Week 13: Tree-based Methods for Classification

MATH-517 Statistical Computation and Visualization

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What is Classification?

Given data on predictor variables (covariates/features) $X \in \mathbb{R}^p$ and a **categorical response variable** $Y \in \{0, \ldots, J-1\}$, build a model for

- predicting the value of the response from the predictors
- understanding the relationship between predictors and the response

Examples:

- X: diagnostic measurements and Y: presence/absence of disease
- X: credit score, age, marital status and Y: loan defaults (yes/no)

Classification Methods:

- Linear discriminant analysis (1930')
- Logistic regression (1944)
- Nearest neighbors classifiers (1951)

What is Classification?

Given data which are realization from

$$(X_1, Y_1), \ldots, (X_N, Y_N) \quad \text{i.i.d.},$$

the goal is to assign probabilities

$$\pi_k(x) = P(Y = k \mid X = x), \quad \text{for } k = 0, \ldots, J-1$$

where x can be a newly observed predictor (prediction)

$\Rightarrow$ similar to the regression function $m(x) = \mathbb{E}(Y \mid X = x)$

The Bayes Classifier

- A classifier $\mathcal{C} : \mathbb{R}^p \to \{0, \dots, J-1\}$ assigns to a predictor X a class, i.e., its prediction for the corresponding Y
- The quality of a classifier can be measured by the expected 0-1 loss

$$P\{\mathcal{C}(X_{new}) \neq Y_{new}\}$$

- The optimal classifier wrt this loss is the **Bayes classifier**

$$\mathcal{C}_{Bayes}(x) = \underset{0 \leq k \leq J-1}{\arg\max}\, \pi_k(x)$$

$\Rightarrow$ the lowest risk is obtained by classifying x to the most probable class

In practice, $\pi_k(\cdot)$ (depends on the joint df of (X, Y)) needs to be estimated and plugged into the classifier $\mathcal{C}_{Bayes}$

Let's estimate it non-parametrically while imposing some structural assumptions

Tree-based Methods

Predict y from a feature vector $x \in \mathbb{R}^p$ by dividing the feature space into (non-overlapping) rectangles $A_1, \ldots, A_m$

$\Rightarrow$ works if y is discrete (classification) or continuous (regression)

Rectangles can be achieved by making successive binary splits on the predictors $X_1, \ldots X_p$

- choose a variable X_j, $j = 1, \ldots p$
- divide up the feature space according to

