Analysis of exponential decay models

MATMEK-4270

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Recap - Finite differencing of exponential decay

The ordinary differential equation

$$u'(t) = -au(t), \quad u(0) = I, \quad y \in (0, T]$$

where $a > 0$ is a constant.

Solve the ODE by finite difference methods:

- Discretize in time:

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N_t - 1} < t_{N_t} = T$$

- Satisfy the ODE at N_t discrete time steps:

$$\begin{aligned} u'(t_n) &= -au(t_n), & n &\in [1, \ldots, N_t], \text{ or} \\ u'(t_{n+\frac{1}{2}}) &= -au(t_{n+\frac{1}{2}}), & n &\in [0, \ldots, N_t - 1] \end{aligned}$$

Finite difference algorithms

- Discretization by a generic θ-rule

$$\frac{u^n - u^{n-1}}{\triangle t} = -(1-\theta)au^{n-1} - \theta u^n$$

$$\begin{cases} \theta = 0 & \text{Forward Euler} \\ \theta = 1 & \text{Backward Euler} \\ \theta = 1/2 & \text{Crank-Nicolson} \end{cases}$$

Note $u^n = u(t_n)$

- Solve recursively: Set $u^0 = I$ and then

$$u^n = \frac{1-(1-\theta)a\triangle t}{1+\theta a\triangle t}u^{n-1} \quad \text{for } n > 0$$

Analysis of finite difference equations

Model:

$$u'(t) = -au(t), \quad u(0) = I$$

Method:

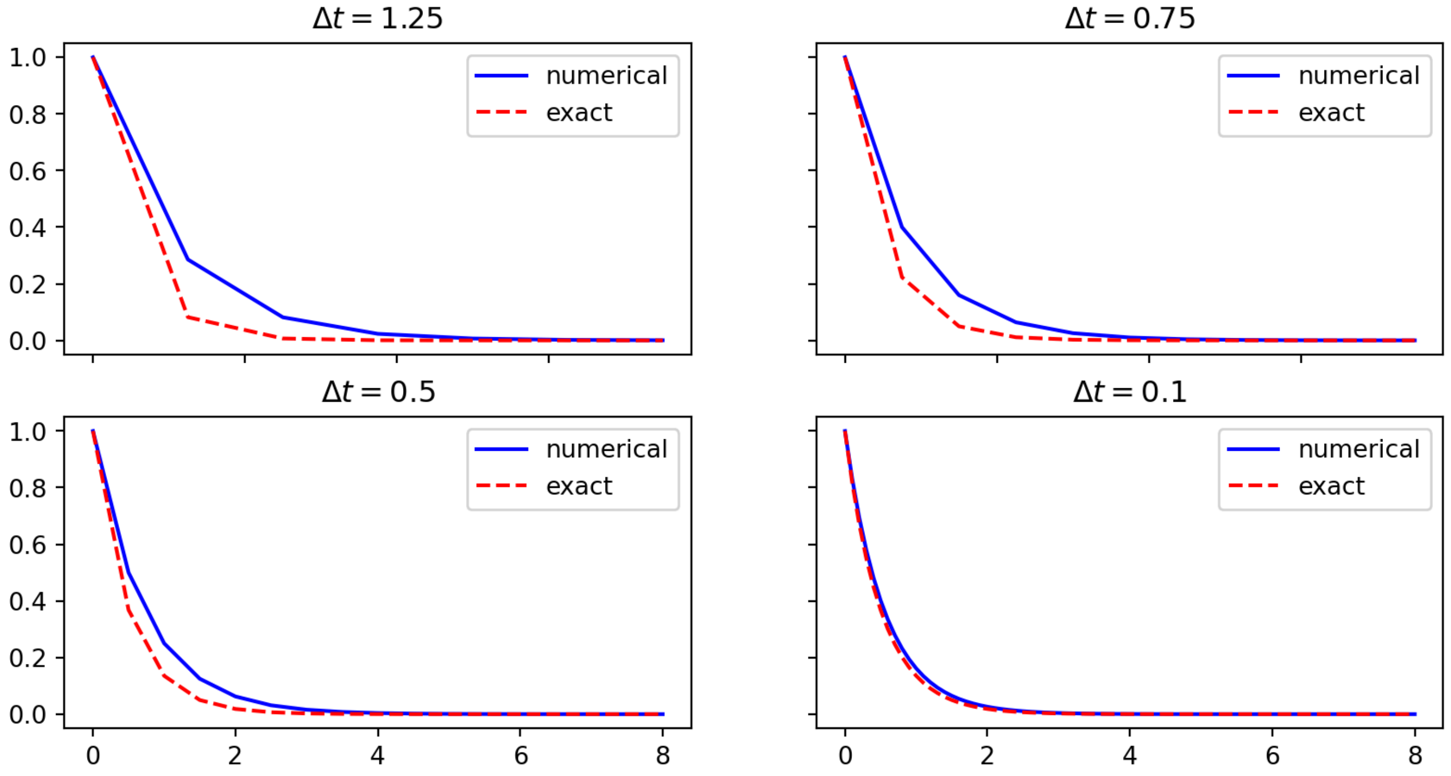
$$u^{n+1} = \frac{1 - (1-\theta)a\Delta t}{1 + \theta a\Delta t} u^n$$

