Function approximation with Chebyshev polynomials and in 2 dimensions

MATMEK-4270

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Short recap

We want to find an approximation to $u(x)$ using

$$u(x) \approx u_N(x) = \sum_{k=0}^{N} \hat{u}_k \psi_k(x)$$

- Least squares method
- Galerkin method
- Collocation method (Lagrange interpolation)

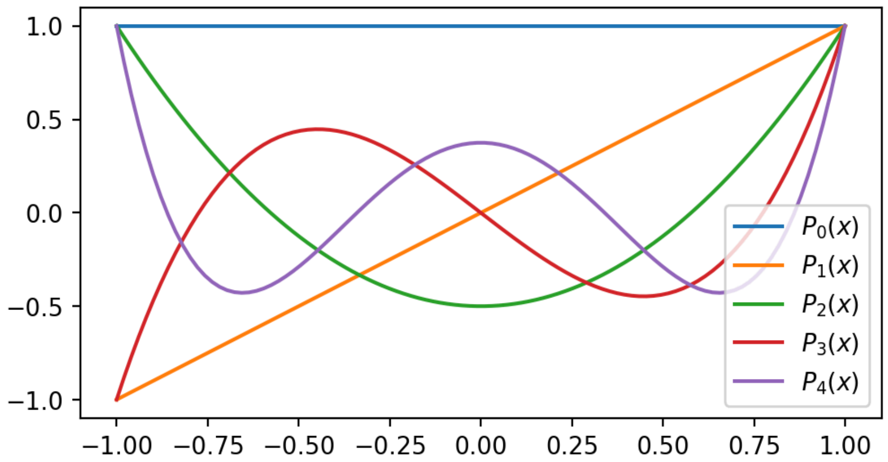
- ψ_j is a basis function
- $\{\psi_j\}_{j=0}^N$ is a basis
- $V_N = \text{span}\{\psi_j\}_{j=0}^N$ is a function space
- $\{\hat{u}_k\}_{k=0}^N$ are the unknowns

The variational methods make use of integrals over the domain. The $L^2(\Omega)$ inner product and norms are (in 1D, where $\Omega = [a, b]$)

$$(f, g)_{L^2(\Omega)} = \int_\Omega f(x) g(x) \, dx \quad \text{and} \quad \|f\|_{L^2(\Omega)} = \sqrt{(f, f)_{L^2(\Omega)}}$$

Legendre polynomials form a good basis for $\mathbb{P}_N$

$$
\begin{aligned}
P_0(x) &= 1,\\
P_1(x) &= x,\\
P_2(x) &= \frac{1}{2}(3x^2-1),\\
&\vdots\\
(j+1)P_{j+1}(x) &= (2j+1)xP_j(x) - jP_{j-1}(x).
\end{aligned}
$$



The Galerkin method to approximate $u(x) \approx u_N(x)$ with Legendre polynomials:

Find $u_N \in V_N (= \text{span}\{P_j\}_{i=0}^N = \mathbb{P}_N)$ such that

$$(u - u_N, v)_{L^2(\Omega)} = 0, \quad \forall\, v \in V_N$$

- Insert for $v = P_i$ and $u_N = \sum_{j=0}^N \hat{u}_j P_j$ and solve to get $\hat{u}_i = \frac{(u,P_i)}{\|P_i\|^2}, i = 0, 1, \ldots, N$
- Requires mapping if $\Omega \neq [-1, 1]$
- The Galerkin method is also referred to as a **projection** of $u(x)$ onto V_N

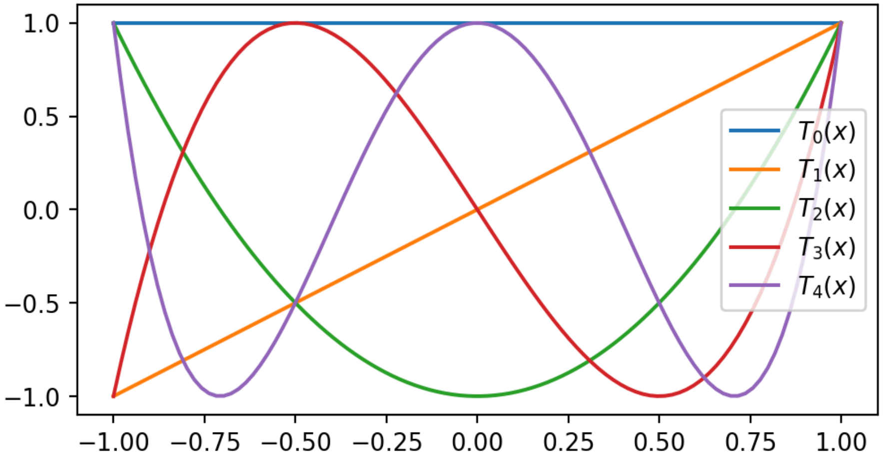
Chebyshev polynomials

The Chebyshev polynomials are an often preferred alternative to Legendre:

$$T_k(x) = \cos(k \cos^{-1}(x)), \quad k = 0, 1, \ldots \quad x \in [-1, 1]$$

As recursion:

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ &\vdots \\ T_{j+1}(x) &= 2xT_j(x) - T_{j-1}(x). \end{aligned}$$



For $T_N(x)$ all extrema points (max and mins) and all roots are, respectively

$$x_j = \cos\left(\frac{j\pi}{N}\right), \qquad j = 0, 1, \ldots, N$$

$$x_j = \cos\left(\frac{(2j+1)\pi}{2N}\right), \qquad j = 0, 1, \ldots, N-1$$

Chebyshev polynomials as a basis

The Chebyshev polynomials $\{T_j\}_{j=0}^N$ also form a basis for $\mathbb{P}_N$. However, the Chebyshev polynomials are **not orthogonal** in the $L^2(-1,1)$ space!

$$(T_i, T_j)_{L^2(\Omega)} \neq \|T_i\|^2 \delta_{ij}$$

The Chebyshev polynomials are, on the other hand, orthogonal in a special **weighted** inner product space.

We define the weighted $L^2_\omega(\Omega)$ inner product as

$$(f, g)_{L^2_w(\Omega)} = \int_\Omega f(x) g(x) \omega(x) d\Omega,$$

which is more commonly written as $(f, g)_\omega$. The weight function $\omega(x)$ is positive (almost everywhere) and a weighted norm is

$$\|u\|_\omega = \sqrt{(u, u)_\omega}$$

Function approximations with Chebyshev polynomials

The Chebyshev polynomials are orthogonal if $\omega(x) = (1-x^2)^{-1/2}$ and $x \in [-1, 1]$. We get

$$(T_i, T_j)_\omega = \|T_i\|_\omega^2 \delta_{ij}$$

where $\|T_i\|_\omega^2 = \frac{c_i \pi}{2}$ and $c_i = 1$ for $i > 0$ and $c_0 = 2$.

The Galerkin method for approximating a smooth function $u(x)$ is now:

Find $u_N \in \mathbb{P}_N$ such that

$$(u - u_N, v)_\omega = 0, \quad \forall \, v \in \mathbb{P}_N$$

We get the linear algebra problem by inserting for $v = T_i$ and $u_N = \sum_{j=0}^N \hat{u}_j T_j$

$$\sum_{j=0}^N (T_j, T_i)_\omega \hat{u}_j = (u, T_i)_\omega \rightarrow \hat{u}_i = \frac{(u, T_i)_\omega}{\|T_i\|_\omega^2}, \qquad i = 0, 1, \ldots, N$$

The least squares method

The least squares method is also similar, using $E_\omega = \|e\|_\omega^2$:

Find $u_N \in \mathbb{P}_N$ such that

$$\frac{\partial E_\omega}{\partial \hat{u}_j} = 0, \quad j = 0, 1, \ldots, N$$

We get the linear algebra problem using

$$\frac{\partial E_\omega}{\partial \hat{u}_j} = \frac{\partial}{\partial \hat{u}_j} \int_{-1}^{1} e^2 \omega dx = \int_{-1}^{1} 2e \frac{\partial e}{\partial \hat{u}_j} \omega dx$$

Insert for $e(x) = u(x) - u_N(x) = u(x) - \sum_{k=0}^{N} \hat{u}_k T_k$ and you get exactly the same linear equations as for the Galerkin method.

