Finite-volume based inversion of the wave equation for elastography applications

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BIOQIC Seminar

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27.11.2018

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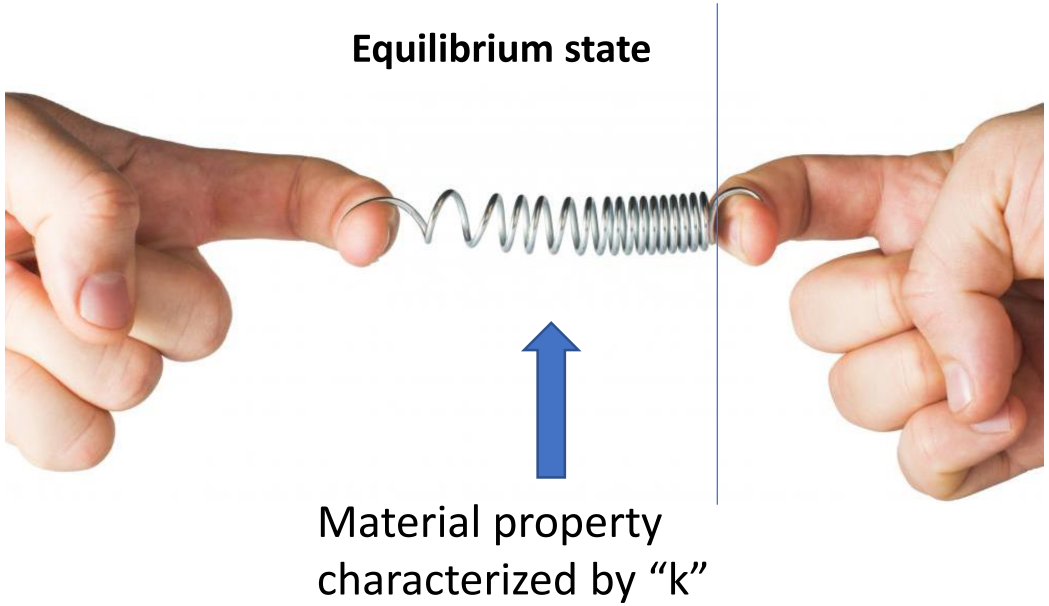
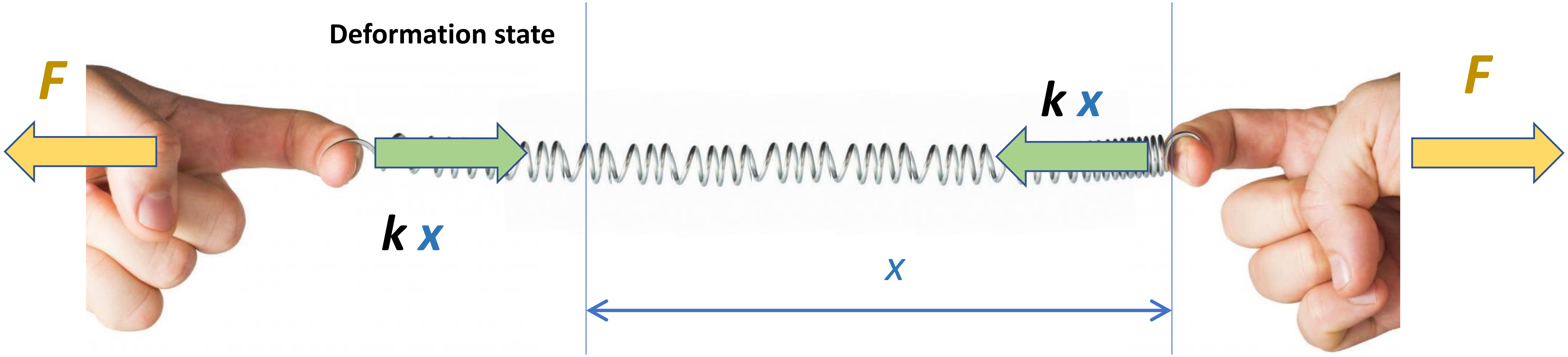
Elasticity from scratch

Elasticity 0D-1D



Elasticity 0D-1D



$$F + kx = 0$$

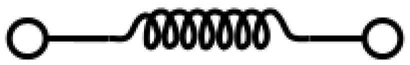
Hooke's Law = Static Equilibrium
(pure elasticity, no inertia)

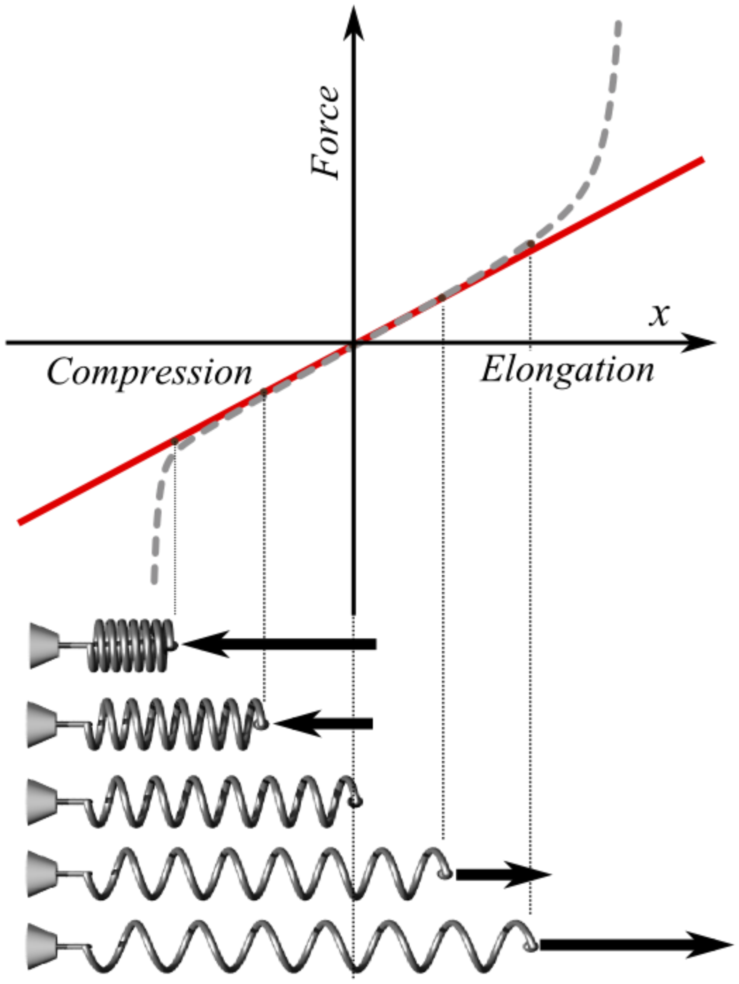
(Linear) Elasticity 0D-1D

$$F + kx = 0$$

Valid in "**linear regime**"
= small |x|

k



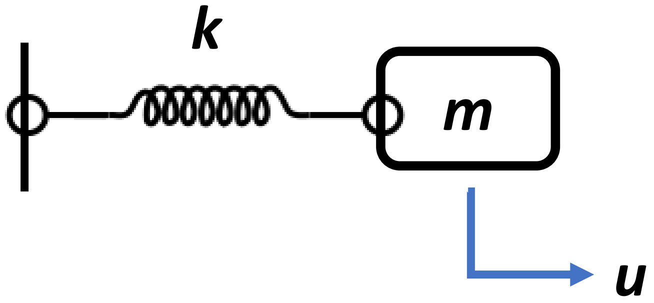


Figures from Wikipedia

(Linear) Elasticity 0D-1D

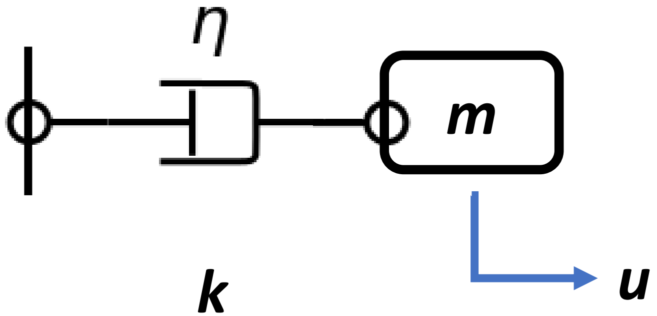
More models … → Equations of motion

spring



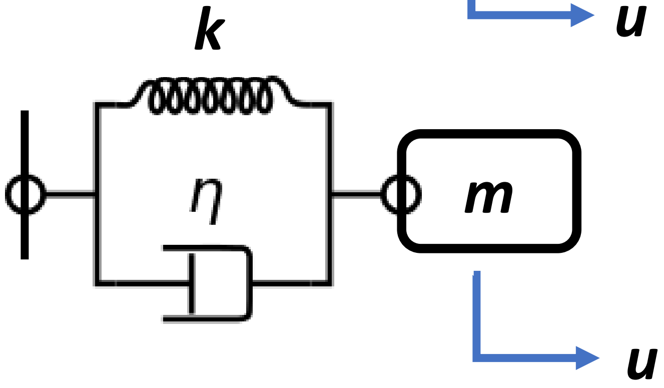
$$m\frac{d^2u}{dt^2} + ku = F_{ext}$$

dashpot



$$m\frac{d^2u}{dt^2} + \eta\frac{du}{dt} = F_{ext}$$

spring & dashpot in parallel (Kelvin-Voigt model)



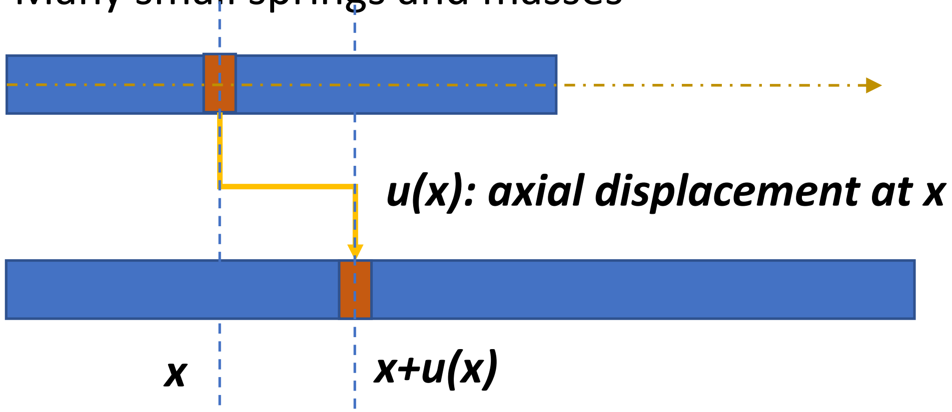
$$m\frac{d^2u}{dt^2} + \eta\frac{du}{dt} + ku = F_{ext}$$

Figures edited from Wikipedia

(Linear) Elasticity 1D

Elastic rod model:

