# Week 10: Extreme Value Theory

## MATH-516 Applied Statistics

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Section 1

# 1. Introduction

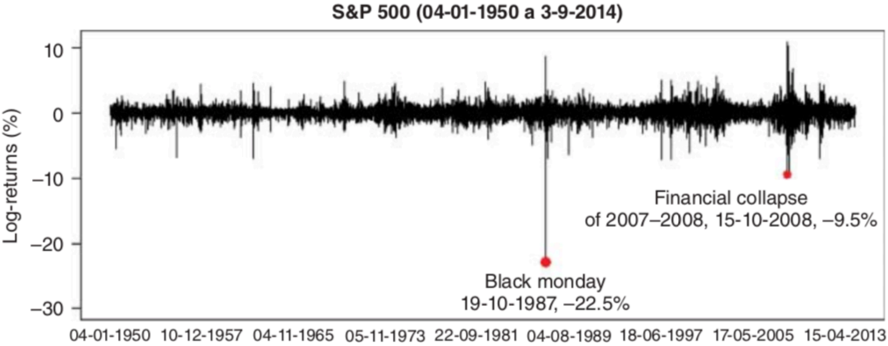
# 1.1 Motivation for modelling extreme events

## Modelling extremes in environmental sciences

- **Temperatures** $\rightarrow$ heat waves (Europe, 2003 $\rightarrow$ 40'000 deaths and €13.1 billion of crop damages)
- **Water heights** $\rightarrow$ floods (hurricane Harvey, 2017 $\rightarrow$ 107 deaths and \$125 billion in damages)
- **Concentrations of air pollutants** $\rightarrow$ health problems

# 1.2 Motivation for modelling extreme events

## Modelling extremes in finance




Growing areas of application include: insurance, athletic records, networks

# 1.3 Basic problem

- Let $X$ be a random variable of interest with cdf $F$
- We are interested in cases where $X$ is "extremely" large or "extremely" low, i.e.,

$$\Pr(X > x) \text{ when } x \text{ is large, or } \Pr(X < x) \text{ when } x \text{ is low}$$

Therefore, we require accurate inference on the tails of $F$. But...

- There are very few observations in the tails of the distribution $\rightarrow$ standard techniques can result in severely biased estimates
- We often require estimates that are beyond the observed values

$\rightarrow$ **Rely on the extreme value paradigm**: base tail models on asymptotically-motivated distributions!

# 1.4 How bad does it get?

We want to study the worst case scenario

Two classical approaches

- Block-maxima: $\max(X_1, \dots, X_n)$ (maximum over, e.g., a year)
- Peaks over threshold: $X|X > u$ for a large threshold $u$

Section 2

# 2. Block-maxima Approach

# 2.1 Notations

- Let $X_1, X_2, \dots$ be iid random variables with distribution function $F$
- We seek approximations to the distribution of the maximum of the $X_i$
- Let $M_n = \max(X_1, \dots, X_n)$ be the worst-case value in a sample of $n$ values. Clearly

$$\mathbb{P}(M_n \leq x) = \mathbb{P}(X_1 \leq x, \dots, X_n \leq x) = F^n(x)$$

- It can be shown that, almost surely, $M_n \stackrel{n\to\infty}{\longrightarrow} x_F$, where $x_F := \sup\{x \in \mathbb{R} : F(x) < 1\} \leq \infty$ is the right endpoint of $F$
- This is not useful, because the distribution is concentrated at $x_F$
- But what about normalized maxima?

