Finite difference methods for the wave equation

MATMEK-4270

Prof. Mikael Mortensen, University of Oslo

The wave equation is a partial differential equation (PDE)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

We will consider the time and space domains: $t \in [0, T], x \in [0, L]$.

- The wave equation is an initial-boundary value problem!
- Two initial conditions required since two derivatives in time
- Two boundary conditions required since two derivatives in space
- The solutions are waves that can be written as $u(x + ct)$ and $u(x - ct)$

Wave solution with different boundary conditions

Boundary conditions

Dirichlet (Fixed end)

$$u(0,t) = u(L,t) = 0$$

The wave will be reflected, but u will change sign. A nonzero Dirichlet condition is also possible, but will not be considered here.

Neumann (Loose end)

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0$$

The wave will be reflected without change in sign. A nonzero Neumann condition is also possible, but will not be considered here.

Boundary conditions continued

Open boundary (No end)

$$\frac{\partial u(0,t)}{\partial t} - c\frac{\partial u(0,t)}{\partial x} = 0$$

$$\frac{\partial u(L,t)}{\partial t} + c\frac{\partial u(L,t)}{\partial x} = 0$$

The wave will simply pass undisturbed and unreflected through an open boundary.

Periodic boundary (No end)

$$u(0,t) = u(L,t)$$

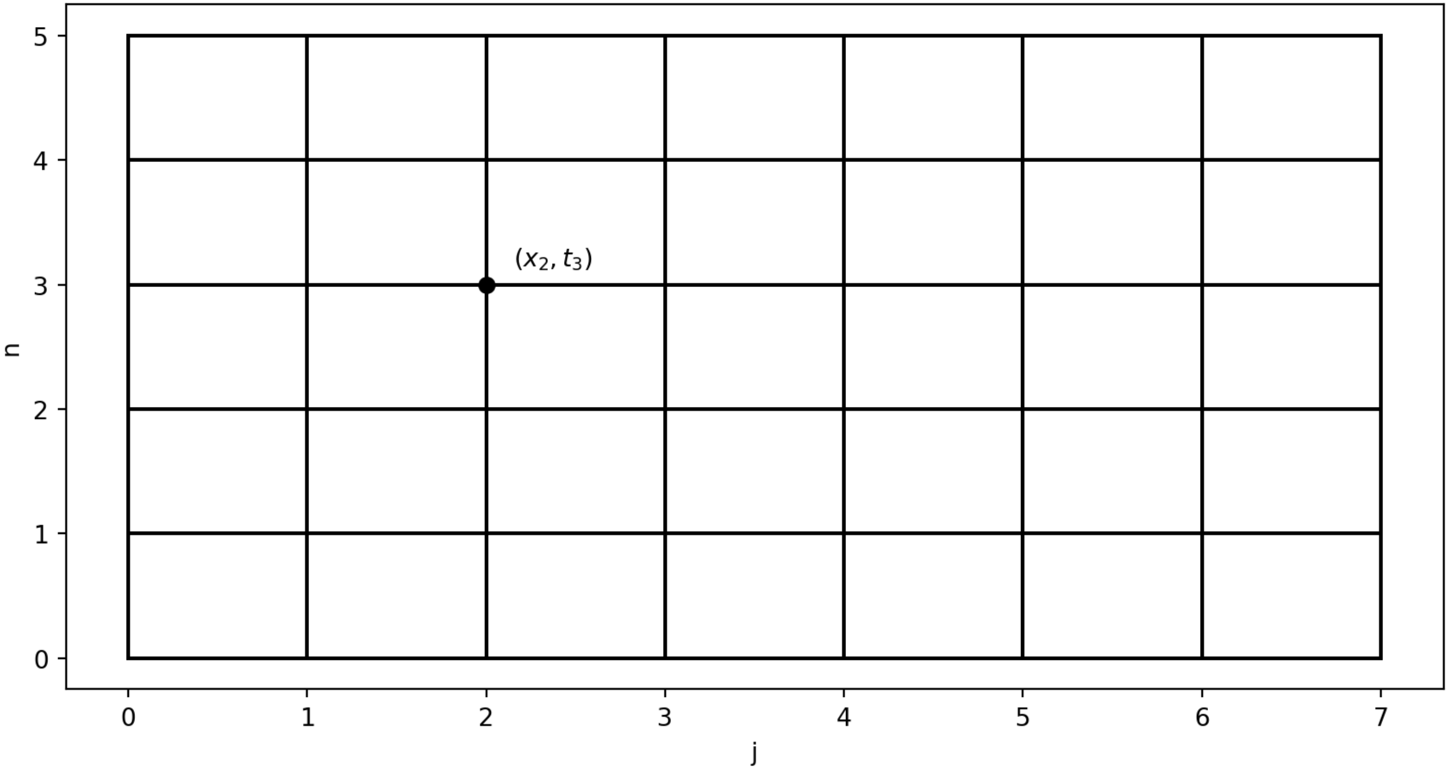
The solution repeats itself indefinitely.

Discretization

The simplest possible discretization is uniform in time and space

$$t_n = n\Delta t, \quad n = 0, 1, \ldots, N_t$$

$$x_j = j\Delta x, \quad j = 0, 1, \ldots, N$$



A mesh function in space and time is defined as

$$u_j^n = u(x_j, t_n)$$

The mesh function has one value at each node in the mesh. For simplicity in later algorithms we will use the vectors