Mapping to reference domain

With a physical domain $x \in [a, b]$ and a reference $X \in [-1, 1]$, we now have the basis function

$$\psi_i(x) = T_i(X(x)), \quad i = 0, 1, \ldots, N$$

and the inner product to compute is

$$(u(x) - u_N(x), \psi_i(x))_\omega = \int_a^b (u(x) - u_N(x))\psi_i(x)\omega(x)dx = 0, \quad i = 0, 1, \ldots, N$$

As for Legendre we use a change of variables $x \to X$, but there is also a weight function that requires mapping

$$\omega(x) = \tilde{\omega}(X) = \frac{1}{\sqrt{1 - X^2}}$$

The mapped problem becomes

for all $i = 0, 1, \ldots, N$:

$$\sum_{j=0}^{N} \overbrace{\int_{-1}^{1} T_j(X) T_i(X) \tilde{\omega}(X) \cancel{\frac{dx}{dX}}\, dX}^{\|T_i\|^2 \delta_{ij}} \hat{u}_j = \overbrace{\int_{-1}^{1} u(x(X)) T_i(X) \tilde{\omega}(X) \cancel{\frac{dx}{dX}}\, dX}^{(u(x(X)), T_i)_\omega}$$

and finally (using $\|T_i\|_\omega^2 = \frac{c_i \pi}{2}$)

$$\hat{u}_i = \frac{2}{c_i \pi} (u(x(X)), T_i)_{L^2_\omega(-1,1)}, \quad i = 0, 1, \ldots, N$$

The procedure is exactly like for Legendre polynomials, but with a weighted inner product using $L^2_\omega(-1,1)$ instead of $L^2(-1,1)$.

The weighted inner product requires some extra attention

$$(f, T_i)_\omega = \int_{-1}^{1} \frac{f(x(X))T_i(X)}{\sqrt{1 - X^2}} dX$$

Since $T_i(X) = \cos(i \cos^{-1}(X))$ a change of variables $X = \cos\theta$ leads to $T_i(\cos\theta) = \cos(i\theta)$. Using the change of variables for the integral:

$$(f, T_i)_\omega = \int_{\pi}^{0} \frac{f(x(\cos\theta))T_i(\cos\theta)}{\sqrt{1 - \cos^2\theta}} \frac{d\cos\theta}{d\theta} d\theta.$$

Insert for $1 - \cos^2\theta = \sin^2\theta$ and swap both the direction of the integration and the sign:

$$(f, T_i)_\omega = \int_{0}^{\pi} f(x(\cos\theta))T_i(\cos\theta) d\theta.$$

Weighted inner product continued

$$(f, T_i)_\omega = \int_0^\pi f(x(\cos\theta))T_i(\cos\theta)d\theta.$$

Using $T_i(\cos\theta) = \cos(i\theta)$ we get the much simpler integral

$$(f, T_i)_\omega = \int_0^\pi f(x(\cos\theta))\cos(i\theta)d\theta.$$

Using this integral, we get the Chebyshev coefficients

$$\hat{u}_i = \frac{2}{c_i\pi}\int_0^\pi u(x(\cos\theta))\cos(i\theta)d\theta, \quad i = 0, 1, \ldots, N$$

Lets try this with an example.

Implementation of the weighted inner product

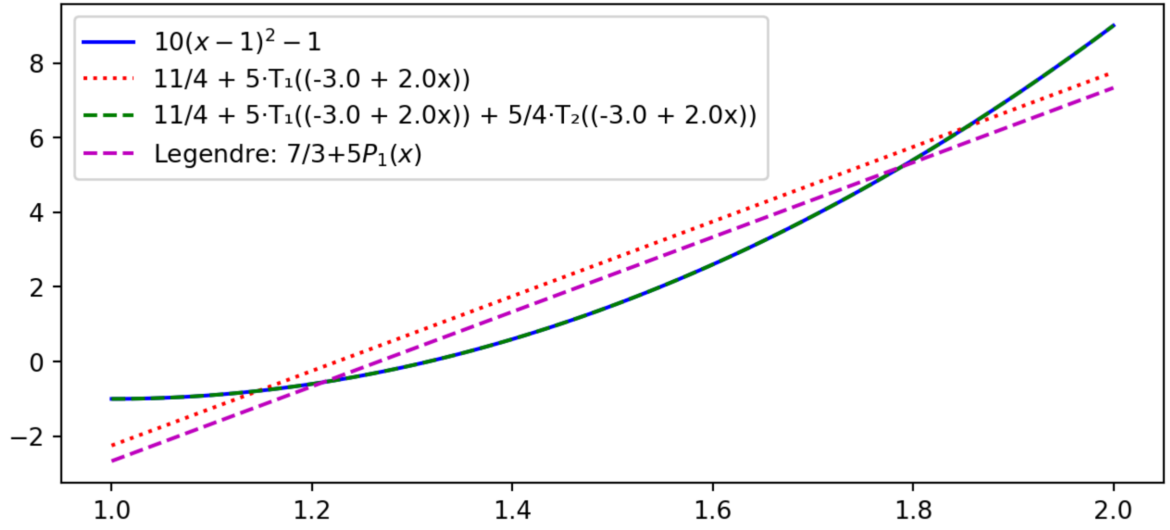
```python
x = sp.Symbol('x', real=True)
k = sp.Symbol('k', integer=True, positive=True)

Tk = lambda k, x: sp.cos(k * sp.acos(x))
cj = lambda j: 2 if j == 0 else 1

def innerw(u, v, domain, ref_domain=(-1, 1)):
    A, B = ref_domain
    a, b = domain
    # map u(x(X)) to use reference coordinate X.
    # Note that small x here in the end will be ref coord.
    us = u.subs(x, a + (b-a)*(x-A)/(B-A))
    # Change variables x=cos(theta)
    us = sp.simplify(us.subs(x, sp.cos(x)), inverse=True) # X=cos(theta)
    vs = sp.simplify(v.subs(x, sp.cos(x)), inverse=True)  # X=cos(theta)
    return sp.integrate(us*vs, (x, 0, sp.pi))
```

Note

We use the Sympy function `simplify` with `inverse=True`, which is required for Sympy to use that $\cos^{-1}(\cos x) = x$, which is not necessarily true.