# 2.2 Limiting Behaviour of Sums or Averages

- We are familiar with the central limit theorem
- Let $X_1, X_2, \ldots$ be iid with finite mean $\mu$ and finite variance $\sigma^2$. Let $S_n = X_1 + \ldots + X_n$. Then

$$\mathbb{P}\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq x\right) \xrightarrow{n\to\infty} \Phi(x)$$

where $\Phi$ is the cdf of the standard normal distribution

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du$$

- More generally, the limiting distributions for appropriately normalized sample sums are the class of $\alpha$-stable distributions; Gaussian distribution is a special case

# 2.3 Limiting Behaviour of Sample Maxima

- Let $X_1, X_2, \ldots$ be iid from $F$ and let $M_n = \max(X_1, \ldots, X_n)$
- Suppose we can find sequences of real numbers $a_n > 0$ and $b_n$ such that $(M_n - b_n)/a_n$, the sequence of normalized maxima, converges in distribution, i.e.,

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) = F^n(a_n x + b_n) \xrightarrow{n\to\infty} H(x)$$

  for some non-degenerate df $H(x)$

- If this condition holds we say that $F$ is in the **maximum domain of attraction** of $H$, abbreviated $F \in MDA(H)$
- Such an $H$ is determined up to location and scale, i.e., will specify a unique type of distribution

# 2.4 Generalized Extreme Value Distribution

- The Generalized Extreme Value (GEV) distribution has df

$$H_\xi(x) = \begin{cases} \exp\left\{-\left(1+\xi x\right)^{-1/\xi}\right\} & \xi \neq 0, \\ \exp\left\{-e^{-x}\right\} & \xi = 0, \end{cases}$$

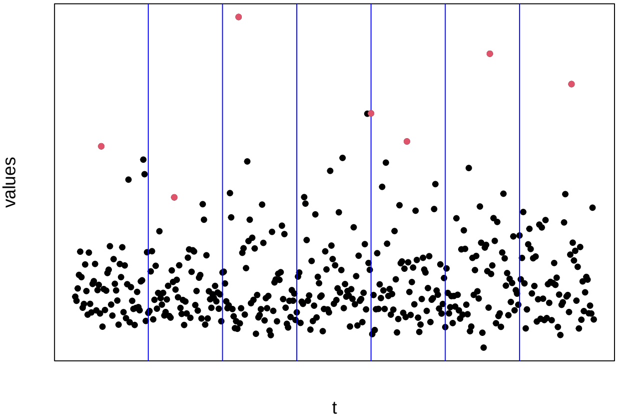
where $1 + \xi x > 0$ and $\xi$ is the shape parameter

Note that this parametrization is continuous in $\xi$. For

$$\begin{aligned} &\xi > 0,\ H_\xi \text{ is equal in type to classical Fréchet df} \\ &\xi = 0,\ H_\xi \text{ is equal in type to classical Gumbel df} \\ &\xi < 0,\ H_\xi \text{ is equal in type to classical Weibull df} \end{aligned}$$

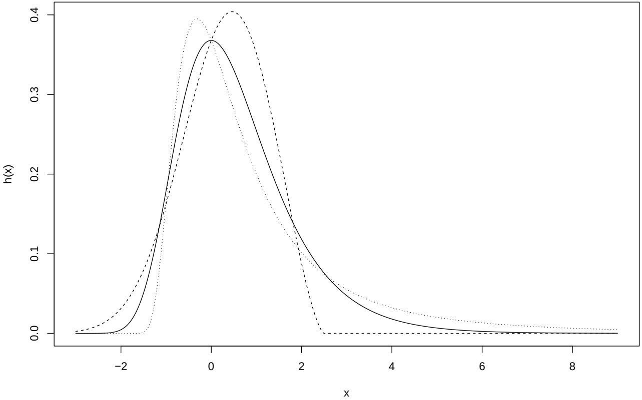
- We introduce location and scale parameters $\mu$ and $\sigma > 0$ and work with $H_{\xi,\mu,\sigma}(x) := H_\xi\left(\frac{x-\mu}{\sigma}\right)$. Clearly $H_{\xi,\mu,\sigma}$ is of type $H_\xi$

# 2.5 Block maxima




$\rightsquigarrow$ *generalized extreme value limit distribution* for rescaled maxima

# 2.6 GEV densities




Solid line corresponds to $\xi = 0$ (Gumbel); dotted line is $\xi = 0.4$ (Fréchet); dashed line is $\xi = -0.4$ (Weibull). $\mu = 0$ and $\sigma = 1$

