1 Example working (as a guideline of code/documentation to generate)

1.1 Problem definition

Coordinates: (x, t), so solution function is $u(x, t)$

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

Dirichlet on left boundary:

$$u(0, t) = 0$$

Dirichlet on bottom boundary:

$$u(x, 0) = e^{-(x-3)^2}$$

Domain: $x \in [0, 6]$, $t \in [0, 3]$

1.2 Solution method

1.2.1 Stencil

Use stencil with values: $u_{i-1,\,j}$ and $u_{i,\,j-1}$ with Taylor series expanded about $u_{i,\,j}$. The unknown in the stencil is $u_{i,\,j}$.

1.2.2 Approximations

Term	$u_{i-1,\,j}$	$u_{i,\,j}$	Desired
u	1	1	0
$\Delta x u_x$	-1	0	1

$$\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Solving this, we get $a = -1$, $b = 1$.

Method will be used on the computational domain, with coordinates (p, q), where nodes are spaced by 1 in each direction, so $\Delta x = 1$.

$$\frac{\partial u}{\partial p} = u_{i,\,j} - u_{i-1,\,j}$$

$$\frac{\partial u}{\partial q} = u_{i,\,j} - u_{i,\,j-1}$$

On a simple grid physical domain with computational domain axes being linearly scaled x and t axes:

$$\frac{\partial u}{\partial x} = \frac{1}{\Delta x}\frac{\partial u}{\partial p}$$

$$\frac{\partial u}{\partial t} = \frac{1}{\Delta t}\frac{\partial u}{\partial q}$$

Of course, $\frac{1}{\Delta x}$ and $\frac{1}{\Delta t}$ would be computed once and reused.

On a more complex physical domain with a coordinate transform applied:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial x}$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial t}$$

where the derivatives from the coordinate transforms would be previously calculated at each point in the mesh.

1.2.3 Substituting equation

With the simple computational domain:

$$\frac{1}{\Delta t}(u_{i,\,j} - u_{i,\,j-1}) + c\frac{1}{\Delta x}(u_{i,\,j} - u_{i-1,\,j}) = 0$$

Then solve for $u_{i,\,j}$:

$$(\frac{1}{\Delta t} + c\frac{1}{\Delta x})u_{i,\,j} = \frac{1}{\Delta t}u_{i,\,j-1} + c\frac{1}{\Delta x}u_{i-1,\,j}$$

$$u_{i,\,j} = \frac{1}{\frac{1}{\Delta t} + c\frac{1}{\Delta x}}\left(\frac{1}{\Delta t}u_{i,\,j-1} + c\frac{1}{\Delta x}u_{i-1,\,j}\right)$$

With the complex domain:

$$\frac{\partial u}{\partial p}\frac{\partial p}{\partial t} + \frac{\partial u}{\partial q}\frac{\partial q}{\partial t} + c\frac{\partial u}{\partial p}\frac{\partial p}{\partial x} + c\frac{\partial u}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\frac{\partial u}{\partial p}\left(\frac{\partial p}{\partial t} + c\frac{\partial p}{\partial x}\right) + \frac{\partial u}{\partial q}\left(\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x}\right) = 0$$

$$\left(\frac{\partial p}{\partial t} + c\frac{\partial p}{\partial x}\right)(u_{i,\,j} - u_{i-1,\,j}) + \left(\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x}\right)(u_{i,\,j} - u_{i,\,j-1}) = 0$$

$$\left(\frac{\partial p}{\partial t} + c\frac{\partial p}{\partial x} + \frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x}\right)u_{i,j} = \left(\frac{\partial p}{\partial t} + c\frac{\partial p}{\partial x}\right)u_{i-1,\,j} + \left(\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x}\right)u_{i,\,j-1}$$

$$u_{i,j} = \frac{1}{\frac{\partial p}{\partial t} + c\frac{\partial p}{\partial x} + \frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x}}\left(\left(\frac{\partial p}{\partial t} + c\frac{\partial p}{\partial x}\right)u_{i-1,\,j} + \left(\frac{\partial q}{\partial t} + c\frac{\partial q}{\partial x}\right)u_{i,\,j-1}\right)$$