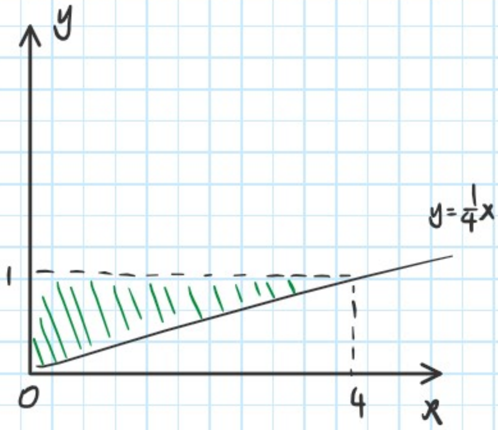
$\mu_{Y|x} = E[Y|x] = \sum_y y \cdot h(y|x)$

Homework 6



Exercise 1

Consider two random variables X and Y with joint pdf:

$$f(x,y) = \frac{x+y}{4}, \qquad 0 < y < 1, \quad 4y > x, \quad x > 0$$

a) (0.5 pt) Find an expression for $E[X|Y=y]$.

b) (1 pt) Find an expression for $Var[X|Y=y]$.

a). $f(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

$f_Y(y) = \int_0^{4y} \frac{x+y}{4} dx = 3y^2$

$\Rightarrow f(x|y) = \dfrac{x+y}{12y^2}$ ✓

$E[X|Y=y] = \int_0^{4y} x \cdot f(x|y)\, dx = \int_0^{4y} \frac{1}{12y^2}(x^2 + xy)\, dx = \frac{1}{12y^2}\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 y\right)\Big|_0^{4y} = \frac{22y}{9} \qquad 0 < y < 1$

这里代入X的范围是(0,4y)

b). $Var[X|Y=y] = E[X^2|Y=y] - (E[X|Y=y])^2$

$E[X^2|Y=y] = \int_0^{4y} x^2 f(x|y)\, dx = \int_0^{4y} \frac{1}{12y^2}(x^3 + x^2 y)\, dx = \frac{1}{12y^2}\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 y\right)\Big|_0^{4y} = \frac{64y^2}{9}$

$\Rightarrow Var[X|y] = \frac{64}{9}y^2 - \left(\frac{22}{9}y\right)^2 = \frac{92}{81}y^2$

Exercise 2 ★

Let X and Y have a bivariate normal distributions with marginal distributions: $X \sim N(45, 5^2)$, $Y \sim N(50, 6^2)$, and $\rho_{XY} = -0.5$. Show standardization and other work by hand!

a) (0.5 pts) What is P[X > 50]?

b) (0.5 pts) What is P[Y < 40]?

c) (0.5 pt) Given that X = 35, find P[Y > 55].

d) (0.5 pt) Given that Y = 68, find P[X < 36].

e) (0.5 pt) Find $P[X \geq Y]$.

f) (1 pt) Find $P[2X + 10Y > 600]$.

a). $P[X>50] = P\left[\frac{X-45}{5} > \frac{50-45}{5}\right] = P[Z>1] = 1-\text{pnorm}(1) = 0.1587$

b). $P[Y<40] = P\left[\frac{Y-50}{6} < \frac{40-50}{6}\right] = P[Z<-\frac{5}{3}] = \text{pnorm}(-\frac{5}{3}) = 0.0478$

c). $\because\ Y|X=x \sim N\left(\mu_Y + \rho\frac{\sigma_Y}{\sigma_X}(x-\mu_X),\ (1-\rho^2)\sigma_Y^2\right)$

$\therefore\ Y|X=35 \sim N(56, (3\sqrt{3})^2)$

$\therefore\ P[Y>55|X=35] = P\left[\frac{Y-56}{3\sqrt{3}} > \frac{55-56}{3\sqrt{3}} \middle| X=35\right] = P\left[Z > -\frac{1}{3\sqrt{3}} \middle| X=35\right] = 0.576$

d). $\because\ X|Y=y \sim N\left(\mu_X + \rho\frac{\sigma_X}{\sigma_Y}(y-\mu_Y),\ (1-\rho^2)\sigma_X^2\right)$

$\therefore\ X|Y=68 \sim N(37.5, \frac{75}{4})$ $(\sqrt{\frac{75}{4}})^2$ $\boxed{注意\ \sigma^2}$

$\therefore\ P[X<36|Y=68] = P\left[\frac{X-37.5}{\sqrt{\frac{75}{4}}} < \frac{36-37.5}{\sqrt{\frac{75}{4}}} \middle| Y=68\right] = P\left[Z < \frac{-3}{\sqrt{75}} \middle| Y=68\right] = 0.365$

二元概率

e). $P[X>Y] = P[X-Y>0] = P[W>0]$

$E[W] = E[X-Y] = \mu_X + \mu_Y = -5$

$Var[W] = Var[X-Y] = 5^2+6^2-2\cdot(-\frac{1}{2})(5\cdot 6) = \sqrt{91}^2$

$P[W>0] = P\left[\frac{W--5}{\sqrt{91}} > \frac{5}{\sqrt{91}}\right] = P\left[Z > \frac{5}{\sqrt{91}}\right] = 0.3$

二元概率

f). $P[2X+10Y>600] = P[W>600]$

$E[W] = 2\mu_X + 10\mu_Y = 590$

$Var[W] = 4\sigma_X^2 + 100\sigma_Y^2 + 40\cdot\rho\cdot\sigma_X\cdot\sigma_Y = 3100 = (10\sqrt{31})^2$

$P[W>600] = P\left[Z > \frac{600-590}{10\sqrt{31}} = \frac{1}{\sqrt{31}}\right] = 0.428$

Exercise 3 ☆ IID

Suppose that at Curtis Orchard:

- All apple weights are independent, and apple weights of the same type are *i.i.d.*
- The weight of Fuji apples is normally distributed with a mean of 150 grams and a standard deviation of 5 grams. Let F denote the weight of a randomly selected Fuji apple.
- The weight of Honeycrisp apples is normally distributed with a mean of 140 grams and a standard deviation of 7 grams. Let H denote the weight of a randomly selected Honeycrisp apple.

