Homework 8

1. A system is described by the following transfer function

$$G(s) = \frac{X(s)}{U(s)} = \frac{1}{2s+1}.$$

(a) Find an ordinary differential equation that represents this system. (Assume zero initial conditions)

(b) Find a state-space representation for this system.

(c) Find the frequency response function.

(d) A sinusoidal forcing function of the form $u(t) = F_0 \sin(\omega t)$ is applied to this system. Find the steady-state response of the system.

a).

$$\because G(s) = \frac{1}{2s+1}$$

$$\therefore 2\dot{x}(t) + x(t) = u(t)$$

b).

let $z = x_1$, $where\ x_1 = x$

$$\Rightarrow \dot{z} = \frac{u(t) - x_1}{2} = -\frac{1}{2}z + \frac{1}{2}u$$

c).

$$G(j\omega) = \frac{1}{2j\omega + 1}$$

$$\Rightarrow M(\omega) = \frac{1}{\sqrt{1+4\omega^2}}$$

$$\Rightarrow \phi(\omega) = 0 - \tan^{-1}\frac{2\omega}{1}$$

$$\Rightarrow x_{ss}(t) = \frac{U_0}{\sqrt{1+4\omega^2}}\sin(\omega t + \phi_0 - \tan^{-1} 2\omega)$$

d).

$$2\dot{x}(t) + x(t) = F_0\, sin\, \omega t$$

$$G(j\omega) = \frac{1}{1+j2\omega} \Rightarrow \begin{cases} \Rightarrow M(\omega) = \dfrac{1}{\sqrt{1+4\omega^2}} \\ \Rightarrow \phi(\omega) = 0 - tan^{-1} 2\omega \end{cases}$$

$$\Rightarrow x_{ss} = \frac{F_0}{\sqrt{1+4\omega^2}} \cdot sin(\omega t - tan^{-1} 2\omega)$$

2. Given a system with the following state-space representation:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u$$

(a) Determine the transfer function that represents this system. (Assume zero initial conditions and x_1 is the output)

(b) Determine the frequency response for this system. What are the mathematical expressions for $|G(j\omega)|$ and $\angle G(j\omega)$?

(c) What is the magnitude and phase of the steady state response when the input is $\sin(2t)$?

a).

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u \Rightarrow \ddot{x}(t) + \dot{x}(t) = u(t) \Rightarrow G(s) = \frac{1}{s^2 + s}$$

b).

$$G(j\omega) = \frac{1}{-\omega^2 + j\omega}$$

$$\Rightarrow M(\omega) = |G(j\omega)| = \frac{1}{\sqrt{\omega^4 + \omega^2}}$$

$$\Rightarrow \phi(\omega) = \angle G(j\omega) = 0 - tan^{-1}\frac{\omega}{-\omega^2} = tan^{-1}\frac{1}{\omega}$$

$$\Rightarrow frequency\ response = \frac{U_0}{\sqrt{\omega^4 + \omega^2}} \cdot sin\left(\omega t + tan^{-1}\frac{1}{\omega}\right)$$

c).

$$x_{ss}(t) = \frac{1}{\sqrt{2^4 + 2^2}} \cdot sin\left(2t + tan^{-1}\frac{1}{2}\right) = \frac{1}{2\sqrt{5}} sin\left(2t + tan^{-1}\frac{1}{2}\right)$$

$$maganitude = \frac{1}{2\sqrt{5}}$$

$$phase = tan^{-1}\frac{1}{2}$$

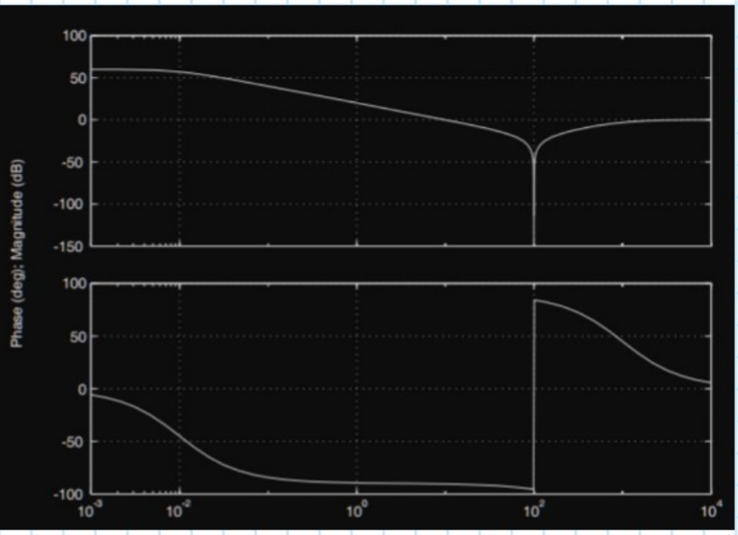
3. The Bode plot for the system with transfer function

