Function approximation with global functions

MATMEK-4270

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Important topics of todays lecture

- Approximation of smooth functions $u(x)$ using **global** basis functions
- Variational methods
  - The least squares method
  - The Galerkin method
- The collocation method
- Runge's phenomenon
- Legendre polynomials
- Spectral convergence

Function approximation with global functions

Consider a generic function

$$u(x), \quad x \in [a, b],$$

where $u(x)$ can by anything, like $u(x) = \exp(\sin(2\pi x))$, $u(x) = x^x$, etc., etc.

We want to find an approximation to $u(x)$ using

$$u(x) \approx u_N(x) = \sum_{k=0}^{N} \hat{u}_k \psi_k(x)$$

where $\psi_k(x)$ is a **global** basis function defined on the entire domain $[a, b]$.

In the approximation

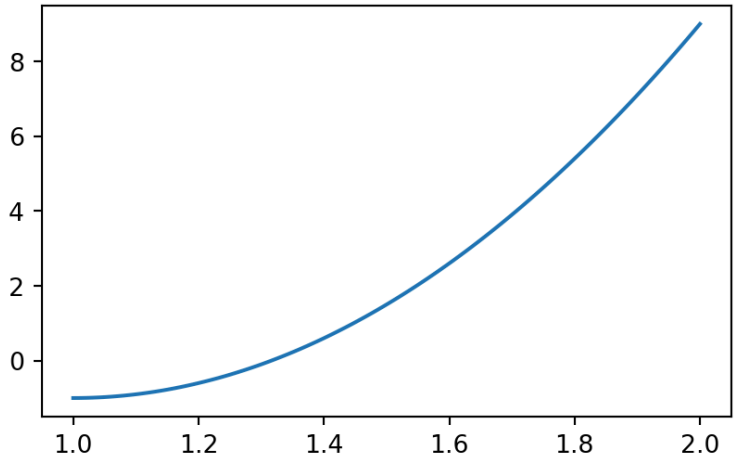
$$u_N(x) = \sum_{k=0}^{N} \hat{u}_k \psi_k(x) \tag{1}$$

- $\psi_k(x)$ is a basis function
- $\{\psi_k\}_{k=0}^N$ is a basis
- $\{\hat{u}_k\}_{k=0}^N$ are expansion coefficients (the unknowns)
- $V_N = \text{span}\{\psi_k\}_{k=0}^N$ is a function space (which is a vector space)

For example, $V_N = \text{span}\{x^k\}_{k=0}^N$ is the space of all polynomials of order less than or equal to N. (More commonly referred to as $\mathbb{P}_N$)

If we say that $u_N \in V_N$, then we mean that u_N can be written as (1) and that u_N is a vector in the vector space (or function space) V_N.

Example: $u(x) = 10(x-1)^2 - 1$ for $x \in [1, 2]$



Let $V_N = \{1, x\}$ be the space of all straight lines.

What is the best approximation $u_N \in V_N$ to the function $u(x)$?

How do we decide what is best? All we know is that

$$u_N(x) = \hat{u}_0 + \hat{u}_1 x.$$

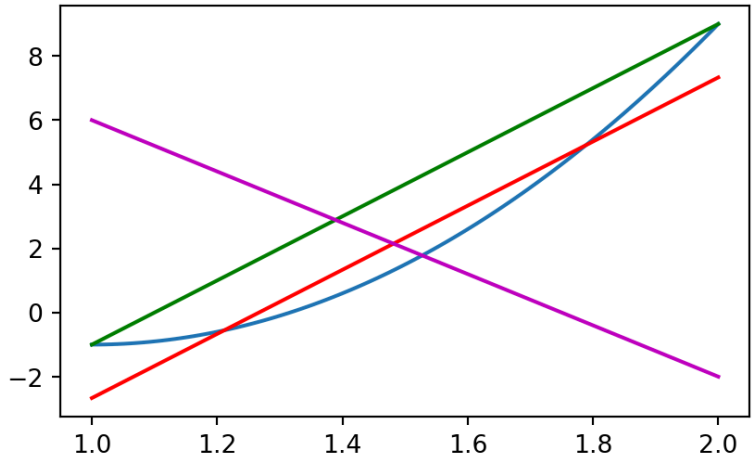
How to find the **best** $\{\hat{u}_0, \hat{u}_1\}$?

How to find the best approximation?

What options are there? Who decides what is **best**?

The three methods covered in this class are

- The least squares method
- The Galerkin method
- The collocation method



The first two are **variational** methods, whereas the third is an **interpolation** method.

Each method will give us $N+1$ equations for the $N+1$ unknowns $\{\hat{u}_k\}_{k=0}^{N}$!

- A variational method derives equations by using integration over the domain
- A collocation method derives equations at different points in the mesh

Variational methods

Since a variational method derives equations by integrating over the domain, we need to define a special notation. The L^2 inner product is defined as

$$(f, g) = \int_\Omega f(x) g(x) d\Omega,$$

for two real functions $f(x)$ and $g(x)$ (complex functions have a slightly different inner product). The symbol Ω here represents the domain of interest. For our first example $\Omega = [1, 2]$.

> **Note**
>
> Sometimes the inner product is written as $(f, g)_{L^2(\Omega)}$, in order to clarify that it is the L^2 inner product on a certain domain. Normally, the $L^2(\Omega)$ subscript will be dropped.

The $L^2(\Omega)$ norm

The inner product is used to define an $L^2(\Omega)$ **norm**

$$\|u\| = \sqrt{(u, u)}$$

The norm gives us a measure for the length or size of a vector.

The $L^2(\Omega)$ error norm

If we define the pointwise error as

$$e(x) = u(x) - u_N(x)$$

then the $L^2(\Omega)$ error norm

$$\|e\| = \sqrt{(e, e)} = \sqrt{\int_\Omega e^2 dx}$$

gives us a measure of the error over the entire domain

Back to our example - how do we find the best possible $u_N \approx u$?

If we use the error

$$e(x) = u(x) - u_N(x)$$

then the L^2 error norm $\|e\|$ represents the error integrated over the entire domain.

Idea!

Why don't we find u_N such that it minimizes the L^2 error norm!

$\longrightarrow$ The Least squares method!

The least squares method for function approximation

We want to approximate

$$u_N(x) \approx u(x)$$

using a function space V_N for u_N. The pointwise error in u_N is defined as

$$e(x) = u(x) - u_N(x)$$

Define the square of the L^2 error norm

$$E = \|e\|^2 = (e, e)$$

The least squares method is to find $u_N \in V_N$ such that

$$\frac{\partial E}{\partial \hat{u}_j} = 0, \quad j = 0, 1, \ldots, N.$$

The least squares method

Find $u_N \in V_N$ such that

$$\frac{\partial E}{\partial \hat{u}_j} = 0, \quad j = 0, 1, \ldots, N$$

This gives us $N+1$ equations for $N+1$ unknowns $\{\hat{u}_j\}_{j=0}^N$.

But how can we compute $\frac{\partial E}{\partial \hat{u}_j}$?

$$\frac{\partial E}{\partial \hat{u}_j} = \frac{\partial}{\partial \hat{u}_j} \int_\Omega e^2 dx = \int_\Omega 2e \frac{\partial e}{\partial \hat{u}_j} dx$$

Insert for $e(x) = u(x) - u_N(x) = u(x) - \sum_{k=0}^N \hat{u}_k \psi_k$

$$\frac{\partial E}{\partial \hat{u}_j} = \int 2 \left(u - \sum_{k=0}^N \hat{u}_k \psi_k \right) \overbrace{\frac{\partial}{\partial \hat{u}_j} \left(u - \sum_{k=0}^N \hat{u}_k \psi_k \right)}^{-\psi_j} dx$$

If we now set $\frac{\partial E}{\partial \hat{u}_j} = 0$, then

$$\frac{\partial E}{\partial \hat{u}_j} = -\int 2\left(u - \sum_{k=0}^{N} \hat{u}_k \psi_k\right)\psi_j dx = 0$$

$$\longrightarrow \int u\psi_j dx = \sum_{k=0}^{N} \int \psi_k \psi_j dx \hat{u}_k, \quad j = 0, 1, \ldots, N$$

gives us $N+1$ linear equations for the $N+1$ unknown $\{\hat{u}_k\}_{k=0}^{N}$

Back to the example: $u(x) = 10(x-1)^2 - 1$ for $x \in [1, 2]$

Find $u_N \in \mathbb{P}_1$ (meaning find $\hat{u}_0$ and $\hat{u}_1$) such that

$$\frac{\partial E}{\partial \hat{u}_0} = 0 \text{ and } \frac{\partial E}{\partial \hat{u}_1} = 0$$

Two equations for two unknowns.