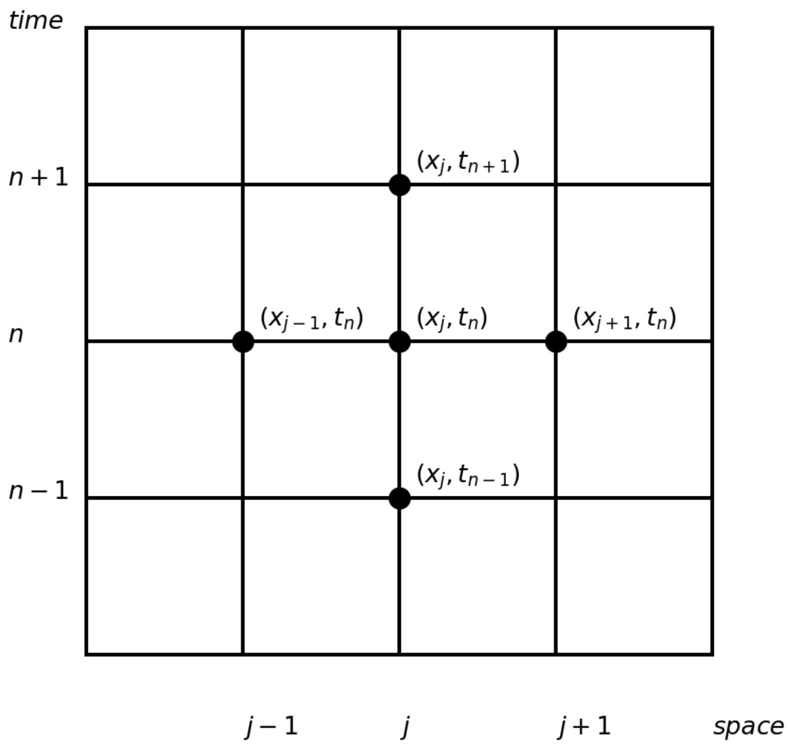
$$u^n = (u_0^n, u_1^n, \ldots, u_N^n)^T,$$

which is the solution vector at time t_n.

A second order accurate discretization of the wave equation is

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

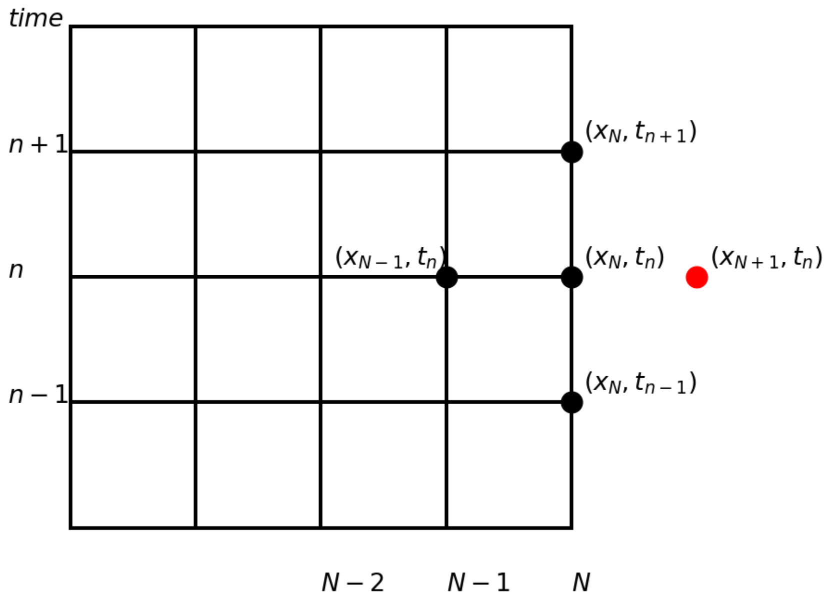
The finite difference stencil makes use of 5 neighboring points



$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

Can only be used for **internal points**

The finite difference stencil is not used at the spatial boundary



$$\frac{u_N^{n+1} - 2u_N^n + u_N^{n-1}}{\Delta t^2} = c^2 \frac{u_{N+1}^n - 2u_N^n + u_{N-1}^n}{\Delta x^2}$$

- Used at the boundary the regular stencil will contain a **ghost node**
- But at the boundary we use boundary conditions and do not solve the PDE!

We use a marching method in time

1. Initialize u^0 and u^1
2. for n in range($1, N_t - 1$):
   - for j in range($1, N - 1$):
     - $u_j^{n+1} = 2u_j^n - u_j^{n-1} + \left(\frac{c\Delta t}{\Delta x}\right)^2 (u_{j+1}^n - 2u_j^n + u_{j-1}^n)$

and apply the chosen boundary conditions.

All the indices makes it a bit messy. Lets make use of a differentiation matrix for the spatial dimension! And the Courant (or CFL) number

$$\overline{c} = \frac{c\Delta t}{\Delta x}$$

Use differentiation matrix to simplify the notation

We define the second differentiation matrix without the scaling $1/(\Delta x)^2$ such that

$$D^{(2)} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & & \cdots \\ \vdots & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

and thus row $0 < j < N$ of $D^{(2)}u^n$ becomes

$$(D^{(2)}u^n)_j = u^n_{j+1} - 2u^n_j + u^n_{j-1}$$

The vectorized marching method becomes

1. Initialize u^0 and u^1
2. for n in range($1, N_t - 1$):
   - $u^{n+1} = 2u^n - u^{n-1} + \underline{c}^2 D^{(2)} u^n$
   - Apply boundary conditions to u_0^{n+1} and u_N^{n+1}

- The boundary step can **often**, but not always, be incorporated into the matrix $D^{(2)}$
- Very easy to vectorize using the matrix vector product!

PDE solvers (of time-dependent problems) should use memory carefully

Note

- At any time we only need to store three vectors: u^{n+1}, u^n and u^{n-1}.
  - Memory requirement = $3(N+1)$ floating point numbers
- Storing all time steps requires $(N_t+1) \times (N+1)$ floating point numbers
- Not a huge problem for our case, but for 2 or 3 spatial dimensions it is very important!

Implementation - A low-memory marching method needs to update solution vectors

1. Allocate three vectors u^{nm1}, u^{n}, u^{np1}, representing u^{n-1}, u^{n}, u^{n+1}.
2. Initialize u^0 and u^1 by setting $u^{nm1} = u^0, u^n = u^1$
3. for n in range($1, N_t - 1$):
   - $u^{np1} = 2u^n - u^{nm1} + \underline{c}^2 D^{(2)} u^n$
   - Apply boundary conditions to u_0^{np1} and u_N^{np1}
   - Update to next iteration:
     - $u^{nm1} \leftarrow u^n$
     - $u^n \leftarrow u^{np1}$

In Python

Set up solver

```
import numpy as np
from scipy import sparse
import sympy as sp
x, t = sp.symbols('x,t')
N = 100
Nt = 500
L = 2
c = 1 # wavespeed
dx = L / N
CFL = 1.0
dt = CFL*dx/c
xj = np.linspace(0, L, N+1)
unm1, un, unp1 = np.zeros((3, N+1))
D2 = sparse.diags([1, -2, 1], [-1, 0, 1], (N+1, N+1))
u0 = sp.exp(-200*(x-L/2+t)**2)
```