Many small springs and masses

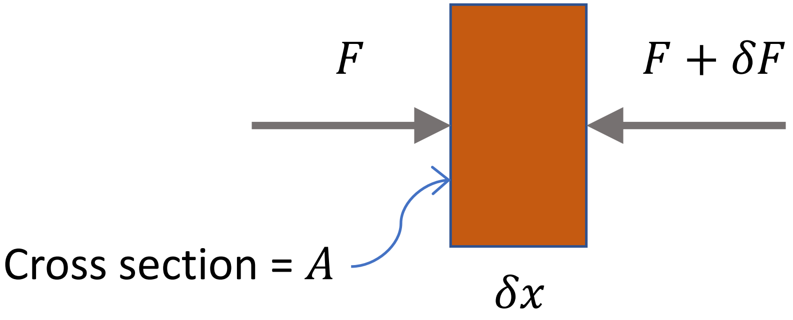


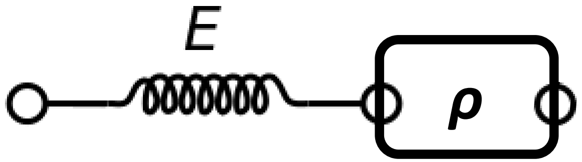
σ : Stress

$e = \frac{\delta u}{\delta x}$: Strain = Elongation rate at point x

$\sigma = Ee$: Constitutive Law

Balance of forces over one single element





$$\frac{F_E}{A} = \frac{\delta\sigma}{\delta x}$$

Elastic forces

$$\frac{F_\rho}{A} = \rho\frac{d^2u}{dt^2}$$

Inertia forces

(Linear) Elasticity 1D

Elastic rod model:

Many small springs and masses

Model valid at each point x between x=0 and x=L, and at any t from 0 to "infinity"

$$(\rho A)(x)\frac{\partial^2 u}{\partial t^2}(x,t) - \frac{\partial}{\partial x}\left(EA(x)\frac{\partial u}{\partial x}(x,t)\right) = 0$$

$$u = 0 \qquad at\ x = 0$$

$$E\frac{\partial u}{\partial x} = P \qquad at\ x = L$$

Boundary conditions

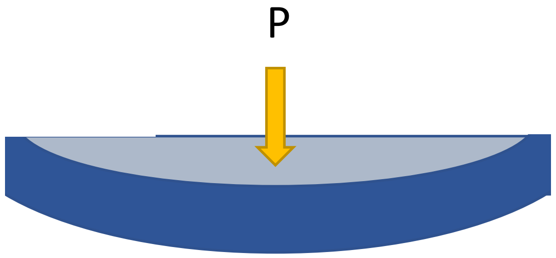
$$u = 0 \qquad when\ t = 0$$

$$\frac{\partial u}{\partial t} = 0 \qquad when\ t = 0$$

Initial conditions

(Linear) Elasticity 1D

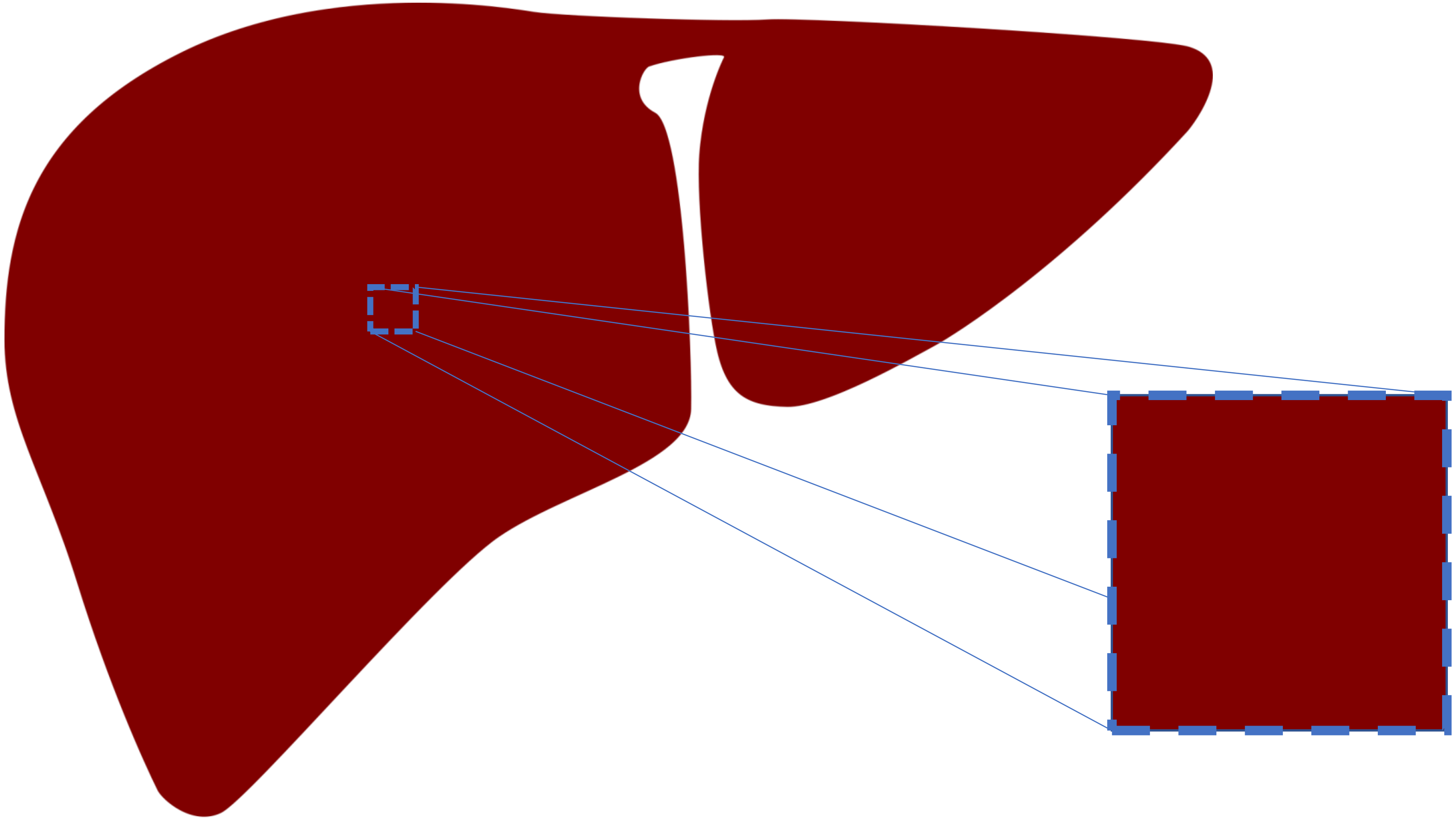
Axial deformation



Bending deformation

Torsional deformation

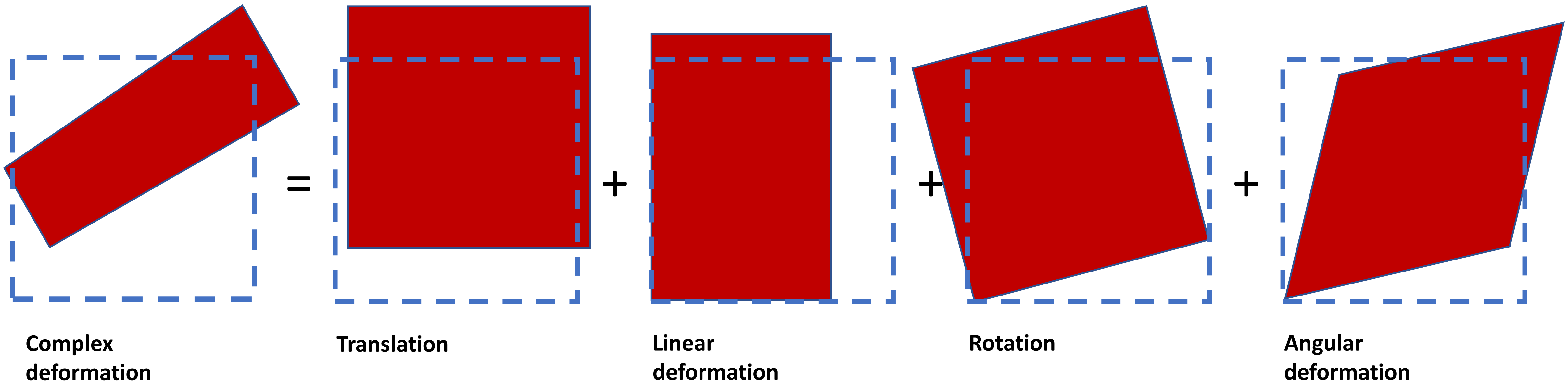
Elasticity 3D



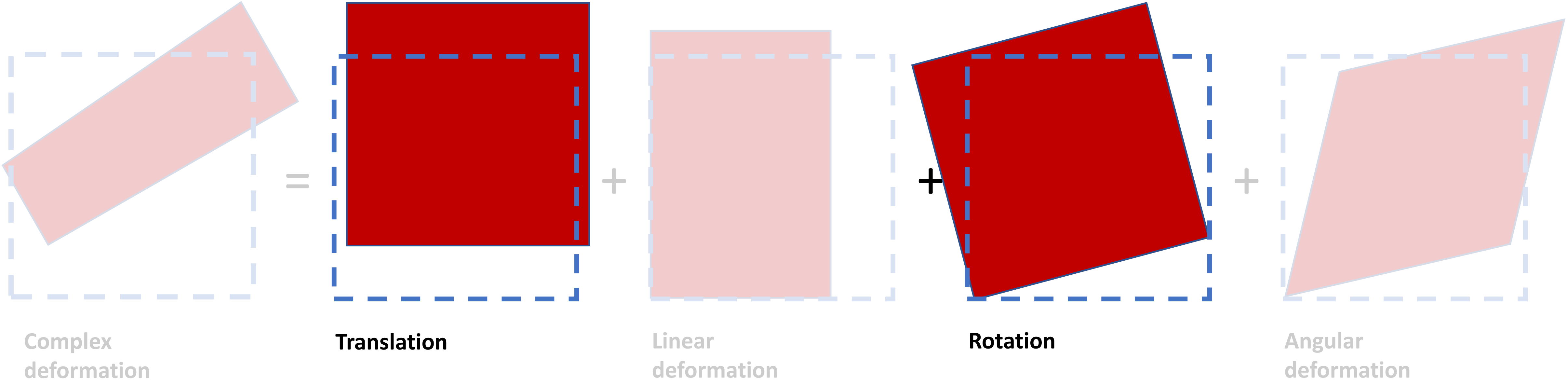
Small portion of tissue
= 1 (differential) element