Implement the 2 equations in Sympy

```
1  def inner(u, v, domain=(-1, 1), x=x):
2      return sp.integrate(u*v, (x, domain[0], domain[1]))
3
4  u0, u1 = sp.symbols('u0,u1')
5  err = u-(u0+u1*x)
6  E = inner(err, err, (1, 2))
7  eq1 = sp.Eq(sp.diff(E, u0, 1), 0)
8  eq2 = sp.Eq(sp.diff(E, u1, 1), 0)
9  display(eq1)
10 display(eq2)
```

$$2u_0 + 3u_1 - \frac{14}{3} = 0$$

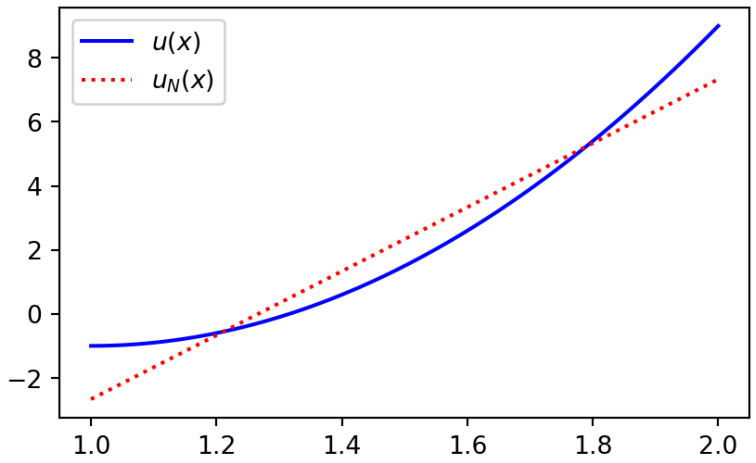
$$3u_0 + \frac{14u_1}{3} - \frac{26}{3} = 0$$

```
1  uhat = sp.solve((eq1, eq2), (u0, u1))
2  print(uhat)
```

```
{u0: -38/3, u1: 10}
```

The LSM best fit equation is

```
uN = -38/3 + 10x
```



Linear algebra approach

$$\underbrace{\int u\psi_j dx}_{b_j} = \sum_{k=0}^{N} \underbrace{\int \psi_k \psi_j dx}_{a_{jk}} \underbrace{\hat{u}_k}_{x_k}, \quad j = 0, 1, \ldots, N.$$

$$\boldsymbol{b} = A\boldsymbol{x} \longrightarrow \boldsymbol{x} = A^{-1}\boldsymbol{b}$$

Assemble **mass matrix** $a_{jk} = \int \psi_k \psi_j dx$ and vector $b_j = \int u\psi_j dx$

```
A = sp.zeros(2, 2)
b = sp.zeros(2, 1)
psi = (sp.S(1), x)
u = 10*(x-1)**2-1
for i in range(2):
    for j in range(2):
        A[i, j] = sp.integrate(psi[i]*psi[j], (x, 1, 2))
    b[i] = sp.integrate(psi[i]*u, (x, 1, 2))
print(A)
print(A**(-1) @ b)
```

```
Matrix([[1, 3/2], [3/2, 7/3]])
Matrix([[-38/3], [10]])
```

Same result as before!

The Galerkin method is a more popular variational method for solving differential equations

Here the error is required to be orthogonal to the basis functions (for the L^2 inner product)

$$(e, \psi_j) = 0, \quad j = 0, 1, \ldots, N.$$

Again $N + 1$ equations for $N + 1$ unknowns.

Not as much work as the least squares method, and even easier to implement in Sympy:

```
1 eq1 = sp.Eq(inner(err, 1, domain=(1, 2)), 0)
2 eq2 = sp.Eq(inner(err, x, domain=(1, 2)), 0)
3 display(eq1)
4 display(eq2)
```

$$-u_0 - \frac{3u_1}{2} + \frac{7}{3} = 0$$

$$-\frac{3u_0}{2} - \frac{7u_1}{3} + \frac{13}{3} = 0$$

```
1 sp.solve((eq1, eq2), (u0, u1))
```

```
{u0: -38/3, u1: 10}
```

The Galerkin formulation of the problem is

Find $u_N \in V_N$ such that

$$(e, v) = 0, \quad \forall\, v \in V_N$$

Here v is often referred to as a **test** function, whereas u_N is referred to as a **trial** function.

Alternative formulation:

Find $u_N \in V_N$ such that

$$(e, \psi_j) = 0, \quad j = 0, 1, \ldots, N.$$

> **Note**
>
> In order to satisfy $(e, v) = 0$ for all $v \in V_N$, we can insert for $v = \sum_{j=0}^{N} \hat{v}_j \psi_j$ such that
>
> $$\sum_{j=0}^{N} (e, \psi_j) \hat{v}_j = 0.$$
>
> In order for this to always be satisfied, we require that $(e, \psi_j) = 0$ for all $j = 0, 1, \ldots, N$.

Linear algebra problem for the Galerkin method

We have the $N+1$ equations

$$(e, \psi_i) = 0, \quad i = 0, 1, \ldots, N.$$

Insert for $e = u - u_N = u - \sum_{j=0}^{N} \hat{u}_j \psi_j$ to get

$$(u - \sum_{j=0}^{N} \hat{u}_j \psi_j, \psi_i) = 0, \quad i = 0, 1, \ldots, N$$

and thus the linear algebra problem $A\boldsymbol{x} = \boldsymbol{b}$:

$$\sum_{j=0}^{N} \underbrace{(\psi_j, \psi_i)}_{a_{ij}} \underbrace{\hat{u}_j}_{x_j} = \underbrace{(u, \psi_i)}_{b_i}, \quad i = 0, 1, \ldots, N$$

Orthogonality is a big deal!

If we choose basis $\{\psi_j\}_{j=0}^N$ such that all basis functions are **orthonormal**, then

$$a_{ij} = (\psi_j, \psi_i) = \delta_{ij} = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ if } i \neq j. \end{cases}$$

and thus

$$A\boldsymbol{x} = \boldsymbol{b} \rightarrow \boldsymbol{x} = \boldsymbol{b} \quad \text{or} \quad \hat{u}_j = (u, \psi_j), \quad j = 0, 1, \ldots, N$$

If the basis functions are merely orthogonal, then

$$(\psi_j, \psi_i) = \|\psi_i\|^2 \delta_{ij},$$

where $\|\psi_i\|^2$ is the squared L^2 norm of ψ_i. We can still easily solve the linear algebra system (because A is a diagonal matrix)

$$\hat{u}_i = \frac{(u, \psi_i)}{\|\psi_i\|^2} \quad \text{for } i = 0, 1, \ldots, N$$

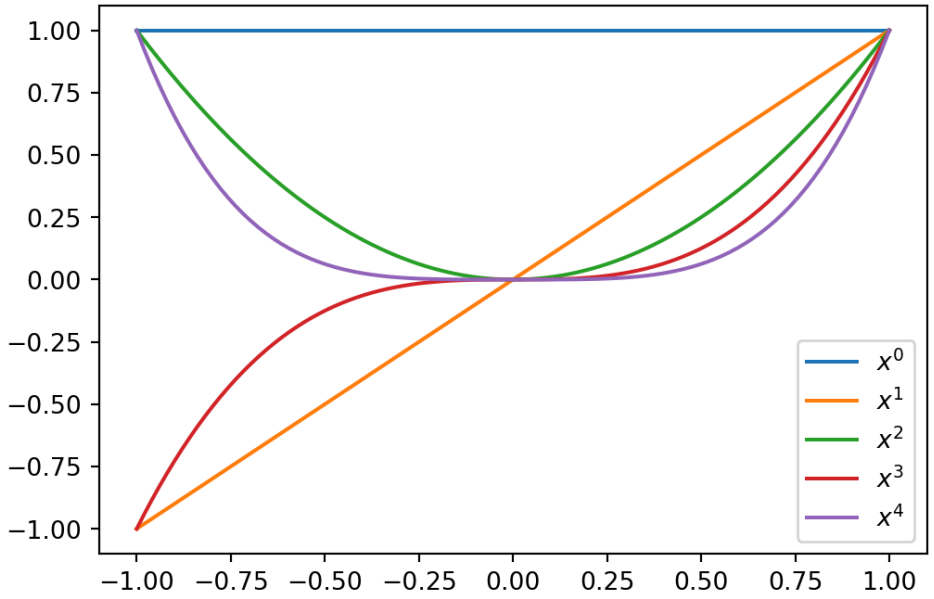
Polynomial basis functions

The monomial basis

$$\{x^n\}_{n=0}^{N},$$

is a basis for $\mathbb{P}_N$. However, it is not a good basis. This is because the basis functions are **not orthogonal** and the mass matrix A is **ill conditioned**. In short, this means it is difficult to solve $A\boldsymbol{x} = \boldsymbol{b}$ with good accuracy using finite precision computers.

