Let

$$Q(\beta_0,\beta_1)=\sum_{i=1}^{n}K\Big(\frac{X_i-x}{h}\Big)\big(Y_i-\beta_0-\beta_1(X_i-x)\big)^2,$$

where K is a kernel and $h>0$ is the bandwidth. Let $K_i:=K\Big(\frac{X_i-x}{h}\Big)$.

Lets derive with regard to β_0 and let it be equal to 0:

$$\frac{\partial Q}{\partial \beta_0}=\sum_{i=1}^{n}2K_i\big(\beta_0+\beta_1(X_i-x)-Y_i\big)=0$$

Divide by 2:

$$\sum_{i=1}^{n}K_i\big(\beta_0+\beta_1(X_i-x)-Y_i\big)=0$$

Expand the summation:

$$\sum_{i=1}^{n}K_i\beta_0+\sum_{i=1}^{n}K_i\beta_1(X_i-x)-\sum_{i=1}^{n}K_iY_i=0$$

Factor out β_0 and β_1:

$$\beta_0\sum_{i=1}^{n}K_i+\beta_1\sum_{i=1}^{n}K_i(X_i-x)=\sum_{i=1}^{n}K_iY_i$$

Lets derive with regard to β_1 and let it be equal to 0:

$$\frac{\partial Q}{\partial \beta_1}=\sum_{i=1}^{n}2K_i(X_i-x)\big(\beta_0+\beta_1(X_i-x)-Y_i\big)=0$$

Divide by 2:

$$\sum_{i=1}^{n}K_i(X_i-x)\big(\beta_0+\beta_1(X_i-x)-Y_i\big)=0$$

Expand the summation:

$$\sum_{i=1}^{n}K_i(X_i-x)\beta_0+\sum_{i=1}^{n}K_i(X_i-x)\beta_1(X_i-x)-\sum_{i=1}^{n}K_i(X_i-x)Y_i=0$$

Factor out β_0 and β_1:

$$\beta_0\sum_{i=1}^{n}K_i(X_i-x)+\beta_1\sum_{i=1}^{n}K_i(X_i-x)^2=\sum_{i=1}^{n}K_i(X_i-x)Y_i$$

We then get the following system

$$\begin{cases} \beta_0 \sum_{i=1}^n K_i + \beta_1 \sum_{i=1}^n K_i(X_i - x) = \sum_{i=1}^n K_i Y_i \\ \beta_0 \sum_{i=1}^n K_i(X_i - x) + \beta_1 \sum_{i=1}^n K_i(X_i - x)^2 = \sum_{i=1}^n K_i(X_i - x)Y_i \end{cases}$$

The solution for $\hat{\beta}_0$:

$$\widehat{\beta}_0 = \frac{\sum_i K_i(X_i - x)^2 \sum_i K_i Y_i - \sum_i K_i(X_i - x) \sum_i K_i(X_i - x)Y_i}{\sum_i K_i \sum_i K_i(X_i - x)^2 - (\sum_i K_i(X_i - x))^2}$$

Substituting the $S_{n,k}(x)$ notation:

$$\hat{m}(x) = \widehat{\beta}_0 = \frac{(nh)^2 \sum_{i=1}^n \big(S_{n,2}(x) - S_{n,1}(x)(X_i - x)\big) \frac{K_i}{nh} Y_i}{(nh)^2 \big(S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2\big)} = \sum_{i=1}^n w_{n,i}(x) Y_i$$

so

$$w_{n,i}(x) = \frac{\big(S_{n,2}(x) - S_{n,1}(x)(X_i - x)\big) K\left(\frac{X_i - x}{h}\right)}{nh\big(S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2\big)}$$

Finally, we can check that the weights sum to 1:

$$\sum_{i=1}^n w_{n,i}(x) = \frac{S_{n,2}(x) \sum_i K_i - S_{n,1}(x) \sum_i K_i(X_i - x)}{nh(S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2)} = \frac{nh(S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2)}{nh(S_{n,0}(x)S_{n,2}(x) - S_{n,1}(x)^2)} = 1$$

0.1 Computational part

1) as grow N the h_{AMISE} error diminish because with a bigger a N we generally get a better estimation of simga and theta so the error get smaller