4.6 FRACTALS, JULIA SETS, AND MANDELBROT SETS

Fixed point iteration and Newton's method can be used for problems defined in the complex plane as well. Consider, for example, the problem of finding a fixed point of $\varphi(z) \equiv z^2$. It is easy to predict the behavior of the fixed point iteration $z_{k+1} = z_k^2$. If $|z_0| < 1$, then the sequence z_k converges to the fixed point 0. If $|z_0| > 1$, then the iterates grow in modulus and the method diverges. If $|z_0| = 1$, then $|z_k| = 1$ for all k. If $z_0 = 1$ or $z_0 = -1$, the sequence quickly settles down to the fixed point 1. On the other hand, if, say, $z_0 = e^{2\pi i/3}$, then $z_1 = e^{4\pi i/3}$ and $z_2 = e^{8\pi i/3} = e^{2\pi i/3} = z_0$, so the cycle repeats. It can be shown that if $z_0 = e^{2\pi i\alpha}$ where α is irrational then the sequence of points z_k never repeats but becomes dense on the unit circle. The sequence of points z_0, $z_1 = \varphi(z_0)$, $z_2 = \varphi(\varphi(z_0))$, . . . is called the **orbit** of z_0 under φ.

If $\varphi(z)$ is a polynomial function, then the set of points z_0 for which the orbit remains bounded is called the **filled Julia set** for φ, and its boundary is called the **Julia set**. Thus the filled Julia set for z^2 is the closed unit disk, while the Julia set is the unit circle. These sets are named after the French mathematician Gaston Julia. In 1918, he and Pierre Fatou independently investigated the behavior of these sets, which can be *far* more interesting than the simple example presented above. Their work received renewed attention in the 1980s when the advent of computer graphics made numerical experimentation and visualization of the results easy and fun [33].

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Figure 4.13 displays the filled Julia set for the function $\varphi(z) = z^2 - 1.25$. The boundary of this set is a **fractal**, meaning that it has dimension neither one nor two but some fraction in between. The figure is also **self-similar**, meaning that if one repeatedly zooms in on a small subset of the picture, the pattern still looks like the original.

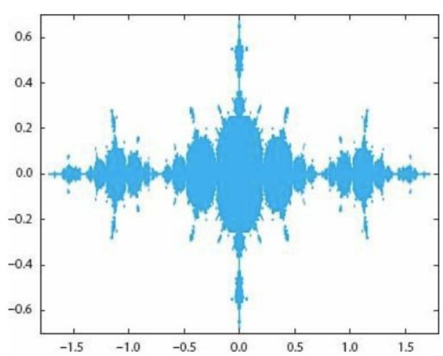


Figure 4.13. Filled Julia set for $\varphi(z) = z^2 - 1.25$.

The plot in figure 4.13 was obtained by starting with many different points z_0 throughout a rectangular region containing the two fixed points of φ, and running the fixed point iteration until one of three things occurred: Either $|z_k| \geq 2$, in which case we conclude that the orbit is unbounded and z_0 is not in the set; or z_k comes within 10^{-6} of a fixed point and stays within that distance for 5 iterations, in which case we conclude that the iterates are converging to the fixed point and z_0 is in the set; or the number of iterations k reaches 100 before either of the previous two conditions occur, in which case we conclude that the sequence z_k is not converging to a fixed point but does remain bounded and hence z_0 is again in the set. Note that since $|z_{k+1}| \geq |z_k|^2 - 1.25$, if $|z_k| \geq 2$, then $|z_{k+1}| \geq 4 - 1.25 = 2.75$, $|z_{k+2}| \geq 2.75^2 - 1.25 \approx 6.3$, etc., so that the sequence really is unbounded. The code to produce this plot is given in figure 4.14.

```
phi = inline('z^2 - 1.25');   % Define the function whose fixed points we seek.
fixpt1 = (1 + sqrt(6))/2;     % These are the fixed points.
fixpt2 = (1 - sqrt(6))/2;

colormap([1 0 0; 1 1 1]);     % Points numbered 1 (inside) will be colored red;
                              %  those numbered 2 (outside) will be white.
M = 2*ones(141,361);          % Initialize array of point colors to 2 (white).

for j=1:141,                  % Try initial values with imaginary parts between
  y = -.7 + (j-1)*.01;        %   -0.7 and 0.7
  for i=1:361,                % and with real parts between
    x = -1.8 + (i-1)*.01;     %   -1.8 and 1.8.
    z = x + 1i*y;             % 1i is the MATLAB symbol for sqrt(-1).
    zk = z;
    iflag1 = 0;               % iflag1 and iflag2 count the number of iterations
    iflag2 = 0;               %   when a root is within 1.e-6 of a fixed point;
    kount = 0;                % kount is the total number of iterations.

    while kount < 100 & abs(zk) < 2 & iflag1 < 5 & iflag2 < 5,
      kount = kount+1;
      zk = phi(zk);           % This is the fixed point iteration.

      err1 = abs(zk-fixpt1);  % Test for convergence to fixpt1.
      if err1 < 1.e-6, iflag1 = iflag1 + 1; else, iflag1 = 0; end;

      err2 = abs(zk-fixpt2);  % Test for convergence to fixpt2.
      if err2 < 1.e-6, iflag2 = iflag2 + 1; else, iflag2 = 0; end;

    end;
    if iflag1 >= 5 | iflag2 >= 5 | kount >= 100,   % If orbit is bounded, set
      M(j,i) = 1;                                  %   point color to 1 (red).
```

