# Homework 4

1. The free motion of a mass-spring-damper system is governed by the differential equation

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0,$$

where $x(t)$ denotes the displacement of the mass from the rest position.

(a) Suppose $m = 2$ kg, $b = 4\ \frac{\text{kg}}{\text{s}}$, and $k = 10\ \frac{\text{kg}}{\text{s}^2}$. If the mass is given a nonzero initial velocity, will its position oscillate over time? Explain your response.

(b) What if $m = 1$ kg, $b = 3\ \frac{\text{kg}}{\text{s}}$, and $k = 2\ \frac{\text{kg}}{\text{s}^2}$?

(c) What if $m = 2$ kg, $b = 4\ \frac{\text{kg}}{\text{s}}$, and $k = 2\ \frac{\text{kg}}{\text{s}^2}$?

(d) In each of these above problems, what physical units should the roots of the characteristic equation have?

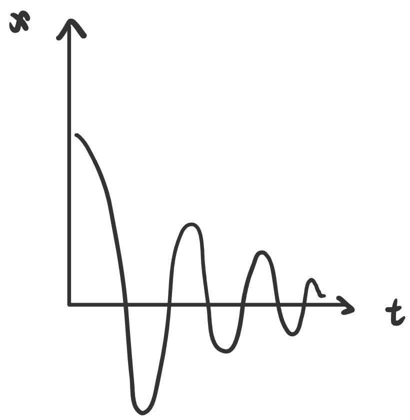
a). $\because 2\ddot{x} + 4\dot{x} + 10x = 0$

$\ddot{x} + 2\dot{x} + 5x = 0$

$\therefore \lambda^2 + 2\cdot\underbrace{\tfrac{1}{\sqrt{5}}}_{\xi}\cdot\underbrace{\sqrt{5}}_{\omega_n}\lambda + \sqrt{5}^2 = 0$

$\therefore 0 < \xi < 1$ which belongs to under-damped

$\therefore$ It will oscillate




b). $\because \ddot{x} + 3\dot{x} + 2x = 0$

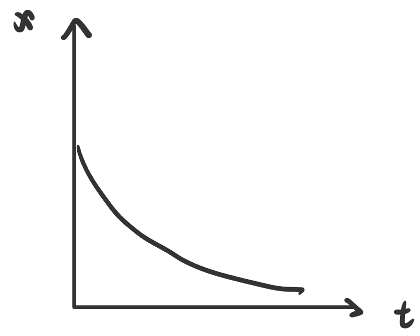
$\therefore \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2) = 0$

$\therefore \lambda_1 = -1,\ \lambda_2 = -2,\ \xi > 1$

$x_1 = e^{-t},\ x_2 = e^{-2t}$

$\therefore x = c_1 e^{-t} + c_2 e^{-2t}$

$\therefore \lim_{t\to\infty} x = 0$ and No oscillate

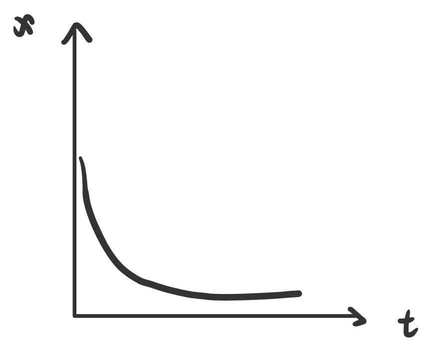



c). $\because 2\ddot{x} + 4\dot{x} + 2x = 0$

$\therefore (\lambda+1)^2 = 0,\ \xi = 1$

$\therefore$ critical damped

$x = c_1 e^{-t} + c_2 t e^{-t}$ No oscillate




d). the unit is $\boxed{\left(\frac{1}{s}\right)}$ because $x$'s unit is m

$\frac{dx}{dt} = \lambda e^{\lambda t}$ is m/s

So $\lambda$'s unit is 1/s

2. Suppose the mass in the system from the last problem is driven by an additional external force

$$F(t) = (2\text{ N})\sin\left(2\,\frac{\text{rad}}{\text{s}}\cdot t\right).\text{ where 'N' and 'rad/s' are units}$$