**Examples**: Rainfall or financial data (usually $\xi > 0$), temperature data (usually $\xi < 0$), and Gaussian data ($\xi = 0$)

# 2.7 Fisher–Tippett Theorem (1928)

**Theorem**: If $F \in \mathsf{MDA}(H)$ then $H$ is of the type $H_{\mu,\sigma,\xi}$ for some $\mu, \sigma, \xi$

Stated differently: "If suitably normalised maxima converge in distribution to a non-degenerate limit, then the limit distribution must be an extreme value distribution"

**Remark 1**: Essentially, all commonly encountered continuous distributions are in the maximum domain of attraction of an extreme value distribution

**Remark 2**: We can always choose normalizing sequences $a_n$ and $b_n$ so that the limit law $H_\xi$ appears in standard form (without relocation or rescaling)

## 2.8 Fisher–Tippett: Examples

**Recall**: $F \in \text{MDA}(H_\xi)$, iff there are sequences $a_n$ and $b_n$ with

$$\mathbb{P}\left\{(M_n - b_n)/a_n \leq x\right\} = F^n\left(a_n x + b_n\right) \overset{n\to\infty}{\longrightarrow} H(x)$$

- The **exponential distribution**

$$F(x) = 1 - e^{-\lambda x}, \lambda > 0, x \geq 0$$

is in MDA($H_0$) (Gumbel). Take $a_n = 1/\lambda$, $b_n = (\log n)/\lambda$

- The **Pareto distribution**

$$F(x) = 1 - \left(\frac{\kappa}{\kappa + x}\right)^\alpha, \quad \alpha, \kappa > 0, \quad x \geq 0,$$

is in MDA($H_{1/\alpha}$) (Fréchet). Take $a_n = \kappa n^{1/\alpha}/\alpha$, $b_n = \kappa n^{1/\alpha} - \kappa$

# 2.9 When does $F \in \text{MDA}(H_\xi)$ hold?

## 2.9.1 Fréchet Case: $(\xi > 0)$

- Gnedenko (1943) showed that for $\xi > 0$

$$F \in \text{MDA}(H_\xi) \iff 1 - F(x) = x^{-1/\xi} L(x)$$

for some slowly varying function $L(x)$

- A function $L$ on $(0, \infty)$ is slowly varying if

$$\lim_{x \to \infty} \frac{L(tx)}{L(x)} = 1, \quad t > 0$$

**Summary:** If the tail of the distribution function $F$ decays like a power function, then the distribution is in $\text{MDA}(H_\xi)$ for $\xi > 0$

**Examples:** Heavy-tailed distributions such as Pareto, Burr, log-gamma, Cauchy, and $t$-distributions as well as various mixture models. Not all moments are finite

# 2.10 When does $F \in \mathsf{MDA}(H_\xi)$ hold?

## 2.10.1 Gumbel Case: $F \in \mathsf{MDA}(H_0)$

- The characterization of this class is more complicated. Essentially, it contains distributions whose tails decay roughly exponentially and we call these distributions light-tailed. All moments exist for distributions in the Gumbel class
- Examples are the normal, log-normal, exponential, and gamma

## 2.11 Using Fisher–Tippett on data: Block Maxima Method

If you are given $n$ values, use the limiting distribution to model $M_n$:

$$\mathbb{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) \approx H_{0,1,\xi}(x)$$

or

$$\mathbb{P}(M_n \leq y) = H_{\xi,b_n,a_n}(y)$$

- All that's left is to estimate three parameters: $\xi$, $b_n$, and $a_n$
- Need repeated values of $M_n \Rightarrow$ required data is a multiple of $n$

The values $b_n$ and $a_n$ are equivalent to the parameters $\mu$ and $\sigma$ in the formula, respectively

# 2.12 ML Inference for Maxima

We have block maxima data $\mathbf{y} = \left(M_n^{(1)}, \dots, M_n^{(m)}\right)^\top$ from $m$ blocks of size $n \rightarrow$ want to estimate $\theta = (\xi, \mu, \sigma)^\top$