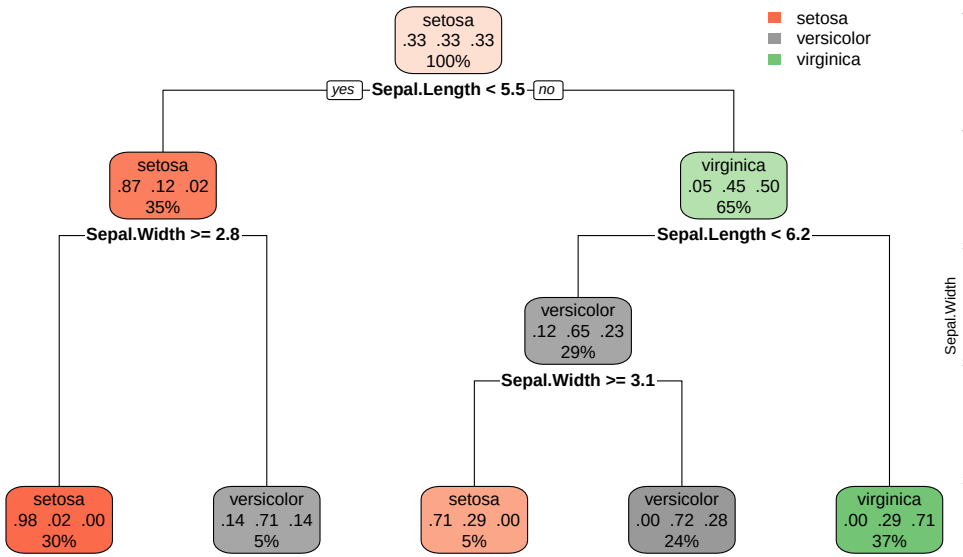
$$X_j \leq s \text{ and } X_j > s$$

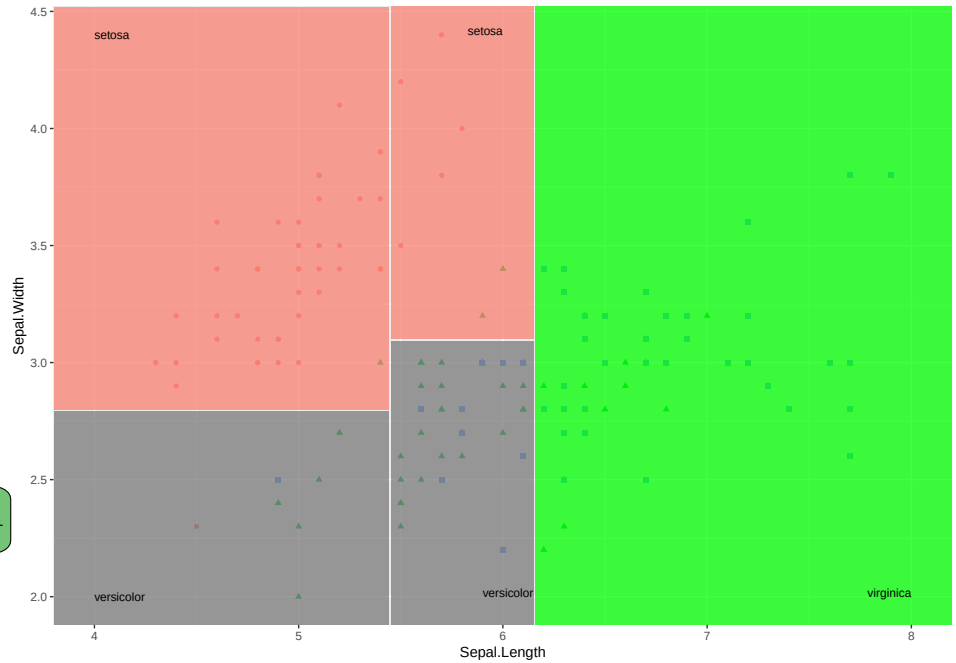
- proceed in each half

Questions: How to choose the splits? When to stop growing the tree?