Solve by marching method

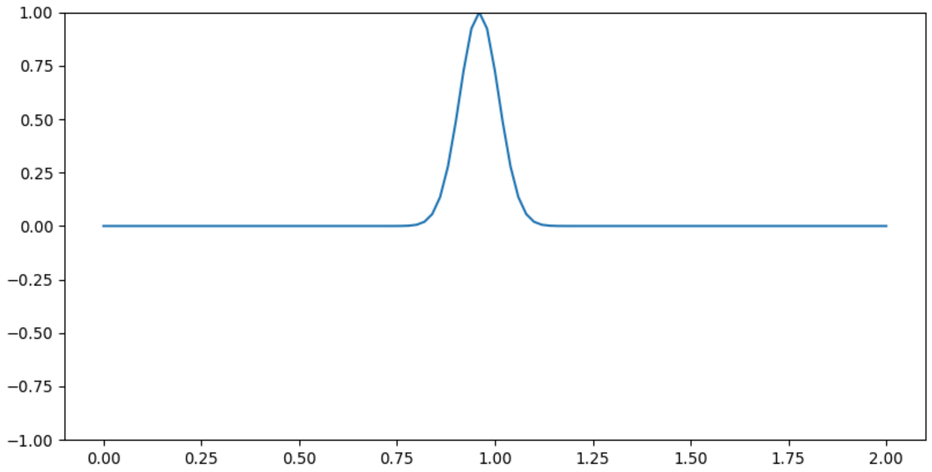
```
unm1[:] = sp.lambdify(x, u0.subs(t, 0))(xj)
un[:] = sp.lambdify(x, u0.subs(t, dt))(xj)
for n in range(Nt):
  unp1[:] = 2*un - unm1 + CFL**2 * D2 @ un
  unp1[0] = 0
  unp1[-1] = 0
  unm1[:] = un
  un[:] = unp1
```

Store results at intermediate intervals for plotting

```
unm1[:] = sp.lambdify(x, u0.subs(t, 0))(xj)
un[:] = sp.lambdify(x, u0.subs(t, dt))(xj)
plotdata = {0: unm1.copy()}
for n in range(Nt):
  unp1[:] = 2*un - unm1 + CFL**2 * D2 @ un
  unp1[0] = 0
  unp1[-1] = 0
  unm1[:] = un
  un[:] = unp1
  if n % 10 == 0:
    plotdata[n] = unp1.copy()
```

For example every tenth time step. Normally you do not need every time step to get a good animation.

Create animation after the simulation is finished




```python
def animation(data):
  from matplotlib import animation
  fig, ax = plt.subplots()
  v = np.array(list(data.values()))
  t = np.array(list(data.keys()))
  save_step = t[1]-t[0]
  line, = ax.plot(xj, data[0])
  ax.set_ylim(v.min(), v.max())
  def update(frame):
    line.set_ydata(data[frame*save_step])
    return (line,)
  ani = animation.FuncAnimation(fig=fig, func=update, frames=len(data), blit=True)
  ani.save('wavemovie.apng', writer='pillow', fps=5) # This animated png opens in a browser
```

How to implement the initial conditions?

To initialize a mesh function u^0, we write

$$u^0 = I(x)$$

which represents

$$u_j^0 = I(x_j), \quad \forall\, j = 0, 1, \ldots, N$$

```
u0 = sp.exp(-200*(x-L/2+t)**2)
unm1[:] = sp.lambdify(x, u0.subs(t, 0))(xj)
```

How about the second condition $\frac{\partial u}{\partial t}(x, 0) = 0$?

Just like for the vibration equation there are several options. If you have an analytical solution $I(x, t)$ that is time-dependent:

$$u^1 = I(x, \Delta t)$$

```
un[:] = sp.lambdify(x, u0.subs(t, dt))(xj)
```

If you do not have $I(x,t)$, then what?

How to fix $\frac{\partial u}{\partial t}(x,0) = 0$, option 1

Use a forward difference

$$\frac{\partial u}{\partial t}(x,0) \approx \frac{u^1 - u^0}{\Delta t} = 0, \quad \text{such that} \quad u^1 = u^0$$

Only first order accurate, but still a possibility.

Use a second order forward difference

$$\frac{\partial u}{\partial t}(x,0) \approx \frac{-u^2 + 4u^1 - 3u^0}{2\Delta t} = 0, \quad \text{such that} \quad u^1 = \frac{3u^0 + u^2}{4}$$

Second order accurate, but **implicit**

How to implement $\frac{\partial u}{\partial t}(x, 0) = 0$, option 2

Use a second order central differece

$$\frac{\partial u}{\partial t}(x, 0) = \frac{u^1 - u^{-1}}{2\Delta t} = 0, \quad \text{such that} \quad u^1 = u^{-1}$$

and the PDE at $n = 0$

$$u^1 = 2u^0 - u^{-1} + \underline{c}^2 D^{(2)} u^0$$

Insert for $u^{-1} = u^1$ to obtain

$$u^1 = u^0 + \frac{\underline{c}^2}{2} D^{(2)} u^0$$

Second order accurate and **explicit**

How to fix boundary conditions?

We will consider 4 different types of boundary conditions

Dirichlet	$u(0,t)$ and $u(L,t)$
Neumann	$\frac{\partial u}{\partial x}(0,t)$ and $\frac{\partial u}{\partial x}(L,t)$
Open	$\frac{\partial u}{\partial t}(0,t) - c\frac{\partial u}{\partial x}(0,t) = 0$ and $\frac{\partial u}{\partial t}(L,t) + c\frac{\partial u}{\partial x}(L,t) = 0$
Periodic	$u(L,t) = u(0,t)$

Note

Accounting for boundary conditions very often takes more than 50 % of the lines of code in a PDE solver!