We construct a **log-likelihood** by assuming we have independent observations from a GEV with density $h_\theta$,

$$l(\theta; \mathbf{y}) = \log \left\{ \prod_{i=1}^{m} h_\theta \left( M_n^{(i)} \right) \mathbb{1}_{\left\{1+\xi\left(M_n^{(i)}-\mu\right)/\sigma>0\right\}} \right\}$$

and (numerically) maximize this w.r.t. $\theta$ to obtain the MLE $\hat{\theta} = (\hat{\xi}, \hat{\mu}, \hat{\sigma})^\top$

# 2.13 ML Inference for Maxima

- When $\xi > -0.5$, maximum likelihood estimator obeys the standard theory. In particular
  - standard errors can be computed from inverse of the observed information matrix
  - likelihood ratio test applies to nested models
- If $\xi \leq -0.5$, Bayesian methods may be preferable (this is very rare in practice!)

Clearly, when defining blocks, **bias** and **variance** must be traded off

- we reduce bias by increasing the block size $n$
- we reduce variance by increasing the number of blocks $m$

# 2.14 Return Levels

- Aim: What is the 40-year return level $R_{365,40}$?
- We define a rare **stress** $R_{n,k}$, the $k$ $n$-block return level, as

$$\mathbb{P}(M_n > R_{n,k}) = \frac{1}{k}$$

i.e., it is the level that is exceeded in one out of every $k$ $n$-blocks, on average

In extreme value terminology, $R_{n,k}$ is the **return level** associated with **return period** $1/k$

If $M_n$ are yearly maxima, then $R_{n,k}$ represents the level that is expected to be exceeded once every $k$ years

- We use the approximation

$$R_{n,k} \approx H^{-1}_{\xi,\mu,\sigma}\left(1 - \frac{1}{k}\right) = \mu + \frac{\sigma}{\xi}\left[\left\{-\log(1 - \frac{1}{k})\right\}^{-\xi} - 1\right]$$

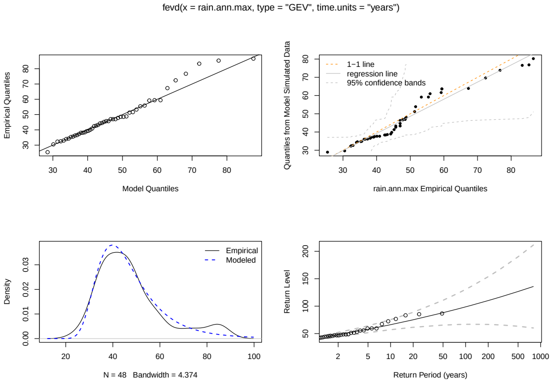
The interest is then in estimating this functional of the unknown parameters of our GEV model for maxima of $n$-blocks

# 2.15 GEV in practice: Daily rainfall in south-west England

```
library(ismev)
library(extRemes)

data(rain) #from ismev
years <- rep(1:48, rep(c(365,365,366,365), times = 12))[-17532] #period 1914 to 1962
rain.ann.max <- unlist(lapply(X = split(rain,years), FUN = max)) # annual maxima

mod <- fevd(rain.ann.max, type="GEV", time.units="years")
plot(mod)
```



# 2.16 GEV in practice: Daily rainfall in south-west England

```r
mod $results$par #gives parameters of the GEV
```

```
  location      scale      shape 
40.7830335  9.7284060  0.1072355 
```

```r
# suggests heavy-tailed model here, but only point estimates
# What about building confidence intervals?
ci.fevd(mod, alpha=0.05, type="parameter")
```

```
fevd(x = rain.ann.max, type = "GEV", time.units = "years")

[1] "Normal Approx."

         95% lower CI   Estimate 95% upper CI
location   37.6941916 40.7830335   43.8718754
scale       7.3991505  9.7284060   12.0576614
shape      -0.1055497  0.1072355    0.3200207
```

```r
# the CI includes 0, so not sure we're that heavy-tailed
gev.rl <- return.level(x = mod, return.period = c(10,100,1000),
                       do.ci = TRUE, alpha = 0.05)
gev.rl
```

```
fevd(x = rain.ann.max, type = "GEV", time.units = "years")

[1] "Normal Approx."

                       95% lower CI  Estimate 95% upper CI
10-year return level       56.67333  65.54301     74.41268
100-year return level      66.85346  98.63615    130.41884
1000-year return level     58.37286 140.34002    222.30718
```