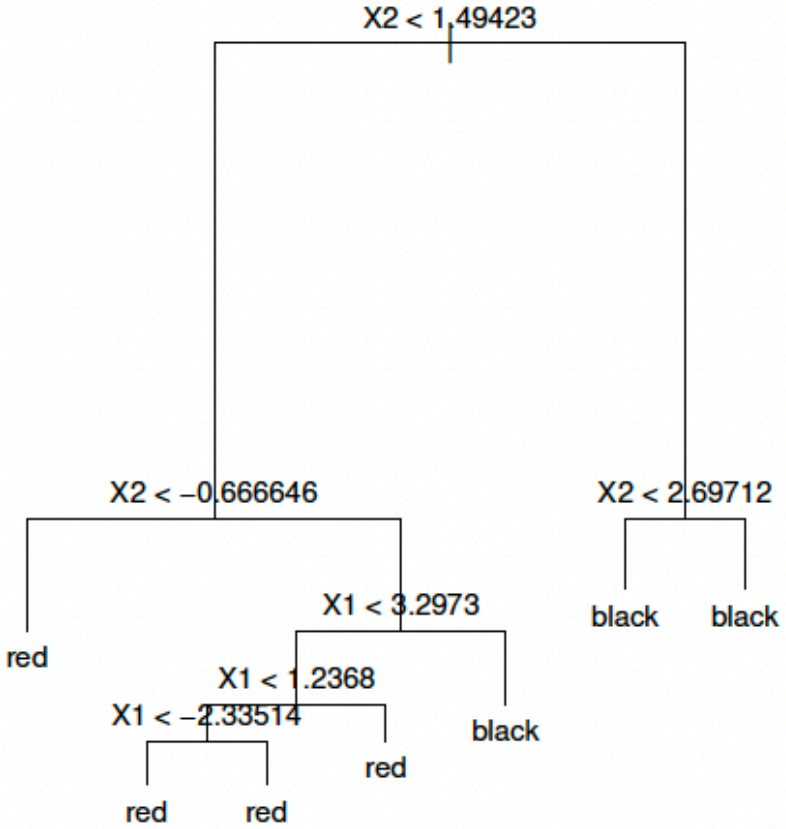
Classification Tree: Example

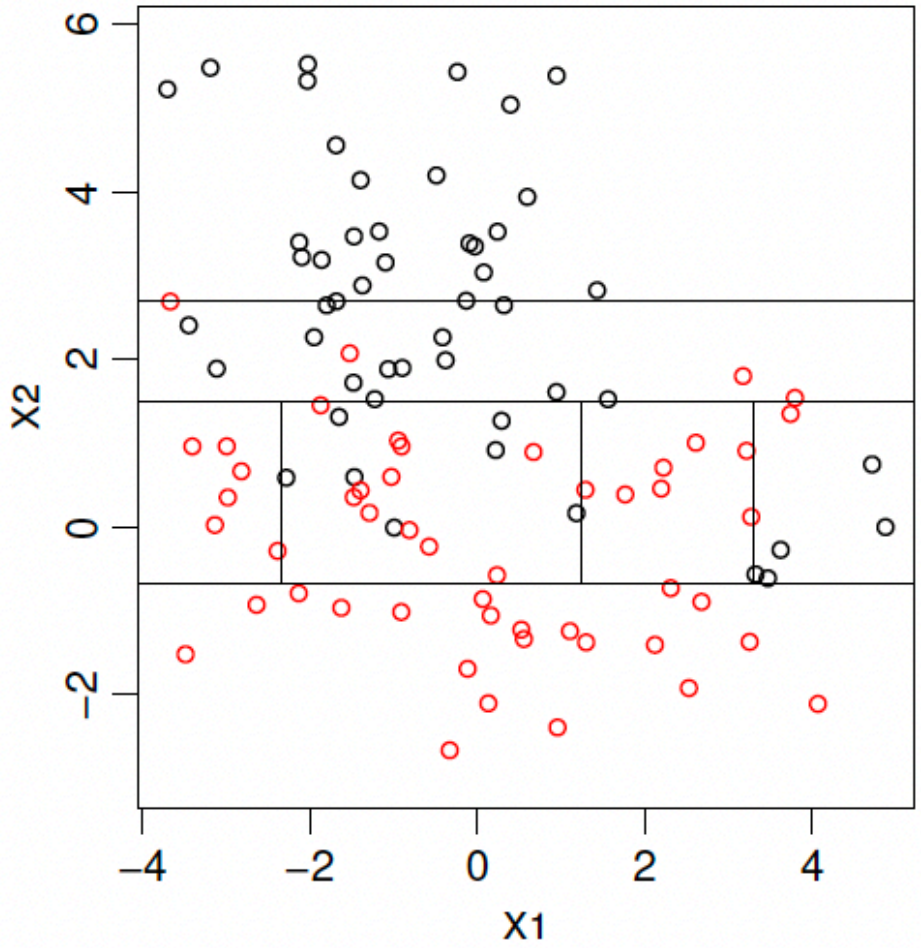
The iris dataset with four features (petal/sepal length and width) and three species





Classification Tree: Simulated Example





Terminology

- each split is called a **node**
- a terminal node is called a **leaf**
- interior nodes lead to **branches**

Classification Trees

Classification trees are popular because they are interpretable and (perhaps) mimic the way (some) decisions are made

A classification tree can be thought of as defining m regions (rectangles) $A_1, \dots A_M$, each corresponding to a leaf of the tree

- each A_m is assigned a class label $c_m \in \{0, \dots J-1\}$ by **majority vote** (the most common class in that region)
- then a new point $x_+ \in \mathbb{R}^p$ is classified by

$$T(x_+) = \sum_{m=1}^{M} c_m \cdot \mathbb{I}_{\{x_+ \in A_m\}} = c_m \text{ such that } x_+ \in A_m \subset \mathbb{R}^p$$

Finding out which region a given point x belongs to is easy since the regions A_m are defined by a tree: just scan down the tree

Tricky part: get a data-driven estimate of the partition: splitting variables? split points?

Predicted Class Probabilities

We can get the predicted class for new points, but also the **predicted class probability**

For each class $k = 0, \ldots, J-1$, we can estimate the probability that the class label is k given that the feature vector lies in region A_m, $P(Y = k \mid X \in A_m)$ by

$$p_{mk} = \hat{p}_k(A_m) = \frac{1}{n_m} \sum_{x_i \in A_m} \mathbb{I}_{\{y_i = k\}}$$

the proportion of points in the region A_m that are of class k, where $n_m = \#\{(x_i, y_i) \mid x_i \in A_m\}$

The predicted class (by majority vote) can be expressed as

$$c_m = \underset{k=0,\ldots,J-1}{\arg\max}\ p_{mk}$$

How to Grow a Tree?

The **CART algorithm** [1] estimates the tree model

$$T(x) = \sum_{m=1}^{M} c_m \cdot \mathbb{I}_{\{x \in A_m\}}$$

using a greedy approach (local optimality/stage) based on binary splits

Starting at the top, for each coordinate $j \in \{1, \dots, p\}$ we look for the best binary split defining

$$A_1(j, s) = \{x \in \mathbb{R}^p : x_j \leq s\} \quad \text{and} \quad A_2(j, s) = \{x \in \mathbb{R}^p : x_j > s\}$$

$\Rightarrow$ The values of $j \in \{1, \dots, p\}$ and $s \in \mathbb{R}$ are found by minimizing

$$\min_{j,s} \{Q_1(T) + Q_2(T)\}$$

where $Q_m(T)$ is a **node impurity measure** (loss function)

[1] Breiman et al. (1984), "Classification and Regression Trees"

Node Impurity Measures for Classification

Recall that p_{mk} is the proportion of training observations in A_m that are from class k