Try with $u(x) = 10(x-1)^2 - 1, x \in [1, 2]$

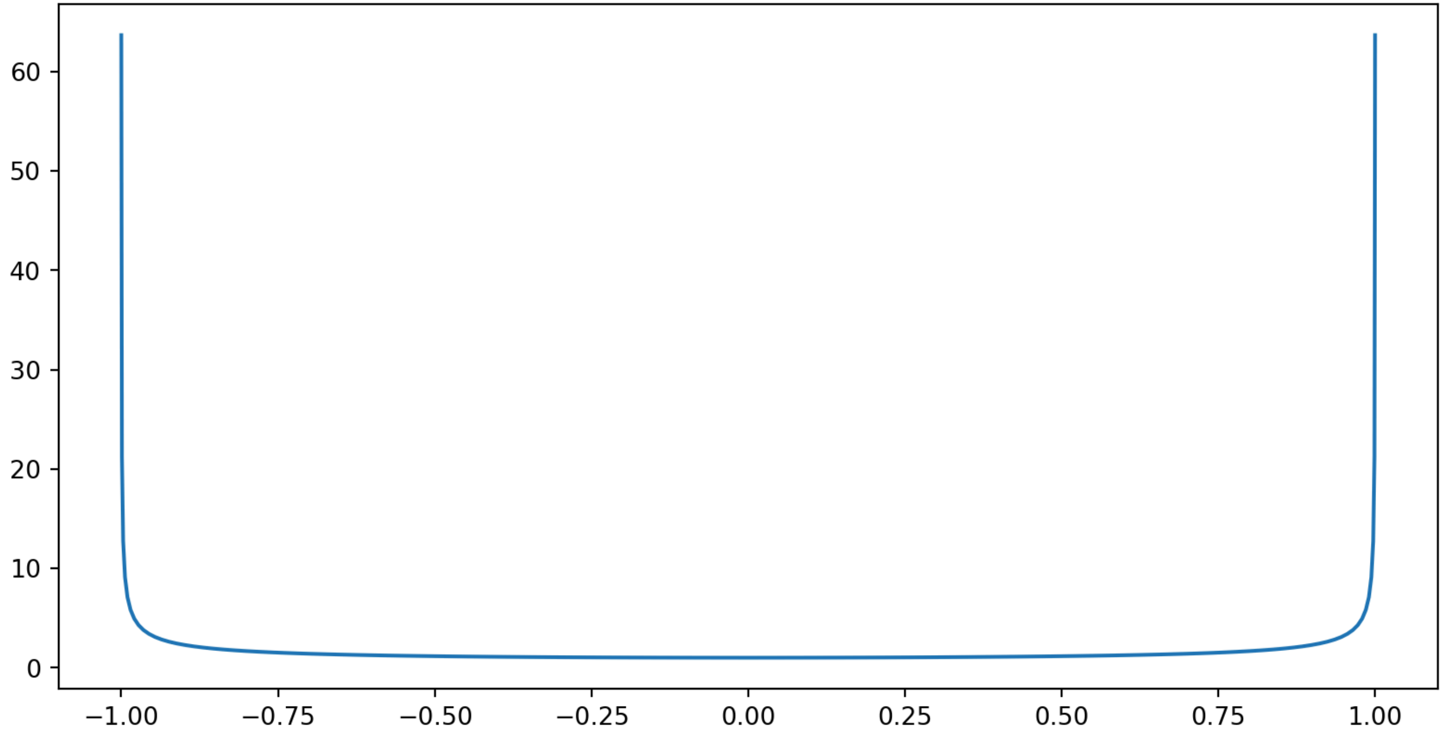
```python
from numpy.polynomial import Chebyshev
u = 10*(x-1)**2-1
uhat = lambda u, j: 2 / (cj(j) * sp.pi) * innerw(u, Tk(j, x), (1, 2))
plt.figure(figsize=(8, 3.5))
xj = np.linspace(1, 2, 100)
uhj = [uhat(u, j) for j in range(6)]
C2, C3 = Chebyshev(uhj[:2], domain=(1, 2)), Chebyshev(uhj[:3], domain=(1, 2))
plt.plot(xj, sp.lambdify(x, u)(xj), 'b')
plt.plot(xj, C2(xj), 'r:'); plt.plot(xj, C3(xj), 'g--')
plt.plot(xj, 7/3+5*(-1+2*(xj-1)), 'm--')
plt.legend(['$10(x-1)^2-1$', f'{C2}', f'{C3}', 'Legendre: 7/3+5$P_1(x)$']);
```



Different from Legendre for the linear profile. But not by much. Why is it different?

The weight function favours the edges

$$\omega(x) = \frac{1}{\sqrt{1 - x^2}}$$



So the weighted Chebyshev approach has smaller errors towards the edges.

Try more difficult function with numerical integration

$$u(x) = e^{\cos x}, \quad x \in [-1, 1]$$

Use numerical integration and change of variables

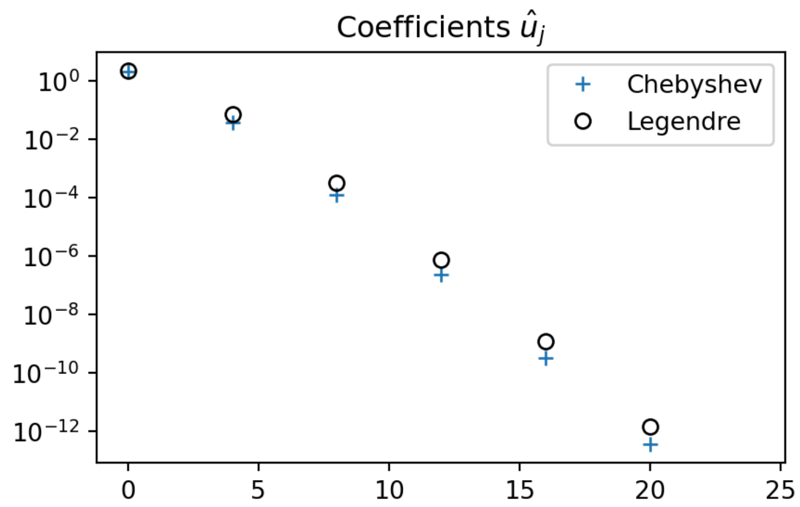
```
from scipy.integrate import quad
def innerwn(u, v, domain, ref_domain=(-1, 1)):
    A, B = ref_domain
    a, b = domain
    us = u.subs(x, a + (b-a)*(x-A)/(B-A)) # u(x(X))
    us = sp.simplify(us.subs(x, sp.cos(x)), inverse=True) # X=cos(theta)
    vs = sp.simplify(v.subs(x, sp.cos(x)), inverse=True)  # X=cos(theta)
    return quad(sp.lambdify(x, us*vs), 0, np.pi)[0]
u = sp.exp(sp.cos(x))
#uhat = lambda u, j: 2 / (cj(j) * sp.pi) * innerw(u, Tk(j, x), (-1, 1)) # slow
uhatn = lambda u, j: 2 / (cj(j) * np.pi) * innerwn(u, Tk(j, x), (-1, 1))
```

Remember, we are computing for $i = 0, 1, \ldots, N$

$$\hat{u}_i = \frac{2}{c_i \pi} \int_{-1}^{1} u(x(X)) T_i(X) \tilde{\omega}(X) dX = \frac{2}{c_i \pi} \int_{0}^{\pi} u(x(\cos\theta)) \cos(i\theta) d\theta$$

Compare with Legendre

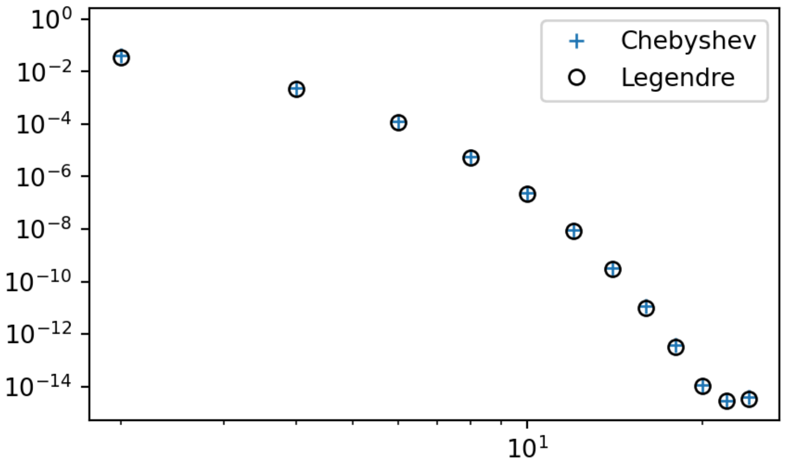
```python
# Compute Chebyshev coefficients:
N = 25
uc = [uhatn(u, n) for n in range(N)]

# Compute Legendre coefficients:
from numpy.polynomial import Legendre
def innern(u, v):
    uj = lambda xj: sp.lambdify(x, u)(xj)*v(xj)
    return quad(uj, -1, 1)[0]
uhatj = lambda u, j: (2*j+1) * innern(u, Legendre.b
ul = [uhatj(u, n) for n in range(N)]

plt.figure(figsize=(5, 3))
plt.semilogy(np.arange(0, N, 2), uc[::2], '+',
             np.arange(0, N, 2), ul[::2], 'ko', fil
plt.title('Coefficients $\hat{u}_j$')
plt.legend(['Chebyshev', 'Legendre']);
```



Very similar convergence. Chebyshev coefficients are slightly smaller than Legendre. How about the L^2 error?