# 2.17 GEV in practice: Daily rainfall in south-west England

What is the 10–period **return level** $R_{365,10}$? i.e., the level that is exceeded once every $10$ years, on average

$$\hat{R}_{365,10} \approx \hat{H}^{-1}_{\hat{\xi},\hat{\mu},\hat{\sigma}}(1-1/k) = 40.78 + 9.73\frac{\left[\{-\log(1-1/10)\}^{-0.11} - 1\right]}{0.11}$$

$\approx$ 65.62 mm is the estimated value of daily rainfall that can be exceeded once every 10 years

**Confidence intervals**:

- Rely on the normal approximation of the distribution of MLE + Delta method
- Rely on parametric or non-parametric bootstrap

Section 3

# 3. Non-stationary Extremes

# 3.1 Modelling Issues

Extreme value data usually show:

- Short term dependence (storms for example); clustering effect and extremal index $\rightarrow$ not covered in this short course about EVT
- Seasonality (due to annual cycles in meteorology)
- Long-term trends (due to gradual climatic change)
- Dependence on covariate effects
- Other forms of non-stationarity

For temporal dependence, there is a sufficiently wide-ranging theory which can be invoked. Other aspects have to be handled at the modelling stage

# 3.2 Non-stationarity Example: Daily mean temperature in Lausanne

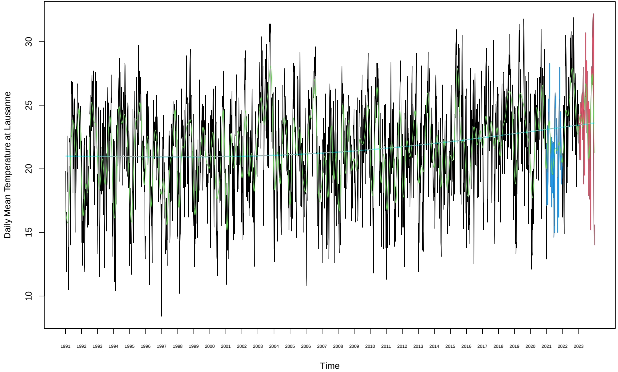



Figure 1: Daily mean temperature in Lausanne

# 3.3 Non-stationarity Example: Dailymean temperature during summer

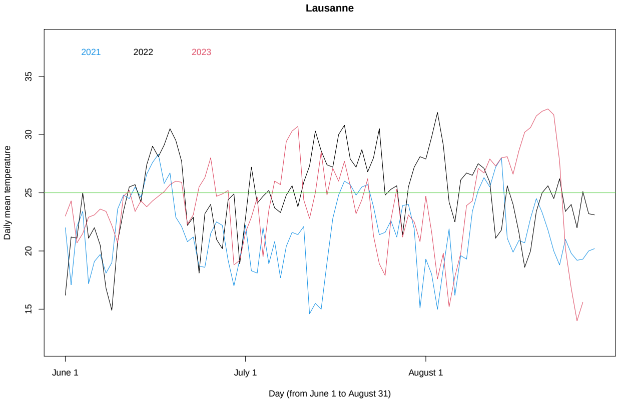



Figure 2: Daily mean temperature during Summer

# 3.4 Non-stationarity

Model trends, seasonality and covariate effects by parametric or nonparametric models for the usual extreme value model parameters

