MATH-517: Assignment 3

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1.Theoretical Exercise

1.1.

The problem is a weighted least squares problem at x:

$$(\hat{\beta}_0(x), \hat{\beta}_1(x)) = \arg\min_{\beta_0,\beta_1\in\mathbb{R}} \sum_{i=1}^{n} (Y_i - \beta_0 - \beta_1(X_i - x))^2 \, K\left(\frac{X_i - x}{h}\right).$$

with

$$X = \begin{pmatrix} 1 & X_1 - x \\ \vdots & \vdots \\ 1 & X_n - x \end{pmatrix}, \quad W = \mathrm{diag}(K\left(\frac{X_1 - x}{h}\right), \dots, K\left(\frac{X_n - x}{h}\right)), \quad Y = \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix}.$$

The problem can be rewritten as

$$(\hat{\beta}_0(x), \hat{\beta}_1(x)) = \arg\min_{\beta_0,\beta_1\in\mathbb{R}} (Y - X\beta)^\top W (Y - X\beta),$$

The weighted least squares solution is

$$\begin{pmatrix} \hat{\beta}_0(x) \\ \hat{\beta}_1(x) \end{pmatrix} = (X^\top W X)^{-1} X^\top W Y.$$

Let $e_1 = (1, 0)^\top$. Then

$$\hat{m}(x) = \hat{\beta}_0(x) = e_1^\top (X^\top W X)^{-1} X^\top W Y = \sum_{i=1}^{n} w_{ni}(x)\, Y_i,$$

where $w_{ni}(x)$ is the i-th entry of the row vector $e_1^\top (X^\top W X)^{-1} X^\top W$.

1.2.

Using the notation

$$S_{n,k}(x) = \frac{1}{nh} \sum_{i=1}^{n} (X_i - x)^k K\left(\frac{X_i - x}{h}\right)$$

we can write

$$X^\top W X = nh \begin{pmatrix} S_{n,0}(x) & S_{n,1}(x) \\ S_{n,1}(x) & S_{n,2}(x) \end{pmatrix}.$$

The inverse is

$$(X^\top W X)^{-1} = \frac{1}{nh(S_{n,0}(x) S_{n,2}(x) - S_{n,1}(x)^2)} \begin{pmatrix} S_{n,2}(x) & -S_{n,1}(x) \\ -S_{n,1}(x) & S_{n,0}(x) \end{pmatrix}.$$

then $X^\top W$ is the $2 \times n$ matrix

$$X^\top W = \begin{pmatrix} K\left(\frac{X_1-x}{h}\right) & \cdots & K\left(\frac{X_n-x}{h}\right) \\ (X_1-x)K\left(\frac{X_1-x}{h}\right) & \cdots & (X_n-x)K\left(\frac{X_n-x}{h}\right) \end{pmatrix}.$$

the product $(X^\top WY)^{-1}X^\top W$ is then

$$\frac{1}{nh(S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2)} \begin{pmatrix} (S_{n,2}(x) - (X_1-x)S_{n,1}(x))K\left(\frac{X_1-x}{h}\right) & \cdots \\ (-S_{n,1}(x) + (X_1-x)S_{n,0}(x))K\left(\frac{X_1-x}{h}\right) & \cdots \end{pmatrix}$$

The i-th entry of the first row is then

$$w_{ni}(x) = \frac{1}{nh}\frac{S_{n,2}(x) - (X_i-x)S_{n,1}(x)}{S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2}K\left(\frac{X_i-x}{h}\right).$$

1.3.

The sum of the weights is

$$\sum_{i=1}^n w_{ni}(x) = \sum_{i=1}^n w_{ni}(x) = \frac{1}{nh}\,\frac{S_{n,2}(x)\sum_{i=1}^n K\left(\frac{X_i-x}{h}\right) - S_{n,1}(x)\sum_{i=1}^n (X_i-x)K\left(\frac{X_i-x}{h}\right)}{S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2}.$$

Since $\sum_{i=1}^n K\left(\frac{X_i-x}{h}\right) = nh\,S_{n,0}(x)$ and $\sum_{i=1}^n (X_i-x)K\left(\frac{X_i-x}{h}\right) = nh\,S_{n,1}(x)$, We have

$$\sum_{i=1}^n w_{ni}(x) = \frac{S_{n,2}(x)S_{n,0}(x) - S_{n,1}(x)^2}{S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2} = 1.$$

2. Practical Exercise