ME 340 Dynamics of Mechanical Systems

State-Space Representation

How to solve higher order ODEs?

- Typically ODEs can be written as first-order **vector** ODEs
  - ⋆ typically used for numerical simulations
- In general, a Nth order homogenous nonlinear system can be written as

$$\dot{x} = f(x)$$

  where the **state** x is a $N \times 1$ vector
- If the non-homogenous terms are present (i.e. when $u(t)$ in an ode are present), then the ode system can be written as

$$\dot{x} = f(x, u)$$

  where the **state** x is a $N \times 1$ vector
- In general, a Nth order homogeneous linear system can be written as

$$\dot{x} = Ax$$

  where the **state** x is a $N \times 1$ vector and A is a $N \times N$ matrix
- If the non-homogeneous terms are present (i.e. when $u(t)$ in an ode are present), then the ode system can be written as

$$\dot{x} = Ax + Bu$$

  where the **state** x is a $N \times 1$ vector

Examples

- Writing Nth order ODEs as 1st order *vector* ODEs.
  - ⋆ best learnt through examples:
    - mass spring damper ODE: $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u(t)$
      - Define $x_1 = x, x_2 = \dot{x}$
      - This implies

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -2\zeta\omega_n x_2 - \omega_n^2 x_1 + u(t)\end{aligned}$$

$$\Leftrightarrow \begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix} = \begin{pmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix} + \begin{pmatrix}0\\ 1\end{pmatrix}u(t)$$

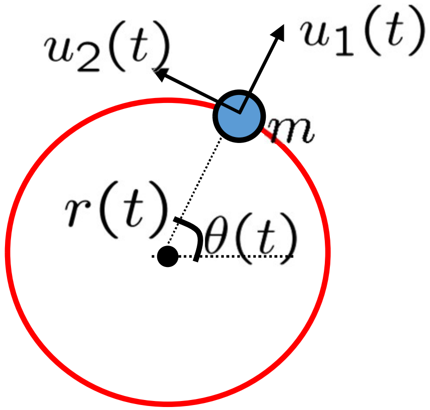
      - note the vector equation has the form: $\dot{\mathbf{x}} = A\mathbf{x} + bu$

    - pendulum example ODE: $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$
      - Define $\theta_1 = \theta, \theta_2 = \dot{\theta}$
      - This implies

$$\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix} = \begin{pmatrix}\theta_2\\ -\frac{g}{l}\sin\theta_1\end{pmatrix}$$

      - note the vector equation has the form: $\dot{\mathbf{\Theta}} = f(\mathbf{\Theta})$
  - ⋆ This vector space is called state space or phase space

Example: satellite around earth



- Using Lagrange dynamics method to obtain the model

$$\mathbf{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i}$$

  ⋆ 2 DOF:$q_1 =$ radius r and $q_2 =$ angle θ,

  – $p = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} \Rightarrow \dot{p} = \begin{pmatrix} -r\dot{\theta}\sin\phi + \dot{r}\cos\theta \\ r\dot{\theta}\cos\theta + \dot{r}\sin\theta \end{pmatrix}$

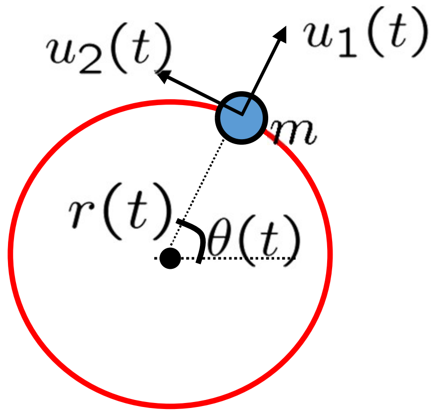
  ⋆ $V = -\beta/r$, $T = \frac{1}{2}m|\dot{p}|^2 = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$, Virtual Work: $\delta W_{nc} = u_1\delta r + u_2 r\delta\theta$