L^2 error - $\|e\| = \sqrt{\int_{-1}^{1} e^2 dx}$ (not weighted)

```
def L2_error(uh, ue, space=Legendre):
    xj = np.linspace(-1, 1, 1000)
    uej = sp.lambdify(x, ue)(xj)
    err = []
    for n in range(0, len(uh), 2):
        uj = space(uh[:(n+1)])(xj).astype(float)
        err.append(np.sqrt(np.trapz((uj-uej)**2, dx=xj[1]-xj[0])))
    return err

errc = L2_error(uc, u, Chebyshev)
errl = L2_error(ul, u, Legendre)

plt.figure(figsize=(5, 3))
plt.loglog(np.arange(0, N, 2), errc, '+',
           np.arange(0, N, 2), errl, 'ko', fillstyle='none')
plt.legend(['Chebyshev', 'Legendre']);
```



Function approximations in 2D

We can approximate a two-dimensional function $u(x, y)$ using a two-dimensional function space W_N

In 2D we will try to find $u_N(x, y) \in W_N$, which implies:

$$u(x, y) \approx u_N(x, y) = \sum_{i=0}^{N} \hat{u}_i \Psi_i(x, y),$$

- $\Psi_i(x, y)$ is a two-dimensional basis function
- $\{\Psi_i\}_{i=0}^{N}$ is a basis
- $W_N = \text{span}\{\Psi_i\}_{i=0}^{N}$ is a 2D function space.

It is more common to use one basis function for each direction

There are not all that many two-dimensional basis functions and a more common approach is to use one basis function for the x-direction and another for the y-direction

$$u_N(x,y) = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} \hat{u}_{ij} \psi_i(x) \varphi_j(y).$$

Note

The unknowns $\{\hat{u}_{ij}\}_{i,j=0}^{N_x,N_y}$ are now in the form of a matrix. The total number of unknowns: $N+1 = (N_x+1)\cdot(N_y+1)$.

The most straightforward approach is to use the same basis functions for both directions. For example, with a Chebyshev basis

$$u_N(x,y) = \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} \hat{u}_{ij} T_i(x) T_j(y).$$

Two-dimensional function spaces

We can define two one-dimensional function spaces for the two directions as

$$V_{N_x} = \mathrm{span}\{\psi_i\}_{i=0}^{N_x} \quad \text{and} \quad V_{N_y} = \mathrm{span}\{\varphi_i\}_{i=0}^{N_y}$$

with a 2D domain Ω created as Cartesian products of two 1D domains:

$$I_x = [a, b] \quad \text{and} \quad I_y = [c, d] \rightarrow \Omega = I_x \times I_y$$

A two-dimensional function space can then be created as

$$W_N = V_{N_x} \otimes V_{N_y}, \quad (x, y) \in \Omega.$$

W_N is the **tensor product** of V_{N_x} and V_{N_y}

Similarly,

$$\Psi_{ij}(x, y) = \psi_i(x)\varphi_j(y)$$

Ψ_{ij} is the tensor product (or outer product) of ψ_i and φ_j.

The tensor product is a Cartesian product with multiplication

Consider the Cartesian product of the two sequences $(1, 2, 3)$ and $(4, 5)$ and compare with the tensor product

Cartesian product:

$$(1,2,3) \times (4,5) = \begin{bmatrix} (1,4) \\ (1,5) \\ (2,4) \\ (2,5) \\ (3,4) \\ (3,5) \end{bmatrix}$$

Tensor product:

$$(1,2,3) \otimes (4,5) = \begin{bmatrix} 1 \cdot 4 \\ 1 \cdot 5 \\ 2 \cdot 4 \\ 2 \cdot 5 \\ 3 \cdot 4 \\ 3 \cdot 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 8 \\ 10 \\ 12 \\ 15 \end{bmatrix}$$

Tensor product of functions

Cartesian product:

$$(\psi_0, \psi_1) \times (\varphi_0, \varphi_1) = \begin{bmatrix} (\psi_0, \varphi_0) \\ (\psi_0, \varphi_1) \\ (\psi_1, \varphi_0) \\ (\psi_1, \varphi_1) \end{bmatrix}$$

Tensor product:

$$(\psi_0, \psi_1) \otimes (\varphi_0, \varphi_1) = \begin{bmatrix} \psi_0 \cdot \varphi_0 \\ \psi_0 \cdot \varphi_1 \\ \psi_1 \cdot \varphi_0 \\ \psi_1 \cdot \varphi_1 \end{bmatrix}$$

This tensor product is the basis for W_N:

$$\{\psi_0\psi_0, \psi_0\psi_1, \psi_1\psi_0, \psi_1\psi_1\}$$

which can also be arranged in matrix notation $\{\psi_i\varphi_j\}_{i,j=0}^{1,1}$ (i is row, j is column)

$$(\psi_0, \psi_1) \otimes (\varphi_0, \varphi_1) = \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} \begin{bmatrix} \varphi_0 & \varphi_1 \end{bmatrix} = \begin{bmatrix} \psi_0 \cdot \varphi_0, \psi_0 \cdot \varphi_1 \\ \psi_1 \cdot \varphi_0, \psi_1 \cdot \varphi_1 \end{bmatrix}$$

Example of tensor product basis

Use the space of all linear functions in both x and y directions

$$V_{N_x} = \text{span}\{1, x\} \quad \text{and} \quad V_{N_y} = \text{span}\{1, y\}$$

Cartesian product

$$(1, x) \times (1, y) = \begin{bmatrix} (1,1) \\ (1,y) \\ (x,1) \\ (x,y) \end{bmatrix}$$

Tensor product

$$(1, x) \otimes (1, y) = \begin{bmatrix} 1 \\ y \\ x \\ xy \end{bmatrix}$$

Numpy naturally arranges the outer product into matrix form:

```python
y = sp.Symbol('y')
Vx = np.array([1, x])
Vy = np.array([1, y])
W = np.outer(Vx, Vy)
print(W)
```

```
[[1 y]
 [x x*y]]
```

We have a function space and a basis, now it's time to approximate $u(x, y)$

The variational methods require the $L^2(\Omega)$ inner product

$$
\begin{aligned}
(f, g)_{L^2(\Omega)} &= \int_\Omega fg \, d\Omega, \\
&= \int_{I_x} \int_{I_y} f(x, y) g(x, y) dx dy.
\end{aligned}
$$

Note

The first line is identical to the definition used for the 1D case and is valid for any domain Ω, not just Cartesian product domains. The only difference for 2D is that f and g now are functions of both x and y and the the integral over the domain is a double integral.