Problem setting

How good is this method? Is it safe to use it?

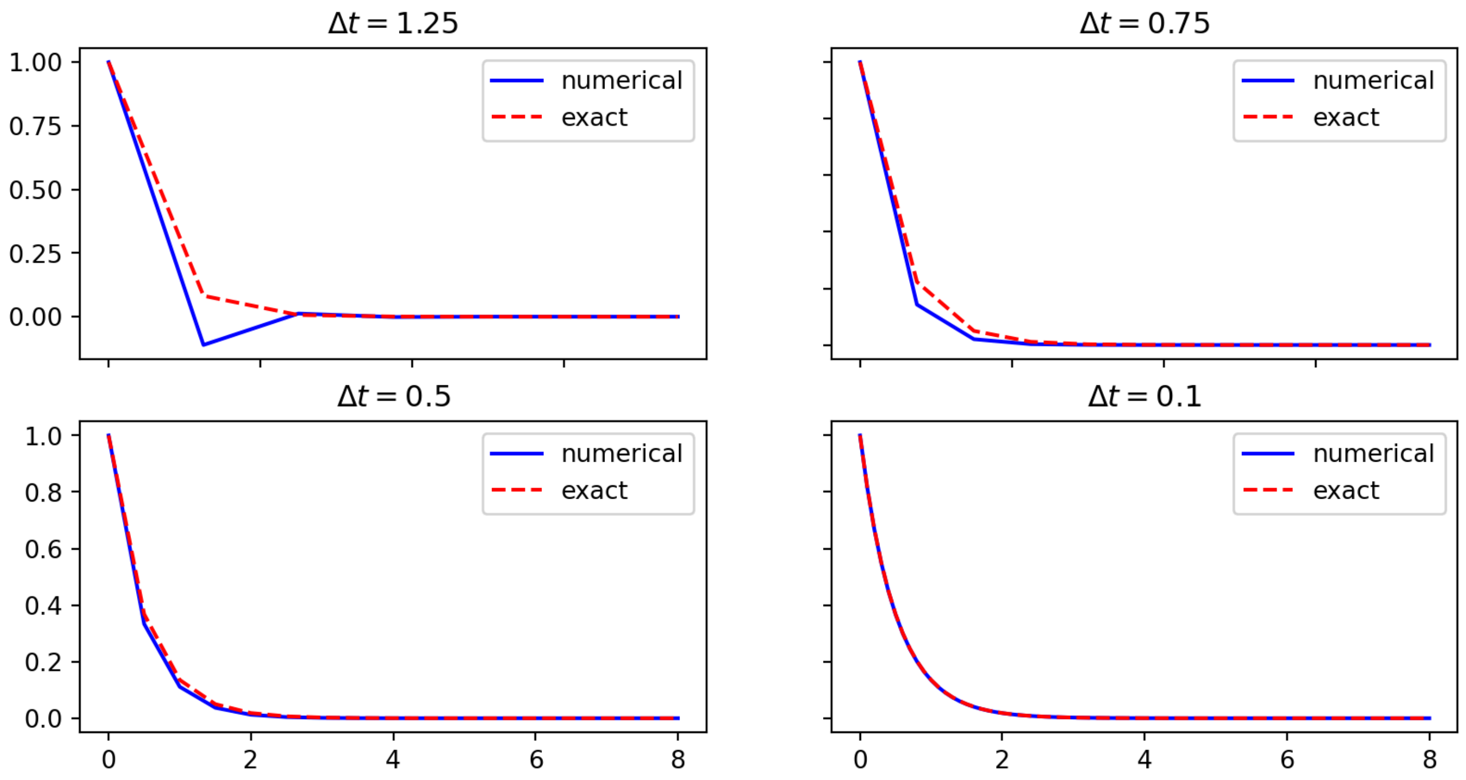
Encouraging numerical solutions - Backwards Euler