Find $x(t)$ if $m = 1$ kg, $b = 3\,\frac{\text{kg}}{\text{s}}$, and $k = 2\,\frac{\text{kg}}{\text{s}^2}$, $x(0) = -\frac{3}{10}$ m, and $\dot{x}(0) = \frac{4}{5}\,\frac{\text{m}}{\text{s}}$. Does the position of the mass oscillate in this case?

$\because\ m\ddot{x}(t) + b\dot{x}(t) + kx(t) = F(t)$

$\therefore\ \ddot{x} + 3\dot{x} + 2x = 2\cdot\sin(2t)$

For fundamental solution

$\lambda^2 + 3\lambda + 2 = 0$

$(\lambda+1)(\lambda+2) = 0$

$\therefore\ \lambda_1 = -1,\ \lambda_2 = -2$

$x_1 = e^{-t},\ x_2 = e^{-2t}$

For particular solution

$x_3 = C_3\cos(2t) + C_4\sin(2t)$

$\dot{x}_3 = 2C_4\cos(2t) - 2C_3\sin(2t)$

$\ddot{x}_3 = -4C_4\sin(2t) - 4C_3\cos(2t)$

$\therefore\ -4C_4\sin(2t) - 4C_3\cos(2t) + 3\left[2C_4\cos(2t) - 2C_3\sin(2t)\right] + 2\left[C_3\cos(2t) + C_4\sin(2t)\right] = 2\sin(2t)$

$\therefore\ \begin{cases} C_3 = -\frac{3}{10} \\ C_4 = -\frac{1}{10} \end{cases}$ ✓

$\therefore\ x_3 = -\frac{3}{10}\cos(2t) - \frac{1}{10}\sin(2t)$

$\therefore\ x = C_1e^{-t} + C_2e^{-2t} - \frac{3}{10}\cos(2t) - \frac{1}{10}\sin(2t)$

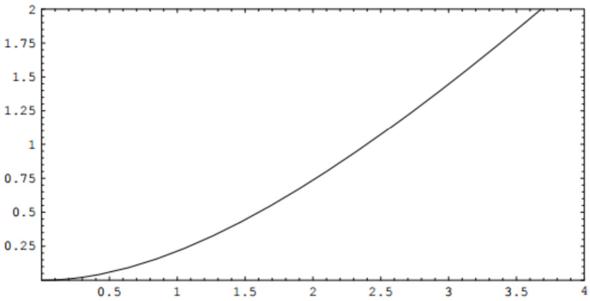
$\therefore\ \boxed{x = e^{-t} - e^{-2t} - \frac{3}{10}\cos(2t) - \frac{1}{10}\sin(2t)}$

$\therefore\ \lim_{t\to\infty} x \neq 0$ $\quad\therefore$ $\boxed{\text{It does oscillate}}$

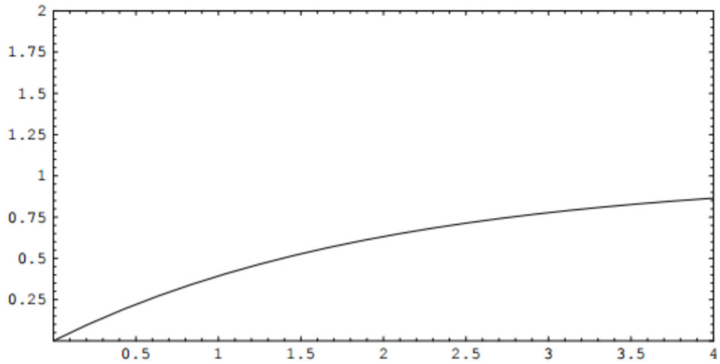
3. The first order system

$$\tau \dot{x} + x = u(t)$$

is given the ramp input $u(t) = t$; the resulting output $x(t)$ looks like…




… and the derivative $\dot{x}(t)$ of this output looks like…




Find the value of the time constant $\tau$ to the nearest integer.