Galerkin for 2D approximations

We want to approximate

$$u(x,y) \approx u_N(x,y)$$

The Galerkin method is then: find $u_N \in W_N$ such that

$$(u - u_N, v) = 0, \quad \forall\, v \in W_N \tag{1}$$

In order to solve the problem we just choose basis functions and solve (1). For example, use Legendre polynomials in both x and y-directions.

$$V_{N_x} = \text{span}\{P_i\}_{i=0}^{N_x}, \quad \text{and} \quad V_{N_y} = \text{span}\{P_j\}_{j=0}^{N_y}$$

$$W_N = V_{N_x} \otimes V_{N_y} = \text{span}\{P_i P_j\}_{i,j=0}^{N_x, N_y}$$

We now compute $(u - u_N, v)$ using

$$v = P_m(x)P_n(y) \quad \text{and} \quad u_N = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} \hat{u}_{ij}P_i(x)P_j(y)$$

It becomes a bit messy, with 4 different indices:

$$\int_{-1}^{1}\int_{-1}^{1}\left(u - \sum_{i=0}^{N}\sum_{j=0}^{N}\hat{u}_{ij}P_i(x)P_j(y)\right)P_m(x)P_n(y)dxdy$$

Note that the unknown coefficients $\hat{u}_{ij}$ are independent of space and we can simplify the double integrals by separating them into one integral for x and one for y. For example

$$\int_{-1}^{1}\int_{-1}^{1} P_i(x)P_j(y)P_m(x)P_n(y)dxdy = \underbrace{\int_{-1}^{1} P_i(x)P_m(x)dx}_{a_{mi}}\underbrace{\int_{-1}^{1} P_j(y)P_n(y)dy}_{a_{nj}}$$

Breaking down $(u - u_N, v)$

$$\text{With} \quad v = P_m(x)P_n(y) \quad \text{and} \quad u_N = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} \hat{u}_{ij}P_i(x)P_j(y)$$

$$(u - u_N, v) = 0 \to \int_{-1}^{1}\int_{-1}^{1}\left(u - \sum_{i=0}^{N}\sum_{j=0}^{N} \hat{u}_{ij}P_i(x)P_j(y)\right)P_m(x)P_n(y)dxdy = 0$$

$$(u, v) = \int_{-1}^{1}\int_{-1}^{1} u(x,y)P_m(x)P_n(y)dxdy = u_{mn}$$

$$(u_N, v) := \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} a_{mi}a_{nj}\hat{u}_{ij}$$

$$(u - u_N, v) = 0 \longrightarrow \boxed{\sum_{i=0}^{N_x}\sum_{j=0}^{N_y} a_{mi}a_{nj}\hat{u}_{ij} = u_{mn}}, \quad (m,n) = (0,\ldots,N_x) \times (0,\ldots,N_y)$$

Solve the linear algebra problem

$$
\begin{aligned}
\sum_{i=0}^{N_x} \sum_{j=0}^{N_y} a_{mi} a_{nj} \hat{u}_{ij} = u_{mn}, \quad (m, n) \in (0, \ldots, N_x) \times (0, \ldots, N_y) \\
\longrightarrow A\hat{U}A = U
\end{aligned}
$$

Can solve for U with the vec-trick ($\text{vec}(A\hat{U}A^T) = (A \otimes A)\text{vec}(\hat{U})$)

$$
\begin{aligned}
(A \otimes A)\text{vec}(\hat{U}) &= \text{vec}(U) \\
\text{vec}(\hat{U}) &= (A \otimes A)^{-1}\text{vec}(U)
\end{aligned}
$$

However, since A here is a diagonal matrix and we only have one matrix $(A\hat{U}A)$ it is actually much easier to just avoid the vectorization and solve directly

$$
\hat{U} = A^{-1} U A^{-1}.
$$

Example:

$$u(x,y) = \exp(-(x^2 + 2(y-0.5)^2)), (x,y) \in [-1,1] \times [-1,1]$$

Find $u_N \in W_N = V_{N_x} \otimes V_{N_y}$ using Legendre polynomials for both directions. With Galerkin: find $u_N \in W_N$ such that

$$(u - u_N, v) = 0 \quad \forall\, v \in W_N$$

1. Find the matrix $U = \{u_{ij}\}_{i,j=0}^{N_x,N_y}$, $u_{ij} = (u, P_i P_j)$
2. Find the matrix $A = \{a_{ij}\}_{i,j=0}^{N_x,N_y}$, $a_{ij} = \|P_i\|^2 \delta_{ij}$
3. Compute $\hat{U} = A^{-1} U A^{-1}$

```python
import scipy.sparse as sparse
from scipy.integrate import dblquad
ue = sp.exp(-(x**2+2*(y-sp.S.Half)**2))
uh = lambda i, j: dblquad(sp.lambdify((x, y), ue*sp
N = 8
uij = np.zeros((N+1, N+1))
for i in range(N+1):
    for j in range(N+1):
        uij[i, j] = uh(i, j)
A_inv = sparse.diags([(2*np.arange(N+1)+1)/2], [0],
uhat_ij = A_inv @ uij @ A_inv
```

Evaluate the 2D solution

We have found $\{\hat{u}_{ij}\}_{i,j=0}^{N_x,N_y}$, so now we can evaluate

$$u_N(x,y) = \sum_{i=0}^{N_x}\sum_{j=0}^{N_y} \hat{u}_{ij} P_i(x) P_j(y)$$

for any x, y, preferably within the domain $[-1,1]\times[-1,1]$.

How to do this?

A simple double for-loop will do, or on matrix-vector form to avoid the for-loop. Use $\boldsymbol{P_x} = (P_0(x), \ldots, P_{N_x})$ and $\boldsymbol{P_y} = (P_0(y), \ldots, P_{N_y}(y))$

$$\boldsymbol{P_x} \hat{U} \boldsymbol{P_y}^T = \begin{bmatrix} P_0(x) & \ldots & P_{N_x}(x) \end{bmatrix} \begin{bmatrix} \hat{u}_{0,0} & \cdots & \hat{u}_{0,N_y} \\ \vdots & \ddots & \vdots \\ \hat{u}_{N_x,0} & \cdots & \hat{u}_{N_x,N_y} \end{bmatrix} \begin{bmatrix} P_0(y) \\ \vdots \\ P_{N_y}(y) \end{bmatrix}$$

Evaluate for a computational mesh

It is very common to compute the solution on a 2D computational Cartesian grid $\boldsymbol{x} = (x_0, x_1, \ldots, x_{N_x})$ and $\boldsymbol{y} = (y_0, y_1, \ldots, y_{N_y})$:

$$\boldsymbol{x} \times \boldsymbol{y} = \{(x, y) | x \in \boldsymbol{x} \text{ and } y \in \boldsymbol{y}\}$$

$$u_N(x_i, y_j) = \sum_{m=0}^{N} \sum_{n=0}^{N} \hat{u}_{mn} P_m(x_i) P_n(y_j).$$