$I = 1, a = 2, \theta = 1, \Delta t = 1.25, 0.75, 0.5, 0.1.$



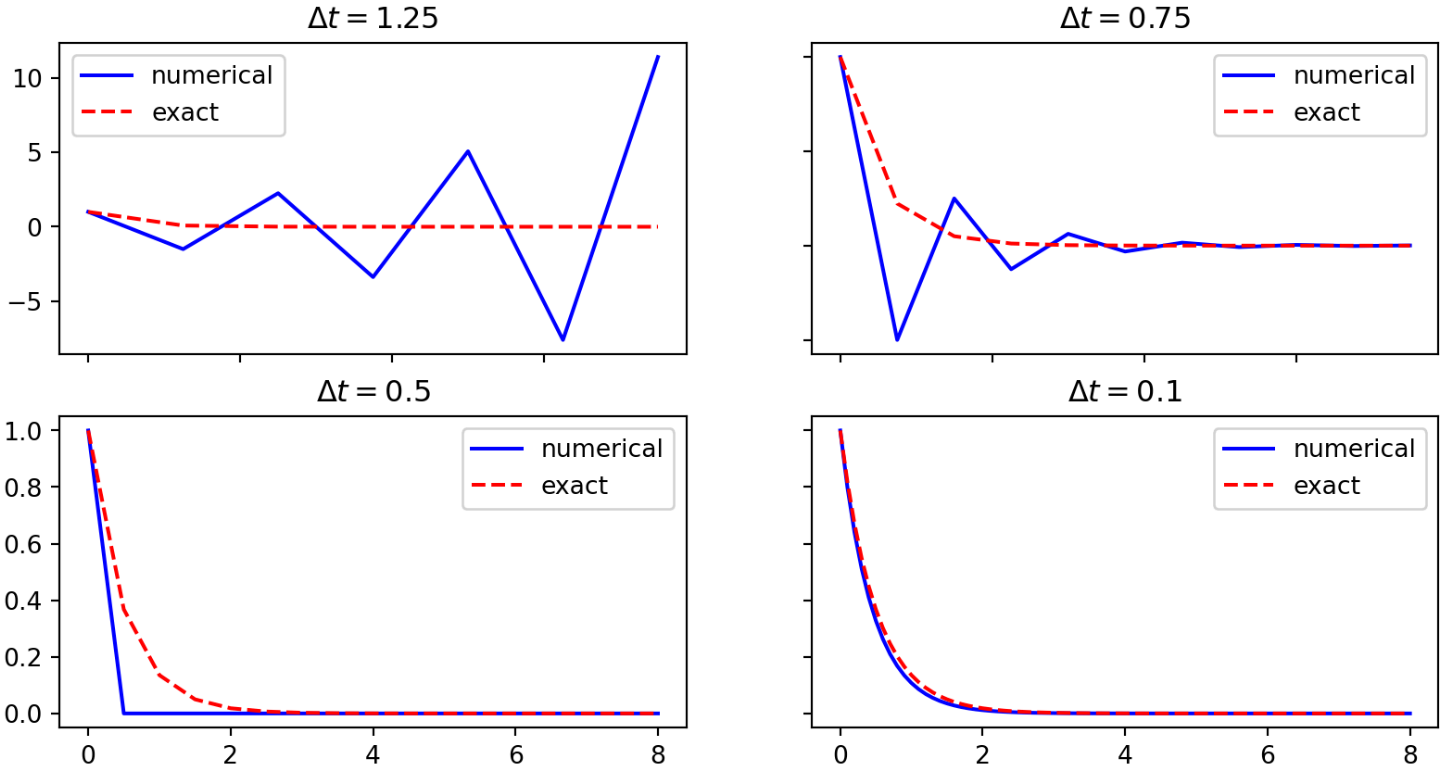
Discouraging numerical solutions - Crank-Nicolson

$I = 1, a = 2, \theta = 0.5, \Delta t = 1.25, 0.75, 0.5, 0.1.$



Discouraging numerical solutions - Forward Euler

$$I = 1, a = 2, \theta = 0, \Delta t = 1.25, 0.75, 0.5, 0.1.$$



Summary of observations

The characteristics of the displayed curves can be summarized as follows:

- The Backward Euler scheme *always* gives a monotone solution, lying above the exact solution.
- The Crank-Nicolson scheme gives the most accurate results, but for $\Delta t = 1.25$ the solution oscillates.
- The Forward Euler scheme gives a growing, oscillating solution for $\Delta t = 1.25$; a decaying, oscillating solution for $\Delta t = 0.75$; a strange solution $u^n = 0$ for $n \geq 1$ when $\Delta t = 0.5$; and a solution seemingly as accurate as the one by the Backward Euler scheme for $\Delta t = 0.1$, but the curve lies *below* the exact solution.
- Small enough Δt gives stable and accurate solution for all methods!

