3.2 Uniform, Exponential, Gamma Distributions

Exponential Distribution

Consider a Poisson process with an expected number of occurrences, λ, in a given interval.

Let W be the waiting time until the first occurrence. Then W follows an **exponential distribution**.

(also the waiting time between occurrences)

Instead, if we count the number of these occurrences, X,

X ~ Pois(λ).

Exponential Distribution: $X \sim Exp(\theta)$

$$\boxed{f(x) = \frac{1}{\theta} e^{-x/\theta} \quad , \quad 0 \leq x < \infty}$$

$$E[X] = \theta$$

$$Var[X] = \theta^2$$

Alternatively, $f(x) = \lambda e^{-x\lambda} \quad , \quad 0 \leq x < \infty$

Finding E[X] and σ^2 for an exponential

$$M(t) = \int_0^\infty e^{tx}\left(\frac{1}{\theta}\right)e^{-x/\theta}\,dx = \lim_{b\to\infty}\int_0^b \left(\frac{1}{\theta}\right)e^{-(1-\theta t)x/\theta}\,dx$$

$$= \lim_{b\to\infty}\left[-\frac{e^{-(1-\theta t)x/\theta}}{1-\theta t}\right]_0^b = \frac{1}{1-\theta t}, \qquad t < \frac{1}{\theta}.$$

$$M'(t) = \frac{\theta}{(1-\theta t)^2} \qquad M''(t) = \frac{2\theta^2}{(1-\theta t)^3}.$$

$E[X] = M'(0) = \theta$

$E[X^2] = M''(0) = 2\theta^2$

$Var[X] = \theta^2$

Memoryless Property

The **exponential** and **geometric** distributions are **memoryless**.

Exponential Distribution

- Radioactive decay
- How long a fly will stay on a table until it takes off?

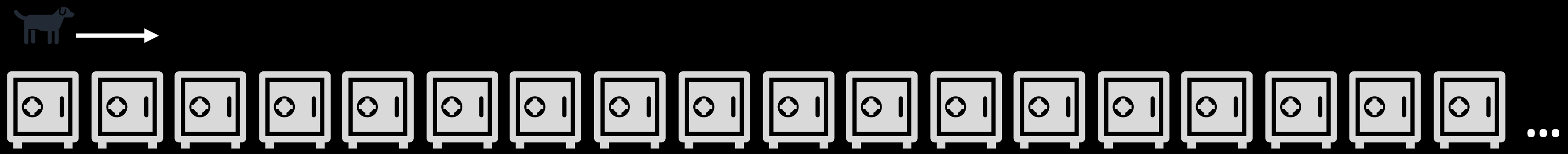
Geometric Distribution

- How many more times do I need to roll a die until my first success

Memoryless Property Example

Chloe walks down an infinite hallway of safes.

- Each safe has a different code (1000 possibilities).
  - $P[correct\ code] = 1/1000$
- Chloe only tries one code per safe
- Let X be the number of safes Chloe **still** needs to try before she successfully opens one. $E[X] = 1000$



Memoryless Property

Discrete Memorylessness (Geometric):

if X is the **total** number of trials required for the first success,

$$P[X > m + n \mid X > m] = P[X > n]$$

Continuous Memorylessness (Exponential):

if X is the **total** time required for the first success,

$$P[X > t + s \mid X > t] = P[X > s]$$

Note: Most random variables/phenomena are **<u>not</u>** memoryless.
We generally obtain and update information over time.
e.g. (**<u>not</u>** memoryless):

- A car engine's remaining life (how many miles it has left).
- The amount of time left until class ends.

Notes: memoryless property and Bayes Rule:

$$P[X > m + n \mid X > m] = P[X > n]$$

Suppose $X \sim Exp(\theta = 4)$

What is P[X > 9]?

What is P[X > 9 | X > 5]?