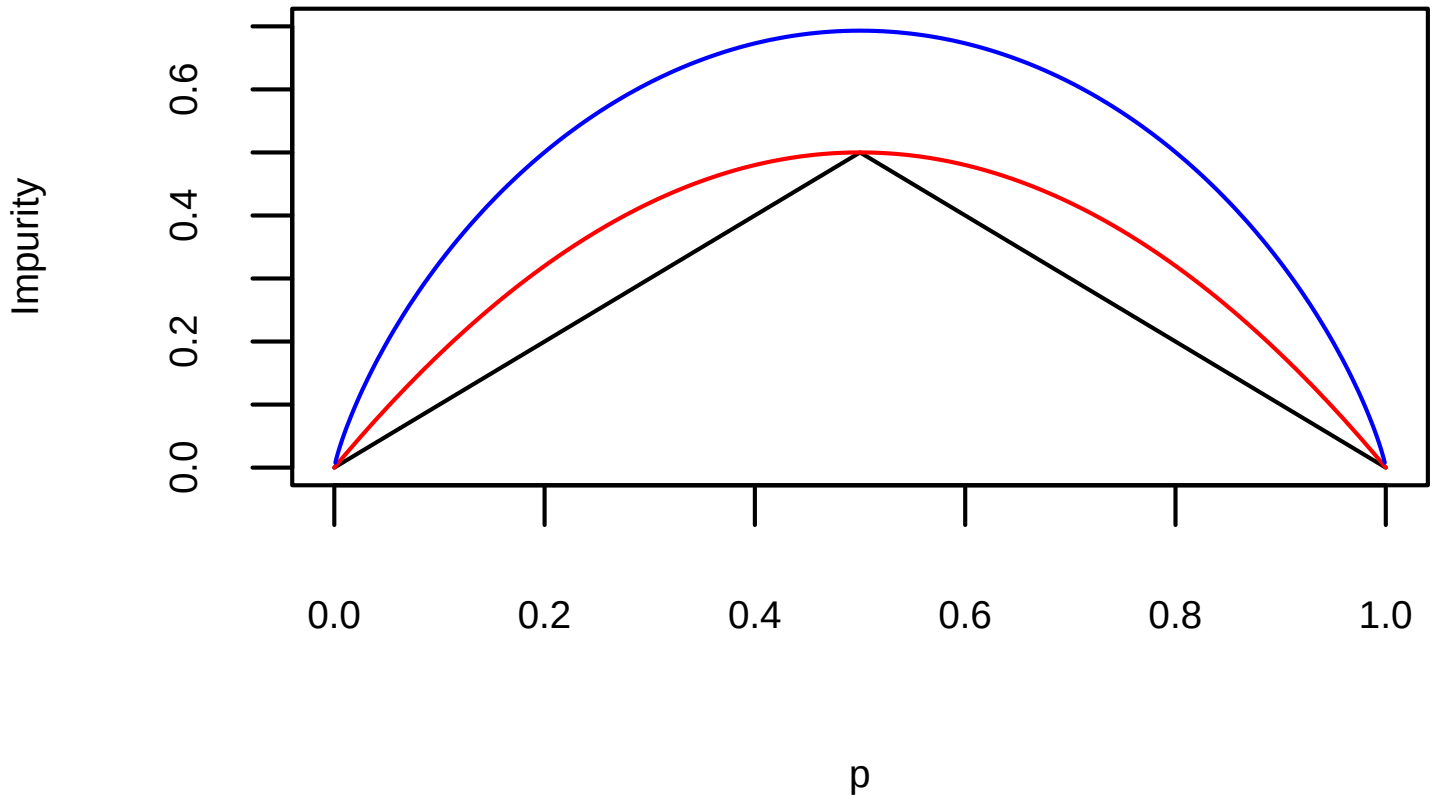
- misclassification error: $\frac{1}{n_m} \sum_{i:x_i \in A_m} \mathbb{I}_{\{y_i \neq c_m\}} = 1 - p_{mc_m}$
- Gini index: $\sum_{k \neq k'} p_{mk} p_{mk'} = \sum_{k=0}^{J-1} p_{mk}(1 - p_{mk})$
- Cross-entropy (or deviance): $-\sum_{k=0}^{J-1} p_{mk} \log(p_{mk})$

For two classes ($J = 2$)

- misclassification error: $1 - \max(p, 1-p)$ (black)
  - is non-differentiable (bad for numerical optimization)
- Gini index: $2p(1-p)$ (red)
- Cross-entropy (or deviance): $-p \log(p) - (1-p) \log(1-p)$ (blue)

Growing a tree is based on either the Gini index or cross-entropy

Why not minimize the misclassification error?



- Gini index and cross-entropy are more sensitive to small changes: going from $p = 0.8$ to $p = 0.9$ is better than going from $p = 0.1$ to $p = 0.2$ (these are equal changes for the misclassification error)

$\Rightarrow$ the Gini index and the cross-entropy will favour pure nodes with $p_{mk} \approx 0$ or $p_{mk} \approx 1$

How large should we grow the tree?

- very large tree might overfit the data
- small tree might not capture the important structure

$\Rightarrow$ Tree size is a tuning parameter reflecting the model's complexity

Pruning:

- built a large tree T_0, stopping only when the number of observations in each leaf is small (for ex. 5)
- prune this large tree, i.e., collapse some of its leaves into the parent nodes (backward elimination)

Alternative to pruning: grid search for the optimal maximal depth of the tree by cross-validation (minimizing the misclassification rate)

Pruning, by how much?