Problem setting

We ask the question

- Under what circumstances, i.e., values of the input data I, a, and Δt will the Forward Euler and Crank-Nicolson schemes result in undesired oscillatory solutions?

Techniques of investigation:

- Numerical experiments
- Mathematical analysis

Another question to be raised is

- How does Δt impact the error in the numerical solution?

Exact numerical solution

For the simple exponential decay problem we are lucky enough to have an exact numerical solution

$$u^n = IA^n, \quad A = \frac{1-(1-\theta)a\Delta t}{1+\theta a\Delta t}$$

Such a formula for the exact discrete solution is unusual to obtain in practice, but very handy for our analysis here.

Note

An exact dicrete solution fulfills a discrete equation (without round-off errors), whereas an exact solution fulfills the original mathematical equation.

Stability

Since $u^n = IA^n$,

- $A < 0$ gives a factor $(-1)^n$ and oscillatory solutions
- $|A| > 1$ gives growing solutions
- Recall: the exact solution is *monotone* and *decaying*
- If these qualitative properties are not met, we say that the numerical solution is *unstable*

For stability we need

$$A > 0 \quad \text{and} \quad |A| \leq 1$$

Computation of stability in this problem

$A < 0$ if

$$\frac{1 - (1-\theta)a\Delta t}{1 + \theta a\Delta t} < 0$$

To avoid oscillatory solutions we must have $A > 0$

$$\Delta t < \frac{1}{(1-\theta)a}, \theta < 1$$

- Always fulfilled for Backward Euler ($\theta = 1 \rightarrow 1 < 1 + a\Delta t$ always true)
- $\Delta t \leq 1/a$ for Forward Euler ($\theta = 0$)
- $\Delta t \leq 2/a$ for Crank-Nicolson ($\theta = 0.5$)

Computation of stability in this problem

$|A| \leq 1$ means $-1 \leq A \leq 1$

$$-1 \leq \frac{1-(1-\theta)a\Delta t}{1+\theta a\Delta t} \leq 1$$

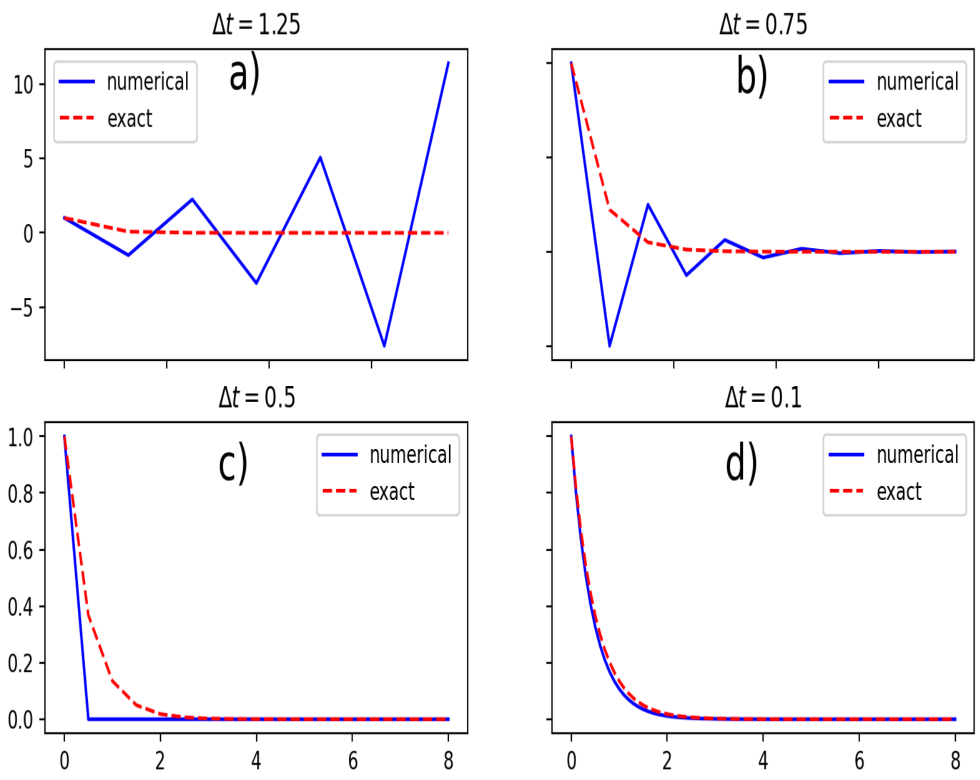
-1 is the critical limit (because $A \leq 1$ is always satisfied):