Dirichlet boundary conditions

We need to fix $u(0, t) = I(0)$ and $u(L, t) = I(L)$ and start by fixing this at $t = 0$

$$u_0^0 = I(0) \quad \text{and} \quad u_N^0 = I(L)$$

Next, we compute

$$u^1 = u^0 + \frac{\underline{c}^2}{2} D^{(2)} u^0$$

Here, if the first and last rows of $D^{(2)}$ are set to zero, then $u_0^1 = u_0^0$ and $u_N^1 = u_N^0$.

Next, for $n = 1, 2, \ldots, N_t - 1$

$$u^{n+1} = 2u^n - u^{n-1} + \underline{c}^2 D^{(2)} u^n$$

Again, if the first and last rows of $D^{(2)}$ are zero, then $u_0^{n+1} = u_0^0$ and $u_N^{n+1} = u_N^0$ for all n. The boundary values remain as initially set at $t = 0$.

Dirichlet boundary conditions summary

Set $u^0 = I(x)$ and define a modified differentiation matrix

$$\tilde{D}^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & & \cdots \\ \vdots & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now boundary conditions will be ok at all time steps simply by:

1. Initialize u^0 and compute $u^1 = u^0 + \frac{\underline{c}^2}{2}\tilde{D}^{(2)}u^0$. Set $u^{nm1} = u^0, u^n = u^1$
2. for n in range($1, N_t - 1$):
   - $u^{np1} = 2u^n - u^{nm1} + \underline{c}^2\tilde{D}^{(2)}u^n$
   - Update to next iteration: $u^{nm1} = u^n; u^n = u^{np1}$

Dirichlet boundary conditions summary

Note

It is also possible to do nothing with $D^{(2)}$ and simply fix the boundary conditions after updating all the internal points

1. Initialize $u^{nm1} = u^0$ and compute $u^n = u^1 = u^0 + \frac{c^2}{2} D^{(2)} u^0$.
2. Set $u_0^{nm1} = u_0^n = 0$ and $u_N^{nm1} = u_N^n = 0$.
3. for n in range$(1, N_t - 1)$:
   - $u^{np1} = 2u^n - u^{nm1} + \underline{c}^2 D^{(2)} u^n$
   - Set $u_0^{np1} = 0$ and $u_N^{np1} = 0$
   - Update to next iteration: $u^{nm1} \leftarrow u^n; u^n \leftarrow u^{np1}$

Note

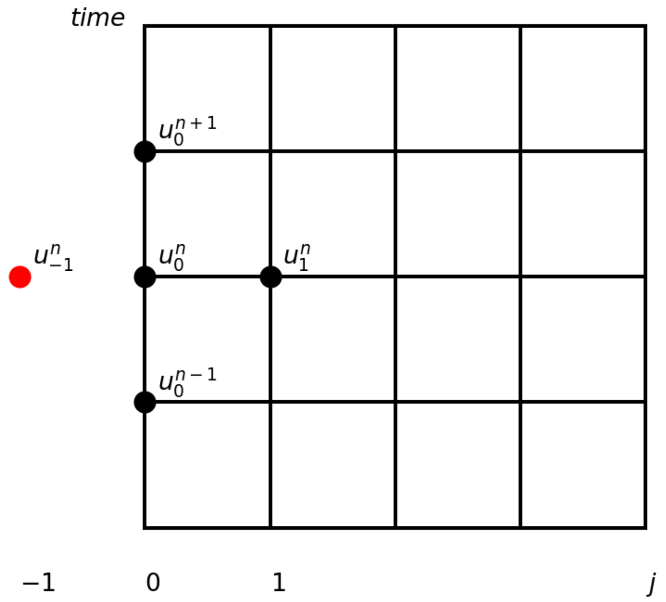
Regular, unmodified $D^{(2)}$, where the first and last rows are completely irrelevant.

Neumann boundary conditions

We need to fix $\frac{\partial u}{\partial x}(0, t) = 0$ and $\frac{\partial u}{\partial x}(L, t) = 0$. We already have $u^0 = I(x)$.

A second order central scheme at $x = 0$ is using ghost cell at $j = -1$

$$\frac{\partial u}{\partial x}(0, t_n) = \frac{u_1^n - u_{-1}^n}{2\Delta x} = 0 \rightarrow u_{-1}^n = u_1^n$$



Use the ghost cell and the PDE to fix the Neumann condition

The PDE at the left hand side $j = 0$ using ghost cell:

$$u_0^{n+1} = 2u_0^n - u_0^{n-1} + \underline{c}^2(u_1^n - 2u_0^n + u_{-1}^n)$$

Insert for $u_{-1}^n = u_1^n$ and obtain

$$u_0^{n+1} = 2u_0^n - u_0^{n-1} + \underline{c}^2(2u_1^n - 2u_0^n)$$

Note

Second order accurate and **explicit**. Can be implemented by modifying $D^{(2)}$!

Neumann at $x = L$ is the same

$$\frac{\partial u}{\partial x}(L, t_n) = \frac{u^n_{N+1} - u^n_{N-1}}{2\Delta x} = 0 \rightarrow u^n_{N+1} = u^n_{N-1}$$

The PDE at the right hand side $j = N$ using ghost cell:

$$u^{n+1}_N = 2u^n_N - u^{n-1}_N + \underline{c}^2(u^n_{N+1} - 2u^n_N + u^n_{N-1})$$

Insert for $u^n_{N+1} = u^n_{N-1}$ and obtain

$$u^{n+1}_N = 2u^n_N - u^{n-1}_N + \underline{c}^2(2u^n_{N-1} - 2u^n_N)$$

And for $n = 1$ we similarly get

$$u^1_0 = u^0_0 + \frac{\underline{c}^2}{2}(2u^n_1 - 2u^n_0) \quad \text{and} \quad u^1_N = u^0_N + \frac{\underline{c}^2}{2}(2u^n_{N-1} - 2u^n_N)$$

Neumann summary

Set $u^0 = I(x)$ and define a modified differentiation matrix

$$\tilde{D}^{(2)} = \begin{bmatrix} -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & & \cdots \\ \vdots & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Now boundary conditions will be ok at all time steps simply by:

1. Initialize u^0 and compute $u^1 = u^0 + \frac{c^2}{2}\tilde{D}^{(2)}u^0$. Set $u^{nm1} = u^0, u^n = u^1$
2. for n in range$(1, N_t - 1)$:
   - $u^{np1} = 2u^n - u^{nm1} + \underline{c}^2\tilde{D}^{(2)}u^n$
   - Update to next iteration: $u^{nm1} \leftarrow u^n; u^n \leftarrow u^{np1}$