The basis functions are shown below for $x \in [-1, 1]$

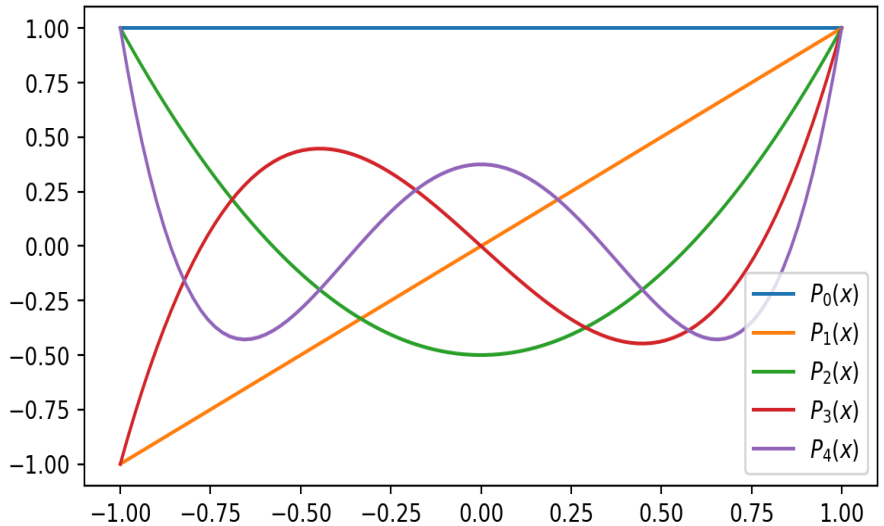


Legendre polynomials are much better basis functions

Legendre polynomials are defined on the domain $\Omega = [-1, 1]$ as the recursion

$$
\begin{aligned}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2}(3x^2 - 1), \\
&\vdots \\
(j+1)P_{j+1}(x) &= (2j+1)xP_j(x) - jP_{j-1}(x).
\end{aligned}
$$

The first 5 Legendre polynomials are plotted on the right



The Legendre polynomials are L^2 orthogonal on the reference domain $\Omega = [-1, 1]$

$$(P_j, P_i)_{L^2(-1,1)} = \frac{2}{2j+1}\delta_{ij}.$$

Orthogonality in the L^2 inner product space makes the Legendre polynomials very popular!

Using $\|P_j\|^2 = \frac{2}{2j+1}$ we get that

$$\hat{u}_j = \frac{(u, P_j)}{\|P_j\|^2} = \frac{2j+1}{2}(u, P_j)$$

Note

This requires that the domain of $u(x)$ is $[-1, 1]$

If the physical domain $[a, b]$ is different from the reference domain $[-1, 1]$, then we need to map

Many well-known basis functions work only on a given reference domain

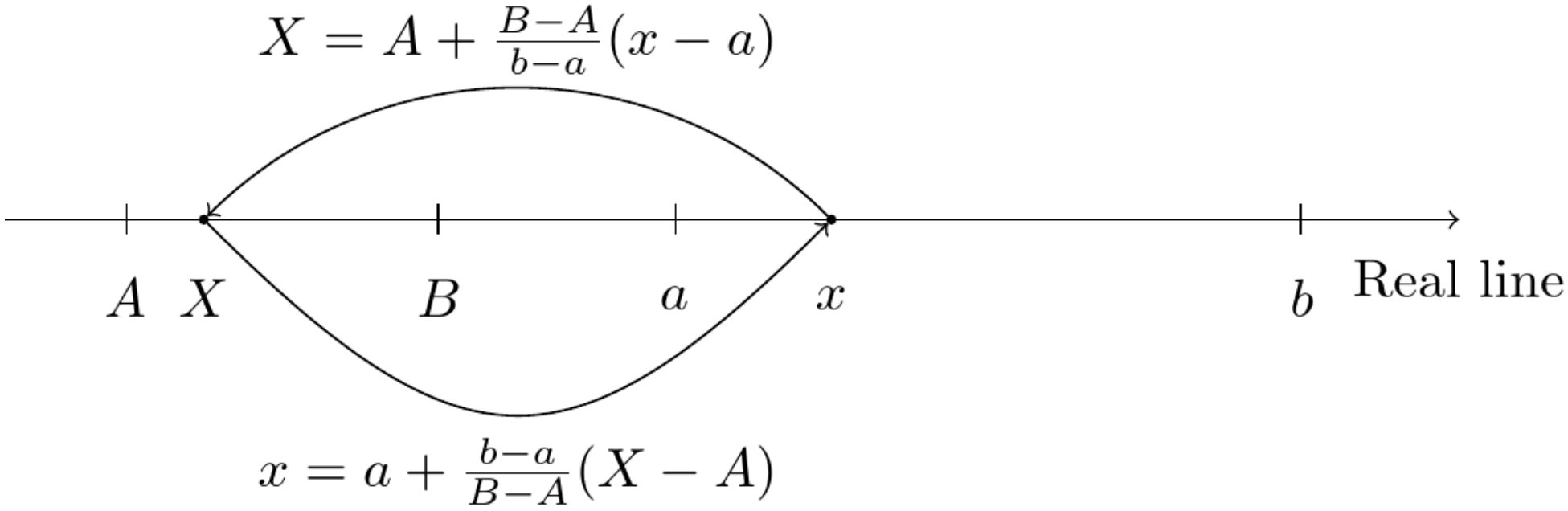
- Sines and cosines $[0, \pi]$
- Legendre polynomials $[-1, 1]$
- Bernstein polynomials $[0, 1]$
- Chebyshev polynomials $[-1, 1]$

Let X be the coordinate in the computational (reference) domain $[A, B]$ and x be the coordinate in the true physical domain $[a, b]$. A linear (affine) mapping, from X to x (and back) is then

$$X \in [A, B] \quad \text{and} \quad x \in [a, b]$$

$$x = a + \frac{b - a}{B - A}(X - A) \quad \text{and} \quad X = A + \frac{B - A}{b - a}(x - a)$$

Affine map



- Equations are defined in the real domain $[a, b]$
- Equations are solved in the computational domain $[A, B]$

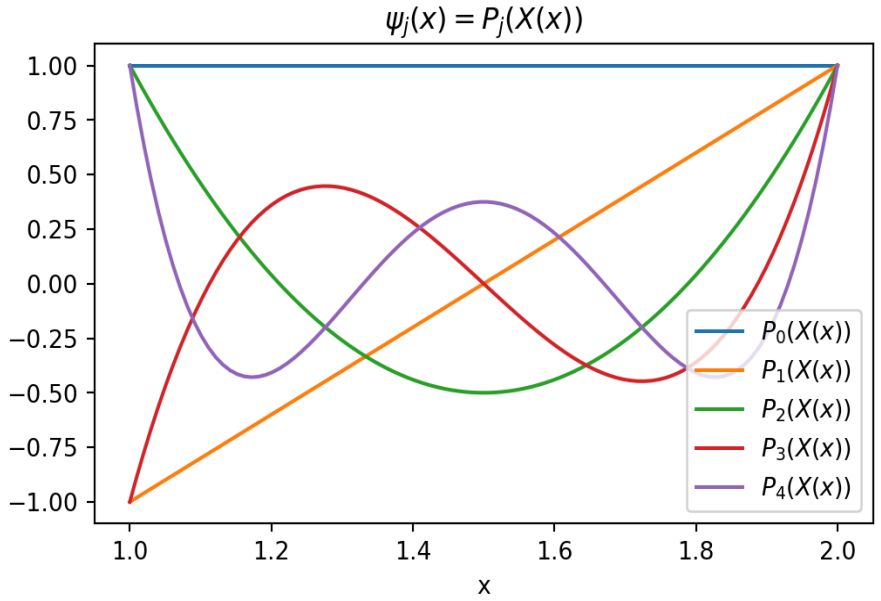
The mapping makes it possible to use Legendre polynomials in any domain $[a, b]$

