ME 340 Dynamics of Mechanical Systems

Lagrangian Dynamics
Part 4

Non-conservative forces

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i$$

- The examples that we have seen do not involve non-conservative forces.
- Example: damping force, motor torque, friction, external forces
- Generalized *non-conservative forces*
  - Work done by force depends on the path (NOT only on the end points)
- When applying Lagrangian equations, an essential step (Step 4) is to derive generalized non-conservative forces.

Non-conservative forces

- To determine non-conservative generalized force Q_i
  - For each r_j compute $\frac{\partial r_j}{\partial q_i}$, and for each θ_k compute $\frac{\partial \theta_k}{\partial q_i}$

$$Q_i = \sum_{j=1}^{N} F_j \frac{\partial r_j}{\partial q_i} + \sum_{k=1}^{M} \tau_k \frac{\partial \theta_k}{\partial q_i}$$

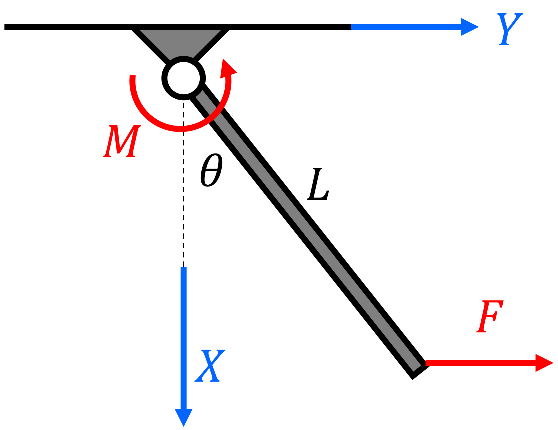
- Or
  - For each r_j compute $dr_j = \sum_{i=1}^{L} \frac{\partial r_j}{\partial q_i} dq_i$, and for each θ_k compute $d\theta_k = \sum_{i=1}^{L} \frac{\partial \theta_k}{\partial q_i} dq_i$
  - Compute

$$dW_{nc} = \sum_{j=1}^{N} F_j dr_j + \sum_{k=1}^{M} \tau_k d\theta_k$$

  - Q_i is the coefficient of dq_i in the expression of dW_{nc}.

Non-conservative torques

- Example:



$$\theta_1 = \theta \qquad \frac{\partial \theta_1}{\partial q_1} = \frac{\partial \theta_1}{\partial \theta} = 1$$

$$r_1 = \begin{bmatrix} L\cos(\theta) \\ L\sin(\theta) \end{bmatrix} \qquad \frac{\partial r_1}{\partial q_1} = \frac{\partial r_1}{\partial \theta} = \begin{bmatrix} -L\sin(\theta) \\ L\cos(\theta) \end{bmatrix}$$

$$Q_1 = M_1 . \frac{\partial \theta_1}{\partial \theta} + F_1 . \frac{\partial r_1}{\partial \theta} = M.1 + \begin{bmatrix} 0 \\ F \end{bmatrix} . \begin{bmatrix} -L\sin(\theta) \\ L\cos(\theta) \end{bmatrix}$$

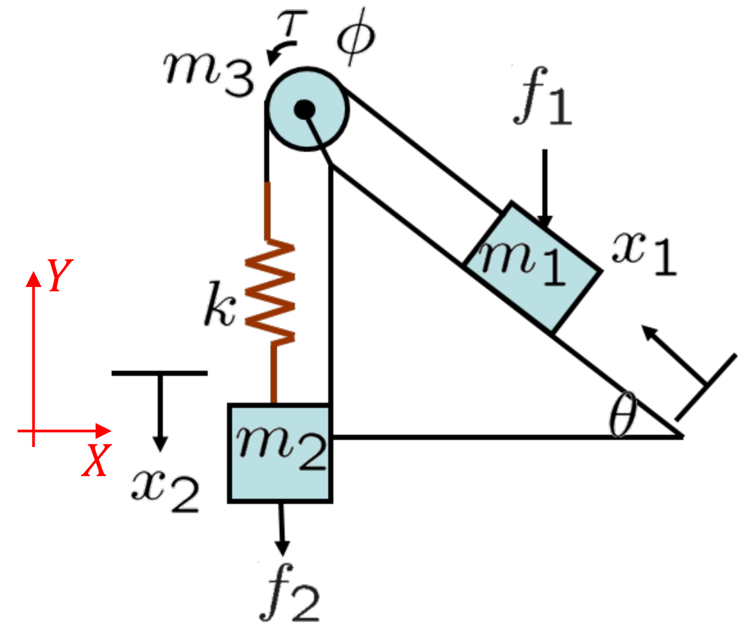
$$\boxed{Q_1 = M + FL\cos(\theta)}$$

Wedge example with non-conservative forces

- Generalized non-conservative forces

$$r_1 = \begin{bmatrix} -x_1\cos\theta \\ x_1\sin\theta \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix};$$