Example 1 (textbook)

Example 3.2-2 Customers arrive in a certain shop according to an approximate Poisson process at a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 5 minutes for the arrival of the first customer? Let X denote the waiting time in minutes until the first customer arrives, and note that $\lambda = 1/3$ is the expected number of arrivals per minute. Thus,

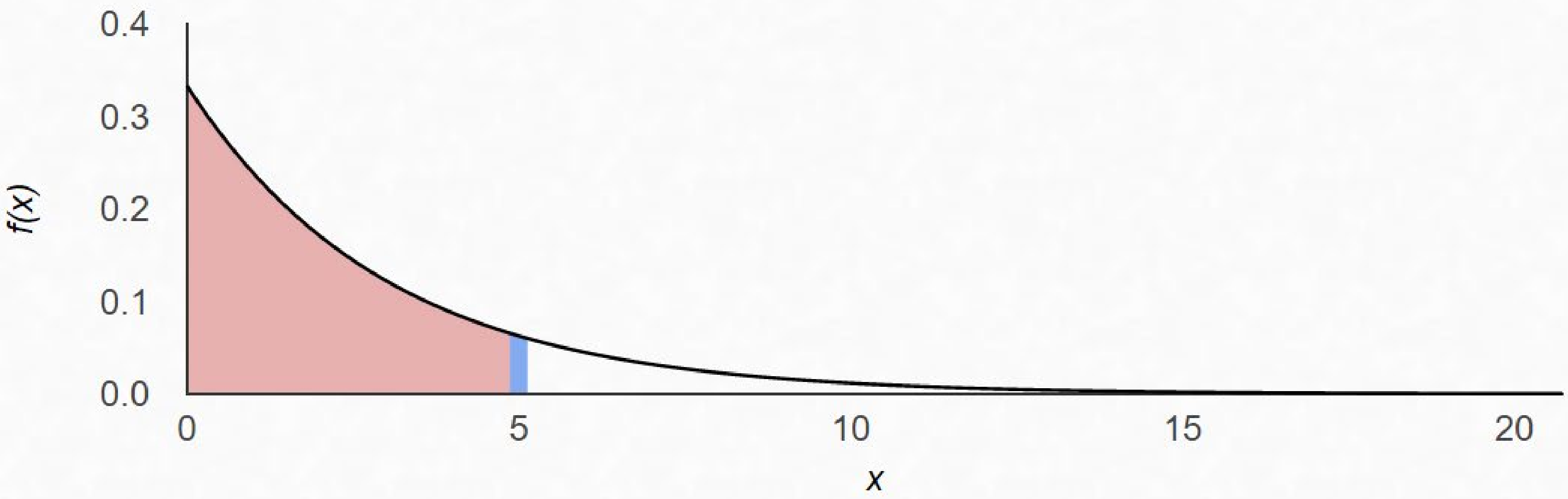
$$\theta = \frac{1}{\lambda} = 3$$

$$f(x) = \frac{1}{3}e^{-(1/3)x}, \qquad 0 \le x < \infty.$$

$$P(X > 5) = \int_5^\infty \frac{1}{3}e^{-(1/3)x}\,dx = e^{-5/3} = 0.1889.$$

Rate $\lambda =$ 1/3

$x =$ 5 P(X < x) = 0.81112



Example 2

Suppose the length of time, X, between occurrences follows an exponential distribution with mean $= 5$ sec.

- A) Give the pdf and support (sample space) of X.

- B) What is the probability that it will take more than 10 seconds for the first occurrence (to happen)? (0.135)

Example 3

Suppose an electronic component has a lifespan which can be modeled as an exponential distribution, with mean = 500 hours.

A) Find the pdf and cdf of this distribution

B) Find $P[X > x]$ ($e^{-x/500}$)

C) If this component has already lasted 200 hours, find the probability that it will last at least 600 hours total. ($e^{-4/5}$)

Example 4 (Exponential Distribution Derivation)

Given a Poisson Process with rate λ:

Let X denote the # of occurrences in a time of length w.

- What is distribution of X? $X \sim Poisson(\quad)$.
- What is $P[X = 0]$? (no occurrences in an interval of length w)

Let W represent the *waiting time* until the 1st occurrence in this Poisson Process.

$$
\begin{aligned}
F(w) = P(W \le w) &= 1 - P(W > w) \\
&= 1 - P(\text{no occurrences in } [0, w]\,) \\
&= 1 - e^{-\lambda w},
\end{aligned}
$$

$$F'(w) = f(w) = \lambda e^{-\lambda w}$$

The Gamma Distribution

Consider a Poisson process with rate λ:

Let a random variable, X, denote the waiting time until the αth occurrence.