The basis functions $\psi_j(x)$ are simply

$$\psi_j(x) = P_j(X(x)) \quad X \in [-1, 1], \quad x \in [a, b]$$

$$X(x) = -1 + \frac{2}{b-a}(x-a)$$

```
xj = np.linspace(1, 2, 100)
A, B = -1, 1
a, b = 1, 2
Xj = -1 + (B-A)/(b-a)*(xj-a)
plt.figure(figsize=(6, 3.5))
legend = []
p = np.zeros(100)
for n in range(5):
    l = sp.legendre(n, x)
    p[:] = sp.lambdify(x, l)(Xj)
    plt.plot(xj, p)
    legend.append(f'$P_{n}(X(x))$')
plt.title(r'$\psi_j(x)=P_j(X(x))$')
plt.xlabel('x')
plt.legend(legend);
```



The mapping complicates the inner product

The Galerkin method is defined on the true domain using $L^2([a,b])$

$$(u(x) - u_N(x), \psi_j(x))_{L^2([a,b])} = \int_a^b (u(x) - u_N(x))\psi_j(x)dx = 0, \quad j = 0, 1, \ldots, N.$$

Insert for $u_N = \sum_{k=0}^{N} \hat{u}_k \psi_k(x)$ and rearrange

$$\sum_{k=0}^{N} \int_a^b \psi_k(x)\psi_j(x)\, dx\, \hat{u}_k = \int_a^b u(x)\psi_j(x)dx, \quad j = 0, 1, \ldots, N.$$

So far just regular Galerkin.

Now introduce $\psi_j(x) = P_j(X(x))$ and integrate with a change of variables $x \to X$. The new integration limits are then $X(a) = -1$ and $X(b) = 1$

$$\sum_{j=0}^{N} \int_{-1}^{1} P_k(X)P_j(X) \cancel{\frac{dx}{dX}}\, dX\, \hat{u}_k = \int_{-1}^{1} u(x(X))P_j(X) \cancel{\frac{dx}{dX}}\, dX, \quad j = 0, 1, \ldots, N$$

We get the linear system with inner products over the reference domain

$$\sum_{j=0}^{N} \int_{-1}^{1} P_k(X) P_j(X)\, dX\, \hat{u}_k = \int_{-1}^{1} u(x(X)) P_j(X)\, dX, \quad j = 0, 1, \ldots, N$$

which can be written as

$$\sum_{k=0}^{N} \underbrace{(P_k(X), P_j(X))_{L^2(-1,1)}}_{\frac{2}{2j+1}\delta_{kj}} \hat{u}_k = (u(x(X)), P_j(X))_{L^2(-1,1)}$$

such that

$$\hat{u}_j = \frac{2j+1}{2} (u(x(X)), P_j(X))_{L^2(-1,1)} \quad j = 0, 1, \ldots, N$$

Implementation of the mapped inner product

The following inner product is valid for any mapping

```
def inner(u, v, domain, ref_domain=(-1, 1)):
  A, B = ref_domain
  a, b = domain
  X = a + (b-a)*(x-A)/(B-A)
  us = u.subs(x, X) #  u(x(X))
  return sp.integrate(us*v, (x, A, B))
```

For example, for our $u(x) = 10(x-1)^2 - 1$ in the domain $\Omega = [a, b] = [1, 2]$ we get

```
uhat = lambda u, j: (2*j+1) * inner(u, sp.legendre(j, x), (1, 2))/2
u = 10*(x-1)**2-1
u0, u1 = uhat(u, 0), uhat(u, 1)
print(u0, u1)
print(f"uN(x) = {u0}+{u1}X(x)")
```

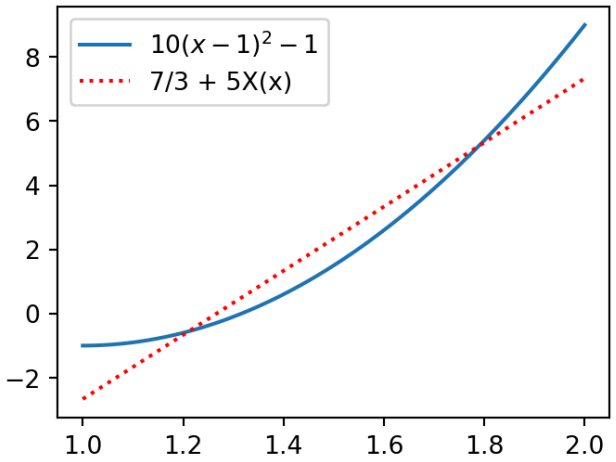
```
7/3 5
uN(x) = 7/3+5X(x)
```

The Legendre polynomials use the reference coordinate X, whereas the true function $u_N(x)$ is a function of x from the true space.

> **Note**
>
> The `inner` product implemented here is using Sympy's `integrate` function, which may be too slow or not find the solution at all. It is normally better to use scipy.integrate.quad

Plot the Legendre approximation

```
plt.figure(figsize=(4, 3))
xj = np.linspace(1, 2, 100)
Xj = -1 + 2/(b-a)*(xj-a)
plt.plot(xj, sp.lambdify(x, u)(xj))
plt.plot(xj, uhat(u, 0) + uhat(u, 1)*Xj, 'r:')
plt.legend(['$10(x-1)^2-1$', f'{uhat(u, 0)} + {uhat(u, 1)}X(x)']);
```



The result is exactly the same as was found with the monomial basis $\{1, x\}$.

Tip

Numpy has a complete module dedicated to Legendre polynomials

Summary of variational methods

We want to approximate $u(x)$ with $u_N(x) \in V_N$

$$u_N(x) \approx u(x)$$

using the inner product

$$(f, g) = \int_a^b f(x)g(x)dx$$

Least squares method

Find $u_N \in V_N$ such that

$$\frac{\partial E}{\partial \hat{u}_j} = 0, \quad j = 0, 1, \ldots, N$$

where $E = (u - u_N, u - u_N)$.

Minimizing the L^2 error norm.

Galerkin method

Find $u_N \in V_N$ such that

$$(u - u_N, v) = 0, \quad \forall\, v \in V_N$$

Or equivalently

$$(u - u_N, \psi_j) = 0, \quad j = 0, 1, \ldots, N$$

Error orthogonal to the test functions

The collocation method

Takes a very different approach than the variational methods, but the objective is still to find $u_N \in V_N$ such that

$$u_N(x) = \sum_{j=0}^{N} \hat{u}_j \psi_j(x)$$

The collocation method requires that for some $N+1$ **chosen** mesh points $\{x_j\}_{j=0}^{N}$ the following $N+1$ equations are satisfied

$$u(x_i) - u_N(x_i) = 0, \quad i = 0, 1, \ldots, N.$$

Inserting for $u_N(x_i)$ we get the $N+1$ linear algebra equations

$$\sum_{j=0}^{N} \hat{u}_j \underbrace{\psi_j(x_i)}_{a_{ij}} = u(x_i) \longrightarrow \sum_{j=0}^{N} a_{ij} \hat{u}_j = u(x_i), \quad i = 0, 1, \ldots, N$$

The Lagrange interpolation method described in lecture 7 is a collocation method

The Lagrange basis functions $\ell_j(x)$ are defined for points $\{x_j\}_{j=0}^N$ such that

$$\ell_j(x) = \prod_{\substack{0\leq m\leq N \\ m\neq j}} \frac{x - x_m}{x_j - x_m} \quad \text{and} \quad \ell_j(x_i) = \delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Hence the matrix $a_{ij} = \psi_j(x_i) = \delta_{ij}$ and we can simply use the coefficients

$$\hat{u}_j = u(x_j) \quad \text{such that} \quad u_N(x) = \sum_{j=0}^{N} u(x_j)\ell_j(x)$$

There is no integration and the method is often favored for its simplicity. There is a problem though. How do you choose the collocation points?!

> **Note**
>
> The Lagrange polynomial here is using all $N+1$ mesh points. This is different from lecture 7, where we only used a few mesh points close to the interpolation point.