$$r_2 = x_2, F_2 = f_2; \phi = \frac{x_1}{R}, \text{ torque } = \tau$$



- Method 1

$$\frac{\partial r_1}{\partial x_1} = \begin{bmatrix} -\cos\theta \\ \sin\theta \end{bmatrix}; \frac{\partial r_1}{\partial x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}; \frac{\partial r_2}{\partial x_1} = 0; \frac{\partial r_2}{\partial x_2} = 1; \frac{\partial \phi}{\partial x_1} = \frac{1}{R}; \frac{\partial \phi}{\partial x_2} = 0$$

$$Q_1 = F_1\frac{\partial r_1}{\partial x_1} + F_2\frac{\partial r_2}{\partial x_1} + \tau\frac{\partial \phi}{\partial x_1} = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix} \cdot \begin{bmatrix} -\cos\theta \\ \sin\theta \end{bmatrix} + f_2 \cdot 0 + \tau\frac{1}{R} = -f_1\sin\theta + \frac{\tau}{R}$$

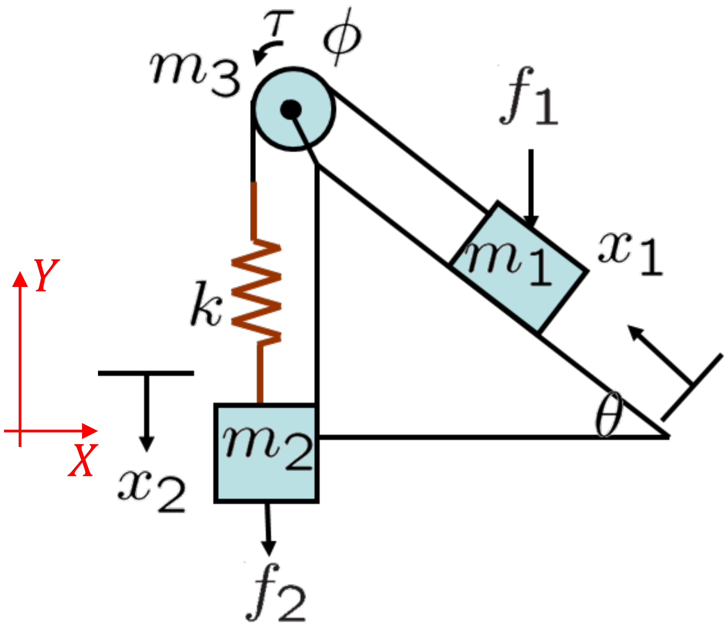
$$Q_2 = F_1\frac{\partial r_1}{\partial x_2} + F_2\frac{\partial r_2}{\partial x_2} + \tau\frac{\partial \phi}{\partial x_2} = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} + f_2 + 0 = f_2$$

Wedge example with non-conservative forces

- Generalized non-conservative forces

$$r_1 = \begin{bmatrix} -x_1 \cos\theta \\ x_1 \sin\theta \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix};$$

$$r_2 = x_2, F_2 = f_2; \phi = \tfrac{x_1}{R}, \text{ torque } = \tau$$



- Method 2

$$dr_1 = \begin{bmatrix} -dx_1 \cos\theta \\ dx_1 \sin\theta \end{bmatrix}; dr_2 = dx_2; d\phi = \tfrac{1}{R} dx_1$$

$$\begin{aligned} dW_{nc} &= F_1 dr_1 + F_2 dr_2 + \tau d\phi = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix} \cdot \begin{bmatrix} -dx_1 \cos\theta \\ dx_1 \sin\theta \end{bmatrix} + f_2 dx_2 + \frac{1}{R} \tau dx_1 \\ &= -f_1 \sin\theta dx_1 + f_2 dx_2 + \frac{1}{R}\tau dx_1 = \left(-f_1 \sin\theta + \frac{\tau}{R}\right) dx_1 + f_2 dx_2 \end{aligned}$$