Open boundary

The wave simply disappears through the boundary

$$\frac{\partial u}{\partial t}(0,t) - c\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial t}(L,t) + c\frac{\partial u}{\partial x}(L,t) = 0$$

As for Neumann there are several ways to implement these boundary conditions. The simplest option is to solve the first order accurate

$$\frac{u_0^{n+1} - u_0^n}{\Delta t} - c\frac{u_1^n - u_0^n}{\Delta x} = 0$$

such that

$$u_0^{n+1} = u_0^n + \frac{c\Delta t}{\Delta x}(u_1^n - u_0^n)$$

Second order option

$$\frac{u_0^{n+1} - u_0^{n-1}}{2\Delta t} - c\frac{-u_2^n + 4u_1^n - 3u_0^n}{2\Delta x} = 0$$

Solve for the boundary node u_0^{n+1}

$$u_0^{n+1} = u_0^{n-1} + \frac{c\Delta t}{\Delta x}(-u_2^n + 4u_1^n - 3u_0^n)$$

Nice option, but difficult to incorporate in the $D^{(2)}$ matrix, since there is no way to modify the first and last rows of $D^{(2)}$ such that

$$u_0^{n+1} = 2u_0^n - u_0^{n-1} + \underline{c}^2(D^{(2)}u^n)_0$$

Second second order option

Use central, second order scheme

$$\frac{u_0^{n+1} - u_0^{n-1}}{2\Delta t} - c\frac{u_1^n - u_{-1}^n}{2\Delta x} = 0$$

and isolate the ghost node u_{-1}^n:

$$\color{red}{u_{-1}^n} = u_1^n - \frac{1}{\underline{c}}(u_0^{n+1} - u_0^{n-1})$$

Use regular PDE at the boundary that includes the ghost node:

$$u_0^{n+1} = 2u_0^n - u_0^{n-1} + \underline{c}^2(u_1^n - 2u_0^n + \color{red}{u_{-1}^n})$$

This gives an equation for u_0^{n+1} that fixes the open boundary condition:

$$u_0^{n+1} = 2(1-\underline{c})u_0^n - \frac{1-\underline{c}}{1+\underline{c}}u_0^{n-1} + \frac{2\underline{c}^2}{1+\underline{c}}u_1^{n_1}$$

Open boundary conditions

Left boundary:

$$u_0^{n+1} = 2(1-\underline{c})u_0^n - \frac{1-\underline{c}}{1+\underline{c}}u_0^{n-1} + \frac{2\underline{c}^2}{1+\underline{c}}u_1^{n_1}$$

Right boundary:

$$u_N^{n+1} = 2(1-\underline{c})u_N^n - \frac{1-\underline{c}}{1+\underline{c}}u_N^{n-1} + \frac{2\underline{c}^2}{1+\underline{c}}u_{N-1}^{n_1}$$

Both **explicit** and second order. But not possible to implement into the matrix such that

$$u^{n+1} = 2u^n - u^{n-1} + \underline{c}^2 D^{(2)} u^n$$

Implementation open boundaries

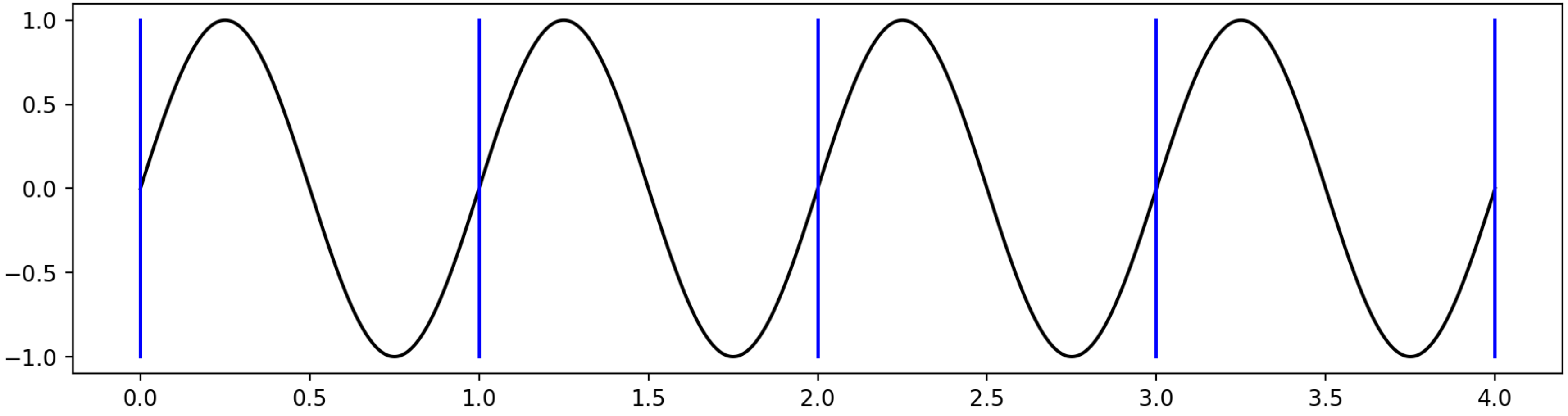
1. Initialize u^0 and compute $u^1 = u^0 + \frac{\underline{c}^2}{2} D^{(2)} u^0$. Set $u^{nm1} = u^0, u^n = u^1$
2. for n in range($1, N_t - 1$):
   - $u^{np1} = 2u^n - u^{nm1} + \underline{c}^2 D^{(2)} u^n$
   - $u_0^{np1} = 2(1 - \underline{c}) u_0^n - \frac{1-\underline{c}}{1+\underline{c}} u_0^{nm1} + \frac{2\underline{c}^2}{1+\underline{c}} u_1^{nm1}$
   - $u_N^{np1} = 2(1 - \underline{c}) u_N^n - \frac{1-\underline{c}}{1+\underline{c}} u_N^{nm1} + \frac{2\underline{c}^2}{1+\underline{c}} u_{N-1}^{nm1}$
   - Update to next iteration: $u^{nm1} \leftarrow u^n; u^n \leftarrow u^{np1}$

Note

There is no need to use a modified $D^{(2)}$. The two updates of u_0^{np1} and u_N^{np1} will overwrite anything computed in the first step.