$$\frac{X(s)}{G(s)} = \frac{s^2 + 0.002s + 10000}{(s + 0.01)(s + 1000)}$$

is shown in the adjoining figure:

Estimate the steady-state output $x_{ss}(t)$ corresponding to the input

$$f(t) = 0.0001 \sin 0.0001t + \sin 100t + \sin 10000t.$$



100

$$\frac{X(s)}{G(s)} = \frac{s^2 + 0.002s + 10000}{(s + 0.01)(s + 1000)} = U(s) = \mathcal{L}(f(t))$$

$$f(t) = 0.0001\, sin\, 0.0001t + sin\, 100t + sin\, 10000t$$

from the graph we see

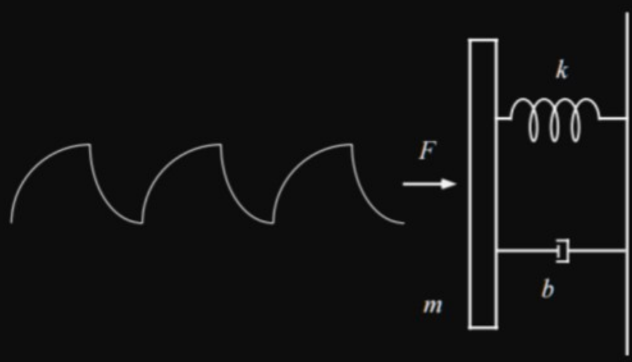
$$at\ \omega\ = 0.0001, 20\, lg\, M(\omega) = 55 \Rightarrow M(\omega) = 10^{2.75}, \phi(\omega) = -5°$$

$$at\ \omega\ = 100, 20\, lg\, M(\omega) = -150 \Rightarrow M(\omega) = 10^{-7.5}, \phi(\omega) = 80°$$

$$at\ \omega\ = 10000, 20\, lg\, M(\omega) = 0 \Rightarrow M(\omega) = 1, \phi(\omega) = 5°$$

$$\therefore x_{ss}(t) = 10^{2.75} \cdot sin(10^{-4}t - 5°) + 10^{-7.5} \cdot sin(100t + 80°) + sin(10^4t + 5°)$$

4. A simple device for harnessing energy from the North Sea, graveyard of the damned, is depicted in the adjoining figure. Waves force the plate periodically; this forcing may be approximated by a sinusoid with period 2π seconds. The motion of the plate generates electricity to help light the city of Copenhagen.



(a) Assuming the system to be underdamped, derive an expression for its resonant frequency in terms of $m, b,$ and k.

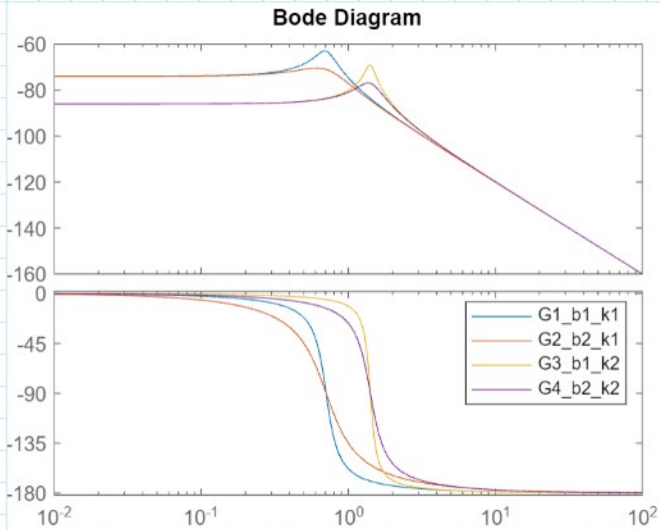
(b) Let $m = 10000$ kg. You, the engineer, have the freedom to choose both a value for b between 2000 kg/sec and 5000 kg/sec and a value for k between 5000 kg/sec^2 and 20000 kg/sec^2 . Find the pair of values which will maximize the amplitude of the forced oscillation of the plate. Does this choice correspond to tuning the system's resonant frequency to the forcing frequency? Feel free to use MATLAB's bode command to explore the frequency response of the system for different values of b and k; print out and turn in any plots you generate to support your final answer.