$$\Rightarrow Q_1 = -f_1 \sin\theta + \frac{\tau}{R}, \ Q_2 = f_2$$

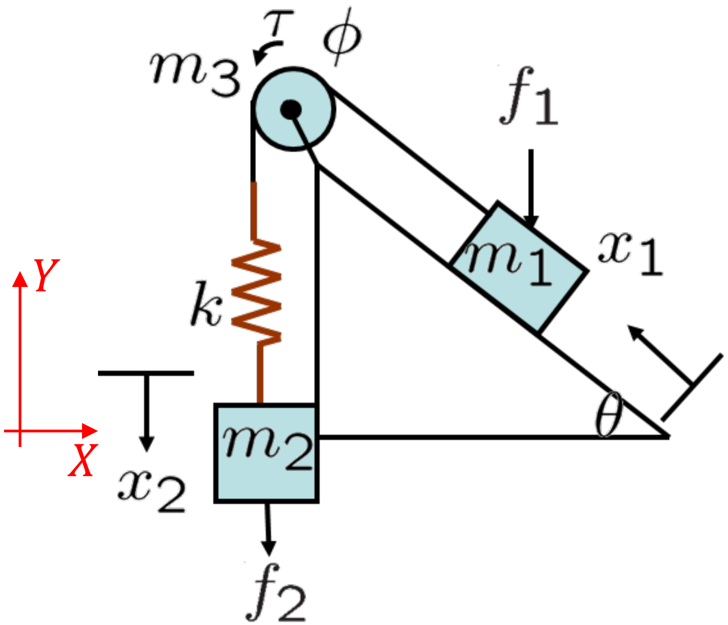
Wedge example with non-conservative forces

- Generalized non-conservative forces

$$r_1 = \begin{bmatrix} -x_1 \cos\theta \\ x_1 \sin\theta \end{bmatrix}, F_1 = \begin{bmatrix} 0 \\ -f_1 \end{bmatrix};$$

$$r_2 = x_2, F_2 = f_2; \phi = \frac{x_1}{R}, \text{ torque } = \tau$$

- $Q_1 = -f_1 \sin\theta + \frac{\tau}{R},\ Q_2 = f_2$



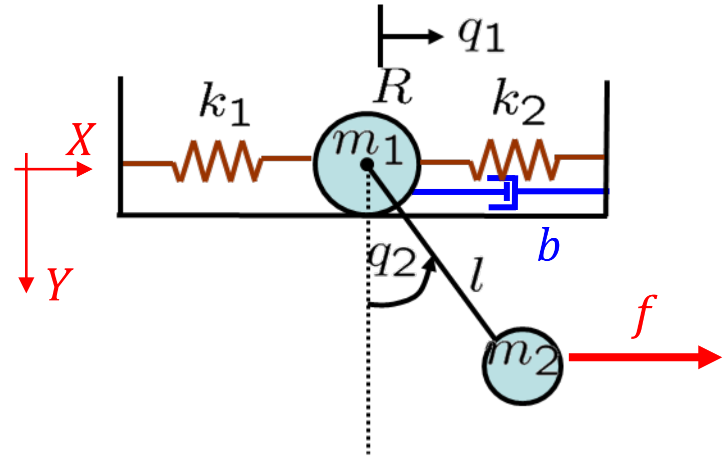
- The equations of motion are

$$\left(m_1 + \frac{m_3}{2}\right)\ddot{x}_1 + m_1 g \sin\theta + k(x_1 - x_2) = -f_1 \sin\theta + \frac{\tau}{R}$$

$$m_2 \ddot{x}_2 - m_2 g + k(x_2 - x_1) = f_2$$

Example: disk + pendulum

新例题



- Two masses, two springs
- Rotation without slippery
- No external forces/torques

- DOF: 2 q_i的个数
- Generalized coordinates
  - Translational position of mass 1 and angular position of mass 2
- Kinetic energy: $T = T_{\text{disk}} + T_{\text{pendulum}}$
  - Disk

$$T_{\text{disk}} = \frac{1}{2}m_1\dot{q}_1^2 + \frac{1}{2}\left(\frac{1}{2}m_1R^2\right)\left(\frac{\dot{q}_1}{R}\right)^2 = \frac{3}{4}m_1\dot{q}_1^2$$