For any subtree $T \subset T_0$ that can be obtained by pruning T_0, we define the **cost-complexity pruning**:

$$C_\lambda(T) = \text{err}_T + \lambda|T|, \quad \lambda \geq 0$$

where $|T| = \#$ leaves in T and err_T is the misclassification rate

For any value of λ we need to find the tree T_λ minimizing $C_\lambda(T)$

$\Rightarrow$ done efficiently by slowly pruning the tree, i.e., constructing the sequence of pruned trees that slowly increase the misclassification rate

- successively delete the terminal node in the fully grown tree that yields the smallest increase of the misclassification rate

Choice of λ : trade-off between goodness-of-fit and complexity

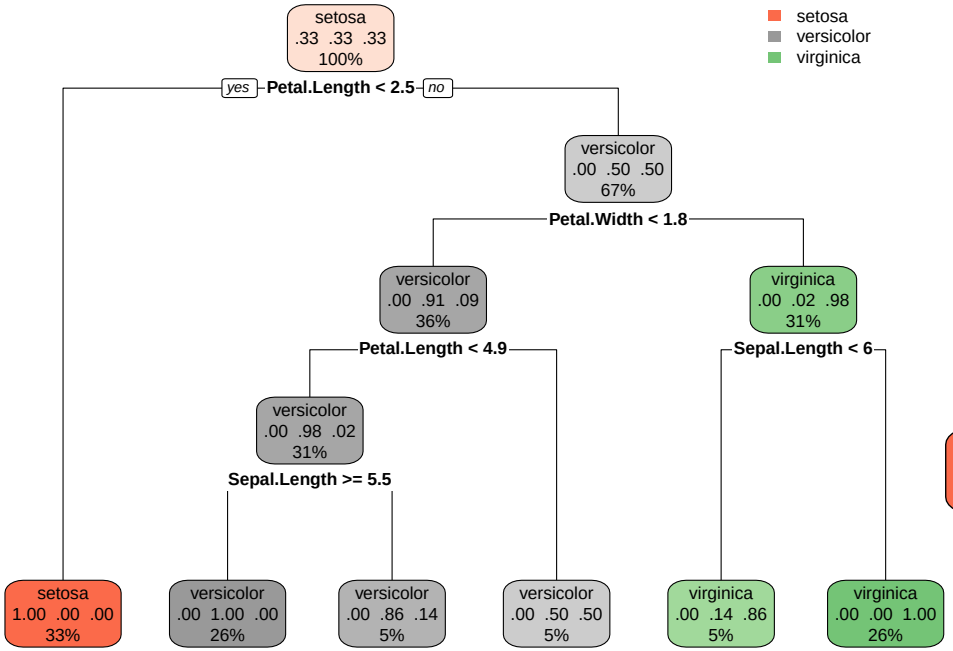
- a larger size means smaller bias and high variance
- a smaller tree means larger bias and smaller variance

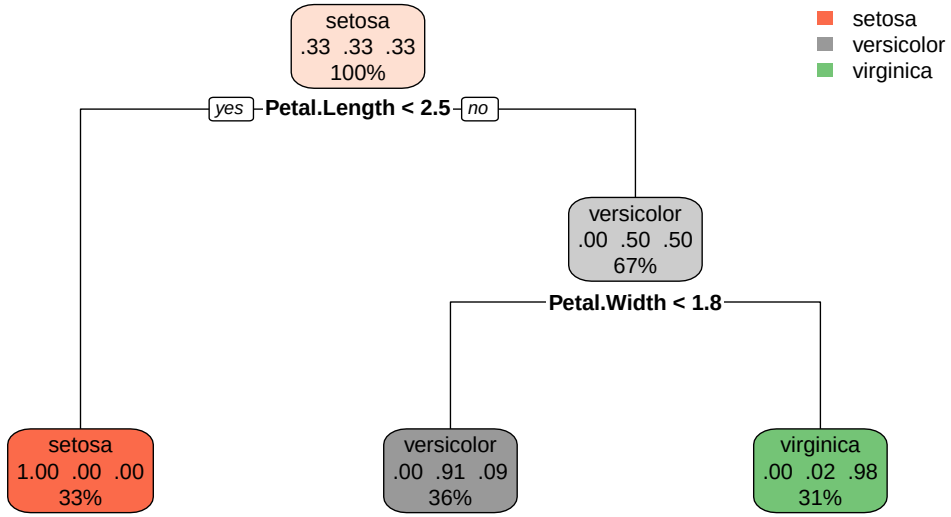
$\Rightarrow$ the value of λ will be chosen by K-fold CV error rates

Pruning: Example

Left: fully grown classification tree (using Gini index)

Right: pruned tree found by CV (using misclassification error)





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- Pros of classification trees?
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  - missing values are dealt with by "surrogate splits" (exploit correlations between covariates)
  - model free and easy to interpret
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- Pros of classification trees?
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  - missing values are dealt with by "surrogate splits" (exploit correlations between covariates)
  - model free and easy to interpret
  - qualitative covariates are easily handled
- Cons of classification trees?
  - classification accuracy is not great
    - tend to have high variance: small change in the training data can produce big changes in the estimated tree

$\Rightarrow$ this can be fixed if we are willing to give up interpretability

How to Fix This?