Liver figure from commons.Wikimedia.org

Elasticity 3D

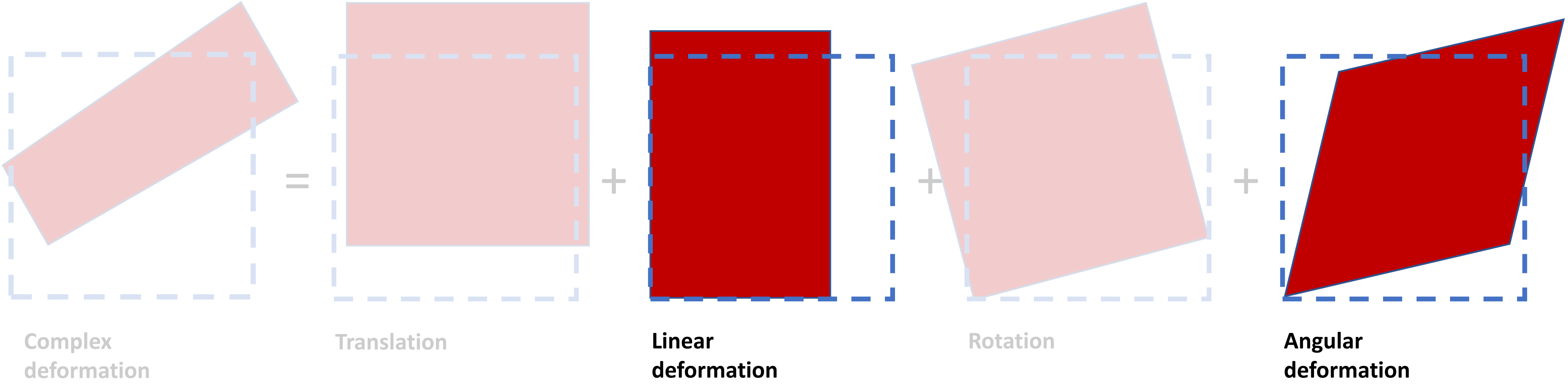


Elasticity 3D



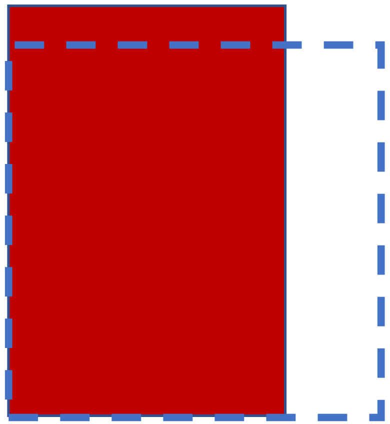
Rigid body motion (no deformation)

Elasticity 3D



Deformation

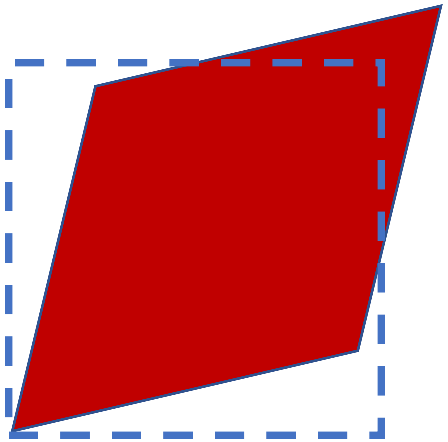
Elasticity 3D



Linear deformation

displacement

$$u = (u_x, u_y, u_z)$$



Angular deformation

Dilatation and/or compression

associated with

$$\nabla \cdot u = div(u) = \sum_{i=1}^{3} \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}$$

Shear

associated with crossed derivatives

$$\frac{\partial u_x}{\partial y}, \frac{\partial u_y}{\partial x}, \frac{\partial u_z}{\partial y}, \frac{\partial u_y}{\partial z}, \frac{\partial u_x}{\partial z}, \frac{\partial u_z}{\partial x}$$

(Linear) Elasticity 3D

In 3D, we can express different type of deformations using the (linear) **strain tensor**

$$e(u) = \begin{bmatrix} \frac{\partial u_x}{\partial x} & \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) & \frac{\partial u_y}{\partial y} & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) & \frac{\partial u_z}{\partial z} \end{bmatrix} = \frac{1}{2}(\nabla u + \nabla u^T)$$

$$div(u) = tr(e(u))$$

(Linear) Elasticity 3D

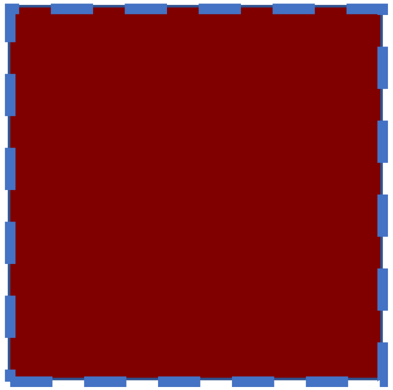
And a constitutive law relates the elastic forces and their deformations in the **stress-strain** constitutive law

$$\sigma(u) = Ce(u) = \sum_{kl} C_{ijkl}\, e_{kl}(u)$$

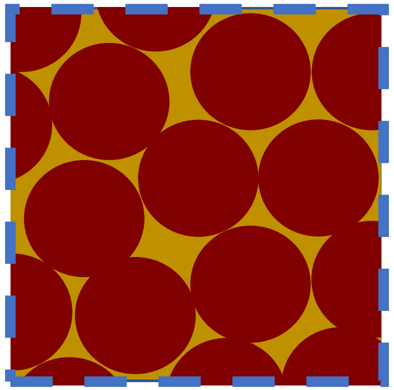
← Generalized **Hooke's Law**

= Linear constitutive Law

where C is the Elasticity Tensor, and can represent different types of tissues



Isotropic



Anisotropic
With no preferred microstructural orientation

Anisotropic
With one preferred microstructural orientation

(Linear) Elasticity 3D

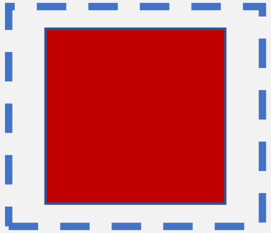
And a constitutive law relates the elastic forces and their deformations in the **stress-strain** constitutive law

$$\sigma(u) = Ce(u)$$

$$= \lambda \, div(u) \, I + 2\mu \, e(u)$$

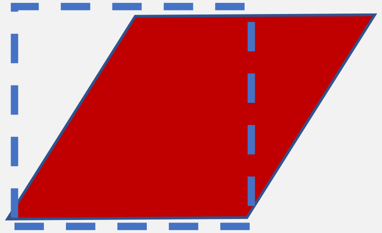
1st Lamé coefficient

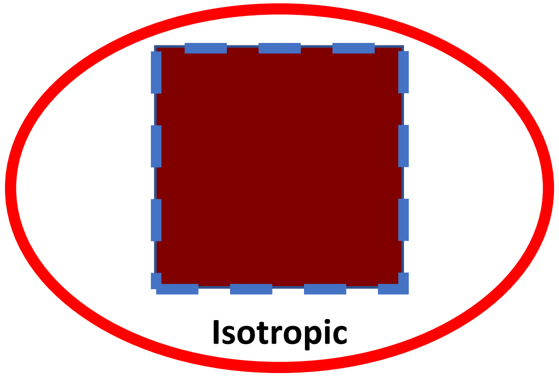
opposition to volumetric changes



2nd Lamé coefficient or **Shear modulus**

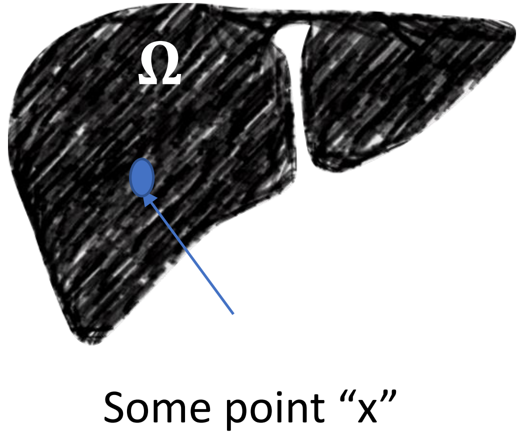
opposition to shear deformations





(Linear) Elasticity 3D

The equation of motion now yields

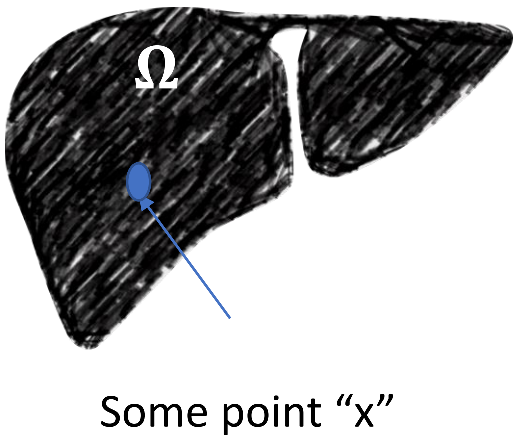


$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - div\, \sigma(u) = F_{ext} & \forall\, x \in \Omega, \forall\, t > 0 \\ \sigma(u) = Ce(u) & \forall\, x \in \Omega, \forall\, t > 0 \end{cases}$$

+ initial conditions
+ boundary conditions

(Linear) Elasticity 3D

The equation of motion now yields



$\lambda = \lambda(x)$

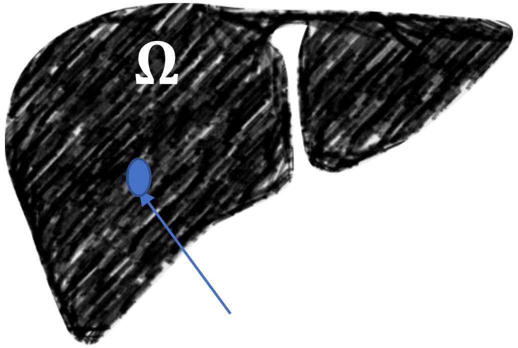
$\mu = \mu(x)$

$$\rho \frac{\partial^2 u}{\partial t^2} - div\left(\lambda div(u) I + 2\mu\, e(u)\right) = F_{ext} \qquad \forall\, x \in \Omega, \forall\, t > 0$$

+ initial conditions
+ boundary conditions

(Linear) Elasticity 3D

The equation of motion now yields



Some point "x"

$\lambda = \lambda(x)$

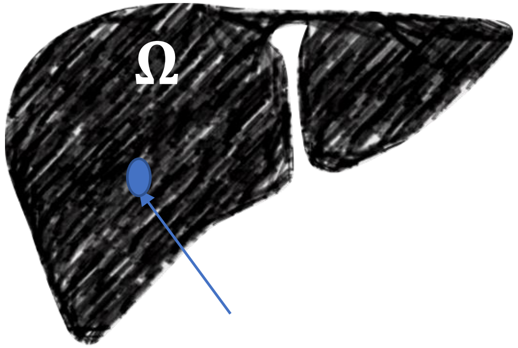
$G = G(x)$

$$\left\{ \rho \frac{\partial^2 u}{\partial t^2} - div\,(\lambda div(u) I + 2G\ e(u)) = F_{ext} \qquad \forall\, x \in \Omega, \forall\, t > 0 \right.$$