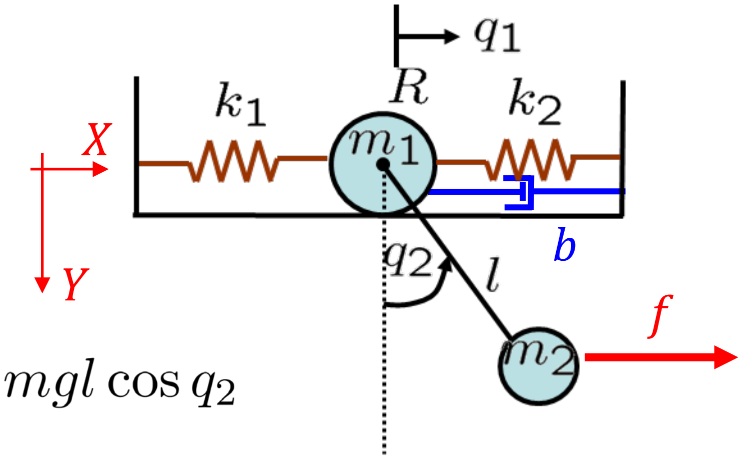
  - Pendulum

$$r = \begin{pmatrix} q_1 + l\sin q_2 \\ l\cos q_2 \end{pmatrix} \Rightarrow \dot{r} = \begin{pmatrix} \dot{q}_1 + l\dot{q}_2\cos q_2 \\ -l\dot{q}_2\sin q_2 \end{pmatrix}$$

$$T_{\text{pendulum}} = \frac{1}{2}m_2|\dot{r}|^2 = \frac{1}{2}m_2(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_2 l\cos q_2 + l^2\dot{q}_2^2)$$

Example: disk + pendulum

- Two masses, two springs
- Rotation without slippery
- No external forces/torques



- Potential energy $V = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 q_1^2 - mgl\cos q_2$
- Generalized forces (non-conservative):

$$r_1 = \begin{bmatrix} q_1 \\ 0 \end{bmatrix} \qquad r_2 = \begin{bmatrix} q_1 + l\sin q_2 \\ l\cos q_2 \end{bmatrix}$$

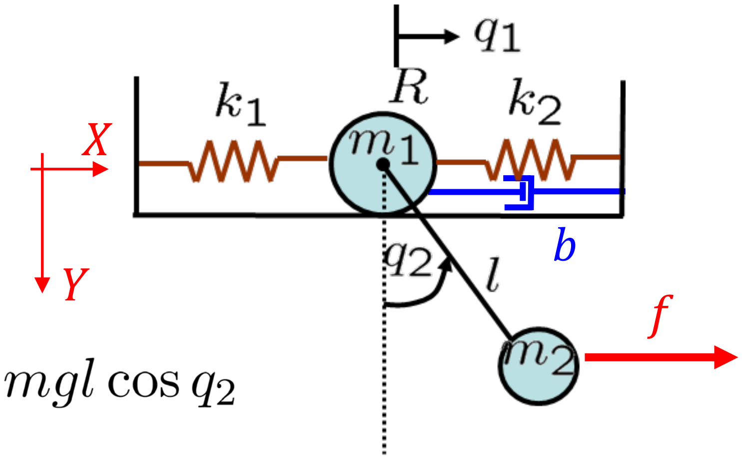
$$\frac{\partial r_1}{\partial q_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \frac{\partial r_1}{\partial q_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \frac{\partial r_2}{\partial q_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \frac{\partial r_2}{\partial q_2} = \begin{bmatrix} l\cos q_2 \\ -l\sin q_2 \end{bmatrix}$$

$$Q_1 = F_1 \frac{\partial r_1}{\partial q_1} + F_2 \frac{\partial r_1}{\partial q_1}$$
$$Q_1 = \begin{bmatrix} -b\dot{q}_1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} f \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$Q_1 = -b\dot{q}_1 + f$$

$$Q_2 = F_1 \frac{\partial r_1}{\partial q_2} + F_2 \frac{\partial r_1}{\partial q_2}$$
$$Q_2 = \begin{bmatrix} -b\dot{q}_1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} f \\ 0 \end{bmatrix} \cdot \begin{bmatrix} l\cos q_2 \\ l\sin q_2 \end{bmatrix}$$
$$Q_2 = fl\cos q_2$$

Example: disk + pendulum

- Two masses, two springs
- Rotation without slippery
- No external forces/torques



- Potential energy $V = \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2 q_1^2 - mgl\cos q_2$
- Two equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial V}{\partial q_1} = Q_1$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_2}\right) - \frac{\partial T}{\partial q_2} + \frac{\partial V}{\partial q_2} = Q_2$$