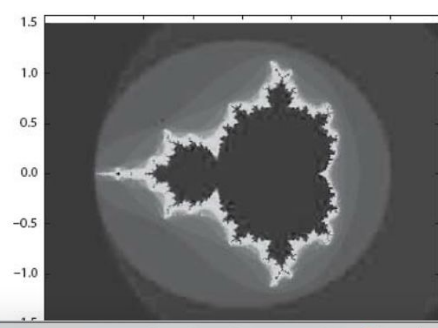
```
      M(j,i) = 1;                          %   point color to 1 (red).
    end;
  end;
end;

image([-1.8 1.8],[-.7 .7],M),  % This plots the results.
axis xy                        % If you don't do this, vertical axis is inverted.
```

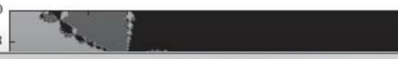
Figure 4.14. MATLAB code to compute the filled Julia set for $\varphi(z) = z^2 - 1.25$.

While this small bit of analysis enables us to determine a sufficient condition for the orbit to be unbounded, we really have only an educated guess as to the points for which the orbit is bounded. It is possible that the iterates remain less than 2 in modulus for the first 100 steps but then grow unboundedly, or that they appear to be converging to one of the fixed points but then start to diverge. Without going through further analysis, one can gain confidence in the computation by varying some of the parameters. Instead of assuming that the sequence is bounded if the iterates remain less than 2 in absolute value for 100 iterations, one could try 200 iterations. To improve the efficiency of the code, one might also try a lower value, say, 50 iterations. If reasonable choices of parameters yield the same results, then that suggests that the computed sets may be correct, while if different parameter choices yield different sets then one must conclude that the numerical results are suspect. Other ways to test the computation include comparing with other results in the literature and comparing the computed results with what is known analytically about these sets. This comparison of numerical results and theory is often important in both pure and applied mathematics. One uses the theory to check numerical results, and in cases where theory is lacking, one uses numerical results to suggest what theory might be true.

constant, one finds that some of these sets are **connected** (one can move between any two points in the set without leaving the set), while others are not. Julia and Fatou simultaneously found a simple criterion for the Julia set associated with $\varphi(z)$ to be connected: It is connected if and only if 0 is in the filled Julia set. In 1982, Benoit Mandelbrot used computer graphics to study the question of which values of c give rise to Julia sets that are connected, by testing different values of c throughout the complex plane and running fixed point iteration as above, with initial value $z_0 = 0$. The astonishing answer was the **Mandelbrot set**, depicted in figure 4.15. The black points are the values of c for which the Julia set is connected, while the shading of the other points indicates the rate at which the fixed point iteration applied to $z^2 + c$ with $z_0 = 0$ diverges.



Newton's method can also be applied to problems in the complex plane, and plots of points from which it is or is not convergent are just as beautiful and interesting as the ones obtained from fixed point iteration. Figure 4.16 shows the behavior of Newton's method when applied to the problem $f(z) \equiv z^3 + 1 = 0$. This equation has three roots: $z = -1$, $z = e^{\pi i/3}$, and $z = e^{-\pi i/3}$. Initial points from which Newton's method converges to the first root are lightest, those from which it converges to the second root are darkest, and those from which it converges to the third root are in between. The boundary of this image is also a fractal and one can zoom in on different pieces of the figure to see a pattern like that of the original.



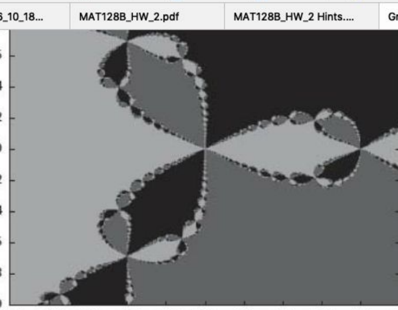


Figure 4.16. Convergence of Newton's method for $f(z) = z^3 + 1$. The lightest colored initial points converge to -1, the darkest colored initial points converge to $e^{\pi i/3}$, and the ones in between converge to $e^{-\pi i/3}$.

Once again, we should stress that further analysis is needed to verify figure 4.16. The figure was obtained by trying many different initial guesses z_0, iterating until 10 consecutive Newton iterates were within 10^{-6} of one of the roots and then coloring that initial guess according to which root was approximated. It is possible that 10 Newton iterates could be within 10^{-6} of one root while later iterates would converge to a different root or diverge. Further analysis can verify the figure, but that is beyond the scope of this book. For an excellent elementary discussion of Julia sets and Newton's method in the complex plane, see [69].