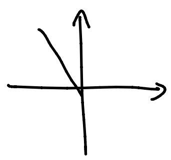
Homework 1

1. Write the complex number $z_1 = -1 + j\sqrt{3}$ in polar coordinates

$$z_1 = -1 + j\sqrt{3} = 2 \cdot e^{j\frac{2\pi}{3}}$$



2. Use polar coordinate representation to show that for any two complex numbers z_1 and z_2
   (a) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\angle\left(\frac{z_1}{z_2}\right) = \angle z_1 - \angle z_2$.
   (b) $|z_1 z_2| = |z_1||z_2|$ and $\angle(z_1 z_2) = \angle z_1 + \angle z_2$

a). Let $z_1 = r_1 \cdot e^{j\theta_1}$ $\quad z_2 = r_2 \cdot e^{j\theta_2}$

$$\left|\frac{z_1}{z_2}\right| = \left|\frac{r_1 e^{j\theta_1}}{r_2 e^{j\theta_2}}\right| = \frac{|r_1 e^{j\theta_1}|}{|r_2 e^{j\theta_2}|} = \frac{|z_1|}{|z_2|}$$

$$\angle\left(\frac{z_1}{z_2}\right) = e^{j\theta_1} / e^{j\theta_2} = e^{j(\theta_1 - \theta_2)} = \angle z_1 - \angle z_2$$

b). $$|z_1 \cdot z_2| = |r_1 \cdot e^{j\theta_1} \cdot r_2 e^{j\theta_2}| = |r_1 e^{j\theta_1}| \cdot |r_2 \cdot e^{j\theta_2}| = |z_1| \cdot |z_2|$$

$$\angle(z_1 \cdot z_2) = e^{j\theta_1} \cdot e^{j\theta_2} = e^{j(\theta_1 + \theta_2)} = \angle z_1 + \angle z_2$$

3. Let $G(s) = \frac{s+100}{s+1}$ be a function in complex variable $s = \sigma + j\omega$. Estimate $|G(s)|$ and $\angle(G(s))$ at
   (a) $s = 10^{-5}$ (b) $s = 10$ (c) $s = j10$ (d) $s = j10^5$

   [Hint: it may be useful to know that for any complex pair z_1, z_2, the following identities hold $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ and $\angle\frac{z_1}{z_2} = \angle z_1 - \angle z_2$.]

a). $G(s) = \dfrac{10^{-5} + 100}{10^{-5} + 1}$ $\quad \therefore |G(s)| = \dfrac{100 + 10^{-5}}{1 + 10^{-5}}$ $\quad \angle(G(s)) = 0$

b). $G(s) = \dfrac{10 + 100}{10 + 1} = \dfrac{110}{11} = 10$ $\quad \therefore |G(s)| = 10$ $\quad \angle(G(s)) = 0$

c). $G(s) = \dfrac{100 + j \cdot 10}{1 + j \cdot 10} = \dfrac{200 - 990j}{101}$ $\quad \therefore |G(s)| = \dfrac{10\sqrt{10^2+1}}{\sqrt{10^2+1}} = 10$ $\quad \angle(G(s)) = \tan^{-1}\dfrac{-990}{200} = -1.37$

d). $G(s) = \dfrac{j \cdot 10^5 + 100}{j10^5 + 1} = \dfrac{100 + 10^{10} - 101 \cdot 10^5 j}{10^{10} + 1}$ $\quad \therefore |G(s)| \approx 1$ $\quad \angle(G(s)) = \tan^{-1}\dfrac{-101 \times 10^5}{100 + 10^{10}} \approx 0.001$

4. Find the real numbers a and b such that $a + jb = \left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)^{105}$.
[Hint: compute the right hand side in polar coordinates].

$\because z = \frac{1}{2} + j\frac{\sqrt{3}}{2} = 1 \cdot e^{\frac{\pi}{3}}$

$\therefore z^{105} = \left(1 \cdot e^{j\frac{\pi}{3}}\right)^{105} = e^{j35\pi} = e^{j\cdot\pi} = -1 + j\cdot 0 \quad \therefore \boxed{a = -1, \ b = 0}$

5. Find all distinct complex numbers z such that $z^2|\bar{z}|^2 = 8(-1 + j\sqrt{3})$. Express each of them in both rectangular form $z = x + jy$ and polar form $z = re^{j\theta}$.