$\because \tau \dot{x} + x = t$

For fundamental solution

$\because \tau \dot{x} + x = 0$

$\tau \lambda = -1$

$\lambda = -\frac{1}{\tau}$

$\therefore \quad x_1 = e^{-\frac{1}{\tau} \cdot t}$

For particular solution

$x_p = a_0 + a_1 t$

$\therefore \tau a_1 + a_0 + a_1 t = t$

$\therefore \begin{cases} a_0 = -\tau \\ a_1 = 1 \end{cases} \Rightarrow x_p = -\tau + t$

$\therefore x = c_1 e^{-\frac{1}{\tau} t} - \tau + t$

$\dot{x} = -\frac{c_1}{\tau} e^{-\frac{1}{\tau} t} + 1$

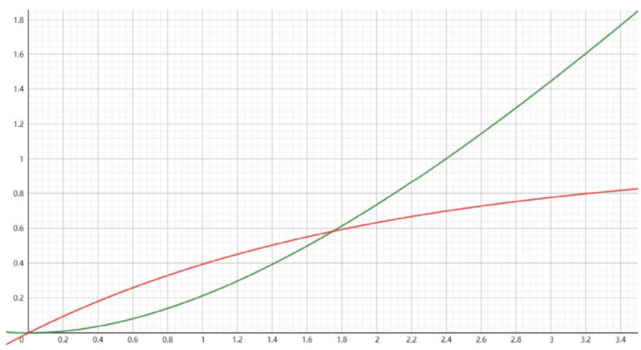
From the graph we know $x(0) = \dot{x}(0) = 0$

$\therefore \begin{cases} -\frac{c_1}{\tau} + 1 = 0 \\ c_1 - \tau = 0 \end{cases}$

$\therefore c_1 = \tau \quad x = \tau e^{-\frac{1}{\tau} t} - \tau + t$

$\because x(3.8) \approx 2$

$\therefore \boxed{\tau \approx 2}$



4. Nearly a decade after his supposed demise at Switzerland's Reichenbach Falls, Professor Moriarty —"organizer of half that is evil and of nearly all that is undetected" in London — plots to exact a sinister revenge from nemesis Sherlock Holmes. On a warm summer's night in 1902, Moriarty kills Scotland Yard's detective Lestrade and plants Holmes' famous pipe and cap near the body. The body is discovered by the London police. Holmes is arrested at Scotland Yard's insistence and is denied access to any particulars from the crime scene.

Despite an absence of motive, the circumstantial evidence incriminating Holmes must be countered with an alibi. Dr. Watson reviews the facts...

(a) Holmes attended an eight o'clock (20:00) concert on the night in question. The great Belgian violinist Eugene Ysaye himself can testify to Holmes' presence in the audience until eleven o'clock (23:00), but no one saw Holmes thereafter until his arrest.

(b) Medical Examiner Jordan measured the temperature T of Lestrade's remains twice at the scene of the crime. Their temperature at midnight (24:00 = 80°F), ~~when Jordan first arrived~~, was 80 °F; their temperature one hour (1h Fa) later was 75 °F. The night air remained a steady $Tair = 70$ °F all the while, as it had since sunset.

A physician by trade, Watson knows that Lestrade's body temperature must have been 98.6 °F at the time of his dispatch. A dilettante mechanical engineer on the side, Watson also knows that convective heat loss would have caused the temperature of Lestrade's body to vary according to the differential equation

$$\frac{dT}{dt} = -\beta(T(t) - T_{air})$$

as the body lay in the street. Here $\beta$ is a constant related to both Lestrade's physical composition and the atmosphere adherent to London's alleyways.