  ⋆ $L = T - V = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) + \beta/r$, $F_r = \frac{\delta W_{nc}}{\delta r}\Big|_{\delta\theta=0} = u_1$, $F_\theta = \frac{\delta W_{nc}}{\delta\theta}\Big|_{\delta r=0} = u_2 r$

- Equations:

$$\begin{aligned} \ddot{r} &= r\dot{\theta}^2 - \beta/r^2 + u_1 \\ \ddot{\theta} &= -2\frac{\dot{r}\dot{\theta}}{r} + \frac{u_2}{r} \end{aligned}$$

Example: satellite around earth



- Define $x_1 = r$, $x_2 = \dot{r}$, $x_3 = \theta$, and $x_4 = \dot{\theta}$
- this implies $\dot{x}_1 = x_2$, $\dot{x}_2 = x_1 x_4^2 - \beta / x_1^2 + u_1$, $\dot{x}_3 = x_4$, and $\dot{x}_4 = -(2x_2x_4)/x_1 + u_2/x_1$,
- That is the state space model is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 x_4^2 - \beta/x_1^2 + u_1 \\ x_4 \\ -(2x_2x_4)/x_1 + u_2/x_1 \end{bmatrix}$$

state-space representation

Examples

- $$(1+2x^2)\ddot{x} + x^2 y + y^2 = u$$
  $$\dot{y} + y^3 x = u$$
  - ⋆ Define $x_1 = x, x_2 = \dot{x}, x_3 = y$
  - ⋆ This implies

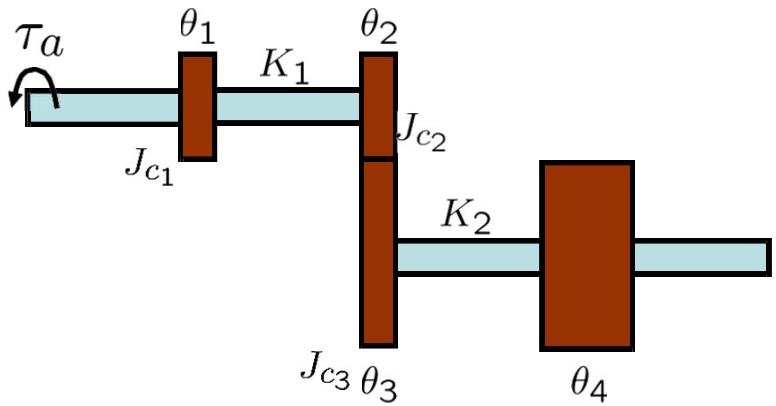
$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= \frac{-x_1^2 x_3 - x_3^2 + u(t)}{1+2x_1^2} \\
\dot{x}_3 &= -x_3^3 x_1 + u
\end{aligned}$$

That is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ \frac{-x_1^2 x_3 - x_3^2 + u(t)}{1+2x_1^2} \\ -x_3^3 x_1 + u \end{bmatrix}$$

- note the vector equation has the form: $\dot{\mathbf{x}} = f(\mathbf{x})$

Example: Gear Train



- From Newton's laws we can obtain ODE that describes this system
  - ⋆ Model Equations:

$$J_{c_1}\ddot{\theta}_1 + K_1\theta_1 = K_1\theta_2$$

$$\left(J_{c_2} + J_3\left(\frac{r_2}{r_3}\right)^2\right)\ddot{\theta}_2 + \left(K_1 + K_2\left(\frac{r_2}{r_3}\right)^2\right)\theta_2 = K_1\theta_1 + K_2\left(\frac{r_2}{r_3}\right)\theta_4$$

$$J_{c_4}\ddot{\theta}_4 + K_2\theta_4 = K_2\left(\frac{r_2}{r_3}\right)\theta_2$$