- Let's think back to CV, and why it gives much better results than the validation set approach
- Validation set: if you pick a different random split, you can get wildly different estimates of test error
- K-fold CV produces much more stable error estimates by averaging over K separate estimates of error
- The idea of **Bagging (Bootstrap AGGregatING)** has a similar motivation: to decrease the variance of a high-variance estimator, we can average across a bunch of estimators

Bagging [2]

For a model $\widehat{f} : x \mapsto \widehat{f}(x) = \widehat{y}$, e.g., $\widehat{f} = \widehat{T}$

- resample the training data $\mathcal{D} = \{(x_i, y_i)\}_{i=1}^{N}$ to create B artificial datasets $\mathcal{D}^{(b)} = \left\{(x_i, y_i)^{(b)}\right\}$
  - $\mathcal{D}^{(b)}$ might have the same size N (sample with replacement: bootstraping)
  - or $\mathcal{D}^{(b)}$ might be smaller than N (sample without replacement: subsampling)
- train a model $\widehat{f}^{(b)}$ on each $\mathcal{D}^{(b)}$
- perform **bagging**: "aggregate" the models $\left\{\widehat{f}^{(b)}\right\}$, i.e., for an input x_+, predict by majority vote:

$$\widehat{y}_+ = \arg\max_k \# \left\{\widehat{f}^{(b)}\left(x_+\right) = k\right\}$$

[2] Breiman (1996) "Bagging predictors"

Bagging

Bias-variance tradeoff of bagging:

- typically reduces variance
  - $f^{(b)}$ are dependent: if they are highly correlated then the variance reduction will be small
- typically increases bias
- generally, the increase in bias is smaller than the reduction in variance

Bagging Trees

Bagging tree algorithm:

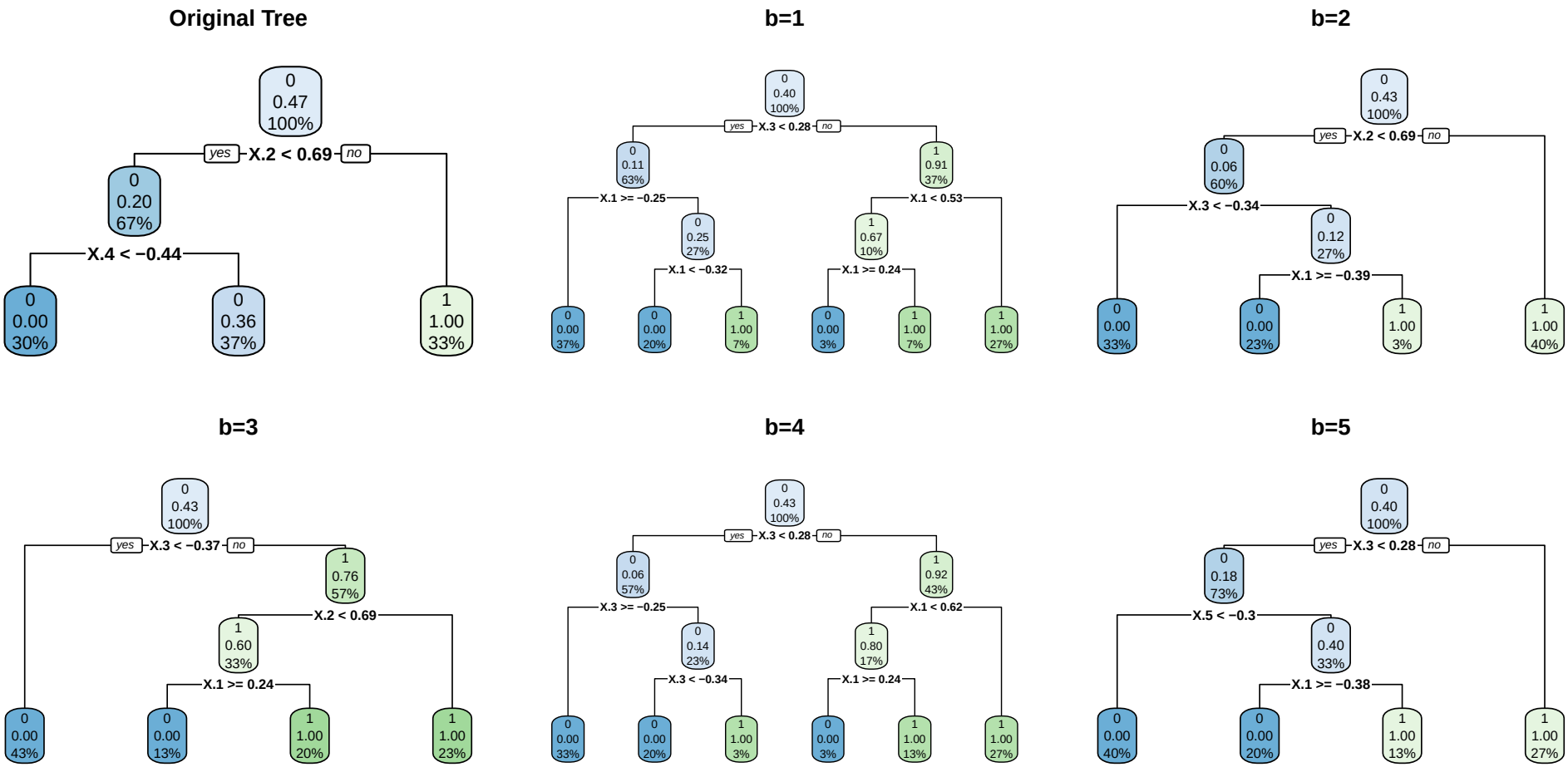
- choose B large (usually 500)
- for $b = 1, \ldots, B$, fit **unpruned trees** $\widehat{T}^{(b)}$ to the bth bootstrap sample (or subsample)
- "aggregate" the trees $\left\{\widehat{T}^{(b)}\right\}$, i.e., for an input x_+, predict by majority vote (from the B trees)