Four nested for-loops, or a triple matrix product

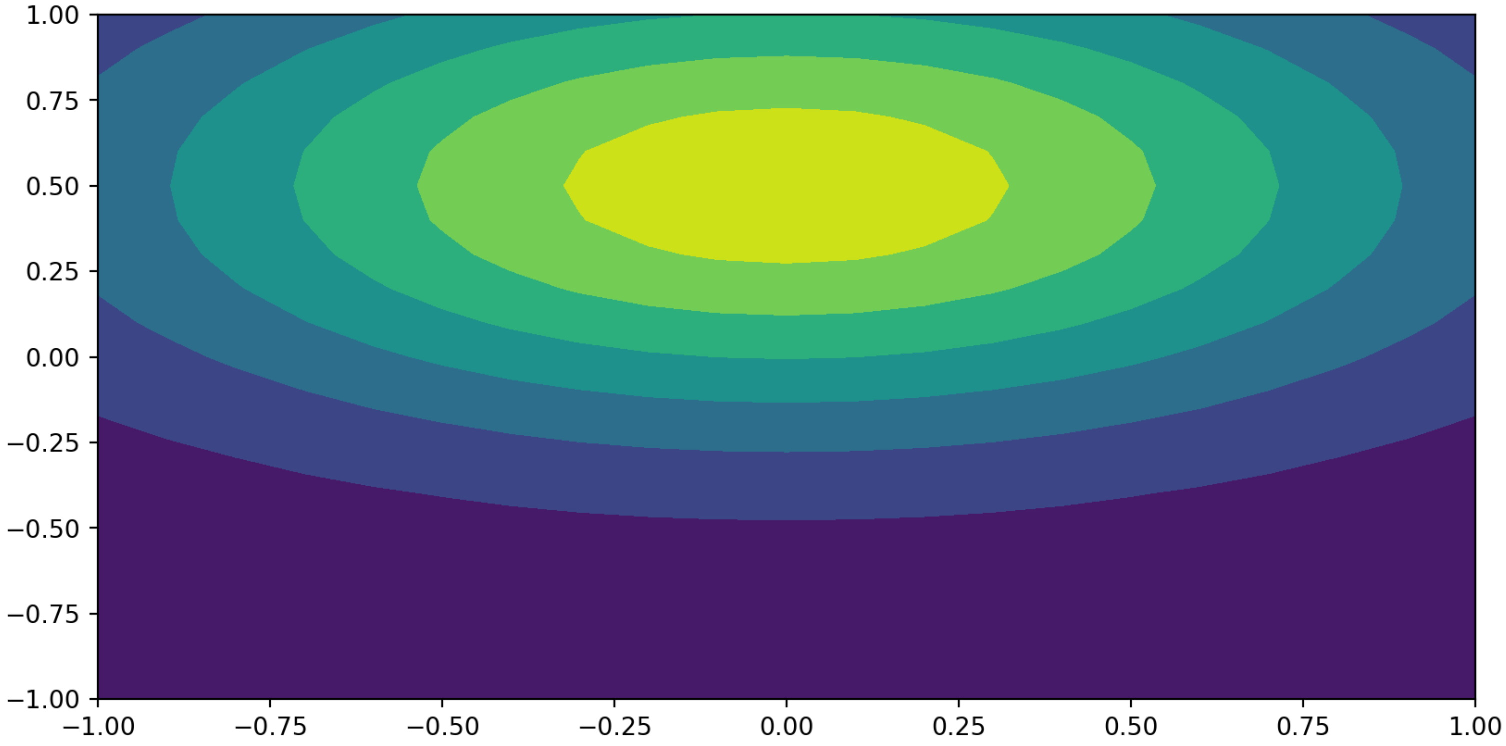
$$\begin{bmatrix} P_0(x_0) & \ldots & P_{N_x}(x_0) \\ \vdots & \ddots & \vdots \\ P_0(x_{N_x}) & \ldots & P_{N_x}(x_{N_x}) \end{bmatrix} \begin{bmatrix} \hat{u}_{0,0} & \cdots & \hat{u}_{0,N_y} \\ \vdots & \ddots & \vdots \\ \hat{u}_{N_x,0} & \cdots & \hat{u}_{N_x,N_y} \end{bmatrix} \begin{bmatrix} P_0(y_0) & \ldots & P_0(y_{N_y}) \\ \vdots & \ddots & \vdots \\ P_{N_y}(y_0) & \ldots & P_{N_y}(y_{N_y}) \end{bmatrix}$$

If $\boldsymbol{P_x} = \{P_j(x_i)\}_{i,j=0}^{N_x,N_x}$ and $\boldsymbol{P_y} = \{P_j(y_i)\}_{i,j=0}^{N_y,N_y}$ this is simply:

$$\boldsymbol{P_x} \hat{U} \boldsymbol{P_y}^T$$

Implement evaluate in 2D

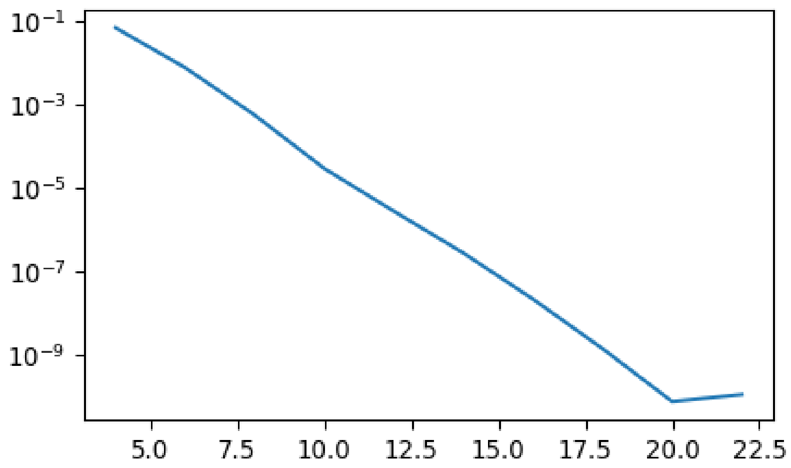
```python
def eval2D(xi, yi, uhat):
    Px = np.polynomial.legendre.legvander(xi, uhat.shape[0]-1)
    Py = np.polynomial.legendre.legvander(yi, uhat.shape[1]-1)
    return Px @ uhat @ Py.T

N = 20
xi = np.linspace(-1, 1, N+1)
U = eval2D(xi, xi, uhat_ij)
xij, yij = np.meshgrid(xi, xi, indexing='ij', sparse=False)
plt.contourf(xij, yij, U)
```



Check accuracy by computing the ℓ^2 error norm

$$\|u - u_N\|_{\ell^2} = \sqrt{\frac{4}{N_x N_y} \sum_{i=0}^{N_x} \sum_{j=0}^{N_y} (u(x_i, y_j) - u_N(x_i, y_j))^2}$$

```python
def l2error(uh_ij):
    N = uh_ij.shape[0]-1
    xi = np.linspace(-1, 1, N+1)
    U = eval2D(xi, xi, uh_ij)
    xij, yij = np.meshgrid(xi, xi, indexing='ij', s
    ueij = sp.lambdify((x, y), ue)(xij, yij)
    return np.linalg.norm(U-ueij)*(2/N)

def solve(N):
    uij = np.zeros((N+1, N+1))
    for i in range(N+1):
        for j in range(N+1):
            uij[i, j] = uh(i, j)
    A_inv = sparse.diags([(2*np.arange(N+1)+1)/2],
    return A_inv @ uij @ A_inv

error = []
for n in range(4, 24, 2):
    error.append(l2error(solve(n)))
plt.figure(figsize=(5, 3))
plt.semilogy(np.arange(4, 24, 2), error);
```



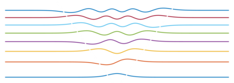
Some helpful tools: Chebfun and Shenfun

Chebfun — numerical computing with functions

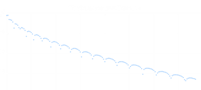
Chebfun is an open-source package for computing with functions to about 15-digit accuracy. Most Chebfun commands are overloads of familiar MATLAB commands — for example `sum(f)` computes an integral, `roots(f)` finds zeros, and `u = L\f` solves a differential equation.

DOWNLOAD (download) BROWSE SOURCE (//github.com/chebfun/chebfun)

```
% Create a ch
ebfun on the
interval [-3,
3]
x = chebfun
('x', [-3
3]);
% Define a po
tential funct
ion
V = abs(x);
```



```
% Create a ch
ebfun f
x = chebfun
('x');
f = exp(-1/(x
+1));
% Plot abs va
ls of Chebysh
ev coeffs of
f
plotcoeffs
```



Demo - Working with Fun

Mikael Mortensen (email: `mikaem@math.uio.no`), Departm

Date: **August 7, 2020**

Summary. This is a demonstration of how the Python module spectral functions in one and several dimensions.