+ initial conditions
+ boundary conditions

(Linear) Elasticity 3D

The equation of motion now yields



Some point “x”

$$\lambda = \lambda(x) \to \infty$$

$$G = G(x)$$

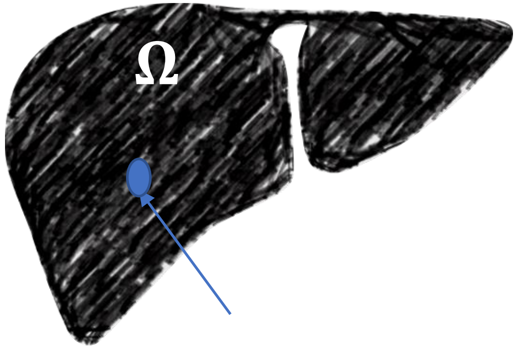
$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - div\,(\lambda div(u) I + 2G\, e(u)) = F_{ext} & \forall\, x \in \Omega, \forall\, t > 0 \\ div(u) = 0 & \forall\, x \in \Omega, \forall\, t > 0 \end{cases}$$

Incompressibility constraint

+ initial conditions
+ boundary conditions

(Linear) Elasticity 3D

The equation of motion now yields



Some point "x"

$\lambda = \lambda(x) \to \infty$

$G = G(x)$

$$\rho \frac{\partial^2 u}{\partial t^2} - div\,(-pI + 2G\ e(u)) = F_{ext}$$

$$div(u) = 0$$

Incompressibility constraint

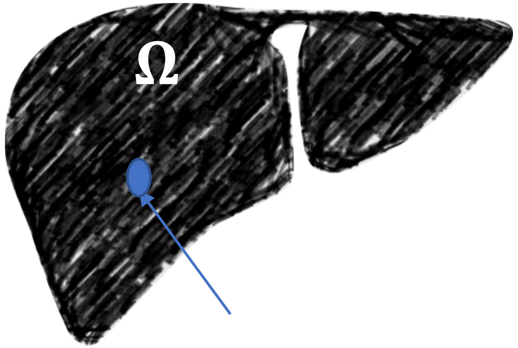
+ initial conditions
+ boundary conditions

$div(u) \to 0$
$\lambda \ \to \infty$

$p = -\lambda div(u)$
is finite

(Linear) Elasticity 3D

The equation of motion now yields



Some point “x”

$\lambda = \lambda(x) \to \infty$

$G = G(x)$

weight

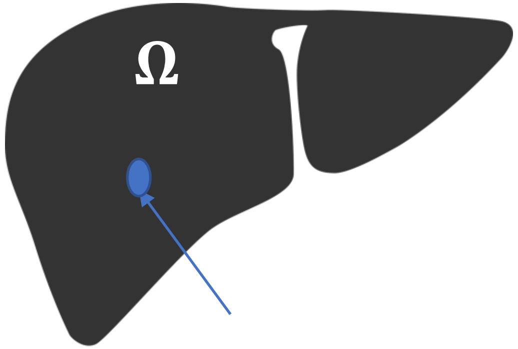
$$\rho \frac{\partial^2 u}{\partial t^2} - div\left(2G\, e(u)\right) + \nabla p = \rho g$$

$$div(u) = 0$$

\+ initial conditions
\+ boundary conditions

(Linear) Elasticity 3D

Particular case:



Some point “x”

$\lambda \approx const. \gg 1\ but\ finite$

$G \approx const.$

$$\rho \frac{\partial^2 u}{\partial t^2} - (\lambda + G)\nabla div(u) + G\Delta u = \rho g$$

$$\Downarrow$$

Navier’s Equation

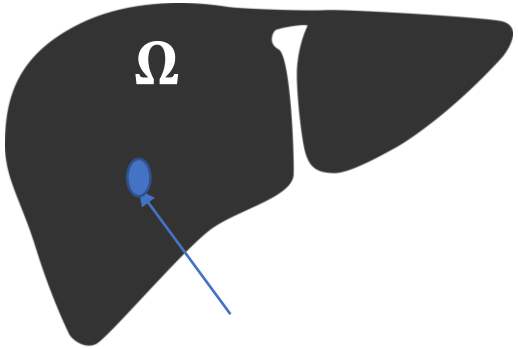
$$\rho \frac{\partial^2 u}{\partial t^2} - (\lambda + 2G)\nabla div(u) + G\nabla \times \nabla \times u = \rho g$$

+ initial conditions
+ boundary conditions

Laplacian operator : $\Delta f = div\nabla f = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

(Linear) Elasticity 3D

Particular case:



Some point “x”

$\lambda \approx const. \gg 1 \; but \; finite$

$G \approx const.$

$$\rho \frac{\partial^2 u}{\partial t^2} - (\lambda + 2G)\nabla div(u) + G\nabla \times \nabla \times u = 0$$

$$u = \nabla\varphi + \nabla \times \Phi$$

Helmholtz-Hodge decomposition
→ Two volumetric WAVES!

$$\frac{\partial^2 \varphi}{\partial t^2} = \sqrt{\frac{\lambda + 2G}{\rho}} \Delta\varphi$$

Compressive waves or P-waves or Acoustic waves [No rotation]

$$\frac{\partial^2 \Phi}{\partial t^2} = \sqrt{\frac{G}{\rho}} \Delta\Phi$$

Shear waves [No volumetric variations]

(Linear) Elasticity 3D

Going back… when coefficients are not constants (heterogeneous media)

$$\rho \frac{\partial^2 u}{\partial t^2} - div\,(\lambda div(u) I + 2G\, e(u)) = \rho g$$

+ initial conditions
+ boundary conditions

$\lambda = \lambda(x)$

$G = G(x)$

$$u(x,\omega) = \int_{-\infty}^{\infty} u(x,t) e^{i\omega t}\, dx$$

Composition of plane waves

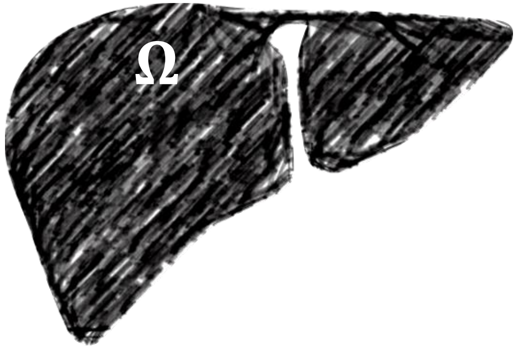
→ Harmonic motion!

→ Now u is complex-valued

(Linear) Elasticity 3D

Going back… when coefficients are not constants (heterogeneous media)

Time-Harmonic motion:



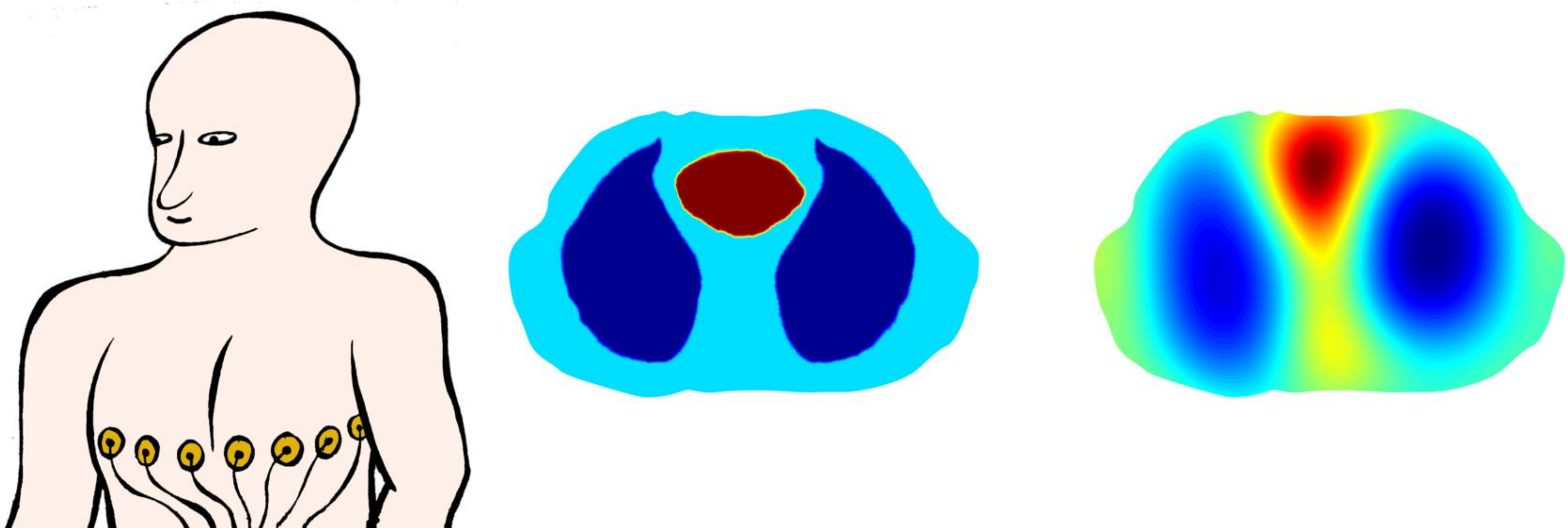
$$\rho\omega^2 u + div\left(\lambda div(u) I + 2G\, e(u)\right) = 0 \quad in\ \Omega$$

+ boundary conditions

$\lambda = \lambda(x)$

$G = G(x)$

In our case, we have "u" and we are interested in the recovering of G, λ and ρ



https://www.helsinki.fi/en/researchgroups/inverse-problems/research/computational-inverse-problems

Some inversion approaches

Inverse Problems in Elasticity

$$\rho\omega^2 u + div\left(\lambda div(u) I + 2G\, e(u)\right) = 0 \quad in\ \Omega$$

Direct Problem:
We have G, λ and ρ
We want u

$$u = A(G, \lambda, \rho)$$

“easy” if well posed
← Stable and unique solution

Inverse Problem:
We have u
We want G, λ and ρ

$$(G, \lambda, \rho) = A^{-1} u$$

Ill-posed
→ instabilities, non-uniqueness
→ challenging!