X follows a Gamma Distribution.

The Gamma Function, Γ

$$\Gamma(t) = \int_0^\infty y^{t-1} e^{-y}\, dy, \qquad 0 < t.$$

← This is the definition of the gamma function

$$\Gamma(t) = \left[-y^{t-1}e^{-y}\right]_0^\infty + \int_0^\infty (t-1)y^{t-2}e^{-y}\, dy$$
$$= (t-1)\int_0^\infty y^{t-2}e^{-y}\, dy \;=\; (t-1)\Gamma(t-1).$$

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\cdots(2)(1)\Gamma(1).$$

$$\Gamma(1) = \int_0^\infty e^{-y}\, dy = 1.$$

When n is an integer $\Gamma(n) = (n-1)!$

Gamma Distribution
X~Gamma(α, θ)

$$f(x) = \frac{1}{\Gamma(\alpha)\theta^{\alpha}} x^{\alpha-1} e^{-x/\theta}, \qquad 0 \leq x < \infty$$

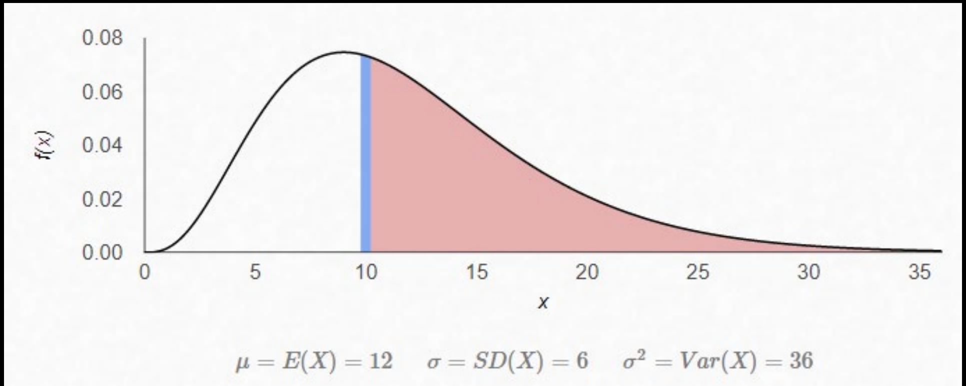
$$E[X] = \alpha\theta$$

$$Var[X] = \alpha\theta^2$$

Gamma Example

Customers arrive in a shop according to a Poisson process with a mean rate of 20 per hour. What is the probability that the shopkeeper will have to wait more than 10 minutes for the arrival of the 4th customer?

$$\int_{10}^{\infty} \frac{1}{\Gamma(4)3^4} x^{4-1} e^{-x/3}\, dx = 0.57$$



What is the probability that the shopkeeper will have to wait between than 5 and 20 minutes for the arrival of the 3rd customer?

Uniform Distribution
$X \sim Unif(a, b)$

A random variable, X, has a **uniform distribution** if its pdf is equal to a constant on its support.

