h(t)$	$X(f)H(f)$	$X(\omega)H(\omega)$
Multiplication	$x(t)h(t)$	$X(f)\star H(f)$	$\frac{1}{2\pi}X(\omega)\star H(\omega)$
Modulation	$x(t)\cos(2\pi\alpha t)$	$0.5[X(f+\alpha)+X(f-\alpha)]$	$0.5[X(\omega+2\pi\alpha)+X(\omega-2\pi\alpha)]$
Derivative	$x'(t)$	$j2\pi fX(f)$	$j\omega X(\omega)$
Times-t	$-j2\pi tx(t)$	$X'(f)$	$2\pi X'(\omega)$
Integration	$\int_{-\infty}^{t}x(t)\,dt$	$\frac{1}{j2\pi f}X(f)+0.5X(0)\delta(f)$	$\frac{1}{j\omega}X(\omega)+\pi X(0)\delta(\omega)$
Conjugation	$x^*(t)$	$X^*(-f)$	$X^*(-\omega)$
Correlation	$x(t)\star\star y(t)$	$X(f)Y^*(f)$	$X(\omega)Y^*(\omega)$
Autocorrelation	$x(t)\star\star x(t)$	$X(f)X^*(f)=\lvert X(f)\rvert^2$	$X(\omega)X^*(\omega)=\lvert X(\omega)\rvert^2$

Fourier Transform Theorems	
Central ordinates	$x(0)=\int_{-\infty}^{\infty}X(f)\,df=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\,d\omega \qquad X(0)=\int_{-\infty}^{\infty}x(t)\,dt$
Parseval's theorem	$E=\int_{-\infty}^{\infty}x^2(t)\,dt=\int_{-\infty}^{\infty}\lvert X(f)\rvert^2\,df=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2\,d\omega$
Plancherel's theorem	$\int_{-\infty}^{\infty}x(t)y^*(t)\,dt=\int_{-\infty}^{\infty}X(f)Y^*(f)\,df=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)Y^*(\omega)\,d\omega$

DTFT 公式

Table 15.1 Some Useful DTFT Pairs

Note: In all cases, we assume $\lvert\alpha\rvert<1$.

Entry	**Signal** $x[n]$	**The** F**-Form:** $X_p(F)$	**The** Ω**-Form:** $X_p(\Omega)$
1	$\delta[n]$	1	1
2	$\alpha^nu[n],\ \lvert\alpha<1\rvert$	$\dfrac{1}{1-\alpha e^{-j2\pi F}}$	$\dfrac{1}{1-\alpha e^{-j\Omega}}$
3	$n\alpha^nu[n],\ \lvert\alpha<1\rvert$	$\dfrac{\alpha e^{-j2\pi F}}{(1-\alpha e^{-j2\pi F})^2}$	$\dfrac{\alpha e^{-j\Omega}}{(1-\alpha e^{-j\Omega})^2}$
4	$(n+1)\alpha^nu[n],\ \lvert\alpha<1\rvert$	$\dfrac{1}{(1-\alpha e^{-j2\pi F})^2}$	$\dfrac{1}{(1-\alpha e^{-j\Omega})^2}$
5	$\alpha^{\lvert n\rvert},\ \lvert\alpha<1$	$\dfrac{1-\alpha^2}{1-2\alpha\cos(2\pi F)+\alpha^2}$	$\dfrac{1-\alpha^2}{1-2\alpha\cos\Omega+\alpha^2}$
6	1	$\delta(F)$	$2\pi\delta(\Omega)$
7	$\cos(2n\pi F_0)=\cos(n\Omega_0)$	$0.5[\delta(F+F_0)+\delta(F-F_0)]$	$\pi[\delta(\Omega+\Omega_0)+\delta(\Omega-\Omega_0)]$
8	$\sin(2n\pi F_0)=\sin(n\Omega_0)$	$j0.5[\delta(F+F_0)-\delta(F-F_0)]$	$j\pi[\delta(\Omega+\Omega_0)-\delta(\Omega-\Omega_0)]$
9	$2F_C\,\mathrm{sinc}(2nF_C)=\dfrac{\sin(n\Omega_C)}{n\pi}$	$\mathrm{rect}\left(\dfrac{F}{2F_C}\right)$	$\mathrm{rect}\left(\dfrac{\Omega}{2\Omega_C}\right)$
10	$u[n]$	$0.5\delta(F)+\dfrac{1}{1-e^{-j2\pi F}}$	$\pi\delta(\Omega)+\dfrac{1}{1-e^{-j\Omega}}$