Intermezzo

<u>On the classification of Partial Differential Equations</u>:

In 2D, we can write it as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = const$$

In analogy to quadratic polynomials, we have that

- If $B^2 - 4AC < 0$ is an **Elliptic** equation → *smooth solutions*
- If $B^2 - 4AC = 0$ is a **Parabolic** equation → *solutions get smoothed as x or y increase*
- If $B^2 - 4AC > 0$ is a **Hyperbolic** equation → *Allow discontinuous and stiff solutions*

Inverse Problems in Elasticity

Direct Problem:
Even for irregular G, λ and ρ
The equation for u is "elliptic" → u is "smooth"

$$\rho\omega^2 u + div\left(\lambda div(u) I + 2G\, e(u)\right) = 0$$

Inverse Problem:
Even for regular u and simplified model (shear only)
The equation for want G is "hyperbolic"
→ G is non-smooth

$$\rho\omega^2 u + div\left(2G\, e(u)\right) = 0$$

$$\rho\omega^2 u + 2G div\, e(u) + e(u)\nabla G = 0$$

In the sense of distributions …

- Looks like a transport (advection) model

$$\alpha G + A\nabla G = f$$

- Numerical solutions require special schemes (some standard discretization may fail)

Inverse Problems in Elasticity

Two approaches

Direct
Uses an explicit relation between u and G, λ, ρ

Pros
- Very Fast
- Not many variables to tune-up in the model

Contras
- Requires many information (full knowledge of u) and/or high quality filtered input data.
- The model is often reduced: Sometimes too simplistic.

Iterative
Minimize the misfit between the data and computational simulations

Pros
- Can be solved using partial information (the more the best)
- Can be applied with complex models

Contras
- Very slow and often requires high-performance computing facilities.
- Complex problems have many coefficients to tune up (missing information)
- Sophisticated non-linear solvers often need fine-tuned parameters

Inverse Problems in Elasticity

Iterative

Minimize the misfit between the data and computational simulations.

The general approach can be established as

$$\min_{G,\lambda,\rho} \left\{ J(G,\lambda,\rho) = \left\| \boldsymbol{u}(G,\lambda,\rho) - u_{exp} \right\|^2 + R(G,\lambda,\rho) \right\}$$

Subject to

$$\rho\omega^2 \boldsymbol{u} + div\left(\lambda div(\boldsymbol{u})I + 2G\, e(\boldsymbol{u})\right) = 0$$

+ boundary conditions

Inverse Problems in Elasticity

Iterative

Minimize the misfit between the data and computational simulations.

The general approach can be established as

Regularization: Penalization to ensure smoothness in coefficients

$$\min_{G,\lambda,\rho} \left\{ J(G,\lambda,\rho) = \left\| \boldsymbol{u}(G,\lambda,\rho) - u_{exp} \right\|^2 + R(G,\lambda,\rho) \right\}$$

Subject to

$$\rho\omega^2\boldsymbol{u} + div\left(\lambda div(\boldsymbol{u})I + 2G\, e(\boldsymbol{u})\right) = 0$$

Need sensitivity respect to G, λ, ρ
→ Adjoint Problem

+ boundary conditions

Inverse Problems in Elasticity

Iterative

Minimize the misfit between the data and computational simulations.

The general approach can be established as a sequence of updates

$$(G,\lambda,\rho)^{k+1} = (G,\lambda,\rho)^{k} - \alpha \frac{\partial J}{\partial(G,\lambda,\rho)}(\boldsymbol{u}^k,\boldsymbol{p}^k)$$

Subject to

Find u and p is computationally very expensive in 3D

$$\rho^k\omega^2\boldsymbol{u^k} + div\left(\lambda^k div(\boldsymbol{u^k})I + 2G^k\, e(\boldsymbol{u^k})\right) = 0$$

\+ boundary conditions

$$\rho^k\omega^2\boldsymbol{p^k} + div\left(\lambda^k div(\boldsymbol{p^k})I + 2G^k\, e(\boldsymbol{p^k})\right) = \boldsymbol{u}(\mathrm{G}^k,\lambda^k,\rho^k) - u_{exp}$$

\+ boundary conditions

Similar for shear-only model

Inverse Problems in Elasticity

<u>Direct</u>

Uses an explicit relation between u and G, λ, ρ

If we could explicitly find some F such that

$$(\boldsymbol{G}, \boldsymbol{\lambda}, \boldsymbol{\rho}) = F(u, \nabla u)$$

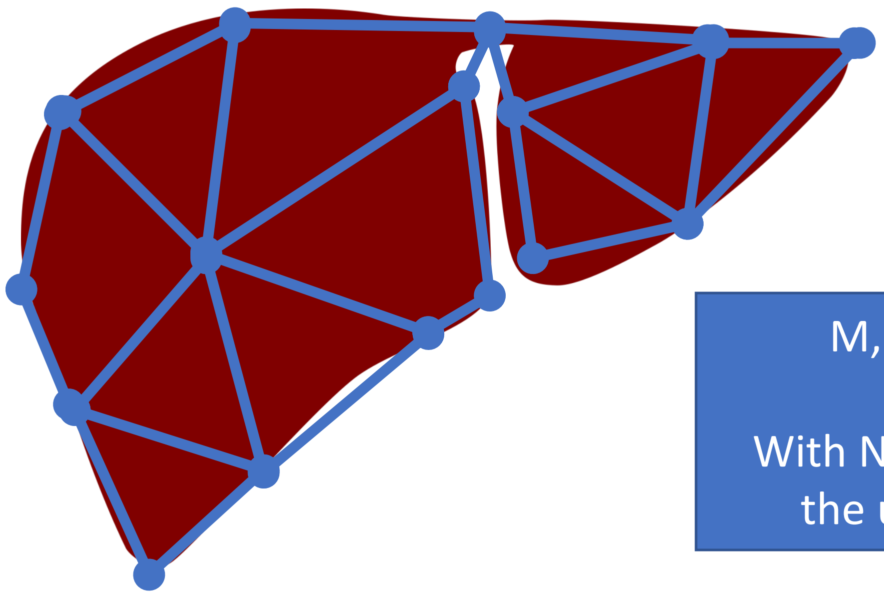
at any point of the tissue by using this model

$$\boldsymbol{\rho}\omega^2 u + div\left(\boldsymbol{\lambda} div(u) I + 2\boldsymbol{G}\, e(u)\right) = 0$$

Then there is a hope … only if u is well-known and noise-free.

FEM solvers

Some iterative approaches use Finite-Element methods to solve PDEs in the full volume, generating a matrix that connects neighboring values



Given material coefficients, find u solution to

$$\int_{\Omega} \rho\omega^2 u \cdot v dx = \int_{\Omega} Ce(u): e(v) dx - \int_{\partial\Omega} Ce(u) n \cdot v \, ds \quad \forall v \in V$$

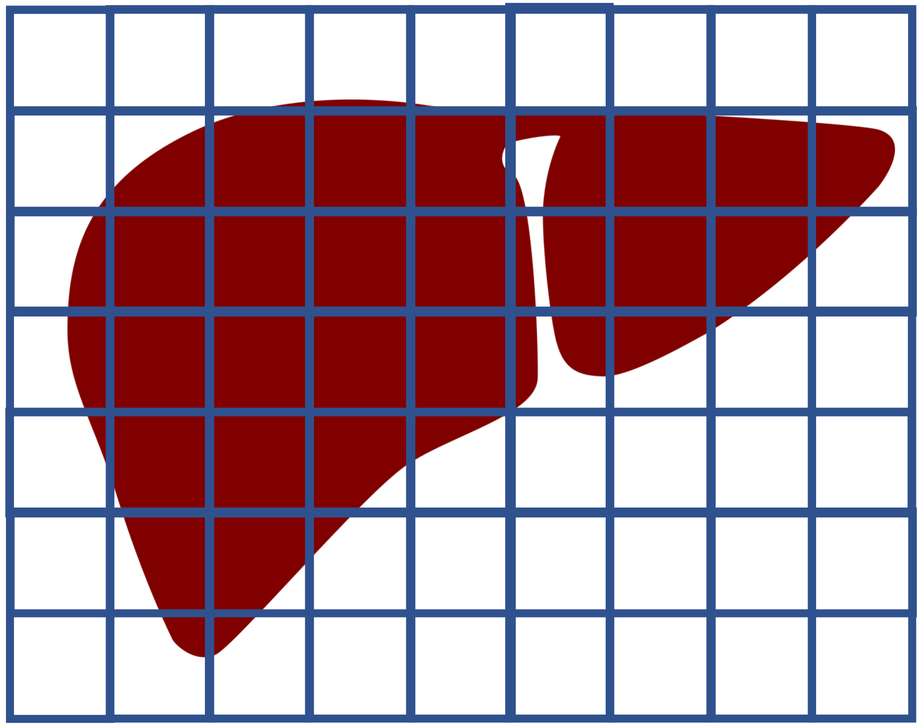
Discretization of whole domain

M,K matrices of size
(3xN) x (3xN)
With N=number of nodes in the unstructured mesh

$$(\omega^2 M - K)u = F$$

FV solvers

Finite Volume solvers also use the whole volume, and generates a matrix that connects neighboring values. However, this type of schemes is well suited for conservation laws (good idea for solving G)



Given material coefficients, find u solution to

$$\int_{\Omega} \rho\omega^2 u dx = \int_{\partial\Omega} Ce(u) n ds$$

Discretization of whole domain

M,K matrices of size
(3xN) x (3xN)
With N=number of voxels

$$\omega^2 Mu - Ku = F$$

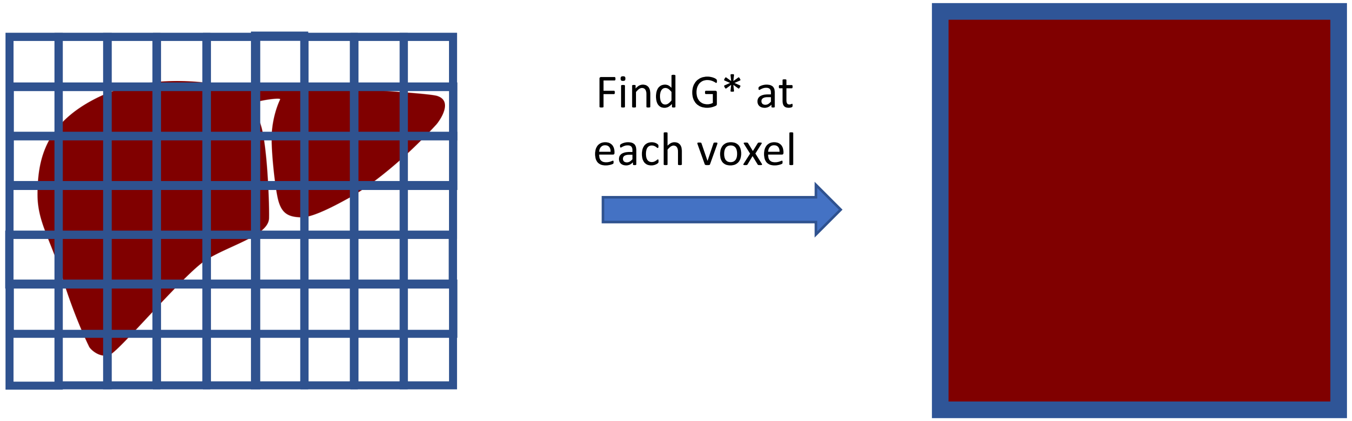
There exist FV methods for unstructured meshes, but they are slower than FEM.