## 3.4.1 Some possibilities for parametric modelling:

- $\mu(t) = \alpha + \beta t$
- $\sigma(t) = \exp(\alpha + \beta t)$
- $\xi(t) = \begin{cases} \xi_1, & t \leq t_0 \\ \xi_2, & t > t_0 \end{cases}$
- $\mu(t) = \alpha + \beta y(t)$

# 3.5 Parameter Estimation

- **Model specification (example):**

$$Z_t \sim \mathsf{GEV}\{\mu(t), \sigma(t), \xi(t)\}$$

- **Likelihood (for complete parameter set $\beta$):**

$$L(\beta) = \prod_{t=1}^{m} h\{z_t; \mu(t), \sigma(t), \xi(t)\},$$

  where $h$ is GEV model density

- Maximization of $L$ yields maximum likelihood estimates
- Standard likelihood techniques also yield standard errors, confidence intervals etc

# 3.6 Model Reduction

- For nested models $\mathcal{M}_0 \subset \mathcal{M}_1$, the deviance statistic is:

$$D = 2\{\ell_1(\mathcal{M}_1) - \ell_0(\mathcal{M}_0)\}$$

- Based on asymptotic likelihood theory, $\mathcal{M}_0$ is rejected by a test at the $\alpha$-level of significance if $D > c_\alpha$, where $c_\alpha$ is the $(1-\alpha)$ quantile of the $\chi^2_k$ distribution, and $k$ is the difference in the dimensionality of $\mathcal{M}_1$ and $\mathcal{M}_0$

# 3.7 Example: Race times[1]

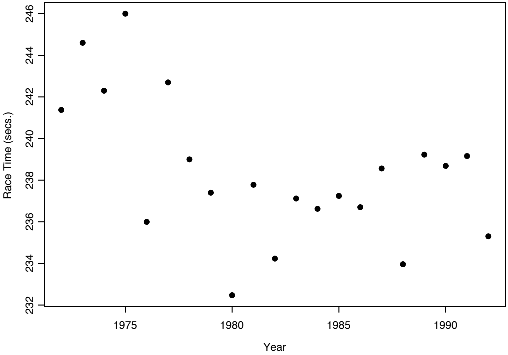



Figure 3: Annual fastest race times for women's 1500m event, with an obvious time trend

[1]From the excellent introductory book: Coles, 2001

# 3.8 Example: Race times

| Model | Log-likelihood | $\hat{\beta}$ | $\hat{\sigma}$ | $\hat{\xi}$ |
|---|---|---|---|---|
| Constant | -54.5 | 239.3 | 3.63 | -0.469 |
| | | (0.9) | (0.64) | (0.141) |
| Linear | -51.8 | $(242.9, -0.311)$ | 2.72 | -0.201 |
| | | (1.4, 0.101) | (0.49) | (0.172) |
| Quadratic | -48.4 | $(247.0, -1.395, 0.049)$ | 2.28 | -0.182 |
| | | (2.3, 0.420, 0.018) | (0.45) | (0.232) |

Quadratic model appears preferable

# 3.9 Example: Race times

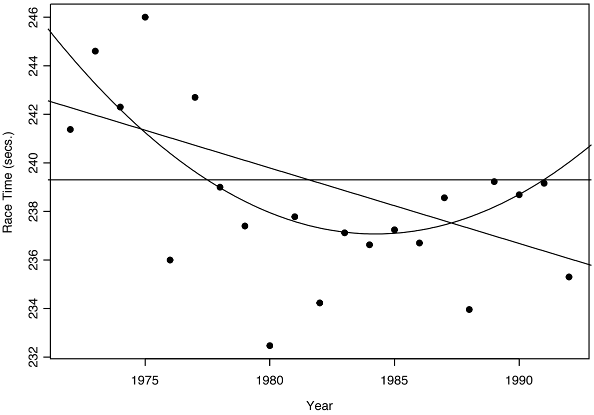


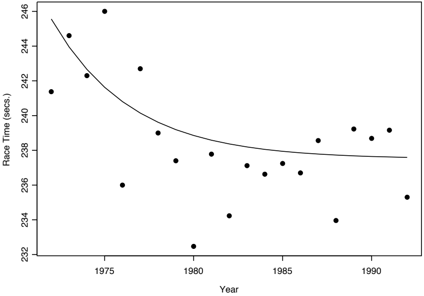

Figure 4: Fitted models for location parameter in women's 1500 metre race times. Note quadratic model would lead to slower races in recent and future events

