MATH-517: Assignment 3

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Theoretical Exercise

Local Linear Regression as a Linear Smoother

Setup: We observe i.i.d. samples (X_i, Y_i) from the model

$$Y_i = m(X_i) + \varepsilon_i, \qquad i \in \{1, \dots, n\}$$

and we estimate m at a target point $x \in \mathbb{R}$ via the local linear estimator we defined in class:

$$(\hat{\beta}_0(x), \hat{\beta}_1(x)) = \arg\min_{\beta_0, \beta_1} \sum_{i=1}^{n} \Big(Y_i - \beta_0 - \beta_1(X_i - x)\Big)^2 K\Big(\tfrac{X_i - x}{h}\Big)$$

where K is a kernel and $h > 0$ a bandwidth. The fitted value is denoted $\hat{m}(x) = \hat{\beta}_0(x)$.

We also define

$$S_{n,k}(x) = \frac{1}{nh} \sum_{i=1}^{n} (X_i - x)^k K\Big(\tfrac{X_i - x}{h}\Big), \quad k \in \{0, 1, 2\}$$

$$T_{n,k}(x) = \frac{1}{nh} \sum_{i=1}^{n} (X_i - x)^k K\Big(\tfrac{X_i - x}{h}\Big) Y_i, \quad k \in \{0, 1\}$$

$$K_i := K\Big(\tfrac{X_i - x}{h}\Big), \quad i = 1, \dots, n$$

and

$$\Delta(x) = S_{n,0}(x) S_{n,2}(x) - S_{n,1}(x)^2$$

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1. Show that $\hat{m}(x)$ can be expressed as a weighted average of the observations

Let

$$X = \begin{bmatrix} 1 & X_1 - x \\ \vdots & \vdots \\ 1 & X_n - x \end{bmatrix}, \quad W = \operatorname{diag}(K_1, \dots, K_n), \quad y = \begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

By weighted least squares, we get

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{bmatrix} = (X^\top W X)^{-1} X^\top W y$$

Computation yields

$$X^\top W X = \begin{bmatrix} \sum_i K_i & \sum_i (X_i - x) K_i \\ \sum_i (X_i - x) K_i & \sum_i (X_i - x)^2 K_i \end{bmatrix} = nh \begin{bmatrix} S_{n,0} & S_{n,1} \\ S_{n,1} & S_{n,2} \end{bmatrix}$$

$$X^\top W y = \begin{bmatrix} \sum_i K_i Y_i \\ \sum_i (X_i - x) K_i Y_i \end{bmatrix} = nh \begin{bmatrix} T_{n,0} \\ T_{n,1} \end{bmatrix}$$

Hence, with $\Delta = \Delta(x)$,

$$(X^\top W X)^{-1} = \frac{1}{nh} \cdot \frac{1}{\Delta} \begin{bmatrix} S_{n,2} & -S_{n,1} \\ -S_{n,1} & S_{n,0} \end{bmatrix}$$

Therefore,

$$\hat{\beta} = \frac{1}{\Delta} \begin{bmatrix} S_{n,2} & -S_{n,1} \\ -S_{n,1} & S_{n,0} \end{bmatrix} \begin{bmatrix} T_{n,0} \\ T_{n,1} \end{bmatrix}, \qquad \hat{m}(x) = \hat{\beta}_0(x) = \frac{S_{n,2} T_{n,0} - S_{n,1} T_{n,1}}{\Delta}$$

Expanding $T_{n,0}, T_{n,1}$ shows

$$\hat{m}(x) = \sum_{i=1}^{n} w_{ni}(x) Y_i$$

so $\hat{m}(x)$ is a linear smoother. $\square$

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2. Explicit weights $w_{ni}(x)$

From the previous definitions and calculations:

$$\hat{m}(x) = \frac{S_{n,2}\,\frac{1}{nh}\sum_i K_i Y_i - S_{n,1}\,\frac{1}{nh}\sum_i (X_i - x) K_i Y_i}{\Delta} = \sum_{i=1}^{n} \left[\frac{1}{nh} \cdot \frac{K_i (S_{n,2} - (X_i - x) S_{n,1})}{\Delta} \right] Y_i$$

Thus,

$$\boxed{w_{ni}(x) = \frac{1}{nh} \frac{K\!\left(\frac{X_i - x}{h}\right)(S_{n,2}(x) - (X_i - x) S_{n,1}(x))}{S_{n,0}(x) S_{n,2}(x) - S_{n,1}(x)^2}}$$

These weights depend only on x, X_i, K, h (not on the Y_i's).

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3. Prove that $\sum_{i=1}^{n} w_{ni}(x) = 1$

We sum the weights:

$$\sum_{i=1}^{n} w_{ni}(x) = \frac{1}{nh} \cdot \frac{1}{\Delta} \left(S_{n,2} \sum_{i=1}^{n} K_i - S_{n,1} \sum_{i=1}^{n} (X_i - x) K_i \right).$$

But

$$\sum_i K_i = nh\, S_{n,0}, \qquad \sum_i (X_i - x) K_i = nh\, S_{n,1}.$$

So,

$$\sum_{i=1}^{n} w_{ni}(x) = \frac{1}{nh} \cdot \frac{1}{\Delta} \left(S_{n,2}\,(nh S_{n,0}) - S_{n,1}\,(nh S_{n,1}) \right) = \frac{S_{n,2} S_{n,0} - S_{n,1}^2}{\Delta} = \frac{\Delta}{\Delta} = 1 \quad \square$$

Conclusion

The local linear estimator can be written as

$$\hat{m}(x) = \sum_{i=1}^{n} w_{ni}(x) Y_i,$$

with weights $w_{ni}(x)$ given in Section 2. By Section 3, they sum to one and do not depend on the responses Y_i.

Practical exercise

Unless stated otherwise, your answer to the practical part should include the following elements:

- Description of the aim of the simulation study.
- Description of the different quantities that intervene in your simulation study. Explain how these quantities (fixed or random) are defined and the reasoning behind the choices you made.
- Description of your findings using appropriate graphics and/or tables (with a caption!) that are well commented in the text.

The code should not appear in the PDF report, unless there is a specific reason to include it.