Our Proposal: a Finite-Volume based scheme

We apply FV directly to the image grid. FV methods are good for Hyperbolic problems.

In contrast to standard FEM, in FV we do not uses a variational formulation, but a direct integration of the model.

And, instead of solving the whole domain, we propose to go at the smallest scale possible: 1 voxel



Finite-Volume based scheme

Direct integration of elasticity equation leads to

$$\int_V \rho\omega^2 u\, dx + \int_V div\left(\lambda div(u)I + 2G\, e(u)\right)dx = 0$$

Assumption: G, λ, ρ are constant voxel-wise.

$$\rho\omega^2 \int_V u\, dx + \lambda \int_V div\,(div(u)I)dx + 2G \int_V div\left(e(u)\right)dx = 0$$

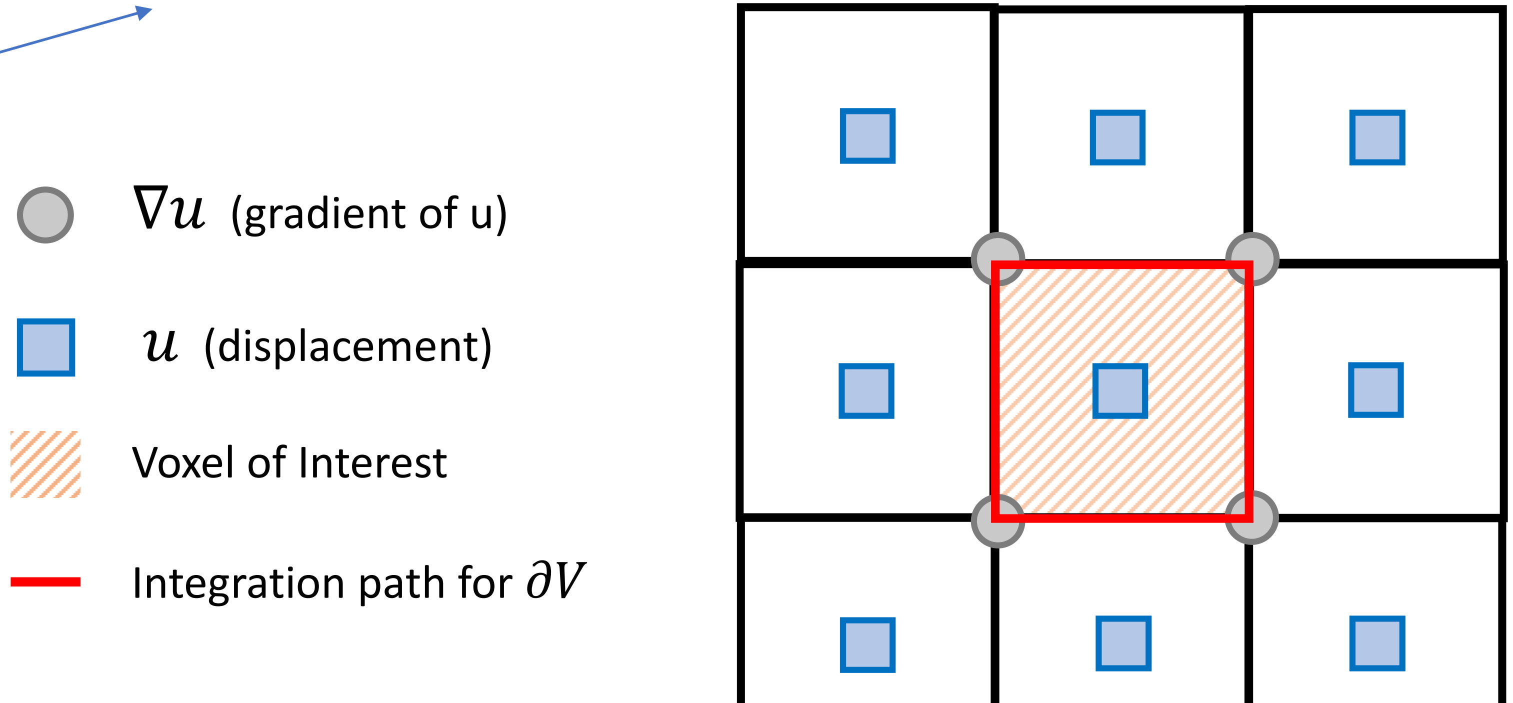
Thus, applying Gauss theorem, we have

$$\rho\omega^2 \int_V u\, dx + \lambda \int_{\partial V} div(u)nds + 2G \int_{\partial V} e(u)nds = 0$$

Finite-Volume based scheme

$$\rho\omega^2 \underbrace{\boxed{\int_V u\,dx}}_{p} + \lambda \underbrace{\boxed{\int_{\partial V} div(u) n ds}}_{q} + 2G \underbrace{\boxed{\int_{\partial V} e(u) n ds}}_{b} = 0$$

These 3 vectors are evaluated for each voxel V
← easy to calculate since we have u in V



Finite-Volume based scheme

$$\rho\omega^2 p_l + \lambda^* q_l + 2G^* b_l = 0$$

A set of 3 simple algebraic equations to solve ρ, λ, G.

And … in an ideal world, we could possibly find all the values we need.

$$\begin{bmatrix} |\mathbf{q}_V|^2 & \mathbf{q}_V^\dagger \mathbf{b}_V & \mathbf{q}_V^\dagger \mathbf{p}_V \\ \mathbf{b}^\dagger \mathbf{q}_V & |\mathbf{b}_V|^2 & \mathbf{b}_V^\dagger \mathbf{p}_V \\ \mathbf{p}_V^\dagger \mathbf{q}_V & \mathbf{p}_V^\dagger \mathbf{b}_V & |\mathbf{p}_V|^2 \end{bmatrix} \begin{bmatrix} \lambda \\ 2\mu \\ \rho\omega^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

But it is too unstable, even for constant ρ

Finite-Volume based scheme

The current model is (without acoustic waves)

$$\rho\omega^2 p_l + 2G^* b_l = 0$$

which yields

$$\boxed{|G^*(\omega)| = \frac{\rho\omega^2}{2}\frac{\|p(\omega)\|_{l^p}}{\|b(\omega)\|_{l^p}}}$$

With the lp-norm

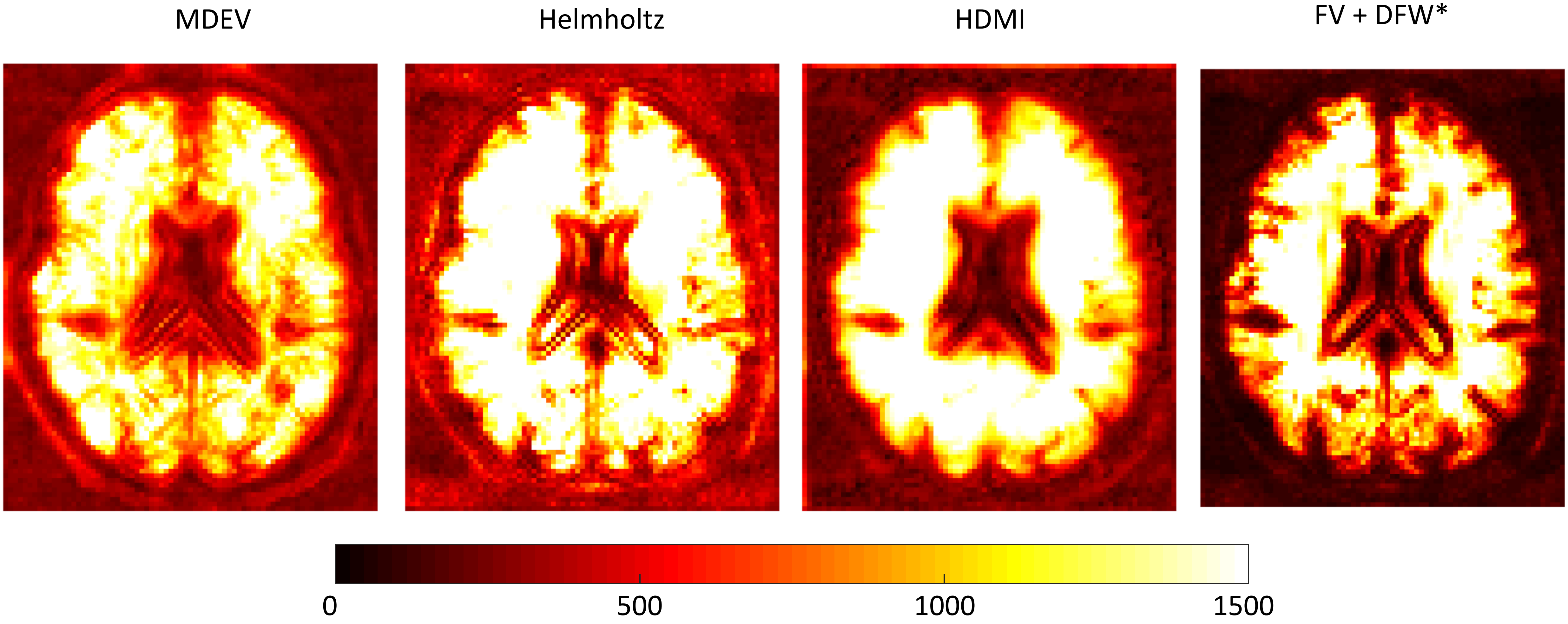
$$\|x\|_{l^p}^p = \left(\sum_{k=1}^{3}|x_k|^p\right)^{1/p}$$

And for multi-frequency:

$$\boxed{|G^*|_{MF} = \frac{\rho}{2}\left(\frac{\sum_j \ \omega_j^{2p}\|p(\omega_j)\|_{l^p}^p}{\sum_j \|b(\omega_j)\|_{l^p}^p}\right)^{1/p}}$$

$$\boxed{\angle G^* = \angle\sum_j\left(p(\omega_j), \bar{b}(\omega_j)\right)}$$

Some results: Absolute value



* Ong et all. “Robust 4D Flow Denoising Using Divergence-Free Wavelet Transform”, MRM 2015. https://people.eecs.Berkeley.edu/~mlustig/Software.html