When did Lestrade die? Can Holmes be exonerated?

$\because T' = -\beta(T-70)$

$\therefore T' + \beta T = 70\beta$

For Fundamental solution

$\because \lambda + \beta = 0$

$\therefore \lambda = -\beta \quad \therefore x_1 = e^{-\beta t}$

For particular solution

$x_p = C_2$

$\therefore C_2 = 70$

$\therefore T = C_1 e^{-\beta t} + 70$

From the question we know $T(0) = 98.6$

$\therefore C_1 + 70 = 98.6 \Rightarrow C_1 = 28.6$

$\therefore T(t) = 28.6e^{-\beta t} + 70$

Let the time between Lestrade died and the first temperature measurement be $a$

$\therefore T(a) = 28.6e^{-\beta \cdot a} + 70 = 80$

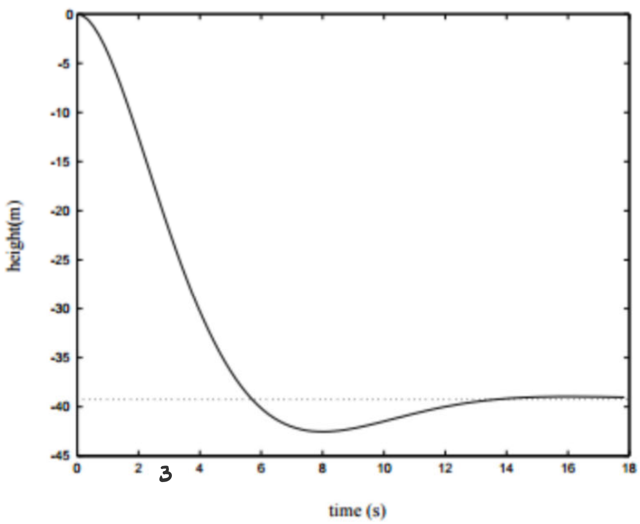
$T(a+1) = 28.6e^{-\beta(a+1)} + 70 = 75$

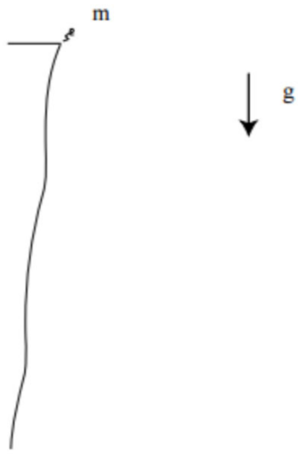
$\therefore \begin{cases} a = \frac{\ln 2.86}{\ln 2} \approx 1.516 \\ \beta = \ln 2 \end{cases}$

$\therefore$ $\boxed{\text{Lestrade die at 22:29 when Holmes is still at concert}}$

5. Mr. Gandalf Grey, known to admirers as “the wizard” decides to try his hand at bungee jumping. He stands at the brink of one of the many high, jagged cliffs in Mordor.

Mr. Frodo Baggins, an ME340 student, observes the wizard Gandalf’s vertical trajectory as a function of time to look like this: ?????




Assume wizard Gandalf’s motion to be governed by a differential equation of the form

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = -mg.$$

Estimate the damping ratio $\zeta$ and undamped natural frequency $\omega_n$ for Gandalf’s bungee experience from the graph.

From the graph we know the first minmum appear at $t = 8s$, $x_{ss} = -39$

For function $\ddot{x} + 2\xi\omega_n\dot{x} + \omega_n^2 x = A$

$$\begin{cases} x_{ss} = \dfrac{A}{\omega_n^2} = -39 \\ t_p = \dfrac{\pi}{\omega_d} = \dfrac{\pi}{\omega_n\sqrt{1-\xi^2}} = 8 \end{cases} \Rightarrow \boxed{\begin{cases} \omega_n = 0.50154 \\ \xi = 0.62203 \end{cases}}$$