Lagrange collocation method for $u(x) = 10(x-1)^2 - 1, x \in [1, 2]$

The approximation using two collocation points (linear function, $u_N \in V_N = \text{span}\{1, x\}$) is

$$u_N(x) = \hat{u}_0 \ell_0(x) + \hat{u}_1 \ell_1(x),$$

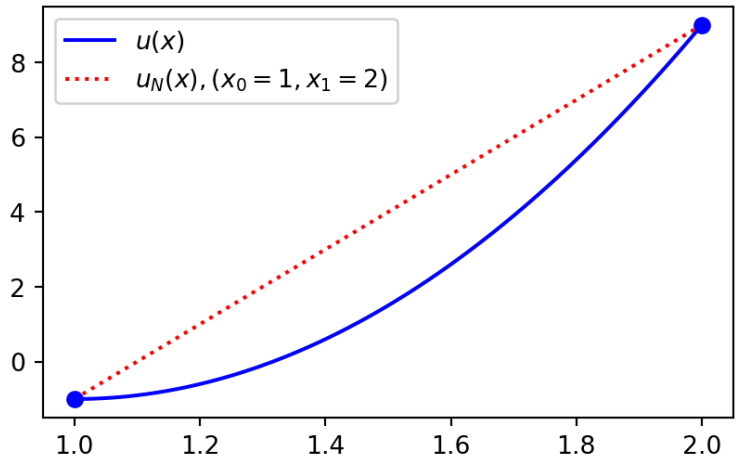
or more simply

$$u_N(x) = u(x_0)\ell_0(x) + u(x_1)\ell_1(x).$$

We can choose the end points $x_0 = 1$ and $x_1 = 2$ and reuse the two functions `Lagrangebasis` and `Lagrangefunction` from lecture 7. The result is then as shown on the next slide

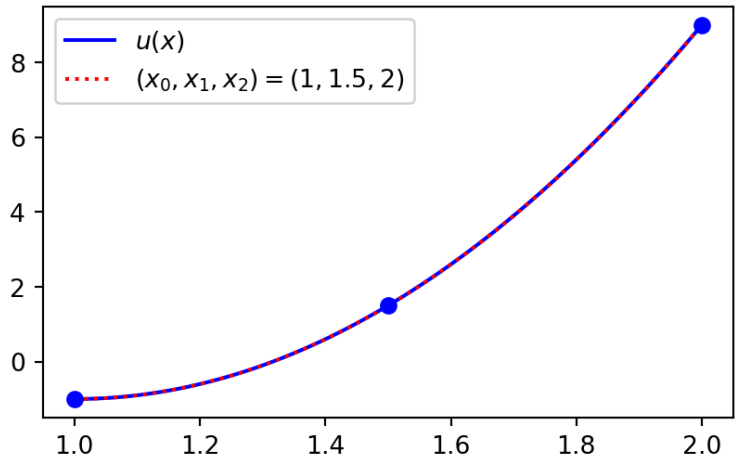
Collocation

```
from lagrange import Lagrangebasis, Lagrangefuncti
xj = np.linspace(1, 2, 100)
u = 10*(x-1)**2-1
xp = np.array([1, 2])
ell = Lagrangebasis(xp)
L = Lagrangefunction([u.subs(x, xi) for xi in xp],
plt.figure(figsize=(5, 3))
plt.plot(xj, sp.lambdify(x, u)(xj), 'b')
plt.plot(xj, sp.lambdify(x, L)(xj), 'r:')
plt.plot(xp, [u.subs(x, xi) for xi in xp], 'bo')
plt.legend(['$u(x)$', '$u_N(x), (x_0=1, x_1=2)$'])
```



Use three points and the approximation is perfect because $u(x)$ is a 2nd order polynomial

```
xp = np.array([1, 1.5, 2])
ell = Lagrangebasis(xp)
L = Lagrangefunction([u.subs(x, xi) for xi in xp],
plt.figure(figsize=(5, 3))
plt.plot(xj, sp.lambdify(x, u)(xj), 'b')
plt.plot(xj, sp.lambdify(x, L)(xj), 'r:')
plt.plot(xp, [u.subs(x, xi) for xi in xp], 'bo')
plt.legend(['$u(x)$', '$(x_0, x_1, x_2) = (1, 1.5,
```



High order interpolation on uniform grids is bad

Lets consider a more difficult function

$$u(x) = \frac{1}{1+25x^2}, \quad x \in [-1,1],$$

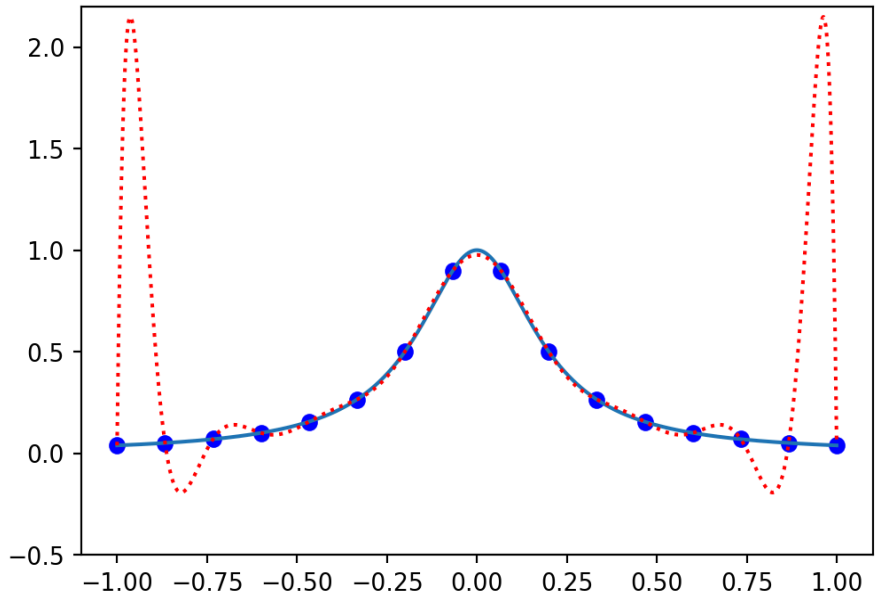
and attempt to approximate it with Lagrange polynomials on a uniform grid.

Use

$$\begin{aligned} N &= 15 \\ x_i &= -1 + \frac{2i}{N}, \quad i = 0, 1, \ldots, N \\ u(x_i) = u_N(x_i) &= \sum_{j=0}^{N} u(x_j)\ell_j(x_i), \quad i = 0, 1, \ldots, N \end{aligned}$$

Implementation

```
u = 1/(1+25*x**2)
N = 15
xj = np.linspace(-1, 1, N+1)
uj = sp.lambdify(x, u)(xj)
ell = Lagrangebasis(xj)
L = Lagrangefunction(uj, ell)
yj = np.linspace(-1, 1, 1000)
plt.figure(figsize=(6, 4))
plt.plot(xj, uj, 'bo')
plt.plot(yj, sp.lambdify(x, u)(yj))
plt.plot(yj, sp.lambdify(x, L)(yj), 'r:')
ax = plt.gca()
ax.set_ylim(-0.5, 2.2);
```



Large over and undershoots in the approximation Lagrange polynomial!

Runge's phenomenon! Interpolation on uniform grids may lead to large over and undershoots.

Why?

Runge's phenomenon

Define the monic polynomial with roots in $\{x_j\}_{j=0}^N$ as

$$p_N(x) = \prod_{j=0}^{N}(x - x_j)$$

It can be shown that the error in the Lagrange polynomial $u_N(x)$ using $\{x_j\}_{j=0}^N$ is

$$u(x) - u_N(x) = \frac{u^{(N)}(\xi)}{(N+1)!}p_N(x)$$

for some $\xi \in [a, b]$.