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b.$$

$$E[X] = \frac{a+b}{2}$$

$$Var[X] = \frac{(b-a)^2}{12}$$

$$f(x) = \frac{1}{b-a}, \qquad a \le x \le b$$

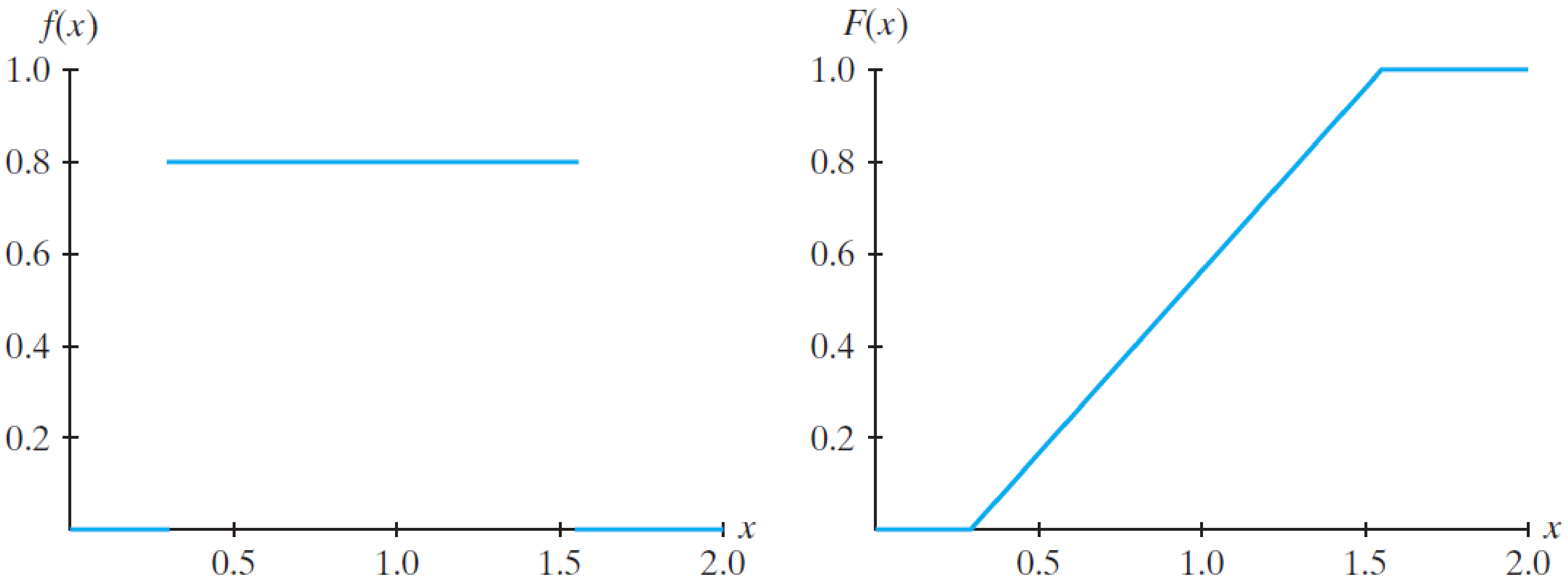


Figure 3.1-1 Uniform pdf and cdf

Uniform Example

Suppose that you arrive at the intersection of Green and Wright, it takes between 0 and 30 seconds for the walk sign to come on. Assume that the time it takes, X, follows a uniform distribution.

What is the probability that it takes between 10 and 20 seconds for the walk sign to come on? $(1/3)$

What are $E[X]$ and $Var[X]$? (15), $(\frac{30^2}{12})$

Transformation Theorem

Theorem Let $U \sim Uniform(0,1)$ and F be a CDF which is strictly increasing. Also, consider a random variable X defined as

$$X = F^{-1}(U).$$

Then,

$$X \sim F \quad (\text{The CDF of} \quad X \quad \text{is} \quad F)$$

Proof:

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(U) \leq x) \\ &= P(U \leq F(x)) \quad (\text{increasing function}) \\ &= F(x) \end{aligned}$$

Transformation Theorem Example 1

Goal: Generate a sample from an exponential Random Variable with parameter $\theta = 1$

Theorem Let $U \sim Uniform(0,1)$ and F be a CDF which is strictly increasing. Also, consider a random variable X defined as

$$X = F^{-1}(U).$$

Then,

$$X \sim F \quad (\text{The CDF of} \quad X \quad \text{is} \quad F)$$

Proof:

$$\begin{aligned} P(X \leq x) &= P(F^{-1}(U) \leq x) \\ &= P(U \leq F(x)) \quad (\text{increasing function}) \\ &= F(x) \end{aligned}$$

$$F(x) = 1 - e^{-x} \qquad x > 0$$

$$U \sim Uniform(0,1)$$

$$\begin{aligned} X &= F^{-1}(U) \\ &= -\ln(1-U) \end{aligned}$$

$$X \sim F$$

U = runif(1; min = 0; max = 1);

X = -log(1-U) OR X = -log(U)

Notes: Transformation Example 2

Use the inverse transform method to simulate n random samples from $f(x) = 3x^2$, $0 < x < 1$.

$F(x) = x^3$, $0 < x < 1$

$F^{-1}(u) = u^{1/3}$

```
n = 10000  #make up a number
u = runif(n)
my_sim = u^(1/3)
windows();hist(my_sim)
x = seq(from=0, to=1, by = 0.01)
lines(x, 3x^2)
```