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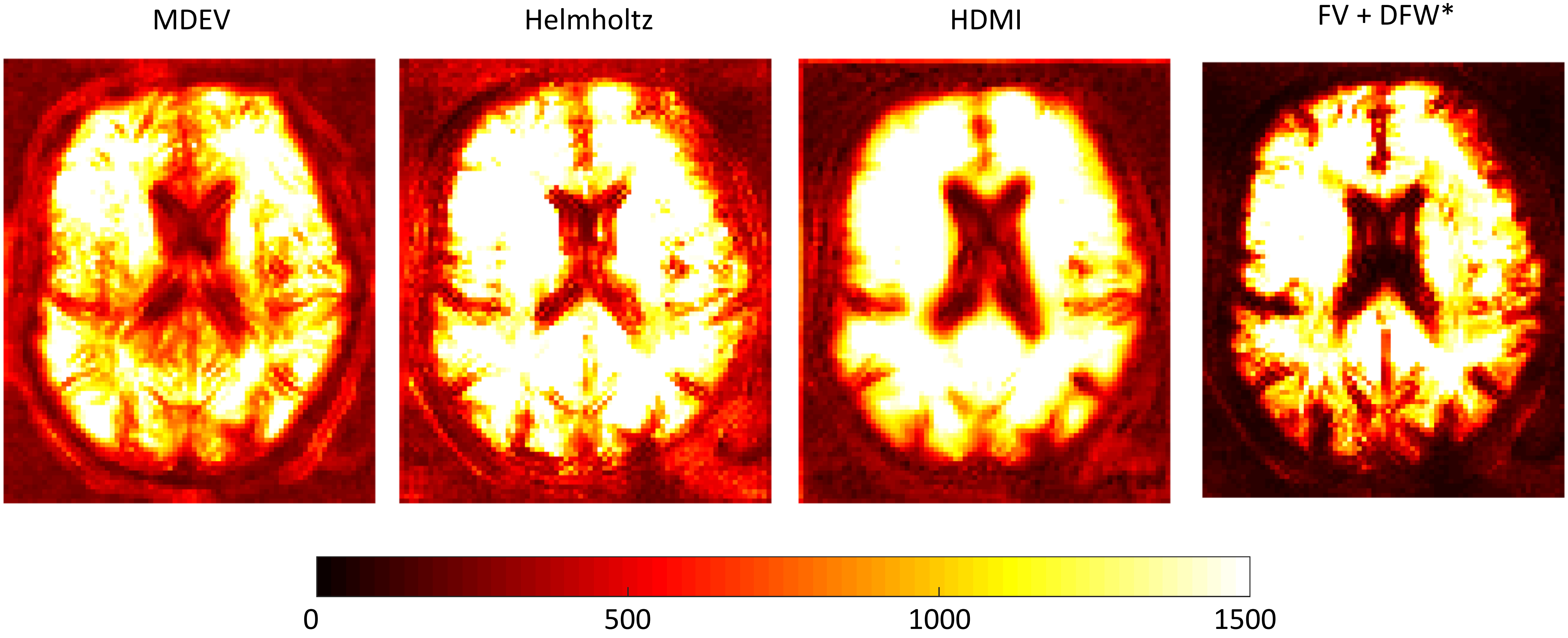

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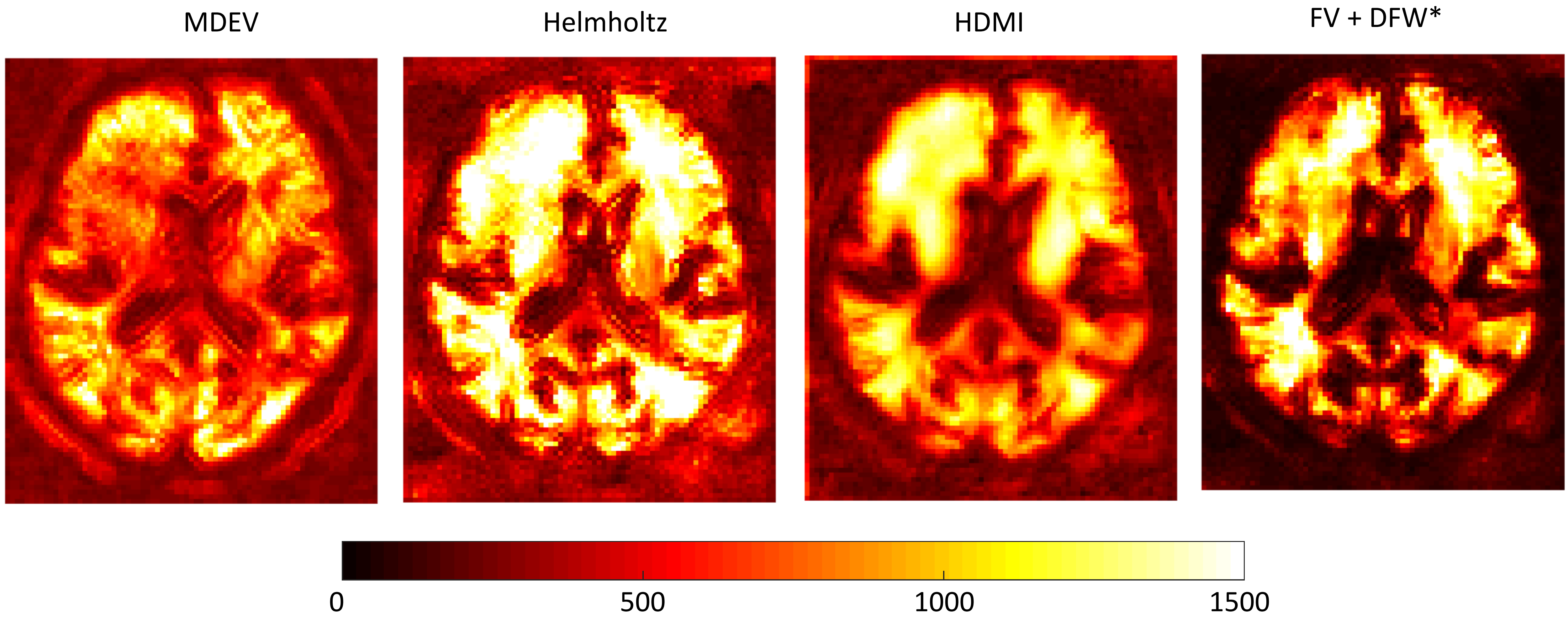
Some results: Absolute value



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https://people.eecs.Berkeley.edu/~mlustig/Software.html

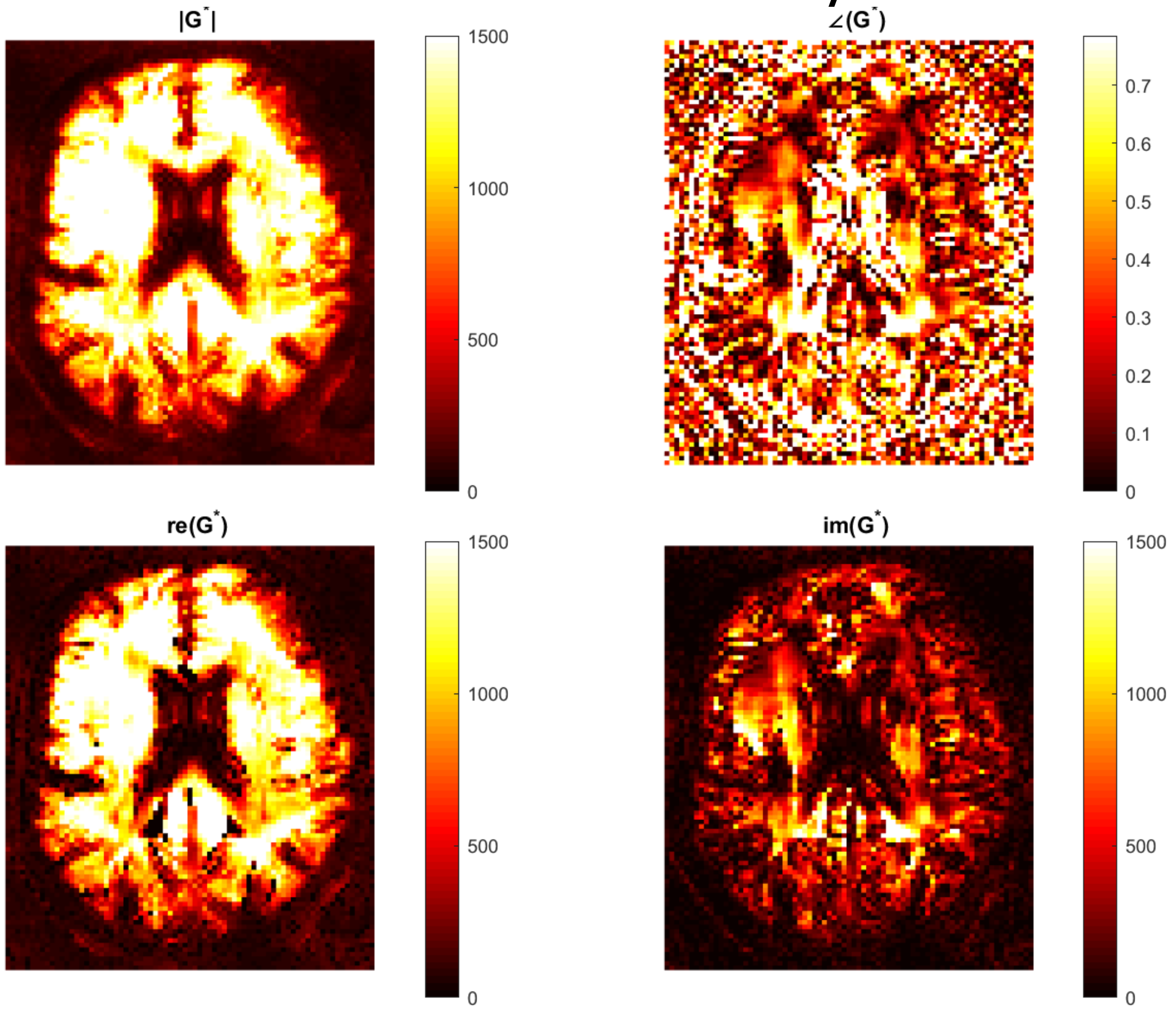
Some results

Average execution time in seconds for different methods
(MDEV solution was already calculated)