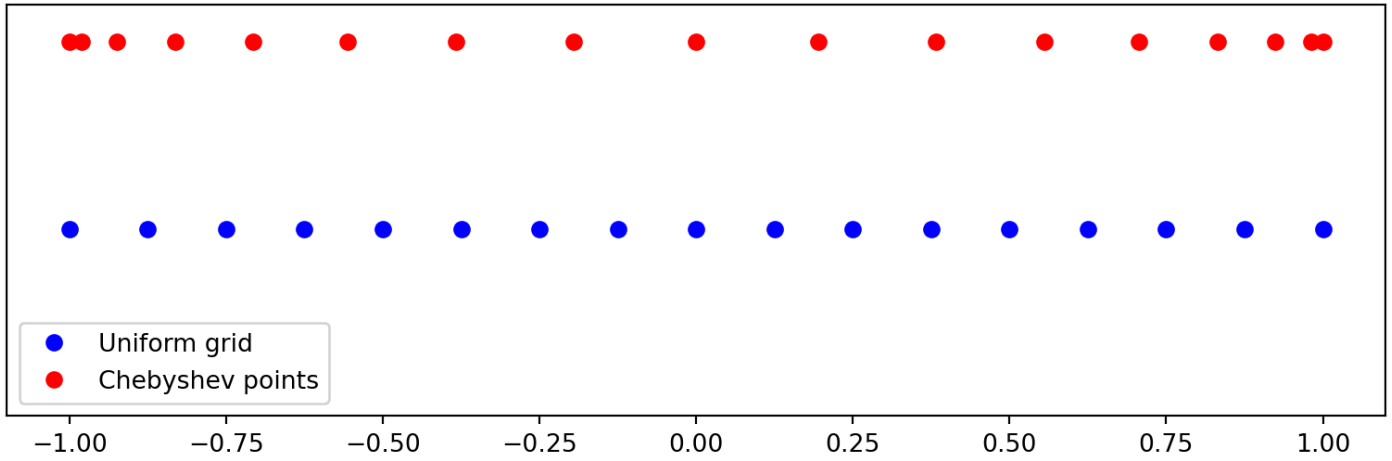
Hence, large errors occur when $p_N(x)$ is large or $u^{(N)}(\xi) = \frac{d^N u}{dx^N}(\xi)$ is large.

Lets look at $p_N(x)$, which is independent of the function $u(x)$!

The monic $p_N(x)$ can be created for any mesh

Lets use both a uniform mesh and Chebyshev points:

$$x_j = -1 + \frac{2j}{N} \quad \text{and} \quad x_j = \cos(j\pi/N), \quad j = 0, 1, \ldots, N$$

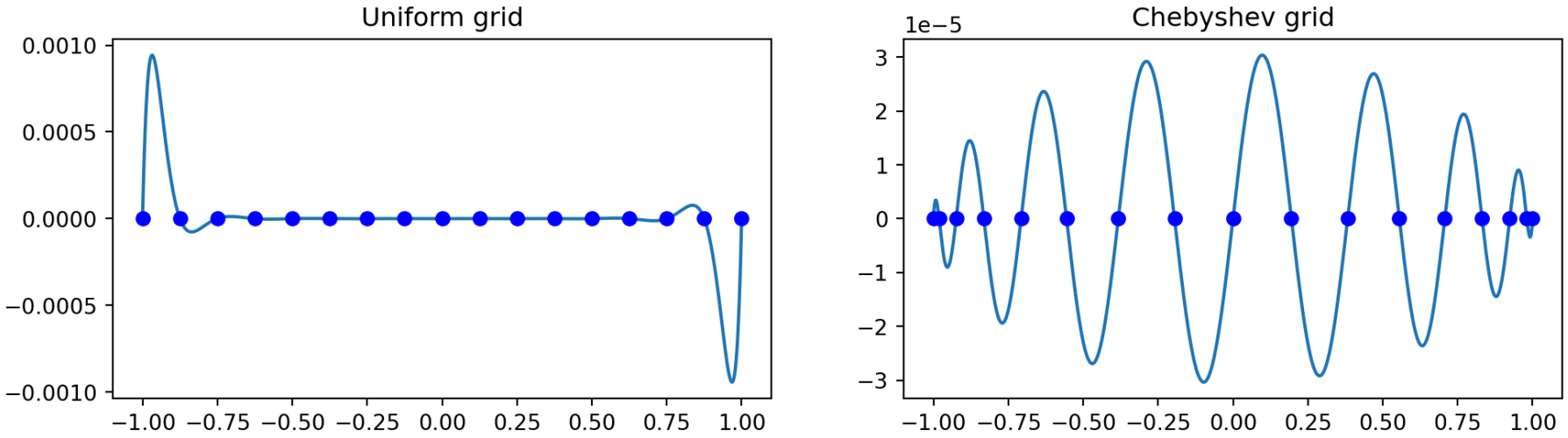


We see that Chebyshev points are clustered near the edges

Lets construct $p_N(x)$ using either the uniform mesh or Chebyshev nodes..

Plot the monic polynomial $p_N(x) = \prod_{j=0}^{N}(x - x_j)$ at uniform and Chebyshev mesh

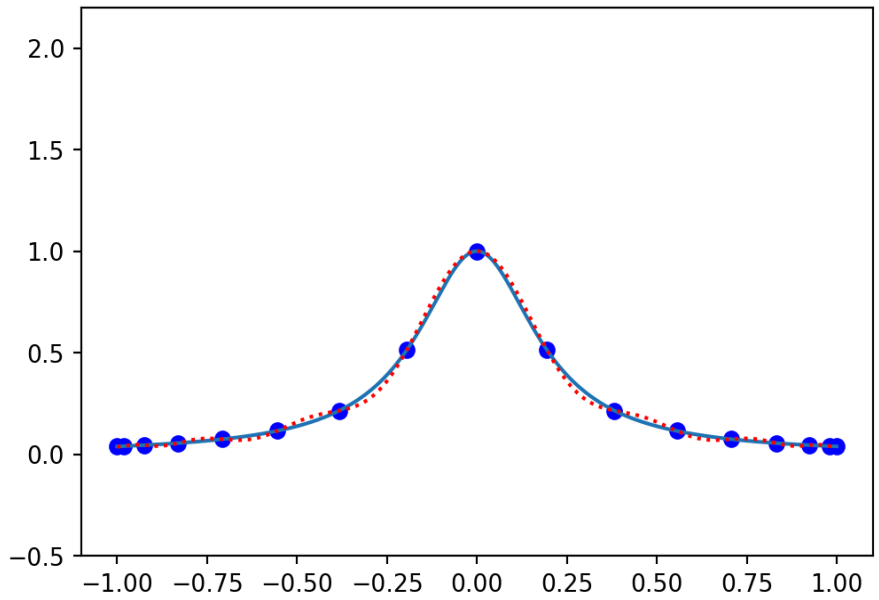
```
xp = np.linspace(-1, 1, N+1); xc = np.cos(np.arange(N+1)*np.pi/N)
pp = np.poly(xp); pc = np.poly(xc)
xj = np.linspace(-1, 1, 1000)
fig, (ax1, ax2) = plt.subplots(1, 2, figsize=(12, 3))
ax1.plot(xj, np.polyval(pp, xj)); ax2.plot(xj, np.polyval(pc, xj))
ax1.plot(xp, np.zeros(N+1), 'bo'); ax2.plot(xc, np.zeros(N+1), 'bo')
ax1.set_title('Uniform grid'); ax2.set_title('Chebyshev grid');
```



- A uniform mesh leads to large oscillations near the edges, whereas Chebyshev points leads to a polynomial with near uniform oscillations.
- The oscillations on the uniform mesh grow even larger when increasing N.

Compute the Lagrange polynomial using Chebyshev points

```
xj = np.cos(np.arange(N+1)*np.pi/N)
uj = sp.lambdify(x, u)(xj)
ell = Lagrangebasis(xj)
L = Lagrangefunction(uj, ell)
plt.figure(figsize=(6, 4))
plt.plot(xj, uj, 'bo')
plt.plot(yj, sp.lambdify(x, u)(yj))
plt.gca().set_ylim(-0.5, 2.2);
plt.plot(yj, sp.lambdify(x, L)(yj), 'r:');
```



Only small oscillations - interpolation is converging using more points!