a).

$$m\ddot{x} + b\dot{x} + kx = u(t)$$

$$\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = \frac{1}{m}u(t)$$

$$\omega_r = \omega_n \cdot \sqrt{1 - 2\xi^2} = \sqrt{\frac{k}{m}} \cdot \sqrt{1 - \frac{b^2}{2mk}}$$



b).

$$m = 10000\ kg$$
$$b_1 = 2000\ kg/s^2$$
$$b_2 = 5000\ kg/s^2$$
$$k_1 = 5000\ kg/s^2$$
$$k_2 = 20000\ kg/s^2$$

$$M(\omega) = \frac{1}{m} \cdot \frac{1}{\sqrt{(\omega_n^2 - \omega^2)^2 + (2\xi\omega_n\omega)^2}} = \frac{1}{m} \cdot \frac{1}{\sqrt{[(\omega_n - 1)^2]^2 + 4\xi^2\omega_n^2}}$$

$\because$ *Time period is* 2π

$$\therefore \omega = 1 \Rightarrow M(\omega) = \frac{1}{m} \cdot \frac{1}{\sqrt{(\omega_n - 1)^4 + 4\xi^2\omega_n^2}} = \frac{1}{\sqrt{\left(\sqrt{\frac{k}{m}} - 1\right)^4 + \frac{b^2}{m^2}}}$$

$\therefore$ *The minimum* $b_1 = 2000, k_1 = 5000$ *will produce the largest amplitude*

5.

$$u_s(t) = 1 = \sin\left(0t + \frac{\pi}{2}\right) \Rightarrow \omega = 0$$

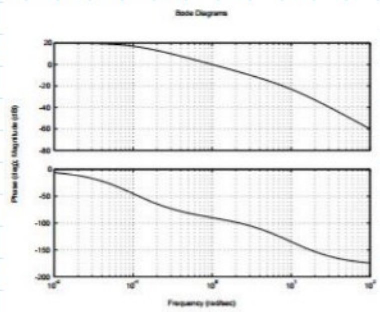
(Ⅰ)

$$20\lg M(\omega = 0) = 20 \Rightarrow M(\omega = 0) = 10$$

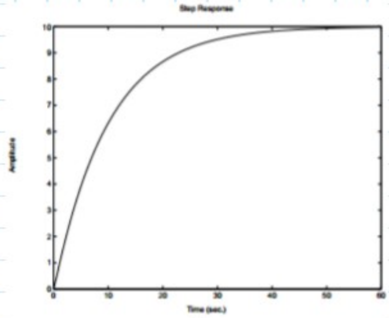
$$\phi(\omega = 0) = 0$$

$$x_{ss}(t) = 10 \cdot \sin\left(\frac{\pi}{2} + 0\right) = 10$$

$\therefore$ Ⅰ matches (e)



(I)



(e)

(Ⅱ)

$$20\lg M(\omega) = -40\lg\omega \Rightarrow M(\omega) = \frac{1}{\omega^2}$$

$$\phi(\omega) = -\pi$$

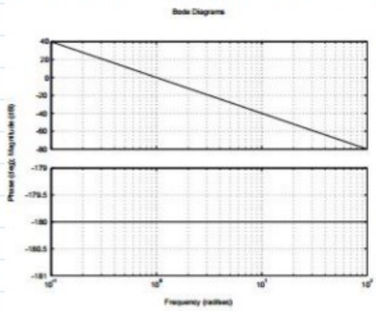
$$\Rightarrow G(j\omega) = \frac{1}{-\omega^2 + j \cdot 0} = -\frac{1}{\omega^2}$$

$$\Rightarrow G(s) = -\frac{1}{s^2} \Rightarrow x(t) = \frac{1}{2}t^2 \cdot u_s(t)$$