Example: Gear Train – State Space Representation

- Let $x_1 = \theta_1, x_2 = \dot{\theta}_1, x_3 = \theta_2, x_4 = \dot{\theta}_2, x_5 = \theta_4, x_6 = \dot{\theta}_4$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} x_2 \\ (-K_1 x_1 + K_1 x_3)/J_{c_1} \\ x_4 \\ -(\left(K_1 + K_2 \left(\frac{r_2}{r_3}\right)^2\right) x_3 + K_1 x_1 + K_2 \left(\frac{r_2}{r_3}\right) x_5)/J_{eq} \\ x_5 \\ (-K_2 x_5 + K_2 \left(\frac{r_2}{r_3}\right) x_3)/J_{c_4} \end{bmatrix}$$

$$y = r_4 x_6$$

where $J_{eq} = (\left(J_{c_2} + J_3 \left(\frac{r_2}{r_3}\right)^2\right))$. that is,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{K_1}{J_{c_1}} & 0 & \frac{K_1}{J_{c_1}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \frac{K_1}{J_{eq}} & 0 & -\frac{(\left(K_1 + K_2\left(\frac{r_2}{r_3}\right)^2\right)}{J_{eq}} & 0 & \frac{K_2\left(\frac{r_2}{r_3}\right)}{J_{eq}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{K_2\left(\frac{r_2}{r_3}\right)}{J_{c_4}} & 0 & -\frac{K_2}{J_{c_4}} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

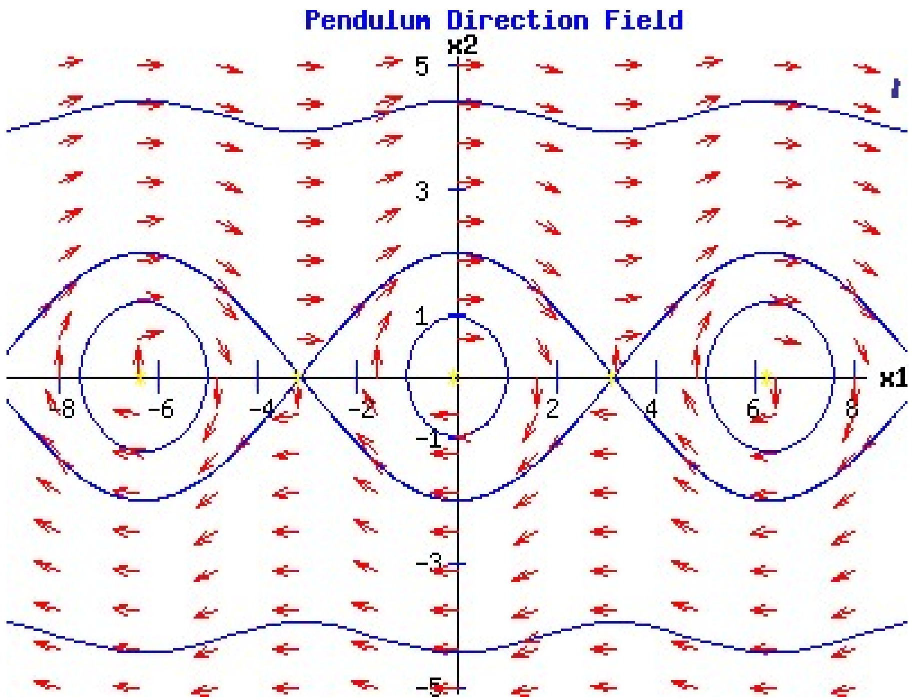
$$y = [\, 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad r_4 \,] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

Phase Portraits

- we can go to each point x and compute the velocity $\dot{x} = f(x)$ at that point and put a corresponding arrow there
- the solution is tangential to these vectors
- plots of trajectories in the state-space or phase space
- phase portait of the pendulum equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\sin x_1 \end{bmatrix}$$

  - ⋆ good visual treatment for second order differential equations
  - ⋆ for each initial condition plot $x_2(t)$ vs $x_1(t)$ for $t > 0$.



First-Order Form of ODEs

Q: what can be done with the First-order form?