The variational method and $u(x) = \frac{1}{1+25x^2}$

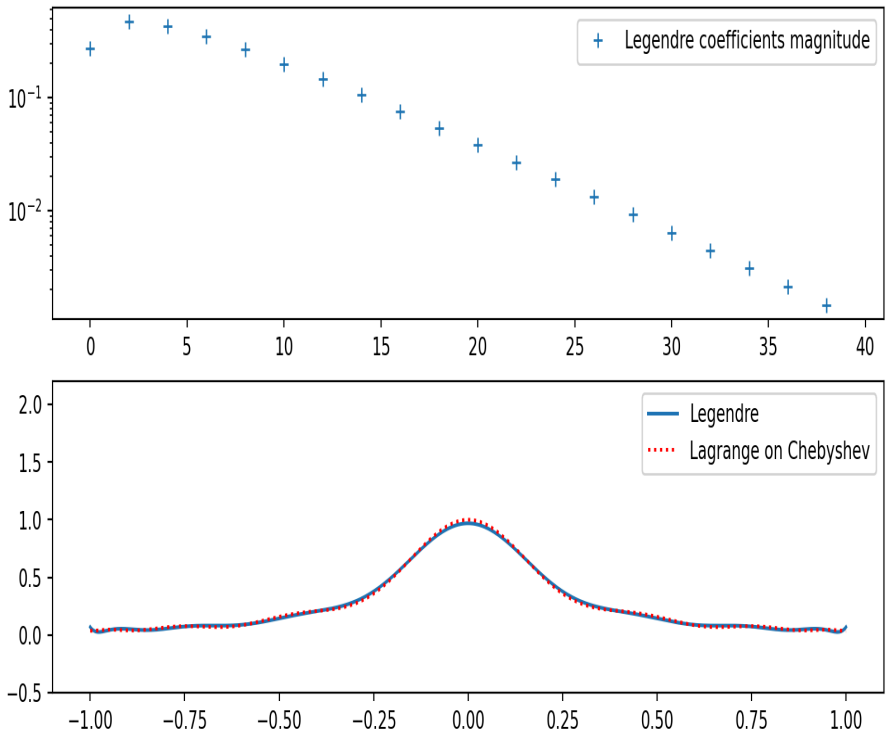
Lets try Legendre polynomials. There is no need for mapping since the domain is $[-1, 1]$. And there are no mesh points! The Legendre coefficients are

$$\hat{u}_j = \frac{2j+1}{2}(u, P_j), \quad j = 0, 1, \ldots$$

```python
# Use scipy to compute integral
from numpy.polynomial import Legendre
from scipy.integrate import quad
def innern(u, v):
    uj = lambda xj: sp.lambdify(x, u)(xj)*v(xj)
    return quad(uj, -1, 1)[0]

u = 1/(1+25*x**2)
uhat = lambda u, j: (2*j+1) * innern(u, Legendre.b
ul = [uhat(u, n) for n in range(40)]

xj = np.linspace(-1, 1, 1000)
uj = sp.lambdify(x, u)(xj)
fig, (ax1, ax2) = plt.subplots(2, 1)
ax1.semilogy(abs(np.array(ul)), '+')
ax2.plot(xj, Legendre(ul[:17])(xj))
ax2.plot(yj, sp.lambdify(x, L)(yj), 'r:')
ax2.set_ylim(-0.5, 2.2);
ax1.legend(['Legendre coefficients magnitude'])
ax2.legend(['Legendre', 'Lagrange on Chebyshev'])
```



Error in Legendre approximations

The smooth function $u(x)$ can be exactly represented as

$$u(x) = \sum_{k=0}^{\infty} \hat{u}_k P_k(x)$$

For all $k \leq N$ the series for $u(x)$ and the series for $u_N(x)$ have exactly the same coefficients $\hat{u}_k$. Hence the error in the approximation u_N is easily computed as

$$e(x) = u(x) - u_N(x) = \sum_{k=N+1}^{\infty} \hat{u}_k P_k(x)$$

Spectral convergence - it can be shown that

$$\|u - u_N\| \sim \|\hat{u}_{N+1}\| \sim e^{-\mu N}, \quad \mu \in \mathbb{R}^+$$

Exponential convergence!

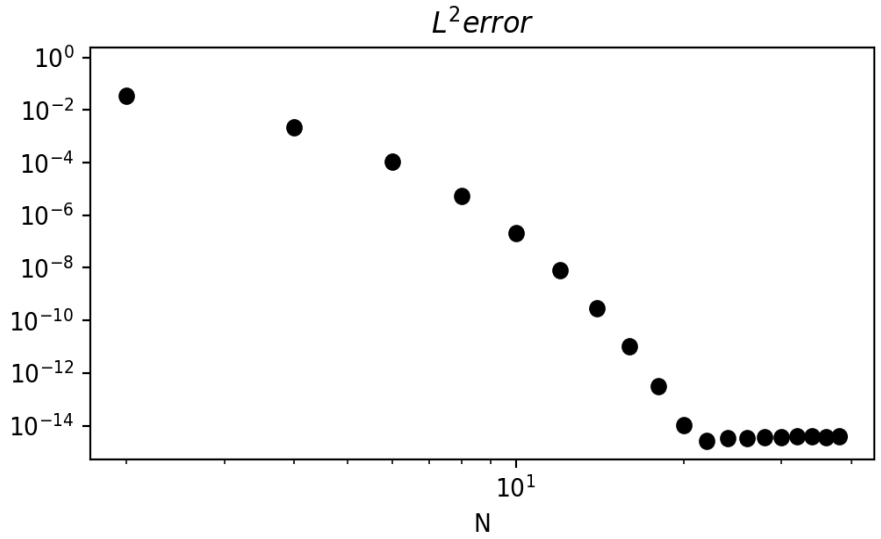
An example of spectral (exponential) convergence

Lets appriximate the smooth function

$$u = e^{\cos(x)}, \quad x \in [-1, 1]$$

and monitor the L^2 error as we increase N.

```
def L2_error(uh, ue):
    uej = sp.lambdify(x, ue)
    err = []
    for n in range(0, 40, 2):
        uf = lambda xj: (Legendre(uh[:(n+1)])(xj)
                         -uej(xj))**2
        err.append(np.sqrt(quad(uf, -1, 1)[0]))
    return err
u = sp.exp(sp.cos(x))
# Compute Legendre coefficients
uh = [uhat(u, n) for n in range(0, 40)]
plt.figure(figsize=(6, 3))
plt.loglog(np.arange(0, 40, 2), L2_error(uh, u), '
plt.title('$L^2 error$');plt.xlabel('N');
```



- Exponentially fast convergence rate!
- Error does not go down further due to machine precision.

Boundary issues

Lets consider a slightly different problem

$$u(x) = 10(x-1)^2 - 1, \quad x \in [0,1]$$

and attempt to find $u_N \in V_N = \text{span}\{\sin((j+1)\pi x)\}_{j=0}^N$ with the Galerkin method.

The sines are orthogonal such that the mass matrix becomes diagonal

$$\int_0^1 \sin((j+1)\pi x)\sin((i+1)\pi x)dx = \frac{1}{2}\delta_{ij}.$$

Hence we can easily get the coefficients with the Galerkin method

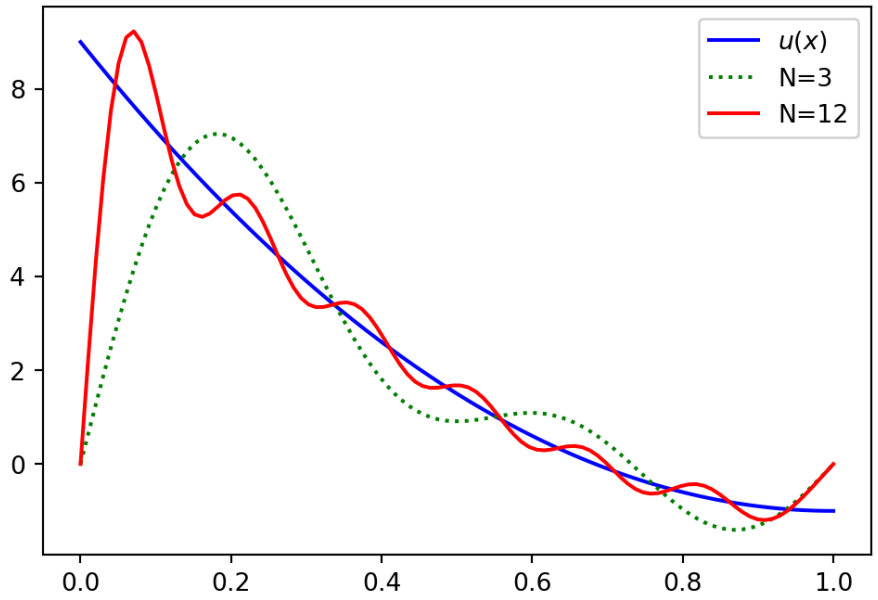
$$\hat{u}_i = 2(u(x), \sin((i+1)\pi x))_{L^2(0,1)}.$$

Implementation

```
u = 10*(x-1)**2-1
uhat = lambda u, i: 2*inner(u, sp.sin((i+1)*sp.pi*
ul = []
for i in range(15):
    ul.append(uhat(u, i).n())

ul = np.array(ul, dtype=float)
def uN(uh, xj):
    N = len(xj)
    uj = np.zeros(N)
    for i, ui in enumerate(uh):
        uj[:] += ui*np.sin((i+1)*np.pi*xj)
    return uj

xj = np.linspace(0, 1, 100)
plt.figure(figsize=(6,4))
plt.plot(xj, 10*(xj-1)**2-1, 'b')
plt.plot(xj, uN(ul[:(3+1)], xj), 'g:')
plt.plot(xj, uN(ul[:(12+1)], xj), 'r-')
plt.legend(['$u(x)$', 'N=3', 'N=12']);
```



What is wrong?