DTFT 性质

Table 15.2 Properties of the DTFT

Property	**DT Signal**	**Result (**F**-Form)**	**Result (**Ω**-Form)**
Folding	$x[-n]$	$X_p(-F)=X_p^*(F)$	$X_p(-\Omega)=X_p^*(\Omega)$
Time shift	$x[n-m]$	$e^{-j2\pi mF}X_p(F)$	$e^{-j\Omega m}X_p(\Omega)$
Frequency shift	$e^{j2\pi nF_0}x[n]$	$X_p(F-F_0)$	$X_p(\Omega-\Omega_0)$
Half-period shift	$(-1)^nx[n]$	$X_p(F-0.5)$	$X_p(\Omega-\pi)$
Modulation	$\cos(2\pi nF_0)x[n]$	$0.5[X_p(F+F_0)+X_p(F-F_0)]$	$0.5[(\Omega+\Omega_0)+X_p(\Omega-\Omega_0)]$
Convolution	$x[n]\star y[n]$	$X_p(F)Y_p(F)$	$X_p(\Omega)Y_p(\Omega)$
Product	$x[n]y[n]$	$X_p(F)\circledast Y_p(F)$	$\frac{1}{2\pi}[X_p(\Omega)\circledast Y_p(\Omega)]$
Times-n	$nx[n]$	$\frac{j}{2\pi}\frac{dX_p(F)}{dF}$	$j\frac{dX_p(\Omega)}{d\Omega}$
Parseval's relation	$\sum_{k=-\infty}^{\infty}x^2[k]=\int_1\lvert X_p(F)\rvert^2\,dF=\frac{1}{2\pi}\int_{2\pi}\lvert X_p(\Omega)\rvert^2\,d\Omega$		
Central ordinates	$x[0]=\int_1X_p(F)\,dF=\frac{1}{2\pi}\int_{2\pi}X_p(\Omega)\,d\Omega \qquad X_p(0)=\sum_{n=-\infty}^{\infty}x[n]$		
	$X_p(F)\Big\vert_{F=0.5}=X_p(\Omega)\Big\vert_{\Omega=\pi}=\sum_{n=-\infty}^{\infty}(-1)^nx[n]$		

DFT 性质

Table 16.1 Properties of the N-Sample DFT

Property	**Signal**	**DFT**	**Remarks**
Shift	$x[n-n_0]$	$X_{\mathrm{DFT}}[k]e^{-j2\pi kn_0/N}$	No change in magnitude
Shift	$x[n-0.5N]$	$(-1)^kX_{\mathrm{DFT}}[k]$	Half-period shift for even N
Modulation	$x[n]e^{j2\pi nk_0/N}$	$X_{\mathrm{DFT}}[k-k_0]$	
Modulation	$(-1)^nx[n]$	$X_{\mathrm{DFT}}[k-0.5N]$	Half-period shift for even N
Folding	$x[-n]$	$X_{\mathrm{DFT}}[-k]$	This is *circular* folding.
Product	$x[n]y[n]$	$\frac{1}{N}X_{\mathrm{DFT}}[k]\circledast Y_{\mathrm{DFT}}[k]$	The convolution is *periodic*.
Convolution	$x[n]\circledast y[n]$	$X_{\mathrm{DFT}}[k]Y_{\mathrm{DFT}}[k]$	The convolution is *periodic*.
Correlation	$x[n]\circledast\circledast y[n]$	$X_{\mathrm{DFT}}[k]Y_{\mathrm{DFT}}^*[k]$	The correlation is *periodic*.
Central ordinates	$x[0]=\frac{1}{N}\sum_{k=0}^{N-1}X_{\mathrm{DFT}}[k] \qquad X_{\mathrm{DFT}}[0]=\sum_{n=0}^{N-1}x[n]$		
Central ordinates	$x[\frac{N}{2}]=\frac{1}{N}\sum_{k=0}^{N-1}(-1)^kX_{\mathrm{DFT}}[k]\ (N\text{ even}) \qquad X_{\mathrm{DFT}}[\frac{N}{2}]=\sum_{n=0}^{N-1}(-1)^nx[n]\ (N\text{ even})$		
Parseval's relation	$\frac{1}{N}\sum_{n=0}^{N-1}\lvert x[n]\rvert^2=\sum_{k=0}^{N-1}\lvert X_{\mathrm{DFT}}[k]\rvert^2$		