Periodic boundary conditions

A periodic solution is a solution that is repeating itself indefinitely. For example $u(x) = \sin(2\pi x)$:

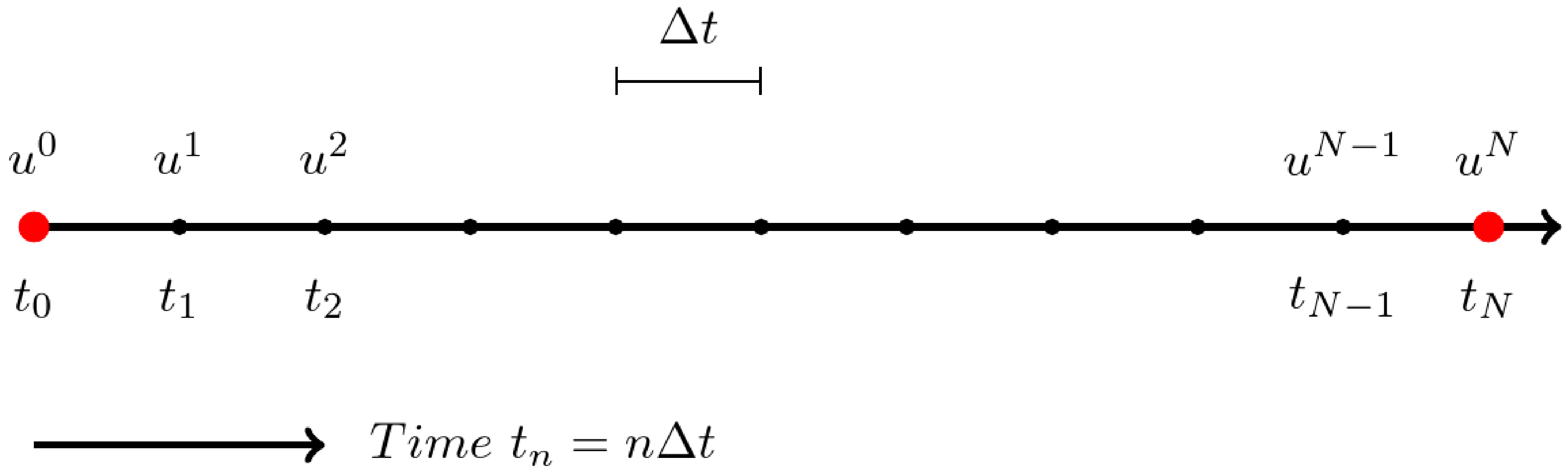


We solve the problem for example for $x \in [0, 1]$, but the actual solution will be like above, with no boundaries.

Note

A periodic domain is also referred to as a domain with no boundaries.

A periodic mesh in time

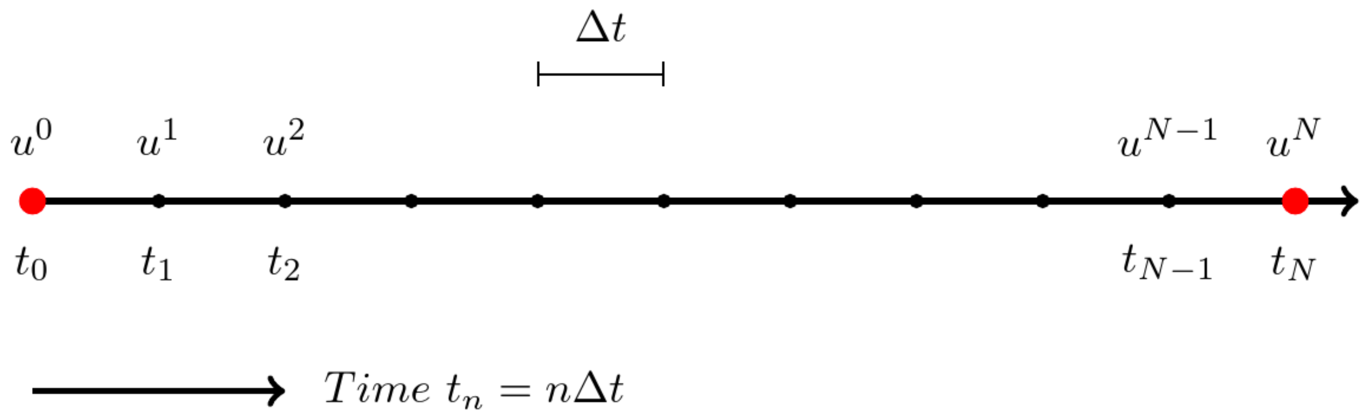


$$u(t_N) = u(0) \quad \text{or} \quad u^N = u^0$$

Note

There are only N unknowns $u^0, u^1, \ldots, u^{N-1}$ for a mesh with $N+1$ nodes.

Consider the discretization of u''



At the left hand side of the domain, the point to the left of u^0 is u^{N-1}

$$u''(0) \approx \frac{u^1 - 2u^0 + u^{-1}}{h^2} = \frac{u^1 - 2u^0 + u^{N-1}}{h^2}$$

At the right hand side of the domain the point to the right of u^{N-1} is $u^N = u^0$

$$u''(t_{N-1}) \approx \frac{u^N - 2u^{N-1} + u^{N-2}}{h^2} = \frac{u^0 - 2u^{N-1} + u^{N-2}}{h^2}$$

Periodic boundary conditions can be implemented in the matrix $D^{(2)} \in \mathbb{R}^{N+1\times N+1}$