All basis functions are zero at the domain boundaries! And as such

$$u_N(0) = u_N(1) = 0$$

Solution, add a boundary function with nonzero boundary values

$$u_N(x) = B(x) + \tilde{u}_N(x) = B(x) + \sum_{j=0}^{N} \hat{u}_j \sin((j+1)\pi x),$$

and use $\tilde{u}_N \in V_N$ with homogeneous boundary values. $B(x)$ can be

$$B(x) = u(1)x + u(0)(1-x)$$

such that

$$u_N(0) = B(0) = u(0) \quad \text{and} \quad u_N(1) = B(1) = u(1)$$

The variational problem becomes: Find $\tilde{u}_N \in V_N$ such that

$$(B + \tilde{u}_N - u, v) = 0 \quad \forall\, v \in V_N$$

Sines with boundary values

Find $\tilde{u}_N \in V_N$ such that

$$(B + \tilde{u}_N - u, v) = 0 \quad \forall\, v \in V_N$$

Insert for $\tilde{u}_N = \sum_{j=0}^{N} \hat{u}_j \sin((j+1)\pi x)$ and $v = \sin((i+1)\pi x)$

$$\left(\sum_{j=0}^{N} \hat{u}_j \sin((j+1)\pi x) + B - u, \sin((i+1)\pi x) \right) = 0, \quad i = 0, 1, \ldots, N$$

and using the orthogonality of the sines we get

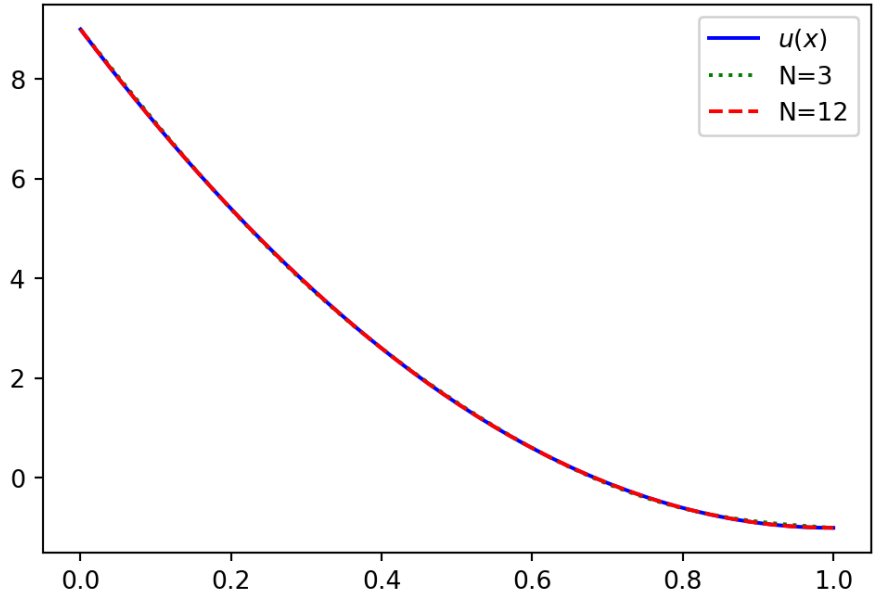
$$\hat{u}_i = 2\left(u - B, \sin((i+1)\pi x)\right) \quad i = 0, 1, \ldots, N$$

Result using boundary function $B(x)$

```
uc = []
B = u.subs(x, 0)*(1-x) + u.subs(x, 1)*x
for i in range(15):
    uc.append(uhat(u-B, i).n())

uc = np.array(uc, dtype=float)
def uNc(uh, xj):
    N = len(xj)
    uj = u.subs(x, 0)*(1-xj) + u.subs(x, 1)*xj
    for i, ui in enumerate(uh):
        uj[:] += ui*np.sin((i+1)*np.pi*xj)
    return uj

plt.figure(figsize=(6,4))
plt.plot(xj, 10*(xj-1)**2-1, 'b')
plt.plot(xj, uNc(uc[:(3+1)], xj), 'g:')
plt.plot(xj, uNc(uc[:(12+1)], xj), 'r--')
plt.legend(['$u(x)$', 'N=3', 'N=12']);
```



How about the Legendre method?

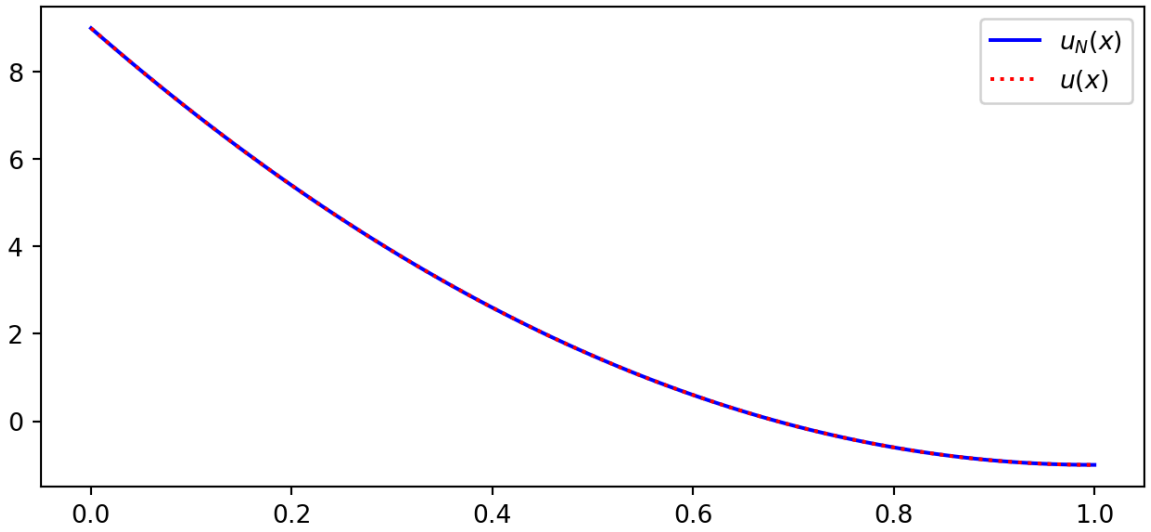
Legendre polynomials $P_j(x)$ are such that the boundary values are nonzero

$$P_j(-1) = (-1)^j \quad \text{and} \quad P_j(1) = 1$$

Hence there is no need for boundary functions or adjustments

```
uhat = lambda u, j: (2*j+1) * inner(u, sp.legendre(j, x), (0, 1))/2
uL = [uhat(u, i) for i in range(6)]
plt.figure(figsize=(8, 3.5))
plt.plot(xj, Legendre(uL, domain=(0, 1))(xj), 'b')
plt.plot(xj, 10*(xj-1)**2-1, 'r:'); plt.legend([r'$u_N(x)$', r'$u(x)$'])
print(uL)
```

```
[7/3, -5, 5/3, 0, 0, 0]
```



Summary

- We have been approximating $u(x)$ by $u_N \in V_N = \text{span}\{\psi_j\}_{j=0}^N$, meaning that we have found $u_N(x) = \sum_{j=0}^N \hat{u}_j \psi_j(x)$
- We have found the unknown expansion coefficients $\{\hat{u}_j\}_{j=0}^N$ using
  - Variational methods
    - The least squares method
    - The Galerkin method
  - Collocation - an interpolation method
- Interpolation on uniform grids is bad for large N due to Runge's phenomenon
- Basis functions are often defined on a reference domain, and problems thus need mapping.
- Some basis functions, like sines, have zero boundary values and need an additional boundary function for convergence.
- Legendre polynomials lead to spectral convergence (for smooth functions $u(x)$).