- ⋆ solve it?
  This is unusual

- ⋆ simulate it?
  Yes! to predict future behavior
  – *given $z(0)$ find $z(t)$ for $t \geq 0$ numerically*

- ⋆ Analysis?
  Yes. analyze the ODE to get qualitative and quantitative info about the solutions; design trade-offs; and control the system. e.g. get an ODE that describes car suspension; analyze $x_c(t)$ w.r.t. different road conditions; study trade-offs between ride-comfort and actuation power/cost; and design active suspension f to achieve a specified ride comfort

Solving First-Order Forms

$$\frac{d}{dt}y = f(y,u) = f(y,t)$$

- Algebraic solution:

$$y(t) = y_o + \int_0^t \frac{dy}{dt}dt$$
$$= y_o + \int_0^t f(y,t)dt$$

- Numerical integration?

Euler's Method

- Simplest

Let $\mathbf{y} = y$ (look at only one variable solution)

$$\frac{dy}{dt} = \frac{y_{n+1} - y_n}{\Delta t}$$

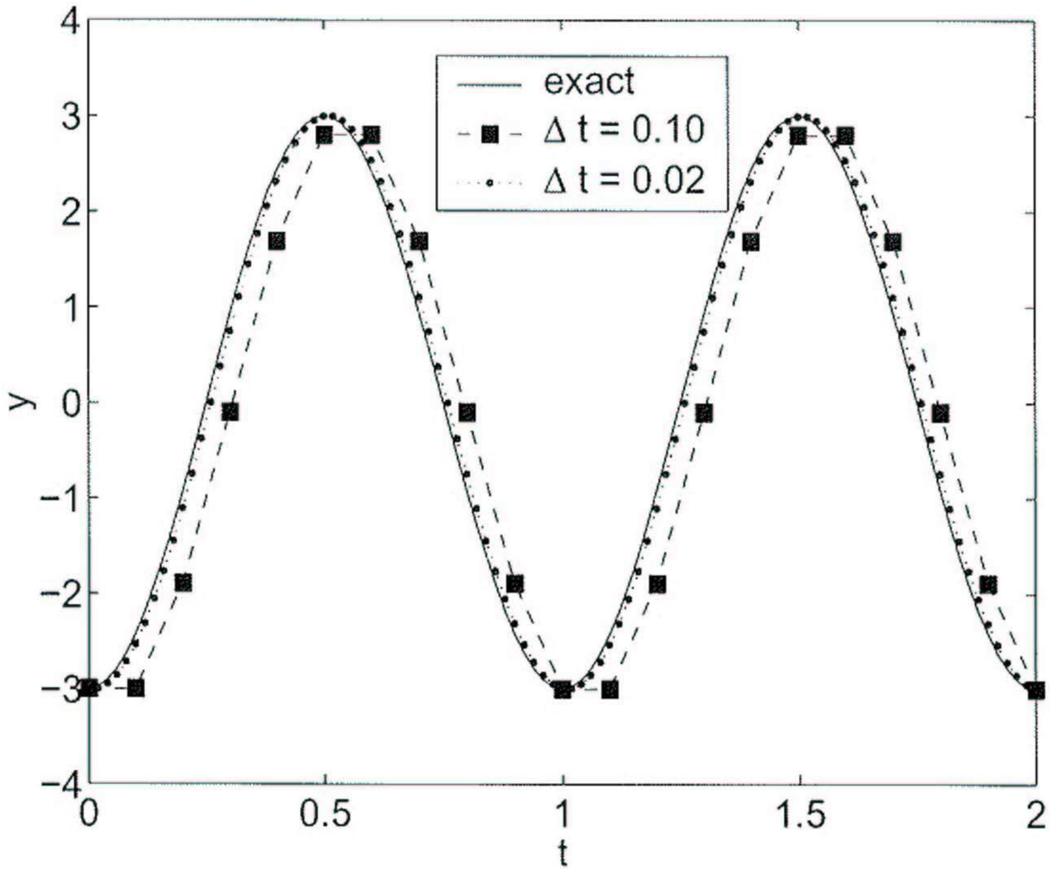
then $\dfrac{y_{n+1} - y_n}{\Delta t} = f(y_n, t_n)$