Always fulfilled for Backward Euler ($\theta = 0$) and Crank-Nicolson ($\theta = 0.5$). For forward Euler or simply $\theta < 0.5$ we have

$$\Delta t \leq \frac{2}{(1-2\theta)a},$$

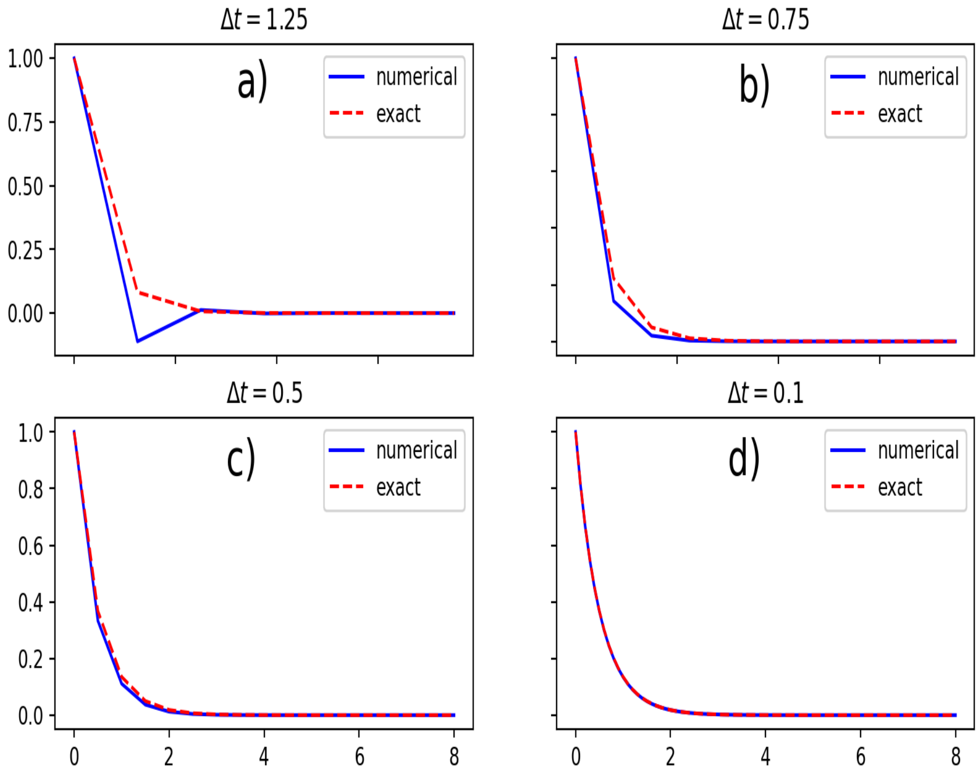
and thus $\Delta t \leq 2/a$ for stability of the Forward Euler ($\theta = 0$) method

Explanation of problems with Forward Euler



a. $a\Delta t = 2 \cdot 1.25 = 2.5$ and $A = -1.5$: oscillations and growth
b. $a\Delta t = 2 \cdot 0.75 = 1.5$ and $A = -0.5$: oscillations and decay
c. $\Delta t = 0.5$ and $A = 0$: $u^n = 0$ for $n > 0$
d. Smaller Δt: qualitatively correct solution

Explanation of problems with Crank-Nicolson



a. $\Delta t = 1.25$ and $A = -0.25$: oscillatory solution

Never any growing solution

Summary of stability

- Forward Euler is *conditionally stable*
  - $\Delta t < 2/a$ for avoiding growth
  - $\Delta t \leq 1/a$ for avoiding oscillations
- The Crank-Nicolson is *unconditionally stable* wrt growth and conditionally stable wrt oscillations
  - $\Delta t < 2/a$ for avoiding oscillations
- Backward Euler is unconditionally stable

Comparing amplification factors

u^{n+1} is an amplification A of u^n:

$$u^{n+1} = Au^n, \quad A = \frac{1-(1-\theta)a\Delta t}{1+\theta a\Delta t}$$

The exact solution is also an amplification:

$$u(t_{n+1}) = e^{-a(t_n+\Delta t)}$$
$$u(t_{n+1}) = e^{-a\Delta t}e^{-at_n}$$
$$u(t_{n+1} = A_e u(t_n), \quad A_e = e^{-a\Delta t}$$

A possible measure of accuracy: $A_e - A$