# 3.10 Example: Race times

Alternative exponential model

$$\tilde{\mu}(t) = \beta_0 + \beta_1 e^{-\beta_2 t}$$

has log-likelihood $-49.5$. Not as good as the quadratic model, though comparison via likelihood ratio test is invalid as models are not nested. Better behaviour for large $t$ suggests a preferable model though



# 3.11 Example: Spatial modelling of rainfall extremes

```r
library(evgam)
library(knitr)

data("COprcp", package = "evgam")
COprcp      <- cbind(COprcp, COprcp_meta[COprcp$meta_row, ])
COprcp$year <- format(COprcp$date, "%Y")
COprcp_gev  <- aggregate(prcp ~ year + meta_row, COprcp, max)
COprcp_gev  <- cbind(COprcp_gev, COprcp_meta[COprcp_gev$meta_row, ])

head(COprcp_gev)
```

```
    year meta_row prcp          id        name       lon     lat   elev
1   1990        1 43.2 USC00050263 ANTERO RSVR -105.8919 38.9933 2718.8
1.1 1991        1 14.7 USC00050263 ANTERO RSVR -105.8919 38.9933 2718.8
1.2 1992        1 44.7 USC00050263 ANTERO RSVR -105.8919 38.9933 2718.8
1.3 1993        1 11.2 USC00050263 ANTERO RSVR -105.8919 38.9933 2718.8
1.4 1994        1 30.5 USC00050263 ANTERO RSVR -105.8919 38.9933 2718.8
1.5 1995        1 26.7 USC00050263 ANTERO RSVR -105.8919 38.9933 2718.8
```

```r
tail(COprcp_gev)
```

```
      year meta_row prcp          id          name       lon   lat   elev
64.24 2014       64 21.6 USW00093058 PUEBLO MEM AP -104.4983 38.29 1438.7
64.25 2015       64 41.9 USW00093058 PUEBLO MEM AP -104.4983 38.29 1438.7
64.26 2016       64 27.7 USW00093058 PUEBLO MEM AP -104.4983 38.29 1438.7
64.27 2017       64 38.4 USW00093058 PUEBLO MEM AP -104.4983 38.29 1438.7
64.28 2018       64 16.8 USW00093058 PUEBLO MEM AP -104.4983 38.29 1438.7
64.29 2019       64 42.4 USW00093058 PUEBLO MEM AP -104.4983 38.29 1438.7
```

# 3.12 Example: Spatial modelling of rainfall extremes

```r
library(evgam)

fmla_gev <- list(prcp ~ s(lon, lat, k = 30) + s(elev, bs = "cr"),
                 ~ s(lon, lat, k = 20), ~ 1) #formula for each GEV parameter
m_gev    <- evgam(fmla_gev, COprcp_gev, family = "gev") #fit the model
summary(m_gev)
```

```
** Parametric terms **

location
            Estimate Std. Error t value Pr(>|t|)
(Intercept)    28.56       0.26  111.89   <2e-16

logscale
            Estimate Std. Error t value Pr(>|t|)
(Intercept)     2.24       0.02  118.07   <2e-16

shape
            Estimate Std. Error t value Pr(>|t|)
(Intercept)     0.08       0.02    5.08 1.92e-07

** Smooth terms **

location
             edf max.df Chi.sq Pr(>|t|)
s(lon,lat) 19.27     29 178.23   <2e-16
s(elev)     5.19      9  19.39  0.00139

logscale
             edf max.df Chi.sq Pr(>|t|)
s(lon,lat) 13.94     19 211.15   <2e-16
```

# 3.13 Example: Spatial modelling of rainfall extremes