rearrange to get

$$\boxed{y_{n+1} = y_n + f(y_n, t_n)\Delta t}$$

Euler's Method equation

Euler's Method - example



Analytic equation:

$$\frac{dy}{dt} = 6\pi \sin(2\pi t)$$

Initial condition:

$$y(t = 0) = -3$$

Analytic (exact) solution:

$$y(t) = -3\cos(2\pi t)$$

Error: $O(\Delta t)$

Second-order Runga-Kutta(Midpoint) method

- Midpoint method (vs. Euler) improves accuracy because also evaluates derivative at midpoint

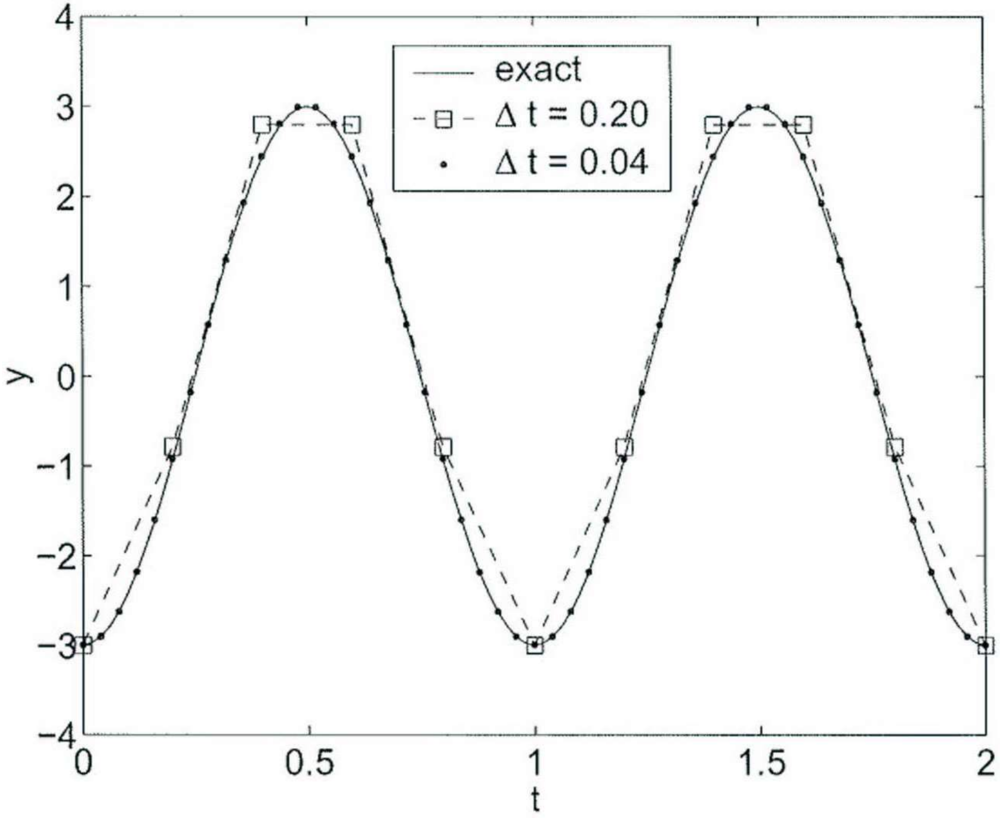
$$t = t_n + \tfrac{1}{2}\,\Delta t$$

- Have 2 equations per time step:

$$y_a = y_n + \frac{1}{2} f(y_n, t_n)\Delta t$$

$$y_{n+1} = y_n + f(y_a, t_n + \frac{\Delta t}{2})\Delta t$$

Midpoint method: example



Fourth-order Runga-Kutta method

- Most popular numerical integrator
- Balances run-time cost and accuracy
- 4 equations per time step