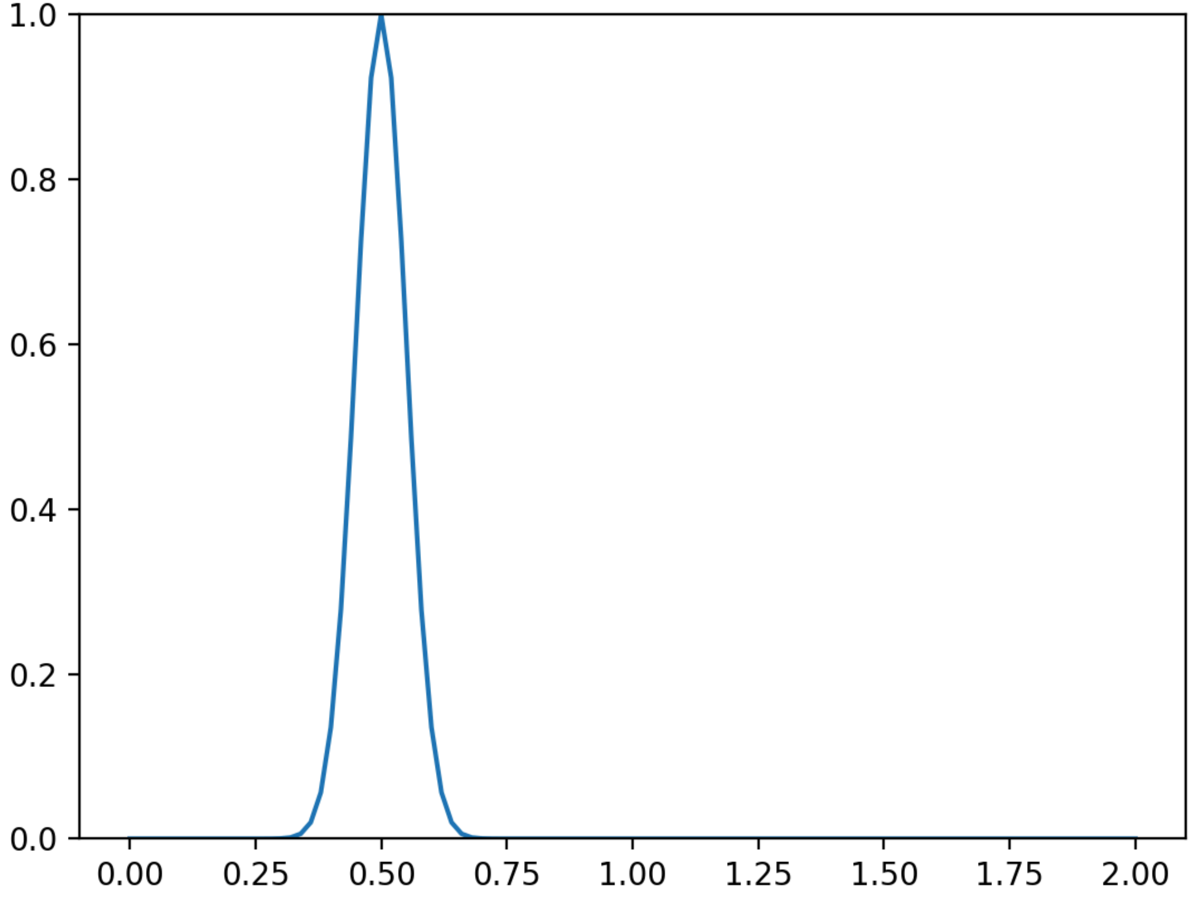
$$\tilde{D}^{(2)} = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & -2 & 1 & 0 & 0 & 0 & \cdots \\ \vdots & & & \ddots & & & & \cdots \\ \vdots & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ \vdots & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Note that the matrix expects $\boldsymbol{u} = (u^0, u^1, \ldots, u^{N-1}, u^N)$, even though $u^0 = u^N$. The last row in $\tilde{D}^{(2)}$ is thus irrelevant, because we wil set $u^0 = u^N$ manually.

Implementation periodic boundaries

1. Initialize u^0 and compute $u^1 = u^0 + \frac{\underline{c}^2}{2}\tilde{D}^{(2)}u^0$. Set $u^{nm1} = u^0, u^n = u^1$
2. for n in range(1, $N_t - 1$):
   - $u^{np1} = 2u^n - u^{nm1} + \underline{c}^2\tilde{D}^{(2)}u^n$
   - $u_N^{np1} = u_0^{np1}$
   - Update to next iteration: $u^{nm1} = u^n; u^n = u^{np1}$

Periodic wave



Properties of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

If the initial condition is $u(x,0) = I(x)$ and $\frac{\partial u}{\partial t}(x,0) = 0$, then the solution at $t > 0$ is

$$u(x,t) = \frac{1}{2}(I(x-ct) + I(x+ct))$$

These are two waves - one traveling to the left and the other traveling to the right

If the initial condition $I(x) = e^{ikx}$, then

$$u(x,t) = \frac{1}{2}\left(e^{ik(x-ct)} + e^{ik(x+ct)}\right)$$

is a solution

Representation of waves as complex exponentials

If the initial condition is a sum of waves (superposition, each wave is a solution of the wave equation)

$$I(x) = \sum_{k=0}^{K} a_k e^{ikx} = \sum_{k=0}^{K} a_k \left(\cos kx + i \sin kx\right)$$

for some K, then the solution is

$$u(x,t) = \frac{1}{2} \sum_{k=0}^{K} a_k \left(e^{ik(x-ct)} + e^{ik(x+ct)}\right)$$

We will analyze one component $e^{ik(x+ct)} = e^{ikx+\omega t}$, where $\omega = kc$ is the frequency in time. This is very similar to the investigation we did for the numerical frequency for the vibration equation.

Assume that the numerical solution is a complex wave

$$u(x_j, t_n) = u_j^n = e^{ik(x_j + \tilde{\omega} t_n)}$$

- How accurate is $\tilde{\omega}$ compared to the exact $\omega = kc$?
- What can be concluded about stability?