Construction

A global spectral function $u(x)$ can be represented on the real l

$$u(x) = \sum_{k=0}^{N-1} \hat{u}_k \psi_k(x), \quad x \in$$

where the domain Ω has to be defined such that $b > a$. The arr cient for the series, often referred to as the degrees of freedom. function and $\psi_k(x)$ is the k'th basis function. We can use any n the chosen basis is then a function space. Also part of the funct when a function space is created. To create a function space T polynomials of the first kind on the default domain $[-1, 1]$, do

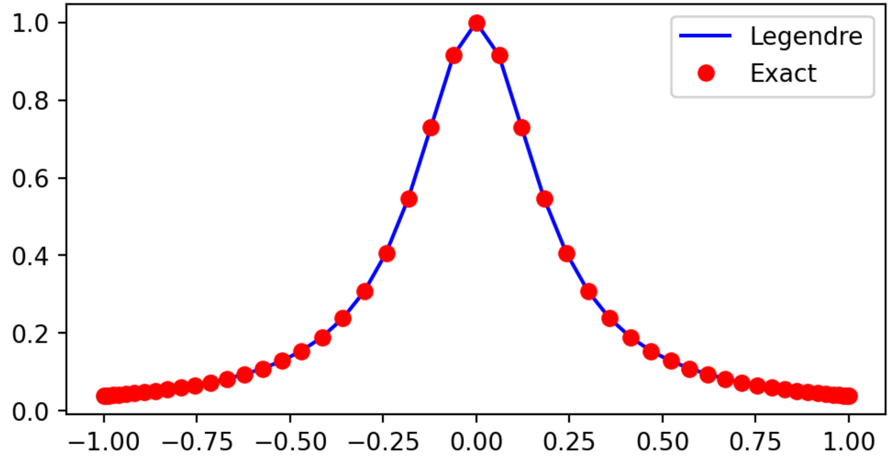
```
from shenfun import *
N = 8
```

Try Shenfun

Approximate with Legendre polynomials through Shenfun and the Galerkin method

$$u(x) = \frac{1}{1+25x^2} \quad x \in [-1, 1] \rightarrow \hat{u}_j = \frac{2j+1}{2}(u, P_j)$$

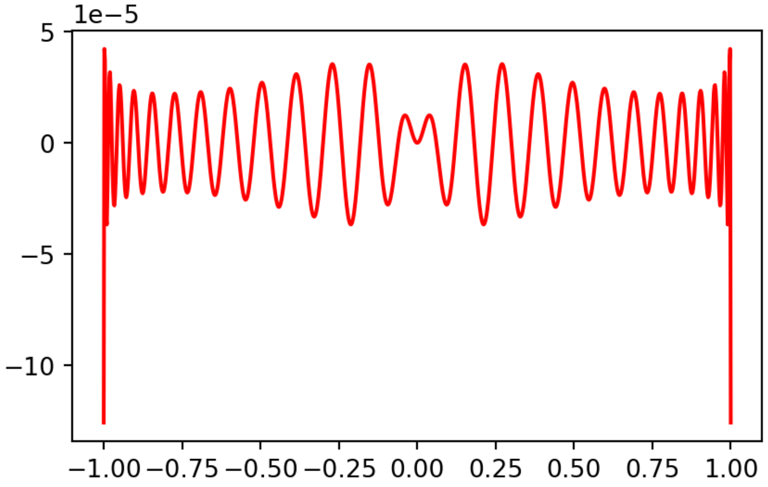
```python
import shenfun as sf
N = 50
ue = 1./(1+25*x**2)
V = sf.FunctionSpace(N+1, 'Legendre', domain=(-1, 1
v = sf.TestFunction(V)
uh = (2*np.arange(N+1)+1)/2*sf.inner(ue, v)
plt.figure(figsize=(6, 3))
plt.plot(V.mesh(), uh.backward(), 'b', V.mesh(), sp
plt.legend(['Legendre', 'Exact'])
```



Note the implementation. Choose FunctionSpace and compute Legendre coefficients using the inner product, with $v = P_j$ as a TestFunction for the function space V.

Plot the pointwise error

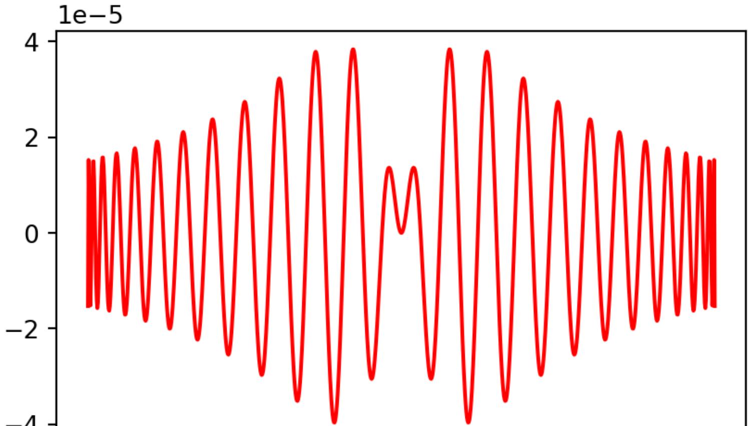
```
xj = np.linspace(-1, 1, 1000)
plt.figure(figsize=(5, 3))
plt.plot(xj, uh(xj)-sp.lambdify(x, ue)(xj), 'r')
plt.ticklabel_format(axis='y', style='sci',
                     scilimits=(-5,-5))
plt.gca().set_yticks([-1e-4, -5e-5, 0, 5e-5]);
```



Note the oscillation in the error that is typical of a spectral method.

Who thinks that Chebyshev can do better?

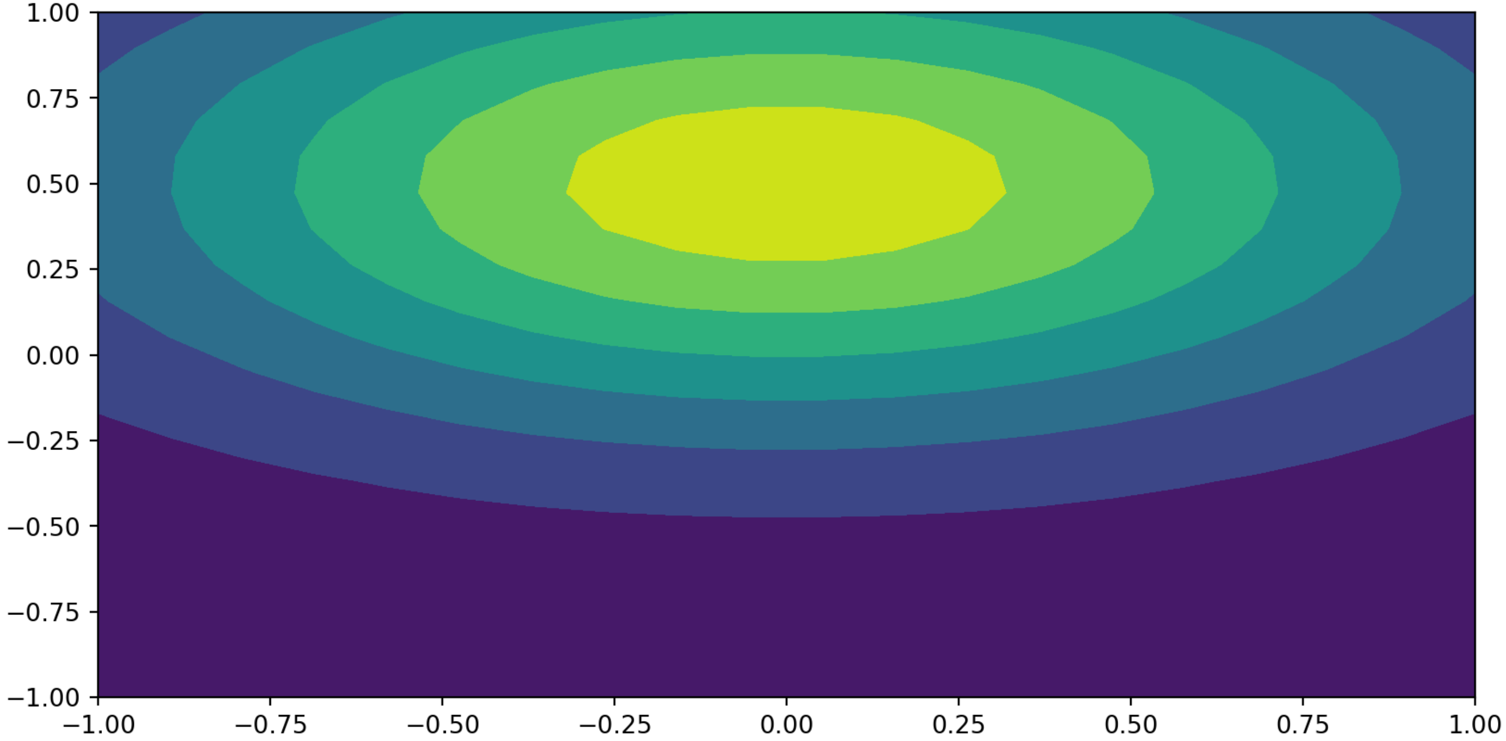
```
V = sf.FunctionSpace(N+1, 'Chebyshev',
                     domain=(-1, 1))
uhc = sf.project(ue, V)
plt.figure(figsize=(5, 3))
plt.plot(xj, uhc(xj)-sp.lambdify(x, ue)(xj), 'r');
```