Show all your work and explicitly state the distributions used.

a) (0.5 pt) Suppose you pick 10 Honeycrisp apples at random. Assuming independence, what is the probability that that the **average** weight of the 10 apples is less than 139 grams?

b) (0.5 pt) Suppose you pick 5 Fuji apples at random. Assuming independence, what is the probability that that the **total** weight of the 5 apples is more than 755 grams?

c) (1 pt) Suppose you pick one Fuji and One Honeycrisp apple at random. What is the probability that the Fuji apple weighs less than the Honeycrisp apple?

d) (1 pt) Suppose you pick 5 Honeycrisp apples and 5 Fuji apples. What is the probability that their total weight is less than 1500g?

① IID

② $F \sim N(150, 5^2)$

③ $H \sim N(140, 7^2)$

独立同分布

a). $H_1, H_2 \cdots H_{10} \overset{iid}{\sim} N(140, 7^2)$, $n=10$

$\bar{H} \sim N(140, \frac{7^2}{10})$

$\bar{H} \sim N(140, (\frac{7}{\sqrt{10}})^2)$

$P[\bar{H} < 139] = P\left[\frac{\bar{H}-140}{\frac{7}{\sqrt{10}}} < \frac{139-140}{\frac{7}{\sqrt{10}}}\right] = P[z < -\frac{\sqrt{10}}{7}] = \boxed{0.3257}$

独立同分布

b). $n=5$

$Y = F_1 + F_2 + F_3 + F_4 + F_5$

$Y \sim [750, (5\sqrt{5})^2]$

$P[Y > 755] = P\left[\frac{Y-750}{5\sqrt{5}} > \frac{755-750}{5\sqrt{5}}\right] = P[z > \frac{1}{\sqrt{5}}] = \boxed{0.3273}$

二元概率

c). $P[F < H] = P[F - H < 0]$

$E[F-H] = \mu_F - \mu_H = 150 - 140 = 10$

$Var[F-H] = \sigma_F^2 + \sigma_H^2 - 0 = \sqrt{74}^2$

$P[F-H<0] = P[W<0] = P\left[\frac{W-10}{\sqrt{74}} < \frac{-10}{\sqrt{74}}\right] = \boxed{0.1225}$

独立同分布 ☆☆☆

d). $W = \sum_{i=1}^{5} F_i + \sum_{i=1}^{5} H_i$

$E[W] = 5\times150 + 5\times140 = 1450$

$Var[W] = 5\times(1^2\times5^2) + 5\times(1^2\times7^2) = 5\cdot5^2 + 5\cdot7^2 = \sqrt{370}^2$

$P[W < 1500] = P\left[\frac{W-1450}{\sqrt{370}} < \frac{50}{\sqrt{370}}\right] = \boxed{0.9953}$

Exercise 4

Let $X_1, X_2, ...X_n \overset{i.i.d}{\sim} f(x) = cx^2, \quad 0 < x < 2.$

a) (0.5 pt) Find a constant, c, that would make this a valid pdf.

b) (1 pt) Evaluate $E[X]$ and $Var[X]$.

c) (0.5 pt) Given a sample of size $n = 50$, evaluate the probability that the sample mean is between 1.5 and 1.6 using the Central Limit Theorem. $P[1.5 < \bar{X} < 1.6]$.

a). $\int_0^2 f(x)\,dx = \int_0^2 cx^2\,dx = \frac{1}{3}cx^3\Big|_0^2 = \frac{8}{3}c = 1 \Rightarrow c = \frac{3}{8}$

b). $E[X] = \int_0^2 x \cdot f(x) = \int_0^2 cx^3 = \frac{c}{4}x^4\Big|_0^2 = 4c = \frac{3}{2}$

$Var[X] = E[X^2] - (E[X])^2 = \frac{32}{5}c - (4c)^2 = \frac{3}{20}$

独立同分布 + 中心极限定理

c). $n=50 \quad P[1.5 \leqslant \bar{X} \leqslant 1.6]$

From b). we know $\mu = \frac{3}{2}$, $\sigma^2 = \frac{3}{20}$

$\because \bar{X} \sim N(\mu, \frac{\sigma^2}{n})$

$\therefore \bar{X} \sim N(\frac{3}{2}, \sqrt{\frac{3}{1000}}^2)$

$P[1.5 \leqslant \bar{X} \leqslant 1.6] = P\left[\frac{1.5-1.5}{\sqrt{\frac{3}{1000}}} < Z < \frac{1.6-1.5}{\sqrt{\frac{3}{1000}}}\right] = P[0 \leqslant Z \leqslant \sqrt{\frac{10}{3}}] = 0.466$