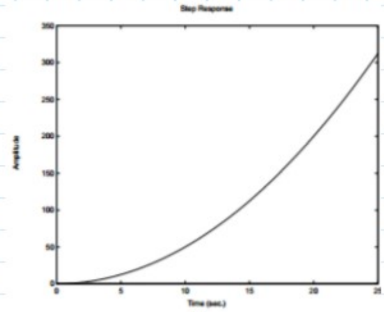
$$x_{ss}(t) = \frac{1}{\omega^2} \cdot \sin\left(\frac{\pi}{2} - \pi\right) = -\frac{1}{\omega^2}$$

$\therefore$ *For* $\omega \to 0, x_{ss}(t) \to \infty$

$\therefore$ Ⅱ *matches (c)*



(II)



(c)

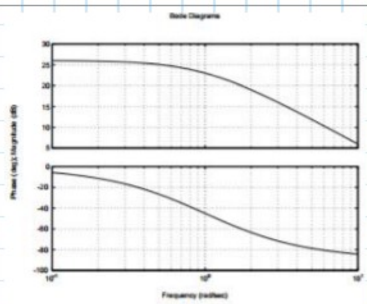
(Ⅲ)

$$20\lg M(\omega = 0) = 25 \Rightarrow M(\omega = 0) = 10^{1.25}$$

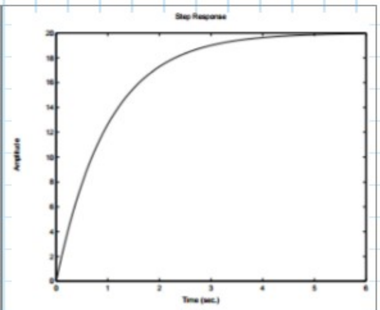
$$\phi(\omega = 0) = 0$$

$$x_{ss}(t) = 10^{1.25} \cdot \sin\frac{\pi}{2} \approx 20$$

$\therefore$ Ⅲ *matches (d)*



(III)



(d)

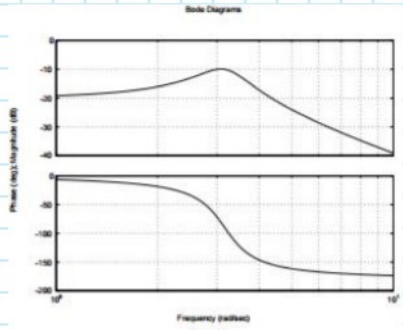
(IV)

$$20\lg M(\omega = 0) = -20 \Rightarrow M(\omega = 0) = \frac{1}{10}$$

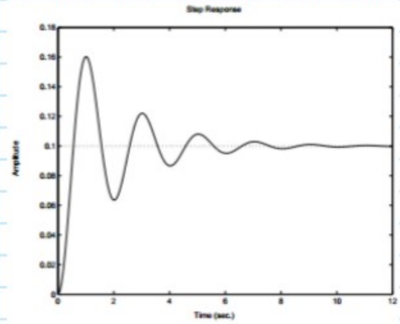
$$\phi(\omega = 0) = 0$$

$$x_{ss}(t) = \frac{1}{10}\cdot \sin\frac{\pi}{2} = \frac{1}{10}$$

$\therefore$ IV $matches\ (b)$



(IV)



(b)

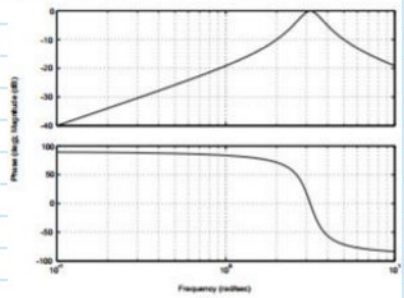
(V)

$$20\lg M(\omega = 0) = -40 \Rightarrow M(\omega = 0) = \frac{1}{100}$$

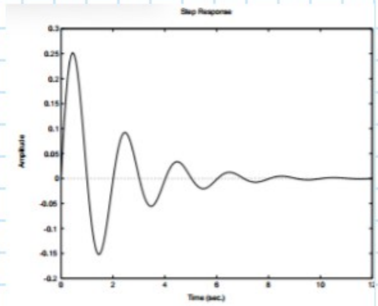
$$\phi(\omega = 0) = 100°$$

$$x_{ss}(t) = \frac{1}{100}\cdot \sin(90° - 100°) \approx 0$$

$\therefore$ V $matches\ (a)$



(V)



(a)

(VI)