Let $z = r \cdot e^{j\theta}$

$\therefore r^2 \cdot e^{2j\theta} \cdot r^2 = 16 \cdot e^{j\frac{2\pi}{3}}$

$r^4 \cdot e^{2j\theta} = 16 \cdot e^{j\frac{2\pi}{3}}$

$\therefore \begin{cases} r^4 = 16 \\ 2\theta = \frac{2\pi}{3} + 2k\pi \end{cases} \Rightarrow \begin{cases} r = \pm 2 \\ \theta = \frac{\pi}{3} + k\pi = \frac{\pi}{3}, \frac{4\pi}{3} \end{cases} \quad \theta \in [0, 2\pi)$

$\therefore$ ① $r = 2, \ \theta = \frac{\pi}{3}$

$z = 2e^{j\frac{\pi}{3}} = 1 + \sqrt{3}j$

② $r = -2, \ \theta = \frac{4\pi}{3}$

$z = -2 \cdot e^{j\frac{4\pi}{3}} = 1 + \sqrt{3}j$

③ $r = -2, \ \theta = \frac{\pi}{3}$

$z = -2e^{j\frac{\pi}{3}} = -1 - \sqrt{3}j$

④ $r = 2, \ \theta = \frac{4\pi}{3}$

$z = 2e^{j\frac{4\pi}{3}} = -1 - \sqrt{3}j$

6. Use Euler's formula $e^{j\theta} = \cos\theta + j\sin\theta$ and the identity $e^{j(\theta_1+\theta_2)} = e^{j\theta_1}e^{j\theta_2}$, and no other trigonometric identities, to prove that

$$(\cos\theta + j\sin\theta)\left(\cos\frac{\theta}{n} + j\sin\frac{\theta}{n}\right) = \cos\left(\frac{(n+1)\theta}{n}\right) + j\sin\left(\frac{(n+1)\theta}{n}\right).$$

$(\cos\theta + j\sin\theta)(\cos\frac{\theta}{n} + j\sin\frac{\theta}{n}) = e^{j\theta} \cdot e^{j\frac{\theta}{n}} = e^{j(\frac{n+1}{n}\theta)} = \cos(\frac{n+1}{n}\theta) + j\sin(\frac{n+1}{n}\theta)$

7. Consider the following differential equation

$$\frac{d^2x(t)}{dt^2}+\frac{5dx(t)}{dt}+6x(t)=0.$$

(a) Find all values of λ such that $x(t)=e^{\lambda t}$ satisfies the above differential equation.

(b) Show that if $x(t)=e^{\lambda t}$ satisfies the above differential equation, then any function of the form $y(t)=ce^{\lambda t}$, where c is a constant real number, also satisfies the above differential function.

a). $\because \ddot{x}+5\dot{x}+6x=0$

$\therefore \lambda^2+5\lambda+6=(\lambda+2)(\lambda+3)=0 \Rightarrow \lambda_1=-2,\ \lambda_2=-3$

b). let $x(t)=c e^{\lambda t}$, $x'(t)=c\cdot\lambda e^{\lambda t}$, $x''(t)=c\lambda^2 e^{\lambda t}$

$\therefore c\cdot\lambda^2\cancel{e^{\lambda t}}+5\cdot c\cdot\lambda\cancel{e^{\lambda t}}+6\cdot c\cdot\cancel{e^{\lambda t}}=0$

$\therefore c[\lambda+2][\lambda+3]=0$ $\qquad\therefore c\cdot e^{\lambda t}$ satisfies the above ODE

8. Consider $z=e^{st}$ where $s=\sigma+j\omega$.

(a) Show that $|e^{st}|=\left|e^{(\sigma+j\omega)t}\right|=e^{\sigma t}$

(b) Deduce that

i. $\lim_{t\to\infty}|e^{\wedge}\{st\}|=0$ for any complex number s such that $real(s)<0$.

ii. $\lim_{t\to\infty}|e^{\wedge}\{st\}|=1$ for any complex number s such that $real(s)=0$.

iii. $\lim_{t\to\infty}|e^{\wedge}\{st\}|=\infty$ for any complex number s such that $real(s)>0$.

a). $|e^{st}|=|e^{(\sigma+jw)t}|=|e^{\sigma t}(\cos wt+j\sin wt)|=|e^{\sigma t}\cdot\sqrt{\cos^2 wt+\sin^2 wt}|=|e^{\sigma t}|=e^{\sigma t}$

b) i). $\lim_{t\to\infty}|e^{st}|=\lim_{t\to\infty}e^{\sigma t}$ $\quad\because \sigma<0$ $\quad\therefore \lim_{t\to\infty}e^{\sigma t}=0$

ii). $\lim_{t\to\infty}|e^{st}|=\lim_{t\to\infty}e^{\sigma t}$ $\quad\because \sigma=0$ $\quad\therefore e^{\sigma t}=1$ $\quad\therefore \lim_{t\to\infty}e^{\sigma t}=\lim_{t\to\infty}1=1$

iii). $\lim_{t\to\infty}|e^{st}|=\lim_{t\to\infty}e^{\sigma t}$ $\quad\because \sigma>0$ $\quad\therefore \lim_{t\to\infty}e^{\sigma t}=\infty$