	Helmholtz	HDMI	FV
avg(time)	0,81	138,41	1,41

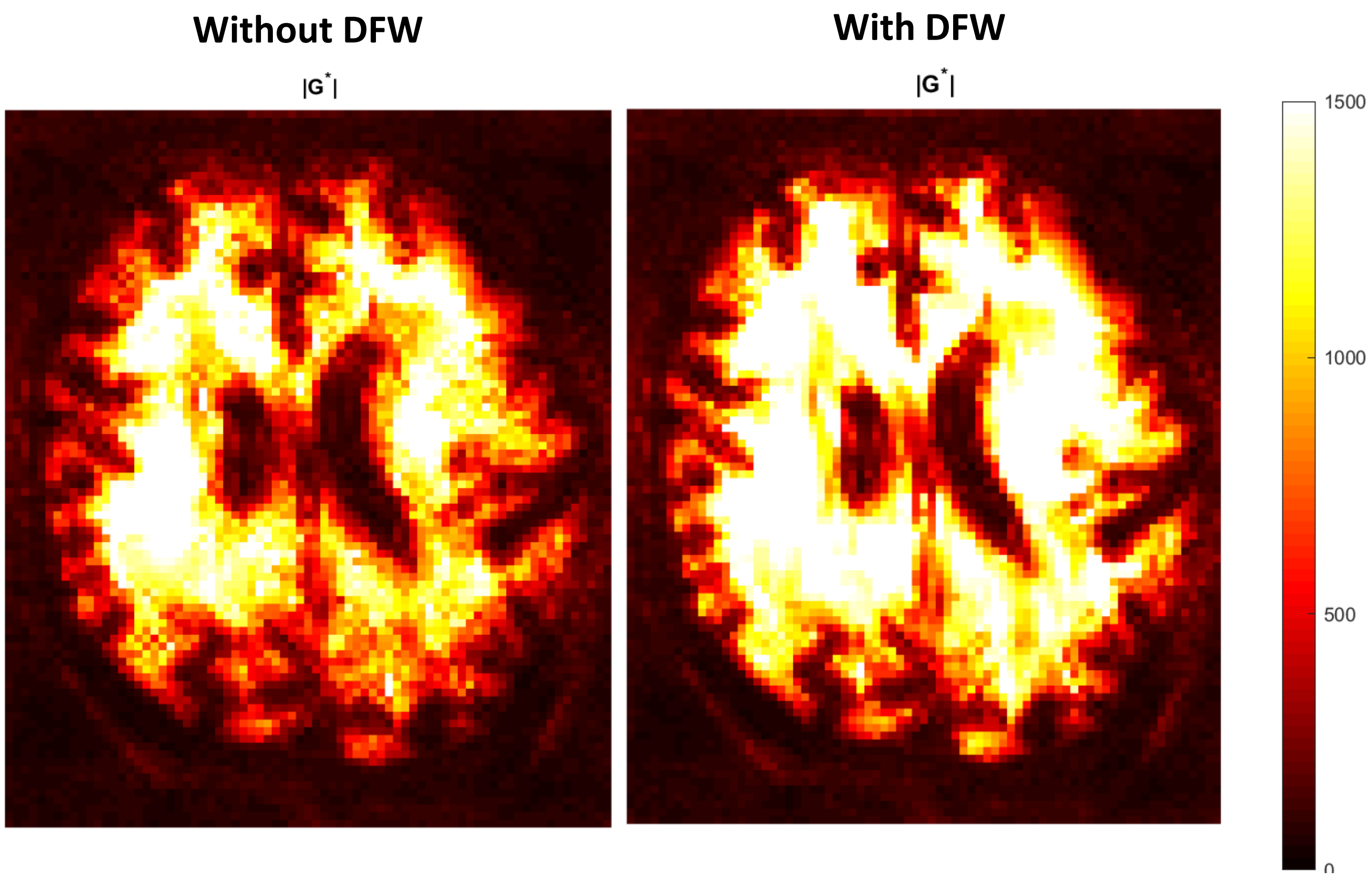
In the MREdge pipeline, all the data is single precision. This reduces the computational cost.

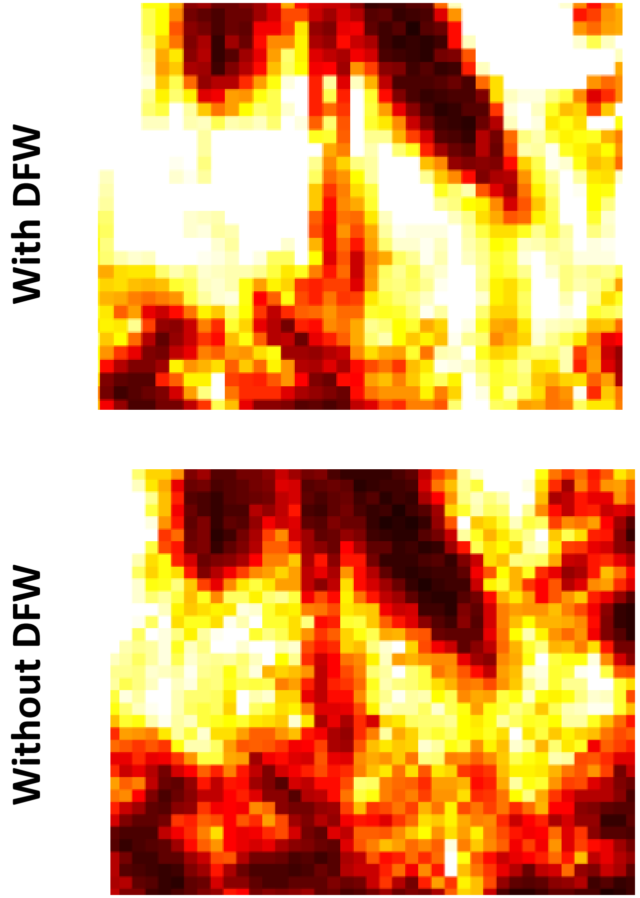
Some results: Absolute value / Phase angle



Some issues

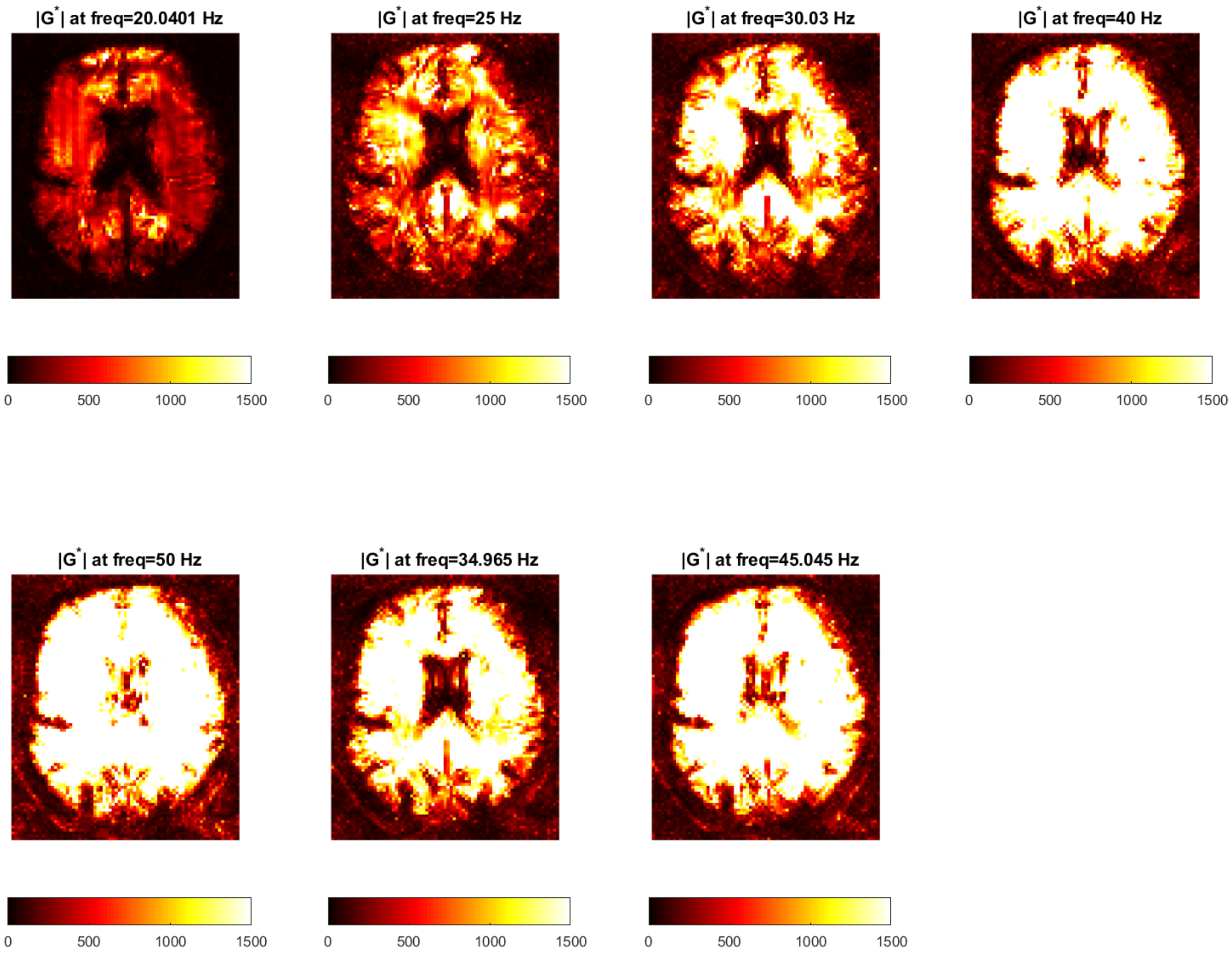
Divergence-Free filter helps to the model (reduce the divergence in the waveforms = reduces compression waves effects)



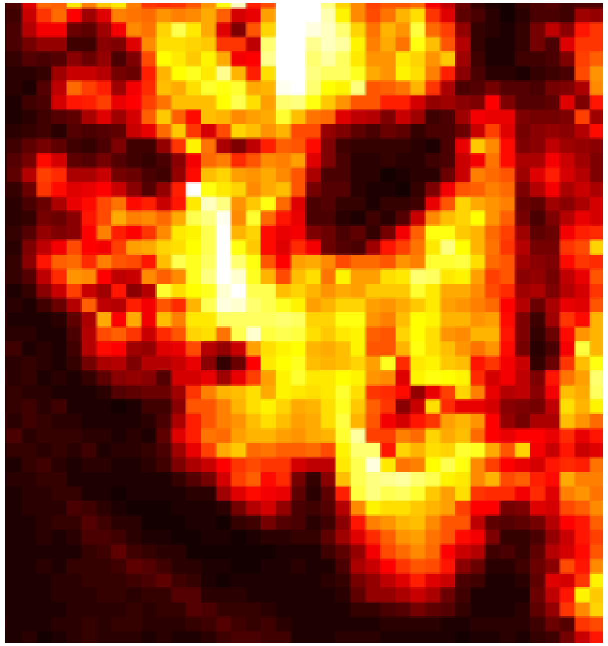


Some issues

Different amplitudes for different frequencies



Numerical artifacts



Final remarks

- We introduced a novel local solver based on FV schemes → only 1 derivative (but 2nd derivative of the model is hidden in boundary integrals, which may introduces instabilities.)
- Fast: Thanks to Eric's advices (and it has been conceived as a unique Matlab file)
- Divergence-Free filters seems to improve some results (to consider in future developments)
- As a local solver, a good delineation of interfaces produce interesting results, at expenses of stability. Magnitude of G* show variations within the tissue ← work in progress
- Wide and narrow band analyses should shed some light about stability.

Vielen Dank!