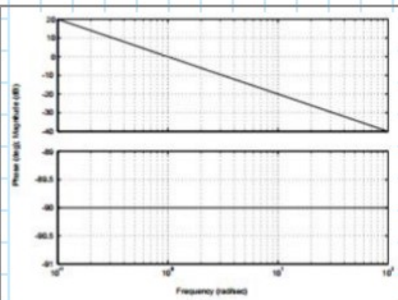
$$20\lg M(\omega) = -20\lg\omega \Rightarrow M(\omega) = \frac{1}{\omega}$$

$$\phi(\omega = 0) = -\frac{\pi}{2}$$

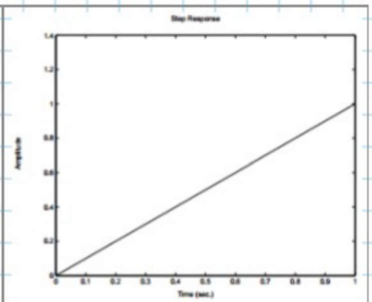
$$\Rightarrow G(j\omega) = \frac{1}{0 - j\omega} \Rightarrow G(s) = -\frac{1}{s}$$

$\Rightarrow x(t)\ is\ first\ order$

$\therefore$ VI matches (f)

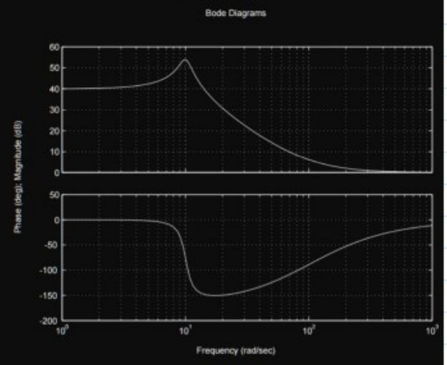


(VI)



(f)

6. The Bode plot for the transfer function $G_1(s) = \frac{s^2+200s+10000}{s^2+2s+101}$ is shown below:

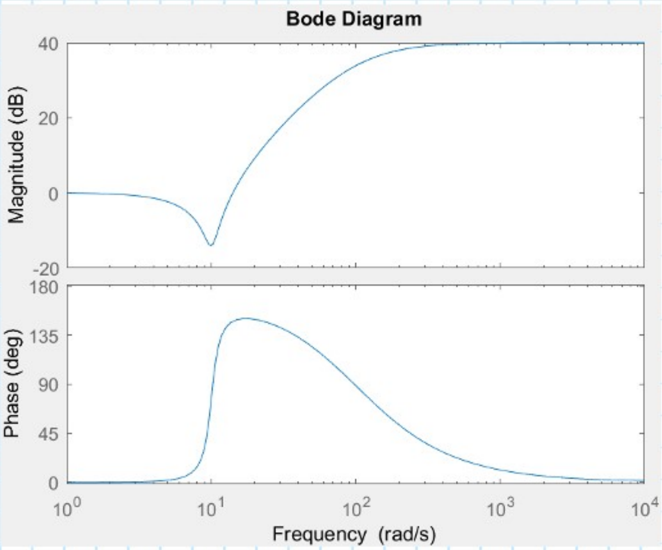


Sketch the Bode plot for the transfer function $G_2(s) = \frac{100(s^2+2s+101)}{s^2+200s+10000}$.

$$G_2(s) = (G_1(s))^{-1} \cdot 100$$

$$20\lg|G_2(j\omega)| = 40 - 20\lg|G_1(j\omega)|$$

Which means the graph for G2 is turning the graph of G1 upside down



7. Sketch the Bode plot for the transfer function

$$G(s) = \frac{X(s)}{F(s)} = \frac{s^3(s+10000)}{\left(s+\frac{1}{10}\right)^2(s+100)}.$$

$$G(j\omega) = \frac{-j\omega(j\omega+100^2)}{\left(j\omega+\frac{1}{10}\right)^2(j\omega+100)}$$

$$20\lg|G(j\omega)| = 60\lg\omega + 10\lg(\omega^2+100^2) - 20\lg\left(\omega^2+\frac{1}{100}\right) - 10\lg(\omega^2+100^2)$$

$$\phi(\omega) = -3\tan^{-1}(\omega) + \tan^{-1}\left(\frac{\omega}{10000}\right) 2\tan^{-1}(10\omega) - \tan^{-1}\left(\frac{\omega}{100}\right)$$