Note that the solution is a recurrence relation

$$u_j^n = e^{ikx_j} e^{i\tilde{\omega} n\Delta t} = (e^{i\tilde{\omega}\Delta t})^n e^{ikx_j}$$

with an amplification factor $A = e^{i\tilde{\omega}\Delta t}$ such that

$$u_j^n = A^n e^{ikx_j}$$

Numerical dispersion relation

We can find $\tilde{\omega}$ by inserting for $e^{ik(x_j+\tilde{\omega}t_n)}$ in the discretized wave equation

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

This is a lot of work, just like it was for the vibration equation. In the end we should get

$$\tilde{\omega} = \frac{2}{\Delta t} \sin^{-1}\left(C \sin\left(\frac{k\Delta x}{2}\right)\right)$$

where the CFL number is $C = \frac{c\Delta t}{\Delta x}$

- $\tilde{\omega}(k, c, \Delta x, \Delta t)$ is the numerical dispersion relation
- $\omega = kc$ is the exact dispersion relation
- We can compare the two to investigate numerical accuracy and stability

Stability

A simpler approach is to insert for $u_j^n = A^n e^{ikx_j}$ directly in

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

and solve for A. We get

$$\frac{\left(A^{n+1} - 2A^n + A^{n-1}\right)e^{ikx_j}}{\Delta t^2} = c^2 A^n \frac{e^{ik(x_j+\Delta x)} - 2e^{ikx_j} + e^{ik(x_j-\Delta x)}}{\Delta x^2}$$

Divide by $A^n e^{ikx_j}$, multiply by Δt^2 and use $C = c\Delta t/\Delta x$ to get

$$A - 2 + A^{-1} = C^2(e^{ik\Delta x} - 2 + e^{-ik\Delta x})$$

continue on next slide

Stability

$$A + A^{-1} = 2 + C^2(e^{ik\Delta x} - 2 + e^{-ik\Delta x})$$

Use $e^{ix} + e^{-ix} = 2\cos x$ to obtain

$$A + A^{-1} = 2 + 2C^2(\cos k\Delta x - 1)$$

This is a quadratic equation to solve for A. Using $\beta = 2(1 + C^2(\cos(k\Delta x) - 1))$ we get that

$$A = \frac{\beta \pm \sqrt{\beta^2 - 4}}{2}$$

We see that $|A| = 1$ for any real numbers $-2 \leq \beta \leq 2$.

For all real numbers $-2 \leq \beta \leq 2$

$$|\beta \pm \sqrt{\beta^2 - 4}| = 2$$

since $|\beta \pm \sqrt{\beta^2 - 4}| = |\beta + i\sqrt{4 - \beta^2}| = \sqrt{\beta^2 + 4 - \beta^2} = 2$

For $|A| \leq 1$ and stability we need $-2 \leq \beta \leq 2$ and thus

$$-2 \leq 2(1 + C^2(\cos(k\Delta x) - 1)) \leq 2$$

Rearrange to get that

$$-2 \leq C^2(\cos(k\Delta x) - 1) \leq 0$$

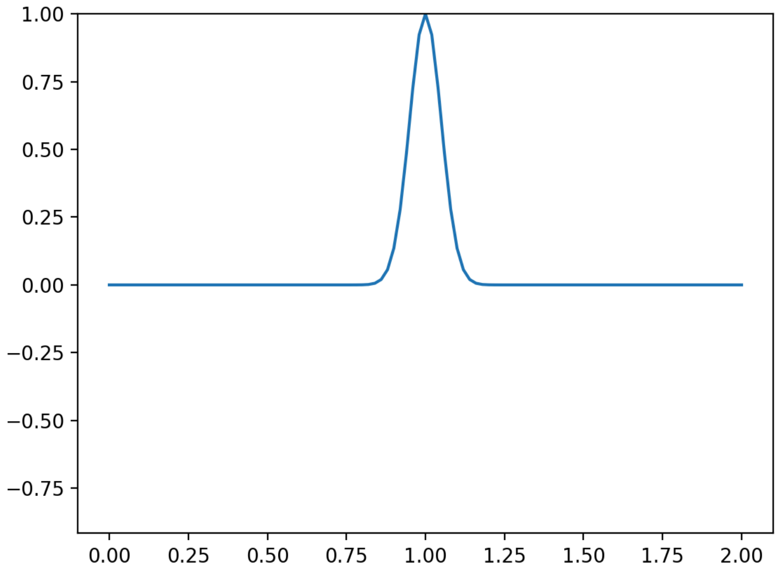
Since $\cos(k\Delta x)$ can at worst be -1 we get that the positive real CFL number must be smaller than 1

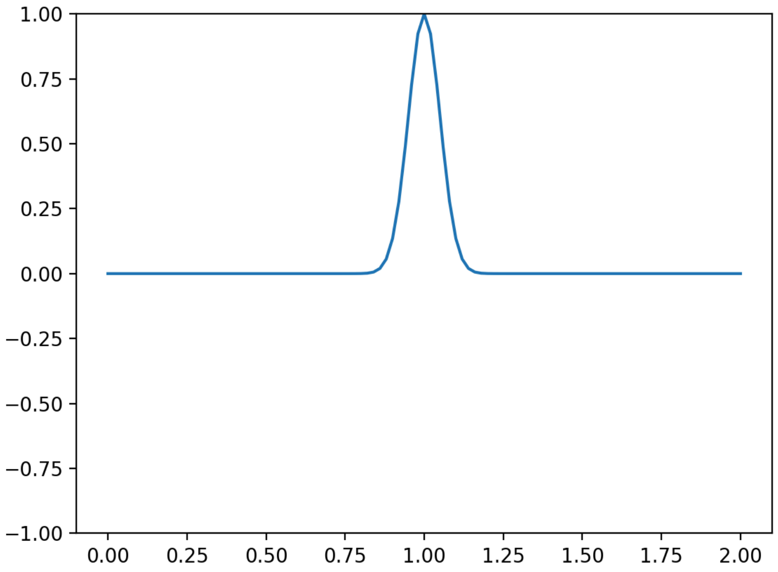
$$C \leq 1$$

Hence (since $C = c\Delta t/\Delta x$) for stability we require that

$$\Delta t \leq \frac{\Delta x}{c}$$

Test Dirichlet solver using CFL=1.01 vs CFL=1.0






```python
unm1[:] = sp.lambdify(x, u0.subs(t, 0))(xj)
un[:] = sp.lambdify(x, u0.subs(t, dt))(xj)
plotdata = {0: unm1.copy()}
CFL = 1.01
for n in range(Nt):
  unp1[:] = 2*un - unm1 + CFL**2 * D2 @ un
  unp1[0] = 0
  unp1[-1] = 0
  unm1[:] = un
  un[:] = unp1
  if n % 10 == 0:
    plotdata[n] = unp1.copy()
```