Shenfun in 2D using tensor products

Approximate $u(x,y) = \exp(-(x^2+2(y-0.5)^2)), (x,y) \in [-1,1]^2$:

```
ue = sp.exp(-(x**2+2*(y-sp.S.Half)**2))
T = sf.FunctionSpace(20, 'Chebyshev')
W = sf.TensorProductSpace(sf.comm, (T, T))
uN = sf.project(ue, W) # projection is Galerkin approximation
xi, yj = W.local_mesh(True, kind='uniform')
plt.contourf(xi, yj, uN.backward(mesh='uniform'))
```



Fast Chebyshev transforms

One of the reasons why Chebyshev polynomials are so popular is the fact that you can transform fast between spectral and physical space

$$u_N(x_i) = \sum_{j=0}^{N} \hat{u}_j T_j(x_i), \quad i = 0, 1, \ldots, N$$

Here

$$\begin{aligned} \text{Physical points} \quad \boldsymbol{u} &= (u_N(x_i))_{i=0}^N \\ \text{Spectral points} \quad \boldsymbol{\hat{u}} &= (\hat{u}_i)_{i=0}^N \end{aligned}$$

Slow implementation: Using $\boldsymbol{T} = (T_j(x_i))_{i,j=0}^{N,N}$ we get

$$\boldsymbol{u} = \boldsymbol{T}\boldsymbol{\hat{u}}$$

which is computed in $(2N+1)(N+1)$ floating point operations, which scales as $\mathcal{O}(N^2)$.

Slow Chebyshev transforms implementation

```
N = 10
T = np.polynomial.chebyshev.chebvander(np.cos(np.arange(N)*np.pi/(N-1)), N)
uhat = np.random.random(N+1)
u = T @ uhat
print(u)
```

```
[ 6.76102216 -1.49078229  1.4606966  -0.46932894  0.95579288 -0.01623504
  0.58018842  0.71007566  0.06310243  1.67617835]
```

Slow because you use $\mathcal{O}(N^2)$ floating point operations and memory demanding because you need a matrix $\boldsymbol{T} \in \mathbb{R}^{(N+1)\times(N+1)}$.

Lets describe a faster way to compute $\boldsymbol{u} = (u_N(x_i))_{i=0}^N$ from $\hat{\boldsymbol{u}} = (\hat{u}_i)_{i=0}^N$

Fast cosine transform

Let

$$x_i = \cos(i\pi/N), \quad i = 0, 1, \ldots, N$$

such that for $i = 0, 1, \ldots, N$:

$$u_N(x_i) = \sum_{j=0}^{N} \hat{u}_j T_j(x_i)$$

$$u_N(x_i) = \sum_{j=0}^{N} \hat{u}_j \cos(ji\pi/N)$$

$$u_N(x_i) = \hat{u}_0 + (-1)^i \hat{u}_N + \sum_{j=1}^{N-1} \hat{u}_j \cos(ji\pi/N)$$

The discrete cosine transform

The discrete cosine transform of type 1 is defined to transform the real numbers $\boldsymbol{y} = (y_i)_{i=0}^{N}$ into $\boldsymbol{Y} = (Y_i)_{i=0}^{N}$ such that

$$Y_i = y_0 + (-1)^i y_N + 2 \sum_{j=1}^{N-1} y_j \cos(ij\pi/N), \quad i = 0, 1, \ldots, N$$

This operation can be evaluated in $\mathcal{O}(N \log_2 N)$ floating point operations, using the Fast Fourier Transform (FFT). Vectorized:

$$\boldsymbol{Y} = DCT^1(\boldsymbol{y})$$

The DCT is found in scipy and we will now use it to compute a fast Chebyshev transform.

Fast Chebyshev transform

We have the DCT^1 for any $\boldsymbol{Y}$ and $\boldsymbol{y}$

$$Y_i = y_0 + (-1)^i y_N + 2\sum_{j=1}^{N-1} y_j \cos(ij\pi/N), \quad i = 0, 1, \dots, N$$

We want to compute the following using the fast DCT^1

$$u_N(x_i) = \hat{u}_0 + (-1)^i \hat{u}_N + \sum_{j=1}^{N-1} \hat{u}_j \cos(ji\pi/N), \quad i = 0, 1, \dots, N \tag{1}$$

Rearrange (1) my multiplying by 2:

$$2u_N(x_i) - \hat{u}_0 - (-1)^i \hat{u}_N = \overbrace{\hat{u}_0 + (-1)^i \hat{u}_N + 2\sum_{j=1}^{N-1} \hat{u}_j \cos(ij\pi/N)}^{DCT^1(\hat{\boldsymbol{u}})_i}$$

$$u_N(x_i) = \frac{DCT^1(\hat{\boldsymbol{u}})_i + \hat{u}_0 + (-1)^i \hat{u}_N}{2}$$

Fast implementation

$$\boldsymbol{u} = \frac{DCT^1(\hat{\boldsymbol{u}}) + \hat{u}_0 + I_m \hat{u}_N}{2}$$

where

$$I_m = ((-1)^i)_{i=0}^N$$

```python
import scipy

def evaluate_cheb_1(uhat):
    N = len(uhat)
    uj = scipy.fft.dct(uhat, type=1)
    uj += uhat[0]
    uj[::2] += uhat[-1]
    uj[1::2] -= uhat[-1]
    uj *= 0.5
    return uj

N = 1000
xi = np.cos(np.arange(N+1)*np.pi/N)
T = np.polynomial.chebyshev.chebvander(xi, N)
uhat = np.ones(N+1)
uj = T @ uhat
uj_fast = evaluate_cheb_1(uhat)
assert np.allclose(uj, uj_fast)
```

Timing of regular transform:

```python
%timeit -q -o -n 10 uj = T @ uhat
```

```
<TimeitResult : 148 µs ± 30.2 µs per loop (mean ± std. dev. of 7 runs, 10 loops each)>
```

Timing of fast transform:

```python
%timeit -q -o -n 10 uj_fast = evaluate_cheb_1(uhat)
```

```
<TimeitResult : 19.5 µs ± 15.3 µs per loop (mean ± std. dev. of 7 runs, 10 loops each)>
```