$$\hat{y}_+ = \arg\max_k \# \left\{\widehat{T}^{(b)}\left(x_+\right) = k\right\}$$

Why does it work well?

- each unpruned tree has low bias but high variance
- the correlation between the trees is typically small when using bootstrap samples

Bagging Trees: Example



Predicted Class Probabilities?

Aim: probability estimate $\hat{\pi}_k(x)$ from the bagging tree

- we can consider the proportion of bootstrapped trees that voted for class k

$$\hat{\pi}_k^{vote}(x) = \frac{1}{B}\sum_{b=1}^{B}\{\widehat{T}^{(b)}(x) = k\}$$

$\Rightarrow$ bad idea...

Suppose we have two classes, and the true probability that $y_0 = 1$ when $X = x_0$ is 0.75

Suppose each of the bagged trees correctly classifies x_0 to class 1

$\Rightarrow \hat{\pi}_1^{vote}(x_0) = 1$, which is wrong!

Instead, we can use each tree's predicted class probabilities: probability bagging

Predicted Class Probabilities?

Instead of just looking at the class predicted by each tree (the classification itself), look at the predicted class probabilities $\hat{\pi}_k^{(b)}(x)$

- Define the bagging estimate of class probabilities:

$$\hat{\pi}_k^{\text{bag}}(x) = \frac{1}{B}\sum_{b=1}^{B}\hat{\pi}_k^{(b)}(x) \quad k = 1, \dots K$$

- Given an input vector x_0, we can classify it according to

$$\hat{y}_0^{\text{bag}} = \underset{k=0,\dots J-1}{\arg\max}\, \hat{\pi}_k^{\text{bag}}(x)$$

$\Rightarrow$ preferred if we want to estimate class probabilities, and it may improve overall classification accuracy (compared to majority vote)

Predicted Class Probabilities?