- Equations of motion

$$\left(\frac{3}{2}m_1 + m_2\right)\ddot{q}_1 + m_2 l\cos(q_2)\,\ddot{q}_2 - m_2 l\sin(q_2)\,\dot{q}_2^2 + (k_1 + k_2)q_1 = -b\dot{q}_1 + f$$

$$\ddot{q}_1\cos(q_2) + l\ddot{q}_2 + g\sin(q_2) = fl\cos q_2$$

Example: cart-pole

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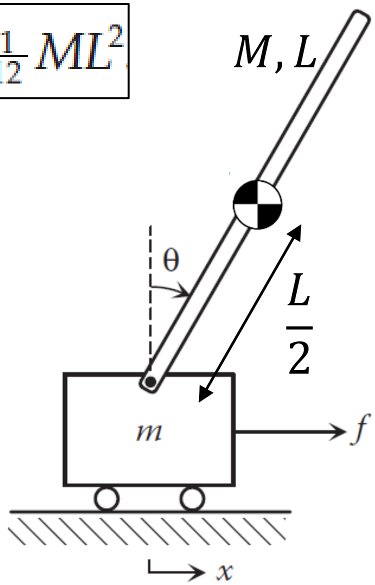
- Kinetic energy:

$$\boxed{I_C = \tfrac{1}{12} ML^2}$$



$$T_{\text{cart}} = \frac{1}{2} m\dot{x}^2 \qquad T_{\text{pendulum}} = \frac{1}{2} M v_C^2 + \frac{1}{2} I_C \dot{\theta}^2$$

$$v_C^2 = \left(\dot{x} + \frac{L}{2}\dot{\theta}\cos\theta\right)^2 + \left(\frac{L}{2}\dot{\theta}\sin\theta\right)^2 = \dot{x}^2 + L\dot{x}\dot{\theta}\cos\theta + \frac{1}{4}L^2\dot{\theta}^2$$

$$T = T_{\text{cart}} + T_{\text{pendulum}} = \frac{1}{2}(m+M)\dot{x}^2 + \frac{1}{6}ML^2\dot{\theta}^2 + \frac{1}{2}ML\dot{x}\dot{\theta}\cos\theta$$

- Potential energy:

$$V_g = Mgh = Mg\frac{L}{2}\cos\theta$$

Example: cart-pole

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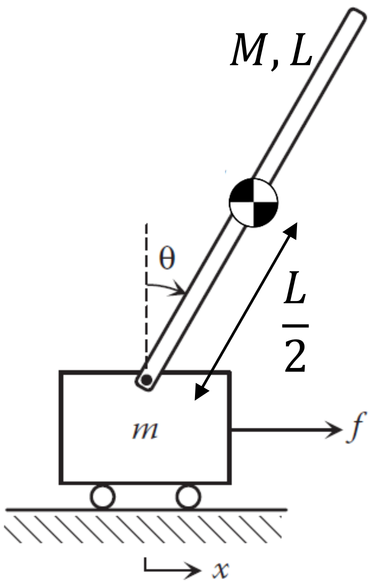

- Non-conservative forces:

$$r_1 = x \qquad \frac{\partial r_1}{\partial x} = 1 \qquad \frac{\partial r_1}{\partial \theta} = 0$$

$$Q_1 = F_1 \frac{\partial r_1}{\partial q_1} = f.1 = f \qquad Q_2 = F_1 \frac{\partial r_1}{\partial q_2} = f.0 = 0$$

- Equations of motion:

$$\frac{\partial T}{\partial \dot{x}} = (m+M)\dot{x} + \frac{1}{2}ML\dot{\theta}\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}}\right) = (m+M)\ddot{x} + \frac{1}{2}ML\ddot{\theta}\cos\theta - \frac{1}{2}ML\dot{\theta}\sin\theta\dot{\theta}$$

$$\frac{\partial T}{\partial x} = 0$$

$$\frac{\partial V}{\partial x} = 0$$

$$\frac{\partial T}{\partial \dot{\theta}} = \frac{1}{3}ML^2\dot{\theta} + \frac{1}{2}ML\dot{x}\cos\theta$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{1}{3}ML^2\ddot{\theta} + \frac{1}{2}ML\ddot{x}\cos\theta - \frac{1}{2}ML\dot{x}\sin\theta\dot{\theta}$$

$$\frac{\partial T}{\partial \theta} = -\frac{1}{2}ML\dot{x}\dot{\theta}\sin\theta$$

$$\frac{\partial V}{\partial \theta} = -\frac{1}{2}MgL\sin\theta$$

Example: cart-pole

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- Equations of motion:

$$(m+M)\ddot{x}+\frac{1}{2}ML\ddot{\theta}\cos\theta-\frac{1}{2}ML\dot{\theta}^2\sin\theta=f$$

$$\frac{1}{3}ML^2\ddot{\theta}+\frac{1}{2}ML\ddot{x}\cos\theta-\frac{1}{2}MgL\sin\theta=0$$