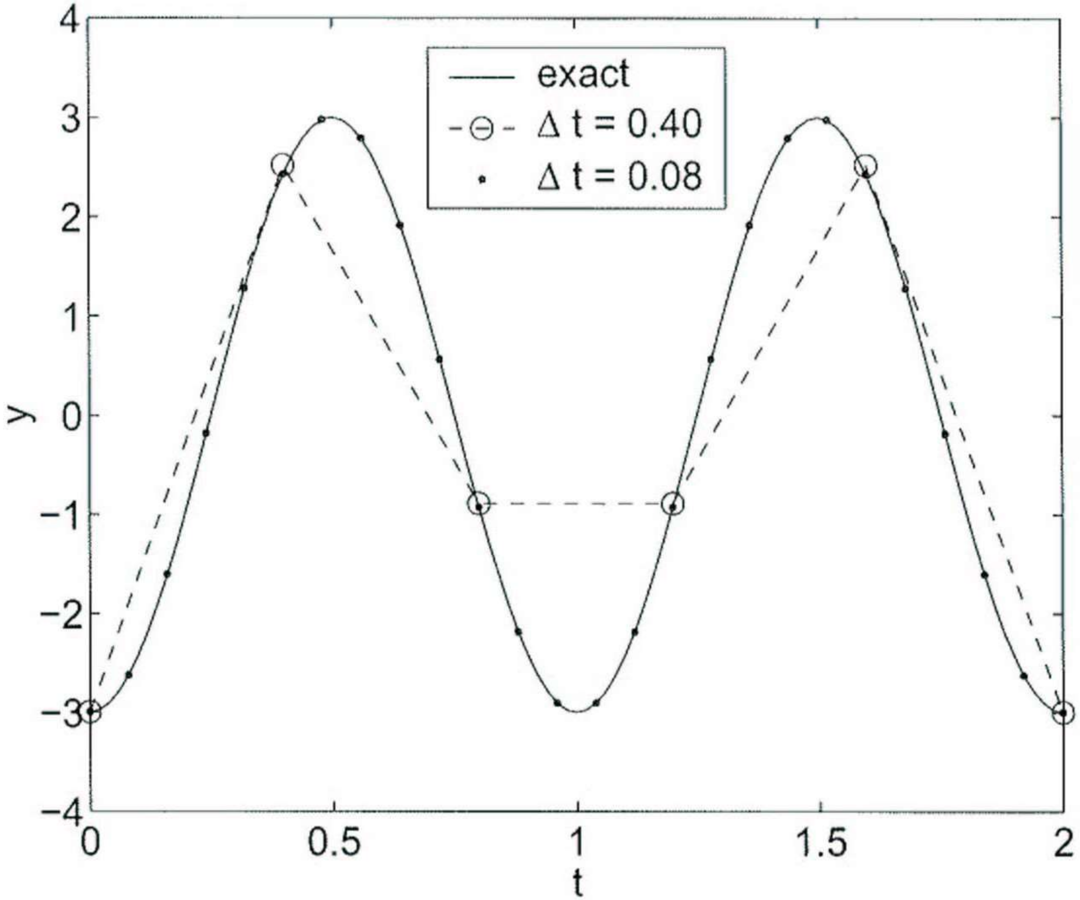
$$y_a = y_n + \frac{1}{2} f(y_n, t_n)\Delta t$$

$$y_b = y_n + \frac{1}{2} f\left(y_a, t_n + \frac{\Delta t}{2}\right)\Delta t$$

$$y_c = y_n + f\left(y_b, t_n + \frac{\Delta t}{2}\right)\Delta t$$

$$y_{n+1} = y_n + \left[\frac{1}{6} f(y_n, t_n) + \frac{1}{3} f\left(y_a, t_n + \frac{\Delta t}{2}\right) + \frac{1}{3} f\left(y_b, t_n + \frac{\Delta t}{2}\right) + \frac{1}{6} f\left(y_c, t_n + \frac{\Delta t}{2}\right)\right]\Delta t$$

Fourth-order Runga-Kutta: example



Error vs time step size

Maximum absolute error = max $|y_n - y(t_n)|$