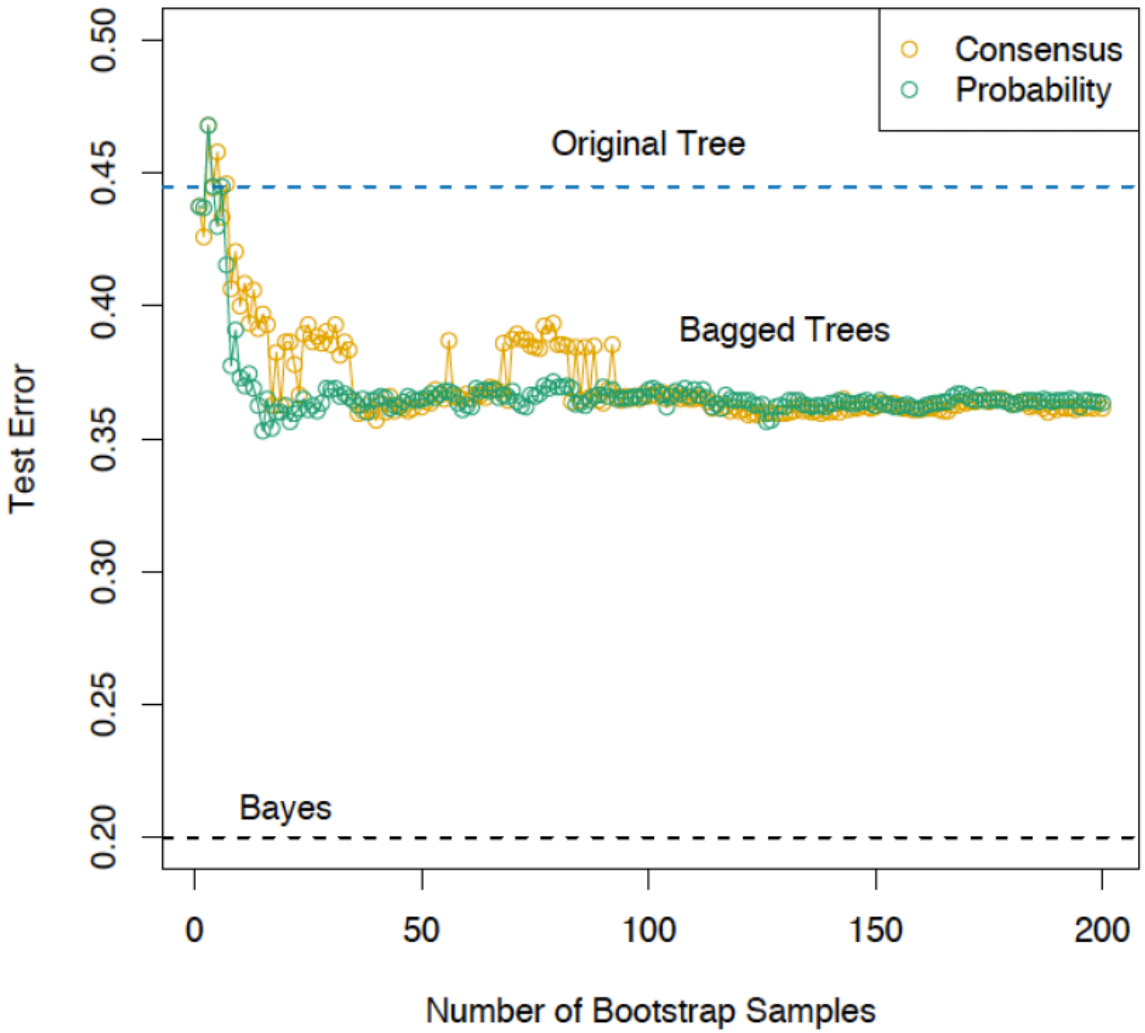


Figure 8.10 from ESL

Improvements? Random Forests (Breiman, 2001)

Random forests extend bagging by incorporating a small tweak

$\Rightarrow$ decrease correlations of bagged trees by making them "more random"

$\Rightarrow$ decreases the variance

Random forest algorithm:

- bootstrap the data B times
- to grow each bagged tree, before performing each split, randomly select m of the p variables to be used for the split
  - the subset of variables changes at each split
  - grow full, unpruned trees
- for prediction: majority vote from the B trees

Random Forests

Intuition: if one variable is much more important than the others then all bagged trees will select this variable for the first split, making these trees similar (hence correlated). Selecting a random subset of m variables for each split avoids this!

Choice of m: $m = \lfloor \sqrt{p} \rfloor$ for classification seems to work well in practice

- e.g., if we have 100 predictors, each split will be allowed to choose from among 10 randomly selected predictors

Note: bagging is a special case of random forests with $m = p$

Pros and Cons of Random Forests

- **Pros**:
  - great predictive performance
  - almost no tuning required
  - out-of-bag (oob) error estimates (no CV)
    - use the $e^{-1}\% \approx 37\%$ data not selected in the bth bootstrap sample to estimate the prediction error from the bth tree
    - can be shown to be equivalent to CV
  - variable importance
    - compute the importance a the jth variable X_j by randomly shuffling its values for the oob data and then measuring the increase in prediction error/decrease in accuracy
    - the higher the increase, the most important is the variable
- **Cons**:
  - lose the interpretability of a single tree

Final Thoughts

- Bagging
  - improves the prediction accuracy for high variance (and low bias) models (such as classification trees) at the expense of interpretability and computational speed
  - consists of independent processes $\Rightarrow$ algorithm is easily parallelizable
  - results in (very) correlated trees $\Rightarrow$ variance reduction is limited
- Random Forests
  - decrease the correlation between bagged trees by considering a random subset of the features/predictors/covariates

  $\Rightarrow$ faster than bagging
  - little theory but consistency was proved and a method to obtain CI was proposed
- We didn't discuss **Boosting** that builds up the ensemble sequentially
  - e.g., to boost trees, we grow small trees, one at a time, at each step trying to improve the model fit in places we've done poorly so far
  - still lose interpretability but like RF and bagging, captures complex structures in the data (vs additive models, e.g., logistic regression)

References

- T. Hastie, R. Tibshirani and J. Friedman (2008) The Elements of Statistical Learning (2nd Edition)
- G. James, D. Witten, T. Hastie and R. Tibshirani (2013) An Introduction to Statistical Learning, with applications in *R*